The generalized Riemann hypothesis for Dirichlet L-functions

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Volume XXVII

Heights, Gram operators and divisors of projected theta periods

Abstract

We compare finite period ideals with complex norms of the signed columns. Gram operators of the first lifts, marked determinants and terminal Fourier coefficients enter the height and common-divisor estimates.

The guide in Volume I lists all twenty-seven volumes.

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141 Complex norms and finite period ideals

141.1 Testing a signed period on rational unitary frames

Fix x on one of the intervals of Section 6.6, and put $E = E_x$, $F = E^+$. Use the matched small endpoint, finite characters and finite class measure of Section 119.6. A small class a has an underlying unimodular Hermitian lattice M_a, stabilizer Γ_a, and a polynomial coefficient space at each member τ of the CM type. Choose an E-basis of the rational space giving a basis of M_a at every place above g_1, g_2, T. Such a basis exists. In any starting basis the condition at a finite place is an open condition on an invertible matrix, and weak approximation on the entries meets these finitely many conditions. Use the dual basis on the opposite coefficient line. These choices write an algebraic output at a as a flag polynomial Q_a times the fixed basis of its determinant line. We evaluate its polynomial part. Restoring a determinant character on a unitary frame multiplies the value by a unit at every place where the frame and its inverse are integral.

Lemma 141.1 (Simultaneous rational frames for the small classes)**.** *At each a there is a finite set $\mathcal{F}_a$ in $U(M_a \otimes E)(F)$, integral on M_a at all three marked rational primes, with the following property. At each of these primes p, and at every coefficient place $w \mid p$, its reductions in the CM coordinates range over all of $\prod_\tau \mathrm{GL}_n(\mathbb{F}_p)$. One set works for all polynomial weights on this Hermitian lattice.*

Proof. The primes split completely in E. For each $v \mid p$ of F the two factors of $E \otimes_F F_v$ are paired by the Hermitian form. An isometry is determined by its matrix on the first factor. On the other it acts by the inverse transpose in paired bases. Unimodularity makes the second lattice the full dual of the first, so every invertible integral matrix on the first preserves both. Choose one of these two factors at each v, and lift any list of invertible residue matrices to integral isometries there.

To meet a prescribed list of residue classes by a single rational isometry, let H be the Hermitian matrix over E in our basis and let $X^* = H^{-1}\bar{X}^{\mathsf{t}}H$. The transformation $X \mapsto (I - X)(I + X)^{-1}$ identifies open subsets of the F-vector space $X^* = -X$ and the unitary group. Its inverse is given by the same formula, wherever $I + g$ is invertible. Any of the chosen local isometries can be moved into this open subset within its residue class. Multiply it on the two paired factors by t, t^{-1} with $t \in 1 + p\mathbb{Z}_p$. Only finitely many t give eigenvalue -1. The inverse transforms of the resulting isometries have open neighborhoods which the transform carries into the prescribed classes. Weak approximation in an F-basis of the skew-Hermitian space meets all these neighborhoods at once. Its rational transform is defined, since its denominator is nonzero at the chosen completions. Taking one such lift of each list gives a finite set. Changing w permutes the factors and may change a chosen factor to its paired one. In integral bases inverse transpose and changes of basis induce bijections on the invertible residue matrices. This proves the assertion for every w. □

For the small weight μ_τ put $m_\tau = \mu_\tau - \mu_{\tau n}\mathbf{1}$ and $h = \max_\tau m_{\tau 1}$. Use the bilinear highest-vector normalization $B = \bigotimes_\tau B_{m_\tau}$. For $g \in \mathcal{F}_a$ let $l_{a,g}$ be the vector in the opposite coefficient fiber such that

$$B(Q, l_{a,g}) = Q((\tau g)_\tau)$$

in the chosen determinant frames. Concretely its polynomial at each embedding is $p_{m_\tau}((\tau g)^{\mathsf{t}} X_\tau)$, where p_m is the product of the leading minors. This is the polynomial identity of Lemma 120.4. Form a test $t_{a,g}$ supported on a by taking the sum of the Γ_a-translates of $l_{a,g}$, with the dual coefficient action and the inverse finite character. At the

other representatives set the value to zero and extend by equivariance. Only the coefficient argument of the test varies. The finite representative, its residue flags and its level character remain those of a.

Proposition 141.2 (Integral generation of the terminal tests)**.** *Let $w \mid p$ be a place of the coefficient field above one of the three marked primes. Suppose $p > h + n$ and $p \nmid |\Gamma_a|$ for each small class. Then over the valuation ring O_w the tests $t_{a,g}$ generate the entire integral small test lattice S_w. For every invariant output table Y, the period with $t_{a,g}$ is exactly $Y(a)((\tau g)_\tau)$ in these frames.*

Proof. In a single factor the dual Weyl lattice ∇_m consists of the flag polynomials with integral coefficients. The Weyl lattice Δ_m lies in it, by its construction from integral row substitutions. The form B_m on this polynomial lattice has monomial weights $\alpha!/H_m$, with H_m the hook product. Each hook is at most m_1+n-1. Thus B_m is integral on $\nabla_m \times \nabla_m$ at w. Lemma 120.4 identifies the dual of ∇_m with Δ_m, giving equality of the two lattices and a perfect pairing. The same holds on their tensor product with the paired determinant lines. In its polynomial realization this tensor product is saturated in the ambient coefficient lattice. Indeed each individual integral flag space is the intersection of a vector subspace with the monomial lattice, hence a direct summand over a discrete valuation ring. Tensoring the splittings proves the assertion.

Each $l_{a,g}$ is integral by its displayed polynomial and our choice of bases. Suppose its evaluations failed to detect some nonzero reduction of a vector in the paired lattice. By saturation the reduced polynomial Q of that vector is nonzero. Each matrix entry at τ has degree at most $m_{\tau 1}$, since its entire matrix column has degree at most $m_{\tau 1}$. The polynomial

$$Q(Y) \prod_{\tau} \det Y_\tau$$

is nonzero over the residue field of O_w, of degree at most $h + 1 < p$ in each variable. A polynomial of degrees strictly less than p in the separate variables and zero on the full grid $\mathbb{F}_p^d$ is zero, even with coefficients in an extension field. Successive use of the one-variable root bound proves this. Thus Q is nonzero on a tuple of invertible residue matrices. Lemma 141.1 supplies a frame giving that tuple. By perfect duality the reductions of the $l_{a,g}$ therefore span the opposite fiber, and Nakayama's lemma lifts this spanning statement to O_w.

The stabilizer trace maps that fiber onto its invariant submodule, since on an invariant vector it is multiplication by the unit $|\Gamma_a|$. Tracing the generators, then summing over the disjoint classes, proves generation of S_w. Contracting an invariant output with each term in the trace gives the same value $B(Y(a), l_{a,g})$ by duality of the character actions. There are $|\Gamma_a|$ terms and the class has measure $1/|\Gamma_a|$, proving the period identity. □

Theorem 141.3 (Determinantal ideals of the actual signed columns at the marked primes)**.** *At every sufficiently large x on the stated intervals, take any finite list of the prescribed signed theta columns with a common small testing space. For each column j let $Y_j = R_1 P_j$ be its actual lowered small table, including all earlier operations. Fix the finite central projector e of Theorem 123.15, allowing also $e = 1$. Write $Z_j = e^{\mathrm{t}} Y_j$, where the transpose is for the finite class pairing. At every $w \mid g_1 g_2 T$, and for every choice of s columns, the fractional ideal of the s-rowed period determinants tested on eS_w is generated by*

$$\det \left[Z_j(a_i)((\tau g_i)_\tau) \right]_{i,j=1}^{s}, \qquad a_i \text{ small classes}, \quad g_i \in \mathcal{F}_{a_i}.$$

It is also generated by the coefficient minors $\det[(Z_j(a_i))_{\alpha_i}]_{i,j=1}^{s}$, in the same polynomial frames, with one joint monomial α_i for each row. Both identities include zero ideals. There is no loss in valuation at any exterior degree in passing between these tests and coefficients.

Proof. The cutoff and stabilizer assertion in the proof of Theorem 127.1 give the hypotheses of Proposition 141.2 for all three primes on these fields and intervals. Indeed the small weight width is bounded by the large weight width by interlacing. The character values and the original column entries lie in the common coefficient field. The projector can have a denominator there. Its transpose exists over the field because the pairings on the opposite finite table spaces are perfect in characteristic zero, by the classwise nondegenerate pairing and stabilizer averaging. Since eS_w is generated by $et_{a,g}$, multilinearity of the determinant proves the first equality of ideals from $\langle Y_j, et_{a,g}\rangle = \langle Z_j, t_{a,g}\rangle$.

For fixed classes use independent matrix variables X_i in each of the s rows of $\det[Z_j(a_i)(X_i)]$. Its coefficients are exactly the asserted determinants, by expansion in those independent variables. Each evaluation is an integral combination of them. For a nonzero determinant polynomial divide it by a coefficient of least valuation. Its reduction is nonzero and its degree in any matrix entry is at most h, since any determinant term uses just one polynomial in each row. Multiplication by one determinant for every row and embedding allows the grid argument in the proposition to find a tuple of invertible residues giving a nonzero value. Choose the rational lifts separately in each row. The resulting determinant attains the least coefficient valuation. If the polynomial is zero, every evaluation and coefficient is zero. Taking all class lists proves the second equality. □

For the saturated projected tests $S_{0,w} = S_w \cap e(S_w \otimes \operatorname{Frac} O_w)$ the distinction between the two integral domains can likewise be kept exact. Let L send the standard generators of a free O_w-module to the tests $t_{a,g}$. Compute $K = \ker((1-e)L)$ over O_w, clearing any scalar denominator of $1-e$ before computing the kernel. Proposition 141.2 gives $L(K) = S_{0,w}$: lift any element of $S_{0,w}$ through L, and the lift belongs to K. Consequently a basis of K, applied as row combinations to the unprojected frame evaluations of Y_j, gives the period minors for $S_{0,w}$ by multilinearity. This computes both choices of integral domain on the given columns. Their minimum valuations still depend on the coefficients of the signed lifts. An estimate for those valuations and an identification with the analytic incoming row are required in Section 6.6.

141.2 A common set of tests for compact norms and marked period ideals

We work on the small definite space at a class a of Section 141.1. Put $\mathcal{S} = \{g_1, g_2, T\}$ and $G = \operatorname{Res}_{F/\mathbb{Q}} U(M_a \otimes E)$. Its real points form a compact group. Give this group Haar measure of total mass one. Each rational point of G gives a frame in the same Hermitian space over F. We shall use one finite set of such frames both for the integral periods and for integration on the compact group. For positive finite integration formulas over the real numbers, see Bayer and Teichmann [1].

Write V for the underlying rational vector space of M_a, and ρ for the action of G on V. In a rational basis let W_d be the space of restrictions to G of polynomials of total degree at most d in the entries of ρ. It is a finite-dimensional rational space of regular functions, unchanged by a rational change of basis. In this section $d \ge 1$.

Lemma 141.4 (The local denominator of compact integration)**.** *Normalized integration on W_d is a rational linear functional I. At a prime p split completely in E, let $K_p = \prod_{v\mid p} \mathrm{GL}_n(\mathbb{Z}_p)$ be the stabilizer of the underlying unimodular lattice. If $p > \max(2d+1, d+n)$, the extension of I to $\mathbb{Q}_p$ is integral on every function in $W_d \otimes \mathbb{Q}_p$ taking integral values on K_p.*

Proof. The rational points of G are dense in $G(\mathbb{R})$. Indeed the Cayley argument of Lemma 141.1 also works at every archimedean place of F. Multiplication by a central unit complex number sufficiently near one avoids eigenvalue -1 there. Weak approximation on the skew-Hermitian

coordinates can include these neighborhoods and any finite list of the open conditions in that lemma. The same Cayley chart shows that $G(\mathbb{R})$ is Zariski dense, since a polynomial vanishing on an open real coordinate box is identically zero.

The translates $R_g f(x) = f(xg)$ preserve the real space of functions $W_d \otimes \mathbb{R}$. With its Haar L^2 inner product, the orthogonal complement of the span of $R_g f - f$, for rational g, consists of the invariant functions. By continuity and density these are the constants. Thus I is the unique functional annihilating the translation differences with $I(1) = 1$. A rational basis and the rational translations give rational linear equations for it. Finitely many of their rows have the full rank, so the solution is rational. Uniqueness for this finite system persists over $\mathbb{Q}_p$.

In paired local bases write Y_v for the matrix on one of the two factors above v. The action on the other factor is its inverse transpose. After any fixed change of basis each entry of ρ is a linear combination of entries of these matrices and their inverse transposes. Write $\Delta(Y) = \prod_{v|p} \det Y_v$. Every $f \in W_d \otimes \mathbb{Q}_p$ has a unique polynomial numerator P_f in $f = \Delta^{-d} P_f$. Its degree in any entry is at most $2d$. Indeed an inverse entry contributes a cofactor divided by a determinant. In one factor, b inverse entries and c forward entries give at most $b + c + (d - b) = d + c$ in each variable after clearing the denominator. A function integral on K_p has P_f integral. Otherwise scale the numerator to a primitive integral polynomial by a positive power of p. Its reduction times Δ is nonzero, with every individual degree less than p. The grid argument of Proposition 141.2 yields an invertible residue tuple on which it is nonzero, contradicting integrality of the original function. The converse follows since Δ is a unit on K_p. The argument uses just one representative of each invertible residue tuple.

For a square variable Y put $\partial_Y = (\partial/\partial Y_{ij})_{ij}$ and $H = \prod_{j=0}^{n-1} (d + j)!/j!$. Then

$$I(f) = \left. \frac{\prod_{v|p} \det(\partial_{Y_v})^d P_f(Y)}{H^{[F:\mathbb{Q}]}} \right|_{Y=0}.$$

To verify the formula, first take $f = 1$. The Fischer squared norm of $(\det Y)^d$ is the hook product of the rectangle (d^n), namely H, by Lemma 92.4. For a right translation $Y_v \mapsto Y_v A_v$ the numerator polynomial is replaced by $P_f((Y_v A_v)_v)/\Delta(A)^d$. The chain rule multiplies its indicated determinant of derivatives by $\prod_v \det(A_v)^d$ at zero, cancelling the denominator. The formula therefore satisfies the same rational translation equations and normalization as I. Uniqueness proves it. The factorial ratios defining H are products of integers at most $d+n-1$. The differential operators have integral coefficients, so an integral numerator gives an integral answer. □

Theorem 141.5 (Positive rational integration weights at the three marked primes)**.** *Suppose each $p \in \mathcal{S}$ exceeds $\max(2d + 1, d + n)$. There is a finite set A_a of rational unitary frames integral at every $p \in \mathcal{S}$, containing the complete residue tests $\mathcal{F}_a$ of Lemma 141.1, and positive rational numbers $\lambda_{a,g}$ such that*

$$\sum_{g \in A_a} \lambda_{a,g} = 1, \qquad I(f) = \sum_{g \in A_a} \lambda_{a,g} f(g) \quad (f \in W_d).$$

Every $\lambda_{a,g}$ is a unit at every prime in $\mathcal{S}$. The same formula integrates W_d after complex scalar extension.

Proof. In $W_d^\vee \otimes \mathbb{R}$ write u_g for evaluation at g. The compact convex hull of the evaluations on $G(\mathbb{R})$ has affine span $u(1) = 1$. Indeed any further affine relation would give a nonconstant element of W_d equal to a constant on the group. Its barycenter I is in the relative interior.

Otherwise a supporting affine functional nonconstant on this hull would attain its maximum at its integral. Continuity and positive Haar mass on every nonempty open set exclude this.

There is a finite set of rational frames integral at the chosen primes whose evaluations have I in the interior of their convex hull relative to the same hyperplane. To see this, take points in the hull on both sides of I along each coordinate axis of that hyperplane, all at a common small distance. Each is a finite convex combination of evaluations. The resulting polytope contains a ball about I. Perturb the finitely many frames to rational frames integral at the chosen primes by the simultaneous approximation in the preceding proof. If all the evaluations move by less than half the radius, every supporting linear functional of norm one still has maximum strictly above its value at I. Separation proves the assertion. The zero-dimensional case requires just one frame. Adjoin $\mathcal{F}_a$ to this set and call the result B. Adjoin also a distinct frame g' for each $g \in B$ congruent to g at all the marked primes. Such frames exist by the same approximation with varying archimedean neighborhoods, since the unitary center has positive real dimension. Call the enlarged set A_a. There are strictly positive real coefficients on all of A_a representing I. In fact assign a sufficiently small positive mass at every point. The barycenter required of the remaining mass is still in the original interior neighborhood.

At each marked p, the functions integral on K_p form a full $\mathbb{Z}_p$-lattice Λ_p in $W_d \otimes \mathbb{Q}_p$. This follows by intersecting its numerator space with the finite monomial lattice. The evaluations from $\mathcal{F}_a$ span $\Lambda_p^\vee$ over $\mathbb{Z}_p$. Indeed their dual lattice is exactly Λ_p by the grid argument in Lemma 141.4, and double duality over a discrete valuation ring gives the assertion. That lemma puts I in their integral span. Write it as $\sum_{g\in B} c_g u_g$ with $c_g \in \mathbb{Z}_p$. Since $p > 2$, write each residue $\bar{c}_g$ as a sum of two nonzero residues: choose the first different from both zero and $\bar{c}_g$. Lift the two to units at g, g'. Their weighted evaluations reproduce I modulo $p\Lambda_p^\vee$, because integral numerators divided by Δ^d take the same residue at g and g'. The old evaluations span $\Lambda_p^\vee$, so changing their coefficients by elements of $p\mathbb{Z}_p$ removes the error. This gives an exact local solution with all coefficients units.

The equations on the coefficients are rational linear equations. Gaussian elimination makes their solution space a rational affine space, since they admit a real solution. Approximate the strictly positive solution at infinity and the constructed unit solutions at the three primes in its free rational coordinates. Positivity and being a local unit are open conditions. The resulting single rational solution gives the asserted weights. □

We now apply the integration formula to the signed columns $Y_j = R_1 P_j$ and $Z_j = e^{\mathrm{t}} Y_j$ of Theorem 141.3. For a small weight set

$$d = \max(1, 2\sum_{\tau} |m_\tau|), \qquad |m_\tau| = \sum_{i=1}^{n} m_{\tau i}.$$

One may take the largest such d for all the common small types in use at x. This is $O(x^C b)$ for a fixed C. Indeed the field degrees and ranks have polynomial bounds and each width is $O(xb + x^{C'})$ by Theorem 127.1. The bounds $g_i > (kb)^8$ and $\log T \gg m x^{\varepsilon}$, with $m \longrightarrow \infty$, put these same three primes above $\max(2d+1, d+n)$ for all sufficiently distant x.

For a complex embedding σ use the transported positive Hermitian form on the small fiber in the highest-vector Fischer normalization. Write r_μ for its complex dimension. We may and do choose the rational basis at each class to be orthonormal and integral at all three marks. Indeed the small definite group in Theorem 123.15 is the isometry group of the standard Hermitian space E^n, as in `Formalism/LocalToGlobal/Columns/StdLatUnimodular.lean`, line 29. Start in the standard rational orthonormal basis. At each split v choose a basis of the first factor of M_a and its paired dual basis in the second. The resulting basis in the product algebra is orthonormal, by unimodularity. A local unitary matrix takes the starting basis to this one.

Generating that same local lattice is an open condition on the matrix. The rational Cayley approximation used in Lemma 141.1 meets all these conditions at once. Its rational matrix is exactly unitary, so the resulting basis is orthonormal at every infinite place. In these frames the representing vector of evaluation at the identity is the tensor of the p_{m_τ}, of norm one by Lemma 120.4, with the unit determinant vector for the separate line. All the tests in what follows use these frames.

Theorem 141.6 (The definite Gram determinant on the same rational tests)**.** *Let G_σ be the Hermitian Gram matrix of s such projected signed columns in the small class metric with masses $1/|\Gamma_a|$. Put*

$$z_{a,g;j} = \langle Y_j, et_{a,g} \rangle, \qquad q_{a,g} = \frac{r_\mu \, \lambda_{a,g}}{|\Gamma_a|}.$$

For $g \in A_a$ use the stabilizer trace defining $t_{a,g}$ in Section 141.1. Then

$$\det G_\sigma = \sum_{\substack{B \subseteq \{(a,g) : g \in A_a\} \\ |B| = s}} \left(\prod_{(a,g) \in B} q_{a,g} \right) \left| \det [\sigma z_{a,g;j}]_{(a,g) \in B, j} \right|^2 .$$

All the weights are positive rational numbers, independent of σ and units at the three marks. At every marked coefficient place the minors in the sum generate the exact determinantal ideal on eS_w.

Proof. Fix a class and an embedding. The product of the small flag modules, meaning their external tensor product over the real places, is an irreducible unitary module of the product of the compact unitary groups. This follows from the irreducibility of each highest-weight representation and the tensor product rule for a direct product. Let l be its functional evaluating at the identity. The average over the group of the positive operators $(l \circ g)^*(l \circ g)$ commutes with the group. Its trace is $\|l\|^2 = 1$, so Schur's lemma makes it $1/r_\mu$ times the identity. Row substitution translates the functional at the identity into matrix evaluation. In the orthonormal bases the transpose and inverse conventions both preserve normalized Haar measure on $U(n)$. Removing the determinant character multiplies each translated functional by a scalar of absolute value one, which cancels in these products.

For vectors P, Q in this fiber, the product of the polynomial value $P(g)$ and the conjugate of $Q(g)$ belongs to $W_d \otimes \mathbb{C}$. Each entry of the complex matrix at a CM embedding is a complex linear combination of entries of $\rho(g)$, by extension of scalars in the restriction of scalars representation. Those real matrix entries are real on $G(\mathbb{R})$, so the same assertion holds for their complex conjugates. The product degree is at most $2 \sum_\tau |m_\tau|$. This also applies at every σ, which permutes the CM embeddings or selects their conjugates. Theorem 141.5 integrates each such product exactly with the same rational weights. Applying the preceding operator identity to the small values of the Z_j, and inserting the class masses, gives the Gram matrix of the array $(\sqrt{q_{a,g}}\, \sigma z_{a,g;j})$. Here the trace identity in Proposition 141.2 identifies each entry before taking absolute values. Cauchy–Binet gives the formula. Finally $\mathcal{F}_a \subseteq A_a$, and every additional frame is integral on the same lattices at the marked primes. Its trace test belongs to S_w, so applying e gives a test in eS_w. The two inclusions of the determinantal ideals follow from Theorem 141.3. The stabilizer orders are units at the marks by Theorem 127.1. The factors in the Weyl dimension formula for r_μ are $(m_{\tau i} - m_{\tau j} + j - i)/(j - i)$, $1 \le i < j \le n$. Both integers in each ratio lie strictly between zero and p at all three primes. The weights $\lambda_{a,g}$ are also units there by construction, proving the assertion about $q_{a,g}$. □

Corollary 141.7 (A simultaneous minor at three prescribed coefficient places)**.** *For each $p \in \mathcal{S}$ choose one coefficient place $w_p \mid p$. For fixed s columns of full rank on the projected tests, there*

are s semilocally integral tests before projection for which the same period determinant z, after projection, attains the least valuation at each of the three w_p. Set $P_{\mathcal{S}} = g_1 g_2 T$, and let M bound all $|\Gamma_a|$ as in Theorem 119.10. For every complex embedding one has

$$|\sigma z|^2 \le (9MP_{\mathcal{S}}^2)^s \det G_\sigma.$$

The logarithm of this comparison factor per exterior degree is $o(b/x^A)$ for every fixed $A > 0$ on the same intervals.

Proof. For w_p, Theorem 141.3 gives rows $t_{i,p} = t_{a,g}$ achieving its minimum. Choose rational integers $0 \le c_p < P_{\mathcal{S}}$ equal to one modulo p and zero modulo the other two primes, by the Chinese remainder theorem. Use $t_i = \sum_p c_p t_{i,p}$ before projection. At w_p the determinant expansion has one term of minimum valuation multiplied by c_p^s. Every other term has one coefficient divisible by p, and its period determinant has valuation at least the same minimum since its rows too are permissible. This proves simultaneous attainment.

At infinity the norm of the linear functional on a small class table given by $t_{a,g}$ is at most $\sqrt{|\Gamma_a|}$. In fact the trace and mass cancel on invariant values, evaluation on that fiber has norm one by unitarity, and taking its class value costs at most $\sqrt{|\Gamma_a|}$ in the mass norm. Thus each combined row functional has norm at most $3P_{\mathcal{S}}\sqrt{M}$. Apply their exterior product to $Z_1 \wedge \cdots \wedge Z_s$. Its norm is at most the product of their norms by the Gram determinant inequality, and the squared norm of the column wedge is $\det G_\sigma$. The upper bounds on the prime sizes in Theorems 7.11 and 9.13 make $\log P_{\mathcal{S}}$ polynomial in x, and Theorem 119.10 gives the same for $\log M$. The asserted comparison follows from $b = \exp(x^{\varepsilon})$. □

The primes in the corollary can be those of Theorem 22.12. Thus on these same E_x the critical-sign summands $e_{r+1}\mathcal{S}_{p,r}(E_x)$ vanish through that theorem's weight range, while the comparison above uses the same three primes for the same signed determinant. The rational frames can introduce denominators away from these primes. The formula determines the Gram matrix of the definite class tables. Comparison with the indefinite cuspidal period and a uniform estimate of the signed coefficients are still needed for Section 6.6.

141.3 All finite period ideals and the definite Gram determinant

Fix the fields, weights and class measures of Section 141.2. Work in a common finite Galois coefficient field $J \subset \mathbb{C}$ and write c for complex conjugation in this embedding. At $w \mid p$ use the valuation $v_w(p) = 1$. Let $Y_1, \dots, Y_s$ be the actual lowered signed columns for a common small type, and set $Z_i = e^{\mathrm{t}} Y_i$. The finite central projector e, which may be the identity, acts on the test side. On each class the bilinear period pairs opposite coefficient spaces by the highest-vector Fischer form with weight $1/|\Gamma_a|$. Denote this sum by $[Y, F]$.

For $D = eS$ or $D = S_0 = S \cap e(S \otimes J)$, define $\mathfrak{a}(D)$ to be the fractional ideal generated by $\det([Y_i, F_j])_{i,j=1}^s$ for $F_1, \dots, F_s \in D$. Use the abbreviations $\mathfrak{a}_{\mathrm{im}} = \mathfrak{a}(eS)$ and $\mathfrak{a}_0 = \mathfrak{a}(S_0)$. We assume the columns are independent on the projected tests, so these ideals are nonzero. They can be computed after localization on local generators of the corresponding lattices, by multilinearity and flatness. Write $b_{\mathrm{im},w} = v_w(\mathfrak{a}_{\mathrm{im}})$ and $b_{0,w} = v_w(\mathfrak{a}_0)$.

There is a conjugate-linear identification C from the algebraic column space to the opposite test space. In standard rational orthonormal coordinates conjugate the coefficients of the flag polynomials and use the dual unit vector of each determinant line. The bilinear Fischer contraction of P with CQ is the Hermitian form of the small fiber, linear in P. In particular

$$\mathcal{D} = \det\left([Z_i, CZ_j]\right)_{i,j=1}^s$$

is an element of J, strictly positive at the chosen complex embedding by the independence hypothesis. Here all finite characters have modulus one, so conjugation changes a column's stabilizer character to the inverse character of a test.

Lemma 141.8 (Conjugating a primitive exterior column into the test lattice)**.** *Fix w. In the direct sum of the class coefficient Weyl lattices at w, every exterior coordinate of $\xi = Z_1 \wedge \cdots \wedge Z_s$ has valuation at least $b_{\mathrm{im},w}$. At the conjugate place cw, conjugation carries the corresponding local Weyl lattice integrally into the opposite dual Weyl lattice at w. Consequently, if $t \in J_w^{\times}$ has $v_w(t) = b_{\mathrm{im},cw}$, then*

$$t^{-1} C\xi \ \in \ \bigwedge^{s} S_{0,w}.$$

Here C on an exterior power acts on every factor.

Proof. Choose a basis at w of the direct sum of the full coefficient lattices. Their integral duals for the classwise contraction are the opposite dual Weyl lattices by Lemma 120.4. They include the dual basis functionals. For any one such functional, take the stabilizer trace on each class, including its inverse level character. This gives an element of S_w. The class-mass pairing of this test with Z_i equals the value of the original functional on Z_i. Indeed each term of the trace has the same pairing on an invariant output, and the number of terms cancels the class mass. This calculation holds also when $p \mid |\Gamma_a|$, since only the unnormalized trace was used in the integral test. Pairing Z_i with a test in S_w equals pairing Y_i with its image under e. Thus every coordinate minor for ξ belongs to $\mathfrak{a}_{\mathrm{im}} O_w$.

For the integral effect of C, start with the Hermitian lattice M_a itself. In standard orthonormal coordinates its conjugate is its bilinear dual: integrality of $\sum_i x_i \overline{y}_i$ for all $x \in M_a$ is equivalent to $y \in M_a$, by unimodularity. After extension of scalars the same equality pairs the factor at cw with the opposite one at w. Thus conjugating an integral basis of a factor at cw gives an integral basis of the opposite lattice at w. The flag construction respects changes of basis in $\mathrm{GL}_n(O_w)$. In particular the leading-minor polynomial and its iterated integral row coefficients defining Δ_m have integral coefficients in the paired frames. Hence conjugation carries Δ_m into the opposite integral polynomial lattice ∇_m there. Exterior determinant lines of any integer power on the two sides pair integrally by the same unimodularity, using the dual line for a negative power. Tensoring and taking the direct sum gives the required integral map in every residue characteristic.

The central idempotent is self-adjoint for the Hermitian class metric. On a common full residue marking the group acts unitarily, and its central coefficients $\sum_{\chi} \chi(1)\chi(g^{-1})$ at g^{-1} are the conjugates of those at g. Restriction to the original level gives Theorem 123.15. In particular $Ce^{\mathsf{t}} = eC$, by the defining bilinear and Hermitian adjoint identities. It follows that $C\xi$ lies rationally in the exterior power of the projected test space. By the first part at cw and the integral map just proved, $t^{-1}C\xi$ belongs to the exterior power of the full ambient test lattice. Inside this lattice the submodule $S_{0,w}$ is the intersection with the rational projected test space. Indeed the invariant table is obtained by the rational stabilizer equations and intersection with the ambient lattice, and S_0 imposes the additional rational equation $eF = F$. This intersection is saturated and hence a direct summand over the valuation ring. A basis extending a basis of the summand shows that the same intersection equality holds in exterior degree s. This proves the assertion. □

Theorem 141.9 (An all-prime inequality for the actual projected definite periods)**.** *For the independent list above one has the fractional ideal memberships*

$$\mathcal{D} \ \in \ \mathfrak{a}_0 \, c(\mathfrak{a}_{\mathrm{im}}) \ \cap \ \mathfrak{a}_{\mathrm{im}} \, c(\mathfrak{a}_0).$$

For every embedding $\sigma : J \hookrightarrow \mathbb{C}$ let G_σ be the positive Gram matrix of $\sigma Z_1, \dots, \sigma Z_s$ in the class metric as in Theorem 141.6. With the absolute norm on fractional ideals, this gives

$$\log N\mathfrak{a}_0 + \log N\mathfrak{a}_{\mathrm{im}} \;\le\; \sum_{\sigma : J \hookrightarrow \mathbb{C}} \log\det G_\sigma .$$

In particular this comparison between finite valuations and complex Gram determinants includes every coefficient prime of the original definite columns, with the specified signs, level and class masses.

Proof. At w consider the alternating functional on s tests given by the determinant of the periods with $Y_1, \dots, Y_s$. On $\bigwedge^s S_{0,w}$ its image is $\mathfrak{a}_0 O_w$ by definition. The action of e on these tests is the identity, so using Z_i gives the same functional. At $C\xi$ its value is $\mathcal{D}$. Lemma 141.8 therefore gives

$$v_w(\mathcal{D}) \ge b_{0,w} + b_{\mathrm{im},cw}.$$

Since $c(\mathcal{D}) = \mathcal{D}$, applying this at the conjugate place gives the second membership. No choice of a single simultaneous generator of the local ideals is needed.

For an embedding σ write $\sigma' = \mathrm{conj}_{\mathbb{C}} \circ \sigma \circ c$. In standard rational orthonormal polynomial coordinates use the direct sum of the complex coefficient spaces over the class representatives, with the rational positive masses and normalized Fischer forms. In this ambient Hilbert space $\sigma\mathcal{D}$ is the Hermitian product of the wedges with coordinate arrays $\sigma\xi$ and $\sigma'\xi$. Indeed conjugating the latter coordinates gives exactly $\sigma(c\xi)$, while the monomial weights in the Fischer pairing and the masses are rational. In particular this computation does not require c to commute with each embedding. Cauchy–Schwarz gives

$$|\sigma\mathcal{D}| \le (\det G_\sigma \, \det G_{\sigma'})^{1/2}.$$

Both determinants are positive, since linear independence persists under the embeddings. The map $\sigma \mapsto \sigma'$ permutes the embeddings. Consequently the sum of their log determinant bounds is $\sum_\sigma \log\det G_\sigma$. At the finite places the first membership gives $\log N\mathfrak{a}_0 + \log N\mathfrak{a}_{\mathrm{im}} \le \log|N_{J/\mathbb{Q}}\mathcal{D}|$ by factorization of fractional ideals. Summing the archimedean inequality gives the result. □

Corollary 141.10 (Uniform cost of using saturated projected tests)**.** *Define $B(D) = (s[J : \mathbb{Q}])^{-1} \log N\mathfrak{a}(D)$ and*

$$H = \frac{1}{s[J : \mathbb{Q}]} \sum_{\sigma : J \hookrightarrow \mathbb{C}} \log\det G_\sigma .$$

On the original fields and intervals,

$$2B(eS) \le H, \qquad 2B(S_0) \le H + \log m_G,$$

where m_G is the order of the two marked finite block groups taken together. For $e = 1$ the latter cost can also be taken to be zero. On the prescribed intervals $\log m_G = o(b/x^A)$ for every fixed $A > 0$.

Proof. The inclusions $m_G eS \subseteq S_0 \subseteq eS$ from Theorem 123.15 give

$$m_G^s \mathfrak{a}_{\mathrm{im}} \subseteq \mathfrak{a}_0 \subseteq \mathfrak{a}_{\mathrm{im}}$$

by evaluating the determinant. Thus $0 \le B(S_0) - B(eS) \le \log m_G$, while the theorem gives $B(S_0) + B(eS) \le H$. These prove both inequalities. For $e = 1$ the two lattices agree. The logarithm of the finite group order is polynomially bounded in x by Theorem 123.15, which

proves the final estimate. In the local normalization above, $\log N\mathfrak{a}(D) = \sum_p \log p \sum_{w|p}[J_w : \mathbb{Q}_p]v_w(\mathfrak{a}(D))$. All three normalized quantities are unchanged on extending the coefficient field: localization and flat base change preserve the period ideals, the rational invariant subspaces and their integral intersections, while each complex embedding contributes with the same multiplicity. The period field is still E_x. □

The right side is the actual definite Gram quantity computed in Theorem 141.6. To use this result in Section 6.6 one still needs a lower valuation estimate encoding the prescribed arithmetic counts, and an upper estimate for these Gram determinants in the specified analytic row, on a family of sufficient rank. The complementary strict cohomology summands in Proposition 6.11 also enter the interval estimate.

141.4 The signal prime count on the signed definite columns

Fix a translate and the ordinary words of Theorem 9.13 on its field E_x. We work with the algebraic class pairing of Theorem 127.1, in the marked elliptic differential. The high set is $T_{\mathrm{hi}} \subset \Phi_0$. Write $n = |\Phi_1|$, $a = \lceil xb \rceil$, $b = \exp(x^{\varepsilon})$, $X' = \exp(x^{\varepsilon/3})$, $b' = xb/X'$. For $1 \le i \le m$, the count uses the weights $e_{i,\tau}$ following (9.1). At an unramified prime inert in K_d and split in K_q put

$$U_p = \sum_{i=1}^{m} \sum_{t \in T_{\mathrm{hi}}} \sum_{\substack{j \ge 1 \\ j \text{ odd}}} [a/p^j - e_{i,\pi^j t}]_+, \qquad \pi = \sigma_p c_E.$$

Its contribution to the uncentered count in this sign is $2U_p \log p$ by (66.1).

Lemma 141.11 (The first row at the actual word weights)**.** *On every sufficiently distant common interval, uniformly in the word, translate and edge, the partition ν_i of the i-th definite kernel satisfies*

$$G_p(\nu_i) \;\ge\; \sum_{t \in T_{\mathrm{hi}}} \sum_{j \text{ odd}} [a/p^j - e_{i,\pi^j t}]_+.$$

Here G_p is the integer in Theorem 120.12.

Proof. Write $d = t_{\mathrm{r}} + 1$, $r = k + (i-1)d$, $s = k + id$. At stage $i-1$ the first highest weight is $u_1 - (k-1)/2 - (i-1)d/2$. Indeed the initial algebraic weight subtracts the half-sum of positive roots, with first coordinate $(k-1)/2$, from u_1. At each step the input determinant shift is $e_{i,\tau} - s/2$ and the output subtracts $e_{i,\tau} - r/2$ from the zero-padded partition. Thus previous entries decrease by $d/2$ per step, and a new entry equals $-e_{i,\tau} + r/2$. These are the shifts in the prescribed polynomial substitution, given in `Formalism/LocalRingCardinality/Params.lean`, line 71. In particular the partition obtained by adding the input shift has first row

$$\nu_{i,\tau,1} = u_1(\tau) + e_{i,\tau} - (r + s - 1)/2.$$

In the polynomial model one has first subtracted the last incoming weight c to trivialize its determinant line. The determinant power in (66.22) is consequently $c + e_{i,\tau} - s/2$. At an edge after the first this is $2H_0 - d$, by the formula for a new entry. At the first edge it is nonnegative by the choices of B_τ, H_0 in (9.1), also at a high place with $c = -\lceil b' \rceil$. This identifies the partition in the operator.

At $t \in T_{\mathrm{hi}}$, $u_1 = a + (k-1)/2$ and $e_{i,t} \ge b'$. At $\mu \in \Phi_1$, u_1 is bounded by a fixed power of x. The ranks and n also have polynomial bounds. Thus $\nu_{i,t,1} \ge a + b'/2$ and $\nu_{i,\mu,1} \le e_{i,\mu} + Cx^B$ for fixed B, C. All $e_{i,\mu}$ on Φ_1 are at least $b/(nX')$. If $M = p^j$ contributes positively at t, then $M < a/e_{i,\pi^j t} \le 2nxX'$. Hence

$$\frac{b'}{2M} \;\ge\; \frac{b}{4n(X')^2} > s + Cx^B$$

eventually. The last inequality follows on taking logarithms, since $x^{\varepsilon} - 2x^{\varepsilon/3}$ dominates every constant multiple of $\log x$. As $d_M(t', s) \geq t'/M - s$, it follows for each active M that

$$d_M(\nu_{i,t,1}, s) - \nu_{i,\pi^j t,1} \;\geq\; a/M - e_{i,\pi^j t}.$$

The positive parts also give the claim for inactive M. Sum the $l = 1$ terms of G_p. Its other terms are nonnegative. □

Let $Y_1, \dots, Y_q$ be columns with this translate and a common small testing space, arising from the full integral initial Weyl tables, with the prescribed signs and coupled last edges. Use S_0, eS, m_G as in Theorem 119.10; here S_0 denotes the saturated projected test lattice. Let $\mathcal{S}$ be any finite set of the unramified primes of the indicated sign, $p \geq 5$, at which the finite class cosets in the path formula have prime-to-p shifts and the good local vacuum of Theorem 66.24. These requirements concern the supplied tests before applying any derivatives. For instance, in the presentation of Theorem 122.4 the primes prime to all its original shift orders Q_i^0 meet the shift condition. One can use representatives equal to the identity at these primes. The proof of that theorem chooses representatives supported at completely split primes outside any prescribed finite set. Thus at p the standard frames are integral frames on the common lattices of consecutive class maps.

We record separately the scalar coefficients that may occur if the finite tests have denominators. In a given column expand the finite coset sections in translates of the good common vacuum, keeping the matrix coset restrictions and the shared coupling label on the last two edges. For a path γ, including one summand of each such expansion, let a_γ be the product of its external scalar weights, its coefficients of translation, and its determinant-character scalars. Include any algebraic change of the marked scalar normalization in this product. Use integral determinant frames at p. Exclude the stabilizer masses and the terminal test projector from a_γ. Zero terms can be omitted. Define

$$L_{i,w} = \max\left(\{0\} \cup \{-v_w(a_\gamma) : \gamma \text{ in column } i\}\right), \qquad w \mid p.$$

These numbers use only the coefficients before applying the theta derivatives. In particular they vanish for integral coefficients. The expansion by normalized finite lattice intertwiners, including the reciprocal indices and the splitting characters, is given in the proof of Theorem 122.13. Scaling an additive differential by c_τ scales a kernel of block degree $|\nu_\tau|$ by $c_\tau^{|\nu_\tau|}$. This explains how to include an algebraic differential rescaling if a different normalization of that line is used.

For each of the m source vertices of an edge, and once for the small endpoint, take $M_v = |\mathrm{GL}_{n_v[E_x:\mathbb{Q}]}(\mathbb{F}_3)|$, where n_v is its rank. Set $D = \prod_v M_v$. In the definitions of $\mathfrak{a}_0, \mathfrak{a}_{\mathrm{im}}$ in Section 141.3 use these q columns. Their valuations are denoted by $b_{0,w}, b_{\mathrm{im},w}$, allowing $+\infty$ for a zero ideal.

Theorem 141.12 (The good signal primes on both projected test lattices)**.** *At each $p \in \mathcal{S}$ and each $w \mid p$, one has*

$$b_{0,w} \geq qU_p - \sum_{i=1}^{q} L_{i,w} - qv_p(D),$$

$$b_{\mathrm{im},w} \geq qU_p - \sum_{i=1}^{q} L_{i,w} - qv_p(Dm_G).$$

For the identity projection one takes $m_G = 1$. The logarithm of Dm_G is $o(b/x^A)$ for every fixed $A > 0$ on the same intervals.

Proof. At an unrefined initial class, any nonzero terminal covector tuple on a path gives an allowed refined input on its Weyl lattice, as in Theorem 73.3. The lattice modification made at T localizes to the original lattice at p since T splits in K_d. Starting at this input, apply the coefficient map on each matrix coset. In the marked differential the rule is to twist the source polynomial by the determinant power, substitute $D^{\mathrm{t}}Y_{[r]}$, and replace D^{α} by the undivided derivative of the corresponding CM section. For a finite coset indicator this is the same rule on its finite sum of translated vacua, by the linearity of differentiation. Its scalar factors are exactly the external factors recorded above. This describes (66.22) at consecutive vertices and the coefficient maps in Theorem 73.1. In the archive the substitution and evaluation are written in `Formalism/LocalToGlobal/KernelLattices/ThDistC.lean`, line 27.

At w, each prime-to-p translated product with its coefficient removed meets the local germ hypothesis. Its ratio to the reference vacuum is an integral elliptic germ by Corollary 65.28 and its proof; this also follows directly by multiplying the individual translation ratios. Theorem 120.12 puts the output in the Weyl lattice times $p^{G_p(\nu_j)}$ at edge j. Changes between two integral frames preserve this lattice. Indeed the Fischer adjoint is transpose substitution on the integral polynomial dual, which is preserved by every integral invertible change of variables. Determinant lines are changed by units. Induction on the edges and Lemma 141.11 give a valuation of at least U_p in every terminal Weyl coordinate before inserting the scalar weights and masses. The inequality involving a real U_p means that each coordinate has at least that valuation. It follows from the integer exponent $\sum_j G_p(\nu_j)$.

Now use the path identity of Theorem 73.1. It sums only paths with the specified initial covector, and keeps each coupling label common to both incident maps. On each path the original scalar product costs at most $L_{i,w}$. Its stabilizer order at each vertex divides M_v, by the faithful integral action and torsion-free level-three kernel used in Theorem 119.10. Clearing D thus clears every path mass, including the endpoint. Summing with the original signs preserves the common bound. The terminal restriction acts integrally on Weyl lattices in matched frames by Theorem 120.6 and Proposition 104.34. The contraction with an integral small test is integral by Fischer duality. This proves the bound for a single entry with tests in S_0. For tests in eS, multiplication by m_G puts every test in the integral small lattice, by the finite character sum for the projector. In each term of a q-row determinant every column occurs exactly once. Adding the entrywise bounds and then taking the ideal of all such determinants proves both inequalities. Finally $\log|\mathrm{GL}_d(\mathbb{F}_3)| \le d^2\log 3$; all vertex ranks, their number and the field degree are polynomial in x. The same polynomial bound for $\log m_G$ is given by Theorem 119.10. □

Corollary 141.13 (The good-prime count and the Gram determinant of the same columns)**.** *Suppose the chosen columns have rank q on the projected tests. Write $\omega_w = [J_w : \mathbb{Q}_p]/[J : \mathbb{Q}]$ and set*

$$L_{\mathcal{S}} = \sum_{p\in\mathcal{S}} \log p \sum_{w\mid p} \omega_w \sum_{i=1}^{q} L_{i,w},$$
$$B_{\mathrm{out}} = \sum_{p\notin\mathcal{S}} \log p \sum_{w\mid p} \omega_w \bigl([-b_{0,w}]_+ + [-b_{\mathrm{im},w}]_+\bigr).$$

Then

$$2\sum_{p\in\mathcal{S}} U_p \log p \;\le\; \frac{\sum_\sigma \log\det G_\sigma}{q[J:\mathbb{Q}]} + \frac{2L_{\mathcal{S}} + B_{\mathrm{out}}}{q} + 2\log D + \log m_G.$$

Proof. For a nonzero fractional ideal $\mathfrak{a}$, its normalized log norm is $\sum_p \log p \sum_{w\mid p} \omega_w v_w(\mathfrak{a})$. This follows from the residue degrees in the ideal norm and $v_w(p) = 1$. Sum both inequalities of Theorem 141.12 with these weights on $\mathcal{S}$; their sum on the other primes is bounded below by

$-B_{\mathrm{out}}$. Theorem 141.9 bounds the sum of the two normalized ideal log norms above by the mean of the log Gram determinants. Divide by q and use $\sum_{w|p}\omega_w = 1$. □

This compares the signal-prime contribution and the complex Gram determinant for one set of actual definite columns. The rank assumption, the costs $L_{\mathcal{S}}, B_{\mathrm{out}}$, and the size of that Gram determinant enter any application to (6.3).

141.5 All active signal primes for the prescribed finite tests

Work with one K_d-oriented field and translate of Theorem 9.22. Let $\mathcal{P}$ consist of the unramified $p \geq 5$ split in K_q, inert in K_d, for which $U_p > 0$. This is a finite set since $p \leq 2nxX'$. Use the marked elliptic differential and the elementary finite tests of the definite column in Theorem 73.1. These tests include the row and flag conditions, the terminal covector condition, and the common vector coset on the last two edges. Stabilizer masses and the small test projector have the normalizations of Theorem 141.12.

Lemma 141.14 (Support and coefficients of the finite tests)**.** *At each $p \in \mathcal{P}$ the elementary finite tests have the standard local lattice vector. On every edge there is a single choice of product basis and class representatives for which their expansions into elliptic products have prime-to-p shifts and coefficients integral above p, simultaneously for $p \in \mathcal{P}$. This holds for each allowed auxiliary character, each matrix coset, and each value of the common vector coset. The coefficients here precede insertion of the stabilizer masses, the small test projector, and any additional scalar linear combination of elementary columns.*

Proof. Write $F = E^+$. At a place v of F an allowed auxiliary character is trivial on all local units unless v splits in E/F, its norm is a rational prime ℓ, and $\ell \notin v^2$. These are the defining conductor restrictions in `Formalism/LocalToGlobal/Divisibility/Tuples.lean`, line 59–66 and `Formalism/LocalToGlobal/Divisibility/Tuples.lean`, line 110–118. The norm condition gives residue degree one and the valuation condition gives ramification index one. At either place of E over a split v the completion is therefore $\mathbb{Q}_\ell$. Since $E/\mathbb{Q}$ is Galois, this forces ℓ to split completely in E. The two flag primes g_1, g_2 and the word's terminal prime T also split completely by construction. Let S_{row} be the selected places at the conductor of the auxiliary character and at g_1, g_2. Every prime below S_{row} or T is split in K_d.

To check that this list includes every modification of the finite vector, consider the matrix test in its position coordinates. At the row places it is a function on lattice cosets. The two integer bounds on denominators and moduli are

$$c_A = \Big(\prod_{P \in S_{\mathrm{row}}} NP\Big)^A, \qquad c_B = T\Big(\prod_{P \in S_{\mathrm{row}}} NP\Big)^B.$$

Away from their support the position lattice and its congruence sublattice agree. At T the extra test is the indicator of the linear congruence on the terminal covectors. On the last two edges the shared variable Y is a vector coset with conditions only at S_{row}. In its Fourier transform the normal covector is multiplied by π_T^{-1}, where $\pi_T\overline{\pi_T} = T$. Hence it too is integral at every $p \in \mathcal{P}$. These are the tests written in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1110–1139 and `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1150–1170. Keeping Y fixed throughout this calculation preserves the coupling. At $p \in \mathcal{P}$ the matrix lattice is consequently the original unramified Hom lattice, with the reference vacuum.

We check the scalar weights as well. At a modified place of residue characteristic ℓ, the flag and block tests take the value zero or a product of residue-field characters, their inverses, and

an additive character of a finite quotient. Every nonzero such character value is a root of unity. The free entries in the unipotent integral are summed over lattice cosets, each with measure a power of ℓ. The Fourier transform of Y has the same form, with a power of ℓ from the coset volume. The determinant changes of variables contribute powers of $(NP)^{1/2}$ and values of the splitting character at ideles supported over ℓ. For the latter, raise the idele to a ray class power to write it as a principal idele times a unit idele on which the finite character is trivial. This can be done since the quotient of the ideal class group with the finite unit conditions is finite. The principal generator is a unit at every prime over p. The algebraic infinity type, after squaring to clear the half weights, is a monomial in its conjugates with integral exponents. Every factor is a p-unit because $E/\mathbb{Q}$ permutes the primes above p. This proves the same unit assertion for the original character value.

There is also a ratio of two compact open volumes on the unipotent radical in the row integral. In a basis adapted to its flag this group acts by upper unitriangular matrices. Conjugation by a diagonal matrix of sufficiently separated uniformizer powers puts both compact subgroups in the group of integral upper unitriangular matrices. Every congruence quotient of this ambient compact group has order a power of ℓ, by counting its strict upper entries. A sufficiently deep congruence kernel intersects each subgroup inside their common intersection, by openness in the radical. The two indices over that intersection are therefore powers of ℓ. This proves the assertion also for the volume ratio and its inverse. The change of variables, finite additive sum, reciprocal radical volume, and terminal Fourier transform occur respectively in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 552–586, `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 941–950 and `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1150–1155. All these factors are integral above p.

It remains to pass from cosets to the vacuum expansion. At a split place let L be the position lattice of the vacuum and refine a coset to $z + L'$ with $L' \subset L$ an open sublattice. On $z + L$ its indicator is

$$[L : L']^{-1} \sum_{\eta \in (L/L')^\vee} \eta(y - z).$$

This follows from the character sum on the finite additive group. The index is a power of ℓ. Hence every term is a position and frequency translate of the vacuum with p-integral coefficient. Its shifts are supported over ℓ. Clearing a common denominator and applying the Chinese remainder theorem identifies these finite local shift classes with a rational class in the period space modulo its lattice. Its order is prime to every member of $\mathcal{P}$.

Apply Theorem 65.11 at each vertex with the whole finite set $\mathcal{P}$ excluded. The representatives move only at other completely split primes. Their lattice intertwiners add shifts there, with root-of-unity factors and inverse square roots of indices supported there. Their character factors are units by the ray class argument above. At p the class lattices at a shared vertex still agree on both incident edges, as in Theorem 122.4. Finally p is unramified and inert in the elliptic CM field K_d. Lemma 65.14 gives one orthonormal product basis per edge generating the trace lattice at all $p \in \mathcal{P}$. The intertwiner to this product lattice again has index prime to every p in the set. In the discriminant-one elliptic model of Section 65.5 prime-to-p translations have integral germs relative to the marked vacuum, including when the translated section vanishes at the mark, by Corollary 65.28. This proves the assertions. □

Theorem 141.15 (The complete signal-prime contribution to the period ideals)**.** *Let column Y_i use an integral initial Weyl table and the prescribed elementary tests. One may use a scalar linear combination of such columns on the same directed tower, with the same weights, for Y_i. Let C_i be a positive integer clearing these additional scalar coefficients. For an elementary*

column or an integral combination take $C_i = 1$. With $\omega_w = [J_w : \mathbb{Q}_p]/[J : \mathbb{Q}]$, one has for every $p \in \mathcal{P}$

$$\sum_{w|p} \omega_w b_{0,w} \geq qU_p - \sum_{i=1}^{q} v_p(C_i) - qv_p(D),$$

$$\sum_{w|p} \omega_w b_{\mathrm{im},w} \geq qU_p - \sum_{i=1}^{q} v_p(C_i) - qv_p(Dm_G).$$

The bounds use the same period ideals in any marked product bases, with the differential and physical coordinates fixed.

Proof. In the bases and representatives of Lemma 141.14, clearing C_i makes every premass path coefficient integral at $w \mid p$, including the character factors. At a common vertex use the integral frames on its Weyl lattice; a unit change of determinant frame preserves it. Theorem 141.12 thus bounds the valuations of the minors in these bases at each individual w by the right sides. This argument applies termwise to the integral combination C_iY_i and then divides by C_i once per column in the minor.

Changing to any other fixed marked product basis multiplies all entries of column i by the same scalar u_i, by Theorem 122.3. Every term of a combination on that tower contains one copy of each directed edge, so it has that same scalar. Every minor is multiplied by $\prod_i u_i$. Enlarge J to contain these scalars if necessary. Since $|\sigma u_i| = 1$ for all embeddings, $|N_{J/\mathbb{Q}} u_i| = 1$, and therefore

$$\sum_{w|p} \omega_w v_w(u_i) = [J : \mathbb{Q}]^{-1} v_p(N_{J/\mathbb{Q}} u_i) = 0$$

at each rational p. Both weighted ideal valuations are therefore preserved. The local degree identity in a tower preserves the averages after extending coefficients. □

Corollary 141.16 (The centered bound for the prescribed column tests)**.** *If these q columns are independent on the projected small tests, the centered sum on the same field and interval satisfies*

$$\Delta_u \leq \frac{\sum_\sigma \log\det G_\sigma}{q[J : \mathbb{Q}]} - A_{\mathrm{sig}}(x) + \frac{B_{\mathrm{data}}}{q} + \frac{2}{q}\sum_{i=1}^{q} \log C_i + 2\log D + \log m_G + o_{\psi,\varepsilon,A}(b/x^A)$$

for every fixed $A > 0$. In B_{data} use $\mathcal{S} = \mathcal{P}$ and the fixed normalization used to evaluate the Gram determinants.

Proof. Theorem 141.15 supplies both lower bounds at every active prime in Theorem 9.22. At the other primes use Proposition 9.23 in the original bases. Sum and apply the product formula inequality of Theorem 141.9 in those bases. The restricted sum of $v_p(C_i)\log p$ is at most $\log C_i$, and the identical assertion applies to D, m_G. Substitution of the centered identity proves the claim. Basis change preserves the determinant of the Gram matrix at each embedding, since it multiplies the columns by scalars of absolute value one. □

141.6 The different and the split tests in the centered bound

Use the q columns of Theorem 141.15. Denote the total derivative degree at embedding τ on edge j of column i by $d_{ij,\tau} = |\nu_{ij,\tau}|$, and let $d_{ij} = \sum_\tau d_{ij,\tau}$. In a linear combination take the common directed tower and weights of that theorem. Let $\mathcal{M}$ be the rational primes below the

row places, the two flag primes, and all terminal primes of the selected columns. Write $\mathcal{T} \subset \mathcal{M}$ for those terminal primes. All of $\mathcal{M}$ splits completely in E by Lemma 141.14. We use the prescribed integral Weyl tables at the incoming endpoint.

Lemma 141.17 (The loss of an undivided derivative on the Weyl lattices)**.** *Work over a coefficient valuation ring, using integral coefficient frames at the two ends of an edge. Suppose each undivided derivative of its class section of the given block degrees has valuation at least t, for a real t. Every coordinate of the resulting edge operator between the source and target Weyl lattices has valuation at least t.*

Proof. Take a generator of the source as a coefficient of the highest polynomial on a product of elementary row matrices with independent parameters. After the determinant twist, its polynomial is the corresponding coefficient of $p_\nu(AX)$, since $\det A = 1$. Pair with an integral dual polynomial P_τ at each target embedding. As in the proof of Theorem 120.12, the differential symbol in that block is $P_\tau([DA_\tau^{\mathsf{t}}, 0])$. Its coefficients in the indeterminates and in the entries of D are integral. The product over embeddings is homogeneous of degrees $d_{ij,\tau}$. After parameter extraction its evaluation is an integral linear combination of precisely the derivatives in the hypothesis. Such generators span the source, and the tensor products of these tests span the integral dual of the target. Testing on a dual basis proves the assertion. A determinant line trivialized in an integral frame contributes a unit under an integral change of frame. □

Here is finite data for a prime $p \in \mathcal{M} \setminus \mathcal{T}$. At each $v \mid p$ in F, let L_v be the standard split position lattice of the given edge. Its $\mathbb{Z}_p$-rank is $D_{jv} = r_j s_j$ for edge ranks r_j, s_j, since $F_v = \mathbb{Q}_p$. For all elementary functions of this edge in this column, divided by the amplitude of the reference position vacuum, choose nonnegative integers a_v, b_v so that their support lies in $p^{-a_v} L_v$ and they are constant on cosets of $p^{b_v} L_v$. Use the matrix-coset indicators appearing in the path sum, with the value of the coupled coset held fixed. Such bounds exist by the denominator and period lattices in Lemma 141.14. For a spherical factor take $a_v = b_v = 0$. Let $c_{ij,p}$ be the maximum of zero and the negatives of the valuations of the nonzero products of these original local values at $v \mid p$, over all the cosets in the finite list and all coefficient places above p. The sum of the corresponding nonnegative maxima place by place also gives an admissible upper bound. This includes the row integral scalars, prior to inserting the class masses.

Label the two places of $K = K_d$ above p by $+, -$. Let ϵ_v be the label of the position factor at v. For $\epsilon \in \{+, -\}$, put

$$h_\epsilon = \max\left(\{0\} \cup \{a_v : \epsilon_v = \epsilon\} \cup \{b_v : \epsilon_v \neq \epsilon\}\right), \qquad J_{ij,p}^{\mathrm{row}} = c_{ij,p} + \sum_{v \mid p} D_{jv} b_v + \frac{d_{ij}}{2}(h_+ + h_-).$$

The lattices, bounds and exponents here refer to the specified column and edge; they can be read off before any differentiation.

At every rational prime put $\rho_p = v_w(\mathfrak{D}_{E/K})$, with $v_w(p) = 1$, independent of $w \mid p$ by normality. Let κ_p be the bound for the marked elliptic differential in Theorem 67.12. For $p \notin \mathcal{M}$ set

$$J_{ij,p}^{\mathrm{vac}} = \tfrac{1}{2} d_{ij} \rho_p + \kappa_p \lfloor d_{ij}/2 \rfloor.$$

Theorem 141.18 (Outside-prime valuations of the two period ideals)**.** *For $p \notin \mathcal{T} \cup \mathcal{P}$ use $J_{ij,p} = J_{ij,p}^{\mathrm{vac}}$ off $\mathcal{M}$ and $J_{ij,p} = J_{ij,p}^{\mathrm{row}}$ on $\mathcal{M}$. In the original normalization, writing $b_{S_0,w} = b_{0,w}$ and $b_{eS,w} = b_{\mathrm{im},w}$, take $m' = 1$ in the saturated case and $m' = m_G$ in the image case. One has*

$$\sum_{w \mid p} \omega_w b_{D',w} \;\geq\; -\sum_{i,j} J_{ij,p} - \sum_i v_p(C_i) - q v_p(D) - q v_p(m')$$

for the saturated and image test lattices, respectively. In particular an unmarked, unramified prime with $\kappa_p = 0$ costs only the explicitly displayed scalar, mass, and projector valuations.

Proof. Fix p and choose class representatives equal to the identity over p, by Theorem 122.4. All class movements occur at other split primes and contribute p-unit multipliers and shifts with zero p-primary component. Also the finite tests and changes of variables at the other marked primes have integral coefficients above p, by the local character, volume and finite Fourier calculations in Lemma 141.14. Choose on each edge the product basis of Theorem 122.14 generating the local Hermitian trace lattice. Its vacuum intertwiner adds shifts of order prime to p and has unit normalization.

Suppose first $p \notin \mathcal{M}$. Every translated elliptic factor now has order prime to p in its translation parameter. Theorem 66.35 bounds the loss at derivative order s by $\kappa_p \lfloor s/2 \rfloor$. In an undivided derivative of degree d_{ij} the orders sum to d_{ij}, so these losses sum to at most $\kappa_p \lfloor d_{ij}/2 \rfloor$. In physical block τ the change of coordinates has entries of valuation at least $-v_w(\bar{\tau}\mathfrak{b})$, since product basis vectors belong locally to $\mathfrak{b}^{-1}\operatorname{Mat}(\mathcal{O}_E)$ and the inverse embedding matrix is the conjugate transpose. Here $\mathfrak{b}\overline{\mathfrak{b}} = \mathfrak{D}_{E/K}$. There are exactly $d_{ij,\tau}$ differentiations in this block. All multinomial coefficients in the chain rule are integral. Thus a bound for the loss above p is

$$H_{ij,w} = \kappa_p \lfloor d_{ij}/2 \rfloor + \sum_{\tau} d_{ij,\tau} v_w(\bar{\tau}\mathfrak{b}).$$

Use τ in place of $\bar{\tau}$ on a conjugated edge. In a common normal coefficient field, conjugation permutes the places above p and preserves their local degrees. For each individual τ the two conjugate ideals consequently have the same weighted average valuation. Their product has valuation ρ_p at every place above p. It follows that $\sum_{w|p} \omega_w H_{ij,w} = J^{\mathrm{vac}}_{ij,p}$.

Suppose next $p \in \mathcal{M} \setminus \mathcal{T}$. For one position coset $z + p^{b_v} L_v$, Fourier expansion on L_v costs exactly the reciprocal index $p^{-D_{jv} b_v}$ before cancellation. Its position translate has order dividing p^{a_v}, and its frequency character has order dividing p^{b_v} modulo the dual lattice. This is the coset character identity in Lemma 141.14. On taking products, the p-parts of the two idempotent components of any resulting shift have orders at most p^{h_+}, p^{h_-}, respectively. This follows by taking the maximum of the exponents on the factors of the direct product. The local basis change is $\mathcal{O}_K \otimes \mathbb{Z}_p$-linear, so these exponents also work for each elliptic coordinate. The conjugate annihilator ideal in Theorem 66.35 exchanges the two components. Since $\kappa_p = 0$, at a place over ϵ the loss of all d_{ij} differentiations is at most $d_{ij} h_{-\epsilon}$. The physical coordinate conversion costs zero here. Indeed $\mathfrak{b}$ in Theorem 64.10 is integral, and its product with its conjugate is a unit at every prime over this unramified p. The two places of K each have weight $1/2$ on averaging over $w \mid p$, by the local degree formula in a tower. Including the original scalar and the Fourier indices gives the bound $J^{\mathrm{row}}_{ij,p}$ on the weighted average.

Apply Lemma 141.17 to every edge at each w, before averaging. Since $p \notin \mathcal{T}$, the terminal refinements localize to the unrefined lattice at each vertex. Thus the identity representatives give matching integral frames at both ends of every composition. The terminal restriction is integral by Theorem 120.6. Each term of a column contains exactly one derivative operator per edge, hence its coordinate loss is at most the sum of the individual bounds plus $v_p(C_i)$. The finite signed sums preserve this lower valuation bound. Clearing D clears the path masses and $m_G e S \subset S_0$ clears the projector. Expand the minors, add over the columns, and take the minimum over the tests. Changing from these product bases to the original ones preserves each weighted ideal valuation by the norm calculation in Theorem 141.15. This proves both inequalities. □

Corollary 141.19 (A finite cost supported on the terminal and row primes)**.** *For independent projected columns let $B_{\mathcal{T}}$ be the sum in Proposition 9.23 restricted to $p \in \mathcal{T}$, in the original*

normalization. On the same intervals let $d_0 = |\operatorname{disc} K|$*. Then*

$$\begin{aligned}\Delta_u \le{}& \frac{\sum_\sigma \log\det G_\sigma}{q[J:\mathbb{Q}]} - A_{\mathrm{sig}}(x)\\ &+ \frac{1}{q}\sum_{i,j}\Big(d_{ij}\log \mathrm{rd}(E) - \tfrac{1}{2}\mathbf{1}_{d_{ij}\ \mathrm{odd}}\log d_0\Big)\\ &+ \frac{2}{q}\sum_{p\in\mathcal{M}\setminus\mathcal{T}}\sum_{i,j} J^{\mathrm{row}}_{ij,p}\log p + \frac{B_{\mathcal{T}}}{q}\\ &+ \frac{2}{q}\sum_i \log C_i + 2\log D + \log m_G + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

Proof. The different tower formula gives $\sum_p \rho_p \log p = \log \mathrm{rd}(E) - \frac{1}{2}\log d_0$. Here 3 splits in K on the selected intervals. Consequently the only nonzero κ_p is $\kappa_{d_0} = 1/2$. Every prime of $\mathcal{M}\cup\mathcal{P}$ is unramified in E and has $\kappa_p = 0$. Twice the sum of the vacuum costs of edge ij is therefore

$$d_{ij}\big(\log \mathrm{rd}(E) - \tfrac{1}{2}\log d_0\big) + \lfloor d_{ij}/2\rfloor \log d_0.$$

This is the displayed discriminant term. At the terminal primes use the finite symbolic bound, which includes the actual input and terminal frames there. At active signal primes use Theorem 141.15; at all others use Theorem 141.18. Sum the two normalized ideal log norms before taking the archimedean bound of Theorem 141.9. For each C_i, D, m_G the sum off the terminal primes is at most its full logarithm. Inserting the centered identity of Theorem 9.22 proves the inequality. □

The loss at unmarked primes in this estimate depends on the field and the degrees alone. The quantitative comparison still involves the size and rank of the signed projected periods and the finite tests at their specified levels.

141.7 The terminal covector congruence in the centered bound

Fix a translate and the common terminal generator π of Corollary 9.15. Thus $T = \pi\bar{\pi}$ splits completely in E, $S_T = \{P : P \mid \pi\}$ contains one place over each place $v \mid T$ of F, and $f = |S_T| = [F:\mathbb{Q}]$. The auxiliary characters are unramified there and T is distinct from the flag primes. We use the elementary column tests of Theorem 141.15. Let r_j, s_j be the ranks at the two ends of edge j. At a fixed pair of unrefined classes write $L_{j,P}$ for the lattice of integral position matrices at P; its reduction consists of all s_j-by-r_j matrices over $\mathbb{F}_T$.

Lemma 141.20 (Fourier expansion with the intermediate covectors fixed)**.** *In the signed path sum one can sum over nonzero terminal covectors at the vertices and use on each edge the indicator of the single affine linear condition*

$$\xi' \bar{X} = \xi$$

on $L_{j,P}$*, for each* $P \in S_T$*. Relative to the unit vacuum, this indicator is a sum of pure frequency translates with common denominator* T^{r_j} *at* P*. Their* T*-primary shift classes are killed by* T *and lie in the component at the conjugate place* $\bar{P}$*.*

Proof. Keep the original classes, their masses, all nonterminal local data and the common vector coset on the last two edges. For an integral matrix path and the specified final covector, successive multiplication backwards gives a unique tuple of intermediate covectors. If any intermediate covector is zero, the initial one is zero and the path is discarded. Conversely every

surviving path occurs once on summing over the nonzero tuples and imposing the displayed equations. This is the path identity of Theorem 73.1. With the tuples fixed the conditions on the separate edges are independent. The matrix-coset terms can be summed into their union because their indicator functions add, the theta section is linear in the finite test, and the iterated map is linear in the section on each edge separately. Chinese remaindering makes this regrouping independent of the conditions at the row places. The finite test at T in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 646 is exactly this covector indicator.

Fix a nontrivial additive character ψ_T of $\mathbb{F}_T$. On $L_{j,P}$ the indicator is

$$T^{-r_j} \sum_{u \in \mathbb{F}_T^{r_j}} \psi_T(\xi' \bar{X} u - \xi u).$$

The sum is one after division if $\xi' \bar{X} = \xi$, and zero otherwise by character orthogonality. Multiplication by $\mathbf{1}_{L_{j,P}}$ supplies the support condition. The coefficient phase is a root of unity. The linear functional $X \mapsto \xi' \bar{X} u$ is a character of $L_{j,P}/TL_{j,P}$, so its frequency is a class in $T^{-1}L_{j,P}^{\vee}/L_{j,P}^{\vee}$. Under the split Hermitian pairing the dual of the position component at P is the component at $\bar{P}$. This proves all the assertions, including the support of each individual shift. □

For column i, let $\lambda_{i,\tau,k}^{\mathrm{in}}$ be the smallest entry of its initial highest weight at embedding τ, and let $\delta_{i,\tau}$ be its terminal normal degree in Corollary 127.2. Define

$$z_i = \prod_{\tau} \tau(\bar{\pi})^{\lambda_{i,\tau,k}^{\mathrm{in}}}, \qquad Z_i^{\mathrm{norm}} = \prod_{\tau} \tau(\bar{\pi})^{-\delta_{i,\tau}}, \qquad \widetilde{Y}_i = (z_i Z_i^{\mathrm{norm}})^{-1} Y_i.$$

The two scalars are units away from T. Use tildes on the valuation of an ideal or on a Gram matrix to indicate that these rescaled columns are used, with the original small test lattices.

Theorem 141.21 (The terminal valuation on both projected period ideals)**.** *For $D' = S_0, eS$, take $m' = 1, m_G$, respectively. One has*

$$\sum_{w|T} \omega_w \widetilde{b}_{D',w} \;\geq\; -\sum_{i,j} \left(f r_j + \tfrac{1}{2} d_{ij} \right) - \sum_i v_T(C_i) - q v_T(Dm').$$

Proof. Choose unrefined class representatives equal to the identity above T, by Theorem 122.4. The small class and its lift then have the standard unrefined frames at T. For each initial residue covector ξ its refined Weyl lattice satisfies $D_{a,\xi} \subset z_i D_a$, where D_a is the unrefined lattice. Indeed the prescribed kernel at P changes the column lattice at $\bar{P}$ by replacing one basis vector by T times that vector, by the two dualizations in the proof of Theorem 99.20. In the polynomial representation with its last weight subtracted, all degrees of that vector are nonnegative. Reinstating the determinant power gives the factor $T^{\lambda_{i,\tau,k}^{\mathrm{in}}}$ at the affected embedding. Their product differs from z_i by a local unit. Thus $z_i^{-1} Y_i$ may be computed with initial values integral in the unrefined frames. At the last class, in those frames the actual restriction is Z_i^{norm} times the integral standard restriction, by Theorem 127.1 and Theorem 120.6. Removing this second scalar allows us to use unrefined large frames all the way through the path sum. There is no choice of a test or a new projection at an intermediate vertex.

We make a single choice of product coordinates at T for each directed edge. The split trace lattice is the orthogonal sum over $v \mid T$ of the position lattice and its dual, paired with each other. For a position basis and its exact dual, the sums of corresponding vectors give a local orthonormal $K \otimes \mathbb{Q}_T$-basis, as in Theorem 122.14. Label its coordinates by v and a matrix-entry index. Approximate this local basis by a rational orthonormal basis agreeing modulo T

at both places of K, using the Cayley approximation in that theorem. The congruence is an open condition. Both the change of basis and its inverse then preserve the individual v-blocks modulo T. In particular an order-T shift has in each new local coordinate the same component modulo the period lattice as under the exact local basis, after identification by these bases. The vacuum intertwiner for this rational basis has index prime to T. Its shifts and the shifts from the other finite primes add no T-primary components.

Fix $w \mid T$ and an embedding τ in the CM type. Let $P_{\tau,w}$ be the prime of E obtained by pulling w back under τ, and put $v = P_{\tau,w} \cap F$. For a coordinate in the v-block, the conjugate annihilator bound of Theorem 66.35 uses the part of the shift at the place of K opposite the one induced by τ, w. If $P_{\tau,w} \notin S_T$, this opposite component of the shift at v is zero modulo the lattice. In fact the position prime there is $\overline{P_{\tau,w}}$, and the lemma puts the entire frequency shift at $P_{\tau,w}$. Hence the undivided derivatives on that coordinate are integral in every order. If $P_{\tau,w} \in S_T$, the same argument permits a loss of one per differentiation, since T kills the primary shift. Here $\kappa_T = 0$.

The physical derivative in a τ-matrix entry is a sum of product derivatives with integral coefficients. For coordinates belonging to $v' \neq v$ these coefficients lie in $T\mathcal{O}_w$. Indeed the coefficient is the conjugate embedded basis entry, which vanishes in those blocks for the exact local basis and agrees with it modulo T for the chosen basis. Conjugation exchanges $P_{\tau,w}$ and its partner over the same v. Each such off-block factor therefore pays the loss of at most one from a differentiation there. To see this in all orders, expand the product of the linear differential operators. In a product coordinate whose primary shift can cost one per derivative, its derivative order counts exactly the individual differentiations landing there. Each arriving from an uncharged block has a factor of T. Terms from charged blocks have integral coefficients, and integral multinomial coefficients preserve the resulting lower bound.

Consequently the loss in a derivative of the edge’s block degrees, apart from the Fourier indices, is at most

$$\sum_{\tau} d_{ij,\tau}\, \mathbf{1}_{P_{\tau,w} \in S_T}.$$

For a conjugated edge evaluate this formula at the conjugate place. There are f Fourier indices from the lemma, costing fr_j in either orientation. Other finite inputs have coefficients integral at T by the split-prime character and volume calculation in Lemma 141.14. At the fixed w, use Lemma 141.17 on the unrefined Weyl lattices, then compose the maps at the common vertices. Taking determinants adds the costs for the columns. The path masses, projector, and coefficient denominators contribute precisely the allowances displayed in the statement.

For each fixed τ, the total local degree weight of $P_{\tau,w} \in S_T$ is $f/[E : \mathbb{Q}] = 1/2$, because T splits completely and local degrees add in the tower $J/\tau(E)/\mathbb{Q}$. The same holds at the conjugate places. Average the preceding bound. A return to the original product bases multiplies the minors by a common product of column phases of rational norm of absolute value one, as proved in Theorem 141.15. This preserves the average and proves the theorem. □

Corollary 141.22 (The centered inequality with explicit finite terms)**.** *At the common prime*

T of the original family, the q independent projected columns satisfy, for every fixed $A > 0$,

$$\begin{aligned}\Delta_u \le{}& \frac{\sum_\sigma \log\det \widetilde{G}_\sigma}{q[J:\mathbb{Q}]} - A_{\mathrm{sig}}(x)\\ &+ \frac{1}{q}\sum_{i,j}\Big(d_{ij}\log \mathrm{rd}(E) - \tfrac{1}{2}\mathbf{1}_{d_{ij}\ \mathrm{odd}}\log d_0\Big)\\ &+ \frac{2}{q}\sum_{p\in\mathcal{M}\setminus\{T\}}\sum_{i,j} J^{\mathrm{row}}_{ij,p}\log p + \frac{1}{q}\sum_{i,j}(2fr_j + d_{ij})\log T\\ &+ \frac{2}{q}\sum_i \log C_i + 2\log D + \log m_G + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

In the first term one may equivalently use the original $\log\det G_\sigma$ *corrected by* $-\sum_{i,\tau}(\lambda^{\mathrm{in}}_{i,\tau,k} - \delta_{i,\tau})\log T$ *at every embedding.*

Proof. Use Theorem 141.21 at T in the proof of Corollary 141.19 and use the rescaled columns throughout. The two scalars removed from each column are units at every other prime, so the previous estimates for both ideals hold there with the same valuations. The T-terms from the two test lattices sum to the displayed terminal cost. The original and rescaled Gram determinants differ by $\prod_i |\sigma(z_i Z_i^{\mathrm{norm}})|^{-2}$, since the class metric and projector are fixed. Now $|\sigma\tau(\bar\pi)| = \sqrt{T}$ for every σ, τ. This gives the asserted correction. □

All terms of this inequality are evaluated on the field and translate of the prime count. Its use in (6.3) requires a uniform bound for their sum and a supply of nonzero projected columns on those intervals.

141.8 The contribution of primes inert in both quadratic fields

Work on the polynomial-conductor fields and intervals of Theorem 9.26. Put $Y = X' = \exp(x^{\varepsilon/3})$. Write T_{hi} for the high set of size h. The extra divisor $\mathcal{D}_u$ of Corollary 68.3 measures divisibility of the signed word periods beyond the primes used in U_p. We estimate a part of it for which the dyadic Frobenius moves every high place to a place of small weight.

Let $G_j = \mathrm{Gal}(C_j/\mathbb{Q})$ with $j = j_x$. For a power of two $M \mid 2^j$ use the cyclic quotient G_j/MG_j of order M. For a prime unramified in E_x let $\bar p$ denote its Frobenius in this quotient. It is also the image of $\pi_p = \sigma_p c_E$, since C_j is totally real. For a function on a finite abelian group normalize its Fourier coefficients so that it equals their sum times the corresponding characters, and write $\widehat{w}(1)$ for the average of w.

Lemma 141.23 (A weighted prime sum on the dyadic quotient)**.** *Let w be real valued on G_j/MG_j with $\sum_\chi |\widehat{w}(\chi)| \le K$. Uniformly in M, w, x,*

$$\sum_{\substack{p\le Y,\ p\ge 5,\ p\ \text{unramified in } E_x\\ p\ \text{inert in } K_q,\ K_d}} w(\bar p)\frac{\log p}{p} = \frac{\widehat{w}(1)}{4}\log Y + O_{\psi,\varepsilon}(K\log^2(2x)).$$

Proof. On unramified primes multiply w by $(1-\xi_q)(1-\xi_d)/4$. The two quadratic fields and the dyadic quotient have pairwise coprime conductors, so their compositum has the direct product Galois group. This follows also by independent variation of the units at the three moduli. The product has constant Fourier coefficient $\widehat{w}(1)/4$, and its sum of absolute Fourier coefficients is at most K. Every character in the expansion belongs to E_x.

For the primitive representative of each character, use $R_\chi(Y)/Y$ from (1.1). Replacing the sum over unramified primes by these expressions costs $O_\psi(K)$. Indeed removing prime powers

of exponent at least two costs at most $\sum_p (\log p)/(p(p-1)) < \infty$ per character. The linear smoothing term costs at most $Y^{-1}\sum_{p\le Y}\log p = O(1)$ by Chebyshev's bound. At $2, 3$ and the primes dividing N_x, the remaining difference costs at most a constant times $\sum_{p\mid 6Fqd_0}\log(p)/(p-1) = O_\psi(1)$. This accounts also for primitive representatives with smaller conductors.

The large-argument estimate in the proof of Theorem 9.26 gives $G_\chi(Y)/Y = o(1)$ uniformly. For the principal character the term remaining in $R_\chi(Y)/Y$ is $\log Y - 1 - \gamma$. For a nonprincipal character it is $-L'(1,\chi)/L(1,\chi)$, bounded in absolute value by $O_\psi(\log^2(2N_x))$ by the zero-free estimate and the logarithmic derivative calculation of Lemma 7.1. Since $N_x \le x^{D_0}$, Fourier summation gives the assertion. □

Theorem 141.24 (A lower bound for the extra divisor of the projected periods)**.** *For any of the high coset choices in Theorem 6.29, the additional term in Corollary 68.3 satisfies, uniformly over translates and the admissible lists of independent signed columns,*

$$\mathcal{D}_u \;\ge\; (\tfrac14 - o_{\psi,\varepsilon}(1))\, hma \log X'.$$

Proof. Let D be the dyadic index used to choose the high set, so $2x^{\varepsilon/8} \le D < 4x^{\varepsilon/8}$. In G_j/DG_j the image of T_{hi} contains at most $3o+1$ points, $o = [K_\psi : \mathbb{Q}]$, since T_{hi} is a union of that many cosets of DA_0. Write S for their difference set. It contains zero and has size at most $(3o+1)^2$. Take $w = 1 - \mathbf{1}_S$. Each point indicator has Fourier coefficient sum in absolute value one, so the lemma applies with $K \le 1 + |S|$ and $\widehat{w}(1) = 1 - |S|/D$.

For a prime counted with this weight, π_p has nonzero image in the dyadic quotient and hence even order. Its cycles on the embedding labels all have that order, since the action is translation in the abelian group. The two negative signs ensure that it preserves the K_q-sign on the K_d-type. Thus $\pi_p T_{\mathrm{hi}} \subset \Phi_0 \setminus T_{\mathrm{hi}}$. Choose in each cycle the half containing at least half of its high places. At least $h/2$ high places are then in the chosen halves.

For each such t, each column and each edge the high partition has $\nu_{t,1} \ge a$ by the word calculation in Lemma 141.11. At $\mu = \pi_p t$ the partition has $\nu_{\mu,1} \le b/(nY) + O(x^B)$ for a fixed B: this place is neither high nor distinguished, so its weight is $\lceil b/(nY)\rceil + O(mH_0+1)$ and its starting first coordinate has polynomial size. The rank s is polynomially bounded as well. Keep only the first high row and the first Frobenius power in the row contribution of Theorem 68.2. Since $d_p(\nu_{t,1}, s) \ge a/p - s$, the contribution of this place is at least $a/p - b/(nY) - O(x^B)$, increasing B if necessary. This is positive throughout $p \le Y$, since $(b/(nY) + x^B)Y/a = o(1)$. The constants are independent of the word, edge and translate.

Here $U_p = 0$. The factor two in $\mathcal{D}_u$ comes from the two integral test lattices. After division by hma, the lower bound from these primes is their sum of $(\log p)/p$ up to an additive loss

$$O\!\left(\frac{(b/(nY) + x^B)Y}{a}\right) = o(1),$$

by the Chebyshev bound again. The lemma gives the more precise lower bound $(1 - |S|/D)\log Y/4 - O_{\psi,\varepsilon}(\log^2(2x))$ for this normalized expression. Since D tends to infinity and $\log^2(2x) = o(\log Y)$, the theorem follows. □

Corollary 141.25 (Aligning the high cosets in the dyadic quotient)**.** *Let $e_2 = [K_\psi \cap C_{j_x} : \mathbb{Q}]$. One may choose the high cosets in the same fields so that the preceding inequality improves to*

$$\mathcal{D}_u \;\ge\; (\tfrac14(1 + e_2^{-1}) - o_{\psi,\varepsilon}(1))hma \log X'.$$

In particular the coefficient is $1/2 - o(1)$ for the conductor-one character.

Proof. Use a generator to identify G_j with $\mathbb{Z}/2^j$. Restriction identifies $A_0 = \mathrm{Gal}(K_\psi C_j/K_\psi)$ with e_2G_j, and $D \mid |A_0|$ on these intervals. In each coset of A_0, denote its common residue modulo e_2 by v, represented in $[0, e_2)$. Inside the quotient modulo e_2D, choose the high residues

$$v + t\, e_2 D/4, \qquad t = 0, 1, 2,$$

and also $t = 3$ in the identity coset. For large x the integer D is divisible by four. These are three or four distinct DA_0-cosets in the required coset, by the restriction identification. This is an allowed selection in Theorem 6.29. The character cancellations use only that T_{hi} is a union of DA_0-cosets, with the prescribed number in each A_0-coset, and that T_2 is balanced modulo D. The sizes and word weights per type are therefore those of Corollary 9.15 on this field.

Now take $M = e_2D/4$ and form S in G_j/MG_j. Its elements are differences $v - v'$, so $|S|$ is bounded in terms of ψ and $S \cap e_2(G_j/MG_j) = \{0\}$. For $\bar{p} \notin S$, the theorem's half-count argument still works. In addition, if $\bar{p}$ is a nonzero multiple of e_2, all high places of any one cycle lie in the same alternating half. Indeed if t' and t are high and $t' = \pi_p^n t$, their projection difference is $n\bar{p}$. It is a multiple of e_2 in S, so it is zero modulo M. The nonzero element $\bar{p}$ has even order in that quotient, forcing n even. Choose the half containing all these high places in each cycle.

The additional $h/2$ high places, beyond the original lower bound $h/2$, can thus be counted with the second weight $\mathbf{1}_{e_2(G_j/MG_j)} - \mathbf{1}_{\{0\}}$. Its mean is $1/e_2 - 1/M$, and its Fourier coefficient sum in absolute value is at most two: a subgroup indicator is the average of the characters annihilating it. Summing the lemma for the first and second weights gives the normalized lower bound

$$\frac{1}{4}\left(1 + \frac{1}{e_2} - \frac{|S|+1}{M}\right)\log Y - O_{\psi,\varepsilon}(\log^2(2x)).$$

Since M grows and $e_2 \le o$, this proves the result. □

These lower bounds enter the centered inequality by subtraction on its right hand side. The archimedean Gram determinants there are calculated on the same projected signed columns. Their required upper bound, the existence of enough independent columns, and the remaining strict cohomology lengths are open steps in the argument for Dirichlet GRH.

141.9 Rational divisors and the height of the signed period columns

Use the common terminal normalization of Corollary 141.22 on the polynomial-conductor family of Corollary 9.27. Let $\widetilde{Y}_1, \dots, \widetilde{Y}_{q_c}$ be the signed columns in this normalization. Write $\widetilde{G}_\sigma$ for the Gram matrix of their projections in the positive class metric at an embedding $\sigma : J \hookrightarrow \mathbb{C}$. Assume the projections are independent when a logarithm of a Gram determinant occurs.

At each unramified $p \ge 5$ inert in K_d let $g_{i,p} = \sum_{j=1}^{m} V_p(\nu_{ij})$. Extend U_p by zero as in Corollary 68.3. Define

$$b_i^{\mathrm{per}} = \prod_p p^{\,g_{i,p} - \lceil U_p \rceil}, \qquad \rho_x = 2\sum_p (\lceil U_p \rceil - U_p)\log p,$$

with both operations restricted to these primes. The data here are calculated before any values of period integrals are known.

Lemma 141.26 (An integer divisor for each individual column)**.** *Each b_i^{per} is a positive integer. Division by it leaves a gain of at least $\lceil U_p \rceil$ at each indicated prime on the individual column in adapted frames, before its specified scalar, stabilizer and projector denominators. For every fixed $A > 0$,*

$$\frac{2}{q_c}\sum_i \log b_i^{\mathrm{per}} = \mathcal{D}_u - \rho_x, \qquad 0 \le \rho_x = o_{\psi,\varepsilon,A}(b/x^A).$$

Proof. On each word the integral gains from the edges give $g_{i,p} \ge U_p$, by Theorem 68.2 and Lemma 141.11. At a prime inert in both quadratic fields this uses only $g_{i,p} \ge 0$. Since $g_{i,p}$ is an integer, all the exponents in the product are nonnegative. They have finite support by Corollary 68.3. The divisibility in adapted frames follows from the edge inclusions on each path in the proof of that corollary. Removing the rational integer subtracts the same valuation at every coefficient place over p. The terminal normalization is a unit at these primes.

Taking logarithms gives the identity from the definition of $\mathcal{D}_u$. A rounding term can occur only when $U_p > 0$, which forces $p \le 2nxX'$ by Theorem 9.22 and the actual word cutoff. Each term before multiplication by $2\log p$ is less than one. Chebyshev's bound gives $\rho_x = O(nxX')$, which is $o(b/x^A)$ since n is polynomial, $X' = \exp(x^{\varepsilon/3})$, and $b = \exp(x^{\varepsilon})$. □

Let G_σ° be the Gram matrix for the projections of $\widetilde{Y}_i/b_i^{\mathrm{per}}$. To specify all other terms, put

$$\begin{aligned} K_u = {} & \frac{1}{q_c}\sum_{i,j}\Big(d_{ij}\log \mathrm{rd}(E_x) - \tfrac{1}{2}\mathbf{1}_{d_{ij}\ \mathrm{odd}}\log d_0\Big) \\ & + \frac{2}{q_c}\sum_{p\in\mathcal{M}\setminus\{T\}}\sum_{i,j} J_{ij,p}^{\mathrm{mat}}\log p + \frac{1}{q_c}\sum_{i,j}\left(2fr_j + \sum_{\tau}\nu_{ij,\tau,1}\right)\log T \\ & + \frac{2}{q_c}\sum_i \log C_i + 2\log D + \log m_G. \end{aligned}$$

Here $f = [E_x^+ : \mathbb{Q}]$, r_j is the source rank on the edge and d_{ij} its total differential degree. The row-prime terms are the finite Fourier costs of Corollary 120.18; C_i, D, m_G clear the given scalar, class mass and projector denominators. For any such family of Gram matrices G use the real number

$$H(G) = \frac{1}{q_c[J:\mathbb{Q}]}\sum_\sigma \log\det G_\sigma + K_u.$$

Theorem 141.27 (A centered bound for the divided columns)**.** *With the specified tests on each E_x, for every fixed $A > 0$,*

$$\Delta_u \le H(G^\circ) - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x + o_{\psi,\varepsilon,A}(b/x^A).$$

The two heights satisfy the exact identity

$$H(\widetilde{G}) - H(G^\circ) = \frac{2}{q_c}\sum_i \log b_i^{\mathrm{per}}.$$

Proof. Projection commutes with multiplication of a column by a rational scalar. At every embedding the Gram determinant is therefore multiplied by $\prod_i (b_i^{\mathrm{per}})^{-2}$. This proves the second identity. At all row and terminal primes, and at primes ramified in E_x, the integers being removed are units. At the other finite places the precise subtraction is accounted for in Lemma 141.26. In particular there is no need to add the integer once more to the scalar denominators in K_u. Corollaries 9.27 and 68.3 give $\Delta_u \le H(\widetilde{G}) - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \mathcal{D}_u + o(b/x^A)$. Insert the two exact identities to obtain the result. □

Corollary 141.28 (A necessary height for independent projected columns)**.** *For the word weights and sparse sets on these intervals one has*

$$\begin{aligned} H(G^\circ) &\ge (\tfrac{1}{2} - o(1))hma\log X', \\ H(\widetilde{G}) &\ge (\tfrac{3}{4} - o(1))hma\log X'. \end{aligned}$$

For the allowed cosets in Corollary 141.25 the second coefficient is at least $\frac{3}{4} + 1/(4e_2) - o(1)$. The lower bounds concern the full quantity H, including K_u.

Proof. By Corollary 9.15, $2\Delta_u/(hmb\delta) = \mathcal{G}_\psi(x) + O(x^{1/2+\varepsilon/2})$, where the notation on the right denotes the fixed weighted sum of the $G_\chi(x)$ in (W). Each fixed $G_\chi(x) = o_\chi(x)$ by Proposition 3.2. Since $0 < \delta \le 1$, $\Delta_u = o(hma)$ uniformly over u. No estimate for zeros inside the critical strip is required beyond that proposition's zero-free boundary argument.

In $A_{\mathrm{sig}}(x)$ the three quadratic logarithmic derivatives are $O_{\psi,\varepsilon}(\log x)$ by Theorem 5.6. Also $\log n = O_\psi(\log x)$ and $\delta = |T_2|/n = o(1)$ by the representative bound in the proof of Theorem 9.26. Its displayed formula for the complementary main term therefore gives

$$A_{\mathrm{sig}}(x) = (\tfrac{1}{2} + o(1))hma \log X'.$$

The bound of Corollary 9.27 makes $W_{d_0}(u) = o(hma)$. Rearranging the centered inequality proves the first claim. The exact height difference, the rounding bound, and Theorem 141.24 now give the second. Apply Corollary 141.25 for the last coefficient. □

For comparison with the precision actually needed, the size conditions preceding (6.3) give $\mathcal{B}/(hmb) \ll_{\psi,\varepsilon} x^{1/2+2\varepsilon}$. Indeed the three contributions after division are bounded by $(2x/m)(1 + m^2/k)$, $2(n/h)mx^{\varepsilon}$, and $(h_2/h)(k/n^2)x^{2\varepsilon}$. Use $h \ge nx^{-\varepsilon/8}/2$, $h_2 \le n$, the stated bounds for m, k, and $m^2/k \le m/x$. Thus $x^{5\varepsilon}\mathcal{B}/(hma) \ll x^{-1/2+7\varepsilon}$. Proving a sufficiently precise upper bound for the divided Gram determinants at this scale still requires information on the actual signed periods.

141.10 First prime powers in the actual projected columns

Take the polynomial-conductor intervals of Theorem 9.26. Fix a node x and one of its translates u. Work with the ordinary word, its integral Weyl table and its projected small test lattice eS as in Theorem 141.15. Write $L = 2nx\exp(x^{\varepsilon/3})$ and let $\mathcal{P}$ consist of the unramified rational primes $5 \le p \le L$ which split in K_q and are inert in K_d. In the sum defining U_p in Section 141.4, denote the summands of a given odd exponent j, summed over all edges and high embeddings, by $U_{p,j}$. For $Z = x^{3/4}$ put

$$H_p = \sum_{\substack{j \ge 3 \text{ odd} \\ p^j \le Z}} U_{p,j}, \qquad g_p = \lfloor H_p \rfloor, \qquad B_p = \lceil hma/p \rceil.$$

All three numbers are computed from the input word. In particular $H_p = 0$ except possibly at the at most $x^{1/4}$ rational primes $p \le x^{1/4}$.

Lemma 141.29 (The omitted odd prime powers)**.** *At every node and translate,*

$$0 \le \sum_{p \in \mathcal{P}} (U_p - U_{p,1} - H_p)\log p \ll hma\, x^{-1/2}\log(2x).$$

Proof. The defining ramp is nonnegative and bounded by a/p^j per edge and high embedding. Put $P = Z^{1/3}$. For $p > P$ the entire sum of p^{-j} for $j \ge 3$ is at most $2p^{-3}$. Integral comparison of $\sum_{n>P}(\log n)n^{-3}$ gives $O(P^{-2}\log(2P))$. For $p \le P$ the first omitted odd power exceeds Z. Its tail has sum at most $2/Z$, so this range contributes $O(P\log(2P)/Z)$. Both bounds are $O(x^{-1/2}\log(2x))$. Multiplication by hma proves the claim without a hypothesis on primes in a short interval. □

For this fixed count choose any nonempty finite list of its ordinary signed columns having the word weights just used. One column suffices to define the quantities below. In a common normal coefficient field J of degree d let $\mathfrak{a}_c$ be the fractional ideal generated by all the pairings

of column c with eS. The zero ideal is allowed. Use C_c, D, m_G from Theorem 141.15, including the stabilizer factors and the denominator of the small projector. For $p \in \mathcal{P}$ define

$$\begin{aligned} t_{c,p} &= d^{-1} v_p(N_{J/\mathbb{Q}}\mathfrak{a}_c) + v_p(C_c D m_G), \\ v_p^* &= \min\Big\{B_p,\ \max\Big(0, \min_c t_{c,p} - g_p\Big)\Big\}, \\ N(x,u) &= \prod_{p\in\mathcal{P}} p^{dv_p^*}. \end{aligned}$$

Here $v_p(N0) = +\infty$. Thus $N(x,u)$ is defined even when an entire projected table vanishes. At a prime where all entries vanish the exponent used is exactly dB_p.

Theorem 141.30 (An integer upper comparison for one translate)**.** *The exponents dv_p^* are nonnegative integers. For the actual field and word at the node put*

$$M(x,u) = \frac{2}{Q\delta}\left(A_{\mathrm{sig}}(x) - W_{d_0}(u) - 2\sum_{p\in\mathcal{P}} H_p \log p\right).$$

Uniformly in the translate and the chosen columns,

$$\mathcal{G}_\psi(x) \ \le\ \frac{4\log N(x,u)}{dQ\delta} - M(x,u) + O_{\psi,\varepsilon}(x^{1/2+2\varepsilon}).$$

Enlarging J preserves $\log N(x,u)/d$. At any chosen subset $\mathcal{E} \subset \mathcal{P}$ one can use $v_p^* = B_p$ without reading a projected coefficient there. The resulting increase in the right side is at most

$$O_{\psi,\varepsilon}\left(x^{1+\varepsilon}\sum_{p\in\mathcal{E}}\frac{\log p}{p}\right) + O_{\psi,\varepsilon,A}(x^{-A}) \qquad (A>0).$$

Proof. The norm of a nonzero fractional ideal is positive rational. For its normalized valuation one has $d^{-1}v_p(N\mathfrak{a}_c) = \sum_{w|p}[J_w : \mathbb{Q}_p]v_w(\mathfrak{a}_c)/d$, where $v_w(p) = 1$. This follows from the usual product of residue degrees with integral ideal exponents. Apply Theorem 141.15 on the image lattice to the single column c. At every active prime it gives $t_{c,p} \ge U_p$. For a zero column this holds with infinite left side. Hence $t_{c,p} - g_p \ge U_{p,1}$ at every active prime. The other bound $B_p \ge U_{p,1}$ follows by dropping every threshold at the first power. At an inactive prime the first-power term is zero. We obtain $U_{p,1} \le v_p^*$ at all primes. All finite valuations in the definition have denominator dividing d, and the upper bound is an integer. This proves the integrality assertion. Ideal extension multiplies its norm valuation by the relative degree, since the local degrees add in a tower. Division by the new d therefore preserves all $t_{c,p}$, including for the zero ideal.

Insert the bound for $U_{p,1}$ and Lemma 141.29 in the exact centered identity of Theorem 9.26. The signs give $-A_{\mathrm{sig}}(x)+W_{d_0}(u)$ there. Multiplication by $2/(Q\delta)$ gives the displayed main term. Here $Q = hmb$, $a \le 2xb$ and $\delta \ge x^{-\varepsilon}$, so the normalized higher-power error is $O_{\psi,\varepsilon}(x^{1/2+2\varepsilon})$. The remainder $o(b/x^A)$ in that identity can use any fixed A. Finally Corollary 9.15 compares $2\Delta_u/(Q\delta)$ with $\mathcal{G}_\psi(x)$ at this individual translate, with an error $O_{\psi,\varepsilon}(x^{1/2+\varepsilon/2})$.

Replacing an exponent by its upper bound adds at most $4(Q\delta)^{-1}\sum_{\mathcal{E}}(hma/p+1)\log p$. In the term from the ceiling, the sum of logarithms is at most $L\log(2L)$. Since n grows polynomially and $b = \exp(x^\varepsilon)$, $L\log(2L)/(Q\delta) = O_{\psi,\varepsilon,A}(x^{-A})$ for every fixed A. This gives the claimed allowance for exceptional primes. □

Corollary 141.31 (Integral tests at one translate per scale)**.** *Use the nodes and weights of Theorem 6.40 on the polynomial-conductor intervals. At each node x_j choose one translate and*

let $N_j, d_j, M_j, Q_j, \delta_j$ denote the data of Theorem 141.30 there. Put $E_k = (\sigma-\theta)^{-k-1}$ and select an integer $H \geq 1$ with

$$HE_k \;\geq\; 1 + \sum_j d_j \sum_{p\in\mathcal{P}_j} B_{j,p}\log p.$$

For $l = 0, 1, 2$ form the positive rational integers

$$P_l = \prod_j N_j^{\lceil 4Hw_{lj}/(d_jQ_j\delta_j)\rceil}.$$

If for every rational frequency and arbitrarily distant common intervals at that frequency one has, simultaneously,

$$H^{-1}\log P_l - \sum_j w_{lj}M_j \leq B_\tau E_k \qquad (l = 0, 1, 2),$$

every open-strip zero of the family has real part at most θ, for $1/2 + 4\varepsilon \leq \theta < 1$. The choice of H uses only finite input data and the positive error allowed in the test.

Proof. Since $0 \leq \log N_j \leq d_j \sum_{p\in\mathcal{P}_j} B_{j,p}\log p$, rounding all exponents up changes each normalized sum by a number between zero and E_k. Multiply the node inequality by the positive weights. Equivalently, interpolate the resulting node majorants of $\mathcal{G}_\psi$ by Lemma 6.37. The interpolation error is $O_{\psi,\varepsilon}(x^{1/2+4\varepsilon})$ on a mesh with fixed constant. Its integral and that of the node error against the three kernels are $O_{\psi,\varepsilon}(E_k)$, by the gamma integral in Theorem 6.40. That theorem's pole argument now gives the conclusion. No identification between the coefficient fields of different nodes is needed to multiply their rational integers. □

The finite exponents retain the original signed projections and the ramified prime in the centering term. The estimate still needed is an upper bound on the logarithms of these integers after that centering, uniform along the specified intervals.

141.11 The projected period at primes between the square root and the scale

On the polynomial-conductor intervals fix the two auxiliary primes q, d_0 of Theorem 19.15. Let $\mathcal{P}_0$ be the rational primes prime to $2Fqd_0$ which split in K_q and are inert in K_d. Put $\mathcal{I}_x = \{p \in \mathcal{P}_0 : \sqrt{x} < p \leq x\}$. We give a comparison requiring projected integral coefficients only at these primes. The translate in those coefficients can be chosen separately at each prime.

Write $w_{\mathbf{1}} = 1,\ w_\eta = 1/(3o+1)$ for the other characters of A/A_0, and use the primitive representatives $\alpha_{ij\eta}$ of Theorem 6.23. Let $c_\psi(n)$ be as in Lemma 6.37. For these fixed auxiliary primes define the real constants

$$D^* = \sum_{\alpha_{ij\eta}\neq\mathbf{1}} (-1)^j w_\eta \operatorname{Re}\frac{L'(1,\alpha_{ij\eta})}{L(1,\alpha_{ij\eta})},$$

$$H^* = \sum_{p\in\mathcal{P}_0}\log p \sum_{\substack{v\geq 3\\ v\ \text{odd}}} \frac{c_\psi(p^v)}{p^v}.$$

The second series converges absolutely. Put

$$M_0(x) = x(\log x - 1 - \gamma - D^* - 4H^*) - 4x\sum_{\substack{p\in\mathcal{P}_0\\ p\leq\sqrt{x}}}\frac{c_\psi(p)\log p}{p}.$$

Proposition 141.32 (The centered first-power sum in a single range)**.** *Uniformly on these common intervals,*

$$\mathcal{G}_\psi(x) = 4\sum_{p\in\mathcal{I}_x} c_\psi(p)(x/p-1)\log p - M_0(x) + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}).$$

Proof. Form $\sum_{i,j,\eta}(-1)^j w_\eta \operatorname{Re} G_{\alpha_{ij\eta}}(x)$. The $i=j=0$ terms are exactly $\mathcal{G}_\psi(x)$. There are $3o$ other terms, each bounded by $O(x^{1/2+\varepsilon/2})$ by the auxiliary twist estimate on the common interval. In the defining ramps, $q, d_0 > x^{401}$ and so neither prime occurs. Coprimality of their conductors with F makes the sum over i,j at p^v equal to the multiplier $(1+\xi_q(p)^v)(1-\xi_d(p)^v)$. It is four exactly at odd v and primes with the indicated two signs and is zero elsewhere. The primes dividing $2F$ and the prime three split in K_d on sufficiently distant intervals, by its prescribed congruences. The remaining coefficient is $c_\psi(p^v)$. The principal correction is $-x\log x+(1+\gamma)x$ and the sum of the other linear corrections is xD^*. This gives the expression with all odd powers directly from the primitive character formula.

For $p\le\sqrt{x}$ at the first power, replacing $x/p-1$ by x/p costs $O(\sqrt{x}\log(2x))$ after summation since $c_\psi(n)<2$. The odd powers $v\ge 3$ contribute $4xH^*+O(x^{1/3}\log^2(2x))$. Indeed there are at most $x^{1/3}\log_2 x$ pairs with $p^v\le x,\ v\ge 3$, so the discarded unit in each ramp costs at most the stated error. For $p^v>x$, the tail of $\sum p^{-v}\log p$ is $O(x^{-2/3}\log(2x))$ by the two-range estimate in Lemma 141.29, now with cutoff $Z=x$. Substitution proves the formula. In this formula the arguments of the ramps are x. The ramified prime which appears at the much larger word arguments in Theorem 9.26 exceeds this cutoff. □

For $p\in\mathcal{I}_x$ let z_p denote the A-coordinate of its Frobenius. With T, T_2 of Theorem 6.29, set $k_p = h\delta c_\psi(p)$.

Lemma 141.33 (A translate-independent integral multiplicity)**.** *The integer k_p counts the high embeddings carried to the distinguished set by the first power at p, for every translate u. In particular $0\le k_p\le h$.*

Proof. In the product coordinates of Theorem 6.23, the required condition on $t\in T$ is $t+z_p\in u+T_2$. The sign pair is $(0,1)$, so the coordinate change in that theorem gives this condition also for odd ψ. Expanding the indicator on A gives the count

$$\frac{1}{n}\sum_{\theta\in\widehat{A}}\theta(z_p-u)\left(\sum_{t\in T}\theta(t)\right)\left(\sum_{z\in T_2}\overline{\theta(z)}\right).$$

If θ is nontrivial on A_0, the product of the two sums vanishes by the subgroup invariance and residue balance in Theorem 6.29. Otherwise the second sum is h_2, $\theta(u)=1$, and the first sum is hw_θ by the prescribed proportions. On these unramified primes the surviving Fourier sum is $c_\psi(p)$, since conjugate terms pair. Multiplication by hh_2/n proves the assertion. □

Put $X'=\exp(x^{\varepsilon/3})$, $a=\lceil xb\rceil$ and define two nonnegative integers from the common input at $p\in\mathcal{I}_x$ by

$$G_p=\left\lfloor m(h-k_p)\,(a/p-2b/(nX'))\right\rfloor,\qquad B_p=\lceil mk_pa/p\rceil.$$

Here x is sufficiently large. Choose any finite list of column and translate pairs from the actual ordinary words with these parameters. For a pair (u,c) write $t_{u,c,p}$ for the normalized valuation of its projected entry ideal, including C_cDm_G, in Theorem 141.30. Use one coefficient field of degree d for this finite list. At each p any subset of the pairs may be read. Put

$$a_p^*=\min\left\{B_p,\ \max(0,\min_{(u,c)\text{ read at }p} t_{u,c,p}-G_p)\right\},\qquad N_0(x)=\prod_{p\in\mathcal{I}_x}p^{da_p^*},$$

taking the minimum of an empty list to be $+\infty$. If the whole list is empty use $d = 1$.

Theorem 141.34 (A prime-by-prime choice of projected entries)**.** *The number $N_0(x)$ is a positive integer, and uniformly over these choices of actual projected columns,*

$$\mathcal{G}_\psi(x) \le \frac{4\log N_0(x)}{dQ\delta} - M_0(x) + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}).$$

Replacing any selections by the empty list on a set $\mathcal{E} \subset \mathcal{I}_x$ raises the right side by at most $O_\psi(\sum_{p\in\mathcal{E}} \log p)+O_{\psi,\varepsilon,A}(x^{-A})$ for every fixed $A > 0$. The three integral tests of Corollary 141.31 hold as a sufficient condition with N_0, M_0 in place of N, M, summing the rounding allowance only over $\mathcal{I}_x$ with the B_p of this subsection.

Proof. All target embeddings at the first power for the chosen signs belong to Φ_1. On its nondistinguished part the word threshold is at most $2b/(nX')$ eventually. In fact it differs from $b/(nX')$ by at most $2+2mH_0$, which is polynomial in x, as in Theorem 9.22. At a distinguished place the threshold lies between b and $b + \sqrt{b}$ by Corollary 9.15. Since $a/p \ge b$ on $\mathcal{I}_x$, there are $h - k_p$ nondistinguished ramps at every edge, each at least $a/p - 2b/(nX') > 0$. The other k_p ramps are at least $(a/p - b - \sqrt{b})_+$. Thus the full word sum U_p for *each* translate is at least

$$G_p + mk_p(a/p - b - \sqrt{b})_+.$$

This also holds if $U_p = 0$, when both terms on the right vanish. At active primes the single-column estimate of Theorem 141.15 applies to each $t_{u,c,p}$. At inactive primes the maximum with zero suffices for the distinguished lower bound. Indeed if a read translate is inactive, the displayed lower bound forces both $G_p = 0$ and $mk_p(a/p - b - \sqrt{b})_+ = 0$, independently of the translate. The upper cutoff B_p also dominates the latter quantity. This proves

$$mk_p(a/p - b - \sqrt{b})_+ \;\le\; a_p^* \;\le\; mk_p a/p + 1.$$

The exponents become integers after multiplication by d by the ideal norm formula of Theorem 141.30.

Using $k_p = h\delta c_\psi(p)$, $Q = hmb$, and $a \ge xb$, the lower inequality bounds $c_\psi(p)(x/p - 1)$ above by $a_p^*/(Q\delta) + c_\psi(p)/\sqrt{b}$. The sum of the errors times $\log p$ is $O_{\psi,\varepsilon,A}(x^{-A})$ for any fixed A, since $p \le x$. Proposition 141.32 gives the asserted upper bound. At a prime switched to its upper cutoff the increase is at most $mk_p(b + \sqrt{b}) + 1$ before normalization, by the same two inequalities. After division by $Q\delta$ and summation this gives the stated cost. Finally the ceiling and logarithm argument of Corollary 141.31 applies to these integers with exactly the same positive kernels and interpolation error. □

The geometric term now reads integral coefficients only in $(\sqrt{x}, x]$, with prime-dependent choices of translates on the same field. The outstanding upper bound compares its logarithm with $M_0(x)$ to square-root accuracy after averaging. The two node inequalities above by themselves allow a total excess of order $x\log(2x)$.

141.12 How many primes the integral comparison must detect

Fix ψ, ε and the polynomial-conductor intervals of Section 141.11. At a node put

$$s_p(x) = \frac{B_p - a_p^*}{Q\delta}, \qquad R(x) = 4\sum_{p\in\mathcal{I}_x} [c_\psi(p) - s_p(x)]_+ \log p, \qquad H_0(x) = \frac{4\log N_0(x)}{dQ\delta} - M_0(x).$$

Thus $s_p(x) = 0$ when no projected column improves the elementary upper bound B_p. We measure how often this can happen if the proposed upper comparison is to hold.

Lemma 141.35 (The excess in the upper comparison)**.** *Uniformly over all choices of entries, on every node,*

$$H_0(x) = \mathcal{G}_\psi(x) + R(x) + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}).$$

On the interior $2Y \le x \le Y^{3/2}/2$ *one also has*

$$4 \sum_{p \in \mathcal{I}_x} c_\psi(p) \log p = (1 + o_{\psi,\varepsilon}(1))x.$$

Both estimates are unconditional.

Proof. Write $B_p = mk_p a/p + \eta_p$ with $0 \le \eta_p < 1$. Since $a = xb + O(1)$, $k_p = h\delta c_\psi(p)$, the normalized upper cutoff differs from $c_\psi(p)x/p$ by at most $c_\psi(p)/(bp)+1/(Q\delta)$. The lower bound for a_p^* in Theorem 141.34 consequently gives

$$0 \le s_p(x) \le c_\psi(p)(1 + b^{-1/2}) + 1/(Q\delta).$$

Replacing $c_\psi(p) - s_p(x)$ by its positive part costs $O_{\psi,\varepsilon,A}(x^{-A})$ after summing with $\log p$, for any fixed A. Subtract the prime sum of Proposition 141.32 from the definition of H_0 and use the cutoff estimate just proved. This gives the first formula.

For the second, during a given interval keep the auxiliary primes fixed. Consider the nonnegative ramp sum

$$F(y) = 4 \sum_{p \in \mathcal{P}_0} \log p \sum_{v \ge 1 \text{ odd}} c_\psi(p^v)(y/p^v - 1)_+.$$

By the character expansion in Proposition 141.32,

$$F(y) = y \log y - (1 + \gamma + D^*)y + o_{\psi,\varepsilon}(y) \qquad (Y \le y \le Y^{3/2}).$$

Indeed each $\mathcal{G}_\psi(y)/y$ tends to zero for fixed ψ. Every open-strip zero has real part strictly below one, and dominated convergence applies in the absolutely summable explicit formula. The auxiliary twists contribute $O_\psi(y^{1/2+\varepsilon/2})$ uniformly on the interval. The coefficient D^* may depend on the two primes but is constant across this interval.

The function F is convex. For $0 < \eta < 1/4$ its right derivative at an interior x is between the backward and forward difference quotients using steps ηx. For $y \log y$ both such slopes equal $\log x + 1 + O(\eta)$. The linear term is exact and the errors in the slopes are $o_{\psi,\varepsilon}(1)/\eta$, uniformly in the interior. Thus $xF'_+(x) - F(x) = x(1 + O(\eta) + o_{\psi,\varepsilon}(1)/\eta)$. First let Y tend to infinity and then η to zero. A ramp at $n \le x$ contributes exactly one to $x(d/dx)_+(x/n - 1)_+ - (x/n - 1)_+$, also at the endpoint. It follows that $xF'_+(x) - F(x)$ is four times the sum of $c_\psi(p^v) \log p$ for $p^v \le x$, $p \in \mathcal{P}_0$, v odd. The primes below $\sqrt{x}$ at the first power cost $O(\sqrt{x} \log(2x))$, and all higher powers cost $O(x^{1/3} \log^2(2x))$ by the counting estimate in Proposition 141.32. This proves the second formula. □

Write $w_j = \int_U^{3U/2} K_k(u) \lambda_j(e^u)\, du$ for the unmodulated positive weights of Theorem 6.40, and put $L_k = (\sigma - 1)^{-k-1}$.

Theorem 141.36 (A necessary bound on the undetected prime mass)**.** *Fix* $1/2 + 4\varepsilon \le \theta < 1$ *and a uniform mesh constant. There is a number* $\delta_* > 0$*, depending on the fixed family and* θ, ε*, with the following property. On any sequence of common intervals on which*

$$\sum_j w_j H_0(x_j) \le B(\sigma - \theta)^{-k-1},$$

one has

$$\sum_j w_j R(x_j) \ll_{\psi,\varepsilon,\theta,B} Y^{-\delta_*} L_k, \qquad 4\sum_j w_j \sum_{p\in\mathcal{I}_{x_j}} c_\psi(p)\log p = (1+o_{\psi,\varepsilon}(1))L_k.$$

Constants may also depend on the mesh constant. For any fixed $0<\lambda\le 1$, the fraction of this last weighted mass on which $s_p(x_j)\le(1-\lambda)c_\psi(p)$ is $O_{\psi,\varepsilon,\theta,B}(\lambda^{-1}Y^{-\delta_*})$. In particular the integral tests cannot hold along such intervals with only the elementary upper cutoffs at every node.

Proof. Let Φ_ψ be the transform in Lemma 2.3. The closed disk about σ of radius $\sigma-1$ lies in $\operatorname{Re} s\ge 1$, where there are no poles. The possible poles are locally finite in $\operatorname{Re} s>0$, and the transform is holomorphic at one. Compactness therefore allows a strictly larger pole-free closed disk of radius $r<\sigma$. Cauchy's inequality gives $|A_k(\sigma)|\ll_\psi r^{-k}$, at this single real center and for every k.

Apply Lemma 6.37 pointwise on each mesh cell with the interpolation weights, and integrate against K_k. It bounds the weighted sum of the nodal $\mathcal{G}_\psi(x_j)$ *from below* by the integral of $\mathcal{G}_\psi(e^u)$ on the window minus $O((\sigma-1/2-4\varepsilon)^{-k-1})$. The tail proof in Theorem 6.40 identifies that integral with $A_k(\sigma)+O_\psi(\sigma^{-k})$. Also $\sum_j w_j x_j^{1/2+\varepsilon}\ll(\sigma-\theta)^{-k-1}$ since adjacent node ratios are at most two on distant windows. Combining the hypothesis and Lemma 141.35 gives

$$0\le\sum_j w_j R(x_j)\ll_{\psi,\varepsilon,\theta,B} r^{-k}+(\sigma-\theta)^{-k-1}.$$

Both r and $\sigma-\theta$ exceed $\sigma-1$. As $k=\lceil 5\sigma\log Y/4\rceil$, division by L_k gives the asserted positive power saving.

For the denominator of the fraction, $\sum_j\lambda_j(x)x_j = x$ exactly. Thus $\sum_j w_j x_j = \int_U^{3U/2} K_k(u)e^u\,du = (1+o(1))L_k$. The omission of nodes outside the interior in the second assertion of the lemma is harmless here, both for x_j and for the prime mass, which is $O(x_j\log(2x_j))$ on the whole interval. Indeed an endpoint node outside the interior, or any such node near either end, has a hat function supported within $O(1)$ in log scale of one of the two endpoints, since adjacent ratios are at most two. On those strips $x_j\log(2x_j)\ll e^{2u}$ wherever its hat is nonzero. The tail estimates in Lemma 2.4 remain $O(\sigma^{-k})$ upon shifting their cutoff by a fixed distance in u, since this changes the exponential majorants in that proof only by constant factors. The sum of all the hats is one. Consequently the interior estimate of Lemma 141.35 gives the second formula after summation. Each prime in the stated exceptional set contributes at least $\lambda c_\psi(p)$ to the positive part defining R. Division of the first formula by the second gives the fraction. For the unmodulated test the three-weight hypothesis at $\tau=0$, $l=0$ is twice the hypothesis here, up to the upward rounding in Corollary 141.31. The rounding is nonnegative. This proves the last assertion for the integral tests as well. □

This gives a quantitative obstruction on the very projected lattices used for the comparison. A proof using these integral tests must recover almost all the indicated prime mass, to the displayed precision. The statement supplies a necessary condition on the valuations and leaves its verification on the period columns open.

141.13 The common divisor at every prime in the middle range

Keep the actual polynomial-conductor fields and the set $\mathcal{I}_x$ of Section 141.11. Here k is the initial rank of a theta word. Write $L_u\subset\Phi_1$ for the h_2 distinguished embeddings at translate u. At an edge i of a word w, write ν_i for its partition tuple and s_i for its target rank. Define

$g_{u,w,p} = \sum_{i=1}^{m} V_p(\nu_i)$, where V_p is the maximum over the two orientations of each even Frobenius cycle in Theorem 68.2.

Theorem 141.37 (Saturation before division of the period columns)**.** *Fix ψ and $0 < \varepsilon < 1/52$. At every sufficiently large node and every $p \in \mathcal{I}_x$, for every actual word and every column on that word allowed in Theorem 141.34,*

$$t_{u,c,p} \;\geq\; g_{u,w,p} \;\geq\; \frac{mkh_2 b}{2p} \;>\; G_p + B_p.$$

Consequently the capped exponents in that theorem satisfy $a_p^* = B_p$, for every choice of columns or translates at each prime.

Proof. The word ranks and the number of parts are bounded by powers of x. On each of the m edges the first k parts at $\tau \in L_u$ equal $b + O(x^C)$. On Φ_0 all parts except possibly the first have size $O(bx^C/X' + x^C)$, with $X' = \exp(x^{\varepsilon/3})$. The first is bounded by $a + O(bx^C/X' + x^C)$. These estimates follow from the partition recurrence in the proof of Theorem 145.9. In detail, padding by zeros and subtracting the return shift cancels the base weight on each newly introduced part. On the original parts the two shifts at successive edges leave only half the rank difference. Their total and the offsets in the word letters are polynomial. Inserting the initial parts in (9.1) gives the stated bounds. In particular the *sum* of the parts at every $\mu \in \Phi_0$ is at most $2a$, since the polynomial number of errors is $o(b)$.

On the indicated signs the permutation $\pi = \sigma_p c_E$ interchanges Φ_0 and Φ_1. Choose Φ_1 as the source half on every cycle. For each $\tau \in L_u$, at the destination $\mu = \pi\tau$ retain just the k indices from τ, of exponent one and row numbers $1, \dots, k$. For each such index the list entry on edge i is at least

$$\max\{0, \lceil (\nu_{i,\tau,l} - s_i(p-1))/p \rceil\} \;\geq\; \frac{b - O(x^C)}{p} - s_i \;\geq\; \frac{3b}{4p}.$$

The last inequality holds uniformly because $p \leq x$ and $b = \exp(x^{\varepsilon})$. The sum of the largest k entries is at least the sum of these k entries. The penalty for choosing them is at most the sum of all parts at μ, hence at most $2a$. Since $pa/(kb) = O(x^2/k) = o(1)$, their net contribution is at least $kb/(2p)$. Distinct τ's yield distinct destinations. All other destinations contribute nonnegatively, by the empty choice. Summing first over destinations and then over the edges proves the second inequality.

For the first inequality, the operator at edge i maps the integral Weyl lattice into $p^{V_p(\nu_i)}$ times the next one, by Theorem 68.2. The integral frames match at consecutive vertices. Clearing the coefficient denominators, the finite stabilizers and the ordinary projector therefore gives a multiple of $p^{g_{u,w,p}}$ on every path, also after taking the signed sum and pairing with each integral test. These are exactly the denominators included in $t_{u,c,p}$. Corollary 68.3 verifies the same assertion at every coefficient place over p in the adapted frames. In an arbitrary marked product frame the scalar change has absolute norm one, as proved in Theorem 141.15. Thus the normalized rational valuation satisfies the asserted inequality in either frame. A zero column has valuation $+\infty$.

Finally the definitions of the two cutoffs give $G_p + B_p \leq mha/p + 1$. Since $h \leq n$, $h_2 \geq nx^{-\varepsilon}$, $a = xb + O(1)$, and $k \geq x^{9/2+10\varepsilon}$, the ratio of $mkh_2 b/(2p)$ to $mha/p + 1$ tends uniformly to infinity. This also proves the assertion for an empty list, by the convention in the definition of a_p^*. □

Corollary 141.38 (Size of the undivided comparison on the common intervals)**.** *On $2Y \le x \le Y^{3/2}/2$ one has $H_0(x) = (1+o(1))x$. For the unmodulated weights of Theorem 141.36,*

$$\sum_j w_j H_0(x_j) = (1+o(1))(\sigma-1)^{-K-1},$$

where $K = \lceil 5\sigma \log(Y)/4 \rceil$ is the order of the kernel. In particular the proposed upper test for the undivided columns fails on every sufficiently distant interval for any fixed $\theta < 1$ in the range of that theorem.

Proof. The preceding theorem gives $s_p(x) = 0$ for every $p \in \mathcal{I}_x$. Both identities of Lemma 141.35 now give the interior asymptotic since $\mathcal{G}_\psi(x) = o_\psi(x)$. For the weighted statement its proof and that of Theorem 141.36 give $4\sum_j w_j \sum_{p \in \mathcal{I}_{x_j}} c_\psi(p)\log p = (1+o(1))(\sigma-1)^{-K-1}$. The evaluation of this last sum there uses only the prime mass and the mesh. Also $\sum_j w_j x_j = (1+o(1))(\sigma-1)^{-K-1}$, and $\sup_{x \ge Y}(|\mathcal{G}_\psi(x)| + x^{1/2+\varepsilon})/x$ tends to zero. Apply the first identity of the lemma at every node. Finally $((\sigma-\theta)/(\sigma-1))^{K+1}$ tends to infinity. □

The failed comparison reads the valuations before removal of the rational divisors in Lemma 141.26. The centered count in Section 6.6 and the valuations after this division are separate open questions.

141.14 The residual valuation of a divided period column

We remove the divisor whose effect was exhibited in Theorem 141.37. On the same fields and at the same nodes, choose any finite list of signed projected columns. Each list member c is attached to one word $w = w(c)$ at one translate u. Scalar combinations within a fixed word and its physical weights have their prescribed denominator C_c. Write d for the common coefficient field degree, and $e_{u,c,p} = t_{u,c,p} - g_{u,w,p}$ for the residual valuation, with value $+\infty$ for a zero column. By Lemma 141.26 the integer

$$b^{\mathrm{per}}_{u,w} = \prod_p p^{g_{u,w,p} - \lceil U_p(u) \rceil}$$

divides each cleared column in the adapted frames. Here the product is over the unramified primes $p \ge 5$ inert in K_d, with the extension convention of that lemma. Division is by a rational scalar on the entire column. Let $t^\circ_{u,c,p}$ be the valuation with the original denominator corrections on this divided column. In the definition of a^*_p use t° in place of t, calling the resulting exponent a°_p. Write $e_p = \min_{(u,c)} e_{u,c,p}$ over the entries read at p, with the same empty-list convention. All sums in this subsection run over $p \in \mathcal{I}_x$.

Lemma 141.39 (The word threshold after centering)**.** *Put $A_p = Q\delta c_\psi(p)(x/p - 1)$. Then $A_p \ge 0$, and uniformly on the common intervals,*

$$\frac{1}{Q\delta} \sum_p \log p \, \max_u \left| \lceil U_p(u) \rceil - G_p - A_p \right| \ll_{\psi,\varepsilon} x^{\varepsilon} \log(2x).$$

Proof. By Lemma 141.33 exactly $k_p = h\delta c_\psi(p)$ high embeddings at each edge have distinguished first destination. There the threshold lies in $[b, b+\sqrt{b}]$. Since the positive part is 1-Lipschitz and $x/p \ge 1$, replacing each such ramp by $b(x/p-1)$ costs at most $\sqrt{b} + 1/p$. At each of the other $h - k_p$ destinations the threshold lies in $[0, 2b/(nX')]$, and $a/p \ge b$. Subtracting the unrounded expression defining G_p leaves between zero and $2mhb/(nX')$ from these destinations. The two roundings cost at most two altogether. Every remaining odd power $j \ge 3$ contributes at most

mha/p^j, and hence all of them together contribute at most $2mha/p^3$. These bounds apply to each translate with the same constants. Thus the absolute difference in the lemma is bounded by

$$mk_p(\sqrt{b}+1/p)+2mhb/(nX')+2+2mha/p^3.$$

After division by $Q\delta = hmb\delta$ and summation with $\log p$, the first three terms are $O_{\psi,\varepsilon,A}(x^{-A})$ for every fixed $A>0$. This uses $\delta \ge x^{-\varepsilon}$ and the superpolynomial growth of b and X'. For the last term use $a/b \le 2x$ and $\sum_{n>\sqrt{x}} n^{-3}\log n \ll x^{-1}\log(2x)$, by integration. This proves the bound. □

Theorem 141.40 (The excess remaining after the rational division)**.** *The number $N^\circ(x) = \prod_p p^{da_p^\circ}$ is a positive integer. Uniformly over the chosen words, translates and subsets of entries read at individual primes,*

$$\frac{\log N^\circ(x)}{dQ\delta} = \sum_p c_\psi(p)(x/p-1)\log p + \sum_p \min\{c_\psi(p), e_p/(Q\delta)\}\log p + O_{\psi,\varepsilon}(x^\varepsilon \log(2x)).$$

The same formula holds if every word at a given translate is divided by the common integer $B_u = \gcd_w b_{u,w}^{\mathrm{per}}$. In particular either normalization gives

$$\frac{4\log N^\circ(x)}{dQ\delta} - M_0(x) = \mathcal{G}_\psi(x) + 4\sum_p \min\{c_\psi(p), e_p/(Q\delta)\}\log p + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}),$$

using the integer for the chosen normalization on the left.

Proof. Removing the word divisor subtracts exactly $g_{u,w,p} - \lceil U_p(u)\rceil$ from the normalized ideal valuation. Thus $t^\circ_{u,c,p} = \lceil U_p(u)\rceil + e_{u,c,p}$, where $e_{u,c,p} \ge 0$. This also holds for a zero column, in the extended sense. The finite norm valuations have denominator dividing d, so that capping at B_p and taking the maximum with zero gives a nonnegative exponent in $d^{-1}\mathbb{Z}$.

Put $C_p = Q\delta c_\psi(p)$. The unrounded upper cutoff is mk_pa/p; since $a = xb + O(1)$,

$$|B_p - (A_p + C_p)| \le 1 + mk_p/p.$$

After normalization and summation the right side is $O_{\psi,\varepsilon,A}(x^{-A})$ for any fixed A. For a nonempty selection with finite minimum, replace $\lceil U_p(u)\rceil - G_p$ by A_p before taking the minimum. The change is at most the maximum absolute error in the lemma. Taking the positive part and capping do not enlarge this error. Replacing the cap by $A_p + C_p$ contributes at most the cutoff error above. As $A_p, e_p \ge 0$, the resulting exponent is $A_p + \min(C_p, e_p)$. If the list is empty or only zero columns are read, both prescriptions take their upper cutoffs and only the cutoff error occurs. Sum over the primes and apply the lemma.

By Theorem 145.3 every quotient $b_{u,w}^{\mathrm{per}}/B_u$ divides the same D_x, for all u, w, with $\log D_x \ll x^C X'$. Thus the valuation using B_u lies between $t^\circ_{u,c,p}$ and $t^\circ_{u,c,p} + v_p(D_x)$. This bound persists under the prime-dependent minima and cutoffs, including when the list is empty. The increase of the normalized logarithm is bounded by $\log D_x/(Q\delta) = O_{\psi,\varepsilon,A}(x^{-A})$. This proves the first formula for both choices. Proposition 141.32 gives the second. □

The nonnegative excess in the last display yields the same sufficient three upper tests as Corollary 141.31 after replacing its integer and center by N°, M_0, and using its rounding procedure with the cutoffs B_p on $\mathcal{I}_x$. Indeed the displayed error integrates to at most the error allowed there. Even an exact evaluation $e_p = 0$ at every prime would recover the centered prime sum to this precision. An upper bound on that sum, or independently on the centered logarithm of the divided columns, is still required.

142 Gram operators in the first theta lifts

142.1 The Petersson pairing as a global Bessel period

We identify the linear functionals measured by Theorem 133.20. Write $S = U \rtimes H$ for the same Bessel group, $[S] = S(F)\backslash S(\mathbb{A}_F)$, $J = K(M)$, $J_S = J \cap S(\mathbb{A}_{F,f})$, and $j_S = \mathrm{vol}_{S_f}(J_S)$. Give S_∞ its previous Haar measure, with the compact H_∞ of mass one. Use the product of this and the finite Haar measure on $[S]$, with counting measure on $S(F)$. Write $g_0 \in G_\infty$ for the tuple of stationary real points. At each real place put $c_{\lambda_v}(h) = \langle w_v, \pi_v(h) w_v \rangle$, where here w_v denotes the unit coefficient vector. Put

$$c_\infty(h) = \prod_v c_{\lambda_v}(h_v), \qquad C(h) = d_\pi c_\infty(h), \qquad \lambda_* = \prod_v (d_v d_{\pi_v}/m_v)^{1/2}.$$

Here $m_v > 0$ is the full real integral in Theorem 132.12, and $d_\pi = \prod_v d_{\pi_v}$. Let v_* be the tensor product of the real value vectors used in Proposition 131.6, and write $\sigma = \sigma_\infty$ for the unitary Bessel action on their value space.

Lemma 142.1 (The compact Bessel quotient at principal level)**.** *The map $(s, c) \mapsto (s, h_c)$ induces a disjoint union*

$$[S]/J_S = \coprod_{c \in \mathcal{C}} \Gamma_c \backslash S_\infty.$$

Its integration rule is j_S times the sum of the real quotient integrals. In particular it is a compact quotient. For $x = (x_c)_c \in \mathcal{H}_X$ the prescription

$$W_x(s, h_c) = \langle x_c, \sigma(s) v_* \rangle$$

defines a continuous function on this union, conjugate linear in x.

Proof. Given a finite point $uh \in U_f H_f$, write $h = \eta h_c k_H$ by the definition of $\mathcal{C}$. Density of $U(F)$ in U_f, proved in Lemma 133.1, permits a rational unipotent u_0 with $h^{-1} u_0^{-1} u h \in J$. Thus uh is in $u_0 \eta h_c J_S$. If two representatives are equivalent, their rational middle component equates their H-classes, so the classes coincide. Its stabilizer at that class is trivial: it preserves a positive definite integral lattice and is principal at an odd split mark, as in Proposition 131.25. The remaining rational stabilizer is exactly $\Gamma_c = U(F) \cap h_c J h_c^{-1}$. Take a measurable real fundamental domain for each such stabilizer. Its product with $h_c J_S$, over the classes, is a fundamental domain upstairs, up to null boundaries. Product Haar gives the asserted integration rule. The real unipotent quotient is compact: the upper triangular central filtration has successive additive lattices of full rank, as in the covolume calculation of Lemma 133.9. One can choose compact parallelepipeds in these quotients and lift and multiply them in the order of the filtration. Since H_∞ is compact and the class set finite, the whole quotient is compact. Finally σ is trivial on Γ_c by the conductor identity used in Theorem 133.2. The displayed matrix coefficient consequently descends and is continuous there. □

Theorem 142.2 (The period represented by the full series)**.** *Let Y be a bounded continuous J-invariant function in $L^2([G])$ reproduced by the archimedean convolution $Y(g) = \int_{G_\infty} C(h) Y(g(h, 1))\, dh$. For all x in the source of the projected Poincaré map,*

$$\langle \mathcal{P} x, Y \rangle = \kappa_* \int_{[S]} W_x(r)\, Y(r(g_0, 1))\, dr, \qquad \kappa_* = \sqrt{j_M / v_U}\, \frac{\lambda_*}{j_S d_\pi} > 0.$$

The same g_0, κ_* work for all finite translates of the test Y and for $T(e_0 e_1) Y$.

Proof. Use the convention that the first variable of an inner product is conjugate linear. Expanding the real average of Proposition 131.6, changing s to s^{-1}, and taking tensor products gives

$$\langle x_c, F(y)\rangle = \lambda_* \int_{S_\infty} c_\infty(g_0^{-1}s^{-1}y) W_x(s, h_c)\, ds.$$

Inversion preserves Haar on S_∞, since its unipotent radical is unimodular and its compact factor acts with modulus one. The coefficient averages converge absolutely place by place by that proposition. This also verifies the order of the factors in $g_0^{-1}s^{-1}y$.

Unfold one seed in the inner product of Theorem 133.12. Its absolute periodization is in $L^2([G])$, hence its product with $|Y|$ is integrable. Its finite support contributes j_M. Summing the coordinates leaves

$$\langle \mathcal{P}x, Y\rangle = \sqrt{j_M/v_U} \sum_c \int_{\Gamma_c\backslash G_\infty} \langle x_c, F(y)\rangle Y(y, h_c)\, dy.$$

Indeed $Y(y, h_c)$ is left Γ_c-invariant by rational invariance and the principal level. The class weight in the seed is a partition for precisely this quotient.

Insert the preceding integral over s. In the double integral take the quotient by diagonal left translation of Γ_c on $S_\infty \times G_\infty$. The product

$$W_x(s, h_c) c_\infty(g_0^{-1}s^{-1}y) Y(y, h_c)$$

is invariant under this action. Either a fundamental set in the y coordinate or one in the s coordinate gives a fundamental set in the product. The equality of the absolute integrals in the two choices follows by partition into countably many translates and Tonelli. With the fundamental set in the s coordinate the absolute integral is at most

$$\|Y\|_\infty \|c_\infty\|_1 \int_{\Gamma_c\backslash S_\infty} |W_x(s, h_c)|\, ds < \infty.$$

The coefficient is integrable by Proposition 131.6, and the last integral is finite by Lemma 142.1. We may now make the same exchange for the complex integral. After substituting $y = sg_0h$, the inner integral over G_∞ equals $d_\pi^{-1} Y(sg_0, h_c)$ by reproduction. Apply the quotient integration rule to the sum over c and divide by j_S. This gives the formula. Right translations at the marked primes commute with real convolution and preserve level since J is normal in K; their finite linear combinations preserve the stated class of tests. □

Corollary 142.3 (The dimension of the projected period functionals)**.** *On the closed archimedean type space in $L^2([G])^J$, the functionals initially given on bounded continuous tests by*

$$Y \longmapsto \kappa_* \int_{[S]} W_x(r)(T(e_0e_1)Y)(r(g_0, 1))\, dr$$

extend continuously. Their span has dimension at least $(1 - 4b^{-8})D_X$. Allowing also finite translates gives dimension at least $(1 - 2b^{-4})^2 L D_X$, with L as in Theorem 133.20.

Proof. Convolution by C is an orthogonal projection in the right regular action. Indeed $C(h^{-1}) = \overline{C(h)}$; Schur orthogonality applied to the same unit coefficient twice gives $C * C = C$ with the factor d_π used above. Since $C \in L^1$, both identities act on L^2. The range is the type space in question. Continuous, compactly supported, J-invariant functions on $[G]$ are dense in $L^2([G])^J$, by density on the locally compact quotient followed by compact averaging over J. Convolving them by C gives a dense set in its range. These convolved tests are bounded and continuous by dominated convergence and reproduce pointwise by Fubini.

On that dense set the theorem and self-adjointness identify the functionals with $Y \mapsto \langle \mathcal{P}_e x, Y\rangle$. The representing vectors lie in the type space by Theorem 133.12, and yield the asserted extensions there. For a finite translate of the functional apply it to $T(s)^*Y$; its representing vector is $T(s)\mathcal{P}_e x$. The two bounds now follow from Theorems 133.18 and 133.20. All constants and periods use the fields and nodes of those theorems. Estimating the integral lattice of the composite theta lift requires control of its arithmetic normalization and of its values on these functionals. □

142.2 The first theta lift of the projected Poincaré series

Keep $G = U(Y_0)$, and put $G_1 = U(Y_1)$, where the dimensions of the two Hermitian spaces are r and $r + t$, $t = 5r$. Use a tame twist of the shorter construction. Its additional finite level places are disjoint from the two marked primes. Let $K_\Phi(g, h)$, for $g \in G(\mathbb{A}_F)$ and $h \in G_1(\mathbb{A}_F)$, denote the theta series of the Weil translate $\omega(g, h)\Phi$. We use the bilinear convention for the lift, integrating the input form times K_Φ. Its datum on the first edge is the specified joint harmonic times the Gaussian at infinity. It is the point datum of the inclusion modulo the marked and twist places times the vacuum at finite places. In particular in a rational Schrödinger polarization it is Schwartz–Bruhat. The estimates below also apply to arbitrary Schwartz–Bruhat data of this pair.

Lemma 142.4 (Absolute growth of the first theta kernel)**.** *Put $d = fr(r + t)$. In sign coordinates at the real places, write $H(y) = \prod_{v|\infty} \max(1, \|y_v\|_{\mathrm{op}})$. For g_f and h in fixed compact sets and $y \in G_\infty$, the sum of the absolute values in $K_\Phi((y, g_f), h)$ is at most $C_0 H(y)^{2d+2}$. Here C_0 may depend on the node, the datum and the compact sets. On the same family, at every sufficiently large node,*

$$\int_{G_\infty} |C(y)| H(y)^{2d+2}\, dy < \infty.$$

Proof. The underlying symplectic space is the restriction of scalars of the Hermitian Hom space with its alternating trace pairing. A rational Lagrangian has real dimension d after restriction to $\mathbb{Q}$. Choose rational coordinates on it. The rational indices selected by the compact support of the finite data are contained in a fractional lattice in $\mathbb{R}^d$. This can be done uniformly on the sets in the lemma: a finite Schwartz–Bruhat vector in the local Weil representation is fixed by an open subgroup, hence has only finitely many translates under a compact set. The same holds at the finite adeles by the restricted tensor construction. Finitely many compact finite supports lie in a common multiple of the standard compact lattice after clearing denominators. Their sup norms are bounded as well.

At infinity let Q_i and P_i be coordinate multiplication and $(2\pi i)^{-1}$ times coordinate differentiation on $\mathbb{R}^d$, respectively. Under conjugation by the unitary Weil operator these $2d$ infinitesimal Heisenberg operators transform by the underlying symplectic matrix or its inverse. This follows by differentiating the covariance of translations and modulations defining the Weil representation. In the unitary dual pair the symplectic action on the Hom space is $A \mapsto hAy^{-1}$. In sign coordinates $\|y_v^{-1}\|_{\mathrm{op}} = \|y_v\|_{\mathrm{op}}$; the matrices of h_∞ and their inverses are bounded on the chosen compact set. Fixed changes to real symplectic coordinates change the norm by a constant. Thus every coefficient in the transformation is at most a constant times $H(y)$. The determinant splitting characters have absolute value one and do not change conjugation.

For any word of length $j \le 2d + 2$ in Q_i, P_i, its L^2-norm on the transformed datum is at most $C_1 H(y)^j$. Indeed move the unitary operator to the left, expand each factor into a linear combination, and bound the finitely many fixed word norms on the original Schwartz vector. Apply the Euclidean Sobolev inequality with $d+1$ derivatives to the transformed function and to

each of its products by a coordinate to power $d+1$. Every derivative involved is a word covered by this estimate. It follows that the value at $u \in \mathbb{R}^d$ is bounded by $C_2 H(y)^{2d+2}(1+\|u\|)^{-d-1}$. The sum of these last radial weights over a fixed fractional lattice converges, as its ball of radius $R \geq 1$ has $O(R^d)$ points. Finite sums of elementary data follow by addition.

Finally $f \ll_{q_0,d_K} x^{11}\log(2x)$, where d_K denotes the fixed quadratic-field parameter, and $x^5 \leq r \leq 3x^5$. Hence $2d+2+9r^2 < r^8$ eventually. Lemma 132.8, with the unitary bound on $\|y_v\| \leq 2$, bounds $|c_{\lambda_v}(y_v)|\max(1,\|y_v\|)^{2d+2}$ by a node-dependent constant times $(2/\|y_v\|)^{9r^2}$. This last majorant is integrable by Lemma 132.13. Take products over the real places and multiply by the finite scalar d_π. □

Let Q_J denote right averaging over J with probability Haar. For a function of the first variable put

$$(\mathcal{C}B)(g) = \int_{G_\infty} C(y)B(g(y,1))\,dy, \qquad B_h(g) = \overline{K_\Phi(g,h)}.$$

The averaging Q_J permits twists ramified away from M, where the unipotent series still has its original level. Let I_0 be the finite index of the principal level at the marks and the twist places in the standard finite compact of Y_0; it is the normalization of the first edge. Write $\Theta_\Phi u(h) = I_0 \int_{[G]} u(g)K_\Phi(g,h)\,dg$ whenever this integral converges.

Theorem 142.5 (An integral formula for the first projected lift)**.** *For the prescribed first-edge data, every h and every $x \in \mathcal{H}_X$, the integral defining $\Theta_\Phi(\mathcal{P}_e x)(h)$ converges absolutely and*

$$\overline{\Theta_\Phi(\mathcal{P}_e x)(h)} = I_0\kappa_* \int_{[S]} W_x(r_0)(\mathcal{C}\,T(e_0e_1)Q_J B_h)(r_0(g_0,1))\,dr_0.$$

The integral on the right, including the inner real convolution, is absolutely convergent locally uniformly in h. The formula holds also for the unprojected map with the finite projection omitted.

Proof. First use the unprojected series. We verify absolute unfolding before taking the finite average. By the seed formula of Theorem 133.12, the integral of its absolute periodization against $|B_h|$ unfolds by Tonelli. Finite integration is over $h_c J$. The average of $|B_h|$ over this coset, as a function of y, is left Γ_c-invariant, since B_h is automorphic in its first variable and $h_c^{-1}\Gamma_c h_c \subset J$ at finite places. Majorize $\|F(y)\|$ using the absolute value of the integral for F in Theorem 142.2. Moving the fundamental domain from y to s as in that theorem now bounds the absolute integral by a finite sum of constants times

$$\int_{\Gamma_c\backslash S_\infty}\int_{G_\infty} |c_\infty(y)|\,(Q_J|B_h|)(sg_0y,h_c)\,dy\,ds.$$

The majorant here can be integrated on a compact set of representatives for the outer quotient. For those s, $H(sg_0y) \ll H(y)$; all finite arguments remain in the compact coset just used. Lemma 142.4 gives integrability with a uniform bound for h in compact sets. The same reasoning holds after any of the finitely many right translations at M.

For the complex integral, right J-invariance of the series replaces B_h by $Q_J B_h$. To move the finite projection from the first to the second factor in the Petersson pairing, change variables separately for each finite translate and use self-adjointness of the group algebra element. The preceding estimates justify these changes even when B_h itself is outside $L^2([G])$. Repeat the complex unfolding of Theorem 142.2 through the substitution $y' = sg_0y$. The resulting inner integral is $d_\pi^{-1}(\mathcal{C}T(e_0e_1)Q_J B_h)(sg_0,h_c)$ by the definition of $\mathcal{C}$. The quotient rule and normalizations in that theorem give the displayed identity. All estimates are uniform in h on compact sets as asserted. □

Corollary 142.6 (An integer acting on the first theta values)**.** *Write* $|\mathcal{G}|e_0e_1 = \sum_{s\in\mathcal{G}} n_s s$*. At the marked split components on* Y_1 *choose* $k_{1,s}$ *reducing to* $\mathrm{diag}(s, 1_t)$*; use the identity outside the marks. For the first-edge datum and the same twists and nodes one has*

$$|\mathcal{G}|\,\Theta_\Phi(\mathcal{P}_e x)(h) = \sum_s n_s\,\Theta_\Phi(\mathcal{P}x)(hk_{1,s}), \qquad n_s \in \mathbb{Z}, \qquad \sum_s |n_s| \le |\mathcal{G}|^{3/2}.$$

Consequently any additive coefficient lattice containing the listed unprojected translates contains the scaled projected value. The cost $\log|\mathcal{G}|$ *of this integer is* $\ll_{q_0,d_K} x^{21}\log(2x)\log b$ *at the common nodes.*

Proof. Let $k_{0,s}$ lift s on Y_0. At each marked split component the point datum is the indicator, in the finite Schrödinger matrix coordinates, of reduction to the inclusion $\iota : y \mapsto (y, 0)$. Acting by $(k_{0,s}, k_{1,s})$ makes the substitution $A \mapsto k_{1,s}^{-1}Ak_{0,s}$, which preserves this condition because $\mathrm{diag}(s, 1_t)\iota = \iota s$. For completeness, this substitution on a split polarization is the action of the linear change of variables preserving the position subspace. In the Schrödinger model its unitary normalization is the square root of the absolute determinant of the substitution, equal to one on units. The dual component acts by the inverse transpose. The tame determinant character of the pair on the integral elements is the character of $\det(k_{1,s})/\det(k_{0,s})$ on reduction: the twist uses the same tame character on both groups, with inverse on the source. The rank-dependent unramified factors are trivial on units. The two residue determinants agree, so this scalar is one. At the twist places the pair is the identity. The full datum is therefore fixed. The resulting kernel identity reads

$$K_\Phi(g, hk_{1,s}) = K_\Phi(gk_{0,s}^{-1}, h).$$

After the change $g \mapsto gk_{0,s}$ in the lift integral it gives $\Theta_\Phi(T(s)\mathcal{P}x)(h) = \Theta_\Phi(\mathcal{P}x)(hk_{1,s})$. Absolute convergence holds by the theorem and its proof. The integer coefficients follow from Theorem 130.2, which proves the group-algebra assertion independently of the space on which the idempotent acts. The sum of the squares of the coefficients of $e = e_0e_1$ equals its identity coefficient, at most one: take that coefficient in $e^*e = e$ and use the normalized regular trace. Cauchy–Schwarz gives the asserted sum of absolute values after multiplication by $|\mathcal{G}|$. The size estimate for $|\mathcal{G}|$ is in the same theorem. This controls the denominator introduced by this finite projection. The arithmetic size of the unprojected values and their behavior under the remaining lifts are additional parts of the integral comparison. □

142.3 The finite Gram operators in the first and auxiliary lifts

For the first theta lift and for each of the m auxiliary single lifts in the shorter construction, the source has rank r and the target has rank $n' = r + t = 6r$. We compute the finite-place operator formed with the matrix coefficient of their theta datum. This operator occurs in the doubling integral. At each finite place w of F give the stabilizer K_w of the standard source lattice mass one. Give the integral matrix lattice in a split Schrödinger model mass one as well.

Lemma 142.7 (The coefficient at the finite level places)**.** *At a marked or twist place write* q *for the residue cardinality and* $J_w = \ker(K_w \to \mathrm{GL}_r(\mathbb{F}_q))$*. For the inclusion point datum of the first edge, the source matrix coefficient is* $q^{-n'r}\mathbf{1}_{J_w}$*. For an auxiliary lift it is the same at the marked places and is* $v_r(q)\mathbf{1}_{K_w}$ *at a place of its tame modulus, where* $v_r(q) = \prod_{i=1}^{r}(1 - q^{-i})$*.*

Proof. These are split places. In the chosen component write A for an n'-by-r matrix and A_0 for its r rows in the specified inclusion. All data in question are zero outside integral matrices. The point datum imposes $A \equiv \iota$; the block datum at a twist place has value $\chi(\det \overline{A_0})$ if $\overline{A_0}$ is

invertible and zero otherwise. Here χ is the local tame character with the source normalization of the lift. Thus if both A and Ag are in the support, both A_0 and $A_0 g$ are integral with unit determinant. Their quotient gives $g \in K_w$. For the point datum it also gives $g \equiv 1$, and every such g fixes the function. Its squared norm is the measure of one residue class, $q^{-n'r}$. For the block datum, substituting Ag multiplies the function by $\chi(\det \bar{g})$. The inverse determinant character on the source in the Weil action cancels that scalar. The remaining squared norm is the fraction of invertible top blocks, namely $|\mathrm{GL}_r(\mathbb{F}_q)|/q^{r^2} = v_r(q)$. This proves the assertions for the actual matrix coefficients, with zero at all other group elements. □

Away from these places let $L_w = (\mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_w})^r$ in the finite lattice frame. Use the local vacuum of squared norm one. Write $a_w(g)$ for its source matrix coefficient, including the unitary splitting character. In the standard unramified normalization it equals one on K_w and is bi-invariant under K_w.

Lemma 142.8 (The mass away from the integral stabilizer)**.** *For every such place, of residue cardinality q in F, one has*

$$|a_w(g)| \le [L_w : L_w \cap gL_w]^{-n'/2}, \qquad \sum_{gK_w \neq K_w} |a_w(g)| \le 128 q^{-r-2}$$

for $r \ge 2$.

Proof. Recall the lattice normalization of the vacuum. The Hom lattice is self-dual for its Heisenberg commutator pairing, with the additive character scaled by the different; the target and source lattice frames are standard. On each component its trace dual is the inverse different times the Hermitian dual, so the scaling cancels that different. The split or unramified quadratic lattice model suffices at all these places. Indeed E contains both $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-d_K})$ with relatively prime discriminants: $d_K \equiv 7 \bmod 8$. For any rational prime one of the two fields is unramified there, and inertia fixes that field. Complex conjugation acts nontrivially on both, so cannot belong to inertia in the abelian field E. Thus E/E^+ is unramified at finite places.

Here is the coefficient estimate in the self-dual lattice model. The unit vacuum spans the eigenspace for the integral Heisenberg lattice with its vacuum character. To see that this space is a line, use the realization by functions equivariant on the left by that lattice. Right action by a lattice element on the coset of z introduces its commutator with z. Agreement with the character for every lattice element forces z to lie in the dual lattice, which is the lattice itself. Equivariance then determines the function from its one value on the identity coset. The standard compact splitting fixes this unit vector: integral isometries preserve the lattice and act by substitution in this model; the determinant twists away from the marked and tame modulus places are unramified. The lattice character can also be described at dyadic unramified places by using an integral sesquilinear form whose antisymmetrization is the given alternating form. Locally choose an integral element of relative trace one, possible by surjectivity of the trace in an unramified quadratic extension (lift a unit trace from the residue field and divide by that unit). Multiplying a purely imaginary trace coefficient by this element gives the required sesquilinear coefficient. In this model its bicharacter on integral vectors is trivial, and the substitution by an integral isometry preserves it.

Project $\omega(g)$ of the vacuum back onto the vacuum line by averaging the lattice action divided by its character, with probability Haar. The squared norm of this projection is the integral of the corresponding lattice coefficients of the translated vacuum. Such a coefficient vanishes outside the intersection of the original and translated Hom lattices: commuting the translating vector past an element of the translated lattice multiplies it by a nontrivial phase if that vector is outside its self-dual lattice. Inside the intersection the absolute value is at most one. The

integral is at most the reciprocal intersection index. On Hom the source acts by precomposition; in matrix coordinates there are n' copies of the dual source lattice. The index on a dual lattice exchanges the two indices of an intersection. They are equal for g unitary, since its determinant has absolute norm one on the whole local quadratic algebra. Taking a square root gives the first inequality.

Over a discrete valuation ring of residue size Q, the number $A_r(j)$ of sublattices of index Q^j in a free rank r lattice satisfies

$$A_r(j) = \sum_{j_1+\cdots+j_r=j} Q^{\sum_{i=1}^{r}(i-1)j_i} \le 4Q^{(r-1)j}.$$

Triangular normal form over the valuation ring gives the formula, with the diagonal powers in the order for which there are $i-1$ free entries modulo the i-th power. For the bound divide by $Q^{(r-1)j}$ and sum each of the first $r-1$ indices freely. The resulting factor is at most $\prod_{l\ge1}(1-2^{-l})^{-1} < 4$: keep the first three factors and use $\prod_{l\ge4}(1-2^{-l}) \ge 1-\sum_{l\ge4}2^{-l}$. Duality gives the same count for overlattices of prescribed index in the ambient vector space.

At a nonsplit place the coset is determined by gL_w. If the intersection with L_w has index Q^j, it has that same index in gL_w. First choose the intersection, then the overlattice. There are at most $16Q^{2(r-1)j}$ choices. The first inequality gives total mass at most $\sum_{j\ge1}16Q^{-(r+2)j}$, as $n' = 6r$, and $Q \ge q$. At a split place the coset is determined by a single rank r lattice in the chosen component. Write q^u, q^v for the two indices of its intersection with the standard lattice. The other component is the inverse dual, so the index on the full quadratic algebra is q^{u+v}. There are at most $16q^{(r-1)(u+v)}$ possibilities for this pair of indices. The sum away from the integral coset is at most $\sum_{j\ge1}16(j+1)q^{-(2r+1)j}$. Both geometric sums are at most $128q^{-r-2}$ for $r \ge 2$. □

Theorem 142.9 (The finite factor in the first lift and the auxiliary lifts)**.** *Let S_f here denote the set of marked and twist places. For the relevant source representation by unitary right translations T, on vectors fixed by all K_w outside S_f, put*

$$\mathcal{R} = \sum_{g\in G(\mathbb{A}_{F,f}^{S_f})/K^{S_f}} \left(\prod_{w\notin S_f} a_w(g_w)\right) T(g), \qquad K^{S_f} = \prod_{w\notin S_f} K_w.$$

This series converges in operator norm on the indicated space. It is self-adjoint and

$$\|\mathcal{R} - I\|_{\mathrm{op}} \le 512 f\, 2^{-r} =: \varepsilon_r$$

at all sufficiently large nodes, for every choice of the twist. Including the local integrals at the level places multiplies it on level-fixed inputs by the following scalars. For an auxiliary lift with tame set $\mathcal{Q}$ the scalar is

$$c_{\mathrm{aux}} = \frac{M^{-fn'r}}{|\mathcal{G}|} \prod_{w\in\mathcal{Q}} v_r(q_w).$$

For the first edge it is

$$c_{\mathrm{first}} = \frac{M^{-fn'r}}{|\mathcal{G}|} \prod_{w\in\mathcal{Q}} \frac{q_w^{-n'r}}{|\mathrm{GL}_r(\mathbb{F}_{q_w})|}.$$

These are the finite Gram factors prior to the scalar normalizing the lift. A stable-level input on Y_0 fixed by the maximal compact at a twist place is in particular fixed by its principal subgroup.

Proof. View each summand as an operator from the fixed space into the ambient Hilbert space, of norm at most the absolute coefficient. Its value is independent of the right coset

representative. Bi-invariance of the coefficient and left reindexing by K^{S_f} show that the sum maps back into the fixed space. Lemma 142.8 bounds the sum of the absolute coefficients other than the identity by

$$\prod_{w\notin S_f}(1+128q_w^{-r-2})-1\le\exp\left(128f\sum_{n=2}^{\infty}n^{-r-2}\right)-1.$$

Indeed at most f places lie over any rational prime p, with $q_w\ge p$. The first term plus an integral bounds the last series by $2^{-r-2}(1+2/(r+1))$. Now $f2^{-r}\to 0$. Using $e^z-1\le 2z$ for sufficiently small $z\ge 0$ gives the stated bound and absolute convergence. Equivalently the sum is convolution by the bi-invariant integrable coefficient, since every right coset has volume one. Unitarity gives $a_w(g^{-1})=\overline{a_w(g)}$, so inversion in this unimodular group gives self-adjointness.

At each level place integrate the coefficient of Lemma 142.7. On the indicated fixed vectors the integral over J_w acts by its volume $|\mathrm{GL}_r(\mathbb{F}_{q_w})|^{-1}$; on K_w-fixed vectors integration over K_w acts by one. There are f marked places of each residue size p_i. Multiplying these factors proves both formulas. □

Corollary 142.10 (Determinants of the normalized finite factors)**.** *On any nonzero finite-dimensional subspace of the indicated fixed space, the compression of $\mathcal{R}$ has positive determinant, and the absolute logarithm of that determinant divided by the subspace dimension is at most $2\varepsilon_r$. Summing these normalized logarithmic errors for all m auxiliary lifts costs at most $1024mf\,2^{-r}$, uniformly in all their tame sets.*

Proof. For large nodes $\varepsilon_r<1/2$. Every Rayleigh quotient of $\mathcal{R}$ on a unit vector belongs to $[1-\varepsilon_r,1+\varepsilon_r]$ by the theorem. This also holds after orthogonal compression, so the absolute logarithm of each eigenvalue is at most $2\varepsilon_r$. Sum over the eigenvalues and then over the lifts. This estimate controls the finite convolutions after division by their displayed scalars. The real local factor and the integral lattice in the global lift enter separately. □

142.4 The real Gram factor and the source of the bilinear lift

The first lift pairs a form bilinearly with its theta kernel. We identify the two types in this pairing. Inner products are linear in the second variable. Write $c_\lambda(g)=\langle w,\pi_\lambda(g)w\rangle$ for the coefficient of the unit vector on the prescribed highest line.

Lemma 142.11 (The two conjugate spaces of series)**.** *For $e=e_0e_1$ define a linear map on the conjugate complex Hilbert space by*

$$P^+:\overline{\mathcal{H}_X}\longrightarrow L^2([G])^J,\qquad P^+(\bar{x})=\overline{T(e)\mathcal{P}x}.$$

Its singular values are those of $T(e)\mathcal{P}$. Its values are cuspidal and fixed by convolution with $\prod_{v|\infty}d_{\pi_v}\overline{c_{\lambda_v}}$. On their span and its span of finite translates,

$$\langle Z,T(g)Y\rangle=\left(\prod_{v|\infty}c_{\lambda_v}(g_v)\right)\langle Z,Y\rangle\qquad(g\in G_\infty).$$

Proof. Conjugation is antiunitary on the automorphic quotient and commutes with pullback by right translations. It commutes with the finite projection because e has rational coefficients by Corollary 142.6. It also preserves vanishing of unipotent constant terms. Conjugate the reproducing identity of Theorem 133.12 and use Theorem 133.18. The two conjugations preserve the eigenvalues of the Gram operator with multiplicity.

At one place put $b_\lambda = d_\pi \bar{c}_\lambda$. Then $b_\lambda * \delta_g * b_\lambda = c_\lambda(g) b_\lambda$. Indeed the convolution at h equals

$$d_\pi^2 \int \overline{c_\lambda(y)} \langle \pi_\lambda(h) w, \pi_\lambda(y) \pi_\lambda(g) w \rangle \, dy.$$

The formula follows by Schur orthogonality. Each integral converges by Cauchy–Schwarz. At these parameters b_λ is integrable by Lemmas 132.8 and 132.13. Its convolution acts by a self-adjoint projection, by the formula at $g = 1$ and the identity $b_\lambda(h^{-1}) = \overline{b_\lambda(h)}$. Insert the projection on both sides of the matrix entry. Take products over the places. Finite translations commute with this real projection. □

At a real source $U(p, q)$, put $r = p + q$, $n' = 6r$, $l = \min(p, q)$, and $K = U(p) \times U(q)$. Write z_{ai} for the Fock coordinates, where a indexes the target and i the source. The polynomial inner product is $\langle z^\nu, z^\xi \rangle = \mathbf{1}_{\nu=\xi} \nu!$. The Gaussian is included when passing to Weil vectors. We use the following property of a joint harmonic on the prescribed lines:

$$\partial_{ai} \partial_{aj} P = 0 \quad \text{if } i, j \text{ have opposite source signs, for every } a. \tag{$*$}$$

For products of flag minors, it suffices that all factors using a given target index use source indices of one sign. Each monomial then satisfies $(*)$. In the bilinear convention the source parameter of the joint harmonic is $\eta = -\lambda$: the conjugate of the automorphic highest line is highest in the opposite compact order and has all torus weights negated. Thus $c_\eta = \bar{c}_\lambda$. We prove the following estimates for every nonzero P with $(*)$ in that compact highest line. The estimates retain its Fock norm, so they apply at any choice of scale.

Lemma 142.12 (The oscillator coefficient on a positive Cartan subgroup)**.** *Let $a(u)$ be the product of the l disjoint hyperbolic boosts on pairs of opposite source signs, with parameters $u_1 \geq \cdots \geq u_l \geq 0$. Put*

$$H_i = n' + \sum_{a=1}^{n'} (z_{ai_+} \partial_{ai_+} + z_{ai_-} \partial_{ai_-}).$$

If P_1, P_2 satisfy $(*)$, then

$$\langle P_1, \omega(a(u)) P_2 \rangle = \left\langle P_1, \prod_{i=1}^{l} (\cosh u_i)^{-H_i} P_2 \right\rangle.$$

In particular $A_P(g) = \langle P, \omega(g) P \rangle$ is in $L^1(U(p, q))$. If $z \in U(p, q)$ is positive Hermitian and $\log \|z\|_{\mathrm{op}} \leq u$, then

$$\|P\|^2 e^{-(n' l + \deg P) u} \leq A_P(z) \leq \|P\|^2.$$

Proof. A compact source change of basis mixes coordinates only within each sign in each target row, by a unitary linear substitution or its conjugate. It preserves $(*)$ and degree. One paired source boost acts infinitesimally by $E_i - F_i$, where

$$E_i = \sum_a \xi_a z_{ai_+} z_{ai_-}, \qquad F_i = E_i^*, \qquad |\xi_a| = 1.$$

Indeed the source boost or its inverse mixes the two entries of a row with coefficients $\cosh u, \pm \sinh u$. Conjugating the negative coordinate to obtain the Fock polarization makes the mixed term interchange multiplication and its adjoint, differentiation. The commutators of the displayed quadratic operator with multiplication and differentiation give these infinitesimal changes, with phases according to the coordinate bases. Its scalar term is zero: the symmetric ordering of the quadratic multiplication and differentiation terms has no diagonal commutator in two distinct coordinates. The determinant splitting character is one along the boost.

Direct differentiation gives $[F_i, E_i] = H_i$, $[H_i, E_i] = 2E_i$, $[H_i, F_i] = -2F_i$. For V killed by F_i, the boost acts by

$$\exp((\tanh u)E_i)(\cosh u)^{-H_i}V.$$

To verify it, differentiate this vector in u and use $F_i e^{tE_i} = e^{tE_i}(F_i + tH_i + t^2E_i)$. Its derivative is $(E_i - F_i)$ times the vector and its initial value is V. All differentiations hold in Fock norm on compact u-intervals: for two variables the squared norm series of $e^{t\xi zw}$ times a monomial has terms bounded by $|t|^{2j}$ times a polynomial in j, and $|t| < 1$. The same bound allows derivatives and finite tensor products. Uniqueness of the differentiable orbit under the unitary group proves the formula. Distinct boost pairs commute. Property $(*)$ gives $F_iP_1 = F_iP_2 = 0$. Move each nonconstant creation term onto P_1 by its adjoint to obtain the stated pairing.

For $z = ka(u)k^{-1}$, apply the formula to $k^{-1}P$. The H_i are diagonal, and their sum on a monomial of this vector is at most $n'l + \deg P$. Since $\cosh u_i \le e^u$, this proves the two inequalities. For $k_1a(u)k_2$ the same identity with two compact translates bounds $|A_P|$ by $\|P\|^2 \prod_i (\cosh u_i)^{-n'}$. The Cartan integration formula for $U(p,q)$ has radial density proportional to

$$\prod_{i<j} \big(\sinh(u_i - u_j)\sinh(u_i + u_j)\big)^2 \prod_i (\sinh u_i)^{2|p-q|} \sinh(2u_i).$$

The restricted roots $e_i \pm e_j, e_i, 2e_i$ have respective real multiplicities $2, 2|p-q|, 1$ from the complex off-diagonal entries, unpaired complex coordinates and imaginary diagonal entry. The density on the chamber is at most a constant times $\prod_i e^{2(r-1)u_i}$. Since $n' > 2(r-1)$, the preceding bound is integrable. If $l = 0$ the group is compact and the products are empty. □

Theorem 142.13 (A quantitative real factor for the bilinear lift)**.** *Let P be as above, and put $\mu = \eta + \rho - 2\rho_c$, $\alpha = \eta + \rho$ in the positive-root order of η. Let V_1 be the Haar volume of $\log \|g\|_{\mathrm{op}} \le 1$ in $U(p,q)$, and put*

$$\delta = (1 + n'l + \deg P + r(\max \alpha - \min \alpha))^{-1}.$$

The real local factor in the bilinear pairing satisfies

$$\mathfrak{a}(P, \eta) := \int_{U(p,q)} \overline{c_\eta(g)} A_P(g)\, dg \;\ge\; \frac{e^{-1}\|P\|^2 V_1}{\dim V_\mu} (\delta/2)^{2pq} > 0.$$

For fixed real data it is the same for every finite tame twist.

Proof. Give K probability Haar. In polar coordinates $g = kz$, write $d\nu$ for the quotient measure in $z > 0$. It is invariant under conjugation by K. With $M_\mu = 1/\dim V_\mu$, the formula of Proposition 132.7 gives

$$\mathfrak{a}(P, \eta) = M_\mu \int_{z>0} \frac{\langle P, \omega(z)P\rangle}{m_{-\alpha}(z)}\, d\nu(z).$$

To see the identity, truncate at $\log \|z\| \le R$ and use the polar formula in the decreasing order of η. For its compact variable u, orthogonality on the irreducible compact copy generated by P gives

$$\int_K \overline{m_\mu(ku)} \langle P, \omega(k)X\rangle\, dk = M_\mu \langle \omega(u)P, X\rangle.$$

The component of X perpendicular to that compact copy contributes zero. Take $X = \omega(z)P$ and change z to uzu^{-1}. The denominator becomes $m_{-\alpha}(z)$. The remaining compact integral uses $\int_K m_\mu(u)\omega(u^{-1})P\, du = M_\mu P$, again by orthogonality. This proves the truncated formula. Its left side converges absolutely by Lemma 142.12 and $|c_\eta| \le 1$. Its right integrand is positive

by the same lemma and positivity of the principal minors of z. Monotone convergence gives the identity.

On $\log\|z\| \le \delta$, one has $\det z = 1$, and every principal minor of size h is at most $e^{h\delta}$. The weight-minor formula in Proposition 132.7 bounds $m_{-\alpha}(z)$ above by $\exp(r(\max\alpha - \min\alpha)\delta)$. Together with Lemma 142.12 this bounds the ratio in the integral below by $e^{-1}\|P\|^2$. The radial volume of this neighborhood is at least $V_1(\delta/2)^{2pq}$. Indeed scale the Cartan coordinates in the unit chamber by δ. The root values x there belong to $[0,2]$; hence $\sinh(\delta x) \ge \delta x \ge (\delta/2)\sinh x$. The sum of the root multiplicities plus the l radial coordinates is $2pq$. The density in the lemma gives the stated volume ratio, including the compact case. All terms in the defining integral are real-place data; the finite tame twist has real weight zero. □

142.5 Petersson norms of the first and auxiliary theta lifts

Let S, T be one of the pairs of ranks $r, 6r$ in Theorem 142.9. In this subsection S denotes the source Hermitian space. Write $G = U(S)$, $H = U(T)$ and $K_\Phi(g,h) = \theta(\omega(g,h)\Phi)$. The splitting characters and the global additive character are those of the pair. Put

$$a(u) = \langle \Phi, \omega(u,1)\Phi\rangle, \qquad \Theta Y(h) = I\int_{[G]} Y(g)K_\Phi(g,h)\,dg.$$

Here I is the scalar multiplying this lift, including its level index when present. We retain the Haar scalar in the doubling formula. This is needed when comparing lifts on different Hermitian spaces.

Proposition 142.14 (The doubled integral at the chosen split point)**.** *For the data with a full-column-rank support at a marked split component, and the real polynomials of Lemma 142.12, one has $a \in L^1(G(\mathbb{A}_F))$ and*

$$\int_{[H]} \overline{K_\Phi(g',h)}K_\Phi(g,h)\,dh = c_{S,T}\sum_{\gamma\in G(F)} a(g'^{-1}\gamma g).$$

The left integral and the series are absolutely convergent. The positive constant $c_{S,T}$ depends on measures and the normalization of the oscillator inner product. For the same pair and measures it is the same for all tame twists.

Proof. The oscillator representation for $S \oplus S^-$, where the second form is negated, is the tensor product of the one for S and its complex conjugate. In position coordinates this says that the alternating character on the negative summand is the conjugate character. The two global theta sums multiply by summing the two rational coordinates independently. A rational change of polarization preserves that identity by adelic Poisson summation with self-dual measure. Thus the integrand is the doubled theta series at the restriction of (g,g') to the two summands.

We use the convergent unitary Siegel–Weil formula, in S. Yamana, *The Siegel–Weil formula for unitary groups*, Pacific J. Math. 264 (2013), Theorem 2. The doubled source is split of Witt index r, the target has dimension $6r$ and its rational Witt index is at most $3r$. Thus dimension minus Witt index exceeds r. The theta integral converges absolutely, and its defining Siegel Eisenstein series is in its half-plane of absolute convergence: the parameter is $(6r-r)/2 > r/2$. The finite data are Schwartz–Bruhat, and the data at the identity at infinity are polynomials times Gaussians. The formula applies as an identity of functions on the doubled group and hence at (g,g'). In the split doubled space the Siegel cosets index its rational maximal isotropic subspaces L, each once, by extending a basis of such a subspace to a hyperbolic frame. Its summand is evaluation at zero in the position model belonging to L.

Here is the role of the marked point. If the projection of L onto S^{-} has a kernel, choose $0 \neq v \in S(F)$ with $(v,0) \in L$, where rational vectors are taken over E. At a marked split place take its nonzero component in the position coordinates. The functional for L is invariant under the Heisenberg operators in its momentum Lagrangian: in its own model these multiply by characters with value one at zero. In the split matrix realization on the two factors this includes the multipliers

$$(A, A') \longmapsto \psi_w(\xi(Av))$$

for every covector ξ of the target component. Indeed the corresponding momentum tensors are the products of the chosen vector in L with covectors in the dual target component. The evaluation of such a tensor on (A, A') is $\xi(Av)$, with no term in A'. A nonzero fixed scalar in the trace pairing can be absorbed into ξ. Consequently this local functional is supported on $Av = 0$. To see the support assertion directly, for every point with $Av \neq 0$ choose ξ for which the multiplier is different from one. It is constant on a neighborhood. Invariance annihilates each test supported there; partition a compact test support into finitely many such open pieces. For the datum of Lemma 142.7 the marked matrix is injective throughout its support. This remains true after precomposition by the invertible source element g_w. The local term and hence this Siegel summand vanish.

Every remaining L projects isomorphically onto S^{-}. Writing it as a graph, isotropy makes its other projection an isometry because the two forms are opposites. Thus these graphs are indexed by $\gamma \in G(F)$. For the identity graph, the change to diagonal and difference coordinates followed by Fourier transformation in the diagonal coordinate takes the tensor $f(x)\overline{f'(y)}$, at the new zero, to $\int f(x)\overline{f'(x)}\,dx$. Indeed the difference is set to zero, and the zero Fourier coefficient integrates the remaining variable. The same calculation with a rational isometry on one factor gives $\langle \omega(g',1)\Phi, \omega(\gamma g,1)\Phi\rangle$, up to the common measure normalization. Rational determinants have adelic absolute value one; the global splitting and the twisting characters are trivial on rational elements. The possible inverse in the choice of graph index just reindexes the sum. With self-dual position measures the identity-graph pairing is the positive L^2 pairing. Changing its normalization or the quotient Haar multiplies the formula by a fixed positive scalar. The target twisting character cancels between the two kernels, and on the source any splitting change contributes its unitary character to the coefficient a. This proves the assertion about $c_{S,T}$.

Finally a is a product of the local coefficients. Lemma 142.12 gives absolute integrability at infinity, Lemma 142.7 at the level places, and the product bound in Theorem 142.9 at the other places, including the identity coset with mass one. These estimates prove the stated L^1 property. □

Theorem 142.15 (A Petersson estimate for the actual theta lift)**.** *The lift initially defined on compactly supported L^2 functions on $[G]$ extends continuously to $L^2([G])$. For the real line of parameter λ and the fixed spaces at the finite places in Theorem 142.9, put*

$$B_\Phi = |I|^2 c_{S,T}\, c_{\mathrm{lev}} \prod_{v\mid\infty} \mathfrak{a}(P_v, -\lambda_v), \qquad c_{\mathrm{lev}} = \begin{cases} c_{\mathrm{first}} & \text{on the first edge,} \\ c_{\mathrm{aux}} & \text{on an auxiliary edge.} \end{cases}$$

Use the nonzero matched harmonic P_v on each edge. With $\varepsilon_r = 512 f 2^{-r}$, for Y, Z in these source spaces the lift satisfies

$$\langle \Theta Z, \Theta Y\rangle = B_\Phi \langle Z, \mathcal{R} Y\rangle, \qquad (1-\varepsilon_r) B_\Phi \|Y\|^2 \le \|\Theta Y\|^2 \le (1+\varepsilon_r) B_\Phi \|Y\|^2.$$

For $Y = P^{+}\bar{x}$ and its indicated finite translates this extension agrees almost everywhere with the original pointwise bilinear theta integral.

Proof. On compact subsets of the source quotient the functions $g \mapsto K_\Phi(g,\cdot)$ have bounded $L^2([H])$ norm. Indeed its norm squared is the diagonal of Proposition 142.14, locally bounded by local uniform convergence of the Siegel Eisenstein series in its open half-plane of absolute convergence. One can lift each compact subset to a compact set of representatives since the rational group is discrete. The sections are measurable as Hilbert vectors, by joint measurability and separability of $L^2([H])$. Thus the lift of a compactly supported L^2 function exists as a Hilbert-space integral, since such a function is integrable on its support. For two such inputs take their inner product and insert the proposition.

To unfold the resulting series absolutely, the integral of the sum of absolute graph terms times $|Z(g')Y(g)|$ over the two source quotients equals

$$\int_{G(\mathbb{A}_F)} |a(u)| \int_{[G]} |Z(g')Y(g'u)|\, dg'\, du \le \|a\|_1 \|Z\|_2 \|Y\|_2.$$

This uses rational counting measure and right invariance of quotient Haar. It holds by Tonelli before the bound. Unfolding the complex integral now gives

$$\langle \Theta Z, \Theta Y\rangle = |I|^2 c_{S,T} \int_{G(\mathbb{A}_F)} a(u) \langle Z, T(u)Y\rangle\, du.$$

This proves boundedness and the formula on all of L^2 by density. For the conjugated projected series $P^+\bar{x}$ the proof of Theorem 142.5 bounds the integral of the absolute value against the kernel for each h. Its estimate uses absolute seeds and is unchanged by conjugating the input. It applies to the required finite translates by shifting the compact set of finite arguments. Cut off on increasing compact sets exhausting the quotient. Dominated convergence gives the original theta integral for each fixed h; L^2 convergence and a subsequence identify it almost everywhere with the bounded extension.

The real type projection on both inputs compresses $T(u_\infty)$ to $\prod_v c_{\lambda_v}(u_v)$. This is the Schur calculation in Lemma 142.11; equivalently it is the highest-line matrix entry on each copy of the source discrete series. Finite translations preserve this real type. Hence the real integral at v is exactly $\mathfrak{a}(P_v, -\lambda_v) > 0$ by Theorem 142.13. Fubini applies by $a \in L^1$. Integrating over the remaining level and unmarked places gives $c_{\mathrm{lev}}\mathcal{R}$ by Theorem 142.9. The operator norm estimate there gives both inequalities. □

Corollary 142.16 (First-lift rank and comparison across twists)**.** *At the common nodes*

$$\operatorname{rank}(\Theta P^+) = \operatorname{rank}(P^+) \ge (1 - 4b^{-8}) D_X.$$

Its full span of lifted marked translates has dimension at least $(1-2b^{-4})^2 L D_X$*, with* L *as in Theorem 133.20. For two tame twists on the same edge with common real datum, choose any fixed nonzero* d*-dimensional subspace* U *on which both finite-level assertions apply. Write* $D_i(U)$ *for the determinant of the Gram matrix of lift* i *on an orthonormal basis of* U*. Then*

$$\left| \frac{1}{d} \log \frac{D_1(U)}{D_2(U)} - \log \frac{|I_1|^2 c_{\mathrm{lev},1}}{|I_2|^2 c_{\mathrm{lev},2}} \right| \le \log \frac{1+\varepsilon_r}{1-\varepsilon_r}.$$

Proof. All factors of B_Φ are positive for the nonzero normalization I of these lifts. At large nodes $\varepsilon_r < 1/2$, so the lift is injective on the specified source, including its marked translates. Apply Theorems 133.18 and 133.20 after the conjugation of Lemma 142.11. For the last estimate every eigenvalue of the compressed Gram matrix for lift i is between $B_{\Phi_i}(1-\varepsilon_r)$ and $B_{\Phi_i}(1+\varepsilon_r)$. Multiply over eigenvalues. The target Haar constant and every real integral are identical for the two twists and cancel. This is a comparison of Petersson determinants on a common source. To use it in the integral comparison of Section 6.6 one still needs arithmetic control of the images in the target coefficient lattices. □

143 Marked determinants and signed coefficient matrices

143.1 One marked period determinant at all complex embeddings

Fix one of the fields E_x, one translate and $s \geq 1$ signed columns of common small type in Theorem 141.6. Use its projected tests and its mass metric, including the same normalization of the columns on both sides. In particular one may use the divided columns $\widetilde{Y}_j/b_j^{\mathrm{per}}$ of Theorem 141.27. Suppose the projected columns are independent. Then their Gram determinants are positive at every complex embedding: the rational quadrature formula expresses them as positive sums of squared absolute values of algebraic minors, and a nonzero minor stays nonzero at every embedding.

Write $Z(t_1, \dots, t_s)$ for the determinant obtained by pairing the columns with the projections of the indicated small tests. Let $\mathcal{V}$ be the finite list of row tests $t_{a,g}$ in the quadrature formula, with positive rational weights q_t. Put

$$q_* = \sum_{t \in \mathcal{V}} q_t = r_\mu \sum_a |\Gamma_a|^{-1}, \qquad L_0 = g_1 g_2 T, \qquad M = \max(1, \max_a |\Gamma_a|).$$

Fix one coefficient place w_p above each of $p = g_1, g_2, T$. Denote by b_p the least valuation at that place on the image of the integral test lattice under the projector, at exterior degree s. The b_p here are valuations of period ideals; they may include the chosen column denominators. The row functional of each $t \in \mathcal{V}$ has norm at most $\sqrt{M}$.

Lemma 143.1 (Perturbations attaining all three finite minima)**.** *There are tests $u_1, \dots, u_s$ of functional norm at most $3L_0\sqrt{M}$ at every complex embedding such that*

$$v_{w_p} Z(u_1 + L_0 v_1, \dots, u_s + L_0 v_s) = b_p \qquad (p = g_1, g_2, T)$$

for any v_i integral in the small test lattices at all three chosen places. In particular the assertion holds for integral linear combinations of tests of $\mathcal{V}$.

Proof. For each p choose s frame rows attaining b_p, possible by Theorem 141.3. For the first row, the second row, and so on, combine the three lifts by the Chinese remainder coefficients between zero and $L_0 - 1$ equal to one at their own rational prime and zero at the others. These are exactly the tests of Corollary 141.7, giving the displayed norm bound and $v_{w_p} Z(u_1, \dots, u_s) = b_p$. On perturbing the rows, each additional determinant term contains at least one factor L_0. Its remaining rows are all permissible at w_p, so before that factor its valuation is at least b_p. It therefore cannot change the least valuation of the original term. This argument uses only $v_{w_p}(L_0) > 0$, including for ramified coefficient places. □

Theorem 143.2 (One algebraic period determinant at all embeddings)**.** *Let J be a number field of definition for the periods in these tests and put $d = [J : \mathbb{Q}]$. There is one determinant $z \in J$ of the given projected columns such that $v_{w_p}(z) = b_p$ at the three chosen places and, for every $\sigma : J \hookrightarrow \mathbb{C}$,*

$$\frac{s!}{(4q_*)^s} \det G_\sigma \;\leq\; \frac{|\sigma z|^2}{L_0^{2s}} \;\leq\; \bigl((3 + 2d^2)^2 M\bigr)^s \det G_\sigma.$$

All row coefficients used in the construction are rational integers applied to the preceding semilocally integral tests.

Proof. At each embedding select a tuple $(t_1^\sigma, \dots, t_s^\sigma)$ of quadrature tests maximizing the absolute determinant. Its value $B_\sigma > 0$ satisfies

$$B_\sigma^2 \geq \frac{s!}{q_*^s} \det G_\sigma.$$

Indeed the Gram formula sums over unordered subsets of size s. The sum of the products of their weights is at most $q_*^s/s!$, by expanding the s-th power of the sum and retaining only tuples with distinct indices. This bound needs no lower estimate for an individual quadrature weight.

Choose common rows r_i inductively. Before step i, for each σ use the first $i-1$ selected rows and complete them with $t_i^\sigma, \dots, t_s^\sigma$. Their determinant at σ is nonzero, initially by definition. With the other rows fixed call the determinant as a functional of row i by l_σ, in this embedding. We choose

$$r_i = u_i + L_0 \sum_{\tau : J \hookrightarrow \mathbb{C}} c_\tau t_i^\tau, \qquad c_\tau \in \{0, 1, \dots, 2d\}.$$

Choose the c_τ for this step independently and uniformly for the counting argument. Fix σ and the choices other than c_σ. Consecutive values of $l_\sigma(r_i)$ differ by the nonzero number $L_0 l_\sigma(t_i^\sigma)$. At most one choice lies in the open complex disk of radius $L_0|l_\sigma(t_i^\sigma)|/2$ about zero, by the triangle inequality. The fraction failing for any embedding is at most $d/(2d+1) < 1$. Hence a single choice keeps the new determinant at least $L_0/2$ times the previous one in absolute value at each embedding. Iteration gives $|\sigma Z(r_1, \dots, r_s)| \geq (L_0/2)^s B_\sigma$, proving the lower estimate.

The choices are integral at the marks, so the lemma gives all three valuations. Each selected row has functional norm at most $(3 + 2d^2)L_0\sqrt{M}$ by the triangle inequality. Apply the wedge of the row functionals to the wedge of the projected columns at σ. Its norm is at most the product of the functional norms, and the squared norm of the column wedge is $\det G_\sigma$. This proves the upper estimate for the very same $z = Z(r_1, \dots, r_s)$. □

Corollary 143.3 (A uniform comparison for divided signed columns)**.** *There is a finite family $\mathcal{Z} \subset J$ of actual period determinants all attaining the three b_p, such that, with $m_\sigma = L_0^{-s} \max_{z \in \mathcal{Z}} |\sigma z|$,*

$$\frac{s!}{q_*^s} \det G_\sigma \;\leq\; m_\sigma^2 \;\leq\; (16M)^s \det G_\sigma.$$

For the base small level on the polynomial-conductor fields in Theorem 141.27, take $s = q_c$ and the divided columns. With its K_u, define

$$H_{\mathcal{Z}} = \frac{2}{s[J : \mathbb{Q}]} \sum_\sigma \log m_\sigma + K_u.$$

Then $H_{\mathcal{Z}} = H(G^\circ) + O_{\psi,\varepsilon}(x^C)$ for a fixed C, uniformly on that interval. The error is independent of the coefficient-field degree and is $o_{\psi,\varepsilon,A}(b/x^A)$ for every fixed $A > 0$.

Proof. Form the list of determinants with rows $u_i + L_0 \epsilon_i t_i$, where $t_i \in \mathcal{V}$ and $\epsilon_i = \pm 1$ vary independently. The lemma supplies the finite assertions and every row has norm at most $4L_0\sqrt{M}$. At a fixed embedding choose the maximizing tuple in the preceding proof. Average these determinants multiplied by $\prod_i \epsilon_i$ over all choices of the signs. Multilinearity kills every term containing an unperturbed row and leaves precisely $L_0^s Z(t_1, \dots, t_s)$. The maximum of the absolute values is at least the absolute value of that average. The bound for B_σ and the exterior norm upper bound give both inequalities.

After taking logarithms the upper error in $H_{\mathcal{Z}} - H(G^\circ)$ is at most $\log(16M)$; the lower error is at worst $-\log^+ q_*$, where $\log^+ q = \max(0, \log q)$. We verify their sizes. Corollary 141.7 bounds $\log M$ by a fixed power of x. The Weyl dimension formula for r_μ has at most $[E_x : \mathbb{Q}]n^2$ factors, each at most the largest width plus n, where n is the small rank. The widths are $O(xb + x^{C_1})$ and the degrees and ranks are polynomially bounded, as in Theorem 141.6. Thus $\log r_\mu \ll x^{C_2}$.

The class mass is bounded above by the number of small classes. Put $N = [E_x : \mathbb{Q}]n$ and take a cyclotomic conductor N_E for E_x from Corollary 9.27. The full-level bound in the proof

of Lemma 121.17 is $\log h_{\max} \ll N^4 \log(2NN_E)$, obtained by recording the trace-lattice Gram matrix and the matrix of a primitive field generator in a bounded integral basis. Passing to the base residue markings adds at most $L_0^{N^2}$ classes per full-level class. Indeed at each of the marked rational primes the kernel of full residue reduction on the underlying integral lattice lies in its small level, and a residue matrix has at most p^{N^2} possibilities. The small objects and their row frames are those in Section 119.4. The logarithms of N_E, L_0 have polynomial bounds on the interval, by Corollaries 9.27 and 141.7. Hence $\log^+ q_* \ll x^{C_3}$. If additional row levels are imposed, the same exact inequalities hold, with the logarithm of their further index added to this class-count bound. Finally $b = \exp(x^{\varepsilon})$ dominates every fixed power of x. This proves the claimed error. □

Consequently $H_{\mathcal{Z}}$ can replace $H(G^{\circ})$ in Theorem 141.27 at its stated precision on the base level. This compares the archimedean and marked finite tests on the given signed periods. The absolute upper estimate for their values in Section 6.6 and the required independence for the chosen input columns remain to be established.

143.2 Translation averages of the signed theta columns

Fix E_x, a translate, and ordinary words in the directed tower of Theorem 9.13 with a common small test space. Work at one complex embedding. Write V for the dual of this projected test space, with the Hermitian norm of Theorem 141.6. An initial table for a word gives a vector in V by the signed path formula of Theorem 73.1. We compute a family of functions whose values at the origin are precisely these vectors. The other values will locate the cancellation which is possible at the origin.

On edge j choose one reference theta vacuum Θ_j on its principally polarized trace lattice Λ_j. Use orthonormal matrix coordinates for its Hermitian metric at the CM embeddings, so the metric weight on the edge is $\exp(-c_j|z|^2)$, $c_j > 0$. On a conjugated edge use the conjugate complex space and coordinates throughout. The finite expansion of Proposition 122.13 expresses each class section, in the common matrix coordinates, as a finite sum of $U_{a_j}\Theta_j$ times scalars. Here a_j is a torsion class modulo Λ_j, and U is the Hermitian translation of Lemma 66.5. Indeed the lattice intertwiner is a sum over the quotient by the intersection of two lattices, and a finite position condition is a finite character sum. Products of the resulting translations are translations times roots of unity. Choose one lift for each residue a_j, absorbing the change of its scalar in the coefficient. Finitely many such sections have the same multiplier on a sublattice $\Gamma_j = L_j\Lambda_j$. One may take L_j divisible by all the translation orders, since the ratio of two period multipliers on Λ_j is their commutator character.

Lemma 143.4 (Inner products of translated derivatives)**.** *On one edge let*

$$f_{\alpha,a}(w) = \partial_z^{\alpha}(U_w U_a \Theta)(0).$$

With probability Haar on $\mathbb{C}^d/\Gamma$ the functions, interpreted with their common unitary multiplier in the variable w, satisfy

$$\int f_{\alpha,a}(w)\overline{f_{\beta,b}(w)}\,dw = \mathbf{1}_{\alpha=\beta}\mathbf{1}_{a=b}c^{|\alpha|}\alpha!\,\|\Theta\|^2.$$

The norm on the right uses probability Haar and the theta metric on $\mathbb{C}^d/\Lambda$.

Proof. For a holomorphic section H, the derivative at zero is $e^{-c|w|^2/2}\prod_l(\partial_{w_l} - c\bar{w}_l)^{\alpha_l}H(w)$. In the metric on the torus the adjoint of $\partial_{w_l} - c\bar{w}_l$ is $-\partial_{\bar{w}_l}$. The opposite-face boundary terms cancel by the multiplier law. Commuting this lowering operator across the raising operators in

the second factor gives c for a matching coordinate. If a lowering operator remains it kills the holomorphic section. If only raising operators remain, move one in the other direction and use holomorphy of the first section. The surviving factor is therefore $\mathbf{1}_{\alpha=\beta} c^{|\alpha|}\alpha!$ times the inner product of the two undifferentiated sections.

For $a \neq b$ modulo Λ, unimodularity of the polarization gives a period $\lambda \in \Lambda$ whose commutators with U_a and U_b differ. Otherwise $a - b$ would belong to the lattice dual to Λ for the alternating form, which is Λ. The period translation acts unitarily on the common cover and has the two sections as eigenvectors with distinct unitary eigenvalues. Their inner product is zero. A translate has the same metric norm as Θ by translation invariance. Passing to a finite cover preserves the probability integral of its squared magnitude. □

For column i introduce independent matrix variables D_j, one matrix at each CM embedding on each edge. Expand its prescribed path formula with these rules. Keep the actual value of the initial table at the composite covector of each path, including the condition that its components be nonzero. At each edge make the polynomial substitution $X = D_j^{\mathsf{t}} Y_{[r]}$ after the specified determinant twist, as in (66.22). Keep the monomials in D_j unevaluated. Apply the terminal coefficient extraction, the small test and its projection, and include the class masses and all finite scalars. On expanding the theta sections, group together paths with the same tuple $a = (a_j)_j$ of torsion residues. The coefficient on D^{α} in that group is a vector $v_{i,\alpha,a} \in V$, since it is linear in the small test. The coupling index on the last two maps is summed once. These vectors require finite scalar and polynomial operations. In particular, the signed path sum takes place before any absolute values. Any column divisor or normalization can be included in its coefficients.

Put $Y_i(\boldsymbol{w}) = \sum_{\alpha,a} v_{i,\alpha,a} \prod_j f_{\alpha_j,a_j}(w_j)$, summing the finite expansion. Then $Y_i(0)$ is the original projected column. Indeed replacing each monomial in D_j by its undivided derivative is exactly the class operator. This replacement commutes with polynomial operations on the other edge variables and with linear terminal restriction. It is the construction in `Formalism/LocalToGlobal/KernelLattices/ThDistC.lean`, line 27. The variable $\boldsymbol{w}$ translates the sections at the chosen representatives. This definition also makes sense when such a translation requires a change of adelic level to be realized as an automorphic test.

Theorem 143.5 (Covariance for the signed paths)**.** *Write $h = \prod_j \|\Theta_j\|^2$ and $q_\alpha = \prod_j c_j^{|\alpha_j|}\alpha_j!$. For a pair of columns in these common coordinates one has the operator identity*

$$\int Y_i(\boldsymbol{w}) Y_l(\boldsymbol{w})^{*}\, d\boldsymbol{w} = h \sum_{\alpha,a} q_\alpha\, v_{i,\alpha,a} v_{l,\alpha,a}^{*}.$$

The integral is over the product of the common covers. If i, l belong to distinct binary words of Theorem 9.13 at this translate, the integral vanishes. For $i = l$ its range is exactly $\operatorname{span}\{v_{i,\alpha,a}\}$.

Proof. Expand the finite sums and use the lemma on each edge. This proves the formula. On the diagonal its quadratic form is a positive sum of squared absolute values. Its kernel is the orthogonal complement of the indicated span.

To see what happens between words, fix a test at either end of the operator identity, a torsion tuple a, and every differential monomial on the edges after the first. The remaining polynomial in the first matrix is a sum of matrix coefficients of one compact column type. Indeed in rank k the first substitution evaluates an input flag polynomial of the first partition ν at $D_1^{\mathsf{t}} Y_{[k]}$. Right multiplication $D_1 \mapsto D_1 g$ replaces its argument by $g^{\mathsf{t}} D_1^{\mathsf{t}} Y_{[k]}$. The translates of flag polynomials of column weight ν under this action have the Schur type indexed by ν, by the highest-weight realization in Section 66.10 (equivalently by the Cauchy decomposition of matrix polynomials). The transpose convention gives the same indexing at every word.

Changing the two matrix frames on this edge acts by left and right invertible matrices and preserves its isotypic component. Contractions at later vertices and the finite signed sums are linear in these first-edge polynomials.

At a fixed CM embedding the first partition is obtained by subtracting the half-sum of positive roots from the initial parameters and adding the common input determinant shift. This is the weight calculation in Lemma 141.11. The shifts on this edge and the translate are independent of the bits. Formula (9.1) shows that at a differing bit the corresponding entries of the partitions differ by Δ. They therefore index inequivalent representations of $U(k)$. The Fischer inner product on the first matrix is invariant under unitary changes of its columns, by its Gaussian integral formula. Its pairing on inequivalent types vanishes by Schur's lemma. The factors $c_1^{|\alpha_1|}$ merely rescale each homogeneous degree; different degrees already pair to zero. Summing the pairings over the fixed data gives the claimed zero operator. □

Corollary 143.6 (Independence under separate torsion translations)**.** *For s prescribed columns let $V_i = \operatorname{span}\{v_{i,\alpha,a}\}$. There are torsion points $\boldsymbol{w}_i$ of the product of common covers for which $Y_1(\boldsymbol{w}_1), \dots, Y_s(\boldsymbol{w}_s)$ are independent if and only if*

$$\dim \sum_{i \in I} V_i \ \geq |I| \qquad (I \subseteq \{1, \dots, s\}).$$

The orders in these covers may be powers of any one fixed prime.

Proof. The span of all values of Y_i is V_i. Containment in V_i follows from the expansion, and a functional annihilating all the values lies in the kernel of the diagonal covariance. The same reasoning and continuity apply to values on the dense set of prime-power grids. A change of lift in the cover changes a value by a common unit scalar and preserves its line.

Independent choices from the V_i require the displayed inequalities. For sufficiency use induction on s, the case $s = 1$ being immediate. If equality holds for a nonempty proper subset I, choose its vectors by induction and divide the ambient space by their span, which is $\sum_{i \in I} V_i$. The dimension inequalities for $I \cup J$ give the required inequalities for the remaining indices in the quotient. Lift their independent choices. If every proper nonempty subset has strict inequality, choose one nonzero vector in the last space and quotient the ambient space by the line it spans. Each of the other nonempty sums loses at most one dimension and induction applies. Finally express each chosen vector as a finite linear combination of torsion values. The nonzero exterior product expands into exterior products of those values, one of which must be nonzero. □

At the prescribed first and later tests the point in this family is $\boldsymbol{w} = 0$ for every word. Orthogonality in the integral above allows vanishing at that point. It determines the signed coefficient spaces against which the fixed evaluation must be tested.

143.3 An upper bound at the prescribed theta tests

Retain the actual signed coefficient vectors of Theorem 143.5. The arithmetic test evaluates every translated column at the origin. To bound that evaluation, group the terms by their torsion tuple before sampling. Each group has a multiplier on the reference trace lattice itself. Thus the length of the shortest dual vector needed in the estimate comes from the original Hermitian lattice, even when the common cover used to compare different multipliers is large.

Let d_j be the complex dimension of the matrix coordinates at edge j, including all CM embeddings, and put $D = 2 \sum_j d_j$. Write K for the largest total degree $\sum_j |\alpha_j|$ among monomials

of the prescribed columns, taking zero for an empty list of monomials. Write $f_0 = [E_x : K_d]$. Define

$$N = \max\left\{1, \left\lceil 8\sqrt{\frac{\pi\ \mathrm{rd}(E_x)}{2f_0}(K+2D+2)}\right\rceil\right\}, \qquad \beta = \tfrac{3}{2}N^D.$$

For each column i, let $\mathcal{A}_i$ be the finite set of tuples a in its expansion. Write $Y_{i,a}$ for the part of Y_i indexed by a.

Lemma 143.7 (Evaluation for a fixed tuple of torsion shifts)**.** *Let $B : V \to W$ be a linear map to a finite-dimensional Hermitian space. For each i, a,*

$$\|BY_{i,a}(0)\|^2 \le \beta \int \|BY_{i,a}(\boldsymbol{w})\|^2\, d\boldsymbol{w} = \beta h \sum_{\alpha} q_\alpha \|Bv_{i,\alpha,a}\|^2.$$

Here the integral uses probability Haar on the product of the reference tori $\mathbb{C}^{d_j}/\Lambda_j$.

Proof. In one coordinate of W, remove the unitary metric factor $\exp(-\sum_j c_j|w_j|^2/2)$ from the expression for $BY_{i,a}(\boldsymbol{w})$. The result is a sum of raising derivatives of the fixed holomorphic product section $\bigotimes_j U_{a_j}\Theta_j$. All terms have this section's multiplier. The commutator in Lemma 143.4 implies that $K+1$ antiholomorphic derivatives in any common direction annihilate the sum. Indeed each such derivative either removes one raising operator or acts on the holomorphic section and gives zero.

Scale the edge coordinates by $\sqrt{c_j}$ to make the common metric parameter one. For the scaled trace lattice on each edge the alternating form is $\pi^{-1}\operatorname{Im}\langle z, w\rangle$. Unimodularity identifies its Euclidean dual with its multiple by i/π, up to a sign. The lower bound of Theorem 65.3 for the squared length of a nonzero trace vector in the scaled lattice is $2\pi f_0/\operatorname{rd}(E_x)$. Hence the real dual minimum of the product has length at least $\sqrt{2f_0/(\pi\operatorname{rd}(E_x))}$. These are the normalizations in the proof of Theorem 66.34. Changing a unitary multiplier by a character leaves this period lattice and the adjoint and commutator identities intact.

Apply Lemma 66.33 with this minimum, dimension D, and degree K. The grid average of the squared magnitude is at most $3/2$ times its integral. The origin is one of the N^D grid points, and every summand is nonnegative. This proves the inequality in each coordinate of an orthonormal basis. Summation gives the assertion for W. Orthogonality of the derivative indices at fixed a gives the stated integral. □

At an embedding σ use its own small-space metric, finite coefficients $v^{(\sigma)}_{i,\alpha,a}$, vacuum norm product h_σ, and differential factors $q_{\alpha,\sigma}$. To form these coefficients one transports the original scalar coefficients together with the phases of the translated germs. Theorem 65.6 supplies a phase common to every derivative order of each such germ. This also allows the fixed marked product normalization of Proposition 122.13. For $\boldsymbol{a} \in \prod_{i=1}^{s}\mathcal{A}_i$, put

$$E_{\boldsymbol{a},\sigma} = h_\sigma^s \sum_{\alpha_1,\dots,\alpha_s}\left(\prod_{i=1}^{s} q_{\alpha_i,\sigma}\right)\left\|v^{(\sigma)}_{1,\alpha_1,a_1}\wedge\cdots\wedge v^{(\sigma)}_{s,\alpha_s,a_s}\right\|^2, \qquad B_\sigma = \sum_{\boldsymbol{a}}\sqrt{E_{\boldsymbol{a},\sigma}}.$$

The sum over each derivative index can be taken over its finite support. The factors inside the exterior product include the finite masses, coupling signs, terminal restriction and finite projection of the original data.

Theorem 143.8 (A bound for the original exterior period)**.** *For $s \ge 1$, the Gram matrix $G_\sigma(0)$ of the actual columns at the prescribed tests satisfies*

$$\det G_\sigma(0) \ \le\ \beta^s B_\sigma^2.$$

The formula applies in every rank. The right side uses undifferentiated reference vacuum norms and finite coefficient operations.

Proof. Fix $\boldsymbol{a}$ and consider the exterior product of the s corresponding column parts, with a separate variable $\boldsymbol{w}_i$ in each part. With all the other factors fixed, exterior multiplication is a linear map of the remaining vector into $\bigwedge^s V$. Applying the lemma in that variable bounds the squared norm at zero by β times its integral. Repeating this argument and using Fubini bounds the squared norm with every variable zero by β^s times the product-torus integral. Expanding the exterior product by multilinearity, two distinct lists of multi-indices are orthogonal in at least one variable by Lemma 143.4. The diagonal terms give precisely $E_{\boldsymbol{a},\sigma}$.

At the origin, multilinearity also expresses the actual exterior product as the sum of these products over all $\boldsymbol{a}$. The triangle inequality bounds its norm by $\beta^{s/2}B_\sigma$. The squared exterior norm is the Gram determinant, including when all maximal minors vanish. □

Corollary 143.9 (The finite coefficients in the centered estimate)**.** *Use the divided period columns of Theorem 141.27 and include each division in $v^{(\sigma)}_{i,\alpha,a}$. On the polynomial-conductor interval one has $\log\beta = o_{\psi,\varepsilon,A}(b/x^A)$ for every fixed $A>0$, uniformly in the translate and word list. If these columns are independent at the prescribed tests, their centered count satisfies*

$$\Delta_u \le K_u + \frac{2}{s[J:\mathbb{Q}]}\sum_{\sigma}\log B_\sigma - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x + o_{\psi,\varepsilon,A}(b/x^A).$$

Proof. The number of embeddings of E_x, the ranks and the number of edges have polynomial bounds in x. On edge j the total degree is the sum of the sizes of its partitions. Their largest entries are $O(xb+x^C)$, by the construction of the word weights and the rank shifts in Lemma 141.11. Thus D and $\log(2+K)$ are polynomially bounded. The conductor bound of Corollary 9.27 gives the same for $\log \mathrm{rd}(E_x)$, using the conductor–discriminant formula. These bounds are uniform at the conjugate trace tori and independent of s. They give $\log\beta \le C' x^{C''}$, which is $o(b/x^A)$ since $b=\exp(x^\varepsilon)$. Independence makes all B_σ positive by the theorem. Substitute the logarithmic inequality in the definition of $H(G^\circ)$ and apply Theorem 141.27. □

This upper bound evaluates the prescribed tests themselves. It leaves the sizes of the indicated signed finite coefficient sums and vacuum norms to be compared with $A_{\mathrm{sig}}(x)$ at the precision required in Section 6.6.

143.4 The fixed projected period and its coefficient matrices

We evaluate the archimedean contribution at the original small tests on E_x in the product coordinates of Lemma 65.8. The coefficient of the total derivative degree will be explicit. The remaining expression involves positive matrices built from signed polynomials in a finite-dimensional space.

Fix an embedding σ of a common coefficient field J, and s projected columns in the same Hermitian space V. At edge j of column i let n_{ij} be the complex dimension of its product coordinates, k_{ij} its differential degree, and $c_{ij,\sigma}=\pi/A_{ij,\sigma}$ the metric parameter of its marked elliptic curve C^σ_{ij}. Use conjugate coordinates on conjugate edges. Put $\alpha! = \prod_{j,h}\alpha_{jh}!$. The finite expansion of the fixed column is

$$Y_i = \sum_{a\in\mathcal{A}_i}\sum_{\substack{\alpha\\ |\alpha_j|=k_{ij}}} v_{i,a,\alpha}\prod_{j,h} J_{\alpha_{jh}}(U_{a_{jh}}\Theta_{t_{ij}}), \qquad J_m(F)=F^{(m)}(0).$$

Indeed, on each edge expand each class section by the product formula preceding Theorem 122.13. The linear change of matrix coordinates in each term replaces the edge polynomial by a homogeneous polynomial of the same degree k_{ij}, the size of its partitions. Apply the signed path formula of Theorem 143.5 with the same terminal coefficient and the same projection. Collect now by the full tuple a of elliptic translations and by the derivative indices α

in product coordinates. This gives the vectors $v_{i,a,\alpha}$ by polynomial operations, including the chain-rule coefficients and all normalizing scalars. In the last two maps there is one sum over their common intermediate class. The phases of the transported sections are included before collecting coefficients. Thus the displayed identity uses the germ of each prescribed section at its given point. We suppress σ on the vectors and take any required rational column division in them. Empty expansions give zero columns.

For every nonempty $\mathcal{A}_i$ choose positive reals $\mu_i(a)$ with sum one. Form the positive semidefinite operator

$$B_i(\mu_i) = \sum_{a,\alpha} \frac{\alpha!}{\mu_i(a)} v_{i,a,\alpha} v_{i,a,\alpha}^{*} \quad \text{on } V.$$

Here vv^{*} takes h to $v\langle v, h\rangle$, with the inner product linear in its second argument. For any positive semidefinite operators $B_1, \dots, B_s$ define

$$D_s(B_1, \dots, B_s) = [t_1 \cdots t_s] \det_V \left(I + \sum_{i=1}^{s} t_i B_i \right).$$

The square brackets take the coefficient of exactly the specified monomial.

Theorem 143.10 (An upper bound at the fixed projected tests)**.** *Write $G_\sigma = (\langle Y_i, Y_j\rangle)_{i,j}$. For each column set*

$$T_{i,\sigma} = \prod_{j} \left[\left(\frac{2 \exp(L_{C_{ij}^{\sigma}})}{g_{ij,\sigma}(t_{ij}^{\sigma})^2} \right)^{n_{ij}} c_{ij,\sigma}^{k_{ij}} H_{n_{ij}}(k_{ij}) \right],$$

where L_C is from Theorem 68.29 and $H_n(k)$ is from Lemma 66.8. Then

$$Y_i Y_i^{*} \;\le\; T_{i,\sigma} B_i(\mu_i), \qquad \det G_\sigma \;\le\; \left(\prod_i T_{i,\sigma} \right) D_s(B_1, \dots, B_s).$$

The inequalities are valid for every choice of the probabilities, including at a rank-deficient projected table.

Proof. For a single elliptic coordinate z, Hermitian translation and Lemma 68.21 give

$$|(U_w \Theta_t)(z)|^2 \le \exp(c|z|^2) \sup_{C} g^2/g(t)^2 \le \exp(c|z|^2)\, 2\exp(L_C)/g(t)^2$$

with every lift w of the prescribed shift. Take the product over the n coordinates of an edge and denote the holomorphic product by F. Its homogeneous Taylor part of order k has squared Fischer norm $\sum_{|\alpha|=k} |\partial^\alpha F(0)|^2/\alpha!$. The sphere bound in Lemma 66.8 bounds this by $(2e^{L_C}/g(t)^2)^n c^k H_n(k)$. Multiplying over the independent edge variables gives a uniform upper bound $T_{i,\sigma}$ for the sum of the squared jet products divided by $\alpha!$, for each fixed tuple a. For $h \in V$, apply Cauchy–Schwarz to $\langle h, Y_i\rangle$ with factors $\sqrt{\alpha!/\mu_i(a)}$ and their reciprocals. The first squared sum is $\langle h, B_i h\rangle$. The second is at most $\sum_a \mu_i(a) T_{i,\sigma} = T_{i,\sigma}$. This proves the operator inequality.

To see the effect on the determinant, write each B_i as a sum of rank-one operators uu^{*}. Cauchy–Binet, applied to the matrix with columns $\sqrt{t_i}\, u$, shows that D_s is the sum of $\|u_1 \wedge \cdots \wedge u_s\|^2$ over one column from each list. In particular it is nonnegative, additive in each operator, and increasing in each for the positive semidefinite order. For the list $Y_i Y_i^{*}$ its value is precisely $\det G_\sigma$. The second assertion follows from the first. If an original list is empty take its $B_i = 0$. □

For a positive operator B on V let $e_s(B)$ be the sum of all products of s distinct eigenvalues, counted with multiplicity. Put $e_s(B) = 0$ if $s > \dim V$. Define

$$\mathcal{C}(B_1, \dots, B_s) = \inf_{\substack{\lambda_i > 0 \\ \prod_i \lambda_i = 1}} e_s\left(\sum_i \lambda_i B_i\right).$$

This expression can be bounded from above by choosing any positive scalings of product one. The optimization loses at most a fixed constant per column in the logarithmic estimate.

Lemma 143.11 (Capacity of the coefficient matrices)**.** *For all $s \geq 1$,*

$$\frac{s!}{s^s}\mathcal{C}(B_1, \dots, B_s) \;\leq\; D_s(B_1, \dots, B_s) \;\leq\; \mathcal{C}(B_1, \dots, B_s).$$

If $\mathcal{C} > 0$, its logarithm is the infimum of a convex function of the real variables $\log \lambda_i$ with zero sum.

Proof. Write $n = \dim V$ and assume $s \leq n$, the other case being immediate. The polynomial $P(t) = e_s(\sum_i t_i B_i)$ is homogeneous of degree s. Its coefficients are nonnegative by the same rank-one expansion, and $[t_1 \cdots t_s]P = D_s$. Every positive evaluation therefore bounds $D_s \prod t_i$ from above. On writing $t_i = e^{y_i}$, the logarithm is the log of a sum of exponentials of affine functions. Its Hessian is the covariance matrix of their exponents with their normalized positive weights, hence is positive semidefinite whenever $P \neq 0$.

For the reverse bound we give Gurvits's capacity argument [2]. The polynomial $\det(zI + \sum_i t_i B_i)$ is nonzero when all its variables have positive imaginary part, since the imaginary part of its quadratic form is positive definite. A polynomial with this nonvanishing property is called stable. Its derivatives and real specializations are stable or zero. For a derivative in one variable of degree $m > 0$, its leading coefficient, as a polynomial in that variable, is the locally uniform limit upon setting that variable to iR and dividing by $(iR)^m$ as $R \to \infty$. It is nonzero as a polynomial and hence has no zeros in the product of upper half-planes by Hurwitz's theorem. The univariate degree thus stays m there, and the logarithmic derivative has strictly negative imaginary part by factoring the polynomial into linear terms. This proves the derivative assertion. Real specialization follows by taking a limit from the upper half-plane. The version of Hurwitz used on a product follows by restricting to complex lines through a putative zero inside a small polydisk and using the univariate result. Since P is the coefficient of z^{n-s}, differentiation in z and setting $z = 0$ give its stability up to the zero case.

Let P now denote any homogeneous stable polynomial of degree q in q variables with nonnegative real coefficients. Its capacity is $\inf_{t>0} P(t)/\prod t_i$, equivalent by homogeneity to the infimum at $\prod t_i = 1$. Assume its capacity is positive. Freeze the first $q-1$ variables at positive real values t'. The resulting polynomial $R(x)$ has only nonpositive real roots by stability, real coefficients, and positivity on $x > 0$. Its degree m is positive since a positive capacity excludes a constant in x. A root at zero of multiplicity at least two would likewise force capacity zero. If zero is a simple root, $\inf_{x>0} R(x)/x = R'(0)$. If $R(0) > 0$, write $R(x) = R(0)\prod_{\nu=1}^{m}(1 + x/b_\nu)$ with $b_\nu > 0$. The arithmetic–geometric mean bounds it above by $R(0)(1 + xR'(0)/(mR(0)))^m$. Minimizing this upper bound divided by x gives

$$R'(0) \;\geq\; g_m \inf_{x>0} R(x)/x, \qquad g_1 = 1, \qquad g_m = ((m-1)/m)^{m-1} \quad (m \geq 2).$$

The same inequality holds for the simple-root case. The numbers g_m decrease with m, so $g_m \geq g_q$. Divide by the product of the fixed t' and take their infimum. The derivative at zero retains capacity at least g_q times the old capacity. It is again stable and homogeneous

with nonnegative coefficients. Repeat in all variables. The final scalar is the coefficient of their product and $\prod_{q=1}^{s} g_q = s!/s^s$. This proves the lower bound, also valid when the capacity is zero. □

Corollary 143.12 (The finite matrix term on the period interval)**.** *On the polynomial-conductor interval of Theorem 141.27 take its $s = q_c$ divided columns. Form $B_{i,\sigma}$ from the actual projected coefficient vectors above and put $\mathcal{C}_\sigma = \inf_{\mu_i} \mathcal{C}(B_{1,\sigma}(\mu_1), \dots, B_{s,\sigma}(\mu_s))$. Set*

$$L_{d_0} = \log(\sqrt{d_0}/(2\pi)) - \gamma_{\mathrm{E}} + D_{\xi_{d_0}}, \qquad \bar{k} = \frac{1}{s} \sum_{i,j} k_{ij}.$$

If the columns are independent at the prescribed tests, then, uniformly on these fields and translates, for every fixed $A > 0$,

$$\begin{aligned} \Delta_u \ &\le K_u + \bar{k} L_{d_0} + \frac{1}{s[J:\mathbb{Q}]} \sum_{\sigma} \log \mathcal{C}_\sigma - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x \\ &\quad + o_{\psi,\varepsilon,A}(b/x^A). \end{aligned}$$

Proof. The two preceding results bound $\det G_\sigma^\circ$ by $\mathcal{C}_\sigma \prod_i T_{i,\sigma}$, also after the infimum since its prefactor is independent of all probabilities. A positive determinant gives $\mathcal{C}_\sigma > 0$. For each edge, Theorem 65.22 gives mean $\log c_{ij,\sigma} = L_{d_0}$ over J. The elliptic curves and nonzero marks in the product presentation are the transported copies of the marked CM model; complex conjugation permutes its embeddings. By Lemma 68.26 and Theorem 68.29, the mean log of the first factor inside the square brackets defining $T_{i,\sigma}$, including its exponent, is

$$n_{ij}\big(\log 2 + 1 + \log(\sqrt{d_0}/(2\pi)) + D_{\xi_{d_0}} - 2b_K(t_{ij})\big).$$

Here $b_K(t_{ij}) \ge 0$. The upper bound after dropping this nonpositive term is $O_{\psi,\varepsilon}(n_{ij} x^{\varepsilon/4})$ by Corollary 65.23 and its bounds on $d_0, D_{\xi_{d_0}}$. Also $\log H_{n_{ij}}(k_{ij}) \le n_{ij}(1 + \log(1 + k_{ij}/n_{ij}))$ by the sphere lemma. The sum of the coordinate dimensions along a word and the log of its largest degree plus two are bounded by powers of x, uniformly in the word, by the rank and weight bounds in Corollary 143.9. Thus, after division by s, all these upper errors together are $O_{\psi,\varepsilon}(x^C)$ for some C depending only on the fixed parameters. Since $b = \exp(x^\varepsilon)$ this is $o(b/x^A)$. Insert the height bound into Theorem 141.27. □

Consequently the archimedean term at these tests has an upper bound in terms of the signed finite coefficient matrices and the explicit degree contribution $\bar{k} L_{d_0}$. Section 6.6 still requires a sufficiently small bound relative to $A_{\mathrm{sig}}(x)$, a nonzero determinant at the fixed tests, and the exceptional strict-length estimate on this same interval.

143.5 A metric bound for the fixed signed coefficients

In Corollary 143.12 the signed coefficient vectors come from the original projected tests. We express their contribution using a single positive Hermitian operator at each embedding. This also gives an upper bound from any chosen orthogonal decomposition of the actual test space.

Lemma 143.13 (Selecting a degree in a positive determinant)**.** *Let B be a positive semidefinite Hermitian operator in finite dimension. For every integer $s \ge 1$,*

$$e_s(B) \ \le \ \inf_{t>0} \frac{\det(I + tB)}{t^s} \ \le \ \frac{e^s s!}{s^s}\, e_s(B).$$

The constant is independent of the dimension.

Proof. Write $P(t) = \det(I + tB) = \sum_{j=0}^{r} d_j t^j$, where r is the rank and $d_j = e_j(B) > 0$ for $0 \le j \le r$. All roots of P are negative real numbers. Every positive evaluation bounds its degree-s term from above. For $r < s$ let $t \to \infty$ to obtain zero. Suppose $s \le r$. The ratios $b_j = j d_j / d_{j-1}$ decrease weakly with j. Indeed, by Rolle's theorem each derivative of P also has all its roots negative, with multiplicities allowed. The log of a positive polynomial with such a factorization has nonpositive second derivative on the positive real axis and at zero. Applied to $P^{(j-1)}$ at zero this gives $(j! d_j)^2 \ge (j-1)! d_{j-1} (j+1)! d_{j+1}$, as required. Take $t = s/b_s$. For $j > s$, each ratio of successive $t^j d_j$ is at most s/j, and for $j \le s$ the ratio is at least s/j. Multiplying the inequalities starting from $t^s d_s$ in either direction gives

$$\frac{t^j d_j}{t^s d_s} \le \frac{s!}{s^s} \frac{s^j}{j!}.$$

Sum over $j = 0, \dots, r$ and use the exponential series. □

Fix the embedding and abbreviate its test space to V, of dimension n. Retain the complete vectors $v_{i,a,\alpha}$ of Section 143.4, including the division of each column when using G_σ°. For a positive definite Hermitian operator H define

$$d_{ia}(H) = \sum_{\alpha} \alpha! \, \langle v_{i,a,\alpha}, H v_{i,a,\alpha} \rangle,$$

$$\Psi_\sigma(H) = \operatorname{tr} H - \log\det H - n + s + 2 \sum_i \log\left(\sum_{a \in \mathcal{A}_i} \sqrt{d_{ia}(H)} \right).$$

Theorem 143.14 (One Hermitian metric for the signed coefficient matrices)**.** *Let $D_{\mathrm{mix}} = \inf_\mu D_s(B_1(\mu_1), \dots, B_s(\mu_s))$, with one probability distribution on the original tuples for each word as in Theorem 143.10. Then*

$$D_{\mathrm{mix}} \;\le\; \mathcal{C}_\sigma \;\le\; \exp\Big(\inf_{0 < H \le I} \Psi_\sigma(H) \Big) \;\le\; \frac{e^s s!}{s^s} \mathcal{C}_\sigma \;\le\; e^s D_{\mathrm{mix}}.$$

The infimum of Ψ_σ over all positive definite operators has the same value. A zero coefficient column gives zero in all three expressions in exponential scale.

Proof. Assume each word has some nonzero coefficient. Define

$$Q = \inf_{\mu,\, \lambda_i > 0} \frac{\det(I + \sum_i \lambda_i B_i(\mu_i))}{\prod_i \lambda_i}.$$

Write $\lambda_i = t \lambda_i'$, with $\prod_i \lambda_i' = 1$. For fixed λ' and μ apply Lemma 143.13 to $\sum_i \lambda_i' B_i(\mu_i)$. Taking the remaining infima gives $\mathcal{C}_\sigma \le Q \le (e^s s!/s^s) \mathcal{C}_\sigma$. Lemma 143.11 gives the two inequalities involving D_{mix}.

For any positive semidefinite W,

$$\log\det(I + W) = \min_{H > 0} \{ \operatorname{tr}(H(I+W)) - \log\det H - n \}.$$

Indeed the difference between the right-hand expression and the left-hand side is the sum of $u - 1 - \log u$ over the positive eigenvalues u of $(I+W)^{1/2} H (I+W)^{1/2}$. It is nonnegative and equality holds for $H = (I+W)^{-1} \le I$. In $\log Q$ insert this identity and take the infima in the opposite order, all being infima over independent sets. Once H and μ are fixed, minimizing in the i-th scalar gives

$$\inf_{\lambda_i > 0} \{ \lambda_i \operatorname{tr}(H B_i(\mu_i)) - \log \lambda_i \} = 1 + \log \operatorname{tr}(H B_i(\mu_i)).$$

Next use $\operatorname{tr}(HB_i(\mu_i)) = \sum_a d_{ia}(H)/\mu_i(a)$. Cauchy–Schwarz gives the minimum, in the sense of an infimum, as $(\sum_a \sqrt{d_{ia}(H)})^2$. For positive d_{ia} use weights proportional to their square roots; zero terms can be assigned arbitrarily small positive total mass. Substitution gives $\log Q = \inf_H \Psi_\sigma(H)$, including the value $-\infty$. The same proof starting with $0 < H \le I$ gives the restricted formula since that set contains every minimizer $(I+W)^{-1}$. For a zero column the mixed coefficient vanishes and in Q its scalar can be taken arbitrarily large. This gives the asserted limiting interpretation. □

Corollary 143.15 (An estimate using any chosen test metric)**.** *In the setting of Corollary 143.12 with independence at the fixed tests, take any $0 < H_\sigma \le I$ on each actual projected test space. Then*

$$\begin{aligned}\Delta_u \ &\le K_u + \bar{k}L_{d_0} + \frac{1}{s[J:\mathbb{Q}]}\sum_\sigma \Psi_\sigma(H_\sigma) - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x \\ &\quad + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

In particular let Π_b be the orthogonal projections of any decomposition of V, of ranks m_b, and let $0 < h_b \le 1$. For $H = \sum_b h_b \Pi_b$ the entire metric term is

$$E_{ia,b} = \sum_\alpha \alpha! \, \|\Pi_b v_{i,a,\alpha}\|^2,$$

$$\Psi_\sigma(H) = s + \sum_b m_b(h_b - 1 - \log h_b) + 2\sum_i \log \sum_a \left(\sum_b h_b E_{ia,b}\right)^{1/2}.$$

Proof. Insert $\log \mathcal{C}_\sigma \le \Psi_\sigma(H_\sigma)$ in Corollary 143.12. The spectral decomposition of the chosen H gives the displayed expression, keeping the projection and all path sums in each vector before taking its squared norm. For comparison with the exact mixed coefficient, the previous theorem shows that optimization over all metrics costs at most one in the normalized log per column. This is $o(b/x^A)$ on the indicated intervals for every fixed A. □

The finite metric problem is convex along the positive definite matrix paths $H(t) = H_0^{1/2}(H_0^{-1/2}H_1H_0^{-1/2})^t H_0^{1/2}$. Indeed diagonalizing the middle matrix expresses each $d_{ia}(H(t))$ as a sum of exponentials of linear functions of t with nonnegative coefficients. Positive such sums are log-convex by Hölder's inequality. Their square roots and the sums of these square roots remain log-convex by the same inequality. The trace of $H(t)$ is convex by the same diagonalization, and $\log\det H(t)$ is affine. This gives the asserted convexity for nonzero columns. Estimating these signed coefficient energies at the required size and proving independence at the original tests remain arithmetic questions.

144 Terminal Fourier components in the period height

144.1 The connected half of the terminal Fourier index

Use the common terminal prime T and the unrefined integral Weyl lattices of Theorem 120.14. Thus $T \ge 5$ is good and splits completely in the fields of the terminal test. On an edge with source matrix rank r, the local condition at each chosen terminal half is $\xi' \bar{X} = \xi$, where the two covectors at its vertices are nonzero. The factor T^{-r} in Lemma 141.20 is given by the actual normalized average over $u \in \mathbb{F}_T^r$. We calculate its contribution before summing over the intermediate covectors.

Lemma 144.1 (Summing a connected frequency block)**.** *Fix a coefficient place $w \mid T$, one such block, and its output covector ξ'. Suppose the terminal frequencies in that block belong*

to the connected CM half at w. In a rational orthonormal product basis adapted to their subspace, the complete multiplier with the denominator T^r on the product vacuum has integral undivided derivatives in all the constrained coordinates. This remains true term by term after the translations at the other finite primes. The choices of bases can be imposed independently on all blocks over T.

Proof. At the residue position lattice, $u \mapsto (\bar{X} \mapsto \xi' \bar{X} u)$ injects $\mathbb{F}_T^r$ into its dual matrix space: some entry of ξ' is a unit, and that row of $\bar{X}$ can vary freely. Lift ξ' to a primitive integral covector and take a basis of the domain of u. The resulting r rank-one functionals extend to an integral basis of the dual lattice. Indeed projection to the row with that unit entry gives a left inverse after division by the unit. Take the dual basis on the position side. Pairing corresponding vectors in these two bases gives an orthonormal basis in the split Hermitian model, as in the proof of Theorem 120.14. Do this at every required block and approximate the combined local basis to precision T^2 by a rational orthonormal basis. The Cayley approximation of Theorem 122.14 permits this open congruence at both halves. Its vacuum intertwiner has index prime to T.

In these product coordinates the r modulation directions are in distinct elliptic coordinates modulo the integral periods, each generating one pure CM T-line. We verify the phases needed when their terms are summed. At each coordinate choose the primitive period ω_1 of Lemma 68.41 to agree with the chosen local generator after multiplication by T, to precision T^2; explicitly prescribe $2\omega_1$ modulo T^2 times the period lattice. The primitive vector lifting argument in that lemma permits precisely these two split congruences. The difference from the original local modulation, after division by T, lies in T times the local period lattice. On the orbit of the T-local vacuum under all these modulations this difference acts trivially. It fixes the vacuum and its pairings with all points of denominator T are integral. The local Heisenberg cocycle is trivial for these pairs as well: the split form pairs integral coordinates with the dual coordinates, and the two factors of T cancel. This uses the product coordinates generating the trace lattice at T.

Here is the same check for the rational lift at the other primes. In a scalar product coordinate write the period $\omega_1 = \delta l$, with $l \in \mathcal{O}_K$ in the standard elliptic lattice $\delta\mathcal{O}_K$. The rational lift of the generator is $2l/T$. Its Hermitian norm $4l\bar{l}/T^2$ is an even integer, since the opposite component of l is divisible by T^2. Thus the corrected translation $W(2jl/T)$ in the proof of Theorem 65.13 agrees with $U_{2j\omega_1/T}$. At primes other than T the lift is integral, including its factor two at the dyadic prime. Its period action on the reference vacuum is trivial there in the corrected convention; the correction is trivial on pairs of integral periods. The product-coordinate directions have zero mutual pairing. Equivalently, on this orbit the physical modulation for a generator is a scalar times the indicated cyclic translation, and any change of scalar compatible with the group law is a character. Such a character only shifts the Fourier label. This verification accounts for the possible phases before any arithmetic triangle inequality is used.

The terminal multiplier on the reference product is therefore the product of r complete cyclic projections S_{k_j} of Lemma 68.41, with the additive labels supplied by ξ. The remaining factors are the original scalar vacua. Operators supported at other finite primes commute with the entire terminal multiplier, by the factorization of the finite Heisenberg action. A prime-to-T translation applied after a cyclic projection can be represented on its common cover by a point of prime-to-T order. Indeed the map from the common cover to the original curve has kernel of order T, so gives an isomorphism on prime-to-T torsion. At T choose a lift that is integral in the common period lattice. On reducing a rational lift modulo that lattice, the period acts by the character of the specified section, a root of unity. This gives exactly the

$U_{b_0}S_{k_j}$ of Theorem 68.42, up to a unit. Thus all undivided derivatives are integral, including those at a zero label. The other-prime vacuum intertwiners are finite sums of such translations with coefficients integral at T. The unit comparison in Lemma 120.13 permits returning to a common locally integral frame. □

Theorem 144.2 (The terminal cost in the prescribed Weyl lattices)**.** *For an edge j on column i and a coefficient place $w \mid T$ put $B_w = \{\tau : P_{\tau,w} \in S_T\}$, using the conjugate coefficient place on a conjugated edge. Remove the separately specified nonterminal scalar factors, and include the Fourier denominators at T. Each matrix entry of the terminally restricted class operator between the integral Weyl lattices has valuation at least*

$$-\sum_{\tau\in B_w}(r_j+\nu_{ij,\tau,1}).$$

The bound holds for the complete indicator on the original matrix lattice with its specified covectors.

Proof. By the split position and frequency identification in Lemma 141.20, the primary shift at v lies over the conjugate of its selected position prime. Consequently it is in the étale CM half at the coefficient place when $P_{\tau,w} \in S_T$, and in the connected half otherwise. This is the conjugate annihilator computation of Theorem 141.21. The map $\tau \mapsto v = P_{\tau,w} \cap F$ is bijective.

Keep the Fourier expansion in the blocks indexed by B_w. It costs T^{-r_j} at each such block. Fix one tuple of its frequencies. Use the bases of Lemma 144.1 in all connected blocks. In each étale block with nonzero frequency choose the first dual coordinate equal to its primitive rank-one functional as in Theorem 120.14. These choices are simultaneous since they are in distinct local blocks. The connected blocks with their denominators now have integral derivatives of all orders. In an étale block only its first coordinate can cost one in valuation per derivative, and none do if its frequency is zero.

In a physical block τ each derivative matrix is an integral combination of regular operators and, for a matching nonzero étale frequency, the extra term $U_\tau\partial_a$ in its first coordinate, where U_τ has rank one. The regular operators are ordinary derivatives on the integral coordinates and $T\partial_a$ on the first coordinates with a possible loss. This decomposition follows by reduction of the adapted basis at T. Derivatives entering such a first coordinate from a different block have coefficients in $T\mathcal{O}_w$. Each flag minor is affine along the rank-one direction U_τ, even after the source matrix substitution. A Weyl test has $\nu_{ij,\tau,1}$ flag minors in each product. Expanding in the separate rank-one variables therefore leaves at most that many extra derivatives in block τ. This is precisely the differential argument of Theorem 120.14, now on the product with the connected blocks already summed. It applies to each integral source generator and dual test. Restoring the denominator on the étale blocks and summing their finitely many terms proves the claim. □

Corollary 144.3 (A smaller terminal term on the common intervals)**.** *In Theorem 141.27 the terminal summand of K_u can be replaced by*

$$\frac{\log T}{q_c}\sum_{i,j}\left(fr_j+\sum_{\tau}\nu_{ij,\tau,1}\right).$$

Here $f=[E_x^+:\mathbb{Q}]$ and q_c is the number of columns in the specified independent list. The saving in the upper bound for Δ_u is $(f\log T/q_c)\sum_{i,j}r_j$.

Proof. For a fixed τ, the two conjugate halves at T are exchanged by conjugation of the coefficient places; equivalently the pullbacks $P_{\tau,w}$ cover the primes of E_x over T with their

degree weights. Exactly half of this weight has $P_{\tau,w} \in S_T$. The theorem thus contributes at most $\frac{1}{2}(fr_j + \sum_\tau \nu_{ij,\tau,1}) \log T$ per edge in the averaged denominator estimate. The two factors in the Gram calculation give the asserted coefficient. Each bound was uniform in the covectors and in the coupled intermediate class. The nonarchimedean triangle inequality therefore carries it through the prescribed class sums and compositions. The same projector denominator, scalar division by b_i^{per}, and archimedean matrix in Theorem 141.27 then give the claim on each of its original fields and intervals. □

The corollary reduces the terminal contribution for any independent list admitted in that theorem. Establishing the required projected rank and uniform upper bound for the resulting Gram height remains necessary in Section 6.6.

144.2 Cancellation of the explicit terminal index in the height

Fix the nonzero covectors at the vertices of a path as in Lemma 141.20. This time perform the full cyclic Fourier sums of Lemma 144.1 at every terminal position. For a given class, choose its rational orthonormal basis adapted to the r_j frequency directions at each of the f positions on edge j. The construction of that lemma applies to both CM halves. It identifies the terminal section on each of the fr_j constrained coordinates with one S_k; the good-reduction orientation was used only in evaluating its finite valuation. Expand the other local operators on the product vacuum as finite sums of translations at primes distinct from T. Return the scalar for each choice of basis to the original normalization according to Theorem 122.3. It has absolute value one at all complex embeddings.

Lemma 144.4 (Norm and pointwise bound for the complete projection)**.** *Let $h = \|\Theta_t\|_{\mathrm{av}}^2$ on the original normalized elliptic curve, with the metric density $\exp(-c|z|^2)$, $c = \pi/A$, and probability Haar measure. For every k, the average squared norm of S_k on its elliptic period lattice is h/T. The same holds for $U_b S_k$ for any torsion lift b. At every complex embedding these normalized factors, formed with the original finite Fourier multipliers, satisfy*

$$|(U_b S_k)(z)|^2 \le e^{c|z|^2}\, \frac{B_K h}{T}, \qquad B_K = e \max\{1, 2\sqrt{|D_K|}/\pi\},$$

using the area and reference norm at that embedding. The corresponding bound for an unprojected translated elliptic factor is $e^{c|z|^2} B_K h$.

Proof. On the common cover by Γ of Lemma 68.41, all T summands have squared norm h by translation invariance and invariance of Haar averages under finite covers. They are mutually orthogonal: the unitary periods from the original lattice act on them by distinct characters. Consequently the normalized sum with coefficient magnitudes $1/T$ has squared norm h/T. Its metric density descends to $\mathbb{C}/\Lambda'$. Both lattices have area A; translating the function again preserves its average norm.

For transport under embeddings we give the scalar calculation. Take $N = 2M$ as in Theorem 68.35, with a common M divisible by T killing every shift under consideration, enlarging it to kill a lift from the common cover if necessary. Under an embedding the rational quotient in that theorem transports exactly when $[w/(2N)]$ is transported. For $w = 2\omega_1/T$ this point is $[\omega_1/(TN)]$. Its conjugate is represented by a primitive period ω_1' divided by TN, since its exact order is TN. To lift a primitive residue pair modulo TN, complete it to a unimodular matrix modulo each prime power by elimination at a unit entry, and combine the elementary matrices by the Chinese remainder theorem. Lifting them gives an integral unimodular matrix. Semilinearity for CM preserves the condition of belonging to the prescribed pure line at level T^2, with the prime transported. Use the multiples jw' of this same representative $w' = 2\omega_1'/T$ in all the

terms, and a representative b' transporting $[b/(2N)]$. The composition phase is transported as well:

$$\exp(\pi i E(b,w)) = e_{N^2}([b/(2N)],[w/(2N)])^2.$$

The defining multiplier for $U_b U_{jw}$ uses this phase or its inverse to power j. Functoriality of the Weil pairing and additivity of division points thus give a common translate of a full cyclic sum on the conjugate curve. The character ζ_T^{-k} conjugates to another T-th root of unity. This proves the norm assertion there including its relative phases.

For the pointwise estimate write either elliptic period lattice as γI, where I is a fractional $\mathcal{O}_K$-ideal. Every nonzero $\alpha \in I$ satisfies $|\alpha|^2 = |N_{K/\mathbb{Q}}\alpha| \geq N(I)$, since $(\alpha)I^{-1}$ is integral. The area of I is $\sqrt{|D_K|}N(I)/2$. Thus the shortest period has square length at least $2A/\sqrt{|D_K|}$, uniformly for the old ideal and for the new ideal $\mathfrak{p}^{-1}\bar{\mathfrak{p}}I$. Choose a disk of squared radius $r^2 = \min(A/\pi, A/(2\sqrt{|D_K|}))$, which embeds up to its boundary in the elliptic torus. At any chosen center apply the mean-value inequality to the squared modulus of the holomorphic Hermitian translate. Its integral over the disk is at most $e^{cr^2}A$ times its squared average norm. Division by πr^2 bounds the metric value there by B_K times that norm. Undoing the metric gives the formulas. □

For the exact expansion at an embedding σ, let a index the remaining data of a summand of column i: the covectors, the class path with its coupled intermediate vertex, and the translations outside T. Allow separate indices for different choices of product coordinates. After expanding the original homogeneous differential and the prescribed projection, write $v'_{i,a,\alpha}$ for the resulting coefficient vector, including the frame scalar, class masses and signs. Use the divided columns if the Gram matrix is G°. The scalar factors differentiated in that summand are the complete $U_b S_k$ and the unprojected $U_b \Theta_t$ just constructed. The Fourier coefficient $1/T$ is already inside each S_k. For positive $\mu_i(a)$ of sum one put

$$B'_i = \sum_{a,\alpha} \frac{\alpha!}{\mu_i(a)} v'_{i,a,\alpha}(v'_{i,a,\alpha})^*.$$

An empty sum gives zero. Let n_{ij}, k_{ij} be the number of elliptic coordinates and the prescribed total derivative degree of edge j. Set $h_{ij,\sigma}$ equal to the squared average norm of its original marked elliptic factor, $c_{ij,\sigma} = \pi/A_{ij,\sigma}$, and

$$Q_{i,\sigma} = \prod_j (B_K h_{ij,\sigma})^{n_{ij}} c_{ij,\sigma}^{k_{ij}} H_{n_{ij}}(k_{ij}), \qquad d_i^T = f \sum_j r_j.$$

Here $H_n(k)$ is the explicit sphere factor of Lemma 66.8. Use the corresponding factor norms in the product if the elliptic marks differ by coordinate.

Theorem 144.5 (Archimedean estimate after the terminal sums)**.** *For the prescribed Gram matrix at any embedding and a list of s columns, the coefficient matrices just defined satisfy*

$$\det G_\sigma \;\leq\; T^{-\sum_i d_i^T} \left(\prod_i Q_{i,\sigma} \right) D_s(B'_1, \dots, B'_s).$$

Proof. Exactly $f r_j$ scalar factors on edge j are projected, since the fixed nonzero ξ' gives r_j independent functionals at each position. Their product, including the unconstrained coordinates, is bounded in squared magnitude by

$$T^{-f r_j} (B_K h_{ij,\sigma})^{n_{ij}} \exp(c_{ij,\sigma}\|z\|^2)$$

by Lemma 144.4. Lemma 66.8 bounds the sum of the squared derivatives of degree k_{ij} divided by their multi-index factorials by this constant times $c_{ij,\sigma}^{k_{ij}} H_{n_{ij}}(k_{ij})$. Taking the product over the

separate edge variables bounds the corresponding sum for each fixed a by $T^{-d_i^T}Q_{i,\sigma}$. Vector Cauchy–Schwarz with weights $\alpha!/\mu_i(a)$, as in Theorem 143.10, gives $Y_iY_i^* \le T^{-d_i^T}Q_{i,\sigma}B_i'$. Monotonicity and multilinearity of the mixed determinant D_s give the claimed bound. □

Corollary 144.6 (The explicit terminal Fourier index cancels)**.** *Write $K_u^{(0)}$ for K_u in Theorem 141.27 with its displayed terminal summand omitted. For an independent list of q_c divided projected columns, the upper bound in that theorem on the same fields and intervals holds with its $H(G^\circ)$ replaced by*

$$K_u^{(0)} + \frac{\log T}{q_c}\sum_{i,j,\tau}\nu_{ij,\tau,1} + \frac{1}{q_c[J:\mathbb{Q}]}\sum_{\sigma:J\hookrightarrow\mathbb{C}}\log\left(\left(\prod_i Q_{i,\sigma}\right)D_{q_c}(B_1',\dots,B_{q_c}')\right).$$

Proof. Use Corollary 144.3 at the finite places. The terms involving fr_j there sum to $q_c^{-1}\sum_i d_i^T\log T$. Theorem 144.5 subtracts exactly this number in the averaged logarithm of the Gram determinant. The stated first-row term and the displayed coefficient bound give the result. □

The matrices B_i' still include all allowed intermediate covectors and depend on T. Bounding their contribution and exhibiting the required rank for the original signed projection are separate arithmetic questions. The cancellation here concerns the specified terminal normalization in the height on E_x.

144.3 Rank of the terminal derivative table on nonzero covectors

Let $p = T$ and use a connected terminal frequency block in Lemma 144.1. Keep the nonzero output covector ξ' fixed. After choosing the basis of its r-dimensional frequency subspace, the input covector is a linear functional on the Fourier parameter u. Write $k = (k_1,\dots,k_r) \in \mathbb{F}_p^r$ for its coordinates, so that its original coefficient is $\exp(-2\pi i\sum_h k_h u_h/p)$. Use the compatible cyclic translations in the proof of that lemma. The input condition at the vertex is $k \neq 0$. First consider a single product term from the translations at the primes other than T.

Lemma 144.7 (The moment of a connected Fourier projection)**.** *For each scalar coordinate h there is a nonzero residue β_h, depending on the differential and the frequency generator, such that for every prime-to-p translate U_b from the common cover and all $n \ge 0$, $k_h \in \mathbb{F}_p$,*

$$\overline{J_n(U_bS_{k_h})} = \overline{J_0(U_bS_{k_h})}\,(\beta_hk_h)^n.$$

We use $0^0 = 1$. The factor of order zero is a unit for $k_h \neq 0$. It is a unit also at $k_h = 0$ if the image of b on $\mathbb{C}/\Lambda'$ differs from the mark t'.

Proof. Write $z' = \alpha z$ and $S_0 = c\Theta_{t'}'$ as in the proof of Theorem 68.42. In the connected orientation $v(\alpha) = 1$ and c is a unit. The points $[\lambda_k]$ in Lemma 68.41 are in the étale p-line on the new curve and $[\lambda_k] = k[\lambda_1]$. Indeed the period pairing with the chosen frequency identifies Λ/Γ with the character group, and $\Gamma \subset \Lambda'$. Use this identification after scaling the curve by α. The prime-to-p point b contributes zero to the additive quantity L_1 in Theorem 68.33, where the annihilator can be chosen with p-part exactly p. Multiplication of its residue formula by $\overline{\alpha/p}^{\,n}$ gives the assertion with

$$\beta_h = \overline{\alpha/p}\,L_1(\alpha\lambda_1).$$

This is nonzero by full étale order. The scalar relating the two translated sections multiplies both the derivative and the value by the same unit, so the equality retains the original phase. At

label zero the point on the new curve has prime-to-p order. Injectivity of prime-to-p reduction and the order-zero divisor calculation in Theorem 68.33 give the last assertion. □

Fix prime-to-p displacements b_h. For a finite list $\mathcal{A}$ of multi-indices use the actual rectangular table

$$M_{\alpha,k} = \prod_{h=1}^{r} J_{\alpha_h}(U_{b_h} S_{k_h}), \qquad \alpha \in \mathcal{A}, \quad k \neq 0.$$

Define $R(0) = 0$ and $R(m) = 1 + ((m-1) \bmod (p-1))$ for $m > 0$, and apply R coordinatewise. Put $\mathcal{B} = R(\mathcal{A}) \subseteq \{0, \dots, p-1\}^r$.

Theorem 144.8 (Rank on the full set of nonzero terminal covectors)**.** *If each b_h misses its mark at label zero, then the rank of the reduction of M is*

$$|\mathcal{B}| - \mathbf{1}_{\{0,p-1\}^r \subseteq \mathcal{B}}.$$

The unshifted product vacuum satisfies the hypothesis. For arbitrary b_h, restrict to the labels $(\mathbb{F}_p^{\times})^r$. On these columns the residue rank is the number of distinct componentwise classes of α modulo $p-1$. In either case some minor of the stated size is a unit at this coefficient place when the rank is positive.

Proof. By Lemma 144.7, scaling a row by $\prod_h \beta_h^{-\alpha_h}$ and a column by the inverse of its product of order-zero values leaves the matrix of the functions $k \mapsto k^{\alpha}$. These scalings are invertible on the indicated labels. On all of $\mathbb{F}_p$ one has $X^m = X^{R(m)}$ as functions. The monomials with each exponent in $\{0, \dots, p-1\}$ form a basis of all functions on $\mathbb{F}_p^r$ over the residue field. For $r = 1$ this is polynomial interpolation at p distinct elements. Tensor products give the result for every r.

On deleting the origin the only functions in the kernel are multiples of its indicator, whose unique expansion in this basis is $\prod_h (1 - X_h^{p-1})$. It lies in the span of the selected monomials precisely when all its 2^r monomials are selected, proving the first formula. At the unshifted point each new mark is nonzero two-torsion. On the multiplicative torus the same interpolation uses the exponents $0, \dots, p-2$ in each coordinate and positive or zero powers reduce modulo $p-1$. The ordinary rank criterion over the residue field gives a nonzero minor; the original matrix is integral, so this minor lifts to a unit. □

Corollary 144.9 (All homogeneous orders at one terminal block)**.** *Assume the order-zero unit hypotheses of Theorem 144.8. If the rows run over all monomials of one degree $d > 0$, the residue rank on $k \neq 0$ is the number of $b \in \{0, \dots, p-1\}^r$ for which*

$$0 < |b| \leq d, \qquad |b| \equiv d \pmod{p-1}.$$

For $d < p$ it is $\binom{d+r-1}{r-1}$. For $d \geq r(p-1)$ it is $(p^r - 1)/(p-1)$. Degree zero has rank one. More generally, for any prescribed linear combinations of these monomial rows with integral coefficients, their residue rank is the dimension of the span of the corresponding reduced polynomials evaluated on $k \neq 0$, after replacing each variable by $\beta_h X_h$.

Proof. For $d > 0$ every reduced monomial has at least one positive exponent. Conversely one can increase any positive exponent of b by $d - |b|$ when this is a nonnegative multiple of $p-1$. This proves the counting formula and shows that its span excludes the delta function at zero. For $d < p$ no exponents change. For $d \geq r(p-1)$, every nonconstant reduced monomial of the given scalar weight occurs. On the punctured vector space all functions satisfying $F(tk) = t^d F(k)$ have dimension $(p^r - 1)/(p-1)$, one arbitrary value on each projective

line. Extend such functions by zero at the origin. Their unique reduced polynomials have zero constant term, and only monomials of weight $d \bmod (p-1)$ occur by uniqueness under scalar substitution. This proves the large-degree formula. In the last assertion the same unit factor from the order-zero values multiplies every row in a given column, so the polynomial computation also applies to the stated linear combinations. □

These are the covectors in the terminal indicator $\xi' \bar{X} = \xi$. When two adjacent class operators share a covector, the global period contracts their two tables in that index. The rank calculation supplies integral minors before that contraction; the rank of the contracted, signed projection in Section 6.6 requires its actual coefficients.

144.4 The signed covector sum on a connected terminal block

Fix a field, word, edge and terminal prime $p = T$ of Theorem 9.13. Work at a coefficient place where the chosen terminal frequency block is connected. On a conjugated edge make this choice on its conjugate. Keep the output covector $\xi' \neq 0$ and the unrefined frames fixed. The input at this block ranges over $k \in \mathbb{F}_p^r \setminus \{0\}$, where r is the incoming rank. Let κ be the residue field of a coefficient valuation ring containing the finitely many scalars in the calculation. Bars denote reduction to this field.

Use the complete cyclic projections and their Fourier labels in Lemma 144.7. In a product term γ of the outgoing section let $b_{\gamma,h}$ be the prime-to-p displacement in its h-th constrained coordinate. Set

$$W_\gamma(k) = \prod_{h=1}^{r} \overline{J_0(U_{b_{\gamma,h}} S_{k_h})}.$$

Any phase multiplying the whole translated product and depending on k is included in $W_\gamma(k)$ by its residue. This convention keeps the same phase in every differential order. The formulas below also allow $W_\gamma(k) = 0$.

Lemma 144.10 (The Weyl symbol in the input covector)**.** *In the adapted basis one can take the following reduction simultaneously for all input covectors. Evaluate an integral Weyl symbol on the translated product at this edge. Leave the unconstrained product derivatives as formal commuting variables D^0. Its residue after evaluation in the r constrained coordinates is*

$$W_\gamma(k)\, Q(D^0, k), \qquad \deg_k Q \le \nu_{\tau,1},$$

where τ is the matching physical block and $\nu_{\tau,1}$ is the first row length of its partition. The polynomial Q is obtained by integral polynomial substitution and reduction. One may then apply any integral linear functional to the coefficients in the complementary derivatives.

Proof. Extend a primitive lift of ξ' to a basis of the dual target space over $\mathbb{Z}_p$. Tensor this basis with a basis of the source. It gives a basis of the dual position matrix lattice, whose first r vectors are the functionals $X \mapsto \xi' X u_h$. The dual position basis consists of the dual simple tensors. In either basis the first r matrices have one common tensor factor. Consequently every linear combination of these matrices has rank at most one. This also holds after transposition. Use these choices in the product basis construction of Lemma 144.1, with the same approximation modulo p^2. The rational basis is fixed as k varies. In the physical chain rule the matrices of the first r directions reduce to $U_1, \dots, U_r$ with this common factor in the matching block. They reduce to zero in every other block. Indeed reduction selects the position or the dual position component of the split basis at the given embedding. Distinct local blocks are independent summands of the split lattice. Taking the tensor basis on both sides ensures the rank property for either choice of component.

For a monomial in the product derivatives, Lemma 144.7 replaces each constrained derivative variable in its residue by $\beta_h k_h$, with the common factor $W_\gamma(k)$ taken only once. Linearity gives this rule for the original symbol, including at a zero label. Thus in the matching physical matrix the substitution is

$$D_\tau^0 \longmapsto D_\tau^0 + \sum_{h=1}^{r} U_h \beta_h k_h.$$

The correction is an outer product with one factor constant over the polynomial ring, so it has rank at most one. By Lemma 141.17 a source generator and a dual test use $P_\tau([D_\tau A_\tau^{\mathsf{t}}, 0])$, followed by coefficient extraction in the elementary parameters of A_τ. Right multiplication and zero padding preserve the common outer factor. Each flag minor is affine in k because two columns from the outer product give zero in its determinant expansion by columns. Expansion by rows gives the same conclusion when the common factor is on the other side. Every term in the flag polynomial has $\nu_{\tau,1}$ minors. The symbols at the other embeddings are constant in k. The product, parameter extraction and integral linear combinations give the asserted degree bound. All reductions take place in the polynomial coefficients before evaluating complementary derivatives, which proves the last assertion. □

To apply the formula to a contraction, choose integral coordinates for the incoming table at the shared vertex. In expanding the sum, the index γ can record a coordinate of that table, a product term of the outgoing section, and any fixed coupling at the other marked primes. Write $Y_{\gamma,i}(k)$ for the incoming values for column i, including any assigned signed scalars. The adjacent map is allowed to depend arbitrarily on k. The expansion uses the unrefined stabilizer masses of Theorem 73.1. For reduction, take a common scalar for the finite list of incoming values and another for all evaluations of the complementary derivative monomials and remaining coefficients. Choose them so the two lists are integral. Such scalars exist by finiteness. They give the same overall normalizing factor in every summand. The assertion concerns the resulting integral matrix and its specified normalization. In particular it supplies no bound for the cost of those two scalars.

Theorem 144.11 (Contraction on the lines of the covector space)**.** *For each line L in $\mathbb{F}_p^r$ choose $v_L \in L \setminus \{0\}$. For $0 \le e \le D = \nu_{\tau,1}$ form the signed moments*

$$B_{\gamma,i,e}(L) = \sum_{a \in \mathbb{F}_p^\times} a^e W_\gamma(a v_L)\, \overline{Y_{\gamma,i}(a v_L)}.$$

For a fixed outgoing Weyl row j, expand the polynomial of Lemma 144.10 after the complementary evaluations as $\sum_{e=0}^{D} Q_{j,\gamma,e}(X)$ in its homogeneous parts. The residue of the matrix after summing over the entire shared covector in this block has entry

$$\sum_{\gamma, L} \sum_{e=0}^{D} Q_{j,\gamma,e}(v_L)\, B_{\gamma,i,e}(L).$$

For several edge shapes the same formula uses their individual degree bounds. It applies with any remaining indices fixed, and subsequently to their original finite linear combinations with integral coefficients. A nonzero minor of this matrix gives a minor with unit determinant in the normalized integral matrix.

Proof. The indicator of the edge is $\xi' \bar{X} = k$ in the fixed frames. By its complete Fourier expansion every outgoing product term has precisely the cyclic scalar labels used in the lemma. All dependence on the same covector from the previous edge is in the values $Y_{\gamma,i}(k)$. Holding the common coupling fixed until the two incident factors have been multiplied retains the signs

of that summand. Apply the lemma after the indicated denominator clearing. For $k = av_L$ the homogeneous part of degree e takes the value $a^e Q_{j,\gamma,e}(v_L)$. Summing over a, then over the lines, uses each admissible k exactly once. Replacing v_L by tv_L multiplies the two factors by t^e and t^{-e}. This proves the identity independently of the representatives. The minor assertion follows by reduction of the determinant. □

Corollary 144.12 (Visible moments at the prescribed word weights)**.** *On the original common intervals $D + 1 < p - 1$ for every sufficiently large x, uniformly in the chosen word, edge and place. On a fixed line let s_L be the dimension over κ of the span of the functions $a \mapsto W_\gamma(av_L)$ for a list of outgoing product terms. When a common scalar incoming table on this line varies, all normalized Weyl rows in this list together have residue rank at most $\min(p-1, s_L(D+1))$, with D the maximum first row length on the list. Here any additional coefficients varying with a must be included in the functions defining s_L.*

For one unshifted product and $D + 1 < p - 1$, the common kernel of the $D + 1$ moments on the line has dimension $p - D - 2$. Its intersection with the ordinary augmentation condition $\sum_a Y(av_L) = 0$ has dimension at least $p - D - 3$. Every member of this intersection has an integral lift satisfying the same augmentation condition before reduction.

Proof. By Lemma 141.11 and the weight formulas following (9.1), $D \ll xb + x^C$ for a fixed C, where $b = \exp(x^\varepsilon)$. Theorem 93.17 gives $\log p = (17 + o(1))m \log b$ and $m \to \infty$. Hence $(D + 1)/(p - 1) \to 0$. On a single line the row functions in the theorem lie in the sum, over $0 \le e \le D$, of a^e times the indicated s_L-dimensional space. The dimension of that sum proves the upper bound.

For the unshifted product every $W_\gamma(av_L)$ is nonzero by Theorem 144.8. Multiplication by this weight is an invertible diagonal operator. The functions $1, a, \dots, a^D$ are independent on $\mathbb{F}_p^\times$, since a polynomial of degree at most $D < p - 1$ vanishing there is zero. Subtracting the rank $D+1$ of the moment map from $p-1$ gives the kernel dimension. One more linear equation removes at most one dimension. Lift all but one value arbitrarily and take the last to be the negative of their sum. This lifts the given residue vector with exact augmentation. □

The moments concern the actual shared index on one outgoing block. On that block even the unshifted integral test can lose many incoming residue directions after contraction. To obtain the projected rank in Section 6.6 one must establish survival for the prescribed incoming columns with their actual weights and the remaining class sums.

144.5 Scalar orbits through the signed theta chain

Fix $p = T$, the word and the terminal generator of Theorem 9.13. Use the original initial tables whose values sum to zero on each nonzero scalar orbit at each $v \mid T$, with the other coordinates held fixed. Extend these tables by zero at any zero covector. The local character decomposition in Corollary 128.26 is integral at T. We compute a gain in the finite estimate each time such an input passes through a terminal incidence condition in the chain.

At a fixed coefficient place $w \mid p$ let ω denote the Teichmüller lift of $\mathbb{F}_p^\times$, and write $d = p-1$. Use the convention $H(ak) = \omega(a)^t H(k)$ for a scalar character of an incoming table. We take $1 \le t < d$. For a fixed nontrivial additive character ψ put

$$g(t) = \frac{d - t}{d}, \qquad G_t(c) = \sum_{a \in \mathbb{F}_p^\times} \omega(a)^t \psi(-ac).$$

One has $G_t(0) = 0$ and $v_w(G_t(c)) = g(t)$ for $c \neq 0$, in the normalization $v_w(p) = 1$. Here is a direct check of the sign convention in Theorem 126.12. Changing $c \neq 0$ changes the sum

by a root of unity, so use the additive terms ζ^a for a primitive p-th root ζ. Let $\lambda = \zeta - 1$, with $v_w(\lambda) = 1/d$ by $\prod_{a=1}^{p-1}(1-\zeta^a) = p$. The coefficient of λ^n after expanding $(1+\lambda)^a$ is $\sum_{a=1}^{p-1}\omega(a)^t\binom{a}{n} \in \mathbb{Z}_p$. For $0 \le n < d-t$ its reduction vanishes by summing powers of exponents from t to $t+n < d$ on $\mathbb{F}_p^\times$. For $n = d-t$ the reduction is $-1/n!$. Thus the term of index $d-t$ has uniquely smallest valuation. The assertion at zero follows by summing the nontrivial scalar character.

Proposition 144.13 (The orbit sum at each edge of a fixed character)**.** *At each terminal place prescribe a nontrivial scalar character χ_v of the initial table, in the convention just given. At every vertex choose representatives for the lines of nonzero covectors. At the terminal vertex choose the actual marked covector for its line. On a local edge with consecutive representatives η, η' and source dimension r, the full signed period in this character may be computed by summing over the intermediate lines and using the local multiplier*

$$\begin{aligned}\mathcal{K}_{\eta,\eta',\chi_v}(X) &= \sum_{a\in\mathbb{F}_p^\times} \chi_v(a)\,\mathbf{1}_{\eta'\bar{X}=a\eta} \\ &= p^{-r}\sum_{u\in\mathbb{F}_p^r} \psi(\eta'\bar{X}u)\sum_{a\neq 0}\chi_v(a)\psi(-a\eta u)\end{aligned}$$

on the integral matrix lattice. The terms with $\eta u = 0$ vanish. For one product vacuum in the adapted coordinates of Lemma 144.4, the complete local section for this multiplier has average squared norm

$$(p-1)p^{-r}\prod_{h=1}^{r} h_h$$

on the constrained coordinates, where h_h is the corresponding original elliptic average squared norm. This holds at every complex embedding and after any common translation on these coordinates.

Proof. Write the covector at the j-th vertex as $a_j\eta_j$. Its incidence with the next vertex is $\eta_{j+1}\bar{X}_j = (a_j/a_{j+1})\eta_j$. Take $a_m = 1$ at the marked final covector. Substitution of $s_j = a_j/a_{j+1}$ is a bijection on the m scalar variables, and $\chi_v(a_0) = \prod_{j=0}^{m-1}\chi_v(s_j)$. Thus all scalar sums separate with exactly the first multiplier displayed above. Perform the substitution independently at each terminal place. The path formula of Theorem 73.1 has already used the unrefined stabilizer masses. The matrix cosets at other places, their signed coefficients, and the coupling coset common to the last two edges are fixed during this substitution. The theta section is linear in each local test. Summing over the other data now gives the stated formula for the signed period. The finite Fourier expansion of the indicator gives the second equality. Scalar character orthogonality removes the terms specified.

For the complex norm the section is $\sum_{a\neq 0}\chi_v(a)\prod_h S_{a\eta_h}$. Work on the product of the common covers for the cyclic projections. In each coordinate the p frequency translates of the vacuum are orthogonal with equal squared norm h_h, by their distinct period characters. Their normalized finite Fourier transform therefore gives p orthogonal sections of squared norm h_h/p. Distinct a give distinct label vectors since $\eta \neq 0$. Taking tensor products and then summing squared norms proves the formula. Translations are unitary on the common covers. At a complex embedding the finite Fourier characters and their phases transport as in Lemma 144.4. The scalar characters continue to have absolute value one. □

Theorem 144.14 (Divisibility from the incoming scalar orbits)**.** *Use the unrefined integral Weyl lattices of Theorem 144.2. For column i and edge j let B_w be the embeddings with étale frequency at the effective coefficient place on that edge, and $v(\tau) = P_{\tau,w} \cap E_x^+$. Suppose the*

incoming table has valuations at least h in the common lattice and scalar characters $\chi_v(a) = \omega(a)^{t_v}$. After summing all input covectors, the terminal part of this edge has entries of valuation at least

$$h - \sum_{\tau \in B_w} (r_j + \nu_{ij,\tau,1} - g(t_{v(\tau)})).$$

The separately specified nonterminal scalars and class masses are excluded here. For every separately zero-sum input the same bound holds with $1/(p-1)$ at each occurrence of $g(t_v)$. These gains add through all the consecutive edges of the original signed period.

Proof. Keep the output covectors and the data away from T fixed. In each étale frequency block use the second formula of Proposition 144.13. For each input line the incoming value at its representative is integral after the common input bound is removed. At a nonzero frequency pairing the scalar sum contributes precisely $g(t_v)$ in valuation. The reciprocal p^{r_j} contributes $-r_j$. Its remaining modulation $X \mapsto \psi(\eta' \bar{X} u)$ has rank one, as a frequency matrix, and is independent of the input scalar. Keep the complete cyclic projections in the connected blocks. There each term with a fixed input scalar is integral in every constrained differential order by Lemma 144.1. The sum with the character coefficients remains integral.

The adapted bases of Theorem 144.2 apply to the product of these choices. At an étale block only a single rank-one direction can cost a valuation of one per derivative, up to $\nu_{ij,\tau,1}$ derivatives. The scalar Gauss factors multiply the entire differential symbol. Their valuation gain survives that estimate and the return to the common frame. Summing over the remaining frequencies and input lines by the valuation inequality proves the bound.

For a fixed matrix coset the coefficient map reads the zero-extended table at $\xi' \bar{X}$. Scaling ξ' by a scales this pullback by a, so scalar types and separate zero sums propagate, including at a fixed coupling coset. Conjugation preserves nontriviality of the character. Induction adds the gains along the chain. The masses and denominators are those of Theorem 141.21.

The scalar projectors at w have unit denominators, powers of $p-1$. They preserve the input bound and the refined lattice on each line, since a scalar preserves the covector kernel. The zero-sum condition kills the trivial character at each place. Every remaining summand gains $g(t_v) \geq 1/(p-1)$, and addition preserves this bound. □

Corollary 144.15 (A gain for the divided projected columns)**.** *For a list of q_c independent projected columns with separately zero-sum initial tables, one may subtract*

$$\frac{f \log T}{q_c (T-1)} \sum_{i,j} 1$$

from the upper bound for their contribution to Δ_u in Corollary 144.6. If all initial tables have the quadratic scalar character at every terminal place, one may subtract $(f \log T/(2q_c)) \sum_{i,j} 1$ instead.

Proof. Scaling by z_i^{-1} as in Theorem 141.21 puts the initial values in the unrefined lattice with the same scalar type. Apply the final lowering and column division of Theorem 141.27. For each τ the places with étale frequency have total normalized degree $1/2$ by Corollary 144.3. Summing their gains for both period ideals and dividing by q_c gives the saving. The quadratic exponent is $(T-1)/2$ at every completion. Theorem 144.5 supplies the archimedean bound. □

The determinant estimate assumes independence. Section 6.6 still requires that independence and the full height bound on the prescribed projections.

144.6 Mean Gauss valuations in the projected period height

Keep the terminal prime $p = T$, the word, the marked generators, and the integral frames of Theorem 144.14. Put $F = E_x^+$ and $f = [F : \mathbb{Q}]$. For a given column fix a place v of F above p. Its residue field is $\mathbb{F}_p$. The scalar action on nonzero input covectors is by $\mathbb{F}_p^\times$. Use a finite Galois coefficient field J containing E_x, the test data and all scalar character values. The probability of a coefficient place $w \mid p$ is $[J_w : \mathbb{Q}_p]/[J : \mathbb{Q}]$. This is the weighting in the global height, and enlarging J gives the same average by the tower formula for local degrees.

At the effective coefficient place of an edge let $\mathcal{E}_v$ be the event that the block belonging to v has étale frequency. For a nontrivial scalar character χ let $g_w(\chi) = (p - 1 - t)/(p - 1)$, where $\chi = \omega_w^t$, $1 \le t < p - 1$, in the convention of Proposition 144.13. The valuation of p is one. On a conjugated edge use the effective place and character of that edge in both the event and g_w.

Lemma 144.16 (The smallest mean on a set of coefficient places of weight one half)**.** *One has* $\Pr(\mathcal{E}_v) = 1/2$*. If* χ *has order* $m > 1$*, then*

$$\mathbb{E}_w[\mathbf{1}_{\mathcal{E}_v} g_w(\chi)] \ge A(m).$$

Here $A(2) = 1/4$*, and for* $m > 2$*,*

$$A(m) = \frac{1}{m\varphi(m)} \sum_{\substack{1 \le a < m/2 \\ (a,m)=1}} a \;\ge\; \frac{1}{12}.$$

These are the exact lower bounds when only the marginal distribution of $g_w(\chi)$ *and the weight of the event are known.*

Proof. For fixed w, v there is a unique embedding τ in the CM type with $\tau^{-1}w \cap F = v$, by complete splitting and the bijection in Theorem 144.2. Conjugation on J fixes the totally real field $\tau(F)$ pointwise. Thus the same τ occurs for the conjugate place. On $\tau(E_x)$ it is CM conjugation, so the two pullbacks to E_x are the partners above v. Exactly one belongs to S_T. Conjugation preserves the local degree. The same pairing applies when the effective place at an edge is conjugated. This gives the half weight for each individual v.

Fix a generator of $\mathbb{F}_p^\times$. Its value under χ is a primitive m-th root. Since $m \mid p - 1$, p splits completely in $\mathbb{Q}(\zeta_m)$. Reduction at all its places takes that value through the primitive m-th roots in $\mathbb{F}_p$, each once. The total local degree over every such place in J is the same. Thus $g_w(\chi)$ is uniformly distributed on $\{a/m : 1 \le a < m,\ (a, m) = 1\}$. For $m > 2$ precisely half of these values lie below $1/2$, by pairing a with $m - a$. Among events of weight $1/2$ the integral is smallest when it puts all its mass on those values. For $m = 2$ the function is identically $1/2$.

For the numerical bound let μ be the Möbius function, $\mathrm{rad}(l) = \prod_{q \mid l} q$, and let $\omega(l)$ count the distinct prime divisors of l. For $m > 2$ the endpoint $m/2$ makes no contribution and inversion gives

$$m\varphi(m)A(m) = \sum_{d \mid m} \mu(d) d \sum_{1 \le b \le m/(2d)} b.$$

Writing $u = m/d$, the inner sum equals $(u^2 + 2u)/8$ for even u and $(u^2 - 1)/8$ for odd u. The quadratic terms together give $m\varphi(m)/8$. For odd m, the remaining terms sum to $-\prod_{q \mid m}(1 - q)/8$. For $4 \mid m$, each squarefree $d \mid m$ leaves even u, and the remaining terms vanish by $\sum_{d \mid m} \mu(d) = 0$. For $m = 2l$ with $l > 1$ odd, the even-u linear terms vanish by the same cancellation over the divisors of l. The odd-u corrections then give $2\prod_{q \mid l}(1 - q)/8$. Consequently

$$8A(m) = \begin{cases} 1 - (-1)^{\omega(m)} \mathrm{rad}(m)/m^2, & m \text{ odd}, \\ 1, & 4 \mid m, \\ 1 + (-1)^{\omega(l)} \mathrm{rad}(l)/l^2, & m = 2l,\ l > 1 \text{ odd}. \end{cases}$$

In both odd arguments l or m is at least three and $\operatorname{rad}(l) \le l$. This proves the lower bound. It is attained by the displayed marginal formula when $m = 6$. □

Theorem 144.17 (A terminal saving independent of the order of a pure scalar character)**.** *Let q_c divided projected columns be independent as in Corollary 144.6. Suppose each initial table belongs to a nontrivial scalar character at each $v \mid T$, of order m_{iv} on column i. From the upper bound in that corollary one may subtract*

$$\frac{2\log T}{q_c}\sum_{i,j}\sum_{v\mid T} A(m_{iv}) \;\ge\; \frac{f\log T}{6q_c}\sum_{i,j} 1.$$

Here j runs over the edges of the word for column i. If the columns can be paired with matching edges, terminal halves and effective places, and inverse scalar characters at every v, the saving is at least $(f\log T/(2q_c))\sum_{i,j} 1$.

Proof. At a coefficient place w, the improvement at v from Theorem 144.14 is $\mathbf{1}_{\mathcal{E}_v} g_w(\chi_{iv})$ in valuation. That theorem applies to the complete edge after the incoming scalar sum, with the common class at two consecutive edges held fixed until their contributions are multiplied. The finite bounds therefore add over edges. Averaging over w, multiplying by $\log T$, and including both factors in the Gram bound gives the prefactor $2\log T/q_c$, as in Corollary 144.15. Lemma 144.16 bounds each remaining average. There are exactly f places v because T splits completely in F.

For a matching pair one has $g_w(\chi) + g_w(\chi^{-1}) = 1$ at the same effective place, including for quadratic χ. The event is the same at the paired edges, hence its mean contribution summed over the two columns is $1/2$. The saving per column averaged over that pair follows after the same factor of two in the Gram bound. □

Corollary 144.18 (Bounds in terms of the scalar support of the prescribed input)**.** *For an independent list with separately zero-sum initial tables, let s_{iv} be the number of distinct scalar characters occurring at v in column i, over a splitting field. In the first bound of Theorem 144.17 each $A(m_{iv})$ may be replaced, for a general input, by*

$$\max\left\{\frac{1}{2(T-1)},\; \frac{1}{3136\,s_{iv}^2}\right\}.$$

For a single supported character the larger bound of the theorem applies directly.

Proof. The projectors onto scalar characters have denominators dividing a power of $T-1$, which is a unit at T. They preserve the integral lower bound of the input. Each component propagates through the incidence condition with its scalar type as in Theorem 144.14. For their sum use the minimum of the gains over the $s = s_{iv}$ supported characters at this place. A nonzero column has $s \ge 1$ and augmentation excludes the trivial character. Every g_w is at least $1/(T-1)$, which proves the first option after averaging over the half event.

For $0 < y < 1$ and any character order $m \ge 2$, its marginal distribution gives

$$\Pr\{g_w(\chi) \le y\} = \frac{\#\{1 \le a \le my : (a,m) = 1\}}{\varphi(m)} \le y + \frac{2^{\omega(m)}}{\varphi(m)}.$$

Indeed Möbius inversion replaces each integer part by its argument with an error of at most one per squarefree divisor. The probability is zero if $my < 1$. For all m, $\sqrt{m}\,2^{\omega(m)}/\varphi(m) \le 6$. At a prime power q^a the factor is at most $2\sqrt{q}/(q-1)$, which is less than one for $q \ge 7$. The product of the three possible factors at $2, 3, 5$ is $\sqrt{30} < 6$. Thus in both ranges $\Pr\{g_w(\chi) \le y\} \le 7\sqrt{y}$.

Set $y = (28s)^{-2}$. The union of the exceptional events for all supported characters has weight at most $1/4$. The étale event has weight $1/2$, leaving weight at least $1/4$ where all the gains exceed y. Its contribution to the mean is at least $y/4 = 1/(3136s^2)$. Linearity of the average and the valuation inequality for the sum prove the claim. □

These bounds retain the original scalar supports and the same integral projections. The archimedean coefficient term and the first-row terms in Corollary 144.6, as well as independence for the columns needed in Section 6.6, remain in the comparison.

144.7 The CM half in the terminal character average

Use the actual CM field $K = \mathbb{Q}(\sqrt{-D})$ of the common period construction, with D the auxiliary prime congruent to seven modulo sixteen. Its discriminant is $-D$. Write λ for its primitive quadratic Dirichlet character of conductor D. The CM embeddings of E_x extend one chosen embedding of K by Theorem 64.10. The terminal prime is $T = \pi\bar{\pi}$ with $\pi \in \mathcal{O}_K$, by Theorem 9.13. In particular the condition $P_{\tau,w} \mid \pi$ in Theorem 144.2 depends only on the restriction of the effective coefficient place w to K. It specifies one of the two primes of K above T, the same one for all τ on this edge. Denote this event by $\mathcal{E}$.

Lemma 144.19 (The joint distribution of the scalar exponents and the CM half)**.** *At one place $v \mid T$ let the nonempty support of a separately zero-sum input consist of characters ρ^e for e in a set $I \subseteq \{1, \dots, m-1\}$, where ρ generates the group spanned by the support and has order $m \mid T-1$. Write $[k]_m$ for the least nonnegative residue modulo m and put*

$$H_I(a) = \min_{e \in I} \frac{[ea]_m}{m} \qquad (a \in (\mathbb{Z}/m\mathbb{Z})^\times).$$

Then the mean $\mathbb{E}_w[\mathbf{1}_{\mathcal{E}} \min_{e\in I} g_w(\rho^e)]$ in the terminal estimate is

$$\begin{cases} \dfrac{1}{2\varphi(m)} \displaystyle\sum_{(a,m)=1} H_I(a), & D \nmid m, \\ \dfrac{1}{\varphi(m)} \displaystyle\sum_{\substack{(a,m)=1 \\ \lambda(a)=\epsilon}} H_I(a), & D \mid m. \end{cases}$$

The sums use one complete residue system. In the second line the sign is determined at any single coefficient place w_0 on the chosen half by $\epsilon = \lambda(m g_{w_0}(\rho))$. If $I = -I \bmod m$, either line equals the first expression.

Proof. Let $a(w) = m g_w(\rho)$, a unit residue as in Lemma 144.16. The definition of the Gauss valuation uses the negative of the scalar character exponent modulo $T-1$. Multiplying that exponent by e and taking its positive fractional representative gives $g_w(\rho^e) = [ea(w)]_m/m$. These numerators are nonzero since the characters in the support are nontrivial.

The prime T splits completely in both K and $\mathbb{Q}(\zeta_m)$. Its primes in the compositum of the two fields are thus indexed simply transitively by the Galois group. Each receives the same total local-degree weight from the coefficient field. The restriction to K fixes $\mathcal{E}$, and the restriction to $\mathbb{Q}(\zeta_m)$ fixes $a(w)$. If $D \nmid m$, then D is unramified in $\mathbb{Q}(\zeta_m)$, whereas it ramifies in K. Hence the intersection is $\mathbb{Q}$, and these two uniform choices are independent. If $D \mid m$, then K is the quadratic subfield of $\mathbb{Q}(\zeta_D)$ cut out by λ. For example, the quadratic Gauss sum at D has square $-D$ and transforms by λ. A Galois element sending the prime at w_0 to the prime at w fixes K exactly when its exponent on ζ_m is in $\ker \lambda$. The inverse of that exponent multiplies $a(w_0)$ to give $a(w)$. This proves the coset formula and the sign rule. For inverse-stable support

$H_I(-a) = H_I(a)$. The two cosets are interchanged by -1, since K is imaginary, proving the last assertion. The argument uses the effective place throughout, so it applies equally to a conjugated edge. □

Theorem 144.20 (The exact mean for a pure scalar character at the terminal prime)**.** *For a character of order $m > 1$ the mean $G = \mathbb{E}_w[\mathbf{1}_{\mathcal{E}} g_w(\chi)]$ is $1/4$ when $D \nmid m$. For $D \mid m$ it is*

$$G = \frac{1}{4} - \frac{\epsilon h(K)}{w_K \varphi(m)} \prod_{\substack{q \mid m \\ q \text{ prime}, q \neq D}} (1 - \lambda(q)),$$

where $h(K)$ is the class number, $w_K = |\mu(K)|$, and ϵ is as in the lemma for $\rho = \chi$. In particular every character of even order has $G = 1/4$. For every order one has, uniformly in v, the word and the edge,

$$|G - 1/4| \le \frac{h(K)}{w_K (D-1)} \ll \frac{\log(2D)}{\sqrt{D}}.$$

Proof. Use $I = \{1\}$ in the lemma. The mean of a/m over all unit residues is $1/2$ by symmetry, including $m = 2$. For $D \mid m$, replace the indicator of the coset by $(1 + \epsilon\lambda(a))/2$. Its correction to $1/4$ is $\epsilon/(2m\varphi(m))$ times $\sum_{1 \le a \le m,\ (a,m)=1} a\lambda(a)$.

Put $B_\lambda = D^{-1} \sum_{a=1}^{D} a\lambda(a)$. For R a multiple of D, periodicity and $\sum_{a=1}^{D} \lambda(a) = 0$ give $\sum_{b=1}^{R} b\lambda(b) = R B_\lambda$. Inclusion-exclusion for the distinct prime divisors of m other than D now gives

$$\frac{1}{m} \sum_{\substack{1 \le a \le m \\ (a,m)=1}} a\lambda(a) = B_\lambda \prod_{\substack{q \mid m \\ q \text{ prime}, q \neq D}} (1 - \lambda(q)).$$

Indeed a squarefree divisor u of those primes contributes $\mu(u) u \lambda(u) \sum_{b \le m/u} b\lambda(b)/m = \mu(u)\lambda(u) B_\lambda$. The identity $B_\lambda = -2h(K)/w_K$ is the imaginary quadratic class number formula at zero. In detail, $L(0, \lambda) = -B_\lambda$ follows by splitting the Dirichlet series into Hurwitz zeta functions and using $\zeta(0, a/D) = 1/2 - a/D$. The factorization $\zeta_K = \zeta L(\,\cdot\,, \lambda)$, $\zeta(0) = -1/2$, and the analytic class number formula $\zeta_K(0) = -h(K)/w_K$ give the asserted identity.

The prime two splits in K, as $-D \equiv 1 \bmod 8$. Therefore $2 \mid m$ kills the correction whenever $D \mid m$. For an odd m divisible by D, each factor at $q \neq D$ has magnitude at most two, while $\varphi(q^a) \ge 2$ for every odd prime power. The remaining denominator $\varphi(D^a)$ is at least $D - 1$. This gives the first uniform bound. For the second, Minkowski gives an integral ideal of norm at most $2\sqrt{D}/\pi$ in each class of K. The number of quadratic ideals of norm b is at most the divisor function of b, by comparison of the split, inert and ramified Euler factors with those of $\zeta(s)^2$. Summing up to R gives at most $R(1 + \log R)$ ideals, by counting pairs of positive integers with product $\le R$. This bounds $h(K)$ as required. □

Corollary 144.21 (The terminal contribution on the original common intervals)**.** *For an independent list of q_c divided projected columns whose input is in one nontrivial scalar character at every terminal place, one may subtract at least*

$$\frac{f \log T}{q_c} \left(\frac{1}{2} - \frac{2h(K)}{w_K (D-1)} \right) \sum_{i,j} 1$$

from the upper bound of Corollary 144.6. If every order is even or is prime to D, the coefficient in parentheses may be taken to be $1/2$. For the family of Theorem 19.15 it is at least $1/2 - o(1)$ uniformly as the intervals tend to infinity. For a general separately zero-sum input the allowable subtraction on any block is obtained by multiplying the finite sum in Lemma 144.19 by $2 \log T/q_c$.

Proof. Apply Theorem 144.14 at each effective coefficient place and average the gains as in Theorem 144.17. For the pure inputs use the exact means just calculated. There are $f = [E_x^+ : \mathbb{Q}]$ blocks per edge, and the two period factors in the determinant give the stated multiplier. For a sum of characters use the integral scalar idempotents at T and the smallest of the supported gains on that block. The joint-distribution lemma evaluates its average. In the selection in Theorem 19.15, $D \equiv -1$ modulo all odd primes up to a parameter tending to infinity, so the prime D tends to infinity as well. The preceding uniform class number estimate proves the limiting statement on these same fields. □

This determines the averaged scalar-character factor of the terminal Gauss sums. The first-row derivative costs, the archimedean coefficient matrices and the required projected rank still enter the bound for the full period determinant.

144.8 Compact frame pairings in the projected period

At a fixed archimedean embedding, the upper bound of Theorem 144.5 uses the Fischer inner products of the polynomials in the differential variables. We compute these inner products with the initial forms and the terminal tests kept in the formula. This extends the scalar calculation at the identity frame in Theorem 66.30. It also gives cross terms between paths with different intermediate coefficient maps.

For one embedding of the CM type let $r \le s$ be consecutive ranks. Give the space H^- of input flag polynomials the norm

$$\|Q\|^2 = \int_{U(r)} |Q(u)|^2 \, du,$$

with Haar probability, and use the corresponding norm in rank s. Restriction to the unitary group determines a polynomial, by Zariski density. A determinant twist on this group preserves the norm. If the input has column weight λ, let n be the twist giving the partition $\nu = \lambda + (n, \dots, n)$ used in the differential symbol. The function $Q \det^n$ extends to a polynomial of this column weight on the full matrix space. In (66.22), view the differential matrix as an independent s-by-r matrix Z. Its polynomial operator is

$$(C(Z)Q)(Y) = (Q \overset{n}{\det})(Z^{\mathsf{t}} Y_{[r]}).$$

Write H^+ for the compact coefficient space of column weight $(\nu, 0, \dots, 0)$ in rank s, and $d_+ = \dim H^+$. For several CM embeddings use the tensor products of these spaces and maps, with a separate Z_τ at each embedding. Set

$$c = \prod_{\tau} \prod_{h=1}^{r} (r - h + 1)_{\nu_{\tau,h}}, \qquad (a)_k = \prod_{l=0}^{k-1} (a + l), \qquad (a)_0 = 1.$$

Throughout this subsection A^* is the Hermitian adjoint in the stated metrics.

Lemma 144.22 (The covariance of one polynomial substitution)**.** *Give the entries of the matrices Z_τ independent standard complex Gaussian distributions of density $\pi^{-1} e^{-|z|^2}$. For every endomorphism A of H^-, one has*

$$\mathbb{E}\, C(Z) A C(Z)^* = \frac{c}{d_+} \operatorname{tr}(A)\, I_{H^+}.$$

Proof. First use one embedding and $A = QQ^*$. For $Y \in U(s)$ the columns of $Y_{[r]}$ are orthonormal. Each row of $Z^{\mathsf{t}} Y_{[r]}$ is a standard Gaussian row of length r by unitary invariance, and

different rows are independent. Write this square Gaussian matrix as uB by Gram–Schmidt, with upper triangular B of positive diagonal. Conditional on B, left unitary invariance makes u Haar. Write $t_h = B_{hh}$. Conditional on the first $h-1$ Gaussian columns, t_h^2 is a sum of $r-h+1$ independent exponential variables of mean one, by projecting the next column to the orthogonal complement. Its moment of order ν_h is $(r-h+1)_{\nu_h}$, as follows from the generating function $(1-z)^{-(r-h+1)}$. The flag identities under upper triangular substitution give

$$|(Q \overset{n}{\det})(uB)|^2 = |Q(u)|^2 \prod_{h=1}^{r} t_h^{2\nu_h}.$$

Successive conditioning from the last column gives $\mathbb{E}|(C(Z)Q)(Y)|^2 = c\|Q\|^2$, independent of Y. Integrating over Y gives the trace of $\mathbb{E}(C(Z)QQ^*C(Z)^*)$.

This operator commutes with left translation by $U(s)$, since changing Y to gY changes Z to $g^{\mathsf{t}}Z$, which has the same law. The column-weight space H^+ is irreducible under this left action. Indeed the Cauchy decomposition of matrix polynomials, used in Section 66.10, pairs one row module with the matching column module. Selecting the right upper-unipotent invariants of the given torus weight selects its unique highest line. Thus Schur's lemma makes the operator scalar, and its trace fixes the scalar. Polarization gives the assertion for $Q_1Q_2^*$; these maps span all endomorphisms. For several embeddings, independence gives the tensor product identity on simple tensors of endomorphisms, which span the full endomorphism space. □

Fix one of the words and the embedding σ. Expand the actual column as in Theorem 144.5, keeping the two last edges' shared class or coupling coset as a single choice. For a summand γ, write x_γ for its initial coefficient vector. At edge j denote the two compact spaces by H_j^-, H_j^+, the latter dimension by d_j, and the constant in the lemma by c_j. Write $M_{\gamma j} : H_j^+ \to H_{j+1}^-$ for the intervening coefficient transport, and $L_\gamma : H_m^+ \to V$ for the final linear map to the actual projected test space. Include the scalar of this summand in L_γ, including its phase, class measure and any column division. One may use an identity map at an intermediate vertex. Determinant characters on frames are included in the maps or in the weights of the adjacent spaces. Use conjugate spaces and conjugate variables on the conjugated factors. More explicitly, conjugation of a product conjugates every preceding coefficient and variable in that product. Moving the bars to these factors gives linear maps between the corresponding conjugate spaces and preserves the Gaussian law and the adjoint formulas.

Group summands only when their ordered scalar elliptic derivative functions and the unitary changes from product variables to physical matrix variables agree on every edge, at this embedding. Write a for such a group. Finer groups, including individual summands, are allowed. The definition permits covectors and adapted bases to vary throughout the original sum. In one group the coefficient polynomial of path γ has the form

$$P_\gamma(Z) = L_\gamma C_m(Z_m) M_{\gamma,m-1} C_{m-1}(Z_{m-1}) \cdots M_{\gamma 1} C_1(Z_1) x_\gamma.$$

In fact keeping the derivatives formal in the class formula gives exactly the determinant multiplication and substitution in (66.22). Expanding translations in the finite local tests changes the scalar elliptic functions to which one applies these derivatives. The remaining coefficient operations are the initial evaluation, the class transports, and the terminal lowering and projection of Theorem 73.1. All are independent of the formal differential variables. An orthonormal product basis from Lemma 65.8 changes the variables by a unitary matrix on the full edge space. One common such change in a group preserves all its Fischer inner products, even when it mixes matrix coordinates.

Theorem 144.23 (The signed coefficient matrix for a group of paths)**.** *Return to the product variables and write the sum of P_γ in one group a as $\sum_\alpha v_{a,\alpha} Z^\alpha$. Its positive semidefinite coefficient matrix is*

$$\begin{aligned} S_a &= \sum_\alpha \alpha!\, v_{a,\alpha} v_{a,\alpha}^* \\ &= \frac{\prod_{j=1}^m c_j}{d_m} \sum_{\gamma,\delta \in a} \operatorname{tr}(x_\gamma x_\delta^*) \left(\prod_{j=1}^{m-1} \frac{\operatorname{tr}(M_{\gamma j} M_{\delta j}^*)}{d_j} \right) L_\gamma L_\delta^*. \end{aligned}$$

In particular the sum retains the cross terms after the signed projection. At a vertex where the two transports are the actions of elements g_γ, g_δ in the same unitary representation of dimension d_j, the trace factor there is its character at $g_\gamma g_\delta^{-1}$ divided by d_j.

Proof. For a standard complex Gaussian entry, $\mathbb{E}[Z^b \bar{Z}^e] = \mathbf{1}_{b=e} b!$ by polar integration. Independence therefore identifies the first line with the expectation of the outer product of the sum of the path polynomials with itself. For a pair of summands start with $x_\gamma x_\delta^*$. Integrate Z_1 by Lemma 144.22. Its value is a scalar multiple of the identity of H_1^+. Transport on the two sides changes that identity to $M_{\gamma 1} M_{\delta 1}^*$. Its trace enters upon integrating Z_2. Each subsequent integration performs the same operation, since the edge variables are independent. The last transports take the last identity to $L_\gamma L_\delta^*$. Multiplying the constants and summing over the ordered pairs proves the formula. For a unitary action its adjoint is the action of the inverse, giving the character assertion. □

Corollary 144.24 (A bound for the prescribed determinant without derivative indices)**.** *For each column i attach its index to the preceding data. Use $Q_{i,\sigma}$ and d_i^T from Theorem 144.5. Put*

$$A_{i,\sigma} = \sum_\gamma \|x_\gamma\| \, \frac{\|L_\gamma\|_{\mathrm{HS}}}{\sqrt{d_{im}}} \prod_{j<m} \frac{\|M_{\gamma j}\|_{\mathrm{HS}}}{\sqrt{d_{ij}}}, \qquad F_i = \prod_{j=1}^m c_{ij}.$$

The endpoint norm uses the given metric on V. For any positive probabilities $\mu_i(a)$ on the groups of each column, the bound of Theorem 144.5 holds with $B_i' = \sum_a S_{ia}/\mu_i(a)$, computed by Theorem 144.23. In particular the actual Gram matrix at the prescribed tests satisfies

$$\det G_\sigma^\circ \;\le\; \prod_i (T^{-d_i^T} Q_{i,\sigma} F_i A_{i,\sigma}^2).$$

Both assertions apply on the original period fields for every translate and every word. Furthermore, for every fixed $B > 0$, uniformly on the intervals of Theorem 141.27,

$$\log F_i = \sum_{j,\tau,h} \nu_{ij,\tau,h} (\log \nu_{ij,\tau,h} - 1) + o_{\psi,\varepsilon,B}(b/x^B),$$

where a zero part contributes zero.

Proof. Within one group the scalar elliptic germs are common. The Cauchy–Schwarz estimate for their derivatives in the proof of Theorem 144.5 therefore applies with the summed coefficient polynomial. Giving each group its own probability yields the first assertion by the same operator inequality and monotonicity of the mixed determinant.

For a single path take the trace of the theorem's identity. Its square root is $\sqrt{F_i}$ times precisely this path's summand in $A_{i,\sigma}$. The jet bound on that path multiplies this square root by at most $(T^{-d_i^T} Q_{i,\sigma})^{1/2}$. Sum over the paths by the triangle inequality for the norm in V. Hadamard's inequality for their resulting column norms proves the displayed bound, including dependent lists.

For $a \ge 1$ and $k \ge 0$, $(a)_k = k!\binom{a+k-1}{a-1}$. The binomial factor lies between one and $(a+k)^{a-1}$. Comparison of the increasing function $\log t$ with its integrals over successive unit intervals gives $\log(k!) = k\log k - k + O(\log(2+k))$ for $k \ge 1$. Thus each log factor differs from the asserted main term by $O(a\log(2+a+k))$, including $k = 0$. The number of embeddings and edges and their ranks are bounded by powers of x. All parts satisfy $k \ll xb + x^C$ on the given words, by Lemma 141.11 and the partition ordering. Since $b = \exp(x^{\varepsilon})$, the total error is bounded by a fixed power of x, which gives the stated uniform error. □

The factorials in this estimate are explicit for the signed columns at the original tests. The endpoint maps, initial forms and class sums still affect their height. Section 6.6 also requires a lower bound for the rank of the evaluated periods and the uniform cohomology estimate on the same intervals.

145 Common divisors of the projected columns

145.1 A common divisor for the signed theta columns

Use the polynomial-conductor intervals of Theorem 141.27. Recall $b = \exp(x^{\varepsilon})$, $X' = \exp(x^{\varepsilon/3})$ and $w'' = b/(nX')$ from the construction of (9.1). At a fixed x and translate u, let w run through the chosen binary words. Write $\nu_{w,j,\tau,l}$ for part l of the partition on edge j at the embedding τ. The two ranks on that edge are $r_j \le s_j$, and $r_* = \max_j r_j$. Let $\mathcal{P}_x$ consist of the unramified primes $p \ge 5$ in E_x which are inert in the quadratic field K_d. Put, as in Lemma 141.26,

$$g_{w,p} = \sum_j V_p(\nu_{w,j}), \qquad b^{\mathrm{per}}_{u,w} = \prod_{p\in\mathcal{P}_x} p^{\,g_{w,p}-\lceil U_p\rceil}.$$

Here U_p depends on u and is zero in the nonsignal case specified in Corollary 68.3. The exponents are nonnegative. We determine the logarithm of their common divisor up to an error smaller than b/x^A for every fixed $A > 0$.

Lemma 145.1 (A uniform cutoff for all row gains)**.** *For all sufficiently distant intervals, throughout each interval and for all translates and words,*

$$g_{w,p} = 0 \qquad (p \in \mathcal{P}_x, \quad p > P_x = 12 r_* n x X').$$

For $p \le P_x$ and any two words at the same translate one has $|g_{w,p} - g_{w',p}| \ll_{\psi,\varepsilon} x^C$ with an exponent independent of p, u, w, w'.

Proof. The first-row formula in Lemma 141.11 is $\nu_{w,j,\tau,1} = u_1(\tau) + e_{j,\tau} - (r_j + s_j - 1)/2$. The least base value in (9.1) and its following weight formulas is at least w'', the rank shifts are bounded by a power of x, and $u_1(\tau) \ge 0$. The largest u_1 is at most $\lceil xb\rceil + O(k)$, and $e_{j,\tau} \le 2b$ eventually. Since w'' grows faster than any fixed power of x, these facts give

$$\min_{w,j,\tau} \nu_{w,j,\tau,1} \ge b/(2nX'), \qquad \max_{w,j,\tau,l} \nu_{w,j,\tau,l} \le 3xb.$$

For two words at this translate every part at the same position differs by at most Δ. Indeed the only varying input entries are the interior u_i, each changing by Δ when its bit changes. Subsequent steps add scalars independent of the word to the old entries and pad by entries independent of the word. This is the recurrence for the ordered weights in the proof of Lemma 141.11.

Fix a single edge of ranks r, s. For one of the allowed choices H of alternating halves of the even cycles of $\pi = \sigma_p c_E$, let μ lie in the opposite halves. For each positive odd a write

$h = \pi^{-a}\mu$. The list for μ in the definition of V_p has entries $D_{l,a} = d_{p^a}(\nu_{h,l}, s)$, $1 \le l \le r$. For integers $M \ge 2,\ s \ge 1,\ t \ge 0$,

$$0 \le d_M(t,s) = \max(0, \lceil (t - s(M-1))/M \rceil) \le \lfloor t/M \rfloor.$$

The upper bound follows by writing $t = Mq + l'$ with $0 \le l' \le M - 1 \le s(M-1)$. Thus the sum of the whole list is at most $3rxb\,p/(p^2-1) \le 6rxb/p$. Every nonempty choice of list entries in the maximum must pay at least $\nu_{\mu,1}$ from the opposite partition. For $p > P_x$ this already exceeds the sum of the list. All maxima are zero, proving the cutoff.

For the variation bound write each sorted-list maximum as a maximum over subsets I of $\sum_{(l,a)\in I} D_{l,a} - \sum_{l \le |I|} \nu_{\mu,l}$, where parts beyond r are zero. Use the same finite list of indices $p^a \le 3xb$ for both words, padding by zeros. The change of $D_{l,a}$ has absolute value at most $\lceil \Delta/p^a \rceil$, by the ceiling formula. The change of the subtracted sum is at most $r\Delta$. Consequently every subset value changes by at most

$$r\Delta + r \sum_{\substack{a \text{ odd} \\ p^a \le 3xb}} \lceil \Delta/p^a \rceil \ \le\ r(\Delta + 2\Delta/p + 1 + \log(3xb)/\log p).$$

Taking maxima preserves a uniform absolute difference bound. Sum over μ, take the maximum over H, and sum over the edges. The numbers of embeddings and edges, the ranks, Δ and $\log(3xb)$ are bounded by powers of x. □

There is also a formula without ceilings for the main term. For one cycle of even length e and a given opposite embedding μ, use the infinite list

$$t_{l,a} = \nu_{\pi^{-a}\mu,l}/p^a \quad (1 \le l \le r, \quad a \ge 1 \text{ odd}).$$

Write T_k for the sum of its k largest entries, $T_0 = 0$, and T_∞ for the sum of all its entries. Set $A_k = \sum_{l \le k} \nu_{\mu,l}$ for $0 \le k \le r$. Define $\overline{V}_p(\nu)$ by taking the same sum over opposite embeddings and maximum over H as for V_p, using at each μ the rational number

$$\max\left\{ \max_{0 \le k < r} (T_k - A_k),\ T_\infty - A_r \right\}.$$

For an empty collection of even cycles take both gains to be zero.

Proposition 145.2 (A finite evaluation of the main term)**.** *In this formula one has*

$$T_\infty = \frac{\displaystyle\sum_{\substack{1 \le a < e \\ a \text{ odd}}} p^{-a} |\nu_{\pi^{-a}\mu}|}{1 - p^{-e}}.$$

For $1 \le k < r$, T_k is obtained from the entries with $a < ke$. For the actual words, $\overline{V}_p(\nu) = 0$ for $p > P_x$, and, for all p,

$$0 \le \overline{V}_p(\nu) - V_p(\nu) \le fr\big(s(1 + \log(3xb)/\log p) + 2\big),$$

where $f = [E_x^+ : \mathbb{Q}]$.

Proof. Group the terms by a mod e. Each such residue and row gives a geometric progression of ratio p^{-e}, proving the sum formula. A positive entry beyond its first k terms has k larger predecessors in the same progression. Zero progressions can be ignored or padded by zeros. This proves the finite rule for T_k. Among subsets of size at least r, the penalty is always A_r,

and their sums increase to the sum of the entire list. This proves that the displayed maximum is the supremum over all finite subsets. The cutoff proof just given also bounds the sum of this larger list by $6rxb/p$.

Increasing a nonnegative summable list by a total amount a_0 increases the supremum by at most a_0. For $p^a \le 3xb$, each $t_{l,a}$ exceeds $D_{l,a}$ by at most s, by the ceiling formula. For $p^a > 3xb$ the old entries are zero. If $J = \lfloor \log(3xb)/\log p \rfloor$, the sum over $a > J$ for any one row is at most $3xb\,p^{-J-1}/(1-p^{-1}) < 2$. Adding these bounds for at most f lists of r rows proves the inequality for each orientation and hence for their maxima. □

Theorem 145.3 (One integer for every theta word at a translate)**.** *At each x, u put $B_u = \gcd_w b^{\mathrm{per}}_{u,w}$. There is a positive integer D_x, valid for every translate, such that*

$$B_u \mid b^{\mathrm{per}}_{u,w} \mid B_u D_x, \qquad \log D_x \ll_{\psi,\varepsilon} x^C X'.$$

For any reference word w_0 in the family the common divisor satisfies the uniform estimate

$$0 \;\le\; \sum_{\substack{p\in\mathcal{P}_x\\ p\le P_x}} \left(\sum_j \overline{V}_p(\nu_{w_0,j}) - U_p \right) \log p - \log B_u \;\ll_{\psi,\varepsilon} x^C X'.$$

On enlarging D_x within this bound, the explicit compact factorials of Corollary 144.24 also have a common factor F_u^- with $F_u^- \mid F_w \mid F_u^- D_x$.

Proof. For $p \in \mathcal{P}_x$, set the exponent of p in D_x equal to the maximum over translates of $\max_w g_{w,p} - \min_w g_{w,p}$. Give other primes exponent zero. By Lemma 145.1 each exponent is polynomially bounded in x and the support is contained in $p \le P_x$. The elementary bound $\sum_{p\le P_x} \log p \le P_x \log(2P_x)$ gives the required logarithmic estimate, with an enlarged C. Subtracting the same $\lceil U_p \rceil$ from every exponent at a translate proves the two divisibilities.

The exponent of p in B_u is $\min_w g_{w,p} - \lceil U_p \rceil$. The difference from the coefficient of $\log p$ in the displayed main term is the sum of three nonnegative quantities,

$$\sum_j (\overline{V}_p(\nu_{w_0,j}) - V_p(\nu_{w_0,j})), \qquad g_{w_0,p} - \min_w g_{w,p}, \qquad \lceil U_p \rceil - U_p.$$

The first is bounded by a power of x by the proposition, the second by the lemma, and the third by one. At primes above the cutoff both gains and U_p vanish, since $0 \le \lceil U_p \rceil \le g_{w,p}$ there. Summation proves the absolute estimate.

For the compact factor take componentwise minima and maxima of the parts over words and form F_u^-, F_u^+ from the same rising factorial products. Each F_w/F_u^- divides F_u^+/F_u^-. At any position this last ratio introduces at most Δ positive integer factors, all at most $r_* + 3xb$. There are polynomially many positions and translates, by Proposition 6.4 and the rank bounds in Corollary 143.9. Multiplying the previous D_x by these ratios for all translates increases its logarithm by at most a polynomial in x. □

Corollary 145.4 (The common division on each projected minor)**.** *Let Y° have columns $\widetilde{Y}_i/b^{\mathrm{per}}_{u,w_i}$ and let Y^B have columns $\widetilde{Y}_i/B_u$, with the same integral period evaluations and the same projection. In any one system of test coordinates at any finite place, write I_h for the fractional ideal generated by the size-h minors. For every $h \ge 1$,*

$$D_x^h I_h(Y^\circ) \;\subseteq\; I_h(Y^B) \;\subseteq\; I_h(Y^\circ).$$

For every independent list of q_c columns in the original positive test metric and at every archimedean embedding,

$$0 \le \frac{1}{q_c} \log \frac{\det G^B_\sigma}{\det G^\circ_\sigma} \le 2 \log D_x = o_{\psi,\varepsilon,A}(b/x^A) \qquad (A > 0 \text{ fixed}).$$

Proof. The change multiplies column i by the positive integer $b^{\mathrm{per}}_{u,w_i}/B_u$ dividing D_x. Each minor is therefore multiplied by a divisor of D_x^h, proving both ideal inclusions even when the ideal is zero. The Gram determinant is multiplied by the square of the product of the column factors. Take logarithms and use $x^{A+C}X'/b \to 0$. The row-gain construction and the common integer use the fields and weights of these very columns. □

This supplies a division compatible with taking linear combinations across words. The finite expression in the theorem computes its size to the precision stated. The comparison of the remaining signed period height with the prime count, its required rank, and the exceptional strict lengths are still needed in Section 6.6.

145.2 Large primes in the common period divisor

Continue with the common integer B_u of Theorem 145.3. We evaluate the contribution of the primes above a fixed power of x using the Frobenius group of the same period field. Put $G = \mathrm{Gal}(E_x/\mathbb{Q})$, $d_G = |G|$, and let S_x contain the ramified rational primes and $2, 3$, with no other primes. Write N_x for the conductor of the field. On these intervals N_x, d_G are bounded by powers of x, by Theorem 9.26 and the field construction. For $g \in G$ put

$$A_g(t) = \sum_{\substack{p \le t,\ p \notin S_x \\ \mathrm{Frob}_p = g}} \frac{\log p}{p}, \qquad y_0 = \exp((\log(2x))^4).$$

Lemma 145.5 (Reciprocal prime sums in the period field)**.** *Uniformly for $t \ge y_0$ and $g \in G$,*

$$A_g(t) = \frac{\log t}{d_G} + C_g + O_{\psi,\varepsilon}(e^{-c\sqrt{\log t}})$$

with some $c > 0$ independent of x, u, g. If χ^* induces the character χ of G, define $D_{\mathbf{1}} = -\gamma$ and $D_\chi = -L'(1,\chi^*)/L(1,\chi^*)$ for $\chi \ne \mathbf{1}$ in this lemma. The constant is

$$C_g = \frac{1}{d_G} \sum_{\chi \in \widehat{G}} \overline{\chi(g)} \left(D_\chi - \sum_{p \in S_x} \frac{\chi^*(p)\log p}{p} - \sum_{p} \frac{\chi^*(p)^2 \log p}{p(p - \chi^*(p))} \right).$$

Proof. The real primitive characters with an auxiliary factor have no exceptional zero near one by Theorem 5.6. The other real primitive characters have conductor dividing the fixed $8F$. Thus the zero-free bound, the zero count and the explicit formula in the proof of Lemma 7.1 apply uniformly on these fields. To check the lower endpoint here, truncate the zero sum in (3.1) at $\exp(\sqrt{\log t})$. The low-zero contribution after division by t is at most

$$O_{\psi,\varepsilon}\left(\log^3(2N_x) \exp\left(-c_1 \min\left\{ \sqrt{\log t},\ \frac{\log t}{\log(2N_x)} \right\} \right) \right).$$

The tail is $O((\log(2N_x) + \sqrt{\log t})e^{-\sqrt{\log t}})$ by the inverse-square ordinate bound of that lemma. Its elementary terms after division by t are $O_{\psi,\varepsilon}((1 + \log t + \log^2(2N_x))/t)$, including the error from the shifted line. Since $\log(2N_x) \ll \log(2x)$ and $\log t \ge (\log(2x))^4$, these bounds give $|G_{\chi^*}(t)| \ll te^{-c_2\sqrt{\log t}}$ for some $c_2 > 0$.

For $h = te^{-c_2\sqrt{\log t}/3}$, take the forward difference quotient of R_{χ^*} from (1.1). It differs from $\sum_{n \le t} \Lambda(n)\chi^*(n)/n$ by at most $(h+1)\log(2t)/t$, by bounding the additional terms with $t < n \le t + h$. The two G_{χ^*}-values in the quotient contribute $O(e^{-c_3\sqrt{\log t}})$ for a smaller $c_3 > 0$. The difference quotient of the main term at the trivial character is $\log t - \gamma + O(h/t)$, and for the nontrivial characters it is exactly D_χ. To pass to unramified primes, remove all powers of

exponent at least two and the first powers in S_x. Every prime in S_x is already below y_0 for large x. The sum of all higher powers of one prime gives the last summand of the formula for C_g. The part with $p^a > t$, $a \ge 2$, in absolute value, is $O(\log(2t)/\sqrt{t})$. For $p \le \sqrt{t}$ the geometric tail is at most $2\log p/t$ per prime. For larger p sum $2\log p/p^2$ by comparison with the integral over all integers. Finally character orthogonality selects $\mathrm{Frob}_p = g$. Averaging over d_G characters preserves the absolute uniform error. □

Choose any one reference word at the fixed translate. Set $S_{j,\tau,k} = \sum_{l=1}^{k} \nu_{w_0,j,\tau,l}$, $S_{j,\tau,0} = 0$, and $S_{j,\tau} = S_{j,\tau,r_j}$. For g inert in K_d write $\pi_g = gc_E$ acting on the embedding labels as in Theorem 68.2. For each even cycle C of this permutation, let C_0, C_1 be its two alternating halves. Put

$$f_{j,g,\mu}(z) = \max_{0\le k\le r_j} \{S_{j,\pi_g^{-1}\mu,k} z - S_{j,\mu,k}\},$$

$$H_g(z) = \sum_{j,C} \max_{b'=0,1} \sum_{\mu\in C_{b'}} f_{j,g,\mu}(z) - \mathbf{1}_{g|K_q=1} \sum_{j,t\in T_{\mathrm{hi}}} [az - e_{j,\pi_g t}]_+.$$

Here $a = \lceil xb \rceil$, and j runs through all edges in the last sum as well. Set $H_g = 0$ when g splits in K_d. Let L_u denote the distinguished small-place set of (9.1), and let O_u consist of the embedding labels outside both T_{hi} and L_u. For g inert in K_d put

$$J_g = \sum_{j,C} \max_{b'=0,1} \sum_{\mu\in C_{b'}\cap O_u} S_{j,\pi_g^{-1}\mu} - \mathbf{1}_{g|K_q=1}\, a \sum_j \#\{t \in T_{\mathrm{hi}} : \pi_g t \notin L_u\}.$$

Give J_g value zero for the other g. All these weights use the original selected field, high set, distinguished set and translate.

Proposition 145.6 (The three prime ranges of the divisor)**.** *There is an exponent C, depending only on the fixed parameters, for which $|J_g| \ll bx^C$. For $z \ge 0$, H_g is continuous and piecewise affine with $H_g(0) = 0$ and Lipschitz constant at most $O(bx^C)$. For any sufficiently large fixed M, uniformly for all g and $x^M < t \le y_0$,*

$$H_g(1/t) = J_g/t + O_{\psi,\varepsilon}(bx^C/X').$$

At primes $p > x^M$ in $\mathcal{P}_x$, the difference between $\sum_j \overline{V}_p(\nu_{w_0,j}) - U_p$ and $H_{\mathrm{Frob}_p}(1/p)$ is $O(bx^C/p^3)$.

Proof. Keep only $a' = 1$ in the lists $t_{l,a'}$ of Proposition 145.2. The largest k entries of the resulting list come from the first k parts at $\pi_g^{-1}\mu$. This gives $f_{j,g,\mu}(1/p)$, including the choice $k = r_j$. The omitted entries have total at most $3r_j xb\, p^{-3}/(1-p^{-2})$ per opposite embedding. The same bound with $ap^{-3}/(1-p^{-2})$ applies to the omitted higher odd powers of each pair in U_p, because all $e_{j,\tau}$ are positive. The halves can be chosen independently on different cycles. The supremum over subsets and maximum over halves change by no more than the uniform error. All multiplicities are polynomial in x, giving the assertion at primes.

The constants in the affine functions have nonpositive sign before taking their maxima, and the choice $k = 0$ is available. Each maximum is Lipschitz with constant bounded by its largest slope. All slopes, summed over their indices, are $O(bx^C)$ by Lemma 145.1. The same bound applies to the subtracted ramps, proving the assertions about H_g and J_g.

At a high place the first part is at least $b/2$ for large x, since u_1 contains $\lceil xb \rceil$. At a place in L_u this lower bound follows from $B_\tau \ge \lceil b \rceil$ in the formula for $e_{j,\tau}$. At the remaining places the full partition size is $O(bx^C/X')$, since $B_\tau \le \lceil b/(nX') \rceil + 1$ there and the other contributions

to the first part are polynomial. These statements use the rank shifts in Lemma 141.11. For $t > x^M$ with M sufficiently large, each incoming partition sum divided by t is less than $b/2$. Thus at the first two kinds of destination every f is zero. At $\mu \in O_u$, choosing $k = r_j$ and dropping the penalty give lower and upper bounds within $S_{j,\mu}$ of $S_{j,\pi_g^{-1}\mu}/t$.

In the subtracted ramps the weights $e_{j,\tau}$ on L_u are at least b. At the remaining places, including the high places, $e_{j,\tau} = O(bx^C/X')$ by the explicit base values. Therefore the ramp vanishes when its destination is in L_u; at the other destinations it differs from a/t by $O(bx^C/X')$. Taking the orientations and the sums proves the middle-range formula, with the polynomial factors absorbed in the exponent C. □

Theorem 145.7 (The absolute size from polynomially many primes and field constants)**.** *For every fixed $A > 0$ one can take a fixed $M = M(\psi, \varepsilon, A)$ and $z_0 = x^M$ such that, uniformly in the translate and the reference word,*

$$\begin{aligned}\log B_u = {} & \sum_{\substack{p \in \mathcal{P}_x \\ p \le z_0}} (g_{w_0,p} - \lceil U_p \rceil) \log p \\ & + \sum_{g \in G} J_g \left(\frac{\log y_0}{d_G} + C_g - A_g(z_0) \right) \\ & + \frac{1}{d_G} \sum_{g \in G} \int_{y_0}^{P_x} H_g(1/t)\, dt + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

The integrals use only finitely many affine pieces and ordinary logarithms. At the same precision the prime series in all the C_g may be truncated at $p \le z_0$, retaining the logarithmic derivatives at one in Lemma 145.5.

Proof. Eventually $z_0 < y_0 < P_x$. By Theorem 145.3 the change from B_u to the divisor of the reference word costs $O(x^C X')$ in the logarithm. Keep its exact exponents at $p \le z_0$. At the other primes the replacement by $\sum_j \overline{V}_p - U_p$ costs $O(x^C X')$ by the nonnegative error estimates in the same theorem. On $z_0 < p \le P_x$ the first-power approximation in the proposition costs in total

$$O\left(bx^C \sum_{n > z_0} \frac{\log n}{n^3} \right) = O\Big(bx^C (1 + \log z_0)/z_0^2 \Big).$$

Take M large enough that this is $o(b/x^A)$.

Up to y_0, replacement by J_g/p costs at most $O(bx^C y_0 \log(2y_0)/X') = o(b/x^A)$. This follows from $\log X' = x^{\varepsilon/3}$, which dominates every fixed power of $\log x$. The remaining middle sum is exactly $\sum_g J_g(A_g(y_0) - A_g(z_0))$. Substitute the lemma at y_0. Its error, even after summation over G, is at most $bx^{C'} e^{-c(\log(2x))^2} = o(b/x^A)$.

On $[y_0, P_x]$ put $W_g(t) = tH_g(1/t)$. The Lipschitz estimate and $H_g(0) = 0$ give $|W_g| \ll bx^C$ and $|W_g'(t)| \ll bx^C/t$ on each smooth piece. The sum over primes in this last range is the Stieltjes integral of W_g against A_g on $(y_0, P_x]$, summed over g. Partial summation with Lemma 145.5 replaces this by the integral in the statement. The constants C_g cancel between endpoints and the integral of the derivative. The error is bounded by

$$O\Big(d_G bx^C (1 + \log P_x) e^{-c(\log(2x))^2} \Big).$$

The pieces are continuous at every junction, so partial summation adds no interior boundary errors. On each piece of $H_g(1/t)$ the integrand is $a_1 + a_2/t$. For each μ the candidates are $r_j + 1$ lines in $1/t$. Their pairwise crossings give a finite subdivision. On each resulting interval the two half-cycle sums cross at most once unless they coincide. Taking the union of these

subdivisions uses a polynomial number of breakpoints in the numbers of edges, embeddings and ranks.

Finally the absolute tail of the prime series in each C_g after z_0 is $O((1+\log z_0)/z_0)$, by comparison with the sum of $\log n/n^2$. All primes of S_x are below z_0 upon enlarging M. Another enlargement absorbs this last error after multiplication by $\sum_g |J_g|$. This gives the claimed accuracy for the original common divisor on every field of the interval. □

The arithmetic information for this divisor now occurs in the primes of size at most x^M, the field constants at one and the explicit integrals. Its value alone gives neither the rank of the signed periods nor the upper bound for their remaining height in Section 6.6.

145.3 The terminal restriction and the two factorial norms

At one translate use the incoming analytic row and the definite word w of (9.1). Both use the same field, embeddings and m edges. Two metric normalizations enter at the last definite edge. We first identify their conversion in the coefficient matrix of Theorem 144.23.

Lemma 145.8 (The endpoint in highest-vector normalization)**.** *Let V_v be a compact flag module of dimension d_v. Use the Hermitian Fischer form in which its highest polynomial p_v has norm one. The squared Haar frame norm of Section 144.8 is $1/d_v$ times this squared Fischer norm. This holds also after a determinant twist and for products over the CM embeddings.*

In the integral frames let $L : V_l \to V_m$ be the actual terminal restriction with its normal-coordinate scalar removed. Here the indices l, m denote the two weights. Write $d_l = \dim V_l$. For endpoint tests $R_\gamma, R_\delta : V_m \to V$ in the same frames and a fixed metric on V, one has

$$\frac{(R_\gamma L)(R_\delta L)^{*,\mathrm{Haar}}}{d_l} = s_w\, R_\gamma R_\delta^{*,\mathrm{F}}, \qquad s_w = \prod_\tau c_\tau^2 \kappa_\tau.$$

The adjoints on the right use the normalized Fischer metric on V_m. The identity holds if both composites on the left are precomposed with the same unitary frame change.

Proof. Evaluation at the identity is the inner product with p_v by Lemma 120.4. The coefficients of p_v are real, so the identity gives the Hermitian version with the form linear in its first entry. Unitary row translation preserves the inner product on the linear variables and hence the Fischer metric on their symmetric tensors. Thus evaluation on unitary frames pairs the vector with the orbit of p_v. The Haar average of the rank-one orthogonal projections onto these unit translates commutes with the unitary action. The flag space is irreducible, as in Lemma 144.22. The average has trace one, so it equals I/d_v. Pairing with an arbitrary vector gives the first assertion. A determinant has absolute value one on the unitary group. Fubini and polarization give the product assertion, with d_v the product dimension.

For a map from V_l to V, changing the domain metric from Fischer to Haar multiplies its adjoint by d_l. Hence the quotient in the display equals $R_\gamma L L^{*,\mathrm{F}} R_\delta^{*,\mathrm{F}}$, with Fischer metrics on both coefficient spaces. Theorem 83.28 identifies $LL^{*,\mathrm{F}}$ with $s_w I$. A common unitary frame change cancels against its adjoint. □

For the endpoint factor in Corollary 144.24, taking the trace gives

$$\frac{\|R_\gamma L\|_{\mathrm{HS,Haar}}}{\sqrt{d_l}} = \sqrt{s_w}\, \|R_\gamma\|_{\mathrm{HS,F}}.$$

The tests R_γ can contain the contraction at the small class, its finite projection and the path scalar from Theorem 73.1. A change of terminal frame can be kept in the polynomial path

or included in this calculation when it is common to the two factors. In the unnormalized column $L_z = Z_w^{\mathrm{norm}} L$, so the squared endpoint has the additional factor $|\sigma Z_w^{\mathrm{norm}}|^2$. In $\widetilde{Y}_w = (z_w Z_w^{\mathrm{norm}})^{-1} Y_w$ it cancels. The factors z_w^{-1} and $1/b_{u,w}^{\mathrm{per}}$ still enter the divided column once each. By Corollary 104.33 the absolute logarithm of s_w, and the sum of its absolute finite log valuations, are $o(b/x^A)$ for every fixed $A > 0$, even with suprema over the words and translates.

Let $D(t) = t \log t - t$ for $t > 0$, with $D(0) = 0$. Let F_w be the product of rising factorials in Corollary 144.24. Define $\mathcal{H}_{\mathrm{in}}$ to be the product, over all real places and edges of the incoming row, of $H(\alpha)H(\beta)H(\gamma)$ in Proposition 104.20. Thus each incoming hook product is taken in Gaussian coordinates of dilation $1/\pi$. Write $h = |T_{\mathrm{hi}}|$, $h_2 = |L_u|$, $a = \lceil xb \rceil$ and $Y = X' = \exp(x^{\varepsilon/3})$.

Theorem 145.9 (The factorials on the given row and word)**.** *Set $N_w = \sum_{j,\tau,l} \nu_{w,j,\tau,l}$, the total definite degree. On the intervals of Theorem 141.27, uniformly in u, w, there is a fixed C such that*

$$N_w = m(ha + kh_2 b) + O_{\psi,\varepsilon}(bx^C/Y + x^C).$$

For every fixed $A > 0$ one has

$$\begin{aligned} \log F_w &= m\{hD(a) + kh_2 D(b)\} + o_{\psi,\varepsilon,A}(b/x^A), \\ \log \mathcal{H}_{\mathrm{in}} &= m(k-1)h_2 D(b) + o_{\psi,\varepsilon,A}(b/x^A), \\ \log \frac{F_w}{(b_{u,w}^{\mathrm{per}})^2 \mathcal{H}_{\mathrm{in}}} &= m\{hD(a) + h_2 D(b)\} - 2\log B_u + o_{\psi,\varepsilon,A}(b/x^A). \end{aligned}$$

Here B_u is the common integer in Theorem 145.3.

Proof. At the first edge the old highest weights are u_l minus the corresponding half-sum entries, of absolute difference $O(k)$ from u_l. The shift at edge j adds $e_{j,\tau} - s_j/2$ to form its partition. For the next input one pads by zeros and subtracts $e_{j,\tau} - r_j/2$. Each old entry consequently decreases by $(s_j - r_j)/2$. A newly introduced entry begins at $-e_{j,\tau} + r_j/2$. This is the recurrence of Lemma 141.11. All rank shifts and jH_0 are polynomial in x, and $e_{j,\tau} = B_\tau + (2j-1)H_0$. It follows that at every edge the original k parts are $u_l(\tau) + B_\tau + O(x^C)$. Every other part is $O(x^C)$, because the two occurrences of B_τ cancel.

At a high place the top part equals $a + O(bx^C/Y + x^C)$. The remaining parts have size $O(bx^C/Y + x^C)$, including the part with initial entry $-\lceil xb/Y \rceil - (k-1)/2$. At a distinguished place the original k parts all equal $b + O(x^C)$. Outside these two sets all parts are $O(bx^C/Y + x^C)$. These bounds follow from the three explicit base values after (9.1) and its interior bit entries. Summing the bounds proves the formula for N_w after absorbing the polynomial number of indices into C.

Apply the last assertion of Corollary 144.24. Replacing a part listed above by a, b, or zero as appropriate changes its D-value by $O(bx^{C'}/Y + x^{C'})$. Indeed the integral of $|\log t|$ on $(0,1)$ is one, and on $[1, 3xb+2]$ the derivative is bounded by $\log(3xb+2)$. The number of terms is polynomial and Y grows faster than every fixed power of x. This proves the first logarithmic estimate.

For the incoming hook products the proof of Proposition 104.20 gives exactly $k-1$ parts $B_\tau + O(x^C)$ per edge and place. The other parts are polynomially bounded. Its hook formula contributes polynomial total error to $m(k-1)\sum_\tau D(B_\tau)$. At the distinguished places $B_\tau = b + O(1)$. At all other places $B_\tau = O(bx^C/Y + x^C)$, whose contribution falls within the stated error. Subtract the resulting two estimates and use the uniform bound $|\log b_{u,w}^{\mathrm{per}} - \log B_u| = o(b/x^A)$ in Theorem 145.3. This proves the last formula. □

Corollary 145.10 (An upper bound for the integral divisor in degree N_w)**.** *For all words and translates on these intervals,*

$$0 \le \log b_{u,w}^{\mathrm{per}} \le N_w(\log P_x + O(1)), \qquad \frac{2\log b_{u,w}^{\mathrm{per}}}{\log F_w} = O_{\psi,\varepsilon}(x^{-2\varepsilon/3}),$$

where $P_x = 12r_*nxY$*.*

Proof. In the subset formula for V_p drop all penalties and keep all entries. At odd exponent j' each source place occurs at most once, and each entry is at most $\nu_{w,j,\tau,l}/p^{j'}$ by Lemma 145.1. Summing gives $g_{w,p} \le N_w p/(p^2-1)$. The signal subtraction is nonnegative, and the cutoff is P_x. For $y \ge 2$, the bound $\sum_{p\le y} \log p/p \le \log y + O(1)$ follows from $\sum_{p\le n} \lfloor n/p \rfloor \log p \le \log(n!)$ and $\sum_{p\le n} \log p = O(n)$, with $n = \lfloor y \rfloor$. The difference $p/(p^2-1) - 1/p$ contributes a convergent prime sum after multiplication by $\log p$. This gives the first estimate.

Let N be the polynomially bounded number of part positions. Since one first part is at least $b/(2nY)$ by Lemma 145.1, we have $\log(N_w/N) \ge \log b - \log Y - O(\log x)$. Each $(r-l+1)_v$ is at least $v!$, whose logarithm is at least $D(v)$ by integral comparison (also for $v = 0$). Convexity now gives $\log F_w \ge N_w(\log(N_w/N) - 1)$. Finally $\log b = x^{\varepsilon}$ and $\log P_x = x^{\varepsilon/3} + O(\log x)$. The second estimate follows. □

The comparison above evaluates the polynomial factors at both sets of prescribed highest weights. A relation between the full period norms must also account for the class sums and for the normalizations accompanying these polynomial factors.

145.4 The leading term of the integral period divisor

On the same fields and intervals, put $Y = X'$, let $o = \mathrm{ord}(\psi)$, and put

$$M_x = mkh_2 b, \qquad c_\psi = \frac{1}{2} - \frac{1}{8 \cdot 2^{v_2(o)}}.$$

We compute the leading term of $\log B_u$ by counting the destinations of the distinguished places under Frobenius. This also gives the sign and size of the comparison with the incoming hook products in Theorem 145.9.

Use $G = \mathrm{Gal}(E_x/\mathbb{Q})$. For g inert in K_d write $\pi_g = gc_E$ acting on the CM type $\Phi = \Phi_0 \sqcup \Phi_1$, where $T_{\mathrm{hi}} \subset \Phi_0$ and $L_u \subset \Phi_1$. Write $O_u = \Phi \setminus (T_{\mathrm{hi}} \cup L_u)$. For each even cycle C of π_g let C_0, C_1 be its alternating halves. Define the integer

$$e_u(g) = \sum_{C \text{ even}} \max_{i=0,1} \#\{t \in L_u \cap C_i : \ \pi_g t \in O_u\}.$$

Set $e_u(g) = 0$ when g splits in K_d. In particular $0 \le e_u(g) \le h_2$.

Lemma 145.11 (The prime sum at the scale of the distinguished places)**.** *Uniformly in u and the reference word on these intervals,*

$$\frac{\log B_u}{M_x \log Y} = \frac{1}{|G|h_2} \sum_{g\in G} e_u(g) + o_{\psi,\varepsilon}(1).$$

Proof. Write $t_0 = \exp((\log(2x))^4)$ and $t_1 = Y\exp(-(\log(2x))^2)$. Eventually $t_0 < t_1 < P_x$. By Theorem 6.29, $h \le n$, $h_2 \ge nx^{-\varepsilon}$, and $k \ge x^{9/2}$. Thus $mha/M_x = O(x^{1+\varepsilon}/k) = o(1)$. The degree formula of Theorem 145.9 gives $N_w = (1+o(1))M_x$. By Theorem 145.3, we may replace B_u by $b_{u,w}^{\mathrm{per}}$, with logarithmic error $o(b/x)$.

At each contributing prime the exponent is between zero and $N_w p/(p^2-1)$ by the proof of Corollary 145.10. At $p \le t_0$ the contribution is therefore $O(M_x(1+\log t_0))$. At $t_1 < p \le P_x$ it is $O(M_x(1+\log(P_x/t_1)))$. Indeed partial summation of the Chebyshev bound $\sum_{p\le t}\log p = O(t)$ gives $\sum_{y<p\le z}\log p/p \ll 1+\log(z/y)$. Both bounds are $o(M_x \log Y)$.

It remains to evaluate the middle range. Denote the first-power weight of Proposition 145.6 by H_g, and the full size of a partition on edge j at μ by $S_{j,\mu}$. For $t_0 < p \le t_1$, all incoming sizes divided by p are smaller than $b/2$, since all sizes are $O(bx^C)$ for a fixed C. At any destination in $T_{\mathrm{hi}} \cup L_u$ the first part is at least $b/2$ eventually by the weight calculation of Theorem 145.9. Its first-power maximum is therefore zero. At a destination in O_u, taking the full list or dropping the penalty bounds its maximum between $S_{j,\pi_g^{-1}\mu}/p - S_{j,\mu}$ and $S_{j,\pi_g^{-1}\mu}/p$.

Each source in L_u has total size $kb + O(x^C)$ at every edge, by the same weight calculation. At a high source the total is $a + O(bx^C/Y + x^C)$. At every ordinary source or destination it is $O(bx^C/Y + x^C)$. For a fixed choice of halves we may therefore keep just the distinguished sources with ordinary destinations, giving each weight kb/p. The total error from all the high sources is at most $mha/p + O(bx^{C'}/Y + x^{C'})$, including the polynomial number of positions. The other substitution errors and the dropped destination penalties are $O(bx^{C'}/Y + x^{C'})$. The subtracted first-power signal is at most mha/p. These bounds hold for any choice of halves. Maximization on each cycle at each edge now selects exactly $e_u(g)$ such sources per edge, since π_g switches the two halves. We obtain

$$H_g(1/p) = \frac{mkb}{p}\, e_u(g) + O\left(\frac{mha}{p} + \frac{bx^{C'}}{Y} + x^{C'}\right) = \frac{mkb}{p}\, e_u(g) + o(M_x/p).$$

The last error is uniform, since $p/Y \le \exp(-(\log(2x))^2)$.

By Proposition 145.6 the higher odd powers add $O(bx^C/p^3) = o(M_x/p)$ uniformly in this range. Replacing the integer row gains and the signal ceiling by their continuous versions costs $O(x^C Y)$ in the total prime sum by Theorem 145.3: the two rounding errors there are separately nonnegative before taking their difference, so the bound also holds on this subinterval. Consequently the middle sum differs from $mkb\sum_{t_0<p\le t_1} e_u(\mathrm{Frob}_p)\log p/p$ by $o(M_x \log Y)$. Here only the unramified primes outside S_x are used, as in Lemma 145.5.

Apply that lemma to $A_g(t_1) - A_g(t_0)$ for each g. The constant cancels. After multiplication by $e_u(g)/h_2$ and summation, the error is $O(|G|e^{-c(\log(2x))^2}) = o(1)$, since $|G|$ is polynomial in x. Finally $\log(t_1/t_0)/\log Y \to 1$, proving the assertion. □

Theorem 145.12 (The leading constant for the common integral divisor)**.** *On the intervals of Theorem 141.27 one has*

$$\log B_u = (c_\psi + o_{\psi,\varepsilon}(1))M_x \log X', \qquad \log b^{\mathrm{per}}_{u,w} = (c_\psi + o_{\psi,\varepsilon}(1))M_x \log X',$$

uniformly for every translate and every word. In particular $c_\psi = 3/8$ for every character of odd order.

Proof. Write $A = \mathrm{Gal}(K_\psi C_{j_x}/\mathbb{Q})$ and $A_0 = \mathrm{Gal}(K_\psi C_{j_x}/K_\psi)$. The two quadratic coordinates are independent of A and of each other by the ramification at q, d_0 in Theorem 9.26. Each sign pair occurs with weight $1/4$, and at fixed signs the translation in A is uniform. Indeed multiplying by c_E adds a fixed element to these coordinates, including a possible component in A. Identify each of Φ_0, Φ_1 with A. Then $|A| = n$, A_0 is cyclic of two-power order, and A/A_0 is cyclic of order o.

First take g inert in K_d and split in K_q. Then π_g switches Φ_0, Φ_1. Every cycle is even and its two halves are its intersections with those sets. All distinguished sources can be selected,

except those sent into T_{hi}. Each such source is sent uniformly through Φ_0 by the group average. The conditional average of $e_u(g)/h_2$ is therefore $1 - h/n = 1 - o(1)$, by Theorem 6.29.

Now take g inert in both quadratic fields. Then π_g preserves each of Φ_0, Φ_1 and translates Φ_1 by a uniform $v \in A$. Only its action there matters. Temporarily count all distinguished sources in the better half of each even cycle, including those sent to L_u. Call the resulting count $e^0(v)$. Removing those destinations changes each maximum by at most the number removed, so

$$0 \le e^0(v) - e_u(g) \le \#\{t \in L_u : t + v \in L_u\}.$$

The average of the right side over v is h_2^2/n, by counting all ordered pairs of distinguished places. From the construction in Theorem 6.29, the original distinguished set has integer representatives in $[0, |A_0|/(8L))$, with $L \to \infty$. Hence $h_2/n = O(1/L + 1/n) = o(1)$, also after translation.

Translate within Φ_1 to view L_u in A_0. That same construction makes it exactly balanced in the D residue classes modulo DA_0, with D a power of two tending to infinity. This means that every such class contains the same number of its elements. Fix the coset of v in A/A_0, and let r be its order there. Visits of a v-cycle to A_0 are spaced by r steps. If r is even, every such cycle is even and all its visits to L_u have the same parity. Thus $e^0(v) = h_2$.

For odd r, multiplication by r permutes A_0, so rv is uniform there. Outside a fraction $1/D$, its subgroup is $2^j A_0$ with $2^{j+1} \mid D$. A v-cycle meeting A_0 then meets it in one coset of $2^j A_0$ and is even. Successive visits differ by its generator rv and are r steps apart. Their parities therefore distinguish the two classes modulo $2^{j+1}A_0$. Balance modulo DA_0 gives equal counts in these classes, so $e^0(v) = h_2/2$. On the exceptional fraction use $0 \le e^0(v) \le h_2$.

The fraction of elements of odd order in the cyclic quotient of order o is $2^{-v_2(o)}$. Averaging the last two cases and restoring the excluded destinations shows that the conditional mean of $e_u(g)/h_2$ when both signs are inert is

$$1 - \frac{1}{2 \cdot 2^{v_2(o)}} + O(1/D + h_2/n).$$

The other two sign pairs contribute zero. The full mean is $c_\psi + o(1)$ uniformly in u. Apply Lemma 145.11 and Theorem 145.3. □

Corollary 145.13 (The integral divisor in the two norm factors)**.** *For the original row and every prescribed word,*

$$\log \frac{F_w}{(b^{\mathrm{per}}_{u,w})^2 \mathcal{H}_{\mathrm{in}}} = -(2c_\psi + o_{\psi,\varepsilon}(1))\, M_x \log X'.$$

For an embedding σ write $A^0_{i,\sigma}$ for the sum of path norm products in Corollary 144.24 on $\widetilde{Y}_i$, before division by b^{per}_{u,w_i}. For q_c words the actual Gram determinant of the divided columns satisfies

$$\begin{aligned} \det G^\circ_\sigma \;&\le\; \exp\bigl(-(2c_\psi + o_{\psi,\varepsilon}(1))\, q_c M_x \log X'\bigr) \\ &\quad \times \prod_{i=1}^{q_c} T^{-d_i^T} Q_{i,\sigma}\, \mathcal{H}_{\mathrm{in}}(A^0_{i,\sigma})^2. \end{aligned}$$

The bound includes lists of deficient rank.

Proof. Since $\log a \ll_\varepsilon x^\varepsilon$, the ratio of $m(hD(a) + h_2 D(b))$ to $M_x \log X'$ is $O(x^{2\varepsilon/3}(xh/h_2 + 1)/k) = o(1)$ by the size inequalities at the start of Lemma 145.11. Theorems 145.9 and 145.12 now give the first formula. Each path norm scales by the column divisor, so $A_{i,\sigma} = A^0_{i,\sigma}/b^{\mathrm{per}}_{u,w_i}$. Substitute into Corollary 144.24 and sum the uniform logarithmic error over the q_c factors. □

Section 6.6 still requires control of the hook, elliptic and class factors together, sufficient rank at the prescribed tests, and the strict local length bound.

145.5 The size of the divisor from five Frobenius powers

At a node and a prime $p \in \mathcal{I}_x$, decompose $\pi = \sigma_p c_E$ into cycles C. Each cycle has even length. Put $T_C = C \cap T_{\mathrm{hi}}$, $L_C = C \cap L_u$. For either all positive odd j, or the truncated set $J_9 = \{1,3,5,7,9\}$, write J for the set of exponents and define

$$F_C(J) = \sum_{t \in T_C} \left(\sum_{\substack{j \in J \\ \pi^j t \notin L_C}} \frac{a}{p^j} + \mathbf{1}_{\pi t \in L_C} (a/p - b)_+ \right),$$

$$R_C(J) = \sum_{\mu \in C \cap \Phi_0} \left(kb \sum_{\substack{j \in J \\ \pi^{-j}\mu \in L_C}} p^{-j} - a \mathbf{1}_{\mu \in T_C} \right)_+ .$$

We suppress J when all positive odd powers are included. The first number uses the sources on Φ_0, the second the sources on Φ_1. Both are explicit in the Frobenius permutation and the two prescribed sets of embeddings.

Theorem 145.14 (The maximizing orientation in the middle prime range)**.** *For all sufficiently large nodes and every $p \in \mathcal{I}_x$, every actual maximizing orientation for $V_p(\nu_i)$ uses Φ_1 as source on each cycle meeting L_u. For every fixed $A > 0$, uniformly in p, u, w, the divisor of the word in Lemma 141.26 satisfies*

$$v_p(b^{\mathrm{per}}_{u,w}) = m \sum_{C \colon L_C \neq \varnothing} (R_C - F_C) + o_{\psi,\varepsilon,A}(b/x^A).$$

Proof. On one edge replace the partition at $t \in T_{\mathrm{hi}}$ by $(a, 0, \ldots)$, the partition at $\tau \in L_u$ by the rectangle with k parts b, and all other partitions by zero. These model rows are allowed to have real entries. The estimates in the proof of Theorem 145.9 give a total absolute row error $O_{\psi,\varepsilon}(bx^{C_0}/X' + x^{C_0})$, also when summed over all the edges and embeddings. Here C_0 is a fixed constant.

For either orientation first pass to the list with entries $\nu_{\pi^{-j}\mu,l}/p^j$. Proposition 145.2 bounds the error by a polynomial on each edge, also on each individual cycle with its orientation fixed, as its proof compares the unsorted lists entry by entry. The contribution at a destination is the supremum over finite subsets of the sum of their entries less the sum of that many leading destination parts, padded with zeros. Perturbing the incoming lists changes this supremum by at most their total absolute change. The cost changes by at most the sum of absolute changes to the destination parts. Each source row appears with total incoming weight at most $\sum_{j \text{ odd}} p^{-j} \le 1$. These facts bound the total change due to the model rows by twice their total absolute error. The resulting errors, including the polynomial loss, are $o_{\psi,\varepsilon,A}(b/x^A)$ for any fixed A.

In the orientation with source on Φ_0, a destination outside L_C can retain its entire list at zero cost. At a destination in L_C, a selection of fewer than k entries pays b for each. At most one first-power entry is nonzero and all entries with $j \ge 3$ are smaller than b, since $p > \sqrt{x}$. A selection of at least k entries costs kb, more than the entire incoming sum $ap/(p^2-1)$. Thus the value is exactly F_C. For source on Φ_1, any nonempty selection pays the same cost, a at destinations in T_C and zero elsewhere. Comparing the whole incoming list with the empty

choice gives R_C. Taking a supremum permits infinite lists by monotone convergence of their entry sums.

If $L_C \neq \varnothing$, the distinct destinations of its first powers show $R_C \geq |L_C|(kb/p - a)$. Also $F_C \leq |T_C|ap/(p^2-1) \leq 2na/p$. The ratios $ap/(kb) = O(x^2/k)$ and $na/(kb) = O(nx/k)$ tend uniformly to zero. Hence $R_C - F_C \geq kb/(2p)$ on each such cycle for large x, a gap exceeding the errors above on every edge. The maxima on different cycles are independent. If L_C is empty the model has $R_C = 0$ and $F_C \geq 0$, so its maximum equals F_C, whether or not the actual model perturbation breaks a tie.

It remains to subtract $\lceil U_p(u) \rceil$. In the word definition replace the distinguished thresholds by b and the other thresholds by zero. At $p^j \leq 3xb$ this costs at most $\sqrt{b} + 2b/(nX')$ per term by the same threshold bounds as in Lemma 141.39. There are polynomially many such terms. Beyond this cutoff the sum of all the model ramps and the original ramps is $O(mh)$, by a geometric tail. The model first and higher powers sum to exactly $m \sum_C F_C$, since $a/p^3 < b$. The total error, including the ceiling, is $o_{\psi,\varepsilon,A}(b/x^A)$. Subtract this from the actual maximized sum over edges to obtain the assertion. □

Corollary 145.15 (Five odd powers determine the normalized logarithm)**.** *Choose a translate $u(p)$ at each $p \in \mathcal{I}_x$. For the common integer B_u of Theorem 145.3,*

$$\frac{1}{Q\delta} \sum_{p \in \mathcal{I}_x} v_p(B_{u(p)}) \log p = \frac{m}{Q\delta} \sum_{p \in \mathcal{I}_x} \log p \sum_{C:\, L_C \neq \varnothing} (R_C(J_9) - F_C(J_9)) + O_{\psi,\varepsilon}\Big(x^{-1/2+(10+1/8)\varepsilon} \log(2x) \Big).$$

The error tends to zero. The same formula holds for any selection of individual word divisors, with the word allowed to depend on the prime as well.

Proof. At a fixed translate $v_p(B_u)$ is the minimum over the words of $v_p(b^{\mathrm{per}}_{u,w})$. The main term of the theorem is independent of the word and its error is uniform. Its contribution to the normalized logarithm is $O_{\psi,\varepsilon,A}(x^{-A})$ for any chosen A, by using a larger exponent in the theorem before summing over $p \leq x$.

Deleting the terms $j \geq 11$ in the reverse sums costs at most $2kbh_2/p^{11}$ in total per edge. For each j a source has a unique destination, and taking the positive part can only reduce the change. The forward tail is at most $2ha/p^{11}$ by the same counting. Both estimates remain valid upon restricting the cycles in the corollary. By integration, $\sum_{n > \sqrt{x}} n^{-11} \log n \ll x^{-5} \log(2x)$. After including m and normalizing by $Q\delta$, the two bounds carry the respective factors $2kn/h$ and $2a/(b\delta)$. From Theorem 6.29 these are $O(x^{9/2+(10+1/8)\varepsilon})$ and $O(x^{1+\varepsilon})$. This gives the displayed error. Its exponent is negative for $0 < \varepsilon < 1/52$. □

This evaluation determines the divisor removed in Section 141.14 over the indicated primes. The logarithmic sum of residual valuations in Theorem 141.40 also involves the images of the actual projected maps.

146 Signed terminal frequencies and their complex norms

146.1 The connected terminal Fourier projection at arbitrary precision

Fix a translate and a place above the terminal prime $p = T$ on the fields of Section 6.6. We evaluate the connected frequency coordinates of Lemma 144.1 in the original signed period. Write $\mathcal{O}$ for the valuation ring of a finite local coefficient field containing the data, enlarged

when necessary, and normalize $v(p) = 1$. Congruences below are modulo $p^M\mathcal{O}$, including when the coefficient field is ramified. For $k \in \mathbb{F}_p$ let $[k]$ be its representative in $\{0, \dots, p-1\}$.

In one scalar coordinate put $F_{b,k} = U_b S_k$, with the complete cyclic projections and prime-to-p displacements of Theorem 68.42. Write $D = d/dz$ in the original frame and $J_i(F) = (D^i F)(0)$. On the quotient elliptic curve C' let $z' = \alpha z$ be the coordinate for its good differential. The connected orientation gives $v(\alpha) = 1$. The displacement for label one gives a point Q on C' of exact p-order whose reduction still has order p. Label k gives $[k]Q$, by the period pairing in Lemma 68.41. Use Z for the elliptic logarithmic derivative in the good differential, as defined in Lemma 68.32.

Lemma 146.1 (A common center for all prime-to-p translates)**.** *The scalar $\theta = \alpha Z(Q)$ is a unit. For every b, k and every $d \geq 0$,*

$$B_d(b,k) = \big((D - \theta[k])^d F_{b,k}\big)(0) \ \in \ p^d\mathcal{O}.$$

In particular the same center $\theta[k]$ works for all displacements at the other primes. The assertion holds when the section vanishes at the origin.

Proof. In the coordinate z' the section is a scalar times the translate of the marked theta on C'. The scalar includes the phase of the translation and has valuation zero in the connected orientation by Theorem 68.42. Let P_0 be the sum of its two-torsion mark and the point induced by b. Its order is prime to p. The point that enters the logarithmic derivative at label k is $P_k = P_0 + [k]Q$. For $k \neq 0$ its reduction is different from the origin because its p-primary part has order p. Lemma 68.32 gives $v(Z(Q)) = -1$, which proves the unit assertion.

We need the first logarithmic derivative to precision p in $\mathcal{O}$. Use the integral extension of C' by the additive group constructed in Lemma 68.32. Over points reducing away from the origin it has an integral coordinate s_0. Addition of a vertical scalar translates that coordinate, and the unique torsion lift $\widehat{P}$ of a torsion point has coordinate $Z(P)$. There is an integral lift g_Q of Q with $s_0 = 0$. Thus $\widehat{Q} = g_Q + \iota(Z(Q))$, where ι denotes the vertical inclusion. The torsion lift of P_0 is integral. Indeed, start with an integral lift in any chart at P_0, multiply it by an order m of P_0 prime to p, and subtract from that lift the resulting integral vertical scalar divided by m. The corrected lift is killed by m. Uniqueness follows from the absence of torsion in the characteristic-zero additive kernel. It also shows that adding torsion lifts lifts the sum. Consequently

$$\widehat{P}_k = \widehat{P}_0 + [k]g_Q + \iota([k]Z(Q)).$$

For $k \neq 0$ the sum of the first two terms is integral and belongs to the s_0 chart over P_k. Hence $Z(P_k) - [k]Z(Q) \in \mathcal{O}$. Returning to z gives

$$(D \log F_{b,k})(0) - \theta[k] = \alpha\big(Z(P_k) - [k]Z(Q)\big) \in p\mathcal{O}.$$

Here $J_0(F_{b,k})$ is a unit by Theorem 68.42.

In a good Weierstrass model for C' the second logarithmic derivative in z' is $-u_{\Lambda'} - \wp(P_k)$, and the subsequent ones are $-\partial^{j-2}\wp(P_k)$ for $j \geq 3$, with ∂ the invariant derivation in that differential. These are integral by the logarithmic derivative calculation of Theorem 68.33. Indeed P_k reduces away from the pole, $u_{\Lambda'}$ is integral on the good model, and its invariant derivation preserves regular functions locally. In the original coordinate the derivative of order j acquires α^j. Thus the logarithmic derivatives at zero of the formal series $e^{-\theta[k]z}F_{b,k}(z)$ belong to $p^j\mathcal{O}$ for all $j \geq 1$. Its undivided derivative of order d is its constant value times the sum over the partitions of a set of d elements, multiplying for each block the logarithmic derivative of its size. This formula follows by differentiating, either adding a singleton or enlarging one

of the existing blocks. Each product lies in $p^d\mathcal{O}$, proving the assertion for nonzero labels without dividing by a factorial. At $k = 0$ the center is zero and Theorem 68.42 gives directly $v(J_d(F_{b,0})) \geq d$, including at a zero of the section. □

For s such connected coordinates use variables $Z_1, \dots, Z_s$, units θ_i, and a tuple of labels $\mathbf{k}$. Write $c_i = \theta_i[k_i]$. For a polynomial f with coefficients in $\mathcal{O}$, its Hasse derivative $f^{[\mathbf{d}]}(c)$ is the coefficient of $X^{\mathbf{d}}$ in $f(c + X)$. It is integral even when ordinary derivatives divided by factorials would be inconvenient to evaluate. For displacements b_i write $B_{i,d} = B_d(b_i, k_i)$ in the corresponding frames.

Theorem 146.2 (The signed terminal calculation modulo every power of p)**.** *At every integer precision $M \geq 1$ one has*

$$f(D_1, \dots, D_s) \prod_i F_{b_i,k_i}(z_i)\Big|_{z=0} \equiv \sum_{|\mathbf{d}|<M} f^{[\mathbf{d}]}(c) \prod_{i=1}^{s} B_{i,d_i} \pmod{p^M\mathcal{O}}.$$

The operator on the left annihilates, modulo p^M, all polynomials in

$$I^M, \qquad I = (p,\ Z_1^p - \theta_1^{p-1} Z_1, \dots, Z_s^p - \theta_s^{p-1} Z_s).$$

For the full set of labels the finite coefficient ring has the description

$$\mathcal{O}[Z_1, \dots, Z_s]/I^M \ \simeq \prod_{\mathbf{k} \in \mathbb{F}_p^s} \mathcal{O}[Z_1, \dots, Z_s]/(p, Z_1 - c_1, \dots, Z_s - c_s)^M,$$

where the centers in each factor use its own label. At fixed labels only the $\binom{s+M-1}{s}$ Taylor coefficients with $|\mathbf{d}| < M$ enter, and the coefficient with index $\mathbf{d}$ is needed only modulo $p^{M-|\mathbf{d}|}$.

Proof. Expand f as $\sum_{\mathbf{d}} f^{[\mathbf{d}]}(c)(Z - c)^{\mathbf{d}}$. The variables act on independent scalar factors, so evaluation of a centered monomial gives exactly $\prod_i B_{i,d_i}$. Its valuation is at least $|\mathbf{d}|$ by the lemma. This proves the congruence, the precision of each coefficient, and vanishing on the M-th power of each fixed-label ideal at that label. Counting monomials of total degree less than M gives the stated binomial coefficient.

For different label tuples the ideals $(p, Z_1 - c_1, \dots, Z_s - c_s)$ are comaximal, since some pair of centers differs by a unit. Their intersection is I. Indeed modulo p the polynomial $Z_i^p - \theta_i^{p-1} Z_i$ is the product of the distinct linear factors $Z_i - \theta_i[k]$ with unit differences. The Chinese remainder theorem in each successive variable identifies the quotient by these polynomials with the product of the evaluation rings $\mathcal{O}/p$ at the tuples. This proof also applies when $\mathcal{O}/p$ has nilpotents. For pairwise comaximal ideals their product is their intersection, and their powers are still pairwise comaximal. Taking the M-th powers thus gives the asserted product decomposition and the common annihilation. □

For example, in one coordinate and for every $n \geq 1$ the calculation at precision two is

$$J_n(F_{b,k}) \equiv c^n J_0(F_{b,k}) + n c^{n-1}\big(J_1(F_{b,k}) - c J_0(F_{b,k})\big) \pmod{p^2\mathcal{O}}.$$

Here $c^0 = 1$ also when $k = 0$. This retains the first correction to the residue formula at the same terminal character.

Corollary 146.3 (Elementary divisors at the prescribed projection)**.** *Fix the frames and a finite matrix of the signed projected columns and tests of Theorem 73.1. Expand each actual entry over its class paths and their shared covectors. At the chosen place evaluate the other factors, and choose a common scalar clearing the coefficients of the remaining polynomials in*

the connected derivatives, including the incoming values, class measures, and any prescribed column divisions. Let H be this integral matrix. In each term apply the theorem to its connected coordinates, and denote the resulting signed matrix by H_M. At every precision M the elementary-divisor valuations of H and H_M, each truncated at M, agree. Zero elementary divisors are assigned valuation $+\infty$. In particular the number with valuation strictly below M is a lower bound on the rank of the original projected columns.

Proof. The integral substitution in (66.22) is a polynomial in the commuting scalar derivatives in the exact chosen frames. After the indicated clearing it remains integral when all other factors have been evaluated. For conjugated edges use the conjugated coefficient data at their corresponding places. Lemma 144.1 gives precisely the complete cyclic projections used above, up to the recorded phases. Apply the congruence to each product term, keeping a shared covector fixed in both adjacent factors until they have been multiplied. Summing with the original coefficients gives $H = H_M$ modulo p^M entry by entry. No error is multiplied by the number of paths in this assertion since $p^M\mathcal{O}$ is an ideal.

Their cokernels after reduction modulo p^M are identical. If ϖ is a uniformizer and $e = v_\varpi(p)$, Smith reduction of a matrix over $\mathcal{O}$ presents this finite cokernel as a sum of modules of lengths $\min(v_\varpi(d_i), eM)$, together with the free target rows reduced modulo p^M. The matrix dimensions are the same, so uniqueness of elementary divisors gives precisely the claimed truncated list on dividing by e. Every divisor of valuation less than M is nonzero in characteristic zero. Clearing by a nonzero common scalar preserves the characteristic-zero rank. □

The calculation applies on the same interval as Corollary 22.23. An estimate for the number of surviving elementary divisors and the height of the projected period over that whole interval is still required in Section 6.6.

146.2 Complex norms on the nonzero terminal frequencies

Fix the terminal prime $p = T = \varpi\bar{\varpi}$ of Theorem 9.13, an output covector, and one of the translated product terms in Lemma 144.1. We work at a complex embedding and keep the point of evaluation at zero. In one constrained elliptic coordinate the lattice is $\Lambda = \delta I$, with I a fractional ideal of the full ring of K_d. Write A for its area, $c = \pi/A$, and Θ for the marked theta section. Its norm uses the Hermitian metric and Haar probability on $\mathbb{C}/\Lambda$. Interchange $\varpi, \bar{\varpi}$ if necessary so that the frequency subgroup is $\varpi^{-1}\Lambda/\Lambda$. Choose its generator w to precision p^2 as in Lemma 68.41. With the translation operators and phases of that lemma, put

$$S_k = p^{-1}\sum_{j=0}^{p-1} e^{-2\pi i k j/p} U_{jw}\Theta, \qquad y_k = J_P(U_b S_k), \qquad J_P(H) = \sum_{n=0}^{K} a_n \partial_z^n H(0).$$

Here $P(Z) = \sum_{n=0}^{K} a_n Z^n$ is any complex polynomial and b is the fixed prime-to-p shift of the product term, lifted to the common cover. Give these polynomials the squared norm $\mu(P) = \|\Theta\|^2 \sum_{n=0}^{K} c^n n! |a_n|^2$.

Lemma 146.4 (Energy on a split cyclic subgroup)**.** *Let λ be the shortest nonzero length of the real dual lattice $\Lambda^\vee$, and put $\rho = \lambda/\sqrt{c}$. Define*

$$E = 50\left(\frac{2\sqrt{K+6}}{\pi\sqrt{p}\,\rho}\right)^6, \qquad N = \max\{1, \lceil 8\sqrt{K+6}/\rho \rceil\}, \qquad B = 3N^2/2.$$

Then

$$\left|\sum_{k\in\mathbb{F}_p} |y_k|^2 - \mu(P)\right| \le E\mu(P), \qquad \max_{k\in\mathbb{F}_p} |y_k|^2 \le (B/p)\mu(P).$$

Also $\rho^2 \ge 2/(\pi\sqrt{|\operatorname{disc} K_d|})$.

Proof. Finite Fourier orthogonality gives $\sum_k |y_k|^2 = p^{-1}\sum_j |J_P(U_bU_{jw}\Theta)|^2$. Commuting the two translations changes each summand inside the absolute value by a scalar of modulus one, common to all derivative orders. The last mean therefore samples the periodic function $g(z) = |J_P(U_zU_b\Theta)|^2$ on $\varpi^{-1}\Lambda/\Lambda$. Its Haar mean is $\mu(P)$. Indeed the metric representative of order n is obtained by applying $(\partial_z - c\bar z)^n$ to the holomorphic section and inserting the metric factor. The adjoint is $-\partial_{\bar z}$. Commuting a lowering operator past a raising operator contributes c. Any leftover lowering operator kills a holomorphic section, so different orders are orthogonal and the squared norm in order n is $c^n n!\|\Theta\|^2$, as in Lemma 143.4.

The Fourier estimate proved in Lemma 66.33 gives $|\widehat{g}(u)| \le \mu(P)(2\sqrt{c(K+6)}/(\pi|u|))^6$ for $0 \ne u \in \Lambda^\vee$. Its hypothesis holds because each lowering operator reduces the raising order by one, so $K+1$ lowering operators kill the whole section. A unitary character in the period multiplier cancels in the metric pairing used in the proof. Averaging retains only the dual of $\varpi^{-1}\Lambda$, which is $\bar\varpi\Lambda^\vee$. Thus the shortest surviving nonconstant frequency has length $s = \sqrt{p}\,\lambda$. In $2^h s \le |u| < 2^{h+1}s$ there are at most $25\cdot 2^{2h}$ such frequencies, by packing disjoint disks of radius $s/2$. Summing the sixth-power bound and using $25\sum_{h\ge 0} 2^{-4h} < 50$ proves the first inequality.

For the pointwise estimate pass to $\Gamma = \bar\varpi\Lambda$. The p sections $U_{jw}\Theta$ on that cover are eigenvectors for a common unitary period translation with distinct eigenvalues. Indeed the pairing of a suitable period of Λ with the generator w is a primitive p-th root. These vectors are orthogonal and have the same norm as Θ for probability measure. Hence $\|S_k\|^2 = \|\Theta\|^2/p$ for every k. By Lemma 68.41, S_k also has a theta multiplier on $\Lambda' = (\bar\varpi/\varpi)\Lambda$, differing from that of S_0 by a unitary character. Its squared metric norm on this torus agrees with the one on the cover. The rotation from Λ to Λ' preserves c and λ. The same adjoint calculation gives mean $\mu(P)/p$ for $|J_P(U_zU_bS_k)|^2$ on the new torus. Lemma 66.33 with the indicated N bounds its mean on the full N-torsion grid by $3\mu(P)/(2p)$. Its N^2 summands include the nonnegative value $|y_k|^2$ at zero, giving the assertion. The identical grid argument on the original torus gives $|J_P(U_b\Theta)|^2 \le B\mu(P)$.

Finally the planar dual is $i\Lambda/A$. If $0 \ne u \in I$, then $(u)I^{-1}$ is an integral ideal, so $|u|^2 \ge N(I)$. The area is $|\delta|^2 N(I)\sqrt{|\operatorname{disc} K_d|}/2$. Consequently $\rho^2 = \min_{0\ne z\in\Lambda}|z|^2/(\pi A)$ has the stated lower bound. □

In one scalar coordinate there is also an estimate after subtracting the mean on $\mathbb{F}_p^\times$. Write $\bar y = (p-1)^{-1}\sum_{k\ne 0} y_k$. Since $\sum_k S_k = \Theta$, the squared norm lost in this subtraction is exactly

$$(p-1)|\bar y|^2 = \frac{|J_P(U_b\Theta) - y_0|^2}{p-1} \le \frac{B(1+p^{-1/2})^2}{p-1}\,\mu(P).$$

Here the triangle inequality uses the two pointwise bounds just proved. Together with the cost $|y_0|^2 \le B\mu(P)/p$ of deleting zero, this gives a lower bound for the actual one-coordinate augmentation, with the original phases of the Fourier sum.

Theorem 146.5 (Rank and norms on nonzero terminal covectors)**.** *For one fixed translated product term let r_0 be the number of constrained coordinates at the chosen split block. In coordinate h use $S_{h,k_h}, b_h, c_h, K_h, E_h, B_h$ as above. On the polynomials $P = \sum_\nu a_\nu Z^\nu$ of degree at most K_h in Z_h, put*

$$\|P\|_{\mathrm F}^2 = \Bigg(\prod_h \|\Theta_h\|^2\Bigg)\sum_\nu |a_\nu|^2 \prod_h c_h^{\nu_h}\nu_h!.$$

Let $T_b P(k)$ be $P(\partial)$ applied at zero to $\prod_h (U_{b_h} S_{h,k_h})(z_h)$. Then

$$\prod_h (1 - E_h - B_h/p)_+ \, \|P\|_{\mathrm{F}}^2 \;\le\; \sum_{0 \neq k \in \mathbb{F}_p^{r_0}} |T_b P(k)|^2 \;\le \prod_h (1 + E_h) \, \|P\|_{\mathrm{F}}^2,$$

where $u_+ = \max(u, 0)$. At the word weights of Theorem 9.13 these factors lie between $1 - \eta$ and $1 + \eta$, with $\eta \le p^{-1+o(1)}$ uniformly as the interval recedes. In particular the evaluation on the nonzero covectors is injective on the prescribed polynomial subspace.

Proof. On one coordinate Lemma 146.4 bounds the squared singular values of the full evaluation map above by $1 + E_h$ in the indicated polynomial metric. Deleting the label zero costs at most $(B_h/p)\mu(P)$ for every input, so the map on $\mathbb{F}_p^\times$ has squared singular values at least $(1 - E_h - B_h/p)_+$, including zero eigenvalues if necessary. On a Cartesian product of label sets the evaluation map is the tensor product of the one-coordinate maps. This follows on monomials by differentiation in separate variables, and extends by linearity. The displayed polynomial metric is the tensor product metric. Diagonalizing the positive Gram operators gives the products of the one-coordinate eigenvalues. The nonzero covectors contain $(\mathbb{F}_p^\times)^{r_0}$ and are contained in the full grid. Adding squared absolute values can only increase the quadratic form. This proves both bounds, including for polynomials with mixed terms.

For the uniform statement take K_h up to the total degree of the prescribed polynomial substitution on the edge. At each CM embedding its degree is the size of its partition and is at most the row rank times the first row length, by the determinant shift in Lemma 141.11. The number of embeddings and all ranks are bounded by fixed powers of x. The first row is $O(x \exp(x^{\varepsilon}) + x^{C})$ by the weight calculation in Corollary 144.12. Also d_0 is polynomially bounded on the conductor intervals of Theorem 19.15, so the discriminant estimate in Lemma 146.4 gives $\rho_h^{-2} \ll x^{C'}$ at every embedding. It follows that $B_h \ll x^{C''} \exp(x^{\varepsilon})$ and $E_h \ll (B_h/p)^3$. The terminal size estimate in Theorem 93.17 gives $\log p \gg m x^{\varepsilon}$ with $m \to \infty$. Hence $r_0 \max_h (E_h + B_h/p) \le p^{-1+o(1)}$. The lower product is at least one minus that sum once all factors are positive. The upper product is at most $\exp(\sum_h E_h)$, which gives the stated error. Restricting a positive definite quadratic form to the prescribed subspace proves injectivity there. □

Corollary 146.6 (Gram determinants and pairing with an incoming table)**.** *Let $P_1, \dots, P_q$ be linearly independent prescribed polynomials for the same product term, with coefficients independent of k. Write G_{F} for their polynomial Gram matrix and G_{term} for the Gram matrix of their values on all nonzero covectors. On the above intervals*

$$\left| \frac{1}{q} \log \frac{\det G_{\mathrm{term}}}{\det G_{\mathrm{F}}} \right| \le 2\eta \qquad (\eta \le 1/2).$$

On the span of the given inputs let T be this evaluation map, and let Π be orthogonal projection onto its image in the space of tables on nonzero covectors, with counting measure. For every incoming table Y one has

$$(1 - \eta)\|\Pi Y\|^2 \le \|T^* Y\|_{\mathrm{F}}^2 \le (1 + \eta)\|\Pi Y\|^2.$$

Proof. Apply the theorem to every linear combination of the P_i. Conjugating the resulting Gram inequalities by $G_{\mathrm{F}}^{-1/2}$ puts each eigenvalue of the relative Gram matrix in $[1 - \eta, 1 + \eta]$. Taking logarithms and using $-\log(1 - \eta) \le 2\eta$ proves the determinant bound. The positive eigenvalues of TT^* are the eigenvalues of T^*T, and TT^* is zero on the orthogonal complement of the image. Its quadratic form on Y is $\|T^* Y\|_{\mathrm{F}}^2$, proving the second assertion. For a bilinear incoming contraction take the complex conjugate of the incoming coefficients in the Hermitian pairing. □

Corollary 9.34 permits these estimates at the very primes for which the full strict groups vanish throughout the weight interval. For the signed period of Section 6.6, one must still bound the projections of the prescribed incoming tables and the cancellation among terms with different shifts and class paths.

146.3 The scalar-orbit projection of the terminal derivatives

In a terminal block the scalar $a \in \mathbb{F}_p^{\times}$ acts on an entire covector by $k \mapsto ak$. The incoming tables in Proposition 144.13 have mean zero on every such orbit. We estimate the part of an outgoing theta table that survives pairing with this space. All translations, phases and derivatives below use the fixed product of Theorem 146.5 at the original point.

Lemma 146.7 (Translation by a short lift of a frequency)**.** *In one scalar coordinate write $y = (J_P(U_b S_k))_{k\in\mathbb{F}_p}$ and $E_+ = E(K+1)$, with E as in Lemma 146.4. There is a unitary operator W on the space of these tables, independent of P, such that*

$$(Wf)(k) = \gamma e^{2\pi i a k/p} f(k+d), \qquad |\gamma| = 1, \quad a \neq 0,$$
$$\|(I-W)y\| \le 4\sqrt{(K+1)/p}\sqrt{1+E_+}\,\|P\|_{\mathrm{F}}.$$

Here $a, d \in \mathbb{F}_p$ and the norm on tables uses counting measure.

Proof. Choose a shortest nonzero $v \in \Lambda$ and set $h = v/\varpi$. Its length is smaller than that of any nonzero period, so $h = aw + \ell$ with $a \not\equiv 0 \pmod p$ and $\ell \in \Lambda$. The disk of radius $|v|/2$ embeds up to its boundary in the torus of area A. Thus $c|h|^2 = \pi|v|^2/(pA) \le 4/p$. In the finite Fourier sum defining S_k, cyclically shifting the summation index gives $U_w S_k = e^{2\pi i k/p} S_k$. A period translation U_ℓ on the summand indexed by j acts by its original period sign times a character of j, by the commutator calculation in Lemma 68.41. Thus it sends S_k to that sign times S_{k+d} for a fixed d. Composing these two translations and commuting past U_b introduces a constant phase independent of k. This proves that $J_P(U_h U_b S_k) = (Wy)(k)$ with the indicated unitary W.

To estimate the difference, let $H_t = U_{th} U_b S_k$ for $0 \le t \le 1$. The translation law in the common coordinate is $U_u H(z) = \exp(-c\bar{u}z - c|u|^2/2) H(z+u)$. Differentiating along this real segment yields $\dot{H}_t = h\partial_z H_t - c\bar{h} z H_t$. Hence $dJ_P(H_t)/dt = J_Q(H_t)$, where $Q(Z) = hZP(Z) - c\bar{h}P'(Z)$. In the factorial metric, $\|ZP\|_{\mathrm{F}} \le \sqrt{c(K+1)}\|P\|_{\mathrm{F}}$ and $\|cP'\|_{\mathrm{F}} \le \sqrt{cK}\|P\|_{\mathrm{F}}$, by shifting each monomial in the norm formula. The complex sampling lemma applies along the segment. Its proof for translated sections uses only unitarity of the multiplier and therefore permits an arbitrary complex displacement. Also $U_{th}U_b$ differs from U_{th+b} by a common phase. The vector of the derivatives in t thus has norm at most $\sqrt{1+E_+}\|Q\|_{\mathrm{F}}$. Integrate from zero to one and insert the bound on $|h|\sqrt{c}$. □

Proposition 146.8 (Loss at a single multiplicative character)**.** *For every character χ of $\mathbb{F}_p^{\times}$, including the trivial character, put $u_\chi(0) = 0$ and $u_\chi(k) = \chi(k)/\sqrt{p-1}$ for $k \neq 0$. If $t = \sqrt{(K+1)/p} \le 1/2$, then*

$$|\langle u_\chi, y\rangle|^2 \le 150(1+E_+)t\,\|P\|_{\mathrm{F}}^2.$$

Proof. For every integer j prime to p, the operator W^j is, up to a unit scalar, a shift by jd and a modulation by $ja \neq 0$. For a nontrivial χ the finite-field translation and modulation bound of Gurevich, Hadani and Sochen therefore gives $|\langle u_\chi, W^j u_\chi\rangle| \le 2/\sqrt{p}$. In their standard split torus the chosen vector is precisely the normalized character on $\mathbb{F}_p^{\times}$ extended by zero, with the counting inner product [3, §I-B and (I.1)]. For the trivial character the same inner product, up

to a phase, is the sum of the nonconstant additive character of frequency ja over all k except 0 and $-jd$, divided by $p-1$. The complete sum vanishes, so its modulus is at most $2/(p-1)$. For $p \mid j$ the unitary bound one suffices.

Write $s = 1 - t$. By unitarity the inverse of $I - sW^*$ is its convergent geometric series. Squaring the norm and grouping pairs of exponents by their difference gives

$$\|(I - sW^*)^{-1}u_\chi\|^2 \le \frac{1}{1-s^2}\left(1 + \frac{4}{\sqrt{p}\,t} + \frac{2s^p}{1-s^p}\right) \le \frac{6}{t}.$$

Indeed $\sqrt{p}\,t \ge 1$, $s^p \le e^{-pt} \le e^{-\sqrt{p}}$, $p \ge 5$, and $1 - s^2 \ge t$. On the other hand the sampling lemma gives $\|y\| \le \sqrt{1+E_+}\|P\|_{\mathrm{F}}$. Applying Lemma 146.7 in $I - sW = tI + s(I - W)$ gives $\|(I - sW)y\| \le 5t\sqrt{1+E_+}\|P\|_{\mathrm{F}}$. Pairing this last vector with $(I - sW^*)^{-1}u_\chi$ recovers the original inner product in absolute value. Cauchy–Schwarz proves the assertion. □

For $D = \mathbb{F}_p^{r_0} \setminus \{0\}$ let A be the orthogonal projection onto the tables with mean zero on each line of nonzero multiples. In particular

$$(Af)(k) = f(k) - (p-1)^{-1}\sum_{a \in \mathbb{F}_p^\times} f(ak).$$

The same a acts on every coordinate. Use T_b for the evaluation map of Theorem 146.5 on D.

Theorem 146.9 (Rank after the scalar-orbit projection)**.** *On every sufficiently distant common interval, at every complex embedding and for one fixed translated product of the marked terminal block,*

$$(1-\eta')\|P\|_{\mathrm{F}}^2 \le \|AT_bP\|^2 \le (1+\eta')\|P\|_{\mathrm{F}}^2, \qquad \eta' \le p^{-1/2+o(1)}.$$

This holds for every linear combination of the prescribed polynomial rows. In particular their rank is preserved by the evaluation followed by scalar augmentation. More generally, on removing the same J specified scalar character types from every line, the lower bound has loss at most $p^{-1+o(1)} + Jp^{-1/2+o(1)}$.

Proof. Let $D_0 = (\mathbb{F}_p^\times)^{r_0}$. Both D_0 and its complement in D are unions of whole scalar orbits. Hence restricting to D_0 commutes with the projections onto individual types, and hence also with deleting them. On D_0 the evaluation is $T_1 \otimes T_R$, where T_1 is scalar evaluation on the first punctured coordinate and T_R is the tensor product for the others. For $r_0 = 1$ use the one-dimensional trivial space as the empty tensor product. Let P_χ be orthogonal projection to the character χ in the first coordinate. It has rank one. Proposition 146.8 gives, in the order of Hermitian forms,

$$T_1^* P_\chi T_1 \le \delta I, \qquad \delta = 150(1 + E_{1,+})\sqrt{(K_1+1)/p}.$$

Write Q_β for the orthogonal type projections of the simultaneous scalar action on the remaining coordinates. These sum to the identity. On the product grid the projection to total type α is $\sum_\chi P_\chi \otimes Q_{\alpha\chi^{-1}}$, since characters multiply on tensor products. Pulling this projection back by $T_1 \otimes T_R$ gives an operator bounded by

$$\delta I \otimes \sum_\chi T_R^* Q_{\alpha\chi^{-1}} T_R = \delta I \otimes T_R^* T_R.$$

This uses positivity of each second factor and therefore applies also to polynomials coupling different coordinates. By the scalar sampling lemma $\|T_R\|^2 \le \prod_{h>1}(1 + E_h)$. The lower

bound on the unprojected product grid in the proof of Theorem 146.5 is $\prod_h (1 - E_h - B_h/p)_+$. Subtract at most $J\delta \prod_{h>1}(1 + E_h)$ for J removed types, then add the nonnegative squared norm on $D \setminus D_0$. The upper bound follows by contractivity of an orthogonal projection and the upper bound on D in the same theorem. In the word range $K_h + 1 = p^{o(1)}$, $E_{h,+} \le p^{-3+o(1)}$, and the lower product is at least $1 - p^{-1+o(1)}$. These estimates are uniform in the word, the shift and the embedding. Taking $J = 1$ and removing the trivial character gives the result for A. □

Corollary 146.10 (Pairing with the full space of separately augmented tables)**.** *Take a fixed product term at all split terminal blocks and impose the zero-mean condition independently at each block, holding the other covectors fixed. Evaluation followed by orthogonal projection to this space has all squared singular values in $[1 - p^{-1/2+o(1)}, 1 + p^{-1/2+o(1)}]$ on the prescribed polynomial subspace with its product factorial metric. The same bounds hold for the Rayleigh quotients of its Gram matrix relative to the polynomial Gram matrix on every independent list of rows.*

Proof. Each block uses its own scalar group. Averaging in independent variables commutes, so the projection onto the common zero-mean space is the tensor product of the single-block projections. The product evaluation also factors. Diagonalizing the positive Gram operators multiplies their eigenvalue bounds. The number of blocks is bounded by a fixed power of x, hence is $p^{o(1)}$ on these intervals. Multiplying the bounds in the theorem preserves the stated error. Restriction to any polynomial subspace preserves the form inequalities. For an incoming vector with the separate zero-mean conditions, moving the orthogonal projection across the pairing leaves that vector fixed. Thus this is the Gram matrix for pairing with an orthonormal basis of the entire stated space. □

The prime choices in Corollary 9.34 apply here. An individual prescribed incoming column can be perpendicular to the span of these projected rows. The estimate in Section 6.6 still requires control of those columns after the intermediate classes, shifts and their coefficients are summed.

146.4 Signed sums of torsion translates at the marked prime

Two translates which have the same displacement on the common elliptic cover must be added with their original phases. We estimate the cross terms for distinct displacements on that cover, still at the origin used to define the period. For one constrained coordinate use $\Gamma = \bar{\varpi}\Lambda$ and $\Xi = \varpi^{-1}\Lambda$. Write $L_b P = (J_P(U_b S_k))_{k \in \mathbb{F}_p}$. We use the same marked section and factorial metric for all b. A translation by a period of Γ acts on every S_k by the same scalar: such a period commutes with all U_{jw} and acts on Θ by its period sign. Thus one can group equal displacements modulo Γ, incorporating the translation phase in their coefficients.

Lemma 146.11 (Inner products for distinct torsion displacements)**.** *Let b, b' be distinct modulo Γ with $M(b - b') \in \Gamma$, and let P, Q have degree at most K. Take a basis h_1, h_2 of Ξ and set $H = \max_i |h_i|$. For every integer $j \ge 1$,*

$$|\langle L_b P, L_{b'} Q\rangle| \le (1 + E(K + j))(MH\sqrt{c(K + j)})^j \|P\|_{\mathrm{F}} \|Q\|_{\mathrm{F}}.$$

One can take $cH^2 \le (4 + \pi\sqrt{|\operatorname{disc} K_d|})/p$.

Proof. Write $e(h, b)$ for the unit scalar in $U_h U_b = e(h, b) U_b U_h$. The associated alternating form, up to sign, is $\operatorname{Im}(\bar{h}b)/A$ in the exponential $e^{2\pi i t}$. The annihilator of Ξ under this pairing

is Γ. Indeed the area normalization makes Λ self-dual for the alternating form, and moving the factor ϖ^{-1} across it conjugates that factor. As $b-b' \notin \Gamma$, for some basis vector h the number $\xi = e(h, b-b')$ is a nontrivial M-th root of unity. In particular $M \geq 2$ and $|\xi - 1| \geq 2\sin(\pi/M) \geq 4/M$.

Form the scalar function of a real variable

$$F(t) = \left\langle \left(J_P(U_{th}U_bS_k)\right)_k, \left(J_Q(U_{th}U_{b'}S_k)\right)_k \right\rangle,$$

using the inner product linear in its first argument. Since $h = aw + \ell$ with $\ell \in \Lambda$, U_h acts on the list $(S_k)_k$ by one unitary matrix, a modulation and a label translation up to a common phase, as in Lemma 146.7. Also $U_{nh} = U_h^n$. Moving this translation past the two shifts and canceling the common unitary in the pairing gives $F(n) = \xi^n F(0)$ for each nonnegative integer n.

In the first argument, differentiation in t applies the polynomial operator $P \mapsto hZP - c\bar{h}P'$. On degree at most $K+j-1$ its norm is at most $2|h|\sqrt{c(K+j)}$, by the factorial norm calculation in that lemma. The same holds in the second argument. Applying the product rule j times and the full-grid sampling bound in Lemma 146.4 at the translated points gives

$$\sup_t |F^{(j)}(t)| \leq (1 + E(K+j))(4|h|\sqrt{c(K+j)})^j \|P\|_{\mathrm{F}}\|Q\|_{\mathrm{F}}.$$

The j-th forward difference of F at zero is $(\xi-1)^j F(0)$. By iterating the fundamental theorem of calculus it is also the integral of $F^{(j)}(t_1 + \cdots + t_j)$ on the unit j-cube. Division by $|\xi-1|^j$ proves the claim.

For the basis take a shortest period $v_1 \in \Lambda$, with length s. It is primitive, so it extends to a lattice basis. Subtract multiples of v_1 from the second vector to obtain v_2 with parallel component of magnitude at most $s/2$. Its perpendicular component has magnitude A/s. The disk argument gives $cs^2 \leq 4$, and the ideal norm bound in Lemma 146.4 gives $s^2/A \geq 2/\sqrt{|\operatorname{disc} K_d|}$. Consequently $c|v_2|^2 \leq 1 + (\pi/2)\sqrt{|\operatorname{disc} K_d|}$. Take $h_i = v_i/\varpi$. □

In each coordinate h, take a nonempty finite set B'_h of displacements in one coset of the M_h-torsion on $\mathbb{C}/\Gamma_h$, with one lift per displacement. For the tuple $\boldsymbol{b} \in \prod_h B'_h$ let $P_{\boldsymbol{b}}$ be the polynomial obtained after summing all coefficients belonging to that tuple. Coefficients may be signed or complex and must be independent of the shared frequency k. Give the collection the squared norm $\|\boldsymbol{P}\|^2 = \sum_{\boldsymbol{b}} \|P_{\boldsymbol{b}}\|_{\mathrm{F}}^2$. Let $S\boldsymbol{P}$ be the sum of the corresponding product evaluations $T_{\boldsymbol{b}}P_{\boldsymbol{b}}$ on the nonzero covectors. We use A for subtraction of the scalar-orbit means as in Theorem 146.9.

Theorem 146.12 (Norm and rank of a signed sum with torsion displacements)**.** *Take the marked primes, embeddings and derivative degrees on the common intervals of Corollary 9.34. The following bounds hold uniformly for the cosets, the sets within them and the grouped coefficient polynomials.*

1. *If $0 < \sigma < 1/2$ is fixed and all $M_h \leq p^{1/2-\sigma}$, then $\big|\|S\boldsymbol{P}\|^2 - \|\boldsymbol{P}\|^2\big| \leq p^{-2\sigma+o(1)}\|\boldsymbol{P}\|^2$.*
2. *If $0 < \sigma < 1/4$ is fixed and all $M_h \leq p^{1/4-\sigma}$, then $\big|\|AS\boldsymbol{P}\|^2 - \|\boldsymbol{P}\|^2\big| \leq p^{-2\sigma+o(1)}\|\boldsymbol{P}\|^2$.*

For uniformly $M_h = p^{o(1)}$ the respective errors are $p^{-1+o(1)}$ and $p^{-1/2+o(1)}$. These estimates also hold for a product of marked blocks, with each covector nonzero and the scalar-orbit projections imposed separately. In each case the evaluated rank of a list of rows equals the rank of its list of grouped coefficient arrays.

Proof. First work in a single coordinate and write $n = |B'_h| \le M^2$. On the direct sum over the shifts, the full-grid map is $L : (P_b)_b \mapsto \sum_b L_b P_b$. Lemma 146.4 bounds each diagonal block of $L^*L - I$ in operator norm by $E(K)$. The off-diagonal blocks have norm at most $u_j = (1 + E(K+j))(MH\sqrt{c(K+j)})^j$ by Lemma 146.11. Using $\sum_{b \ne b'} v_b v_{b'} \le (n-1)\sum_b v_b^2$ on the norms of the inputs gives, for every $j \ge 1$,

$$\|L^*L - I\| \le E(K) + (n-1)u_j.$$

Deleting the label zero costs at most nB/p in the squared operator norm of that evaluation functional. This follows by Cauchy–Schwarz across the n shifts from the pointwise bound, with B of Lemma 146.4. In the punctured coordinate the squared norm of the projection onto any one multiplicative character is bounded, as a form on the direct sum, by $150n(1 + E(K + 1))\sqrt{(K+1)/p}$ times its input form. This is the same Cauchy–Schwarz argument applied to Proposition 146.8.

On the full or componentwise punctured grid the map on the arrays is the tensor product of these one-coordinate direct-sum maps. Indeed on a tensor of lists the product of the sums is the sum over all shift tuples. Such tensors span the array space, with exactly the stated direct-sum polynomial metric. The tensor bounds, restriction to nonzero covectors and scalar-character calculation in Theorems 146.5 and 146.9 therefore apply with the three bounds just obtained. In particular the cost of removing the trivial total character is at most the one-character loss in one coordinate times the upper squared norm in the other coordinates. This follows by summing the orthogonal character projections on those other coordinates, as in the proof of the latter theorem.

For the first range take a fixed integer $j \ge 1/\sigma$. Uniformly $\sqrt{c(K+j)}H \le p^{-1/2+o(1)}$ by the basis bound and the growth estimates of Theorem 146.5. Then $nu_j \le p^{1-2\sigma-j\sigma+o(1)}$, and the zero-label loss is at most $p^{-2\sigma+o(1)}$. For the second range take $j = 2$. The off-diagonal error is at most $M^4 p^{-1+o(1)} \le p^{-4\sigma+o(1)}$, and the character loss is at most $M^2 p^{-1/2+o(1)} \le p^{-2\sigma+o(1)}$. The other errors are smaller. When $M = p^{o(1)}$, taking $j = 2$ gives the two sharper rates as stated. The number of coordinates and blocks is bounded by a fixed power of x, hence is $p^{o(1)}$. Multiplying the upper and lower factors preserves these rates. The relative error tends to zero in each range. Applying the lower inequality to every linear combination of the rows proves the rank assertion, including when the original list is dependent. □

Corollary 146.13 (Displacements at the two auxiliary primes)**.** *Fix an integer $C \ge 1$. Within one fixed product model of a marked block, group the signed terms by the cosets, in the common covers, of the $(g_1 g_2)^C$-torsion in each constrained coordinate. Within each group the ranks of the signed polynomial arrays and of their separately augmented terminal tables agree. The logarithms of the two Gram determinants on any independent list of q arrays differ by at most $qp^{-1/2+o(1)}$. The other displacements, the chosen cosets and the lifts of their base points may vary with the interval.*

Proof. Proposition 6.30 puts each g_i between $(kb)^8$ and twice that quantity, with $b = \exp(x^{\varepsilon})$. Thus $\log(g_1 g_2)^C \ll_C x^{\varepsilon} + \log x = o(\log p)$ by Theorem 93.17. The order is coprime to p. Projection of the common cover to the original elliptic curve has degree p and so is an isomorphism on torsion of this order. Explicitly, multiplication by that order is invertible on its kernel, allowing adjustment of any lift to a unique torsion lift. Differences of the indicated displacements therefore satisfy the coset hypothesis in the same cover as the Fourier labels. The theorem now applies to the signed sums within each group. On an independent list, conjugating the two Gram inequalities by the inverse square root of the polynomial Gram matrix bounds each relative eigenvalue within $p^{-1/2+o(1)}$ of one. Summing their logarithms gives the assertion. □

Corollary 146.14 (Contraction against the actual refined initial class)**.** *Fix a word c, an initial coarse class and one unit-basis matrix coefficient of its infinity type ρ, as in Proposition 86.29. At the first edge fix the output covectors and product coordinates. Collect any set of signed path contributions whose first-edge displacements meet the hypotheses of part 2 of Theorem 146.12 in these coordinates. The nonterminal cosets, the remaining edges and the test scalars may be summed within this collection, keeping each shared index once. For each output test i let $\boldsymbol{P}_i$ be its grouped array, including its arithmetic coefficients and class masses. Let G_{F} be the Gram matrix of these arrays, and let G_{in} be the Gram matrix of the resulting period functionals, restricted to the separately zero-mean tables on this actual initial class and matrix coefficient. Use the dual of the automorphic probability norm on the functionals. Then*

$$(1-\eta)R_cG_{\mathrm{F}} \ \le G_{\mathrm{in}} \le \ (1+\eta)R_cG_{\mathrm{F}}, \qquad R_c = M_H(\dim\rho)(p^k-1)^f,$$

where k is the initial unitary rank, $f = [E_x^+ : \mathbb{Q}]$, and η is the augmented error in that theorem. On summing over distinct initial coarse classes or orthogonal matrix coefficients, the corresponding Gram matrices add with the same bounds.

Proof. By Proposition 86.29 the M_H initial classes at the two flag primes have trivial stabilizers. Each has exactly $N = (p^k-1)^f$ refined classes indexed by the nonzero covectors, by Corollary 9.14. They have equal Haar mass $1/(M_HN)$ in probability measure, since refinement divides each compact open class into equal-measure cosets. Schur orthogonality gives archimedean squared norm $1/\dim\rho$ for the chosen matrix coefficient. Hence the squared norm of an arbitrary table ϕ supported on this class and coefficient is $R_c^{-1}\sum_k|\phi(k)|^2$, where here the summation index runs through the tuples of nonzero covectors. All such values can be prescribed independently. Imposing the zero means restricts this space precisely to the image of the product projection A.

In the path identity of Theorem 73.1 the initial argument is read once at its input refinement. With the next covectors fixed, Lemma 141.20 expresses the first incidence at that input by the cyclic projections in our coordinates. All choices on later edges are independent of the first input label once the common indices are fixed. Summing their coefficients with the actual phases therefore gives on this collection the functional $\phi \mapsto \sum_k(S\boldsymbol{P}_i)(k)\phi(k)$. This formula absorbs the weights of the arithmetic sum into its coefficient array, so the use of counting measure in writing it introduces no second class mass. Independent variations in the unconstrained coordinates may be evaluated first by linearity.

For any table y, the squared dual norm of $\phi \mapsto \sum_k y(k)\phi(k)$ on this subspace is $R_c\|Ay\|^2$. Indeed, in counting measure its Riesz vector is $\bar{y}$, and the zero-mean projection commutes with conjugation. Scaling the source metric by $1/R_c$ scales the dual metric by R_c. Apply the identity to every linear combination of the tests and use Theorem 146.12. Disjoint initial classes have disjoint supports and distinct basis matrix coefficients are orthogonal by Schur orthogonality. Dual norms on their orthogonal direct sum therefore add in squares, as claimed. □

Distinct intermediate classes can still contribute to the same initial class, with different first output covectors, displacements or product models. Their functionals then add before the dual norm is taken. The lower bound for this full sum and its comparison with the incoming analytic row are still required in Section 6.6.

146.5 Inner products of residue pullbacks through different class paths

In Theorem 73.1 the matrices on two paths through different intermediate classes can be compared in their common initial and final frames. We calculate their inner product when the final covector runs over its full table. We then count the markings at which their initial

evaluations can lie on the same line. Throughout, $p = T$ is the split terminal prime already chosen on the interval.

At one half let U, V be vector spaces over $\mathbb{F}_p$ of dimensions k, n. For a nontrivial complex character χ of $\mathbb{F}_p^\times$, let $H_\chi(U^\vee)$ be the space of tables on $U^\vee$ satisfying $h(a\xi) = \chi(a)h(\xi)$ for $a \neq 0$. All tables vanish at zero and use counting measure. For a linear map $C : U \to V$ write $P_C h(y) = h(yC)$. This is an operator into $H_\chi(V^\vee)$. We use a Hilbert–Schmidt inner product linear in the second argument.

Theorem 146.15 (The Gram matrix of two prescribed residue maps)**.** *For any two maps $C, D : U \to V$, one has*

$$\langle P_C, P_D \rangle_{\mathrm{HS}} = \frac{1}{p-1} \sum_{t \in \mathbb{F}_p^\times} \chi(t)\, p^{n - \operatorname{rank}(D - tC)}.$$

For $C \neq 0$ its diagonal entry is $a_C = p^n(1 - p^{-\operatorname{rank} C})/(p-1)$. Suppose C, D are nonzero and projectively distinct as matrices. Let h be the rank of $D - ZC$ over $\mathbb{F}_p(Z)$, and $\delta = \min_{t \neq 0} \operatorname{rank}(D - tC)$. Then

$$\frac{|\langle P_C, P_D \rangle_{\mathrm{HS}}|}{\sqrt{a_C a_D}} \;\leq\; \min\left(1, \frac{h p^{-\delta}}{1 - p^{-1}}\right), \qquad \delta \geq 1.$$

The inner product is zero if the ranks at all nonzero scalars are equal. For $D = tC$, $P_D = \chi(t) P_C$.

Proof. Choose one representative u of each line in $U^\vee$. The functions supported on their respective lines, taking the values $(p-1)^{-1/2}\chi(a)$ at au, form an orthonormal basis. Compute the inner product by applying both operators to this basis and summing over $y \in V^\vee$. A pair of nonzero evaluations contributes only when $yD = tyC$. There is then exactly one such t, and the contribution is $\chi(t)/(p-1)$. For each fixed t, the equation has $p^{n - \operatorname{rank}(D - tC)}$ solutions. Those with $yD = yC = 0$ must be removed for each t. Their number is independent of t and $\sum_{t \neq 0} \chi(t) = 0$, giving the formula. Setting $D = C$ gives the diagonal. Homogeneity proves the assertion for scalar multiples.

A nonzero h-rowed minor of $D - ZC$ is a polynomial of degree at most h. Every specialization of smaller rank is a root, and every specialization has rank at most h because the larger minors vanish identically. Subtract the constant p^{n-h} in the character sum. At most h summands then survive, each of modulus at most $p^{n-\delta}$. The two diagonal entries are at least $p^n(1 - p^{-1})/(p-1)$. Division and the Hilbert–Schmidt Cauchy–Schwarz inequality give the estimate. A zero specialization would make the matrices proportional. The assertion about constant rank follows from the same character sum. □

Corollary 146.16 (A bound for signed sums on the full terminal table)**.** *Fix common endpoints and frames. For a finite list of paths let $C_{i,v} \neq 0$ be the composite residue matrices at the selected halves $v \mid \pi_T$, with f such halves. Prescribe a nontrivial character χ_v at each half. Put $Q_i = \bigotimes_v P_{C_{i,v}}$, $a_i = \prod_v a_{C_{i,v}}$. Let A_i be their coefficient maps between fixed finite-dimensional Hilbert spaces, including all finite signs and assigned class masses. In a fixed coupling on the last two edges the same shared index is used in both maps. Matrices differing only by local nonzero scalars may first be collected, multiplying each coefficient by the corresponding character values. For the resulting list put*

$$d_{ij} = \#\{v : [C_{i,v}] \neq [C_{j,v}]\}, \qquad b = \min\{1, k/(p-1)\}, \qquad e = \max_i \sum_{j \neq i} b^{d_{ij}}.$$

Then

$$(1-e)\sum_i a_i\|A_i\|_{\mathrm{HS}}^2 \;\le\; \Big\|\sum_i Q_i\otimes A_i\Big\|_{\mathrm{HS}}^2 \;\le\; (1+e)\sum_i a_i\|A_i\|_{\mathrm{HS}}^2.$$

In particular if every two tuples differ at least at d halves, a list of M terms has $e\le(M-1)b^d$.

Proof. For the full final table replace the marked covector in the path recursion of Theorem 73.1 by its argument y. The incidence conditions now give the initial covector $(y_vC_{i,v})_v$. A zero entry gives a zero value. The coefficient of a fixed path, including its matrix cosets, is held fixed in this extension. Evaluation at the prescribed marking recovers the original summand. A path with a zero composite at a half can be discarded before this calculation. Across several final classes the calculation applies in each pair of endpoint frames separately.

Inner products of tensor products multiply. Each unequal matrix line therefore contributes a normalized inner product of modulus at most b by Theorem 146.15, since $h\le k$. At the other halves the modulus is one. Expand the squared norm, writing $z_i=\sqrt{a_i}\|A_i\|_{\mathrm{HS}}$. The absolute value of the sum of off-diagonal terms is at most $\sum_{i\neq j} b^{d_{ij}}z_iz_j$. The inequality $2z_iz_j\le z_i^2+z_j^2$ bounds this by $e\sum_i z_i^2$. This also verifies the estimate for arbitrary signed coefficients without an independence assumption on the paths. □

Proposition 146.17 (Markings at which two paths can cancel)**.** *For prescribed C,D at one half, choose y uniformly among nonzero final covectors. The probability that both yC,yD are nonzero and on the same line is*

$$\frac{1}{p^n-1}\sum_{t\neq0}\big(p^{n-\operatorname{rank}(D-tC)}-p^{n-\operatorname{rank}[C\ D]}\big),$$

where $[C\ D]:U\oplus U\to V$ has both sets of columns. For nonproportional C,D the probability is at most $\min\{1,(p-1)p^{-h}+hp^{-\delta}\}$, with h,δ as above. For independent markings at several halves the exact probabilities multiply. For a list of paths the fraction of markings at which any two surviving paths read the same tuple of input lines is at most the sum of these products over unordered pairs.

Proof. A surviving collision determines t uniquely. The common kernel of the two pullbacks has exactly $p^{n-\operatorname{rank}[C\ D]}$ elements, including zero. Removing it gives the numerator. Among nonzero y the probability of annihilating a fixed matrix of rank j is $(p^{n-j}-1)/(p^n-1)\le p^{-j}$. Apply this with $j=h$ at all regular scalars and with $j\ge\delta$ at the at most h exceptional scalars. Independence here refers solely to uniform counting on a product of finite sets. Multiplication gives the product formula, followed by the union bound. At any particular marking outside this union, different surviving tuples define orthogonal evaluation functionals on the homogeneous initial table, since their line supports are disjoint. □

At the particular Bessel marking the formula tests the actual covectors $y_v=(\bar\pi \bmod v)\overline{e_N^\vee}$. A counting bound for the excluded markings leaves their membership to be checked at this prescribed tuple. Even within one surviving tuple of input lines the signed arithmetic coefficients still add before the absolute value is taken.

146.6 Initial lines at the prescribed marking and their heights

Fix $E=E_x$, its $f=[E^+:\mathbb{Q}]$ selected terminal places S, and the ordinary initial frames of the signed period. Put $n=[E:\mathbb{Q}]=2f$ and $p=T=\varpi\bar\varpi$, where S consists of the places above the chosen generator ϖ in the imaginary quadratic field. Thus p splits completely in E,

$\varpi\mathcal{O}_E = \prod_{v\in S} v$, and every complex conjugate of ϖ has absolute value $\sqrt{p}$. We give bounds at the specified marking for the same field and prime used in Corollary 9.34.

Write $\mathrm{red}_S(P)$ for the tuple of projective reductions of an E-point $P = [z]$. At each finite place divide the coordinates by any local scalar that gives them maximum absolute value one before reducing. The resulting line is independent of the choice. Use the absolute projective height

$$H_2(P) = \left(\prod_w \|z\|_w^{n_w} \right)^{1/n},$$

where the finite norms are maxima with $|p|_w = p^{-1}$ at $w \mid p$, the infinite norms are Euclidean, and n_w is the local degree. In particular the weight of a complex place is two. The product formula gives independence from the global representative.

Lemma 146.18 (Separation of reductions at the actual terminal halves)**.** *For distinct projective points P, P', the number $c(P, P')$ of places in S where their reductions agree satisfies*

$$c(P, P') \log p \le n \log \big(H_2(P) H_2(P') \big).$$

In particular reduction on S is injective among points of height strictly less than $p^{1/4}$. The same assertion holds for matrices regarded as points in projective space. In affine coordinates $P = [1 : z_2 : \cdots : z_k]$ with entries in $\mathcal{O}_E$ of absolute value at most B at every embedding, two distinct such points satisfy the stronger bound $c(P, P') \log p \le n \log(2B)$. Here $B \ge 1$.

Proof. For representatives z, z' choose a nonzero minor $m = z_i z'_j - z_j z'_i$. At a finite place, $|m|_w \le \|z\|_w \|z'\|_w$. At an agreeing place in S, reduction after the indicated local scalings gives an additional factor p^{-1}. At each infinite place the determinant inequality on the two coordinate pairs gives the same bound without that factor. Multiply over all places using $\prod_w |m|_w^{n_w} = 1$ and $n_v = 1$ on S. The factor $p^{-c(P,P')}$ proves the assertion. If all $f = n/2$ places agree, the product of the heights is at least $p^{1/2}$. For integral affine points choose an index with $z_j - z'_j \neq 0$. Every agreeing place divides this algebraic integer and its absolute norm is at most $(2B)^n$. This proves the last claim. □

Theorem 146.19 (Short representatives and many separated initial lines)**.** *Let N_* be the cyclotomic modulus of the interval. In the fixed residue coordinates of the initial class every tuple of lines in $\prod_{v\in S} \mathbb{P}^{k-1}(\mathbb{F}_p)$ is represented by an integral row in E^k with every entry, in every complex embedding, of absolute value at most $nN_*\sqrt{p}/2$. Its projective height is at most $\sqrt{k}\, nN_*\sqrt{p}/2$.*

Fix $0 < \theta < 1$ and $k \ge 2$. There is a set of integral affine rows whose distinct reductions differ at more than θf of the actual halves. With

$$L = \left\lfloor \frac{p^{(1-\theta)/2}}{4nN_*} \right\rfloor$$

one can take exactly $(L+1)^{n(k-1)}$ rows. On the common intervals the logarithm of this number, divided by $f(k-1)\log p$, is $1 - \theta + o(1)$, uniformly in x.

Proof. Put $K = \mathbb{Q}(\zeta_{N_*})$, which contains E by Theorem 19.15. The traces to E of the powers of ζ_{N_*} span E over $\mathbb{Q}$. Indeed the powers span K, and trace is surjective because its restriction to E is multiplication by $[K : E]$. Select a rational basis $b_1, \ldots, b_n$ among these traces. They are algebraic integers with $|\sigma b_j| \le [K : E] \le N_*$ in every embedding.

The Chinese remainder theorem supplies an integral lift z of any tuple in $\prod_{v\in S} \mathbb{F}_p$. Expand z/ϖ in the basis b_j, with rational coordinates c_j. Subtract $\varpi \sum_j m_j b_j$ from z, rounding each

c_j to a nearest integer m_j. This preserves every prescribed residue and leaves absolute value at most $\sqrt{p}\sum_j |\sigma b_j|/2$ at every embedding. Choose a nonzero representative of each desired line at each place and apply this construction to each coordinate. The resulting integral row is locally primitive at every selected place. Its finite height factors are at most one, giving the height bound. This argument only uses a rational basis of integral elements.

For the separated family take first entry one and, independently for each other entry, any sum $\sum_j m_j b_j$ with $m_j \in \{0, \dots, L\}$. Linear independence gives the stated count. Every nonconstant entry is bounded by $B = nN_*L$, which is at least one on sufficiently distant intervals. A nonzero difference of entries has norm at most $(2B)^n$, and $2B < p^{(1-\theta)/2}$. Lemma 146.18 gives $c < (1-\theta)f$. Finally n, N_* have polynomial bounds in V, while $\log p = (17 + o(1))m \log b$ with $b = \exp(x^{\varepsilon})$ by Theorem 93.9. Thus $\log(4nN_*) = o(\log p)$ uniformly. Substituting in L proves the logarithmic count. □

Proposition 146.20 (Contraction and the integral image on a prescribed set of lines)**.** *Fix a coarse initial class a, a tuple of nontrivial scalar characters with product χ on $D = (\mathbb{F}_p^{\times})^f$, and a finite set $\mathcal{B}$ of distinct tuples of input lines. Take the coefficient field to contain the character values. For each $B \in \mathcal{B}$ fix any representative q_B in the actual residue frames. For the prescribed final marking and projected test F, set*

$$W_{B,F} = \sum_{\gamma\colon [\eta_\gamma]=B} \chi(t_\gamma) W_{\gamma,F}, \qquad \eta_\gamma = t_\gamma q_B, \quad t_\gamma \in D.$$

The sum is over every surviving path starting at a with the indicated tuple, including all intermediate classes and the coupled last edges of Theorem 73.1. On the homogeneous integral initial tables supported on $\mathcal{B}$ the image of the period as an $\mathcal{O}_J$-module is exactly

$$\sum_{B \in \mathcal{B}} W_{B,F}(D_{a,B}),$$

with $D_{a,B}$ the actual coefficient lattice on the line orbit. On a fixed orthonormal matrix coefficient let $w_{B,F}$ denote the scalar value of the functional there. The squared dual norm on this complex source in the automorphic probability metric is exactly

$$\frac{R_c}{(p-1)^f} \sum_{B \in \mathcal{B}} |w_{B,F}|^2, \qquad R_c = M_H(\dim \rho)(p^k - 1)^f.$$

For multiple tests the identity applies to every linear combination. It therefore gives a positive-semidefinite lower bound on the Gram matrix of the full projected period on the ordinary initial source.

Proof. A homogeneous table on the orbit of q_B has values $\phi(tq_B) = \chi(t) z_B$. The initial class has trivial stabilizer, and its lattice on each orbit is constant as shown before Theorem 99.20. The unit character values make integrality equivalent to $z_B \in D_{a,B}$, including at places dividing $p-1$. Every such choice is allowed on each orbit independently. Nontriviality at each half makes every local scalar sum zero. Substitution in the path identity gives exactly $\sum_B W_{B,F}(z_B)$, proving the image formula. For one orthonormal coefficient, Corollary 146.14 gives the squared source norm $(p-1)^f R_c^{-1} \sum_B |z_B|^2$. Taking the dual norm proves the second formula. Restriction to a subspace can only decrease a dual norm, also for each linear combination of the tests. This is the asserted matrix inequality. □

Corollary 146.21 (Paths with small projective height at the fixed marking)**.** *Work in common residue coordinates at the initial class. Suppose the marked rows of a collection of paths are the reductions of nonzero rows P_i over E of height less than $p^{1/4}$. Distinct projective P_i then*

contribute to disjoint terms of the sum in Proposition 146.20. In particular this holds for compatible integral matrix lifts $A_{i,j}$ giving the prescribed nonzero terminal pullbacks, if the reductions of the projective products are the marked input lines and

$$H_2(y_i)\prod_{j=0}^{m-1} H_2(A_{i,j}) < p^{1/4}.$$

Here y_i lifts the marked terminal row and the matrix height uses the entries as projective coordinates. The same conclusion holds for the entire separated family in Theorem 146.19.

Proof. Matrix multiplication satisfies the maximum-entry norm inequality at finite places by the ultrametric inequality. At infinite places the Euclidean norm for rows and the Frobenius norm for matrices are submultiplicative, by Cauchy–Schwarz. Taking the weighted products bounds the height of the nonzero product row by the displayed left side. The separation lemma gives distinct tuples, and the integral and norm formulas use disjoint input orbits on these tuples. □

Thus all actual tuples admit bounded integral coordinate lifts, and many tuples admit mutually separated smaller ones. The signed coefficient $W_{B,F}$ at each tuple still includes paths of every height reducing to it. A lower estimate for those coefficient sums and their comparison with the lattice indices across the tests are required for Section 6.6.

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