The generalized Riemann hypothesis for Dirichlet L-functions

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Volume XXVI

Local Bessel correlations and relative Gram frequencies

Abstract

We compute position integrals for marked vectors and bound the first relative Gram transform at split and inert finite places. The rational sum puts these estimates on the number fields and intervals of the projected period. Its real factor must be estimated at the corresponding archimedean frequencies.

The guide in Volume I lists all twenty-seven volumes.

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136 Fixed vectors and local position integrals

136.1 A fixed frame vector at a split mark with unit pivots

Use the fields, the shorter interval, the two old primes and the measures of Theorem 134.11. At a marked place w, let O_w be the set of residues with all the arm pivots units in both components. For $J \subseteq \Xi$, let $E(\Xi; J)$ be the energy in (134.6) with the prescribed vector fixed at each place of Ξ, and with $s_w \in O_w$ at every unselected place and at every place of J. Here and below the constraint on $z(s)$ has been dropped. All these energies are sums of nonnegative squared norms indexed by residues and links. Write $E(\Xi) = E(\Xi; \varnothing)$. We use the notation $\Xi \setminus J$ also for the induced choice of vectors at the places left in Ξ.

Theorem 136.1 (Integrated fixed-vector factor on the open cell)**.** *Fix $w \in J$, put $q = q_w$, and write the chosen vector in its prescribed residue component as $d = \ell + cz \neq 0$, with $\ell \in L$ in that component and $c \in \mathbb{F}_q$. Let θ be the terminal entry in the sphere formula (134.13) and put $\beta = (\theta\bar{\theta} - N^2)/N$. If the unrestricted local factor of Corollary 134.13 vanishes for a pair of heads, the restricted factor also vanishes. Otherwise θ, β are integral at w. The restricted factor is the unrestricted one multiplied by*

$$R_w(d,\theta) = \begin{cases} q^{-2a+1-k}\,\delta_w(\beta/q)/\delta_w(\beta), & c \neq 0, \quad \theta_{\mathrm{sel}} \equiv -N \pmod q, \quad \beta \in q\mathcal{O}_{F_w}, \\ 0, & \text{otherwise.} \end{cases} \tag{136.1}$$

Here θ_{sel} is the component of θ in the same copy of the residue field as d. In particular $0 \le R_w \le q^{1-k-2a}$, uniformly in the terminal entry and both heads.

Proof. In the open cell the two contributing heads must be units with the same reductions, and the middle matrix x must be integral. For such heads the two character factors cancel on each residue by Proposition 134.12. With $\bar{x}$ and the pivot reductions fixed, the possible residues are parametrized, each once, by the two independent unipotents in $s = \bar{u}\, w_{\bar{\mathbf{p}}} \bar{x}\, \bar{v}^{-1}$. Consequently the restriction amounts to the probability that $sd = d$ for independent uniform $\bar{u}, \bar{v}$.

Work in the selected component, using the paired dual frame as the basis if it is the second component. The split group in this basis is a general linear group with the flag of L. The radical preserves L, acts trivially on the middle quotient, and contains every shear from the middle space to L. In the other component the inverse adjoint supplies the paired action, so the use of either component gives the full group of shears. For $c = 0$, the nonzero vector $w_{\bar{\mathbf{p}}}\bar{x}\,\bar{v}^{-1}d$ belongs to the dual arm, whereas $\bar{u}^{-1}d$ belongs to L. Equality is impossible. For $c \neq 0$, both $\bar{u}^{-1}d$ and $\bar{v}^{-1}d$ are uniform on $cz + L$, independently. Indeed the shears act transitively on this affine space, so each point has the same number of preimages. Write these points as $cz + \ell_1, cz + \ell_2$. The equation $cz + \ell_1 = w_{\bar{\mathbf{p}}}\bar{x}(cz + \ell_2)$ equates a vector in $L \oplus W$ to one in $W \oplus L^\vee$. It forces $\ell_1 = \ell_2 = 0$ and $\bar{x}z = z$ in the chosen component. These three conditions are also sufficient. The residue count is thus the unrestricted count times $q^{-2a}\mathbf{1}_{\bar{x}z=z}$.

On the integral fiber in (134.13), $xz = \omega - (\theta/N)z$. The extra middle condition is constant on the fiber and says that $\omega_{\mathrm{sel}} \equiv 0$ and $\theta_{\mathrm{sel}} \equiv -N$. The volume of the fiber of integral isometries remains $|\mathrm{GL}_k(\mathbb{F}_q)|$. It remains to compute the measure of integral ω with the indicated congruence. The split, unimodular middle lattice has coordinate vectors $v, \lambda \in \mathcal{O}_{F_w}^k$ in dual bases, so $Q(\omega) = v \cdot \lambda$. Use v for the selected component, interchanging the names in the other orientation. If $v \in q\mathcal{O}_{F_w}^k$, the pairing belongs to $q\mathcal{O}_{F_w}$. For $\beta \in q\mathcal{O}_{F_w}$, the change $v = qv'$ multiplies the ambient measure by q^{-k} and the target measure by q^{-1}. Equivalently one may count preimages of $\beta + q^h\mathcal{O}_{F_w}$ and divide by q^{-h} for $h > \mathrm{ord}_w\,\beta$. The restricted sphere measure is therefore $q^{1-k}\delta_w(\beta/q)$, giving (136.1).

For completeness, if $j = \operatorname{ord}_w \beta \geq 0$, then

$$\delta_w(\beta) = (1 - q^{-k}) \sum_{i=0}^{j} q^{-(k-1)i}.$$

In fact the set of v of minimum coordinate valuation i has measure $q^{-ki}(1 - q^{-k})$. For each such v, the map $\lambda \mapsto v \cdot \lambda$ sends probability Haar measure to probability Haar measure on $q^i \mathcal{O}_{F_w}$, which has density q^i on that ideal. This proves the formula and $\delta_w(\beta/q) \leq \delta_w(\beta)$ when $j \geq 1$. The terminal spheres in question have $\beta \neq 0$ by Lemma 134.7. □

Corollary 136.2 (Reduction to singular minors at every selected mark)**.** *Let $E_{\mathrm{sing}}(\Xi)$ be $E(\Xi)$ restricted to residues $s_w \notin O_w$ at every selected w, and set $B_0 = b^{6k}$. For the choices $|\Xi| < f/4$ at each sufficiently large node,*

$$E(\Xi; J) \leq \prod_{w \in J} q_w^{1-k-2a}\, E(\Xi \setminus J), \tag{136.2}$$

$$\max_{\Xi} B_0^{|\Xi|} E(\Xi) \leq (1 - 2fb^{-4k})^{-1} \max_{\Xi} B_0^{|\Xi|} E_{\mathrm{sing}}(\Xi). \tag{136.3}$$

In particular the right side of (136.3) *bounds* $\max_{\Xi} B_0^{|\Xi|} \mathcal{E}_{\Xi}^{\sharp}$ *for the original cutoff in Theorem 134.11.*

Proof. Fix the terminal entry in (134.13). At every place of J, both the fixed and the unrestricted residue sums impose the same unit and congruence requirements on the two heads. The ratio (136.1) is independent of the heads. For a nonintegral terminal entry at any of these places both contributions are zero. Otherwise the entire contribution for the restricted mask at this terminal entry equals the contribution for $\Xi \setminus J$ times $\prod_{w \in J} R_w(d_w, \theta)$. This is an identity for the full quadratic expression. The unrestricted expression is nonnegative for each terminal entry by its sphere integral of squared norms. We may therefore apply the uniform upper bounds for the ratios and then sum in θ. This proves (136.2).

In $E(\Xi)$, a residue outside the all-singular part belongs to O_w for at least one selected w. Nonnegativity before the expansion in heads bounds its total contribution by $\sum_{w \in \Xi} E(\Xi; \{w\})$. Since $q_w \geq b^{10}$ and $a \geq 1$, we have $B_0 q_w^{1-k-2a} \leq b^{-4k}$. After weighting the energies by $B_0^{|\Xi|}$, each term in this sum is therefore at most $b^{-4k} \max_{\Xi'} B_0^{|\Xi'|} E(\Xi')$. There are at most $2f$ summands, and $2fb^{-4k} < 1$ on the stated interval. Taking a maximum and rearranging gives (136.3). Restoring the discarded condition on $z(s)$ can only decrease the sum of squared norms. □

For the empty choice, $E_{\mathrm{sing}}(\varnothing)$ includes every residue with unit pivots at both marks. The remaining maximal energy thus includes this term as well as the terms having a singular minor at each selected place.

136.2 Summation of rational frequencies on the shorter interval

At a node let E be the CM field, $f = [E : \mathbb{Q}]/2$, and let Φ_v range over one embedding from each conjugate pair. For a finite set T of primes of E, write $\mathcal{O}_{E,T}$ for the ring of elements integral outside T. If $y \in E^\times$, define

$$I_T(y) = \prod_{\mathfrak{p}\in T} (\mathrm{N}\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}\, y}, \qquad D_T(y) = \prod_{\mathfrak{p}\in T} (\mathrm{N}\mathfrak{p})^{\max(0,-\,\mathrm{ord}_{\mathfrak{p}}\, y)}.$$

For a nonzero vector of scales $c = (c_v)_v$ with every $c_v \neq 0$, let $g = (\prod_v |c_v|)^{1/f}$. With $K \geq 0$ define the nonnegative weight on nonzero T-integers

$$W_{T,c,K}(y) = \frac{I_T(y)^{K/2}}{D_T(y)^3} \prod_v |c_v \Phi_v y|^{-K} \min\{1, |c_v \Phi_v y|^{-6}\}. \tag{136.4}$$

Lemma 136.3 (Summability with unequal archimedean scales)**.** *There is an absolute $C \geq 1$ such that, for every E, T, c, K as above,*

$$\sum_{0\neq y\in\mathcal{O}_{E,T}} W_{T,c,K}(y) \leq C^f\, g^{-Kf} \max\{1, g^{-1}\}^{6f}.$$

Proof. Write $(y) = \mathfrak{n}/\mathfrak{d}$ with coprime integral ideals. Since y is integral outside T, $D_T(y) = \mathrm{N}\mathfrak{d}$. The factors in the norm of (y) at primes outside T are at least one, so $I_T(y) \leq \prod_v |\Phi_v y|^2$. This cancels the power K of the embeddings and leaves g^{-Kf}. Put $u_v = |c_v|/g$. Then $\prod_v u_v = 1$, and

$$\min\{1, |c_v \Phi_v y|^{-6}\} \leq \max\{1, g^{-1}\}^6 \max\{1, |u_v \Phi_v y|\}^{-6}.$$

It suffices to sum the product on the right together with $(\mathrm{N}\mathfrak{d})^{-3}$.

For fixed $\mathfrak{d}$, all the points lie in $\mathfrak{d}^{-1}$, which is a union of $\mathrm{N}\mathfrak{d}$ cosets of $\mathcal{O}_E$. The index follows by factoring $\mathfrak{d}$ into prime ideals and taking local indices in the corresponding discrete valuation rings. In each coset embed y as $(u_v \Phi_v y)_v$. A product of half-open squares of side $1/2$ in these complex planes contains at most one point of the coset. Indeed the difference of two distinct points comes from a nonzero algebraic integer, so the product of its squared absolute coordinates is an integer at least one, even after the rescaling. Two points in the same box would have each coordinate difference of modulus at most $1/\sqrt{2}$.

The sum over all squares of the supremum of $\max(1, |z|)^{-6}$ on the square is some absolute constant C_0. On the square with lower corner $(h/2, j/2)$, where $t = \max(|h|, |j|) \geq 3$, the supremum is at most $(2/(t-1))^6$, and there are $8t$ squares of that order. Taking the product of the sums gives C_0^f per coset. Summing over all denominators costs at most $\sum_{\mathfrak{d}} (\mathrm{N}\mathfrak{d})^{-2} \leq \zeta(2)^{2f}$. For the inequality, each rational prime has at most $2f$ prime ideals above it and each has norm at least that prime. Unique ideal factorization then bounds each Euler factor. The nonnegative series may be summed in any order. This proves the lemma with $C = C_0 \zeta(2)^2$. □

Theorem 136.4 (The two frequency scales at an actual node)**.** *Use $m = 2n$, $K = k - n^7$, $M = p_0 p_1$, and the different generator δ_E of Corollary 131.2. Put*

$$\kappa_v = \frac{M|\Phi_v \delta_E|}{2\pi}, \qquad b_{j,v} = \frac{\kappa_v^{2(n-j)}}{N} \quad (1 \leq j \leq n), \qquad Q_E = \frac{M\rho_E}{2\pi}.$$

For either of the choices

$$c_v = \frac{M}{b_{j,v}} \qquad \text{or} \qquad c_v = \frac{2\pi i}{\Phi_v(\delta_E) b_{j,v}},$$

for every j and every finite T, one has

$$\sum_{0\neq y\in\mathcal{O}_{E,T}} W_{T,c,K}(y) \;\le\; C^f\left(\frac{b^{15/2}}{Mw^m}\right)^{Kf} \;\le\; b^{-12kf} \tag{136.5}$$

at all sufficiently large nodes of the same shorter interval. Here w is the geometric mean of the weights in Lemma 133.8. The constants and the threshold are independent of T, j and the particular choice of the prime in that lemma's interval.

Proof. The norm of the different generator is $|\operatorname{disc} E|$, so $(\prod_v \kappa_v)^{1/f} = Q_E$. This quantity is at least one at the stated nodes. The respective values of g^{-1} are

$$\frac{Q_E^{2(n-j)}}{MN}, \qquad \frac{Q_E^{2(n-j)+1}}{MN}.$$

Each exponent is at most $m-1$. The lower endpoint for N is $w^m Q_E^{m-1} b^{-15/2}$. Thus both values are at most $b^{15/2}/(Mw^m) \le b^{-25/2}$, since $M \ge b^{20}$ and $w \ge 1$. Lemma 136.3 gives the first inequality. For the second, $n \le \sqrt{x}$ and $k \ge x^5 - 2\sqrt{x}$ imply $n^7/k \longrightarrow 0$. Hence $(25/2)(k-n^7) - \log C/\log b \ge 12k$ eventually.

The first scale pairs the finite argument $y = (p^{-1} - (p')^{-1})/M$ with the difference of the inverses of the normalized real entries $b_{j,v}\Phi_v p$ and $b_{j,v}\Phi_v p'$. The second arises when sampling a common entry with the additive character defined by the trace divided by δ_E. Indeed at a complex place that trace contributes $2\Re(z\,\Phi_v y/\Phi_v\delta_E)$. In the rescaled variable $b_{j,v}z$, the wave $u \mapsto \exp(u-\bar{u})$ has coefficient $2\pi i\,\Phi_v y/(\Phi_v(\delta_E)b_{j,v})$, up to the choice of the sign in the Fourier transform. At a finite prime the annihilator of the integers is the integers for this pairing, by the definition of the inverse different. □

Corollary 136.5 (Matrix bound for unequal last pivots)**.** *Let P be any finite set of distinct possible nonzero last pivots at a fixed arm position j. Choose T such that $y_{p,p'} = (p^{-1} - (p')^{-1})/M$ is integral outside T for the pairs under consideration. Form the matrix on P with entries $W_{T,c,K}(y_{p,p'})$ for those unequal pairs and zero elsewhere, using $c_v = M/b_{j,v}$. Its operator norm on $\ell^2(P)$ is at most b^{-12kf}. This bound is uniform in the number of pivots and holds also after deleting any set of entries. For the pairs occurring together at the unused marks of Proposition 134.12, one may exclude the primes over those marks from T.*

Proof. For each fixed p, distinct $p' \neq p$ give distinct nonzero $y_{p,p'}$. The same injectivity holds in a fixed column. Thus all row sums and all column sums of the nonnegative matrix are at most the sum in (136.5). If A is its matrix and this bound is e, Cauchy–Schwarz gives $|(Ah)_p|^2 \le e\sum_{p'} A_{p,p'}|h_{p'}|^2$. Summation in p gives $\|Ah\|_2^2 \le e^2\|h\|_2^2$. Deleting entries decreases the row and column sums. Finally the two pivots at an unused mark are congruent units in each component, so their inverses are congruent there as well. At each prime over that mark M has valuation one, since both old primes split completely. Division by M therefore leaves a local integer. □

Estimate (136.5) controls the arithmetic weight. Applying it to the signed integrals in (134.13) requires estimates for the Fourier coefficients of those integrals at these same frequencies.

136.3 Mixed Bessel periods and the discrete-series mass

There are two tests in the outer product of an orbital integral. At every real place retain the full discrete-series vector F_v in the first position and the cut $F_v^\flat$ in the second. Let G_v be the real unitary group at the embedding v of the totally real field F, and set

$$I_v^{\mathrm{mix}}(g) = \int_{G_v} F_v(y) F_v^\flat(g^{-1}y)^* \, dy.$$

Both test vectors and the Haar measure are the ones in Theorem 132.14.

Lemma 136.6 (Absolute convergence on the real open cell)**.** *The integral defining $I_v^{\mathrm{mix}}(g)$ converges absolutely on the whole cell with nonzero leading arm minors, at the nodes of the shorter interval. Its absolute integral is locally bounded there.*

Proof. First take $g = w_{\mathbf{p}}\sigma$ in the normal form of Lemma 134.7. In $y = guc$, integrate first over $u \in U_v$ and then over $\dot c \in U_v\backslash G_v$. On the support of the second test we may take c in a fixed compact set. Indeed the support has bounded $\inf_{s\in S_v} \|sc\|_{\mathrm{op}}$, the subgroup S_v/U_v is compact, and a bounded set in this unitary matrix group has compact closure. Its image in the quotient has finite measure.

We bound the measure of u for which the length $\ell(guc) \le R$, $R \ge 1$. For any such u there is $u' \in U_v$ with $\|u'gu\|_{\mathrm{op}} \le C_1(1+R)$. This follows by writing the element in the infimum as a compact factor times a unipotent and using boundedness of c^{-1}. Order the basis in upper flag order, reversing the order on the dual arm. Then $g^{-1}u'g$ is lower block unitriangular, and u is upper block unitriangular. Both remain unitriangular when the coordinates of the middle block are listed separately. In $A = g^{-1}(u'gu)$, the entry u_{ij} for $i \le j$ equals the minor with rows $1,\dots,i$ and columns $1,\dots,i-1,j$. Indeed the first i rows of the lower factor only use its first i columns, whose leading determinant is one. The indicated minor of the upper factor is u_{ij}. Thus $\|u\|_{\mathrm{op}} \le C_g(1+R)^r$, with the fixed changes of basis absorbed in C_g.

The radical has $2a+1$ diagonal blocks, so $\log u$ is a polynomial of degree at most $2a$ in $u-1$. Haar measure in these logarithm coordinates is a constant times Lebesgue measure on its real Lie algebra. To see this, choose coordinates adapted to the lower central series. By the finite Baker–Campbell–Hausdorff formula, translation adds a constant in the first quotient and has identity diagonal blocks in each later quotient, with polynomial additions from earlier quotients. Its Jacobian is one. Since the real dimension is at most r^2, the measure under consideration is $O_g((1+R)^{2ar^3})$. The bounds just obtained are uniform for $g = w_{\mathbf{p}}\sigma$ in a compact part of the normal-form parameters.

The full vector is bounded by a constant at the node times $\max(1,\ell/b^{6n})^{-3r^3 f}$, by Theorem 132.14. The interval has $f \ge n = a+1$. Summation over dyadic lengths now converges because $3r^3 f > 2ar^3$. The second test is bounded and its norm is constant on the unipotent fibers. This proves the asserted integrability. General elements of the cell differ from their normal forms by left and right unipotents. Equivariance and Haar invariance leave the absolute integral unchanged, and the normal-form parameters vary continuously on the cell. This proves the local bound. □

Theorem 136.7 (One full test in the arithmetic link sum)**.** *Use the same finite set $\mathcal{Q}^a$ of heads and the same terminal set $\mathcal{T}$ as in (134.10) and (134.13). For any of the masks in Sections 134.6 and 136.1, let E_{cut} denote its cut energy and form E_{mix} by replacing I_v by I_v^{mix} at every real place in these formulas. On the same fields and interval,*

$$B_0^{|\Xi|/2}\left|\sqrt{E_{\mathrm{mix}}} - \sqrt{E_{\mathrm{cut}}}\right| \le b^{-k^3 f/3}\sqrt{\mathcal{M}} \qquad (|\Xi| < f/4) \tag{136.6}$$

at all sufficiently large nodes. The finite orbital factors, the sphere densities and the fixed-vector ratios (136.1) *are the same for the mixed expression.*

Proof. Let F be the product of the local full tests and write $F^\flat(y) = \alpha(y)F(y)$, $0 \le \alpha \le 1$, for the product cut. At $z = \gamma^{-1}y$, the difference between the mixed outer product and the cut outer product is $(1-\alpha(y))\alpha(z)F(y)F(z)^*$. Its norm is at most the full-to-cut error norm, since $(1-\alpha(y))\alpha(z) \le 1-\alpha(y)\alpha(z)$. Theorem 132.17 estimates the sum of these absolute errors with the two nonnegative class weights.

Restrict the rational terms to the displayed head and terminal sets, and group them within each (c, s, c') by their middle isometry. These restrictions are invariant under the two rational unipotent actions used to unfold the sum, by Lemma 134.7. Every nonzero cut term is retained. Groups in which the complete cut integral is zero may also be retained, assigning it value zero. In a block the sum over all groups of their error norms is at most $E_{csc'}/v_U$ in the notation of Theorem 132.17. The sum of their squared norms is at most its square. Hence the difference of the two square roots is at most $\sqrt{|\mathcal{C}|}\, b^{-k^3 f/2}$ by Corollary 132.18. This argument applies to every residue mask in the statement, since it only deletes terms from a sum of nonnegative errors. Now $B_0^{|\Xi|/2} \le b^{3kf/4}$ and $\mathcal{M} \ge D_{\mathrm{id}} \ge |\mathcal{C}|$ by Lemma 134.3 at large nodes. This proves (136.6).

For the identification of the mixed expression, unfold each rational double orbit exactly as in Theorem 134.8. The absolute row bound just used, together with absolute convergence of the cut row, justifies removing the two partitions and regrouping. Both tests transform by the same unipotent character, so these changes of variables give the same finite character factors. Lemma 136.6 permits the individual real integrals and their products. Finite levels and Haar measures are as before. Left and right translations by the compact middle frame fixer act on the two real factors by the same unitary actions as for the cut tests. Consequently the Gram products of the archimedean matrices still depend on the terminal entry and the two heads. The finite sphere disintegration of Theorem 134.10, applied to this finite list of matrices, gives precisely the claimed mixed expression. □

At one real place let $R(x)$ act by right translation in the vector-valued quotient Hilbert space of the unipotent character, and put $C_Z(x) = \langle Z, R(x)Z\rangle$, linear in the second argument. Write $d_{\pi,v}$ for the formal degree in the specified real Haar measure and $d_v = \|F_v\|^2$.

Theorem 136.8 (The mixed position integral and its tail)**.** *Set $D_\pi = \prod_v d_{\pi,v}$, $d_B = \prod_v d_v$, and let $\mathcal{B} = \{(x_v) : \max_v \|x_v\|_{\mathrm{op}} \le b\}$. The integral*

$$J_{\mathrm{mix}} = \int_{\prod_v G_v} \prod_v \overline{C_{F_v}(x_v)}\, C_{F_v^\flat}(x_v)\, dx$$

is absolutely convergent and satisfies

$$(1-b^{-r^3 f})^{2f}\, \frac{d_B^2}{D_\pi} \le J_{\mathrm{mix}} \le \frac{d_B^2}{D_\pi}, \tag{136.7}$$

$$\int_{\mathcal{B}^c} \prod_v |C_{F_v}(x_v) C_{F_v^\flat}(x_v)|\, dx \le b^{-r^8/2} \frac{d_B^2}{D_\pi} \tag{136.8}$$

for all sufficiently large nodes.

Proof. Write $c_v(x) = \langle w_v, \pi_v(x) w_v\rangle$ for the unit discrete-series coefficient defining the full test; here w_v denotes a vector. Proposition 131.6 and normalization give $C_{F_v}(x) = d_v c_v(x)$. In the quotient representation set

$$P_v = d_{\pi,v} \int_{G_v} \overline{c_v(x)} R(x)\, dx.$$

The coefficient is integrable by that proposition. More explicitly this integral defines a bounded operator by integration on each vector. It is an orthogonal projection. Indeed $p = d_{\pi,v}\overline{c_v}$ satisfies $p^* = p$ since $c_v(x^{-1}) = \overline{c_v(x)}$, and $p * p = p$ by Schur orthogonality: integrating $c_v(h)c_v(h^{-1}x)$ in h gives $c_v(x)/d_{\pi,v}$. Conjugating gives the stated convolution identity. Integration in any unitary representation respects convolution and adjoints by absolute integrability. The reproduction identity in Theorem 132.14 gives $P_vF_v = F_v$. Thus

$$\int_{G_v} \overline{C_{F_v}(x)} C_{F_v^\flat}(x)\,dx = \frac{d_v}{d_{\pi,v}}\,\|P_vF_v^\flat\|^2.$$

The norm on the right is at most $\sqrt{d_v}$ before squaring; it is at least $(1-b^{-r^3f})\sqrt{d_v}$ by contraction and the local cut error. Absolute convergence follows also from $|C_{F_v^\flat}| \le d_v$. Multiplying proves (136.7).

Put $P = 9r^2$. Lemmas 132.8 and 132.13 give $|c_v(x)| \le \min(1, (2/\|x\|_{\mathrm{op}})^{r^8})$ and $d_{\pi,v}\int (2/\|x\|_{\mathrm{op}})^P dx \le b^{38r^2}$. Hence $d_{\pi,v}\int |c_v| \le b^{38r^2}$; on $\|x\|_{\mathrm{op}} > b$ the latter bound acquires the additional factor $(2/b)^{r^8-P}$. Cover the complement of $\mathcal{B}$ by the f events where one coordinate exceeds b. After dividing by d_B^2/D_π the result is at most $fb^{38r^2f}(2/b)^{r^8-9r^2}$. Here $f \ll_{q_0,d} x^{12}$, $r \ge x^5$, and $b \to \infty$. In particular $fr^2 = o(r^8)$ and $2/b \le b^{-3/4}$ eventually. This proves (136.8). □

The position integral is now evaluated for tests that also occur in the arithmetic energy. Passing from the discrete head sums and their finite orbital factors to this integral still requires control of the intermediate correlations.

136.4 Fourier coefficients of the unramified terminal density

Fix a finite place w of the totally real field. Write O for its integers, π for a uniformizer, $q = |O/\pi O|$, and A for the integers of the quadratic algebra $E \otimes_F F_w$. Assume that N is a unit, the middle lattice is unimodular and A is split or unramified. Put $\eta = 1$ in the split case and $\eta = -1$ in the unramified field case, and set $e = \eta^k$. Give A and O additive volume one. Let δ_w be the integral density of the rank k middle norm Q, with target measure of volume one on O, as in Theorem 134.10. As a function of the terminal entry on the compact additive group A, consider

$$H_w(t) = \delta_w\big((t\bar{t} - N^2)/N\big), \qquad t \in A.$$

The extension by zero to the quadratic local algebra incorporates the integrality of t. This is the terminal factor at the standard places outside S in (134.13). At the unused marked places it is the entire dependence on t in Corollary 134.13 for compatible heads.

Theorem 136.9 (Fourier expansion at a standard finite place)**.** *For $k \geq 4$, H_w is continuous, and its coefficient at the trivial character of A is*

$$\widehat{H}_w(0) = 1 - \eta^{k+1} q^{-(k+1)}. \tag{136.9}$$

For a nontrivial additive character ξ of A, let L be the least integer with $\xi|_{\pi^L A} = 1$. Then

$$|\widehat{H}_w(\xi)| \leq 4q^{-kL}, \qquad \sum_{\xi \neq 0} |\widehat{H}_w(\xi)| \leq 8q^{2-k}. \tag{136.10}$$

The coefficients here are integrals against $\overline{\xi(t)}$. These formulas apply also in residue characteristic two.

Proof. For $s \in O$ the density is

$$\delta_w(s) = (1 - eq^{-k}) \sum_{h \geq 0} e^h q^{-(k-1)h} \mathbf{1}_{s \in \pi^h O}. \tag{136.11}$$

We verify this including the value at zero. Modulo π the norm has $q^{2k-1} - eq^{k-1}$ solutions at a unit target and $q^{2k-1} + e(q-1)q^{k-1}$ at zero. Indeed a nonsingular residue Hermitian form splits as an orthogonal sum of unit lines. In the field case this follows by choosing a vector of nonzero norm and passing to its orthogonal complement. A form all of whose norms vanish has all its pairings zero, by varying the scalar in one coordinate and using the nondegenerate trace pairing. In the split case take dual bases in the two components. For a nontrivial residue additive character the sum over a single line is ηq: in the split case sum over the second variable in uv, and in the field case each nonzero norm has $q + 1$ preimages. Additive orthogonality gives the indicated counts.

Each nonzero residue vector is nonsingular for the scalar norm equation, since the trace pairing is perfect. Each lift of a prescribed next target digit then has q^{2k-1} choices of the next vector digit. For a unit s the density is $1 - eq^{-k}$. For $s \in \pi O$, the primitive part has density $1 + e(q-1)q^{-k} - q^{1-2k}$. The vectors in πA^k contribute only for $s \in \pi^2 O$, and there they contribute $q^{2-2k}\delta_w(s/\pi^2)$ by scaling the vector and target. The first two terms of the series (136.11) for $s \in \pi O$ sum to this primitive part, since $e^2 = 1$. The terms from the third onward are its stated recursive remainder. This proves the series for each nonzero s. At zero one can use the finite congruence densities $q^j \Pr(Q(x) \in \pi^j O)$. Their primitive part is the same for $j \geq 1$, and for $j \geq 2$ their remaining part is q^{2-2k} times the density at precision $j - 2$. The two starting densities are finite and the ratio is less than one. Taking the limit gives the same sum at zero. The series converges uniformly and its terms are locally constant.

For $h \geq 1$ let $V_h = \{t \in A : t\bar{t} \equiv N^2 \bmod \pi^h\}$. Its measure is $(1 - \eta/q)q^{-h}$. In the split case there are $q - 1$ pairs modulo π with the prescribed nonzero product, and in the field case there are $q + 1$ such elements. Each solution has q lifts for a specified next norm digit. Indeed the differential of the norm at a unit is its unit norm times the trace after dividing the increment by that unit. The residue trace is surjective for either algebra. Dividing these counts by $|A/\pi^h A| = q^{2h}$ gives the measure. Substitute $s = (t\bar{t} - N^2)/N$ in (136.11) and integrate over A. One obtains

$$(1 - eq^{-k}) \left(1 + (1 - \eta/q) \sum_{h \geq 1} (eq^{-k})^h \right) = 1 - \eta e q^{-(k+1)}.$$

For nontrivial ξ, its integral against the constant term or against $\mathbf{1}_{V_h}$ for $h < L$ vanishes by translation on $\pi^h A$. For $h \geq L$ the absolute value of the latter integral is at most the measure of V_h. Thus the coefficient is at most

$$|1 - eq^{-k}|(1 - \eta/q)\frac{q^{-kL}}{1 - q^{-k}} \leq 4q^{-kL}.$$

At most q^{2L} characters have conductor exponent L, since they descend to $A/\pi^L A$. Summing over $L \geq 1$ proves (136.10). Fourier inversion here can be seen without a limiting inversion hypothesis: average H_w on cosets of $\pi^j A$ and use finite character orthogonality. These averages converge uniformly to H_w, and their finite expansions exhaust the absolutely summable coefficients. □

Corollary 136.10 (The product and the denominator of a terminal frequency)**.** *For any set $\mathcal{V}$ of the standard places just described, the product of H_w on $\prod_{w \in \mathcal{V}} A_w$ has an absolutely convergent character expansion. Its constant coefficient is $\prod_{w \in \mathcal{V}} (1 - \eta_w^{k+1} q_w^{-(k+1)})$. Relative to this coefficient, the sum of the absolute values of all other coefficients is $O(f 2^{2-k})$, uniformly on the shorter interval. For a character given by a nonzero rational frequency y in the trace pairing divided by δ_E, let T consist of the primes of E over the finite set of places where this character on A_w is nontrivial in $\mathcal{V}$. The absolute value of its coefficient divided by the constant coefficient is at most*

$$I_T(y)^{(k - n^7)/2} D_T(y)^{-3} \tag{136.12}$$

at sufficiently large nodes, with the notation of Section 136.2.

Proof. The local mean is positive and at least $1 - 2^{-k-1}$. After division by the means the nonconstant coefficients have sum $O(q_w^{2-k})$. At most f places lie over any rational prime. For $k \geq 5$, summing m^{2-k} over integers $m \geq 2$ costs $O(2^{2-k})$, by comparison with the integral on $[2, \infty)$ for the terms after $m = 2$. Hence $\sum_w q_w^{2-k} = O(f 2^{2-k}) = o(1)$ at the given nodes. The product of one plus each local bound is at most the exponential of their sum. Expanding finite products and taking the uniformly absolutely convergent limit gives the asserted relative sum and constant coefficient.

Locally the trace pairing divided by the different generator has A_w as its own annihilator. Thus the exponent L is the least integer for which $\pi^L y \in A_w$. If D_w denotes the product of the denominator norms over this place, then $q_w \leq D_w \leq q_w^{2L}$ for a nontrivial character. In the split case one adds the two nonnegative denominator exponents. In the unramified field case the residue norm is q_w^2. By (136.10) the normalized coefficient is at most $8q_w^{-kL} \leq 8D_w^{-k/2}$. Since $D_w \geq 2$ and $n^7 \geq 12$ at large nodes, this is at most $D_w^{-(k-n^7)/2-3}$. The signed index at this place is at least D_w^{-1}, directly from the valuations. Multiplication over the nontrivial places proves (136.12). □

For a compatible pair of heads, (136.12) supplies the finite part of the frequency weight (136.4) at all these places. At a selected mark with a singular minor the test also contains the signed unipotent orbital integrals, so their Fourier coefficients have to be controlled in addition to this density.

136.5 Propagation of a fixed vector at a singular mark

Fix a selected split mark w of residue order q. Work in the component where the fixed vector was prescribed. Its group is $\mathrm{GL}(V)$ over $\mathbb{F}_q$, with the paired inverse adjoint determining the other component. Use the frame $f_i = e_i$ for $i \le a$, $f_n = z$, $n = a+1$, and $T_i = \langle f_1, \dots, f_i \rangle$, $T_0 = 0$. Thus $T_a = L$. Let $\bar{U}$ be the residue unipotent group, and let $\bar{\chi}$ be its level character. Call a residue s *compatible* if

$$\bar{\chi}(u) = \bar{\chi}(s^{-1}us) \quad (u \in \bar{U} \cap s\bar{U}s^{-1}).$$

Lemma 136.11 (Character compatibility on the residue double orbit)**.** *If $K_w(s;g) \neq 0$ for any local orbital parameter g, then s is compatible. For any compatible s there is a linear map A on V such that, for $i < n$,*

$$Af_i - f_{i+1} \in T_i, \qquad Asf_i - sf_{i+1} \in sT_i.$$

This holds even if some of the leading arm minors of s vanish.

Proof. In the local integral of Theorem 134.8, change the two unipotent variables by integral lifts of $u_0, v_0 \in \bar{U}$. Surjectivity of reduction follows from the split upper block matrices. The principal kernel has trivial character, so this gives

$$K_w(u_0 s v_0^{-1}; g) = \overline{\bar{\chi}(u_0)}\, \bar{\chi}(v_0)\, K_w(s;g).$$

Taking $u_0 = sv_0s^{-1}$ gives the necessity of compatibility.

Write $\bar{U} = 1 + \mathfrak{n}$, where $\mathfrak{n}$ consists of the strictly upper blocks in the chosen component, including the strict upper entries on the arm lines. The character of $1 + X$ is of the form $\psi(\Lambda(X))$ for a nontrivial additive character ψ of $\mathbb{F}_q$ and a linear functional Λ. Here is its relevant part. The frame character in Section 130.3 sums the consecutive frame entries in both components under the trace. Its coefficient in the selected component, denoted by c_0, is nonzero: M has valuation one and the trace divided by the different has conductor the integer ring. The contribution to $\Lambda(X)$ from an entry taking a later block into the line e_i is c_0 times the coefficient of e_i in Xf_{i+1}. Indeed the other component is the inverse adjoint. On the first off-diagonal blocks it reads minus the adjoint of X, all higher powers having zero such blocks. Its frame uses entries on the opposite arm and hence has zero coefficient on the entries under consideration. These statements also apply with the two components interchanged.

For $X \in \mathfrak{n} \cap s\mathfrak{n}s^{-1}$, apply compatibility to $1 + tX$ for all $t \in \mathbb{F}_q$. Since ψ is nontrivial, $\Lambda(X) = \Lambda(s^{-1}Xs)$. The two functionals therefore extend to one linear functional on the sum of the two matrix spaces, and a basis extension makes it a functional on all endomorphisms. Write it as $X \mapsto \mathrm{tr}(BX)$ by the perfect matrix trace pairing. Testing maps from each later block into e_i determines the part of Be_i modulo T_i as $c_0 f_{i+1}$. Testing their conjugates by s gives the same assertion for $s^{-1}Bs$. Taking $A = c_0^{-1}B$ proves the claim. □

Theorem 136.12 (A chain of common vectors and the resulting valuations)**.** *Suppose s is compatible and maps a vector of $\lambda f_j + T_{j-1}$ into $\lambda f_j + T_{j-1}$, where $\lambda \neq 0$ and $1 \le j \le n$. In particular this holds if s fixes the prescribed vector of top index j. For every $j \le i \le n$, the two affine sets $\lambda f_i + T_{i-1}$ and $s(\lambda f_i + T_{i-1})$ intersect. The resulting common vectors at the successive levels are linearly independent modulo T_{j-1}.*

Let $\mathfrak{l}$ be the prime of E for the selected component. For any integral local lift with residue s, its integral leading frame determinants satisfy

$$\mathrm{ord}_{\mathfrak{l}} D_i \ge i - j + 1 \quad (j \le i \le a), \qquad \mathrm{ord}_{\mathfrak{l}}(D_n + ND_a) \ge n - j + 1. \tag{136.13}$$

We put $\operatorname{ord}(0) = +\infty$. *In particular the actual head and terminal entry of any contributing orbital factor at this residue satisfy* $\operatorname{ord}_{\mathfrak{l}}(D_a(\theta + N)) \geq n - j + 1$.

Proof. Start with a common vector in the two stated affine sets at j. The map of Lemma 136.11 carries any such common vector at $i < n$ to one at $i+1$: the lower terms on either side map into the preceding levels of the next affine sets, and the top coefficient is still λ. Iteration produces the vectors claimed. Distinct top indices give independence modulo T_{j-1}.

Reduce the entire matrix $C = (h(f_i, gf_l))_{i,l \leq n}$ of an integral lift g in the chosen component. If $j \leq i \leq a$, the source coordinates of the common vectors of levels $j, \dots, i$ give independent vectors in the first i columns. Their images under s lie in L, so pair to zero with every frame row. Thus the reduction of the leading i-square has nullity at least $i - j + 1$. For the last common vector, the image is $\lambda z + \ell$, $\ell \in L$. Pairing with the frame rows gives zero apart from the last entry, which is $-N\lambda$. Add N to the last diagonal entry of C. The modified residue matrix therefore kills the source coordinate vectors of all $n - j + 1$ levels, including the last one with terminal coefficient λ. Its determinant is $D_n + ND_a$ by expansion in that entry. An integral square matrix with an h-dimensional residual kernel has determinant divisible by $\mathfrak{l}^h$: extend a basis of that kernel to a residue basis and lift the basis change to an invertible integral matrix. The h corresponding columns then belong to $\mathfrak{l}$. This proves (136.13). In a contributing local integral some unipotent translates give an integral lift of the residue. They preserve the leading determinants, and $D_n = D_a\theta$ by Lemma 134.7. □

Corollary 136.13 (Terminal fixers suffice at the selected singular marks)**.** *Let j_w be the top index of the chosen vector at each selected mark, and let Ξ_z replace it by z in the same component. For the cut or mixed energies restricted to a singular arm minor at each selected place,*

$$E_{\mathrm{sing}}(\Xi) \leq \left(\prod_{w \in \Xi} q_w^{n - j_w} \right) E_{\mathrm{sing}}(\Xi_z). \tag{136.14}$$

In particular the product costs at most $b^{11a|\Xi|}$ at the same nodes. The empty choice incurs factor one.

Proof. Within a fixed double orbit of $\bar{U}$, the squared norm in the sphere formula is constant as a function of the residue, with all other variables fixed. The reason is the first identity in the proof of Lemma 136.11: its scalar has modulus one and is common to every head in the sum. A double orbit that contributes is compatible. The singular-minor requirement is constant on the orbit because the leading minors are unipotent invariants.

A uniform element of the double orbit of s is obtained as usv^{-1} for independent uniform $u, v \in \bar{U}$. All fibers have equal cardinality by transitivity. For a fixed vector d of top index j and coefficient λ, its $\bar{U}$-orbit is exactly $D = \lambda f_j + T_{j-1}$. For an arm line this follows from the independent upper entries in its column, and for the terminal vector from the free middle-to-arm shears. The probability of fixing d on this double orbit equals $|D \cap sD|/|D|^2$. If this is nonzero, put $h_i = \dim(T_i \cap sT_i)$. The intersection of affine spaces is a translate of $T_{j-1} \cap sT_{j-1}$; the probability is thus $q^{h_{j-1} - 2(j-1)}$. By Theorem 136.12 there is also a match at $z + L$, after division by λ. Its corresponding probability is $q^{h_a - 2a}$. For $j \leq a$, the independent common vectors of levels $j, \dots, a$ give $h_a \geq h_{j-1} + a - j + 1$. The ratio of the two probabilities is therefore at most $q^{a-j+1} = q^{n-j}$. For $j = n$ the two probabilities are equal by the affine orbit formula itself.

The inequality holds separately on each double orbit of positive weight, and trivially on those with no fixed d. Sum the nonnegative contributions, then apply the same comparison

successively at all selected places. Each replacement preserves the requirements at the other places. Finally $q_w \le b^{11}$ and $n - j_w \le a$. □

Thus a bound for terminal fixers can be applied to all prescribed frame vectors with the explicit loss in (136.14). The signed orbital factor at a singular terminal fixer remains in this bound.

136.6 Transversality of the full isotropic arm

At a split mark w write $A = \mathcal{O}_{F_w} \times \mathcal{O}_{F_w}$ and $\bar{A} = \mathbb{F}_q \times \mathbb{F}_q$. Let P_w be the set of residues for which the full a-by-a arm determinant D_a is a unit in both components. Thus P_w contains O_w of Section 136.1. We calculate the integrated factor on P_w while allowing the leading minors of orders strictly less than a to vanish modulo the mark.

Let V_0 be the radical for the three blocks $L, W, L^\vee$, and let U_A be the subgroup of U acting within the two arms and trivially on W. Thus $U = V_0 \rtimes U_A$. For an invertible arm matrix B over the paired algebra, let w_B carry L to $L^\vee$ by B in their dual bases, carry $L^\vee$ back to L by the inverse adjoint, and act identically on W. For $h, h' \in U_A(F_w)$ write $B(h, h'; \mathbf{p})$ for the arm matrix determined by $h w_{\mathbf{p}} (h')^{-1} = w_{B(h,h';\mathbf{p})}$. Define

$$L_w(\bar{B}, \mathbf{p}) = \int_{U_A(F_w)^2} \mathbf{1}_{B(h,h';\mathbf{p}) \in \mathrm{GL}_a(A),\ B(h,h';\mathbf{p}) \bmod \mathfrak{p}_w = \bar{B}} \overline{\chi_w(h)} \chi_w(h')\, dh\, dh'.$$

The arm measures give their principal integral kernels volume one.

Theorem 136.14 (The marked factor with transverse arms)**.** *Let $U_w(\mathbf{p}, \mathbf{p}', \theta)$ denote the entire local factor of the expanded sphere expression (134.13) when $s_w \in P_w$ and no vector is fixed at w. With H_w from Theorem 136.9, extended by zero off A, one has*

$$U_w(\mathbf{p}, \mathbf{p}', \theta) = |\bar{V}_0|^2\, |\mathrm{GL}_k(\mathbb{F}_q)|\, H_w(\theta) \sum_{\bar{B} \in \mathrm{GL}_a(\bar{A})} L_w(\bar{B}, \mathbf{p}) \overline{L_w(\bar{B}, \mathbf{p}')}. \tag{136.15}$$

Imposing a fixed frame vector d multiplies this expression by $R_w(d, \theta)$ of (136.1), with value zero if $\theta \notin A$. The multiplier is independent of both heads. Both formulas continue to hold with $\bar{B}$ restricted to any fixed subset, using that same subset in the sum.

Proof. We first verify the three-block decomposition used over the local ring. An integral isometry whose residue lies in P_w factors uniquely as

$$v_1\, w_B x\, v_2^{-1} \quad (v_i \in V_0(\mathcal{O}_{F_w}),\ B \in \mathrm{GL}_a(A),\ x \in \mathrm{U}(W)(\mathcal{O}_{F_w})).$$

To see this, identify the chosen component with a general linear group and put $J = w_{1_a}$. In $J^{-1} g$ the leading diagonal block of size a is invertible: its determinant is the given arm pairing in that component up to a unit from the bases. The leading square of size $a + \dim W$ is also invertible. By the complementary-minor identity, its determinant is, up to a unit, the determinant of the final a-square of $g^{-1} J$. The latter is the arm pairing in the paired component up to a unit, since the paired action is inverse adjoint. The middle Gram matrix is unimodular here because N is a unit. Block elimination now gives a lower unitriangular factor, a block diagonal factor, and an upper unitriangular factor, all integral with integral inverses. In detail, clear the first block column and row with the inverse of the first pivot block. The determinant of the next pivot block is the ratio of the two invertible leading determinants. Clear the second column and row in the same way. All choices are forced. Multiplying back by J gives the displayed factorization, including its paired component by inverse adjunction. Exactly the same proof works over the residue field, and the factors of any integral lift reduce to its unique residue factors.

Write the two unipotents in the integral defining K_w in the form $v_1 h, v_2 h'$. The product measures give the prescribed Haar normalization: unipotent conjugation by U_A preserves Haar on V_0 (its determinant on the Lie algebra is one), and the principal subgroup is the semidirect product of the two principal subgroups. Since h, h' act trivially on W, the middle of the resulting product is $w_{B(h,h';\mathbf{p})} x$. Write the residue decomposition as $\bar{v}_{1,s} w_{\bar{B}_s} \bar{x}_s \bar{v}_{2,s}^{-1}$. By the

uniqueness just proved, v_1, v_2 must belong to the integral cosets with these reductions, each of volume one. The middle factor must be integral and reduce to $\bar{x}_s$. Conversely these conditions, together with the prescribed reduction of B, suffice. This yields

$$K_w(s; w_{\mathbf{p}} x) = \overline{\bar{\chi}(\bar{v}_{1,s})}\bar{\chi}(\bar{v}_{2,s}) L_w(\bar{B}_s, \mathbf{p})\, \mathbf{1}_{x \text{ integral},\ \bar{x}=\bar{x}_s}.$$

The integral defining L_w is finite for a fixed nonzero head. Indeed the full flag decomposition over the local field at $w_{\mathbf{p}}$ has continuous inverse on the locus where its leading determinants are nonzero. In the compact set of allowed arm matrices these determinants have fixed nonzero values on the orbit, so restricting to that closed level set bounds both arm unipotents. The character is constant on each principal coset.

For fixed $\bar{B}, \bar{x}$ there are $|\bar{V}_0|^2$ distinct residues, given by freely choosing the two outer radicals. Their two character factors cancel in the product for the paired heads. The integral over the fiber of $xz = \omega - (\theta/N)z$ has volume $|\mathrm{GL}_k(\mathbb{F}_q)|$ for integral θ, ω and is zero otherwise, as proved in Corollary 134.13. Integrating the norm sphere therefore gives $H_w(\theta)$, proving (136.15).

For $d = \ell + cz$, $c = 0$ gives no fixed vector: the two inverse radical translates lie in L and the invertible arm matrix sends a nonzero vector of L into $L^\vee$. For $c \neq 0$, the two inverse radical translates independently range uniformly over $cz + L$, since V_0 itself contains the middle-to-arm shears. The equation between them through $w_{\bar{B}}\bar{x}$ forces both arm components to vanish and $\bar{x}z = z$ in the selected component. Its probability is $q^{-2a}\mathbf{1}_{\bar{x}z=z}$, independently of $\bar{B}$. The same change of variable on the split norm sphere as in Theorem 136.1 now gives precisely $R_w(d, \theta)$. This proof was for each $\bar{B}$ separately. □

Corollary 136.15 (Separation of the places with a defective full arm)**.** *Retain the original choice Ξ, $|\Xi| < f/4$. For $J \subseteq \Xi$, let $E_{\mathrm{def}}(\Xi \setminus J; J)$ be the following cut or mixed energy. At the original unselected places impose all unit pivots. At the places of J impose only $s_w \in P_w$. At every place of $\Xi \setminus J$ retain the fixed vector and impose $s_w \notin P_w$. In the energy $E(\Xi)$ of Section 136.1, or in its mixed version, one then has*

$$B_0^{|\Xi|} E(\Xi) \leq \sum_{J \subseteq \Xi} \left(\prod_{w \in J} B_0 q_w^{1-k-2a} \right) B_0^{|\Xi \setminus J|} E_{\mathrm{def}}(\Xi \setminus J; J). \tag{136.16}$$

Each factor in parentheses per place is at most b^{-4k}. The sum of these products over nonempty J is $O(fb^{-4k})$, uniformly on the interval. All the terms still have the original set of more than $7f/4$ places with full pivot congruences from Proposition 134.12.

Proof. Partition the actual residues by the subset J of selected places at which both full arm determinants are units. Each part has nonnegative energy. In its sphere integral keep the terminal entry fixed. At every $w \in J$, Theorem 136.14 identifies the complete local factor, for every pair of heads, with the one omitting the vector condition times $R_w(d_w, \theta)$. The product of these scalars can be taken outside the head sum. The resulting contribution with no vector restrictions on J is nonnegative at each terminal entry, by the squared norm in the sphere formula. Bound the product using $R_w \leq q_w^{1-k-2a}$ and sum the entries and the disjoint parts. This proves (136.16). As $q_w \geq b^{10}$, $B_0 = b^{6k}$ and $a \geq 1$, each weighted factor is at most b^{-4k}. Their nonempty subset sum is at most $(1 + b^{-4k})^{|\Xi|} - 1 = O(fb^{-4k})$, since $fb^{-4k} \to 0$. The original unselected conditions were retained in every part. □

The term with $J = \varnothing$ now has a noninvertible full arm pairing in at least one component at every selected mark. The integral on such residues, as well as the correlations of the real orbital factors at unequal heads, still enter the arithmetic second moment.

136.7 Integral positions at a marked terminal stabilizer

Fix one of the original marked places in the interval of Theorem 134.11. Work in the component in which the vector is specified. In a split basis put $G = \mathrm{GL}_r(F_w)$, $K = \mathrm{GL}_r(\mathcal{O}_w)$, and let J be the kernel of reduction from K to $\bar{G} = \mathrm{GL}(V)$, where $V = \mathbb{F}_q^r$. The residual flag has blocks of dimensions $1^a, d, 1^a$ with $d = k+1$, $r = 2a+d$ and $n = a+1$. Write $V = L \oplus W \oplus L^\vee$, with $z \in W$ the terminal vector. It is nonzero since its norm is a unit. Let Z be the inverse image of its stabilizer in $\bar{G}$. Use the unipotent group U and character $\chi = \chi_w$ of Theorem 134.8. Give J and $U \cap J$ volume one in their respective groups. The Haar measure on $U\backslash G$ is the quotient of these two measures. Both groups are unimodular.

For $y = uj \in UJ$ put $\phi(y) = \chi(u)$ and extend ϕ by zero. The character is trivial on $U \cap J$, so this definition is independent of the decomposition. For a subgroup $J \subset H \subset K$ write $\bar{H} = H/J$, choose lifts $k_d \in H$ of its elements d, and set

$$\Phi_d(y) = \phi(y k_d^{-1}), \qquad C_H(x) = \int_H \phi(hxh^{-1})\, dh.$$

In subscripts of Φ the letter d denotes a residue class. The functions are independent of the lifts, since J is normal in K. Each is right J-invariant, transforms by χ on the left under U, and has support compact modulo U.

Proposition 136.16 (Two correlations of the marked functions)**.** *For an open rational head and middle element $g = w_{\mathbf{p}}\sigma$ of Theorem 134.8 in the class-removed local coordinates, put*

$$I(g)_{d,e} = \int_G \Phi_d(y)\overline{\Phi_e(g^{-1}y)}\, dy \qquad (d, e \in \bar{H}).$$

Then $I(g)_{d,e} = K_w(de^{-1}; g)$. In particular, for two such parameters g, g',

$$\frac{1}{|\bar{H}|}\sum_{d,e\in\bar{H}} I(g)_{d,e}\overline{I(g')_{d,e}} = \sum_{s\in\bar{H}} K_w(s; g)\overline{K_w(s; g')}.$$

For every $x \in G$ the right translates of the same functions satisfy

$$C_H(x) = \int_{U\backslash G} \sum_{d\in\bar{H}} \overline{\Phi_d(y)}\Phi_d(yx)\, d\dot{y}. \tag{136.17}$$

Proof. The identity $\Phi_d(y) = \int_U \chi(u)\mathbf{1}_{uk_dJ}(y)\, du$ follows since the contributing set of u's is a coset of $U \cap J$ and k_d normalizes J. Expand the two factors of $I(g)_{d,e}$. The left cosets uk_dJ and gvk_eJ intersect precisely when $u^{-1}gv \in k_dJk_e^{-1}$. Their intersection then has volume one. For fixed nonzero rational pivots the preimage of this compact coset on the double orbit has compact unipotent coordinates, by the inverse flag elimination in the proof of Theorem 134.8. Thus the calculation also converges with both characters replaced by absolute values. Invert u and v. Haar measure is preserved, and the characters become $\overline{\chi(u)}\chi(v)$, as in the definition of $K_w(de^{-1}; g)$. Each value of de^{-1} occurs $|\bar{H}|$ times in the double sum.

For the right correlation fix d. The first factor restricts $\dot{y}$ to the image of Jk_d. Two elements jk_d represent the same point precisely when their j's differ on the left by $U \cap J$. The quotient measure on this image is the pushforward of dj on J, since $U \cap J$ has volume one. The product in (136.17) is left U-invariant. At $y = jk_d$ its first factor is one, and its second is $\phi(jk_dxk_d^{-1}) = \phi((jk_d)x(jk_d)^{-1})$ by right J-invariance. Sum the integrals over the disjoint cosets Jk_d of H. □

The integral part of the position mass at H will be written

$$A_H = \frac{\int_K |C_H(x)|^2\,dx}{\operatorname{vol}(J)\operatorname{vol}(H)^2}.$$

Let $\mathfrak{n}$ be the space of strictly upper block matrices of the above flag, with the dual arm in reverse order. Write $\bar{\chi}(1+X) = \psi(\Lambda(X))$ on $\mathfrak{n}$, as in Lemma 136.11.

Lemma 136.17 (An exact sum on the residue space)**.** *For $H = K$ or $H = Z$ one has*

$$A_H = \sum_{\mathcal{C}} \frac{1}{|\mathcal{C}|}\left|\sum_{X\in\mathcal{C}\cap\mathfrak{n}} \psi(\Lambda(X))\right|^2. \tag{136.18}$$

Here $\mathcal{C}$ ranges over the nilpotent conjugacy orbits of $\bar{H}$ on $\operatorname{End}(V)$*. The zero orbit contributes one. A contributing orbit has rank at most $2a$.*

Proof. For $y \in K$, membership in UJ is equivalent to reduction into $1+\mathfrak{n}$. Indeed $U\cap K$ surjects onto this group by lifting its upper block entries. Conversely, $y = uj \in K$ forces $u \in K$. Hence $\phi(y)$ on K is the indicated residue character extended by zero. On the residue class of $x \in K$, $C_H(x)/\operatorname{vol}(H)$ is the average of this function on the $\bar{H}$-conjugates of $\bar{x}$. Subtracting the identity translates an orbit of unipotent elements into an orbit of nilpotent endomorphisms. Every point of the orbit occurs equally often in the average. Each residue class of K has volume $\operatorname{vol}(J)$. Squaring and summing therefore gives (136.18). For $X \in \mathfrak{n}$ one has $X(L+W) \subset L$. Its image on the remaining a dimensions adds at most a more dimensions. □

Theorem 136.18 (Integral mass for the terminal vector)**.** *Uniformly at the two marks of the same fields and intervals, for all sufficiently large nodes,*

$$A_Z \le q^{20n^2}, \tag{136.19}$$

$$\frac{\int_K |C_Z(x)|^2\,dx}{\operatorname{vol}(J)\operatorname{vol}(Z)\operatorname{vol}(K)} = \frac{A_Z}{q^r-1} \le 2q^{-r+20n^2}. \tag{136.20}$$

The estimate includes matrices of all residual arm ranks. Even after multiplication by $q^a B_0$, the last bound is at most b^{-3k}, where $B_0 = b^{6k}$.

Proof. Group the nonzero nilpotent matrices by their rank ℓ and by $h = \mathbf{1}_{z\in XV}$, both invariant under $\bar{Z}$. We first count the possible X in $\mathfrak{n}$. Let $S = XV$ and $s = \dim(S\cap L)$. A basis of S can be chosen by taking s vectors in L, including z as an additional vector if $h = 1$, and then choosing $\ell - s - h$ vectors in V. This is possible since $z \notin L$. It gives at most $q^{as+r(\ell-s-h)}$ choices of S, allowing repetitions of a given space. For fixed S, the images of $L+W$ belong to $S\cap L$ and the remaining a basis vectors have images in S. This gives at most $q^{(a+d)s+a\ell}$ maps. Since $r = 2a+d$, the exponent of s cancels. Thus the total number $N_{\ell,h}$ of such X satisfies

$$N_{\ell,h} \le (\ell+1)q^{r(\ell-h)+a\ell}.$$

We next bound every relevant orbit from below. For a nilpotent X with Jordan block lengths $t_1,\dots,t_m$, its commuting algebra has dimension $\sum_{i,j}\min(t_i,t_j)$. Indeed a map between two cyclic Jordan blocks is determined by the image of a generator, which must be killed by the corresponding power of X. Writing $b_j = \#\{i : t_i \ge j\}$ turns this dimension into $\sum_{j\ge1} b_j^2$. Here $b_1 = r-\ell$ and $\sum_{j\ge2} b_j = \ell$, so the dimension is at most $(r-\ell)^2+\ell^2$. The group of invertible commuting maps is contained in this algebra. Also $|\bar{G}| = q^{r^2}\prod_{i=1}^{r}(1-q^{-i}) \ge q^{r^2}/4$. For the last inequality use $q \ge 2$, take the first two factors at $q = 2$, and bound the remaining product below by $1-\sum_{i\ge3}2^{-i} = 3/4$. Orbit–stabilizer gives $|\bar{G}\cdot X| \ge q^{2\ell(r-\ell)}/4$.

Set $D = C_{\bar{G}}(X)$. Applying orbit–stabilizer also to z under $\bar{G}$ and D yields

$$|\bar{Z} \cdot X| = |\bar{G} \cdot X| \frac{|Dz|}{q^r - 1}.$$

If $h = 0$, split $V = V_1 \oplus V_0$ into the blocks of length at least two and the blocks of length one. Each length $t \geq 2$ uses $t \leq 2(t-1)$ dimensions. Thus $\dim V_0 \geq r - 2\ell$. If z has nonzero V_0 component, $\mathrm{GL}(V_0)$ fixing V_1 pointwise already gives at least $q^{r-2\ell} - 1$ values of Dz. Otherwise $z \in V_1 \setminus XV$. Take a linear functional λ which is one on z and vanishes on $XV + V_0$. For $v \in V_0$ the maps $1 + v\lambda$ are invertible, commute with X, and send z to the distinct vectors $z + v$. This proves the same lower bound. At the nodes in question $r - 2\ell \geq 1$. When $h = 1$ we use $|Dz| \geq 1$. We have obtained the two orbit bounds

$$|\bar{Z} \cdot X| \geq \begin{cases} q^{2r\ell - 2\ell^2 - 2\ell}/8, & h = 0, \\ q^{2r\ell - 2\ell^2 - r}/4, & h = 1. \end{cases}$$

In (136.18), the sum of the squared intersection sizes for given (ℓ, h) is at most $N_{\ell,h}^2$. The contribution with $h = 0$ is therefore at most $8(\ell+1)^2 q^{2a\ell + 2\ell^2 + 2\ell}$. For $h = 1$ it is at most $4(\ell+1)^2 q^{-r + 2a\ell + 2\ell^2}$. Include the zero orbit and sum over $1 \leq \ell \leq 2a$. The result is at most $1 + 24a(2a+1)^2 q^{12a^2 + 4a}$, which is at most q^{20n^2} at all sufficiently large nodes. Finally $[K : Z] = q^r - 1$ since $\bar{G}$ acts transitively on the nonzero vectors. This proves (136.20). On our interval $q \geq b^{10}$, $n \leq \sqrt{x}$, and $k \geq x^5 - 2\sqrt{x}$. In particular $20n^2 + a = o(k)$. Thus $2q^{-r + 20n^2 + a} \leq b^{-9k}$ for all sufficiently large nodes. Multiplication by B_0 proves the last assertion. □

Corollary 136.13 costs at most q^a in passing from a selected frame vector to z on the defective arm locus. The same proof applies with the mask $s_w \notin P_w$ of Corollary 136.15, since the full arm minor is invariant under both residual unipotents. At a terminal fixer one may enlarge this residue set to all of $\bar{Z}$ in the sum of squared norms before expanding the squares. Proposition 136.16 identifies the resulting local residue pairing and its companion position correlation. Theorem 136.18 estimates the latter on K. To bound the original arithmetic second moment by this method one must also treat $G \setminus K$ and justify the comparison with the sum over rational heads and the archimedean factors.

136.8 The identity term on integral positions at an unrestricted mark

Continue with $V = L \oplus W \oplus R$, where $R = L^{\vee}$ has its basis in reverse dual order. Write L_i for the span of the first i basis vectors of L. In the notation of Lemma 136.17, every $X \in \mathfrak{n}$ has the form

$$X = \begin{pmatrix} A & B & C \\ 0 & 0 & D \\ 0 & 0 & E \end{pmatrix}.$$

Here A, E are strictly upper triangular a-squares, $C : R \to L$, $B : W \to L$, and $D : R \to W$. Denote the rows of B by β_i and the first column of D by δ. The actual residue character has phase

$$\Lambda(X) = \sum_{i<a} c_i A_{i,i+1} + c_a \beta_a(z) + \eta(\delta) + \sum_{i<a} e_i E_{i,i+1}, \tag{136.21}$$

where the c_i, e_i are nonzero and $\eta \in W^*$ satisfies $\eta(z) \neq 0$. Indeed the first split component of the frame character of Section 130.3 reads the successive left entries and $\beta_a(z)$. In the other component the isometry is the inverse adjoint. On consecutive blocks its off-diagonal part is minus the adjoint of the corresponding block, since every term of higher degree skips a block. In the reverse dual basis this reads the consecutive right entries and the pairing of the first column of D with the paired terminal vector. The two scalar coefficients coming from the conductor are units by Lemma 136.11, and the pairing of the two terminal vectors is the unit $-N$. This proves (136.21) also with the components interchanged.

For a $\bar{G}$-conjugacy class $\mathcal{C}$ of nilpotent matrices set $S(\mathcal{C}) = \sum_{X \in \mathcal{C} \cap \mathfrak{n}} \psi(\Lambda(X))$. The next estimate keeps cancellation inside each class.

Proposition 136.19 (One middle dimension saved in the class sums)**.** *For $1 \le \ell \le 2a$, $d = \dim W = k + 1$, one has*

$$\sum_{\operatorname{rank} \mathcal{C} = \ell} |S(\mathcal{C})| \;\le\; 10 q (a+1)^2 q^{4a^2 + d(\ell - 1)}. \tag{136.22}$$

Proof. Put $b' = \operatorname{rank} B$, $c' = \operatorname{rank} D$ and $T = (a+1)^2 q^{4a^2}$. The image on W has dimension b' in L, and projection of the full image modulo L has dimension at least c'. Hence $b' + c' \le \ell$. For prescribed ranks the number of pairs B, D is at most $q^{(d+a)(b'+c')}$, by factoring each map through a space of its rank. The choices for A, C, E number $q^{2a^2 - a}$. There are at most $(a+1)^2$ rank pairs, and $a(b' + c') \le 2a^2$. Thus the total count at rank ℓ is at most $T q^{d\ell}$, and the count with $b' + c' < \ell$ is at most $T q^{d(\ell-1)}$.

There is another set of size at most $T q^{d(\ell-1)}$: those matrices with η in the row space of B. In that case $b' \ge 1$. A basis for the row space can be chosen starting with η, so $b' - 1$ arbitrary covectors and at most ab' scalar coefficients suffice to specify B. The same bound for D, A, C, E as above gives the asserted count.

First consider all matrices with $\beta_a \neq 0$. Conjugation by $\mathrm{GL}(W)$, acting as the identity on the two arms, preserves this set and each class. Average the phase over these conjugations. The middle part becomes $c_a \beta_a(w) + \mu(\delta)$, with (w, μ) uniform on

$$w \neq 0, \qquad \mu(w) = \eta(z).$$

Indeed $(w, \mu) = (g^{-1} z, \eta g)$, and a basis of $\ker \mu$ together with w shows transitivity, since $\eta(z) \neq 0$. When $\delta = 0$, the average of the additive character has absolute value $1/(q^d - 1)$, by summing a nontrivial linear character over nonzero w. When $\delta \neq 0$, average first over all μ with fixed value on w. This gives zero unless $\delta \in \mathbb{F}_q w$, an event of probability $(q-1)/(q^d - 1)$. Both averages are bounded in absolute value by $2q^{1-d}$. The other entries of the phase are

unchanged. By the triangle inequality the sum of the absolute values of the class contributions from this set is at most $2qTq^{d(\ell-1)}$.

Next suppose $B \neq 0$ and its last nonzero row is $j < a$. Then $B(W) \subset L_j$. If $X(L) \not\subset L_j$, the image of $L + W$ has dimension at least $b' + 1$, which implies $b' + c' < \ell$. These matrices have the smaller count already proved. If β_j is proportional to η, it is a nonzero multiple, so the second smaller count applies.

On the rest of this set, fix B and a class. Let ξ be the coordinate of the $(j+1)$-st left basis vector, extended by zero on $W + R$. For $u \in \ker \eta$ conjugate by $1 + u\xi$, with inverse on the left. Here $Xu = Bu$, $\xi(Xu) = 0$, and ξX vanishes on $L + W$. The new matrix is

$$X' = X + (Bu)\xi - u(\xi X).$$

It belongs to $\mathfrak{n}$, preserves the value of B, and still has $X'(L) \subset L_j$. Indeed $(Bu)\xi$ takes the $(j+1)$-st line into L_j, and the second change sends only R into W. These conjugations permute the set under consideration. The phase changes by $c_j\beta_j(u)$, since the change involving the first right column is killed by $\eta(u) = 0$. The restriction of β_j to $\ker \eta$ is nonzero. Averaging over u makes the class sum on this set vanish.

It remains to treat $B = 0$. If $\ell > c'$, the first smaller count applies. Otherwise $\ell = c' > 0$. Transpose the matrix and reverse the two complete outer flags, moving the old right arm to the left. This bijection carries the matrices to the same block form on the dual space, now with B of rank ℓ and $D = 0$. It carries conjugacy classes to conjugacy classes, and interchanges the two nonzero terminal data in (136.21), up to nonzero scalars. The pairing of the new terminal data is still nonzero. The preceding averaging arguments apply on this restricted set: the middle basis changes preserve $D = 0$, and in the case $j < a$ the whole image of X equals $B(W) \subset L_j$, since both have dimension ℓ. Consequently $\xi X = 0$ and the shears also preserve $D = 0$. The exceptional row-space count holds on this subset as well.

All cancellations and basis averages were taken within a single class. Summing their bounds over classes uses the total counts above. The two middle basis averages cost at most $4qTq^{d(\ell-1)}$ together. The smaller counts from the remaining cases, including the transposed row-space condition, cost at most $4Tq^{d(\ell-1)}$. This proves (136.22). □

Theorem 136.20 (Integral mass with its identity term)**.** *At every original marked place, for all sufficiently large nodes, the full integral position mass with $H = K$ satisfies*

$$0 \le A_K - 1 = \frac{\int_{K\setminus J} |C_K(x)|^2\,dx}{\operatorname{vol}(J)\operatorname{vol}(K)^2} \le q^{-2d+20n^2}. \tag{136.23}$$

Proof. On J the conjugated function in C_K is identically one because J is normal in K. The zero orbit in (136.18) accounts for exactly this integral. At rank $\ell > 0$, every $\bar{G}$-orbit has size at least $q^{2r\ell-2\ell^2}/4$, by the commuting-algebra calculation in the proof of Theorem 136.18. The sum of the squared absolute class sums is bounded by the square of (136.22). Since $r = d + 2a$ and $2\ell^2 - 4a\ell \le 0$ for $\ell \le 2a$, the nonzero ranks together contribute at most

$$800a(a+1)^4 q^{-2d+8a^2+2}.$$

The factor apart from q^{-2d} is at most q^{20n^2} at all sufficiently large nodes. This proves the stated bound. □

Corollary 136.21 (The product of integral masses at the original marks)**.** *Let $\mathcal{W}$ be the $2f$ marked places of Theorem 134.11. At $w \in \Xi$ use the stabilizer $H_w = Z_w$ of z in the prescribed component, and at the other places take $H_w = K_w^{\mathrm{int}} = \mathrm{GL}_r(\mathcal{O}_w)$. Here K_w^{int} denotes the integral*

group at any marked place. Write $M_w(H) = A_H \operatorname{vol}(H)/\operatorname{vol}(K_w^{\mathrm{int}})$. *For the top indices* j_w *of the original selected frame vectors,*

$$B_0^{|\Xi|}\left(\prod_{w\in\Xi} q_w^{n-j_w}\right)\prod_{w\in\mathcal{W}} M_w(H_w) \le b^{-3k|\Xi|}(1+O(fb^{-19k})).$$

Proof. At a selected place apply (136.20) and $n - j_w \le a$. At every other place (136.23) gives $M_w(K_w^{\mathrm{int}}) \le 1 + b^{-19k}$, since $d = k + 1$, $20n^2 = o(k)$ and $q_w \ge b^{10}$. The product of at most $2f$ such factors is bounded by $\exp(2fb^{-19k}) = 1 + O(fb^{-19k})$, using the polynomial bound for f on the interval. □

The product here integrates independent integral positions at the original marked completions. The common rational heads and the real integrals in Theorem 134.8 also enter the projected period. An estimate for that second moment must still account for the nonintegral positions and for those correlations.

136.9 The two middle root coordinates on nonintegral positions

Fix an original split mark. Write $O = \mathcal{O}_w$, choose a uniformizer π, and take an additive character Ψ of F_w of conductor O. In a split integral basis the contribution to $\chi(u)$ of the two simple blocks adjacent to W is

$$\Psi\big((Pz + \eta v)/\pi\big).$$

Here P is the last row of the block $W \to L$, multiplied by its character unit, and v is the first column of the block $R \to W$. The primitive covector η includes the unit multiplying the opposite terminal root, and $c = \eta z \in O^\times$. These statements over O follow from the frame formula of Section 130.3 by inverse adjunction on consecutive blocks, just as for (136.21). The remaining simple entries on the two arms will be denoted by s_i, including their character units. If $y = uj \in UJ$, any two choices of u differ on the right by $U \cap J$. Consecutive blocks add under multiplication in U, so the classes of s_i, P, v modulo the corresponding πO-lattices depend only on y.

Put $t = 0$ if both P and v lie in their πO-lattices. Otherwise let $t \geq 1$ be least with $\pi^{t-1}P, \pi^{t-1}v$ integral. Write

$$R_t = O/\pi^t O, \qquad p = \pi^{t-1}P \bmod \pi^t, \qquad b' = \pi^{t-1}v \bmod \pi^t, \qquad \psi_t(h) = \Psi(\pi^{-t}h).$$

At least one of p, b' is primitive over R_t, meaning that its reduction modulo π is nonzero.

Lemma 136.22 (The finite middle average at precision t)**.** *Let $F_{K,t}$ be the normalized average of the middle character under changes of integral basis in W, acting trivially on both arms. Define $F_{Z,t}$ with the additional condition that the basis change fixes z modulo π. At $t = 0$ both averages equal one. For $t \geq 1$,*

$$F_{K,t} = \frac{\displaystyle\sum_{\alpha \in R_t^\times} \psi_t(-c\alpha - \alpha^{-1}p(b'))}{q^{td}(1 - q^{-d})}, \qquad |F_{K,t}| \leq 2q^{-t(d-1)}. \tag{136.24}$$

If b' is primitive,

$$F_{Z,t} = q^{-(t-1)d} \sum_{\substack{h \in R_t^\times \\ \bar{h}\,\bar{z} = \overline{b'}}} \psi_t(ch + h^{-1}p(b')). \tag{136.25}$$

For $t \geq 2$ the latter average is zero if b' is nonprimitive. In all cases $t \geq 1$ one has $|F_{Z,t}| \leq q^{-(t-1)(d-1)}$. For either average to be nonzero when $t \geq 2$, the two frequencies p, b' must both be primitive, and $\overline{p(b')}/\bar{c}$ must be a nonzero square in $\mathbb{F}_q$. For $F_{Z,t}$ one further needs $\overline{b'} = \bar{h}\bar{z}$ with $\bar{h}^2 = \overline{p(b')}/\bar{c}$.

Proof. The pair $(w, \mu) = (g^{-1}z, \eta g)$ for a uniform integral basis change reduces to the uniform distribution on $\mu(w) = c$ over R_t. To see transitivity, w is primitive and $R_t^d = R_t w \oplus \ker \mu$ because c is a unit. A basis of its kernel is obtained by solving for a coordinate with unit coefficient. Send w to z and a kernel basis to a basis of $\ker \eta$. This gives g over R_t, and any lift of its entries is invertible over O. If g is to fix $\bar{z}$, precisely the pairs with $w \equiv z \bmod \pi$ occur, by the same construction. A transitive action of a finite group gives uniform measure on its orbit. For each primitive w there are $q^{t(d-1)}$ covectors with $\mu(w) = c$.

For the full average insert $q^{-t}\sum_{\alpha \in R_t} \psi_t(\alpha(\mu(w) - c))$ in the unrestricted sum of $\psi_t(pw + \mu b')$ over w, μ. The insertion is the indicator of $\mu(w) = c$: the additive pairing of R_t with itself is nondegenerate by the conductor of Ψ. If b' is primitive, summing over μ requires $\alpha w = -b'$, so that α is a unit and $w = -\alpha^{-1}b'$. The surviving factor from the μ-sum is q^{td}. If p is primitive

instead, sum first over w to obtain the same expression. Dividing by $q^{td}(1-q^{-d})q^{t(d-1)}$, the number of pairs, gives (136.24). The unit sum has at most q^t terms.

For the stabilizer average fix $w \equiv z \bmod \pi$. The average of $\psi_t(\mu b')$ over $\mu(w) = c$ is zero unless $b' = hw$ for some $h \in R_t$; this follows by extending w to a basis and varying the remaining coordinates of μ. When it exists h is unique and the average equals $\psi_t(ch)$. If b' is primitive, h is a unit and $w = h^{-1}b'$. There are $q^{(t-1)d}$ possible w's before this restriction. This gives (136.25), with at most q^{t-1} terms. If b' is nonprimitive and $t \geq 2$, any surviving h lies in πR_t, and p is primitive. For every residue vector l the replacement $w \mapsto w + \pi^{t-1}l$ preserves $b' = hw$, the class of w modulo π, and h. The factor $\psi_t(pw)$ averages to zero over these replacements. At $t = 1$ the elementary bound one suffices.

For $t \geq 2$ translate a unit summation variable h by $\pi^{t-1}s$, $s \in \mathbb{F}_q$. Modulo π^t its inverse changes by $-\pi^{t-1}sh^{-2}$, since $2(t-1) \geq t$. Summing this last digit in either displayed formula gives zero unless $\bar{c} = \overline{p(b')}\bar{h}^{-2}$. This forces both primitive conditions and the asserted square condition. The calculation also holds in residue characteristic two. □

Theorem 136.23 (Middle and arm averages in the actual position correlation)**.** *For $H = K$ or $H = Z$, define on UJ*

$$D_H(y) = F_{H,t} \prod_i R(s_i), \qquad R(s) = \begin{cases} 1 & s \in \pi O, \\ -1/(q-1) & s \in O^\times, \\ 0 & s \notin O. \end{cases}$$

Extend D_H by zero off UJ. For every $x \in G$,

$$C_H(x) = \int_H D_H(hxh^{-1})\, dh. \tag{136.26}$$

In particular, retain in this last integral only contributing conjugates of depth $t \geq T$, where $T \geq 2$. Denote by $\sigma_H(x,T)$ the Haar probability in H that $hxh^{-1} \in UJ$ has $t \geq T$, integral outer simple entries, and satisfies the necessary middle conditions of Lemma 136.22. The absolute value of this part of $C_H(x)/\operatorname{vol}(H)$ is at most

$$\begin{cases} 2q^{-Tk}\sigma_K(x,T), & H = K, \\ q^{-(T-1)k}\sigma_Z(x,T), & H = Z, \end{cases} \qquad k = d-1.$$

Proof. The middle basis changes used in the lemma are contained in H and normalize U and J. The same holds for diagonal units on the outer lines, with the left line next to W and the first right line both held fixed. The successive outer diagonal ratios run independently through $O^\times$: their values recover the outer units recursively from the two fixed values. These changes scale the respective simple entries independently and preserve P, v. The middle changes preserve all the outer simple entries. Averaging a factor $\Psi(su/\pi)$ on $u \in O^\times$ gives $R(s)$, by subtracting its integral on πO from that on O, then dividing by $1 - q^{-1}$. Multiplicative Haar probability on the units equals normalized additive measure there, since multiplication by a unit preserves additive measure.

For fixed x, average $\phi(lhxh^{-1}l^{-1})$ first over the product of these two compact groups of basis changes l. Membership in UJ is preserved. On its support the middle and outer averages just computed give $D_H(hxh^{-1})$. Averaging in h gives $C_H(x)$ by left invariance on H. All integrands are bounded on compact domains. This argument also preserves t because the basis changes are invertible over O. Now apply Lemma 136.22 and $|R(s)| \leq 1$ on the indicated set. □

This calculation applies to every position outside K as well. It detects large denominators in the two simple blocks adjacent to W. A large entry in a block farther above the diagonal may occur with both of those coordinates in πO. Controlling the measure of such positions, and integrating the depth bounds against the remaining local factors, are still required for the full position mass.

136.10 Summing over nonintegral positions at the marked places

Use the split integral lattice $\mathcal{L} = O^r$ and the matrix norm for which the norm of an integral unit matrix is one and $|\pi| = q^{-1}$. For $x \in G$ let m be the integer given by $q^m = \max(\|x\|, \|x^{-1}\|)$, where the matrix norm is the maximum of the absolute values of the entries. For an F_w-linear map T on $\mathcal{L} \otimes F_w$ put

$$p(T) = \operatorname{length}_O\left((T\mathcal{L} + \pi\mathcal{L})/\pi\mathcal{L}\right).$$

This module is finite. A nonzero Smith entry of valuation e contributes $\max(1-e, 0)$ to its length, by diagonalizing over O after clearing denominators. Put $p_+ = p(x-1)$, $p_- = p(x^{-1}-1)$. Write $\mathcal{S}(m, P, Q)$ for the set on which $(m, p_+, p_-) = (m, P, Q)$.

These sets are invariant under conjugation by K and multiplication on either side by J. For the norm this follows by multiplying back by an integral inverse. For the length use $xj - 1 = (x-1)j + (j-1)$ and $jx - 1 = j(x-1) + (j-1)$, with $(j-1)\mathcal{L} \subset \pi\mathcal{L}$. Apply the same calculation to the inverse. Each shell contains finitely many cosets modulo J. Indeed the bounds on the matrix and its inverse give a compact set: the pairs of bounded matrices x, y satisfying $xy = yx = 1$ form a closed subset of a compact set, and projection to x is continuous. The cosets modulo J are open. Nonintegral positions have $m \geq 1$.

Lemma 136.24 (Counting cosets and bounding their conjugacy orbits)**.** *The number $N(m, P, Q)$ of cosets xJ in $\mathcal{S}(m, P, Q)$ meeting U is at most*

$$q^{4a^2(m+1) + d\min(P,Q)}. \tag{136.27}$$

If this number is positive then $P, Q \leq 2a(m+1)$. Every K-conjugacy orbit on G/J meeting U in this shell has cardinality at least

$$\tfrac{1}{4} q^{(d-2a)(P+Q)}. \tag{136.28}$$

Proof. Write $u = 1 + X \in U$ in the three blocks of Section 136.8. Both $u - 1$ and $u^{-1} - 1$ have rank at most $2a$, by the image calculation of Lemma 136.17, which works over F_w. All their entries have valuation at least $-m$. The assertion on P, Q follows from the Smith formula.

The cosets counted in $N(m, P, Q)$ correspond to the cosets of $U \cap J$ in the unipotent shell, hence their count equals its unipotent Haar measure under our normalization. In upper block coordinates this measure is the product of additive measures giving πO volume one: left multiplication is triangular with diagonal one if the coordinates are ordered by increasing block separation, and the product assigns volume one to $U \cap J$.

Set $h = m + 1$. Each coordinate in $\pi^{-m}O$, reduced modulo πO, has q^h possibilities of measure one. The two triangular arm blocks and the full block $C : R \to L$ together cost at most $q^{2a^2 h}$. Let s_B, s_D be the lengths of the images of B, D on their integral source lattices, modulo π times their target lattices. Then $s_B + s_D \leq P$: the image of X on W gives the B-image in L, and projection to W gives the D-image. These two contributions enter the kernel and the image, respectively, of that projection on $(X\mathcal{L} + \pi\mathcal{L})/\pi\mathcal{L}$.

The columns of B take values in a submodule of $(\pi^{-m}O/\pi O)^a$ of length s_B. The rows of D take values in such a submodule of length s_D, since transposition preserves Smith entries. There are at most $q^{a^2 h}$ choices for either submodule: its inverse image in $(\pi^{-m}O)^a$ is a free lattice of rank a, so it has a generators after reduction. Once the two submodules are fixed, their d columns and d rows together give at most $q^{d(s_B + s_D)}$ choices. This bounds the shell measure by $q^{4a^2 h + dP}$. Inversion preserves Haar measure on the unimodular group U and interchanges P, Q, giving (136.27).

For the orbit estimate take a representative $u \in U$. Inside the middle lattice choose a direct summand containing $D(R)$ over F_w of rank $\operatorname{rank} D$, by taking the saturation of its

image after clearing denominators. On an integral complement split off the kernel of B; it is a direct summand since the quotient is torsion free over a discrete valuation ring. We obtain a summand W_0 of rank $d_0 \geq d - 2a$ on which $B = 0$, with $D(R)$ in the complementary vector space. Consequently in an integral basis the matrix of u is $Y \oplus 1_{d_0}$, on a complementary summand and W_0. This integral change of basis preserves both lengths.

An element g of K stabilizes uJ under conjugation precisely when $u^{-1}gu - g \in \pi M_r(O)$, by multiplying the congruence by g on the right. Each column c_0 in the block from W_0 to the complement must satisfy $(Y^{-1} - 1)c_0 \in \pi O^{r-d_0}$, an index q^Q condition by the definition of the image length. Each row r_0 in the opposite block must satisfy $r_0(Y - 1) \in \pi O^{r-d_0}$, an index q^P condition by transposition. In additive probability on $M_r(O)$ these conditions are independent across the d_0 columns and d_0 rows. The measure of the stabilizer is thus at most $q^{-d_0(P+Q)}$. On K, additive measure restricted from the matrix ring is proportional to group Haar: multiplication by an integral invertible matrix preserves it. The additive measure of K is at least $1/4$, as in the calculation of $|\bar{G}|$ in Theorem 136.18. Dividing the two measures proves (136.28). □

On every shell contributing to the position mass one has $P = Q$, by Lemma 136.26 below. The case of separated lengths in the following estimate consequently has zero mass.

For $m \geq 1$ put $l_m = \lceil (m+1)/(2a+1) \rceil$. On a shell with $|P - Q| \geq l_m$, use the full position correlation C_H. On any other nonintegral shell restrict the integral (136.26) to the conjugates of middle depth at least l_m if $H = K$, and at least $l_m + 1$ if $H = Z$. Denote the resulting function on $G \setminus K$ by C_H^*.

Theorem 136.25 (An integrated bound on nonintegral shells)**.** *For $H = K$ or $H = Z$, on the fields and intervals of Theorem 134.11, all sufficiently large nodes satisfy*

$$\frac{\int_{G\setminus K} |C_H^*(x)|^2 \, dx}{\operatorname{vol}(J)\operatorname{vol}(H)\operatorname{vol}(K)} \leq q^{-9k/10}. \tag{136.29}$$

At a selected place this still gives a bound b^{-2k} after multiplication by $B_0 q^a$.

Proof. On the shell in question let F be D_H of Theorem 136.23 with the indicated depth restriction, extended by zero off UJ. This function is right J-invariant. The shell integral in (136.29), with its normalization, equals

$$\frac{1}{[K:H]} \sum_{\mathcal{C}} \frac{\left| \sum_{yJ \in \mathcal{C}} F(y) \right|^2}{|\mathcal{C}|}, \tag{136.30}$$

where $\mathcal{C}$ ranges over H-conjugacy orbits on the cosets in that shell. Indeed $H \subset K$ normalizes J, so its conjugation average on each such orbit is the ordinary finite average. Each coset has volume $\operatorname{vol}(J)$. For any yJ, its H-orbit has size at least the size of its K-orbit divided by $[K : H]$, by orbit–stabilizer. Thus the index in (136.30) absorbs the possible loss from passing to H.

Let ϵ bound $|F|$ on this shell. Its support uses at most $N(m, P, Q)$ cosets across all orbits. Bound the sum of the squared orbit counts by the square of the total count. Lemma 136.24 bounds (136.30) by

$$4\epsilon^2 q^{8a^2(m+1)+2d\min(P,Q)-(d-2a)(P+Q)} \leq 4\epsilon^2 q^{16a^2(m+1)-d|P-Q|},$$

since $P + Q \leq 4a(m+1)$. When $|P - Q| \geq l_m$ we may take $\epsilon = 1$. On all other shells Lemma 136.22 gives $\epsilon \leq 2q^{-kl_m}$, including $l_m = 1$.

There are at most $2a+1$ possible m's with a fixed $l_m = l$ and at most $(2a(2a+1)l+1)^2$ length pairs for each. Uniformly on the interval $16a^2(2a+1) \le k/32$ eventually, since $a < \sqrt{x}$ and $k \ge x^5 - 2\sqrt{x}$. For either kind of shell the last displayed bound is at most $16q^{-31kl/32}$. Sum first over the lengths and then over $l \ge 1$. The sum of $l^2 q^{-31k(l-1)/32}$ is bounded by an absolute constant for large k; the polynomial factor in a is absorbed by $q^{(31/32-9/10)k}$. This proves (136.29). Finally $q \ge b^{10}$, $B_0 = b^{6k}$, and $a = o(k)$, so $B_0 q^{a-9k/10} \le b^{-2k}$ eventually. □

What is left in $C_H - C_H^{*}$ has $|P-Q| < l_m$ and middle depth below the specified threshold. This difference is a correlation on each shell and can interfere with C_H^{*} there. The square root of (136.29) bounds the contribution of C_H^{*} in the triangle inequality for the full L^2 norm. An estimate for the remaining correlation is therefore still needed, as is the comparison with rational heads in the arithmetic second moment.

136.11 Positions for the original open residues

There is an exact relation between the lengths used in Section 136.10. There is also a stronger restriction on the position when the original residue test at an unselected place is retained. Both statements hold at each of the two marked primes.

Lemma 136.26 (The two image lengths and the determinant)**.** *For $x \in \mathrm{GL}_r(F_w)$ and the lengths $P = p(x-1)$, $Q = p(x^{-1}-1)$ defined above,*

$$Q - P = \mathrm{ord}_w \det x.$$

In particular every shell contributing to C_H has $P = Q$.

Proof. Let $\mathcal{L}_x = \{v \in \mathcal{L} : (x-1)v \in \pi\mathcal{L}\}$. The index $[\mathcal{L} : \mathcal{L}_x]$ is q^P by the homomorphism to $(F_w^r)/\pi\mathcal{L}$. If $v \in \mathcal{L}_x$, then $xv = v + (x-1)v$ is integral and $(x^{-1}-1)xv = -(x-1)v \in \pi\mathcal{L}$. Applying the same argument to x^{-1} proves $x\mathcal{L}_x = \mathcal{L}_{x^{-1}}$. With $\mathrm{vol}(\mathcal{L}) = 1$, comparison of volumes gives $q^{-Q} = |\det x| q^{-P}$. Every conjugate of an element of UJ has determinant a unit, proving the last assertion. □

Put $\bar{G} = \mathrm{GL}_r(\mathbb{F}_q)$. As before O_w denotes the residues with all arm pivots units in both split components. On the open double orbit write a canonical representative as $g = w_{\mathbf{p}}\sigma$, with both arm parameters and the middle matrix retained. For $\alpha, \beta \in \bar{G}$ set $s = \alpha\beta^{-1}$ and use the single-residue functions Φ_α, Φ_β of Proposition 136.16 with $H = K$. Their product at the pair of arguments is

$$F_{\alpha,\beta}(g; y) = \Phi_\alpha(y)\overline{\Phi_\beta(g^{-1}y)}.$$

Its y-integral is the original local factor $K_w(s; g)$.

Lemma 136.27 (Integral elimination on opposite flags)**.** *A lift $h \in K$ of any $s \in O_w$ has a unique decomposition*

$$h = u_L \, g_s(h) \, u_R^{-1}$$

in which $u_L, u_R \in U(O)$ and $g_s(h)$ is an integral canonical representative. Its three reductions are determined by s. In addition,

$$UJ \cap h^{-1}UJh = J.$$

Proof. Use the block ordering $1^a, d, 1^a$ with the dual arm reversed. Let w reverse the two arms, acting on the middle by the identity, so that $w^{-1}Uw = U^-$ is the lower block unipotent. The canonical representatives have the form $w\ell$ with ℓ block diagonal. For $w^{-1}h$, the initial determinants at the ends of the first a blocks are, up to basis units, the arm minors in the chosen component. The remaining proper initial block determinants are units by the complementary-minor formula: the complementary minors of $(w^{-1}h)^{-1}$ are the arm minors in the paired inverse-adjoint component, again up to units. Thus block Gaussian elimination gives a lower unitriangular matrix, an invertible block diagonal matrix, and an upper unitriangular matrix. At each stage the next diagonal block is invertible over O, since its determinant is the ratio of two successive unit determinants. Subtracting the corresponding multiples of rows and columns constructs the factors over O. The elimination is forced and commutes with reduction. Multiplying back by w gives the asserted decomposition. The same elimination over F_w proves uniqueness even if the unipotents were originally allowed to be nonintegral.

Every factor in the displayed decomposition normalizes J. The outer unipotents in $U(O)$ normalize UJ, and the middle representative conjugates U to U^-. Thus the intersection assertion reduces, up to integral unipotent conjugation, to $UJ \cap U^-J = J$. If $uj_1 = vj_2$ with

$u \in U,\ v \in U^-$ and $j_i \in J$, then $v^{-1}u \in J$. Its lower-times-upper decomposition is unique over F_w. Elimination on a matrix congruent to the identity gives integral factors reducing to the identity. Consequently $u, v \in J$, as required. □

Theorem 136.28 (The position correlation for two possibly unequal heads)**.** *Suppose $s = \alpha\beta^{-1} \in O_w$. For every pair of canonical parameters g, g', and every $x \in G$,*

$$\int_G F_{\alpha,\beta}(g;y)\overline{F_{\alpha,\beta}(g';yx)}\,dy = K_w(s;g)\overline{K_w(s;g')}\,\mathbf{1}_J(x). \tag{136.31}$$

The measures here give J and $U \cap J$ volume one. In particular integrating in x recovers the actual product of orbital factors, with no restriction on the difference of the two heads beyond the local residue test in those factors.

Proof. Write the unique residue decomposition as $s = \bar{u}_L\,\bar{g}_s\,\bar{u}_R^{-1}$. Suppose $F_{\alpha,\beta}(g;y) \neq 0$. Then $y \in uk_\alpha J$ and $g^{-1}y \in vk_\beta J$ for $u, v \in U$, so that $u^{-1}gv \in k_\alpha J k_\beta^{-1}$ is an integral lift of s. By Lemma 136.27, g, u, v are integral, $\bar{g} = \bar{g}_s$, $\bar{u} = \bar{u}_L^{-1}$ and $\bar{v} = \bar{u}_R^{-1}$. The only possible coset of y is therefore the coset $y_s J$ with $\bar{y}_s = \bar{u}_L^{-1}\alpha$.

Conversely, if g is integral and $\bar{g} = \bar{g}_s$, then on $y_s J$ one has $\overline{g^{-1}y} = \bar{u}_R^{-1}\beta$. Lifting the two residual unipotents shows that both factors are nonzero there. Their product is the unit complex number $\bar{\chi}(\bar{u}_L^{-1})\overline{\bar{\chi}(\bar{u}_R^{-1})}$, independent of g. In all cases, including incompatible heads, this proves

$$F_{\alpha,\beta}(g;y) = K_w(s;g)\,\mathbf{1}_{y_s J}(y),$$

since the coset has volume one. For $y \in y_s J$, yx belongs to $y_s J$ exactly when $x \in J$. Multiplying the two formulas proves (136.31). □

The completed local orbital integral has an equivalent normalization. Give the canonical parameters the measure obtained from Haar measure on the regular set of G by taking out the left and right unipotent measures. This is well defined in the elimination coordinates $u_1 g u_2^{-1}$: Haar invariance makes the measures in u_1, u_2 their Haar measures, with the remaining factor a measure $d\nu(g)$. Equivalently one may pull back the matrix Haar density through the invertible elimination coordinates and use translation invariance in each unipotent variable. For $s \in O_w$,

$$\int |K_w(s;g)|^2\,d\nu(g) = 1. \tag{136.32}$$

Indeed integrate the indicator of the coset of s in K, whose volume is one, in these coordinates. The two unipotents are each restricted to one principal coset when g is integral with residue $\bar{g}_s$, and there are no other contributions. Thus the unsigned double integral is precisely $|K_w(s;g)|^2$ at each g.

Corollary 136.29 (All unselected positions on the same fields)**.** *For every term of the actual marked energy of Theorem 134.11, one may insert in its product of paired orbital factors the position integrals* (136.31) *at all unselected marks simultaneously. Their joint position dependence is exactly $\prod_{w \in \mathcal{W}\setminus\Xi} \mathbf{1}_{J_w}(x_w)$. This applies before summing the rational heads, including in the presence of conditions linking the residues at different places. In the independent completed average with uniform residues, the local open factor relative to all residues is*

$$\rho_w = \frac{|O_w|}{|\bar{G}|} = \frac{(1-q^{-1})^{2a}}{\prod_{i=d+1}^{r}(1-q^{-i})}, \qquad 1 - \frac{4af}{q_*} \ \leq \prod_{w \in \mathcal{W}\setminus\Xi} \rho_w \ \leq 1.$$

Proof. At each unselected mark the actual residue belongs to O_w. Apply the theorem separately to each pair of heads in the expanded finite sum in the sphere formula. The factor on the right of (136.31) depends on x only through its indicated principal subgroup. The normalization of each subgroup makes the new integral identical to the original summand.

For the last statement use (136.32) and count the residue decompositions. The two unipotent factors each have $|\bar{U}|$ choices and the diagonal part has $(q-1)^{2a}|\mathrm{GL}_d(\mathbb{F}_q)|$ choices. They are all distinct by uniqueness. Dividing this product by $|\mathrm{GL}_r(\mathbb{F}_q)|$ and using the count by independent columns gives the stated ratio: the powers of q cancel since $2\dim U + 2a + d^2 = r^2$. It lies between $(1-q^{-1})^{2a} \ge 1 - 2a/q$ and one. There are at most $2f$ places in the product and each $q \ge q_*$. □

Thus nonintegral positions at the unselected marks disappear when one keeps their original open residue tests. The remaining marked places can have defective flags. The continuous local calculation does not replace the rational summation and its archimedean weights in the arithmetic second moment.

136.12 Terminal stabilizers through a growing range of denominators

At a selected split mark let Z still denote the stabilizer of $\bar{z}$ in K, where the bar indicates the fixed residue of the terminal vector in the chosen component. Write

$$C_Z^+(x) = \int_Z \mathbf{1}_{UJ}(hxh^{-1})\,dh.$$

This bounds $|C_Z(x)|$. We estimate its full second moment on the first nonintegral shells. The middle root coordinates can have arbitrary depth on the shells in this estimate.

Lemma 136.30 (A moving terminal vector in the stabilizer of a coset)**.** *Take $u \in U$, written in the three blocks of Lemma 136.24. Denote by $D : R \to W$ its right-to-middle block and let*

$$T_D = D(R) \cap W(O),$$

where $D(R)$ is the image over F_w. This is a saturated sublattice of rank at most a. Let $\mathcal{D} \subset K$ be the group stabilizing uJ under conjugation. If $\bar{z} \notin T_D/\pi T_D$, viewed inside $W(O)/\pi W(O)$, then

$$|\mathcal{D}\bar{z}| \geq q^{d-2a} - 1.$$

For u distributed by unipotent Haar measure on any nonempty shell $\mathcal{S}(m, P, P) \cap U$, the fraction for which $\bar{z}$ belongs to this residue subspace is at most $(q^a - 1)/(q^d - 1)$.

Proof. Choose an integral complement V' to T_D in $W(O)$. On V' let W_0 be the kernel of the other middle block $B : W \to L$. The kernel is saturated, so it has an integral complement in V', and its rank d_0 is at least $d - 2a$. By assumption the projection of $\bar{z}$ to $\overline{V'}$ is nonzero. The complement to W_0 in V' can be chosen so that the W_0-component of $\bar{z}$ is also nonzero. In fact if this component already is nonzero there is nothing to choose. Otherwise change a complementary basis vector having a nonzero coefficient in the projection by adding a primitive vector of W_0. This is still an integral basis and makes the W_0-component nonzero. We use $d - 2a \geq 1$, as holds at these nodes.

Now u is the identity on W_0 and preserves its chosen complement in the full space: B vanishes on W_0 and the image of D is in $T_D \otimes F_w$. The copy of $\mathrm{GL}(W_0)(O)$ acting trivially on that complement commutes with u. Its action on the nonzero residue component of $\bar{z}$ gives $q^{d_0} - 1$ values.

For the fraction assertion conjugate u by integral changes of basis in W acting trivially on the arms. This preserves the shell, U, and its Haar measure: the automorphism preserves $U \cap J$, which has volume one. It carries T_D to its image under the basis change. A uniform integral change sends a fixed nonzero residue vector uniformly over the $q^d - 1$ nonzero vectors. A residue subspace of dimension at most a contains at most $q^a - 1$ of them. Integrate this inequality in u on the finite-measure shell. □

Theorem 136.31 (An unsigned estimate for all positions of bounded denominator)**.** *For a shell $\mathcal{S} = \mathcal{S}(m, P, P)$ meeting UJ one has, at every sufficiently large node,*

$$\frac{\int_{\mathcal{S}} C_Z^+(x)^2\,dx}{\mathrm{vol}(J)\,\mathrm{vol}(Z)\,\mathrm{vol}(K)} \leq 24q^{-d+2a+16a^2(m+1)}. \tag{136.33}$$

Let $\mathcal{B}$ be the union of all shells with $m + 1 \leq k/(1024(a+1)^2)$. Uniformly at either original mark on the same interval,

$$\frac{\int_{\mathcal{B}} C_Z^+(x)^2\,dx}{\mathrm{vol}(J)\,\mathrm{vol}(Z)\,\mathrm{vol}(K)} \leq q^{-9k/10}. \tag{136.34}$$

This includes every defective arm configuration in this range.

Proof. Use (136.30) with $H = Z$ and $F = \mathbf{1}_{UJ}$. For yJ let $\mathcal{D}_y$ be its K-stabilizer and let $\mathcal{C}_K(y), \mathcal{C}_Z(y)$ denote its two conjugacy orbits. Orbit–stabilizer gives the exact formula

$$[K : Z]\,|\mathcal{C}_Z(y)| = |\mathcal{C}_K(y)|\,|\mathcal{D}_y\bar{z}|.$$

Indeed the stabilizer in Z is $\mathcal{D}_y \cap Z$, and its index in $\mathcal{D}_y$ is the last orbit size. That size is constant on each Z-orbit since Z fixes $\bar{z}$.

Put $N = N(m, P, P)$ and $D_0 = q^{2(d-2a)P}/4$. By Lemma 136.24 the K-orbit of each contributing coset has size at least D_0. On the Z-orbits with $|\mathcal{D}_y\bar{z}| \geq q^{d-2a}/2$, the normalized sum of squared intersection counts is at most $2q^{-d+2a}N^2/D_0$. On the other orbits every unipotent representative has $\bar{z} \in \overline{T_D}$, by the preceding lemma. The union of their unipotent representatives is a union of cosets of $U \cap J$, each of volume one. Its total measure is at most $2q^{a-d}N$, by the fraction assertion of the lemma applied to the full shell. This use of measure permits T_D itself to vary with the representative inside a coset. Their contribution to the squared sum is therefore at most $4q^{2a-2d}N^2/D_0$.

The count (136.27) and $P \leq 2a(m+1)$ imply $N^2/D_0 \leq 4q^{16a^2(m+1)}$. Adding the bounds proves (136.33). The equality of the two lengths on every contributing coset is Lemma 136.26. In $\mathcal{B}$ the positive exponent $16a^2(m+1)$ is at most $k/64$. There are at most $2a(m+1)+1$ possible P's for each m, and at most k such exponents m. Since $d = k+1$, $a = o(k)$, and both a and k are polynomially bounded in x, this polynomial factor and q^{2a} are absorbed by $q^{(1-1/64-9/10)k}$ eventually. This proves (136.34). □

Corollary 136.32 (Absolute comparison for a subset of the fixer residues)**.** *For $S \subset \bar{Z}$, let $c_\alpha(x) = \int_{Jk_\alpha} \phi(hxh^{-1})\,dh$. Then the original relative-residue test in the compact position average satisfies*

$$\frac{1}{\mathrm{vol}(J)|\bar{G}|^2}\int_{\mathcal{B}} \sum_{\alpha\beta^{-1}\in S} |c_\alpha(x)c_\beta(x)|\,dx \leq q^{-9k/10}.$$

Thus the same estimate includes the extra defective-arm test. At a selected place the two factors B_0 and q^{n-j_w} together leave a bound b^{-2k}.

Proof. Enlarge S to $\bar{Z}$ in the sum of absolute values and use $|c_\alpha| \leq \int_{Jk_\alpha} \mathbf{1}_{UJ}(hxh^{-1})dh$. The allowed pairs are those for which α, β belong to the same coset $\bar{Z}\gamma$. The sum of the latter integrals in one such coset is $C_Z^+(k_\gamma x k_\gamma^{-1})$, by the change of variable $h = z_0k_\gamma$, $z_0 \in Z$. The sum over pairs is thus bounded by the sum of these squares over the cosets. Conjugation by K preserves $\mathcal{B}$ and Haar measure. As $\mathrm{vol}(J) = 1$, one has $|\bar{G}| = [K : Z]\,\mathrm{vol}(Z) = \mathrm{vol}(K)$. The resulting normalized integral is precisely the one in (136.34). Finally $n - j_w \leq a = o(k)$, $q \geq b^{10}$, and $B_0 = b^{6k}$. □

For denominators beyond this range, the term $16a^2(m+1)$ in (136.33) can exceed k. There the growth of the denominator must enter the saving as well. The exact calculation at unselected places in Theorem 136.28 still holds at every depth.

136.13 A change of variables for the interior rows of a flag

We work directly with the class sums in the position integral. For $H = K$ or Z, an H-conjugacy orbit on G/J will also denote its inverse image $\mathcal{C}$ in G. This set is invariant under multiplication by J on both sides. Indeed it is right J-invariant and invariant under conjugation by $J \subset H$. Its signed intersection count on a fixed shell is

$$S_{\mathcal{C}} = \int_{U \cap \mathcal{C}} \chi(u)\, du.$$

This is exactly the sum in (136.30) for $F = \phi$: each coset meeting U contributes one coset of $U \cap J$, of measure one and constant character. We shall partition this last integral in unipotent matrix entries.

For a scalar, row, column or matrix V put $e(V) = \min\{j \geq 0 : \pi^j V \text{ is integral}\}$. Write the upper unipotent matrix as

$$u = \begin{pmatrix} T & B & C \\ 0 & I & D \\ 0 & 0 & S \end{pmatrix}, \qquad T = I + A, \quad S = I + E.$$

The rows of B are β_i. Write $b_i = e(\beta_i)$, $b_* = \max_i b_i$, $e_D = e(D)$, and let s_B, s_D be the two middle image lengths of Lemma 136.24. The integer $\Delta = P - s_B - s_D$ is nonnegative. The following elementary relations control the entries outside B, D.

Lemma 136.33 (Lengths seen in one left row)**.** *For every left coordinate covector λ, where $b_\lambda = e(\lambda B)$,*

$$\begin{aligned} e(\lambda T) &\leq b_\lambda + \Delta, \\ e\big(\lambda(C - BD)\big) &\leq e_D + 1 + \Delta + b_\lambda, \\ m &\leq 2a\big(b_* + e_D + 1 + \Delta\big). \end{aligned}$$

On $\mathcal{S}(m, P, P)$ the unipotent measure of the set $\Delta \geq L$, for an integer $L \geq 1$, is at most $q^{4a^2(m+1)+d(P-L)}$.

Proof. Take the image M of $u - 1$ in $F_w^r/\pi\mathcal{L}$ on integral source vectors. Projection onto the middle coordinates has image of length s_D; its kernel contains the image M_B on $W(O)$, lying in the left coordinates and of length s_B. The quotient of the kernel by M_B has length Δ and is killed by π^Δ. In particular $\pi^\Delta A$ has columns belonging modulo π to M_B, giving the first assertion. Multiplication by π^{e_D+1} takes M into the kernel. Thus the columns of $\pi^{e_D+1+\Delta}C$ also belong modulo π to M_B. Applying λ, and bounding the row of BD by the norms of λB and D, proves the second assertion. The same argument on all coordinates shows that every entry of $u-1$ has pole at most $b_* + e_D + 1 + \Delta$, because $\pi^{b_*+1} M_B = 0$. The strictly upper block matrix $u - 1$ has $(2a+1)$-st power zero. The finite geometric series for u^{-1} bounds its pole by $2a$ times that of $u - 1$, giving the asserted bound for m. Finally repeat the submodule count in Lemma 136.24 with $s_B + s_D \leq P - L$. For each pair of permitted submodules the d columns and d rows have at most $q^{d(P-L)}$ choices, and all other counts are as in that lemma. □

Fix $j < a$ and let λ read left coordinate $j + 1$. Set

$$R = \max_{i > j} b_i, \qquad \xi = \lambda T, \qquad \gamma = \lambda(C - BD).$$

For a column $b \in L$ write $b = b_{\leq j} + b_{>j}$ by its coordinates. Use the primitive terminal covector η from Section 136.9.

Proposition 136.34 (A shear which tests the j-th row)**.** *Let $L \geq 1$ and fix any stratum specified by the row pole orders b_i, the integer e_D, the shell, and the tests*

$$b_a < L, \qquad e(\xi) \leq b_{j+1} + L, \qquad e(\gamma) \leq e_D + b_{j+1} + L + 1.$$

The index j may be chosen as a function of the b_i. Put $h = R + L + 3$. In the integral of χ over this stratum in any fixed $\mathcal{C}$, the contribution from

$$\beta_j(\ker \eta \cap W(O)) \not\subset \pi^{1-h} O \tag{136.35}$$

is zero. If $b_j \geq R + 6L$, the fraction of the stratum satisfying the opposite inclusion, measured on the whole shell before restricting to $\mathcal{C}$, is at most $2q^{-kL}$.

Proof. For $w \in \pi^h(\ker \eta \cap W(O))$, put $b = Bw$, $v = \lambda b$, and $A_w = I - b_{>j}\lambda/(1+v)$ on the left block. The scalar v lies in πO. Multiply u on the right and left, respectively, by the following matrices:

$$r_w = \begin{pmatrix} I & 0 & 0 \\ w\xi & I & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad l_w = \begin{pmatrix} A_w & 0 & 0 \\ -w\lambda & I + w\lambda B & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Both are in J. In fact $b_{>j} \in \pi^{h-R} L(O)$, $w\lambda B$ has entries in $\pi^{h-b_{j+1}} O$, and $w\xi$ in $\pi^{h-b_{j+1}-L} O$. Direct multiplication shows that $l_w u r_w$ belongs to U. Its five blocks are

$$T' = T + b_{\leq j}\xi, \quad B' = A_w B, \quad C' = A_w C, \quad D' = D - w\gamma, \quad S' = S. \tag{136.36}$$

For example $\lambda(T + b\xi) = (1+v)\xi$, which leaves only $b_{\leq j}\xi$ in the first block. The lower-left block cancels since $(I + w\lambda B)w = (1+v)w$. The change in T is strictly upper triangular as ξ starts in coordinate $j+1$.

This operation preserves the stratum. Rows $i \leq j$ of B are fixed, and every other row is multiplied on the right by the integral unit matrix $(I + w\lambda B)^{-1}$. Thus the individual b_i's are preserved. Also $\lambda T' = \xi$ and $\lambda(C' - B'D') = \gamma$, since $\lambda A_w = \lambda/(1+v)$ and $\lambda B w = v$. The change in D has entries in $\pi^{2-e_D} O$, which preserves its pole order, including the case $e_D = 0$. Both the coset orbit and the shell are preserved by the two principal multipliers. The inverse operation uses $-w$: the new $\lambda B' = \lambda B/(1+v)$ and the new $b_{>j}$ for the argument w is $b_{>j}/(1+v)$. These identities in (136.36) give back the original blocks.

We check preservation of the unipotent Haar measure explicitly. The row $R_1 = \lambda B$ changes to $R_1/(1 + R_1 w)$. On the ball $e(R_1) \leq R$ this is a bijective isometry, with inverse $R_1 \mapsto R_1/(1 - R_1 w)$. Indeed in the difference of two outputs the change of denominator contributes at most q^{R-h} times the norm of the difference of the inputs; the term with common denominator has exactly that norm. Thus balls of each radius are taken to balls of the same radius and additive measure is preserved. Given this row, each other changed row of B is multiplied by an integral unit matrix. For fixed B, the transformation of T is left multiplication by $I + b_{\leq j}\lambda$ in the triangular unipotent group. The transformation of C has determinant of absolute value one on each column. Given B, C, the transformation of D is multiplication by $I + w\lambda B$ followed by a translation. All preserve the relevant measures. The product formula for the measure on U now gives the claim also after restriction to the invariant stratum.

By (136.21) and its integral lift, the change of the phase before division by π is $c_j \beta_j(w)$ modulo πO. Indeed the consecutive entries of T change only at j, where $\xi_{j+1} = 1$. The change in the last row evaluated at z is $-(\beta_a w)(\lambda B z)/(1+v)$, which belongs to πO since $h - b_a - b_{j+1} \geq 1$. The D-term is unchanged because $\eta(w) = 0$. The row β_j itself is unchanged. Average over w with additive probability. The character $w \mapsto \Psi(c_j \beta_j(w)/\pi)$ has integral zero exactly when it is nontrivial, giving (136.35).

For the fraction bound the stated stratum on the whole shell is invariant under integral middle basis changes: BD, T, C are unchanged and the indicated row and matrix pole orders are preserved. If $b_j \geq R + 6L$, the covector $\pi^{b_j}\beta_j$ is primitive. The surviving inclusion requires that it vanish on the integral kernel of η modulo π^{b_j+1-h}, hence modulo π^L, since $b_j + 1 - h \geq 5L - 2 \geq L$. It must therefore be a scalar multiple of η modulo π^L, by choosing a basis adapted to this primitive covector. Under a uniform integral middle basis change the primitive row is uniform amongst $q^{Ld}(1 - q^{-d})$ primitive rows modulo π^L, by extension to an integral basis. There are at most q^L multiples of η. Integrating this probability over the stratum gives the result, since $d - 1 = k$. □

The change of variables uses the principal group on both sides. It applies equally to orbits for the terminal stabilizer and for the full integral group. In using the cancellation one keeps the whole stratum specified in the proposition; cutting it first by additional conditions that vary under (136.36) could change the integral.

136.14 The full position mass at a selected mark

The loss in the elementary count of Lemma 136.24 grows with the denominator exponent m. We use the interior row test to obtain a saving proportional to m on the large shells. All integrals and characters are normalized as in Section 136.7.

Proposition 136.35 (Signed class sums on large shells)**.** *Suppose $m \geq 200n^3$, put $L = \lfloor m/(200n^3) \rfloor$, and fix a shell $\mathcal{S}(m, P, P)$. For $H = K$ and for $H = Z$, the signed class sums of Section 136.13 satisfy*

$$\sum_{\mathcal{C}} |S_{\mathcal{C}}| \leq 10q^{4a^2(m+1)+dP-kL}. \tag{136.37}$$

Here $\mathcal{C}$ runs over the H-orbits on G/J in this shell.

Proof. Write $N_0 = q^{4a^2(m+1)+dP}$, an upper bound for the entire unipotent measure on the shell. Split the integral into the two parts $b_* \geq e_D$ and $b_* < e_D$. On the latter part transpose the full matrix and reverse the flags to exchange the arms. On the dual space the new block B is the transpose of the old D in the reversed arm basis, and the new D is the transpose of the old B. This change preserves unipotent Haar measure, the matrix height, the Smith lengths and the sum $s_B + s_D$. The phase is again (136.21) in the new coordinates with the primitive middle vector and covector exchanged up to units. The orbit sets $\mathcal{C}$ are carried to the corresponding orbit sets for the group induced on the dual space: the image of a right J-coset is a left J-coset, and the whole conjugacy orbit is bi-J-invariant. This gives a bijection of orbit sets by applying the inverse transpose. In particular the use of principal multipliers in Proposition 136.34 is valid in both orientations. We may now use $b_* \geq e_D$ on either part, retaining the strict inequality on the transposed part.

In a given orientation first take $b_* < 8aL$. If $\Delta < L$, Lemma 136.33 would give $m \leq 2a(16a + 1)L < 200n^3L$, contradicting the choice of L. Thus this part has the smaller count with $\Delta \geq L$.

For $b_* \geq 8aL$, treat the case $b_a \geq L$ by averaging the middle basis. In original coordinates this event implies that at least one of the two middle simple blocks has pole order at least L, including when the dominant arm has been transposed. All pole comparisons made so far are invariant under middle basis changes. For K use the full middle group, and for Z use its subgroup fixing the original terminal vector modulo π. These averages preserve each original $\mathcal{C}$. The depth t in Lemma 136.22 is at least $L + 1$. Thus, in the sum of the absolute class contributions, the total from this part is at most $2q^{-kL}N_0$ per orientation.

There remains $b_* \geq 8aL,\ b_a < L$. At some $j < a$ one has

$$b_j \geq R + 6L, \qquad R = \max_{i>j} b_i.$$

Indeed if every increase of the successive maxima, read from the last row to the first, were less than $6L$, the final maximum would be at most $b_a + 6(a - 1)L < 8aL$. Fix the row pole orders and e_D, choosing the least j with this property. If either bound on ξ, γ in Proposition 136.34 fails, Lemma 136.33 implies $\Delta \geq L$. Together with the first case these exceptions have measure at most $q^{-dL}N_0$ per orientation.

On the remaining strata apply Proposition 136.34 inside each orbit. The operations in its proof preserve the row orders, e_D, and hence the orientation choice. After cancellation only the stated congruence on β_j remains. Its fraction on the whole stratum is at most $2q^{-kL}$. Summing over the disjoint choices of the row orders and e_D costs at most $2q^{-kL}N_0$ per orientation. This application uses the full strata defined by the row, column and pole tests. We used $\Delta \geq L$ only to count the parts which failed one of those tests. Adding the bounds for both orientations, with $d = k + 1$, proves (136.37). □

Theorem 136.36 (Complete local mass at the terminal stabilizer)**.** *Uniformly at each original split mark of the shorter interval, at all sufficiently large nodes,*

$$\frac{\int_{\{m\ge 200n^3\}} |C_H(x)|^2\,dx}{\mathrm{vol}(J)\,\mathrm{vol}(H)\,\mathrm{vol}(K)} \le q^{-3k/2} \qquad (H=K,Z), \tag{136.38}$$

$$\frac{\int_G |C_Z(x)|^2\,dx}{\mathrm{vol}(J)\,\mathrm{vol}(Z)\,\mathrm{vol}(K)} \le 2q^{-9k/10}. \tag{136.39}$$

Proof. Apply (136.30) with $F=\phi$. Its numerator on each orbit is $|S_{\mathcal{C}}|^2$. When the orbit meets U, orbit–stabilizer and Lemma 136.24 give $[K:H]|\mathcal{C}| \ge q^{2(d-2a)P}/4$. The sum of squared absolute class sums is bounded by the square of (136.37). Using $P\le 2a(m+1)$, this bounds the normalized integral on one shell by

$$400q^{16a^2(m+1)-2kL}.$$

For fixed $L\ge 1$ there are at most $200n^3$ values of m; each satisfies $m+1\le 400n^3L$ and has at most $2a(m+1)+1$ choices of P. Lemma 136.26 accounts for all the shells with nonzero integrand. On the given interval $k\ge x^5-2\sqrt{x}$ and $n\le\sqrt{x}$, hence $6400a^2n^3=o(k)$. For all large nodes the last bound is at most $400q^{-31kL/16}$. The sum of $Lq^{-31k(L-1)/16}$ is bounded by an absolute constant for large k. The remaining polynomial factor in a,n is absorbed by $q^{(31/16-3/2)k}$, proving (136.38).

For $m<200n^3$ one has $m+1\le k/(1024n^2)$ at sufficiently large nodes, because $n^5=o(k)$. The entire contribution for Z on that range is at most $q^{-9k/10}$ by Theorem 136.31 and $|C_Z|\le C_Z^+$. Adding the first estimate proves (136.39). □

Corollary 136.37 (The complete marked position integral with the selected tests)**.** *Let $\mathcal{W},\Xi$ and j_w be as in Theorem 134.11 and Corollary 136.13. At an unselected place use the relative-residue test $S_w=O_w$, and at a selected place use $S_w=\bar{Z}_w$. With the c_α of Corollary 136.32, put*

$$\mathcal{P}_w(S_w)=\frac{1}{\mathrm{vol}(J_w)|\bar{G}_w|^2}\int_{G_w}\left(\sum_{\alpha\beta^{-1}\in S_w} c_\alpha(x)\overline{c_\beta(x)}\right)dx.$$

The finite sum is taken before integration. At an unselected place $\mathcal{P}_w=\rho_w$. At a selected place it is nonnegative and at most $2q_w^{-9k/10}$. In particular

$$B_0^{|\Xi|}\left(\prod_{w\in\Xi} q_w^{n-j_w}\right)\prod_{w\in\mathcal{W}}\mathcal{P}_w(S_w)\le b^{-2k|\Xi|}\prod_{w\notin\Xi,\ w\in\mathcal{W}}\rho_w. \tag{136.40}$$

When Ξ is empty the two products agree exactly.

Proof. At an unselected place a pair with $s=\alpha\beta^{-1}\in O_w$ can contribute to the integrand only if both conjugates of x belong to UJ for some $h_1\in Jk_\alpha$, $h_2\in Jk_\beta$. The residue of $h_2h_1^{-1}$ is $s^{-1}\in O_w$, since inversion conjugates the leading Hermitian Gram determinants. Apply Lemma 136.27 to $h_1xh_1^{-1}$ and $h_2h_1^{-1}$. It gives $x\in J$. On J both single-coset integrals are one. The normalized result is therefore ρ_w by the count in Corollary 136.29.

At a selected place group the pairs according to their common coset $\bar{Z}\gamma$. The sum at x is exactly $\sum_\gamma |C_Z(k_\gamma x k_\gamma^{-1})|^2$. Invariance of Haar measure under K-conjugation shows that the normalized integral equals the left side of (136.39). This proves convergence and the stated estimate with its normalization. Finally $q_w\ge b^{10}$, $n-j_w\le a=o(k)$, and $B_0=b^{6k}$, so $2B_0q_w^{a-9k/10}\le b^{-2k}$ eventually. Multiply the separate factors. The enlargement to the entire terminal stabilizer is available in the original sum of squared norms, as in Section 136.7. □

Equation (136.40) evaluates or bounds all the marked position factors, on the same fields and without a denominator cutoff. The original arithmetic energy still sums over rational heads coupled to the real orbital integrals. Passing from that sum to position integrals requires a global comparison. The local estimate alone does not give a bound for the integral projected theta periods or for strict cohomology.

136.15 Sampling a common pivot at the unselected places

In passing from the rational sum in (134.13) to an integral, the equality of two rational pivots and the equality of their residues play different roles. We keep the original congruence for unequal rational pivots and complete the additive test on the actual diagonal. The following argument concerns the enlarged product mask of Corollary 134.13.

Write $\mathcal{V} = \mathcal{W} \setminus \Xi$ for the unselected places, $A_w = O_w^{\mathrm{int}} \times O_w^{\mathrm{int}}$ for the paired ring of integers there, and $A_w^\times$ for its group of units. Here $O_w^{\mathrm{int}} = \mathcal{O}_{F_w}$. Give each additive ring volume one. For $p, p' \in E^\times$ put

$$I_w(p,p') = \mathbf{1}_{p,p' \in A_w^\times,\, p \equiv p' \bmod \pi A_w}.$$

Up to the constant $q_w^{2a(2k+2a+1)}|\mathrm{GL}_k(\mathbb{F}_{q_w})|$, the factor from w for two heads $\mathbf{p}, \mathbf{p}'$ and terminal entry t is

$$H_w(t) \prod_{i=1}^{a} I_w(p_i, p_i'),$$

by Corollary 134.13. Put $h_w^* = (1 - q_w^{-k})/(1 - q_w^{1-k})$.

Theorem 136.38 (A positive completion before additive sampling)**.** *In the actual cut or mixed squared energy one obtains an upper bound by making both of the following replacements at the unselected places:*

$$H_w(t) \longmapsto h_w^*\, \mathbf{1}_{A_w}(t),$$
$$I_w(p,p') \longmapsto I_w(p,p') + \mathbf{1}_{p=p',\ p \in A_w \setminus A_w^\times}.$$

The second replacement may be made for any subset of the arm coordinates. At an unequal pair which survives, the two pivots are congruent units in both components. At an equal pair $p = p' = c$ the completed factor is $\mathbf{1}_{A_w}(c)$. Consequently both additive characters on this common entry must be trivial on A_w for its local Fourier coefficient to be nonzero. The same assertion holds for the completed terminal entry.

Compared with the original local integrals over the common entries, the ratio of the product of the new zero coefficients to the old one, when all a arm entries and the terminal entry are integrated, is

$$\prod_{w \in \mathcal{V}} \frac{1 - q_w^{-k}}{(1 - q_w^{1-k})(1 - q_w^{-(k+1)})} (1 - q_w^{-1})^{-2a} = 1 + O(b^{-9}). \tag{136.41}$$

This ratio is for the local zero coefficients in additive measure. The one-sided bound for the coupled rational sum holds before any zero coefficient is taken.

Proof. For fixed t, the kernel on the finite list of heads supplied by the remaining real and finite variables is positive semidefinite. To verify this at the level of the original integral, hold the other residues and middle elements fixed. The matrix indexed by $\mathbf{p}, \mathbf{p}'$ is the Gram matrix of the real tensor $A_{\mathbf{p}}(t)$ multiplied by the corresponding finite scalar $K(s; w_{\mathbf{p}} x)$ from the remaining places. Summing residues or integrating with the positive sphere and stabilizer measures preserves this property. The same reasoning applies to the mixed tensors of Theorem 136.7. The determinant weight and the sphere-density constants outside these matrices are nonnegative.

By (136.11) with split sign, $0 \le H_w(t) \le h_w^* \mathbf{1}_{A_w}(t)$. For each coordinate i the kernel I_w on the list of heads is a sum of rank-one matrices: for each unit residue take the outer product of the indicator of that residue class with itself. The added kernel is likewise positive semidefinite. Partition the list according to the *actual rational value* of p_i and take the rank-one indicator matrices for the values in $A_w \setminus A_w^\times$. Hence every replacement adds a positive semidefinite matrix. Entrywise products of positive semidefinite matrices are positive semidefinite, as is

seen by using tensor products of their Gram vectors. Multiplying entrywise by the other kernels and evaluating the quadratic form on the all-ones vector proves the asserted upper bound one replacement at a time.

The equal and unequal evaluations follow directly from the formula. In the unequal case congruence of the units also gives $(p^{-1}-(p')^{-1})/M \in A_w$, since M has valuation one at each component. On the diagonal, the additive Fourier transform of $\mathbf{1}_{A_w}$ is its annihilator indicator. Under the trace pairing divided by the different generator, the annihilator is A_w itself. This is the defining duality of the different. At the terminal entry the same calculation has the constant h_w^*.

On a common arm entry the original integral is $(1-q_w^{-1})^2$ and the new one is one. On the terminal entry the original integral is $1-q_w^{-(k+1)}$ by (136.9). This proves the product formula. The logarithm of each terminal ratio is $O(q_w^{1-k})$ and that of the arm ratio is $O(a/q_w)$. Now $|\mathcal{V}| \le 2f$ and $12af/q_* \le b^{-9}$ by Theorem 134.11. Exponentiating proves (136.41). □

We record the resulting rational sampling statement with its covolume, since a loss per infinite place would enter the leading constant. Use the additive characters of $E\backslash\mathbb{A}_E$ paired by $\mathrm{Tr}(\,\cdot\,/\delta_E)$. Take ordinary Lebesgue measure at the complex embeddings and volume one on all finite integer rings. Let T be a finite set of primes of E disjoint from the primes belonging to $\mathcal{V}$. If F is an integrable test on the adeles, write $\widehat{F}(y)$ for its transform at the rational frequency y, using the conjugate additive character.

Proposition 136.39 (One-sided sampling after completion)**.** *Suppose $F \ge 0$ is lower semicontinuous and integrable. Outside T its finite part is the product of the additive integer-ring indicators, up to a positive constant independent of the entry. Inside T and at infinity the variables may be coupled. If $\sum_{y\in E}|\widehat{F}(y)|<\infty$, then for any finite set $D\subset E$,*

$$\sum_{u\in D} F(u) \le \frac{1}{c_E}\left(\widehat{F}(0)+\sum_{0\neq y\in \mathcal{O}_{E,T}}|\widehat{F}(y)|\right), \qquad c_E = 2^{-f}\sqrt{|\operatorname{disc} E|}.$$

In particular if, at the scale $c_v = 2\pi i/(\Phi_v(\delta_E)b_{j,v})$ of Theorem 136.4, the nonzero coefficients are at most $B\,W_{T,c,k-n^7}(y)$ for $B\ge 0$, their total contribution here is at most Bb^{-12kf}/c_E.

Proof. The covolume of the diagonal E is c_E. Indeed finite approximation reduces a fundamental domain to the finite integer rings times a fundamental parallelotope of the Minkowski lattice of $\mathcal{O}_E$, which has that volume. The characters of the compact additive quotient are indexed by E for this pairing, by additive adelic duality. Explicitly, on a quotient by a finite open integer ideal the archimedean characters are those of the dual fractional lattice. The trace divided by δ_E identifies the dual of any fractional ideal $\mathfrak{a}$ with $\mathfrak{a}^{-1}$; taking the union over the finite levels gives the rational characters just stated. Compatible finite characters have their values on the diagonal fixed by the archimedean character, by finite approximation.

Periodize F over the diagonal E. Its periodization is lower semicontinuous as the supremum of finite sums of nonnegative translates. It descends to an integrable function on the compact quotient, by Tonelli's theorem. In probability Haar measure its Fourier coefficients are $c_E^{-1}\widehat{F}(y)$, by unfolding the fundamental domain. Absolute summability gives a continuous Fourier series with these coefficients. It agrees almost everywhere with the periodization by uniqueness of Fourier coefficients on a compact abelian group. This uniqueness also follows by density of finite linear combinations of characters in the continuous functions. At zero the lower semicontinuous periodization is at most the value of the continuous representative: a strictly larger value would persist on a nonempty open set of positive Haar measure. Every summand on a finite rational subset is included there. The integer-ring factors force y to be integral outside T. Taking

absolute values of the nonzero terms yields the first estimate. The final one follows from (136.5). □

Theorem 136.38 verifies the required integer-ring factors at the unselected marks when sampling a common coordinate of the actual positive expression. For unequal entries it retains the condition on the inverse difference in Corollary 136.5. To apply the quantitative sampling estimate to the full period, one must still bound the transforms of the factors at the remaining places and at infinity for the intervening arm coordinates.

137 The first relative Gram transform at split places

137.1 The first relative Gram frequency on the principal subgroup

We calculate one part of the Fourier integral left by the first common arm coordinate. In the chosen split component let e be the first left frame vector and ℓ its primitive paired frame covector. Thus $\ell(e) = 0$, and U fixes e and ℓ. The two components of its Gram entry at a relative element g are $\ell g e$ and $\ell g^{-1} e$, using the dual action in the second component. The terminal vector z is primitive with $\ell(z) = 0$ and $\bar{e}, \bar{z}$ are independent. We can choose an integral basis starting with e, z, in which ℓ is the last coordinate. Write

$$\omega_{\alpha,\beta}(g) = \Psi\big(\alpha \ell g e + \beta \ell g^{-1} e\big) \qquad (\alpha, \beta \in F_w),$$

with Ψ of conductor O. This wave is invariant under left and right U.

Lemma 137.1 (Fourier cancellation for an incident vector and covector)**.** *Let $H = K$ or Z, and let $t = \max(0, -\operatorname{ord}\alpha, -\operatorname{ord}\beta)$, using $\operatorname{ord}(0) = +\infty$. For $t = 0$ the average of $\omega_{\alpha,\beta}$ on H equals one. For $t \geq 1$ its absolute value is at most*

$$6q^{-(r-1)t} \quad (H = K), \qquad 8q^{1-(r-1)t} \quad (H = Z). \tag{137.1}$$

Proof. By inversion we may suppose $\operatorname{ord}\beta = -t$. Put $w = ge$, $\mu = \ell g^{-1}$. For g uniform on K, this is the uniform incident pair of a primitive vector and a primitive covector satisfying $\mu(w) = 0$, in every finite quotient $O/\pi^t O$. Indeed extend w to a basis of $\ker\mu$, then add a vector on which μ is one. This gives the required integral change of basis. The basis extension is valid over the finite quotient: a primitive covector has a unit coefficient, and a primitive vector in its kernel extends to a basis there by a unit coordinate. Conversely every basis change has the prescribed incidence. Lifts of an invertible matrix over the quotient are invertible over O.

For Z the additional conditions are independence of $\bar{w}, \bar{z}$ and $\bar{\mu}(\bar{z}) = 0$. To verify sufficiency in the basis construction, adjust z by a multiple of π times a vector on which μ is one, to obtain $z' \equiv z \bmod \pi$ in $\ker\mu$. The independent residues of w, z' allow them to begin a basis of the kernel. Sending e, z to w, z' gives a change of basis in Z. Transitivity gives uniform probabilities, and the number of allowed covectors for a fixed w is independent of w. Thus w is uniform among the primitive vectors for K and among the vectors with $\bar{w} \notin \mathbb{F}_q \bar{z}$ for Z.

Keep w fixed, lifting it integrally. Write $V_w = \{\mu \in \mathcal{L}^* : \mu(w) = 0\}$, a free module of rank $r - 1$. For K integrate $\Psi(\beta\mu(e))$ over the primitive part of V_w. Subtract the integral on πV_w from the integral on V_w. Orthogonality gives the conditional average

$$\frac{\mathbf{1}_{e \in Ow + \pi^t \mathcal{L}} - q^{-(r-1)} \mathbf{1}_{e \in Ow + \pi^{t-1}\mathcal{L}}}{1 - q^{-(r-1)}}.$$

All these tests depend only on reductions at precision t. For $h \geq 1$ the probability that $e \in Ow + \pi^h \mathcal{L}$ is at most $2q^{-(r-1)h}$: $w \bmod \pi^h$ must be a unit multiple of e, out of $q^{rh}(1 - q^{-r})$ primitive vectors. The same bound with $h = 0$ is immediate. The factor $\Psi(\alpha \ell w)$ has modulus one. Applying these bounds in the preceding average proves the first estimate at the ranks in question.

For Z start with $V_w' = \{\mu \in V_w : \mu(z) \in \pi O\}$. The excluded covectors are still πV_w, now of relative measure $q^{-(r-2)}$. In a basis beginning with w, z, orthogonality changes the conditional average to

$$\frac{\mathbf{1}_{e \in Ow + \pi^{t-1} Oz + \pi^t \mathcal{L}} - q^{-(r-2)} \mathbf{1}_{e \in Ow + \pi^{t-1}\mathcal{L}}}{1 - q^{-(r-2)}}.$$

In the first event $e \equiv aw + \pi^{t-1} bz \bmod \pi^t$ requires a to be a unit, even for $t = 1$, since $\bar{e}, \bar{z}$ are independent. There are at most q^{t+1} possible $w \bmod \pi^t$, out of $q^{rt}(1 - q^{1-r})$ eligible vectors. The probability is at most $2q^{1-(r-1)t}$. The second event has probability at most $2q^{-(r-1)(t-1)}$, by counting possible unit multiples as above when $t > 1$, and directly for $t = 1$. Substitution proves the second estimate. At integral frequencies the two arguments of Ψ are integral throughout H. □

For $H = K, Z$ consider the position pairing with the wave in its relative variable. At $x \in J$ its finite expression is

$$\sum_{\gamma,\delta \in \bar{H}} \int_{(U\backslash G)^2} \overline{\Phi_\gamma(y)} \Phi_\gamma(yx) \Phi_\delta(y') \overline{\Phi_\delta(y'x)}\, \omega_{\alpha,\beta}(y(y')^{-1})\, d\dot{y}\, d\dot{y}'.$$

This uses exactly the two quotient positions in (136.17). Inserting the Gram character on their relative group element is well defined on the two quotients by its U-invariance. Denote its integral over $x \in J$, divided by $\operatorname{vol}(J) \operatorname{vol}(H) \operatorname{vol}(K)$, by $B_H(\alpha, \beta)$.

Theorem 137.2 (The principal positions in the first relative transform)**.** *One has*

$$B_H(\alpha, \beta) = \frac{\operatorname{vol}(H)}{\operatorname{vol}(K)} \int_H \omega_{\alpha,\beta}(g)\, \frac{dg}{\operatorname{vol}(H)}. \tag{137.2}$$

In particular $B_H(0, 0) = \operatorname{vol}(H)/\operatorname{vol}(K)$*. For a rational nonzero frequency with split components* α, β*, set*

$$I = q^{\operatorname{ord}\alpha + \operatorname{ord}\beta}, \qquad D = q^{\max(0, -\operatorname{ord}\alpha) + \max(0, -\operatorname{ord}\beta)}, \qquad K_0 = k - n^7.$$

At sufficiently large nodes, the corresponding bound on the same fields is

$$|B_H(\alpha, \beta)| \le B_H(0, 0)\, \min\{1, I^{K_0/2} D^{-3}\}.$$

Proof. All Φ_γ are right J-invariant. At $x \in J$ each pair of factors in one quotient variable is its squared absolute value. Its support projects to Jk_γ, with the quotient integral obtained from ordinary Haar on Jk_γ since $U \cap J$ has volume one. This is the quotient calculation in Proposition 136.16. The wave is invariant under the excluded U-variables, as the group fixes both e and ℓ. Summing the cosets gives the integral of $\omega_{\alpha,\beta}(h_1 h_2^{-1})$ over H^2. One of these compact integrations gives $\operatorname{vol}(H)$, and the x-integration contributes $\operatorname{vol}(J)$, proving (137.2).

For $t = 0$ the weight is at least one before taking the minimum. For $t \ge 1$, $I \ge D^{-1}$, $D \le q^{2t}$, so that $I^{K_0/2} D^{-3} \ge q^{-(K_0+6)t}$. Since $r - 1 = k + 2a$, the bounds in (137.1) are at most $q^{-(K_0+6)t}$ for all $t \ge 1$ at the stated nodes. The estimate by one holds for every normalized average. This proves the bound with the local weight occurring in (136.4). □

The frequency in (137.2) is applied to the relative element $y(y')^{-1}$. Restricting to the principal positions made its distribution a compact group integral. For $x \notin J$ the two factors in each quotient variable are generally different. Their unweighted sum is controlled at the marks by Theorem 136.36 and Corollary 136.37; the additional wave requires an estimate for that weighted pairing.

137.2 The tail of the first relative Gram transform

We now allow the position x in Theorem 137.2 to run over G. Write $v_D = \mathrm{vol}(D)$ for a compact open subgroup D. In the notation of that theorem the integrand, including its normalization, is

$$b_H(x;\alpha,\beta) = \frac{1}{v_J v_H v_K} \int_{H^2} \phi(h_1 x h_1^{-1})\, \overline{\phi(h_2 x h_2^{-1})}\, \omega_{\alpha,\beta}(h_1 h_2^{-1})\, dh_1\, dh_2. \tag{137.3}$$

Indeed, for each of the two quotient positions in (136.17) the first factor restricts the variable to UJk_γ. Choose its representative $h = jk_\gamma$. As in Proposition 136.16, the quotient measure on this part is the pushforward of dj, and the factor at hx is $\phi(hxh^{-1})$. The wave is unchanged by any of the choices in the two U-orbits. Summing over both sets of residue cosets gives (137.3). The compact integrals in this formula are taken before the integration in x.

For an integer $t \geq 1$ define

$$H_t = \{h \in H : he \equiv e,\ \ell h \equiv \ell \bmod \pi^t\}, \qquad H_0 = H, \qquad N_t = [H : H_t].$$

Here e is the first left flag vector, and ℓ is the covector reading the last coordinate of the reversed right flag. The left cosets $H_t h$ are parametrized by the distinct pairs $(h^{-1}e, \ell h) \bmod \pi^t$. Equality of two such pairs means that the quotient $h'h^{-1}$ fixes both initial coordinates to the stated precision. In particular $N_t \leq q^{2rt}$. For $t = \max(0, -\operatorname{ord}\alpha, -\operatorname{ord}\beta)$ the wave in (137.3) is constant on each pair of these cosets. In fact the two entries of the phase are obtained by evaluating the covector of each pair on the vector of the other.

Lemma 137.3 (Signed sums with both end coordinates fixed)**.** *For $H = K, Z$, any $t \geq 0$, $m \geq 200n^3$, and $L = \lfloor m/(200n^3) \rfloor$, the H_t-orbits on the right J-cosets in $\mathcal{S}(m, P, P)$ satisfy*

$$\sum_{\mathcal{C}} \left| \int_{U \cap \mathcal{C}} \chi(u)\, du \right| \leq 10 q^{4a^2(m+1)+dP-kL}.$$

In the integral $\mathcal{C}$ denotes the inverse image in G.

Proof. Every element of H_t normalizes J. Thus each indicated orbit defines a right J-invariant set, stable under conjugation by H_t. It is consequently stable under left multiplication by $J \cap H_t$ as well. Its signed sum on G/J is the displayed integral, since a coset meeting U cuts it in a coset of $U \cap J$, of measure one and constant character.

We check the transformations used in Proposition 136.35 for these smaller sets. All allowed middle basis changes in its proof fix both ends of the flag exactly and belong to H_t. The row shear in Proposition 136.34 uses left and right principal multipliers l_w, r_w. They fix e because $\lambda(e) = 0$ and $\xi(e) = \lambda T e = 0$, since $Te = e$. Their last row is unchanged from the identity, hence they fix ℓ. Both therefore belong to $J \cap H_t$ for every t, including for Z since they act trivially modulo π. These facts preserve each orbit set under the shear.

On the part with dominant opposite arm, transposition followed by reversal exchanges the initial vector and the last covector. The images of the orbit sets under this map remain disjoint. In the reversed coordinates the two shear multipliers again fix both ends exactly and are principal. Pulling them back to the original coordinates gives elements of $J \cap H_t$, with the two sides exchanged. The transformed sets are consequently stable under this operation as well. Middle averaging on this part can be performed in the original coordinates, as in the proof of Proposition 136.35.

For clarity, put $V_0 = q^{4a^2(m+1)+dP}$. In each orientation the strata failing the height or row tests there have total measure at most $q^{-dL} V_0$, by Lemma 136.33. On the strata with a deep terminal middle coordinate the average of Lemma 136.22 bounds the sum of the absolute

contributions by $2q^{-kL}V_0$. On the remaining strata the row shear cancels within each set unless the exceptional covector congruence holds. Its measure on the whole strata is at most $2q^{-kL}V_0$, by the fraction bound of Proposition 136.34. These measure bounds apply to the sum because the sets are disjoint. Adding both orientations and using $d = k + 1$ proves the assertion. □

Theorem 137.4 (A conductor-dependent tail estimate)**.** *At each original split mark, for $H = K, Z$ and all sufficiently large nodes, $\int_G |b_H(x;\alpha,\beta)|\,dx$ is finite. For every pair of local frequencies, with t as above,*

$$\int_{\{m\geq 1200n^3(t+1)\}} |b_H(x;\alpha,\beta)|\,dx \leq q^{-6k(t+1)}. \tag{137.4}$$

Proof. Choose representatives k_i of $H_t\backslash H$. Write $C_D(y) = \int_D \phi(hyh^{-1})\,dh$ and put $A_i(x) = C_{H_t}(k_i x k_i^{-1})/v_{H_t}$. Let ω_{ij} denote the common wave on the two cosets. Then (137.3) is

$$\frac{v_H}{v_J v_K}\frac{1}{N_t^2}\sum_{i,j}\omega_{ij}A_i(x)\overline{A_j(x)}.$$

The normalized sum is bounded in absolute value by $(N_t^{-1}\sum_i |A_i|)^2 \leq N_t^{-1}\sum_i |A_i|^2$. Conjugation by k_i preserves the shell and its measure. On a shell $\mathcal{S}$, integration gives at most

$$\frac{v_H}{v_K}\frac{1}{v_J v_{H_t}^2}\int_{\mathcal{S}} |C_{H_t}(x)|^2\,dx.$$

The second factor is exactly $\sum_{\mathcal{C}} |S_{\mathcal{C}}|^2/|\mathcal{C}|$, where $S_{\mathcal{C}} = \int_{U\cap\mathcal{C}} \chi(u)\,du$. Indeed on a right coset C_{H_t}/v_{H_t} is the average of ϕ over its orbit, and each coset has measure v_J. Lemma 136.24 gives $[K : H_t]|\mathcal{C}| \geq |\mathcal{C}_K| \geq q^{2(d-2a)P}/4$ for any contributing orbit, with $\mathcal{C}_K$ its full K-orbit. The first inequality follows by comparing the stabilizer in H_t with that in K. Since $(v_H/v_K)[K : H_t] = N_t$, the preceding lemma and the square of its bound give

$$\int_{\mathcal{S}} |b_H(x;\alpha,\beta)|\,dx \leq 400N_t q^{8a^2(m+1)+4aP-2kL} \leq 400q^{2rt+16a^2(m+1)-2kL}.$$

The only contributing shells have $P = Q$ by Lemma 136.26. We used $P \leq 2a(m+1)$.

On the tail in the theorem $L \geq 6(t+1)$, and $m+1 \leq 400n^3L$. On this interval $6400a^2n^3 \leq k/16$ and $2r \leq 9k/4$ for all sufficiently large nodes. Thus $2rt \leq 3kL/8$ and the last bound is at most $400q^{-25kL/16}$. For fixed L there are at most $200n^3(800an^3L+1)$ pairs (m,P). Extract $q^{-6k(t+1)}$ using $L \geq 6(t+1)$. The remaining sum is at most

$$400\cdot 200n^3\sum_{L\geq 1}(800an^3L+1)q^{-9kL/16},$$

which is at most one for all large nodes, uniformly in the frequency and the mark. This follows by summing the geometric series and its first moment, since $q \geq 2$ and a, n grow polynomially in the interval parameter while k grows at least as fast as $x^5 - 2\sqrt{x}$. Every bounded range of m has compact closure in G, by the simultaneous bounds on x, x^{-1}, and the compact averages in (137.3) are bounded there. This proves the absolute convergence assertion. □

For nonzero rational frequencies this tail has the decay required in (136.4), since the weight there is at least $q^{-(k-n^7+6)t}$. The integral over J was calculated in Theorem 137.2. A frequency-dependent compact range of nonprincipal positions still lies between these two estimates.

137.3 Frequency decay on all integral positions

Keep the relative wave and the groups of Section 137.2. Put $c_\lambda = \max(0, -\operatorname{ord}\lambda)$, with $c_0 = 0$, and $t = \max(c_\alpha, c_\beta)$. We will estimate the integral of $|b_H|$ over the whole of K, including positions outside J.

Lemma 137.5 (The operator on pairs of end coordinates)**.** *For $t \geq 1$, write the distinct pairs indexing $H_t\backslash H$ as $(w_i, \mu_i) \in (O/\pi^t)^r \times ((O/\pi^t)^r)^*$. On this finite set with probability measure the operator with kernel*

$$\Psi\big(\alpha\mu_i(w_j) + \beta\mu_j(w_i)\big)$$

has norm at most $\min(1, 4q^{t+1-r(c_\alpha+c_\beta)/2})$ on complex square-integrable functions.

Proof. Use counting probability on the full product of the vector and covector spaces over O/π^t. The ambient operator is a product of two finite Fourier operators, interchanging the order of the vector and covector coordinates. For one such operator with coefficient λ, its adjoint times itself averages against the kernel $\mathbf{1}_{u-u' \in \pi^{c_\lambda}(O/\pi^t)^r}$, by orthogonality of additive characters. This operator is q^{-rc_λ} times the orthogonal projection onto functions constant on these cosets. This includes $c_\lambda = 0$. Consequently the norm of the product is $q^{-r(c_\alpha+c_\beta)/2}$.

By the basis calculation in Lemma 137.1, the set of pairs has uniform measure after conditioning on membership. For K its ambient probability equals $(1-q^{-r})q^{-t}(1-q^{-(r-1)})$. Indeed w is primitive, and on the hyperplane $\mu(w) = 0$ the covector is primitive. For Z the probability equals $(1-q^{1-r})q^{-t-1}(1-q^{-(r-2)})$. Here $\bar{w}$ is independent of $\bar{z}$, and inside $\mu(w) = 0$ the additional residue equation $\bar{\mu}(\bar{z}) = 0$ imposes one independent condition. Both probabilities are at least $\frac{1}{4}q^{-t-1}$ at these ranks. Extending a function by zero, applying the ambient operator, and restricting back gives the desired operator after division by the probability of the set. The norm bound follows with this factor. The bound by one also follows directly from Cauchy–Schwarz and the modulus of the kernel. □

Lemma 137.6 (Mean square after fixing the two ends in the residue field)**.** *For $H = K, Z$ and $t \geq 1$, at the original split marks,*

$$\frac{v_H}{v_K}\frac{1}{v_J v_{H_t}^2}\int_K |C_{H_t}(x)|^2\,dx \;\leq\; q^{100n^2}.$$

Proof. For integral x the value of $\phi(hxh^{-1})$ depends only on the reductions of h, x, by Lemma 136.17. The residue image D of H_t equals that of H_1. To lift a residue matrix in H_1, prescribe its first column to be e and its last row to be ℓ exactly, and lift all remaining entries arbitrarily. These prescriptions agree at their intersection and the determinant is a unit. In the case Z the terminal condition is already imposed modulo π. Thus the normalized averages for all $t \geq 1$ coincide on K.

Write $\bar{H}$ for the residue image of H. The required quantity is

$$\frac{|\bar{H}|}{|\bar{G}|}\sum_{\mathcal{C}}\frac{|S_{\mathcal{C}}|^2}{|\mathcal{C}|}, \qquad S_{\mathcal{C}} = \sum_{X \in \mathfrak{n}\cap\mathcal{C}} \psi(\Lambda(X)),$$

where the orbits are now for D. This follows by averaging on each orbit as in (136.18). For X of rank $j > 0$ let E_X be its centralizer in $\bar{G}$ and $\mathcal{C}_K$ its full conjugacy orbit. Let T_X be the size of the E_X-orbit of the list of fixed data, namely (e, ℓ) for K and (e, ℓ, z) for Z, with the dual action on ℓ. Orbit–stabilizer gives

$$\frac{|\bar{H}|}{|\bar{G}|\cdot|\mathcal{C}|} = \frac{[\bar{H} : D]}{|\mathcal{C}_K| T_X} \leq \frac{4q^{2r}}{q^{2rj-2j^2}T_X}.$$

Here $[\bar{H} : D] \le q^{2r}$ counts the pairs of end coordinates. The lower bound on $|\mathcal{C}_K|$ follows from the commuting algebra calculation in Theorem 136.18. Always $j \le 2a$. Put $M = n^2 q^{4a^2}$.

For K, Proposition 136.19 gives $\sum_{\mathcal{C}} |S_{\mathcal{C}}| \le 10 q M q^{d(j-1)}$ also on these refined orbits. To see this, the middle changes of basis fix both ends. Each shear there has the form $1 + w\xi$, with ξ the covector of a left coordinate strictly after the first one and w in the middle space. It fixes e, ℓ. The transposed argument exchanges these two conditions. All averages therefore preserve each refined orbit, and its proof uses counts on the whole stratum for the remaining exceptions.

We give the corresponding argument for Z, writing D_X for the right-to-middle block of X, to distinguish it from the group D. On $D_X \neq 0$ transpose and reverse the flags. In the new coordinates the left middle block B' is nonzero, and the new terminal covector η' is the old fixed z under duality, up to a nonzero scalar. Middle changes preserving η' belong to the transformed group D. They send the new terminal vector z' uniformly over the affine hyperplane $\eta'(w) = \eta'(z') \neq 0$. This follows by extending a basis of $\ker \eta'$ by w. The term involving η' in the phase is constant under this average. If the last row of B' is nonzero, orthogonality on this affine space cancels its contribution unless that row is proportional to η'. In the latter case the row space of B' contains η'.

If the last nonzero row has index $i < a$, use the shear in Proposition 136.19, with $w \in \ker \eta'$. It fixes the two ends and the covector η', extended by zero on the arms. Thus the cancellation in each class again applies. The only exceptions in that proof have either the sum of the two middle ranks strictly below j, or the row space of B' containing η'. Each has size at most $M q^{d(j-1)}$. Indeed the first follows by counting the two rank factorizations as in that proposition, and in the second one row-space basis vector is prescribed. On $D_X = 0$ with $\operatorname{rank} B < j$ the first smaller count applies directly. Consequently the sum of the absolute class contributions from all the parts just considered for Z is at most $12 q M q^{d(j-1)}$. Here the nonvanishing-block condition survives the conjugations because B' remains nonzero under the middle changes and is fixed by the interior-row shear.

It remains to count $D_X = 0$, $\operatorname{rank} B = j > 0$ for Z. Now $XV = B(W) \subset L$. Choose a covector λ supported on W with $\lambda(z) = 1$. For each $v \in W \cap \ker B \cap \ker \lambda$, the transformation $1 + v\lambda$ commutes with X and sends z to $z + v$. It is invertible since $\lambda(v) = 0$. This gives at least q^{d-a-1} images of z. There is a vector $u \in \ker X$ with $\ell(u) = 1$. Indeed subtract from the last right basis vector a middle vector having the same image under X, which exists since $XV = B(W)$. For any covector ν vanishing on $XV + \langle e, z, u \rangle$, the map $1 + u\nu$ is invertible, commutes with X, fixes e, z, and takes ℓ under the dual action to $\ell - \nu$. There are at least q^{r-j-3} choices. Taking the orbit of z first and then the orbit under its stabilizer gives

$$T_X \ge q^{d-a-1+r-j-3} \ge q^{2r-5a-4}.$$

The total count on this part is at most $M q^{dj}$ by the full rank count in Proposition 136.19. The orbit lower bound applies to every class meeting the part.

To sum, use $(u+v)^2 \le 2(u^2+v^2)$ for the two absolute contributions within a class. For the signed smaller sums the bound per rank is at most $4q^{4a}(12qM)^2$ before this factor of two, since $r = d + 2a$ and $2j^2 - 4aj \le 0$. For the last part the count and its bound on T_X give at most $4q^{5a+4}M^2$ per rank. The zero orbit contributes $|\bar{H}|/|\bar{G}| \le 1$. Summing at most $2a$ positive ranks proves the displayed inequality, using $n = a + 1$ and $q \ge 2$. □

Theorem 137.7 (Every integral position at nonintegral frequency). *For $H = K, Z$, uniformly at the original split marks of the common interval and at all sufficiently large nodes, one has, for $t \ge 1$,*

$$\int_K |b_H(x; \alpha, \beta)|\, dx \le 4q^{100n^2 + t + 1 - r(c_\alpha + c_\beta)/2}.$$

In particular for a nonzero rational frequency with local indices I, D as in Theorem 137.2,

$$\int_K |b_H(x;\alpha,\beta)|\,dx \le \min\{1, I^{(k-n^7)/2}D^{-3}\}.$$

The latter statement is asserted for $t \ge 1$. At $t = 0$ the integral is exactly $A_H v_H / v_K$.

Proof. In the finite sum for (137.3) apply Lemma 137.5 to the two coefficient vectors with entries $A_i(x)$ of Theorem 137.4. The resulting bound is the operator norm times $(v_H/(v_J v_K))N_t^{-1}\sum_i |A_i(x)|^2$. Conjugating x by each k_i in its integral over K gives the mean square in Lemma 137.6. This proves the first estimate.

Put $s = c_\alpha + c_\beta$, so $1 \le t \le s$, $D = q^s$, and $I \ge q^{-s}$. On this interval n tends to infinity, and $r = k + 1 + 2a$. Therefore $4q^{100n^2+t+1} \le q^{(r-k+n^7-6)s/2}$ eventually, uniformly in s. The first bound is then at most $q^{-(k-n^7+6)s/2}$, which is at most both one and $I^{(k-n^7)/2}D^{-3}$ since $k > n^7$. At $t = 0$ the wave is identically one, and the last formula follows from the definition of A_H. □

The estimate here gives the frequency weight over all integral positions. At selected marks it does not provide a separate terminal mass factor multiplying this weight. For positions outside K, Theorem 137.4 controls the distant shells. The intervening nonintegral shells, the later Gram coordinates, and the comparison with the arithmetic sum still require estimates.

137.4 Unequal conductors and small nonintegral heights

Fix a position of height m in the sense of Section 136.10, and put $s = 2m + 1$. For $j \geq 1$ let $J(j) = 1 + \pi^j M_r(O)$. For every $h \in H$ the function $f_x(h) = \phi(hxh^{-1})$ depends only on $h \bmod \pi^s$. Indeed, put $y = hxh^{-1}$. For $j_0 \in J(s)$ one has $y^{-1}(j_0 - 1)y \in \pi M_r(O)$ since $\|y\|, \|y^{-1}\| \leq q^m$. Hence $j_0 y j_0^{-1} \in yJ$ and the right J-invariance of ϕ gives the assertion.

Proposition 137.8 (The derivative of the wave on a congruence subgroup)**.** *Suppose $t = \max(c_\alpha, c_\beta) > s$, set*

$$j_0 = \max\{s, \lceil t/2 \rceil\}, \qquad h_0 = t - j_0 \geq 1, \qquad A = \pi^t \alpha, \quad B = \pi^t \beta.$$

In the compact double integral (137.3) one may insert the indicator of

$$A(ge)\ell \equiv Be(\ell g^{-1}) \pmod{\pi^{h_0}}, \qquad g = h_1 h_2^{-1}. \tag{137.5}$$

If $c_\alpha \neq c_\beta$, then $b_H(x; \alpha, \beta) = 0$. If $c_\alpha = c_\beta$, the condition requires

$$(ge, \ell g^{-1}) \equiv (\lambda e, (A/B)\lambda \ell) \pmod{\pi^{h_0}}$$

for a unit λ. For a uniform $g \in H$ the probability of the condition is at most $4q^{1-(2r-2)h_0}$.

Proof. Replace h_1 by $(1 + \pi^{j_0} Y)h_1$, and average in the integral matrix Y with additive probability. Each left multiplier belongs to H and leaves $f_x(h_1)$ fixed by the height bound. The inverse multiplier is congruent to $1 - \pi^{j_0} Y$ modulo π^t, since $2j_0 \geq t$. For fixed g the ratio of the new wave to the old wave is therefore

$$\Psi\Big(\pi^{j_0 - t} \operatorname{tr}\left((A(ge)\ell - Be(\ell g^{-1}))Y\right)\Big).$$

Entrywise orthogonality gives precisely (137.5). If the conductors differ, exactly one of A, B is a unit. Both unscaled rank-one matrices are nonzero modulo π, so the condition is impossible.

For equal conductors read the matrix congruence in a basis starting with e in which ℓ is the last coordinate covector. In its last column all entries of ge below the first one must vanish, and in its first row every entry of ℓg^{-1} before the last must vanish. The corner entry gives the stated relation of the remaining coefficients, which are units by primitivity. Conversely these conditions suffice. There are at most q^{h_0} admissible pairs. The number of pairs occurring uniformly at this precision is at least $\frac{1}{4} q^{(2r-1)h_0 - 1}$ by the counts in Lemma 137.5. Division proves the bound. □

Theorem 137.9 (A terminal saving on small positive heights)**.** *Let $H = K, Z$ at one of the marked split places and let a nonzero rational frequency have $u = c_\alpha + c_\beta \geq 1$. Set $K_0 = k - n^7$ and $W = \min\{1, I^{K_0/2} D^{-3}\}$, with the local indices in Theorem 137.2. At all sufficiently large nodes,*

$$\int_{\{m \geq 1:\ 64r(m+1) \leq n^7 u\}} |b_H(x; \alpha, \beta)|\, dx \ \leq\ q^{-r} W. \tag{137.6}$$

If $c_\alpha \neq c_\beta$, all the heights with $2m + 1 < t$ have identically zero integrand, independently of the inequality in (137.6).

Proof. Put $l = \min(t, s)$. The reductions of H_t and H_l modulo π^s coincide. This is immediate when $t \leq s$. If $t > s$, a matrix of H_s modulo π^s has first column e and last row ℓ. Lift these entries exactly and lift the others arbitrarily. The resulting unit matrix belongs to H_t, retaining the terminal condition modulo π when present. Thus the normalized conjugacy average on the shell is the same for H_t and H_l, since f_x factors through this reduction.

Use the operator of Lemma 137.5 in the coset expression (137.3). After integrating the sum of squares of its coefficients on a shell $\mathcal{S}$, the preceding observation gives the upper bound

$$4q^{t+1-ru/2}\,\frac{v_H}{v_K}\,\frac{1}{v_J v_{H_l}^2}\int_{\mathcal{S}}|C_{H_l}(x)|^2\,dx.$$

To bound the last two factors use the orbit sum in the proof of Theorem 137.4. All contributing shells have $P=Q$, and on each such shell the sum of the absolute signed intersection counts is at most the unsigned count $q^{4a^2(m+1)+dP}$ of Lemma 136.24. Every contributing H_l-orbit has size at least $q^{2(d-2a)P}/(4[K:H_l])$. Since $[H:H_l]\le q^{2rl}$ and $P\le 2a(m+1)$, this proves the explicit bound

$$\int_{\mathcal{S}}|b_H(x;\alpha,\beta)|\,dx\le 16q^{\,t+1-ru/2+2rl+16a^2(m+1)}.$$

We now have $2rl\le 4r(m+1)$ and $16a^2\le r$ on the given interval at large nodes. Under the height condition in the theorem, $8r(m+1)\le n^7u/8$. Also $t+1+\log_q 16\le 6u$, and n tends to infinity. In particular

$$t+1+\log_q 16+5r(m+1)+2r+2(m+1)\le (r-k+n^7-6)u/2$$

for all sufficiently large nodes, using $r\ge k$, $r\ge 2$. It follows that the last shell bound is at most $q^{-2r-2(m+1)}q^{-(K_0+6)u/2}$. The last factor is at most W by $D=q^u$ and $I\ge D^{-1}$. At each m there are at most $2a(m+1)+1$ possible values of P. The sum of $(2a(m+1)+1)q^{-2(m+1)}$ over all $m\ge 1$ is $O(a+1)$, which is at most q^r eventually. This gives (137.6). The exact vanishing is Proposition 137.8. □

Here $q^{-r}\le v_Z/v_K=(q^r-1)^{-1}$, so the last positive-height estimate retains a terminal mass factor together with the frequency weight. For equal conductors the rank-one congruence controls the relative position. Its unweighted probability by itself gives no estimate for a coefficient concentrated on the same congruence. The region between the small-height estimate and Theorem 137.4 remains in the first-coordinate comparison.

137.5 The two end ideals in the first relative transform

There is a part of the first relative transform for which the frequency estimate, including the terminal mass factor, holds at every matrix height. It is determined by two ideals of O. Let $\mathcal{L} = O^r$, with the original ends e, ℓ. For $y \in G$, define

$$I_e(y) = \{\lambda(e) : \lambda \in \mathcal{L}^*,\ \lambda(y-1)\mathcal{L} \subset \pi O\},$$
$$I_\ell(y) = \{\ell(v) : v \in \mathcal{L},\ (y^{-1}-1)v \in \pi\mathcal{L}\}.$$

Each is the image of an O-module under a linear form. At height m each contains $\pi^{m+1}O$, by taking all rows or columns divisible by π^{m+1} and using primitivity. Write δ_e, δ_ℓ for their valuations truncated above at t. The ideals depend only on yJ. For $j \in J$ one has $yj - 1 = (y-1)j + (j-1)$, and $(yj)^{-1} - 1 = j^{-1}(y^{-1}-1) + (j^{-1}-1)$. These identities preserve the row and column conditions. The truncated valuations are constant on H_t-conjugacy orbits on G/J. Indeed the row and column modules are transported under conjugation, and the end coordinates on which they are evaluated are preserved modulo π^t by H_t.

Lemma 137.10 (Movement of both ends by a coset stabilizer)**.** *Let $y \in U$, let $\mathcal{D}_y$ be the stabilizer of yJ in K, and put $d_* = d - 2a$. The size of its orbit on the pair (e, ℓ) mod π^t is at least*

$$q^{d_*(2t-\delta_e-\delta_\ell)}.$$

If $\bar{z} \notin \overline{T_D}$, with T_D the saturated middle sublattice of Lemma 136.30, its orbit on the pair together with $\bar{z}$ has size at least

$$(q^{d_*}-1)q^{d_*(2t-\delta_e-\delta_\ell)}.$$

Proof. Use the decomposition $y = Y \oplus 1$ on an integral complement V_1 and a middle summand W_0 of rank $d_0 \geq d_*$ from Lemma 136.24. Thus $e \in V_1$ and ℓ vanishes on W_0. The element with diagonal blocks one and a single lower block $A : V_1 \to W_0$ stabilizes yJ if $A(Y-1)$ is divisible by π. This follows on multiplying out $y^{-1}gy - g$, which must lie in $\pi M_r(O)$. Each row of A can be chosen independently, and its values on e fill $I_e(y)$. Indeed adjoining the W_0-entries of a row does not change the test or its value on e. Consequently these transformations move e through at least $q^{d_0(t-\delta_e)}$ residues.

Likewise a single upper block $C : W_0 \to V_1$ may be chosen with $(Y^{-1}-1)C$ divisible by π. It fixes e and moves ℓ, in the dual action, by $-\ell C$ on W_0. The entries of this row fill $I_\ell(y)$ independently, giving $q^{d_0(t-\delta_\ell)}$ possible residues. Taking first the vector orbit and then the covector orbit under its stabilizer proves the first bound. Under the extra hypothesis we may choose W_0 and its complement as in Lemma 136.30, so that $\bar{z}$ has nonzero W_0-component. The full group $\mathrm{GL}(W_0)(O)$ acts trivially on V_1, commutes with y, and fixes both ends exactly. It moves that component through $q^{d_0} - 1$ values. Taking the remaining orbit in the stabilizer of the pair proves the second bound. □

For $t \geq 1$ and $0 \leq M \leq 2t$ put

$$\phi_M(y) = \phi(y)\mathbf{1}_{\delta_e(y)+\delta_\ell(y) \leq M}.$$

Let $b_{H,M}$ denote (137.3) with ϕ_M in both factors. This is a term in the original integral obtained by writing each coefficient as $\phi_M + (\phi - \phi_M)$.

Theorem 137.11 (Frequency and terminal mass at all heights with an end-ideal bound)**.** *For a nonzero rational frequency put $u = c_\alpha + c_\beta \geq 1$, $t = \max(c_\alpha, c_\beta)$, and*

$$M = \min\{2t, \lfloor n^7 u/(16d)\rfloor\}, \qquad W = \min\{1, I^{(k-n^7)/2}D^{-3}\}.$$

At every original split mark of the shorter interval, uniformly for all sufficiently large nodes,

$$\int_G |b_{K,M}(x;\alpha,\beta)|\,dx \le W, \qquad \int_G |b_{Z,M}(x;\alpha,\beta)|\,dx \le q^{-r}W.$$

Proof. On a shell $\mathcal{S}(m,P,P)$ let E_H be the normalized mean square of the H_t-conjugacy averages of ϕ_M, namely v_H/v_K times their squared integral divided by $v_J v_{H_t}^2$, with the compact averages taken using ordinary Haar measure. The folded coset identity in the proof of Theorem 137.4 expresses it as

$$E_H = \frac{v_H}{v_K}\sum_{\mathcal{C}} \frac{|S_{\mathcal{C}}|^2}{|\mathcal{C}|},$$

where only the H_t-orbits passing the ideal test are used and $S_{\mathcal{C}}$ is the original signed class sum. For each contributing orbit choose $y \in U$ in its preimage. Write $T_y = [\mathcal{D}_y : \mathcal{D}_y \cap H_t]$. This is the orbit size in the lemma for the pair when $H = K$, and for the pair and $\bar z$ when $H = Z$. The orbit–stabilizer identity is

$$\frac{v_H}{v_K|\mathcal{C}|} = \frac{[H : H_t]}{|\mathcal{C}_K|T_y}.$$

The full orbit has size $|\mathcal{C}_K| \ge q^{2(d-2a)P}/4$, and $[H : H_t] \le q^{2rt}$. Put $T_0 = q^{d_*(2t-M)}$ and $N = q^{4a^2(m+1)+dP}$. Every T_y is at least T_0, and the unsigned unipotent measure of the entire shell is at most N.

For Z, split the tested orbits at $T_y = q^{d_*}T_0/2$. Every unipotent representative of an orbit below this threshold must have $\bar z \in \overline{T_D}$ by the lemma. The total unipotent measure of these orbits is at most $2q^{a-d}N$ by Lemma 136.30 applied on the whole shell. This argument only requires the orbit size to be constant on an H_t-orbit, which follows from its index formula. The subspace $\overline{T_D}$ may depend on the representative. For all larger orbits use their denominator bound and the count N. Squaring these two counts separately shows that the usual bound with $T_y \ge T_0$ gains a factor at most $2q^{-d_*} + 4q^{2a-2d} \le 6q^{-d_*}$. For K use T_0 alone. Since $P \le 2a(m+1)$, we obtain in both cases

$$E_H \le 24q^{16a^2(m+1)+8at+d_*M-\epsilon d_*}, \qquad \epsilon = \mathbf{1}_{H=Z}.$$

Here $2(r-d_*) = 8a$. Lemma 137.5 and integration of its coefficient norm over the shell, as in Theorem 137.7, multiply this bound by at most $4q^{t+1-ru/2}$.

Sum first for $m < 1200n^3(t+1)$. There are at most $1200n^3(t+1)(2400an^3(t+1)+1)$ choices of m, P. Its base-q logarithm is $O(\log(2n)+\log(2t))$ uniformly for $q \ge 2$. At large nodes $16a^2(m+1) \le n^7u/16$, since $a < n$, $m+1 \le 2400n^3t$, $t \le u$, and n tends to infinity. Also $d_*M \le n^7u/16$. The remaining positive exponents $8at+t+1$, the logarithm of the shell count, $\log_q 192$, and the cost $\epsilon(r-d_*) \le 4a$, together are at most $n^7u/16$ eventually, uniformly for $t \ge 1$. Since $r \ge k$, $3n^7/16 \le (r-k+n^7-6)/2$ at these nodes. It follows that the entire bounded range contributes at most $\frac12 q^{-\epsilon r}q^{-(k-n^7+6)u/2} \le \frac12 q^{-\epsilon r}W$.

Finally, on the distant shells the proof of Theorem 137.4 applies to ϕ_M without an increased constant. The estimate of Lemma 137.3 sums absolute values over all H_t-orbits, and the truncation retains whole orbits, as proved above. The sum on the tail is therefore at most $q^{-6k(t+1)}$. This is at most $\frac12 q^{-r}W$, since $u \le 2t$, $W \ge q^{-(k-n^7+6)u/2}$, and $r/k \to 1$. Combining the two ranges proves the theorem. □

The test on both factors is substantive. For example take $y = 1 + \pi^{-m}ef$, where f is a primitive middle covector extended by zero on both arms. Then $I_e(y) = \pi^{m+1}O$ and $I_\ell(y) = O$: the row condition forces divisibility of $\lambda(e)$, whereas the last right basis vector satisfies the column condition and evaluates to one. Thus mixed terms and terms in which both coefficients exceed the ideal bound remain to be estimated.

137.6 Successive row integration at a deep end

Work at an original split mark with $U, J, K, e, \ell, z, \eta$ as in Sections 136.9 and 137.5. The basis vectors of the left arm L are $e_1 = e, \dots, e_a$, and L_j is the span of its first j vectors. Write $y = 1 + X \in U$, use $B : W \to L$ for its middle columns, and let β_j be row j of B. Fix a bounded shell of height m.

For $1 \le j \le a$ let p_j be the least valuation of $\lambda(e_j)$ over integral rows λ vanishing on L_{j-1} with $\lambda X \in \pi\mathcal{L}^*$. Define p_{a+1} in the same way with $\lambda|_L = 0$ and with z in place of e_j. Each ideal contains $\pi^{m+1}O$. In particular all these integers lie between zero and $m + 1$. We have $\min(t, p_1) = \delta_e(y)$.

Lemma 137.12 (A row change preserving all subsequent rows)**.** *Let $H = K$, or let H be the subgroup of K fixing the middle covector η modulo π, extended by zero on the arms. Let H_t impose the two end conditions of Section 137.2. In an H_t-orbit of right J-cosets, integrate the original $\chi(y)$ on U. For each $j = a, a-1, \dots, 1$ one may impose*

$$\beta_j(V_j) \subseteq \pi^{1-p_{j+1}}O \tag{137.7}$$

without changing the integral. Here V_j is an integral direct summand of $W(O)$ of codimension at most $a - j + 2$ determined by the rows strictly below j.

Proof. Fix those lower rows. Take a row κ vanishing on L_j with $\kappa X \in \pi\mathcal{L}^*$, for which $\kappa(e_{j+1})$ has valuation p_{j+1} if $j < a$, or $\kappa(z)$ has valuation p_{a+1} if $j = a$. It depends only on lower rows because κ annihilates L_j. To make a measurable choice, use a fixed list of representatives of integral rows modulo π^{m+2} with the prescribed leading zero entries. The test on κX and a valuation at most $m + 1$ are constant on these residue classes. Choose the first row attaining the least valuation. Let V_j be the common kernel on $W(O)$ of β_i for $i > j$, of $\kappa|_W$, and, when H fixes the covector, of η. This kernel is saturated since its quotient embeds in a vector space over F_w. It is therefore an integral summand of the asserted codimension.

For $w \in V_j$, extended by zero on the arms, put $b = Xw = Bw$. Then $b \in L_j$ and $\kappa w = \kappa b = 0$. The exact identity

$$(1 - w\kappa)y(1 + w\kappa y) = (1 + b\kappa)y =: y' \tag{137.8}$$

follows by multiplication using $yw = w + b$. The left multiplier belongs to H_t. It fixes both ends exactly since $\kappa(e) = 0$ and $\ell(w) = 0$, and in the covector case $\eta(w) = 0$ as well. The product of the two multipliers in left-to-right order is $1 + w\kappa X \in J$. Consequently $y'J$ is conjugate to yJ by the left multiplier. Also $y' \in U$, because b is supported in L_j and κy vanishes there. It has precisely the original rows below j.

For a fixed lower part this is a measure-preserving substitution on the upper rows. Indeed put $\gamma = (\kappa y)|_W$, determined by the lower part. Each upper middle row is transformed by $\beta_i \mapsto \beta_i(1 + w\gamma)$, and $\gamma(w) = 0$. The determinant of this linear map is one. The other upper entries undergo translations depending on the old middle rows and the fixed lower part. The upper block coordinates give product additive Haar measure on U by Lemma 136.24. Equivalently the inverse in (137.8) uses $-w$, since $X'w = b$ and $\kappa y' = \kappa y$, where $X' = y' - 1$. This substitution thus preserves the shell and the chosen orbit in both directions.

The only change in the character (136.21) is the j-th simple entry. If $j < a$, it changes before division by π by the unit c_j times $\beta_j(w)\kappa(e_{j+1})$, since $\kappa y e_{j+1} = \kappa(e_{j+1})$. If $j = a$, the change is $c_a\beta_a(w)\kappa(z)$, since $yz - z \in L$. All earlier simple entries are fixed because κy vanishes on L_j. All right-arm and middle-to-right entries are fixed. Averaging in w on V_j with probability Haar gives zero unless (137.7) holds, by the conductor of Ψ. The substitutions fix all tests previously imposed at indices greater than j, as those tests only read rows below or at their own index. This proves the assertion successively in the stated order. □

Theorem 137.13 (Signed sums at full first-end depth)**.** *Use either group H of Lemma 137.12 and any $t \ge 1$. On $\mathcal{S}(m,P,P)$ and for $0 \le h \le t$,*

$$\sum_{\mathcal{C}:\,\delta_e \ge h} \left| \int_{\mathcal{C}\cap U} \chi(y)\,dy \right| \le 2q^{16n^2(m+1)+d(P-h)}.$$

The condition on δ_e is constant on each of the indicated H_t-orbits.

Proof. The orbit invariance follows from the definition of I_e in Section 137.5. Apply the row lemma within each orbit and bound the sum of absolute values by the unipotent measure surviving all the tests and $p_1 \ge h$.

Consider the module generated by the rows of X modulo $\pi\mathcal{L}^*$ using integral coefficients. Its length is P, by the Smith entries of X and its transpose. Let ν be the length generated by the rows below the left arm. Adding rows $a, a-1, \dots, 1$ increases the length by $p_a, p_{a-1}, \dots, p_1$, respectively. In fact the new quotient is cyclic, and the least integer b such that the row times π^b belongs to the earlier module is exactly p_j. This follows by writing the linear relation as $\lambda X \in \pi\mathcal{L}^*$ and scaling its leading unit. Thus

$$P = \nu + \sum_{j=1}^{a} p_j.$$

This also implies $\pi^{p_j}\beta_j(V_j) \subseteq \pi O$, since all middle components of lower rows are either zero or among the β_i with $i > j$. Together with (137.7) this restricts the values of β_j on V_j to $\pi^{1-\min(p_j,p_{j+1})}O$.

Fix a profile $p_1, \dots, p_{a+1}$. For fixed lower rows, choose a basis splitting V_j inside $W(O)$. The basis change preserves the maximum entry bound and additive measure. Each of its complementary coordinates costs at most q^{m+1} in measure for the row β_j, since πO has measure one. Each coordinate on V_j costs at most $q^{\min(p_j,p_{j+1})}$. The other variable entries of the j-th row are in at most $2a$ outer coordinates. Integrating first row one and then proceeding downwards, these bounds give a factor at most

$$q^{a(3a+1)(m+1)+d\sum_{j=1}^{a}\min(p_j,p_{j+1})}$$

over the bottom rows. All conditional upper bounds used here are uniform in the fixed lower rows.

The bottom rows of X have entries only in the last a columns, comprising the blocks $D : R \to W$ and $E : R \to R$. For fixed ν their total measure within the entry bound is at most $q^{2a^2(m+1)+d\nu}$. Indeed their common row module in $(\pi^{-m}O/\pi O)^a$ has at most a generators, so there are at most $q^{a^2(m+1)}$ possible modules. There are at most a^2 entries of E, and each of the d rows of D then lies in a module of size q^{ν} modulo πO^a.

Requiring $p_{a+1} = v$ reduces this last upper bound by a factor at most $2q^{-(d-a)v}$. To verify it let $T_D = D(R) \cap W(O)$ be the saturated lattice of rank at most a used in Lemma 136.30. Every integral middle covector annihilating it, extended by zero on the arms, annihilates X. Thus $p_{a+1} \ge v$ implies $z \in T_D + \pi^v W(O)$, by a basis splitting this lattice. Integral middle changes of basis preserve the entry bounds, bottom measure and ν. For $v \ge 1$ a primitive vector is uniform under this action on $q^{dv}(1-q^{-d})$ residues, of which at most q^{av} belong to the given submodule. Averaging the indicator gives the bound, also valid trivially for $v = 0$. This calculation is a count on all bottom rows after bounding the upper ones, so it requires no change of basis inside a restricted orbit.

Now

$$\sum_{j=1}^{a}\min(p_j,p_{j+1}) = \sum_{j=1}^{a} p_j - \sum_{j=1}^{a}(p_j - p_{j+1})_+ \le \sum_{j=1}^{a} p_j - p_1 + p_{a+1}.$$

After multiplication the exponent involving d is at most $d(P-p_1)+ap_{a+1}$. There are at most $(m+2)^{a+1}$ profiles, and ν is determined by P and the first a entries. Since $p_{a+1}\le m+1$, $n=a+1$ and $m+2\le 2^{m+1}$, all remaining exponents fit inside $16n^2(m+1)$. This proves the result. □

The additional terminal condition permitted in the row change fixes a covector. The group fixing the original terminal vector acts differently on $1-w\kappa$. Its reduction sends $\bar z$ to $\bar z-\bar w\,\bar\kappa(\bar z)$. This dependence matters when using the same calculation on that side.

137.7 Row cancellation with the terminal vector fixed

Keep $p_1,\dots,p_{a+1}$ and the unipotent coordinates of Section 137.6. At the original vector stabilizer Z the shear used there must also preserve $\bar z$. We account for this condition by adjoining the column z to $X=y-1$. All row modules below are taken with integral coefficients modulo π times the corresponding ambient lattice, also when some entries have denominators.

Lemma 137.14 (The extra column and the terminal orbit)**.** *For $1\le i\le a+1$ let σ_i be the increase in row-module length on adjoining the column z to the rows from i downwards. At $i=a+1$ this means all middle and right rows. Each increase is zero or one. Moreover $\sigma_i=1$ precisely when some integral row λ vanishing on L_{i-1} satisfies $\lambda X\in\pi\mathcal{L}^*$ and $\lambda(z)\in O^\times$.*

For $1\le j\le a$ impose also $\lambda(z)\in\pi O$ in the definition of p_j, calling the new value p_j^Z. Set $p_{a+1}^Z=\max(1,p_{a+1})$. Then

$$\begin{aligned} p_j^Z &= p_j+\sigma_j-\sigma_{j+1},\\ p_{a+1}^Z &= p_{a+1}+\sigma_{a+1},\\ \sum_{j=1}^{a}\min(p_j^Z,p_{j+1}^Z) &\le P-\nu-p_1+\sigma_2+p_{a+1}. \end{aligned}$$

If $\sigma_2=1$, the stabilizer of yJ in K moves the pair of ends at precision t together with $\bar z$ through at least $\frac12 q^{d_*(2t-\delta_e-\delta_\ell+1)}$ positions, where $d_*=d-2a$.

Proof. Project the augmented row module to the old one. The kernel is a submodule of $O/\pi O$ in the last coordinate. Its nonzero elements come exactly from linear combinations of the specified rows satisfying the two tests on λ. The successive cyclic length increments for the augmented matrix are p_j^Z, just as for X in Theorem 137.13. Taking differences gives the first identity. The second follows since $\sigma_{a+1}=1$ precisely when $p_{a+1}=0$. The smallest positive valuation of the bottom evaluation at z under the extra constraint is then one, by multiplying a unit value by π. The last inequality follows by summing the first identities and subtracting at least $p_1^Z-p_{a+1}^Z$ in the positive successive drops.

If $\sigma_2=1$, take a witnessing λ with $\lambda(e)=0$. In the middle summand W_0 of Lemma 137.10, take $v\in W_0(O)$ with $1+\lambda(v)$ a unit. The transformation $g=1+v\lambda$ is integral and invertible. It fixes e,ℓ exactly, and $y^{-1}gy-g=v\lambda X$ belongs to $\pi M_r(O)$, so it stabilizes yJ. Since $\lambda(z)$ is a unit, the distinct residues of these v give distinct images of $\bar z$. At least half the residue vectors are allowed, because an excluded affine hyperplane has fraction q^{-1}, or is empty if λ vanishes on the summand modulo π. Its rank is at least d_*. Multiplying the resulting lower bound in the pair stabilizer by the pair orbit size of Lemma 137.10 proves the assertion. □

Theorem 137.15 (Two end depths in the refined orbital mean square)**.** *Let $H=K$ or Z, $t\ge1$, and $\mathcal{S}=\mathcal{S}(m,P,P)$. With ordinary Haar measure in C_{H_t}, at the original split marks,*

$$\frac{v_H}{v_K}\frac{1}{v_J v_{H_t}^2}\int_{\mathcal{S}}|C_{H_t}(x)|^2\,dx\le 256(t+1)^2q^{64n^2(m+1)+8at}. \tag{137.9}$$

All end ideals and all mixed contributions to the coefficient mean square are included.

Proof. Group the H_t-orbits meeting U by $p=\delta_e,\ s=\delta_\ell$. Put $T_y=[\mathcal{D}_y:\mathcal{D}_y\cap H_t]$ for the full coset stabilizer $\mathcal{D}_y$. This index is constant on each refined orbit. For K assign all orbits to type $\epsilon=0$. For Z assign type $\epsilon=1$ exactly when

$$T_y\ge\tfrac12 q^{d_*(2t-p-s+1)}.$$

In either case we have $T_y \ge \frac{1}{2} q^{d_*(2t-p-s+\epsilon)}$ by Lemma 137.10. For Z every unipotent representative on a type-zero orbit has $\sigma_2 = 0$, by Lemma 137.14.

For K, the sum of the absolute signed class sums in this group of orbits has the bound of Theorem 137.13 with $h = p$. For Z perform the same successive shears using the covectors chosen with $\kappa(z) \in \pi O$. For $j < a$ their leading valuations are p^Z_{j+1}; for $j = a$ their evaluation at z has valuation p^Z_{a+1}. The left multiplier $1 - w\kappa$ now fixes $\bar{z}$ for every integral w. The transformation, measure, and phase identities in Lemma 137.12 apply verbatim. All the substitutions remain in the original refined orbit. Thus the integral on each orbit can be cut by all the resulting lower-row tests. The middle entries of row j, on the summand in that lemma, now cost at most $q^{d\min(p^Z_j, p^Z_{j+1})}$: a relation in the augmented module also gives the required relation on the middle columns of X.

For a fixed profile, the bottom coordinate count is still the one in Theorem 137.13, including the gain $2q^{-(d-a)p_{a+1}}$ with the original bottom depth. We can use that bound because ν here is the length for the unaugmented bottom rows. The last inequality in the preceding lemma shows that the total exponent involving d, after the bottom count, is at most $d(P - p_1 + \sigma_2) + ap_{a+1}$. In the selected group of orbits $\sigma_2 \le \epsilon$ for all surviving representatives. We may sum profiles of both the original and augmented valuations. Each lies in $[0, m+1]$; there are at most $(m+2)^{2a+2}$ choices, and ν is determined by the original profile. The other row and submodule costs are as in that theorem. In both groups H this gives

$$\sum_{\mathcal{C}} |S_{\mathcal{C}}| \le 2q^{20n^2(m+1)+d(P-p+\epsilon)}.$$

We also have $2q^{20n^2(m+1)+d(P-s)}$ for the same sum. Indeed pass to the dual space by inverse transposition and reverse the arm order. This is a group isomorphism, preserving J, right cosets and the two end conditions. The first end valuation there is the original δ_ℓ, by transposing the defining column test on $y^{-1} - 1$. The two Smith lengths are interchanged and both equal P. The unipotent Haar normalization is preserved as $U \cap J$ maps to its counterpart. On consecutive blocks inversion negates the entries, and transposition exchanges the terminal vector and covector up to units. Hence the character has the same form with unit coefficients and with terminal pairing still a unit. The original Z now fixes the new terminal covector modulo π. This is precisely the second group in Theorem 137.13. Its estimate holds also on any subset of the orbits at depth s, since it sums absolute values. This proves the second bound.

Take the smaller of the two bounds, giving $2q^{20n^2(m+1)+d(P-\max(p-\epsilon,s))}$. In the normalized orbital sum for the left of (137.9), the multiplier on $|S_{\mathcal{C}}|^2$ is

$$\frac{[H : H_t]}{|\mathcal{C}_K| T_y} \le 8q^{2rt-2d_*P-d_*(2t-p-s+\epsilon)}$$

by Lemma 136.24. Bounding the sum of squares by the square of the preceding bound leaves, besides $40n^2(m+1) + 4aP + 8at$, the exponent

$$d_*(p+s-\epsilon) - 2d\max(p-\epsilon, s) = -d|p-s-\epsilon| - 2a(p+s-\epsilon) \le 2a.$$

As $P \le 2a(m+1)$, these terms are covered by the exponent in (137.9). There are at most $2(t+1)^2$ groups of orbits. The stated constant includes them and the numerical factors above. □

The possible loss of one residue digit from the vector condition has thus been included in the same estimate as the two end depths. No bound on either depth relative to the Fourier conductor is imposed in (137.9).

137.8 The complete unrestricted mass and its principal term

At zero frequency the full integral for K has principal contribution one in our normalization. We estimate its complement over all heights. Retain B, D, S for the blocks of $y \in U$ in Section 136.13. The right-to-middle block of y^{-1} is $D^- = -DS^{-1}$.

Lemma 137.16 (A saving on every nonprincipal shell)**.** *For the K-orbits of right J-cosets in $\mathcal{S}(m, P, P)$ with $P > 0$,*

$$\sum_{\mathcal{C}} |S_{\mathcal{C}}| \le 6q^{16n^2(m+1)+d(P-1)}.$$

Proof. Partition the unipotent integral into three regions. First take $B \notin \pi M_{a,d}(O)$. On this region use the shear (137.8). At step j the entire middle matrix changes by $B \mapsto B(1 + w\gamma)$, including the unchanged lower rows since those rows vanish on w. Here $\gamma = (\kappa y)|_W$ is integral because κ is integral and $\kappa X \in \pi\mathcal{L}^*$. Also $\gamma(w) = 0$. Thus $1 + w\gamma$ and its inverse are integral, so that the first region is preserved. Every shear stays in the K-orbit of the original coset. The row proof of Theorem 137.13 therefore applies within this region and each orbit, without imposing a positive end depth.

Retain the actual positive drops in the last line of its count. Writing $v = p_{a+1}$, the saving in the exponent of d is

$$v + \sum_{i=1}^{a} (p_i - p_{i+1})_+ \ge \max_{1 \le i \le a} p_i.$$

The residual cost av is already included in $16n^2(m + 1)$. Here at least one $p_i > 0$: the bottom rows of X have zero middle component, while at least one of the top rows has nonzero middle component modulo πO. Thus the full row module has larger length than the bottom one. The first region contributes at most $2q^{16n^2(m+1)+d(P-1)}$.

In the second region $B \in \pi M_{a,d}(O)$ and $D^- \notin \pi M_{d,a}(O)$. Use the inverse-transpose isomorphism and reverse the arms as in Theorem 137.15. The new left middle block is the transpose of D^-, up to the basis reversal. The old test on B becomes the test on the right-to-middle block of the inverse matrix in the new coordinates. This latter block stays fixed under every left row shear. Indeed $y'^{-1} = y^{-1}(1 - b\kappa)$, and $y^{-1}b \in L_j$, so the inverse changes only in the top j rows. Thus both tests of the second region survive the calculation. The same bound as in the first region follows.

In the third region B, D^- both have entries in πO. For each right unitriangular S, the substitution $D \mapsto -DS^{-1}$ preserves additive Haar on the rows of D as $|\det S| = 1$. Thus the total conditional measure allowed for B, D is at most one, using $\mathrm{vol}(\pi O) = 1$ for each coordinate. All other upper entries together cost at most $q^{2a^2(m+1)}$. This estimates the sum over disjoint classes by absolute measure and is within the displayed bound since $P \ge 1$. Adding the three contributions proves the lemma. □

Theorem 137.17 (The unrestricted position mass on the full group)**.** *At every original split mark, uniformly for all sufficiently large nodes of the shorter interval,*

$$0 \le \frac{\int_G |C_K(x)|^2\,dx}{v_J v_K^2} - 1 = \frac{\int_{G\setminus J} |C_K(x)|^2\,dx}{v_J v_K^2} \le q^{-2k+20000n^5}.$$

Proof. The contribution from J is one because J is normal in K and $\phi = 1$ there. A shell with positive mass has $P = Q$. Also $P = 0$ exactly on J, since $P = 0$ means $(x - 1)\mathcal{L} \subset \pi\mathcal{L}$. For $P > 0$, apply the full-orbit formula (136.30) and the lower orbit bound $q^{2(d-2a)P}/4$ from Lemma 136.24. The preceding lemma yields at most

$$144q^{32n^2(m+1)+4aP-2d} \le 144q^{40n^2(m+1)-2d}$$

per shell, using $P \le 2a(m+1)$. When $m < 400n^3$, there are at most $400n^3(800an^3+1)$ such shells. Their sum is at most $q^{-2k+17000n^5}$ eventually, since $d = k+1$ and $m+1 \le 400n^3$.

For $m \ge 400n^3$ apply the signed large-shell bound in the proof of Theorem 136.36, namely $400q^{-31kL/16}$ per shell with $L = \lfloor m/(200n^3) \rfloor$. Now $L \ge 2$. There are at most $200n^3(800an^3L+1)$ pairs of shell indices at each L. Summing the geometric series with its linear factor gives at most q^{-3k} at large nodes, since $62/16 > 3$ and the polynomial in a, n is absorbed by the difference of exponents. Adding proves the stated inequality. On our interval $n^5 = o(k)$, so the bound tends uniformly to zero. □

In particular this calculation determines the zero-frequency principal term before any restriction to a terminal-vector fixer. The exact relative-residue factor ρ_w at an unselected mark is still given by Corollary 136.37.

137.9 The full first relative Fourier bound at the marked places

For the actual first relative coordinate of (137.3) write $\mathcal{B}_H(\alpha,\beta) = \int_G |b_H(x;\alpha,\beta)|\,dx$. The coefficients in this definition are the original ϕ in both variables.

Theorem 137.18 (All positions and both unrestricted end depths)**.** *At every original split mark, for both $H = K$ and $H = Z$, put $u = c_\alpha + c_\beta$, $t = \max(c_\alpha, c_\beta)$, and $K_0 = k - n^7$. For a nonzero rational frequency with $u \geq 1$, at all sufficiently large nodes of the shorter interval,*

$$\mathcal{B}_H(\alpha,\beta) \leq q^{-n^7u/4} \min\{1, I^{K_0/2}D^{-3}\}. \tag{137.10}$$

In particular the factor q^{-r} in addition to the indicated frequency weight is available whenever $n^7u \geq 4r$. If $u = 0$, the values are the zero-frequency masses. They satisfy

$$1 \leq \mathcal{B}_K \leq 1 + q^{-2k+20000n^5}, \qquad \mathcal{B}_Z \leq 2q^{-9k/10}.$$

Proof. On a fixed shell, use the coset-pair expression for $b_H(x;\alpha,\beta)$ in Theorem 137.4. Apply the finite Fourier operator bound of Lemma 137.5 to its two coefficient vectors. Conjugating x by each coset representative preserves the shell and its measure. Thus the integral on the shell is at most $4q^{t+1-ru/2}$ times the left side of (137.9). The latter includes the original coefficients for all end valuations. In particular mixed contributions between different depths in the two factors are controlled by the same operator estimate.

For $m < 1200n^3(t+1)$, sum this bound over at most $1200n^3(t+1)(2400an^3(t+1)+1)$ shells. The total, for all large nodes, is at most

$$\tfrac{1}{2}q^{-ru/2+n^7u/8}.$$

Indeed $64n^2(m+1) \leq 153600n^5u$ since $t \leq u$. The linear terms $8at + t + 1$, the logarithms to base q of the shell count and of $2048(t+1)^2$, and this last multiple of n^5u, together are at most $n^7u/8$ eventually. This is uniform for $u \geq t \geq 1$, since $\log(2t) \ll t$, $q \geq 2$, and $n \to \infty$.

For the other shells Theorem 137.4 gives $q^{-6k(t+1)}$. This is also at most $\frac{1}{2}q^{-ru/2+n^7u/8}$, since $u \leq 2t$ and $r/k \to 1$. Hence the full integral is at most $q^{-ru/2+n^7u/8}$. As $r \geq k$, at large nodes this is bounded by

$$q^{-n^7u/4}q^{-(k-n^7+6)u/2} \leq q^{-n^7u/4}\min\{1, I^{K_0/2}D^{-3}\}.$$

The last inequality uses $D = q^u$, $I \geq D^{-1}$ and $K_0 > 0$. At $u = 0$ the relative wave is identically one on H^2, so $\mathcal{B}_H = \int_G |C_H|^2/(v_J v_H v_K)$. Apply Theorems 137.17 and 136.36. □

Corollary 137.19 (The selected factors in the first rational frequency sum)**.** *Use the field E, shorter interval, selected set Ξ, and indices j_w of Theorem 134.11. Let T contain both primes of E at every $w \in \Xi$ and exclude the unselected marks as in Proposition 136.39. In the common first-coordinate scale of Theorem 136.4, set*

$$V_w(y) = I_w(y)^{K_0/2}D_w(y)^{-3}, \qquad Z_w^0 = \frac{v_{Z_w}}{v_{K_w}} = \frac{1}{q_w^r - 1}.$$

Write $\mathcal{B}_{Z_w}(y)$ for the full first transform at the two local components of the rational frequency. Then

$$\sum_{0\neq y\in\mathcal{O}_{E,T}} W_{T,c,K_0}(y) \prod_{w\in\Xi} \frac{B_0 q_w^{n-j_w}\mathcal{B}_{Z_w}(y)}{Z_w^0 V_w(y)} \leq b^{-7kf} \tag{137.11}$$

at all sufficiently large nodes. The terminal volume in the denominator permits comparison at its original principal scale.

Proof. Both components at a marked split place use an additive character of conductor O; the trace normalization changes the frequencies there only by units. Thus their valuations give I_w, D_w in the rational weight. If $u_w \geq 1$, the theorem gives $\mathcal{B}_{Z_w}(y) \leq V_w(y)$. If $u_w = 0$, it gives $\mathcal{B}_{Z_w}(y) \leq 2q_w^{-9k/10} \leq 1 \leq V_w(y)$. The remaining multiplier per selected place is at most

$$B_0 q_w^{a+r} \leq b^{6k+11(a+r)} \leq b^{18k},$$

using $B_0 = b^{6k}$, $q_w \leq b^{11}$, $n - j_w \leq a$, and $r = k + 1 + 2a$ with $a = o(k)$. There are fewer than $f/4$ selected places. Their product is at most b^{5kf}. The sum of W_{T,c,K_0} itself is at most b^{-12kf} by Theorem 136.4. This proves (137.11), including the empty product. □

The factors at the selected split places in (137.11) are now estimates of the actual first relative transform with no test on end ideals. At the other places the formula still uses the explicit arithmetic weight. Replacing those factors by their actual transforms, and passing through later pivots and unequal rational heads, are additional steps in the projected-period comparison.

137.10 The first transform at standard unmarked split places

We change the level as well as the residue cardinality. At a split place with the standard integral lattice take $G = \mathrm{GL}_r(F_w)$, $K = \mathrm{GL}_r(O)$ and now $J = K$. Both K and $U \cap K$ have volume one. Assume the simple character has additive conductor O, unit coefficients on the arms, and primitive integral terminal vector and covector. Define $\phi_0(uy) = \chi_0(u)$ for $u \in U$, $y \in K$, and extend it by zero off UK. It is well defined since $\chi_0 = 1$ on $U \cap K$. In this normalization $\mathcal{B}_w^0$ is the integral of the absolute value of (137.3) with $H = J = K$ and $\phi = \phi_0$. Indeed the image of K in $U\backslash G$ carries its quotient measure from K with fiber $U \cap K$ of volume one. The relative wave is unchanged on both U-orbits as in that equation.

These hypotheses hold in the integral split model of Theorem 134.8 at all split places outside a finite set at the node. The fixed flag basis and its dual, the form and its inverse, and the class representatives are integral there. The level M is a unit, and the normalization by the different makes the trace character have integer-ring conductor. A fixed nonzero terminal vector and its paired covector are primitive away from finitely many places. The arguments below are uniform over the places satisfying the hypotheses.

Set $P = \operatorname{length}\big(((x-1)\mathcal{L} + \mathcal{L})/\mathcal{L}\big)$, where length is taken on the quotient, and use the corresponding integer for x^{-1}. In particular P is the length of $(x\mathcal{L}+\mathcal{L})/\mathcal{L}$. Both lengths and m are invariant under multiplication by K on either side. For a contributing coset, $\operatorname{ord}\det x = 0$, so the lengths agree by the elementary divisors of x. Write $\mathcal{S}(m,P)$ for such a shell. Except on K one has $1 \le m \le P$: a largest pole occurs in x or x^{-1} and contributes at least its order to the respective length. On a shell meeting U, $P \le 2am$ by applying the Smith formula to $x-1$ at a unipotent representative of rank at most $2a$ after subtracting the identity.

Lemma 137.20 (Signed shell estimates at level one)**.** *Let K_t fix both ends modulo π^t, with $K_0 = K$ in this lemma. Use the valuations δ_e, δ_ℓ of Section 137.5 truncated at t, now with the row and column conditions taking values in the integral lattices. On the K_t-orbits of right K-cosets in $\mathcal{S}(m,P)$, at prescribed end depths p, s,*

$$\sum_{\mathcal{C}} |S_{\mathcal{C}}| \le 2q^{16n^2(m+1)+d(P-\max(p,s))}.$$

For $P > 0$ the unrestricted sum over all these orbits is at most

$$6q^{16n^2(m+1)+d(P-L)}, \qquad L = \left\lceil \frac{P}{2a^2+1} \right\rceil. \tag{137.12}$$

Proof. Here the class sum is $\int_{U\cap\mathcal{C}} \chi_0(y)\,dy$; the intersection with a right coset has volume one if nonempty and the character is constant there. In the row proof of Theorem 137.13 take the row module modulo $\mathcal{L}^*$ and impose $\kappa X \in \mathcal{L}^*$. With p_j the corresponding least valuations, the row length identity $P = \nu + \sum p_j$ is unchanged. In (137.8) the product of the multipliers is $1 + w\kappa X \in K$: it is integral with integral inverse $1 - w\kappa X$, since $\kappa X w = 0$. The left multiplier still fixes both ends. The phase has no division by π, so its row test is $\beta_j(V_j) \subseteq \pi^{-p_{j+1}}O$. The relation modulo the integral lattice gives the second bound $\beta_j(V_j) \subseteq \pi^{-p_j}O$. The measure of their intersection in one coordinate is precisely $q^{\min(p_j,p_{j+1})}$, since O now has volume one. Every discarded outer coordinate costs at most q^m. The row substitution, its inverse, and its Jacobian are the same algebraic identities as before. The bottom module uses $(\pi^{-m}O/O)^a$, and $p_{a+1} = v$ still forces $z \in T_D + \pi^v W(O)$, with the same probability bound. Thus the full count of Theorem 137.13, in this normalization, costs no more than $2q^{16n^2(m+1)+d(P-p)}$. Inverse transposition gives the bound with s, hence the first assertion.

For the second assertion define b as the least nonnegative integer with $\pi^b B$ integral, and define b^- the same way for $D^- = -DS^{-1}$. On $b > a(L-1)$, the projection of the row

module onto its middle coordinates has length at least b by a largest Smith pole of B. The bottom module projects to zero. Hence $\sum_{j=1}^{a} p_j \ge b$ and some $p_j \ge L$. The calculation in Lemma 137.16 saves this maximum in the exponent of d, since $v + \sum (p_j - p_{j+1})_+ \ge \max p_j$. On $b \le a(L-1)$, $b^- > a(L-1)$ use inverse transposition. These two regions are preserved by their respective shears: the tested left block is multiplied on the right by an integral unit matrix, and the opposite right-to-middle inverse block is unchanged. On the remaining region the measure of B, D^- is at most $q^{2a^2 d(L-1)}$, using the substitution $D \mapsto -DS^{-1}$ at fixed S. The other entries cost at most $q^{2a^2 m}$. Since $2a^2(L-1) \le P - L$ by the definition of L, summing the three regions proves (137.12). □

Theorem 137.21 (The full first transform at a standard split place)**.** *Write $u = c_\alpha + c_\beta$, $t = \max(c_\alpha, c_\beta)$ and $K_0 = k - n^7$. Uniformly at the places just described and at all sufficiently large nodes,*

$$1 \le \mathcal{B}_w^0(0,0) \le 1 + q^{-2k+1000n^4},$$
$$\mathcal{B}_w^0(\alpha,\beta) \le q^{-n^7 u/4} \min(1, I^{K_0/2} D^{-3}) \qquad (u \ge 1)$$

for nonzero rational frequencies in the second line. Integral local frequencies give the same value as $(0,0)$.

Proof. We detail the orbit denominator at this level. For a unipotent representative write $y = Y \oplus 1_{d_0}$ as in Lemma 136.24, with $d_0 \ge d_* = d - 2a$. A matrix $g \in K$ stabilizes yK precisely when $y^{-1}gy - g$ is integral. Indeed $y^{-1}gy$ has unit determinant, so integrality also gives an integral inverse. In the two off-diagonal rectangles of g, the individual rows must take $Y - 1$ to integral rows, and the columns must be taken to integral columns by $Y^{-1} - 1$. Each condition has index q^P by the Smith formula. The $2d_0$ conditions are independent in a uniform integral matrix before conditioning on invertibility. Its probability of invertibility is $\prod_{i=1}^{r}(1-q^{-i}) \ge 1/4$. Thus the full orbit size is at least $q^{2d_* P}/4$. In its stabilizer the independent lower and upper rectangular shears of Lemma 137.10 are valid with integral row and column conditions. The same orbit argument gives at least $q^{d_*(2t-p-s)}$ images of the end pair.

The coefficient mean square is the sum of squared class sums divided by orbit sizes, by Haar averaging on these right cosets. At level K_t its inverse orbit sizes are bounded by $[K : K_t]$ divided by the full orbit size and the number of end images under its stabilizer. Here $[K : K_t] \le q^{2rt}$. Using the first bound in the lemma at each p, s, summing at most $(t+1)^2$ pairs and using $P \le 2am$ gives at most

$$16(t+1)^2 q^{40n^2(m+1)+8at}$$

on the shell. The depth terms satisfy $d_*(p+s) - 2d\max(p,s) \le 0$. The estimate of Lemma 137.5 for the relative wave on end pairs involves only K, e, ℓ and its additive conductor. It therefore multiplies the last bound by at most $4q^{t+1-ru/2}$ for $t \ge 1$.

Alternatively use (137.12) and the operator norm bound one, discarding the extra end images in the orbit denominator. The resulting shell estimate is

$$144 q^{2rt+40n^2(m+1)-2dL}.$$

For fixed $L \ge 1$ there are at most $(2a^2+2)^2 L^2$ possible shells and $m+1 \le (2a^2+2)L$. At $t = 0$ their sum is at most $q^{-2k+1000n^4}$ for large nodes: after bounding $40n^2(2a^2+2) \le 160n^4$, sum the geometrically decreasing series with its quadratic factor. The contribution on K is exactly one. This proves the first assertion.

For $t \ge 1$, at heights $m \ge 20(2a^2+1)(t+1)$ one has $L \ge 20(t+1)$. For large nodes both $2rt$ and $40n^2(m+1)$ are at most $kL/8$. The series over these L is therefore at most $q^{-10k(t+1)}$, by

the same shell count. In the remaining range the end-pair operator estimate gives a sum at most $\frac{1}{2}q^{-ru/2+n^7u/8}$. To see this, the exponent $40n^2(m+1)$ is $O(n^4u)$, the other positive exponents are $O(nu)$, and the shell count and $(t+1)^2$ are polynomial in $n, t+1$, since $P \le 2am$. Their logarithms and numerical constants are absorbed uniformly by $n^7u/8$. Adding the distant sum and applying the last exponent comparison of Theorem 137.18 proves the second assertion. □

Corollary 137.22 (Product over the standard unmarked split places)**.** *In $E = \mathbb{Q}(\sqrt{-d}, \mu_{q_1})$ of Corollary 11.30, let $\mathcal{V}_0$ be any set of standard unmarked split places of $F = E^+$ with residue characteristic prime to q_1. Here d in the field denotes the quadratic prime. For a nonzero rational frequency, write u_w, I_w, D_w for its local data. Then*

$$\prod_{w\in\mathcal{V}_0} \mathcal{B}_w^0(y) \le \exp\bigl(O(q_1^{2-k})\bigr) \prod_{\substack{w\in\mathcal{V}_0\\ u_w>0}} \min\bigl(1, I_w^{K_0/2} D_w^{-3}\bigr).$$

The product of the zero-frequency masses lies between one and the same exponential. Its implied constant is absolute.

Proof. At a split place in this set, the residue field of E has the same size q_w as that of F. It contains all q_1-th roots of unity since reduction is injective on roots of order prime to the residue characteristic. For example, in the kernel of reduction the first nonzero term of $(1+h)^{q_1}-1$ has the same valuation as h. Hence $q_w \ge q_1+1$. We have $f = [F:\mathbb{Q}] \le \varphi(q_1) \le q_1$, and at most f places can have any prescribed prime-power residue size. At large nodes $q_w^{-2k+1000n^4} \le q_w^{-k}$, so

$$\sum_{w\in\mathcal{V}_0} q_w^{-2k+1000n^4} \le f \sum_{j>q_1} j^{-k} \le q_1^{2-k}/(k-1).$$

Apply the theorem at the finitely many nonintegral local frequencies and use $1+x \le \exp x$ at the others. This also proves convergence and the assertion at zero frequency. □

This supplies the standard split factors without charging a constant at each of the infinitely many good places. Nonsplit places have a Hermitian relation between their two arms, so the independent middle blocks used in the shell count above require a different treatment there.

138 The first relative Gram transform at inert places

138.1 The first coordinate on an unramified Hermitian lattice

Let A/F_w be an unramified quadratic field extension. Write O_A, O for the rings of integers, $\pi \in O$ for a common uniformizer, and $q = \#(O/\pi)$. Let $\mathcal{L}$ be a self-dual Hermitian O_A-lattice of rank r, with form h linear in the second variable. Write $Q(v) = h(v, v)$. In the standard model of Section 134.4, the first flag vector e lies in an integral hyperbolic pair. We use probability Haar measure on $K = \mathrm{U}(\mathcal{L})$, and take level $J = K$. Fix an additive character ψ of F_w of conductor O, and put $\psi_A = \psi \circ \mathrm{Tr}_{A/F_w}$. The trace pairing on O_A is perfect over O, since its reduction is the nondegenerate trace pairing on a separable quadratic extension. Thus ψ_A has conductor O_A.

For $\alpha \in A$, let $c = \max(0, -\operatorname{ord}_A \alpha)$, taking $c = 0$ for $\alpha = 0$. The first relative Gram entry gives the character

$$\omega_\alpha(g) = \psi_A(\alpha h(e, ge)).$$

Both copies of U preserve this function because U fixes e. Let ϕ_0 be the right K-invariant function supported on UK with $\phi_0(uk) = \chi_0(u)$, where the local Bessel character χ_0 is trivial on $U \cap K$. In (137.3) take both compact integration groups equal to K, and use ϕ_0, ω_α. Denote the resulting integral by $b^0_K(x; \alpha)$. For $x \in K$ both factors involving ϕ_0 are one, so the contribution from K to $\int |b^0_K(x; \alpha)|\, dx$ is $\left| \int_K \omega_\alpha(g)\, dg \right|$.

Lemma 138.1 (The compact orbit of a primitive isotropic vector)**.** *For $c \geq 1$ put $V_c = \mathcal{L}/\pi^c\mathcal{L}$. Let Y_c consist of the vectors of V_c nonzero modulo π whose norm is zero modulo $\pi^c O$. For Haar-uniform $g \in K$, the image of ge is uniform on Y_c. The counting probability $\rho_c = |Y_c|/|V_c|$ satisfies*

$$\rho_c = q^{-c}(1 + (q-1)(-1)^r q^{-r} - q^{1-2r}), \qquad \rho_c \geq (1 - q^{-2})q^{-c}.$$

Proof. For primitive v the map $u \mapsto \mathrm{Tr}\, h(v, u)$ is onto modulo π. Indeed self-duality makes $h(v, \cdot)$ onto O_A, and the residue trace is onto. In passing from precision j to $j+1$ the new norm digit of $v + \pi^j u$ changes by this trace, for $j \geq 1$. Each primitive solution has exactly q^{2r-1} lifts one digit higher. Successive lifts and completeness give an exactly isotropic primitive lift in $\mathcal{L}$.

Given such a vector v, choose f with $h(v, f) = 1$. Choose $b \in O_A$ with $b + \bar{b} = Q(f)$, and replace f by $f - bv$. The trace equation is soluble integrally because the residue trace has a unit in its image. The new f gives a hyperbolic partner of v. The plane splits off orthogonally since its Gram determinant is a unit. Every unimodular Hermitian lattice over O_A has an orthonormal basis. In the residue space some vector has nonzero norm, since otherwise applying the norm to $v + \lambda w$ for all residue scalars λ would give $\mathrm{Tr}(\lambda h(v, w)) = 0$ for all λ, v, w, contradicting nondegeneracy. A lift has unit norm. The norm on $O_A^\times$ is onto $O^\times$: it is onto in the cyclic multiplicative groups of the residue fields, and a correction $1 + \pi^j t$ changes its j-th digit by $\mathrm{Tr}\, t$. Thus the lift can be scaled to norm one. Split off that line and induct. Applying this classification to the two hyperbolic complements gives an integral isometry taking e to v. This proves the orbit assertion, including at places above two.

For the count modulo π, diagonalize the lattice as just proved. For each nontrivial additive character of the residue field, the sum of its values on the norms in one residue coordinate equals $1 + (q+1)(-1) = -q$, by the fiber sizes of the norm. Character orthogonality in the norm equation therefore gives $q^{2r-1} + (q-1)(-1)^r q^{r-1}$ zeros. Remove the zero vector and use the one-digit lifting count above. For the lower bound write a vector in the fixed hyperbolic plane and its complement as $xe + w + yf$. Choose the residue $x \neq 0$ and choose w freely. The equation for y is one surjective trace equation, with q solutions. This gives at least $(1 - q^{-2})q^{-1}$ at the first digit and the asserted lower bound at every precision. □

Theorem 138.2 (The principal first transform at an inert place)**.** *If $c = 0$, the average of ω_α on K is one. If $c \geq 1$, its exact value is*

$$\int_K \omega_\alpha(g)\, dg = \frac{(1-q^{-1})(-1)^{rc}q^{-rc} - \mathbf{1}_{c=1}q^{-2r}}{\rho_c}. \tag{138.1}$$

In particular the absolute value is at most $3q^{-(r-1)c}$. At the nodes of Theorem 137.18, for nonintegral rational frequency this contribution is at most $I_w^{(k-n^7)/2}D_w^{-3}$.

Proof. The assertion at $c = 0$ follows from integrality. For $c \geq 1$ write $\gamma = \pi^c\alpha$, a unit, and take counting probability on V_c. To compute the unnormalized average over $Q(x) = 0 \bmod \pi^c$, insert its indicator

$$q^{-c}\sum_{s\in O/\pi^c} \psi(\pi^{-c}sQ(x))$$

alongside $\psi_A(\pi^{-c}\gamma h(e,x))$. If s is a nonunit, translation by $\pi^{c-1}u$ fixes the quadratic phase. For $c = 1$ this is because $s = 0$ in the quotient. The translation multiplies the linear phase by a nontrivial character of $u \bmod \pi$, so this average vanishes. For a unit s, put $v = s^{-1}\bar{\gamma}e$. Then

$$sQ(x) + \mathrm{Tr}(\gamma h(e,x)) = sQ(x+v)$$

because $Q(v) = 0$. In an orthonormal basis the shifted average is the r-th power of the one-coordinate average $G_c(s) = \mathbb{E}_z\psi(\pi^{-c}sz\bar{z})$. The residue computation gives $G_1(s) = -q^{-1}$. At $c \geq 2$, averaging over the last digit of z forces $z = 0 \bmod \pi$ by trace duality. Substitution of $z = \pi u$ then gives $G_c(s) = q^{-2}G_{c-2}(s)$, with $G_0(s) = 1$. Consequently every unit contributes $(-1)^{rc}q^{-rc}$.

It remains to remove the imprimitive vectors from the average. At $c = 1$ their only contribution is the mass q^{-2r} of zero. At $c \geq 2$ put $x = \pi y$. The norm condition on y has precision $c-2$, while the linear character has conductor exponent $c-1$. Translation of y by $\pi^{c-2}u$ preserves that norm condition and gives zero upon averaging over $u \bmod \pi$. The unit fraction among the s is $1 - q^{-1}$. Dividing the resulting primitive average by ρ_c proves (138.1). The bound follows from the lower bound for ρ_c. At an inert place $I_w = q^{-2c}$ and $D_w = q^{2c}$ for a local pole of order c. Since $r - 1 = k + 2a$, the asserted arithmetic weight absorbs the factor 3 for all sufficiently large nodes. □

Proposition 138.3 (Compression of the first Fourier operator)**.** *Let $K_c = \{g \in K : ge \equiv e \bmod \pi^c\}$ and $c \geq 1$. For $v \in Y_c$ let $A_x(v)$ be the conditional average of $\phi_0(h_1xh_1^{-1})$ over $h_1^{-1}e = v \bmod \pi^c$ with h_1 Haar-uniform in K. Then for every x*

$$|b_K^0(x;\alpha)| \leq \frac{q^{-rc}}{\rho_c}\frac{1}{|Y_c|}\sum_{v\in Y_c}|A_x(v)|^2.$$

These conditional averages are over the left cosets of K_c.

Proof. For $v = h_1^{-1}e$ and $w = h_2^{-1}e$ the relative pairing is $h(e, h_1h_2^{-1}e) = h(v,w)$. The two compact variables are independent; conditioning gives the corresponding quadratic form on Y_c with counting probability and kernel $\psi_A(\alpha h(v,w))$. On V_c with counting probability the squared operator norm of this kernel is q^{-2rc}. Indeed summation of the product of a kernel entry with the conjugate entry in another row gives zero unless the two row indices agree, by trace duality and self-duality of $\mathcal{L}$. The diagonal kernel is one, and the normalized diagonal operator is $1/|V_c|$ times the identity. Extending functions by zero and then restricting the output compresses the ambient operator. Changing the measure from ambient probability to conditional probability introduces the factor ρ_c^{-1}. Applying the operator bound to A_x proves the estimate. □

The proposition retains the two original Bessel coefficients at nonintegral x. Integrating its right side outside K requires a separate estimate of their conditional mean square.

138.2 The first neighbors of an unramified Hermitian lattice

Continue in the standard inert model of Section 138.1. We write $\mathcal{L} = L \oplus W \oplus L^{\vee}$ for the integral decomposition, with $L, L^{\vee}$ dual totally isotropic summands of rank a and W self-dual of rank $d = r - 2a$. Denote by $\mathcal{N}$ the set of positions $x \in \mathrm{U}(\mathcal{L} \otimes_{O_A} A)$ with

$$\mathrm{length}_{O_A}\left((x\mathcal{L} + \mathcal{L})/\mathcal{L}\right) = 1.$$

Right K-cosets in $\mathcal{N}$ are identified with their lattices $x\mathcal{L}$. The function ϕ_0 of the preceding section takes a particularly simple form on these cosets.

Lemma 138.4 (Neighbor lattices and the unipotent orbit)**.** *The set of cosets in $\mathcal{N}$ is one K-orbit of size*

$$N_1 = \frac{q|Y_1|}{q^2 - 1} \geq q^{2r-2}.$$

Its intersection with UK/K has size

$$S_1 = \frac{q^{2a} - 1}{q + 1}.$$

The value of ϕ_0 on each of these S_1 cosets is one.

Proof. For $\mathcal{L}' = x\mathcal{L}$ set $\ell = (\pi\mathcal{L}' + \pi\mathcal{L})/\pi\mathcal{L}$. The length assumption makes ℓ a line in $\mathcal{L}/\pi\mathcal{L}$. It is isotropic since the norm and all pairings on $\mathcal{L}'$ are integral. The intersection $\mathcal{M} = \mathcal{L}' \cap \mathcal{L}$ is the inverse image of $\ell^{\perp}$: for $z \in \mathcal{L}$, orthogonality to ℓ modulo π means $h(z, \mathcal{L}') \subset O_A$, which characterizes membership in the self-dual $\mathcal{L}'$. Lift ℓ to an isotropic vector v with hyperbolic partner f as in Lemma 138.1, and let W_v be their orthogonal complement in $\mathcal{L}$. Then

$$\mathcal{M} = O_A v \oplus W_v \oplus \pi O_A f, \qquad \mathcal{M}^{\#} = \pi^{-1} O_A v \oplus W_v \oplus O_A f.$$

Here the superscript denotes the dual lattice. The form πh on $\mathcal{M}^{\#}/\mathcal{M}$ has hyperbolic basis $\pi^{-1}v, f$ over the residue field of A. Self-dual lattices between $\mathcal{M}$ and $\mathcal{M}^{\#}$ correspond exactly to self-orthogonal lines in this plane: integrality is isotropy and duality takes a line to its perpendicular. In addition to the original line of f, these are the q lines spanned by $\pi^{-1}v + tf$ with $t + \bar{t} = 0 \bmod \pi$. All the resulting lattices are isometric to $\mathcal{L}$ by the orthonormal basis construction in the preceding lemma.

The group K is transitive on the choices of ℓ. Its stabilizer permutes the q lines just found transitively. Indeed the isometry fixing f, W_v and taking v to $v + \pi u f$, with $u + \bar{u} = 0$, adds the residue of u to t. Every trace-zero residue has a trace-zero lift: subtract π times a solution of a trace equation from any lift of that residue. Thus there is just one orbit. Its size follows by dividing $|Y_1|$ by the number $q^2 - 1$ of generators of a residue line and multiplying by q. The lower bound follows from Lemma 138.1.

It remains to determine which neighbors occur with nonzero coefficient. Write a representative of a contributing coset in U as

$$y = \begin{pmatrix} T & B & C \\ 0 & 1 & D \\ 0 & 0 & S \end{pmatrix}, \qquad S = T^{-*}, \qquad HD = -B^{*}S,$$

where H is the Gram matrix on W and * denotes conjugate transpose in the paired bases. The matrix $\pi(y - 1)$ is integral and has rank one modulo π, since its reduced image is ℓ. If the unipotent T is nonintegral, then its inverse must also be nonintegral: an integral inverse of determinant one has an integral inverse by the adjugate identity. Both T, S would therefore

have a pole, and the reduction of $\pi(y-1)$ would have a nonzero image from L in L and a nonzero projection from $L^\vee$ to $L^\vee$. This contradicts rank one. Now T, S are integral and invertible over O_A. A pole in B would give one in D by the displayed relation, producing a nonzero image from W in L and a nonzero projection of the image to W. Thus B, D are integral as well.

Right multiplication by $\operatorname{diag}(T^{-1}, 1, S^{-1})$ makes both diagonal arms identity while keeping the same lattice. Use primes for the resulting off-diagonal blocks. Choose $\theta \in O_A$ with $\theta + \bar{\theta} = 1$. The integral unipotent with blocks B', D' and $C_0 = -\theta (D')^* H D'$ is unitary: the remaining equation is $C_0 + C_0^* = -(D')^* H D'$. Multiplication on the right by its inverse leaves only the upper-right block $C' - C_0$. Consequently the supporting right cosets are indexed by the nonzero rank-one skew-Hermitian matrices $\pi(C' - C_0) \bmod \pi$ on the arms. Every such residue lifts to an integral skew-Hermitian matrix by choosing one triangular half arbitrarily and lifting trace-zero diagonal entries as above. Two such representatives give the same coset exactly when their residues agree. The Bessel character reads the successive flag entries and the terminal entry in W; all those entries vanish on this upper-right block. Right K-invariance thus gives value one on each coset.

For each image line choose a generator v_0 in the left arm. A rank-one skew-Hermitian matrix with that image is $\lambda v_0 v_0^*$, since its adjoint has the same image. It has $q-1$ choices for the nonzero trace-zero scalar λ. The number of lines in the left arm is $(q^{2a}-1)/(q^2-1)$. Their product is S_1. □

Theorem 138.5 (The first inert shell at every conductor)**.** *Give K volume one as above. For integral α, one has*

$$\int_{\mathcal{N}} |b_K^0(x;\alpha)|\, dx = S_1^2/N_1 < q^{-2d}.$$

For $c = -\operatorname{ord}_A \alpha \geq 1$, one has

$$\int_{\mathcal{N}} |b_K^0(x;\alpha)|\, dx \leq S_1 \rho_c^{-1} q^{-rc}.$$

In particular the contribution of $K \cup \mathcal{N}$ is between 1 and $1 + q^{-2d}$ at zero frequency. At a local pole of a nonzero rational frequency its contribution is at most $q^{-n^7 c/2} I_w^{(k-n^7)/2} D_w^{-3}$ at the large nodes of Theorem 137.18.

Proof. For integral α the relative character on K^2 is one, so the compact double average is the squared absolute value of the compact single average. Right K-invariance gives $\phi_0(h_1 x h_1^{-1}) = \phi_0(h_1 x)$. The lattice of $h_1 x$ is uniform on the N_1 neighbors as h_1 varies. The single average is therefore S_1/N_1 for every x in this shell. Each neighbor has volume one, proving the equality. Now $S_1 < q^{2a-1}$ and $N_1 \geq q^{2r-2}$, which give the stated strict inequality.

For $c \geq 1$ integrate Proposition 138.3 over $\mathcal{N}$. The square of a conditional average is at most the conditional average of the squared modulus. After averaging over $v \in Y_c$, the integral of the latter is

$$\int_K \int_{\mathcal{N}} |\phi_0(h_1 x)|^2 \, dx \, dh_1 = S_1,$$

by invariance of $\mathcal{N}$ on the left and the support calculation of the lemma. This proves the second estimate. Adding Theorem 138.2 bounds the positive conductor contribution on the two sets together by $(3 + 2S_1) q^{-(r-1)c} \leq 5 q^{-kc}$, since $r - 1 = k + 2a$ and $c \geq 1$. For large n, $5 \leq q^{(n^7/2 - 6)c}$. Use $I_w = q^{-2c}$ and $D_w = q^{2c}$ to obtain the assertion. □

This calculation uses the original character and accounts for every lattice of length one. At length two an off-diagonal pair B, D can both have poles, so the support on the middle block acquires additional degrees of freedom.

138.3 A lattice length bound for distant inert positions

Use the standard inert model and the measure conventions of Section 138.2. For x in the unitary group put $P = \mathrm{length}_{O_A}((x\mathcal{L}+\mathcal{L})/\mathcal{L})$ and let m be the least nonnegative integer for which $\pi^m x$ and $\pi^m x^{-1}$ are integral. Both invariants depend only on the double coset KxK. Write $\mathcal{S}(m,P)$ for the positions with these invariants. We give two counts, one for the supporting unipotents and one for the size of their compact orbits.

For $y \in U$ write T, B, D, S, C for its blocks in the notation of Lemma 138.4. Define lattices in the left arm and integers by

$$L' = L + TL, \qquad J_B = L' + BW, \qquad h_B = \mathrm{length}(J_B/L), \qquad \delta(y) = P - 2h_B.$$

For a right coset yK with $y \in U$, these are independent of the unipotent representative. Indeed two representatives differ on the right by $U \cap K$; this changes T to TT_0 for T_0 integral upper unitriangular and changes B by TB_0 with B_0 integral. Both lattices stay the same.

Lemma 138.6 (Lengths of the two middle maps)**.** *One has $2h_B \le P$. For $m \ge 1$ and an integer $b \ge 0$ the number of supporting right cosets in $\mathcal{S}(m,P)$ with $\delta(y) \ge b$ is at most*

$$q^{5a^2m + d(P-b)}. \tag{138.2}$$

Here and below a supporting coset is a coset on which ϕ_0 is nonzero.

Proof. Let $\tau = \mathrm{length}(L'/L)$ and $\beta = \mathrm{length}(J_B/L')$. The kernel of the projection of $(y\mathcal{L}+\mathcal{L})/\mathcal{L}$ to the last two blocks contains J_B/L. In the projected module the last block has length τ, because $S = T^{-*}$ and $\det T = 1$. More explicitly, the positive and negative parts of the elementary divisors of T have equal total length, and dual inversion interchanges them. The kernel of this last projection is the image of Du modulo W for $u \in L^\vee$ with $Su \in L^\vee$. Since

$$SL^\vee \cap L^\vee = (TL + L)^{\#}, \qquad HD = -B^*S,$$

its length is β. Indeed the matrix of B in bases of W, L' and its adjoint in the dual bases have identical Smith exponents, and the sums of their pole orders compute the two lengths. Adding the lengths gives $P \ge h_B + \tau + \beta = 2h_B$.

For the count use product Haar measure in the entries of T above its diagonal, all entries of B, and a skew-Hermitian matrix Z of size a. To see the normalization, write $y = u_0 \,\mathrm{diag}(T, 1, S)$. The middle-to-dual block of u_0 is $D_0 = -H^{-1}B^*$, and its upper-right block is $Z - \theta D_0^* H D_0$, with $\theta + \bar\theta = 1$. The product of two such u_0's adds their B's and adds their Z's with a translation depending on the B's. Conjugation by the diagonal sends B to TB and Z to TZT^*; both linear maps are unipotent and preserve volume. For T multiplication is triangular in its entries with diagonal one. Thus the product measure is Haar, and the integral parameter lattices describe $U \cap K$ exactly, of volume one.

The entries of T have poles of order at most m, for a cost at most $q^{a(a-1)m}$. At fixed T the lattice J_B lies between L and $\pi^{-m}L$. The number of possibilities is at most q^{2a^2m}: each submodule of $\pi^{-m}L/L$ is generated by at most a elements, by taking a basis of its inverse image. With J_B fixed, the d columns of B are in J_B, costing at most q^{2dh_B}. Finally $CS^{-1} = Z - \theta D_0^* H D_0$ has entries in $\pi^{-2m}O_A$ since both C and S^{-1} have poles of order at most m. At fixed T, B any two possible Z's differ by a skew-Hermitian matrix in that ball. This costs q^{2a^2m}, as the integral skew-Hermitian matrices are a full O-lattice of rank a^2. On the specified set, $2h_B \le P - b$. Multiplication of the four factors proves (138.2); Haar volume counts right cosets because each intersection with U has volume one. □

Lemma 138.7 (A compact orbit bound from a primitive frame)**.** *On any K-orbit of right cosets in $\mathcal{S}(m,P)$ meeting UK/K, let N be the orbit size. For $m \ge 1$,*

$$N \ge \tfrac{1}{3} q^{2dP-4a^2m}.$$

Proof. Write $A_x = x\mathcal{L} + \mathcal{L}$. Its elementary divisors give s independent primitive vectors $v_1, \dots, v_s$, extendible to an integral basis, and positive integers $\lambda_i \le m$ with $\sum_i \lambda_i = P$, such that $A_x = \mathcal{L} + \sum_i O_A \pi^{-\lambda_i} v_i$. For an orbit meeting U one has $s \le 2a$. Indeed $A_y/\mathcal{L}$ is generated by the image of $y-1$ modulo $\mathcal{L}$, and $y-1$ has rank at most $2a$: the images of the left and middle blocks are in $L \otimes A$ and the remaining block has a columns. A rank-$2a$ lattice image has a basis with at most $2a$ elements.

We record the compact counting argument for arbitrary such frames, allowing degeneracy of their Gram matrix. In the residue field $|\mathrm{U}_j(q)| = q^{j^2} \prod_{i=1}^{j} (1 - (-1)^i q^{-i})$. Indeed the same additive-character count as in Lemma 138.1 gives $q^{2j-1} - (-1)^j q^{j-1}$ vectors of norm one. Choosing successively an orthonormal basis proves the formula. The product is between $2/3$ and 2; use $\sum_{i \ge 1} q^{-2i} \le 1/3$ for the lower bound and $\exp(\sum_{i \ge 1} q^{-(2i-1)}) < 2$ for the upper bound.

Modulo π, let the radical of the span of the s frame vectors have dimension j. A nondegenerate complement in their span of dimension $s-j$ is fixed pointwise, so one can work in its perpendicular, with j fixed isotropic vectors. Complete these to hyperbolic pairs. Concretely choose dual partners with pairings the identity and subtract suitable linear combinations of the fixed vectors to set the partners' Gram matrix to zero. The required equation $C + C^* = D$ for a Hermitian D is soluble by choosing a triangular half and solving the diagonal trace equations. In the resulting three blocks of sizes $j, r-s-j, j$, the isometries fixing the first j vectors pointwise have identity diagonal arms, an arbitrary unitary middle block, $j(r-s-j)$ free entries over the quadratic residue field, and j^2 free scalar digits in the skew-Hermitian upper-right part. These choices follow by first pairing the middle and last blocks to determine the opposite rectangle, then pairing the last block with itself. Hence the residue pointwise stabilizer has size at most $2q^{(r-s-j)^2+2j(r-s-j)+j^2} = 2q^{(r-s)^2}$.

An isometry modulo π^i lifts to one modulo π^{i+1}, in exactly q^{r^2} ways. Correct any lift by $1 + \pi^i X$; the required correction in the Gram matrix is $X^* H' + H' X$ modulo π, with H' the invertible Gram matrix of the full lattice. This map is onto the Hermitian matrices by the same trace equations and has kernel of dimension r^2 over the residue field of F_w. Completeness lifts each finite solution to K. For lifts fixing the frame, two choices differ by such a skew-adjoint X annihilating its s independent residue vectors. The corresponding skew-Hermitian form descends in both entries to the quotient by their span, allowing at most $q^{(r-s)^2}$ choices. Orbit–stabilizer at precision m now gives at least $q^{(2rs-s^2)m}/3$ different images of the ordered frame.

Every $g \in K$ fixing $x\mathcal{L}$ also fixes A_x. In the chosen basis this forces the last $r-s$ entries of gv_i to be divisible by π^{λ_i}, because $\pi^{-\lambda_i} g v_i \in A_x$. At most $q^{2rsm-2(r-s)P}$ matrices of frame images modulo π^m meet those conditions, even when all other requirements are dropped. The images are uniform on their orbit. The probability of fixing $x\mathcal{L}$ is thus at most $3q^{s^2m-2(r-s)P}$. Its reciprocal is N. Use $s \le 2a$ and $r - s \ge d$ to conclude. □

Theorem 138.8 (Unweighted inert shell energy)**.** *For $m \ge 1$, write ϕ_b for ϕ_0 restricted to supporting cosets with $\delta(y) \ge b$, and give K volume one. Then*

$$\int_{\mathcal{S}(m,P)} \left| \int_K \phi_b(gx)\, dg \right|^2 dx \le 3q^{14a^2m-2db}.$$

For the original zero-frequency integral on an entire shell one can take $b = P \bmod 2$.

Proof. On an orbit of N right cosets the compact average is the signed sum of the values on the orbit divided by N; each right coset has volume one. Its energy is the squared modulus of the sum divided by N. By the triangle inequality, the sum of these squared moduli is bounded by the square of the total number counted in Lemma 138.6. Division by the lower bound of Lemma 138.7 proves the estimate. Since $\delta = P - 2h_B$ is nonnegative and has the parity of P, taking $b = P \bmod 2$ includes all supporting cosets. At zero frequency the square of the compact average is exactly $b_K^0(x;0)$. □

Corollary 138.9 (A summable contribution at all heights)**.** *Restrict the original ϕ_0 outside K to the cosets with $\delta(y) \geq \lceil m/(100a^2) \rceil$. Call this restriction f, extended by zero elsewhere. At the large nodes the full unweighted compact average satisfies*

$$\int_G \left| \int_K f(gx)\, dg \right|^2 dx \leq q^{-2d+1500a^4}.$$

Proof. Put $j = \lceil m/(100a^2) \rceil \geq 1$. Since $P = \sum \lambda_i \leq 2am$, for each j there are at most $(100a^2 j)(200a^3 j + 1)$ possible pairs m, P. The theorem bounds each by $3q^{-(2d-1400a^4)j}$. Here $d/a^4 \to \infty$ along the specified nodes. The sum of $j^2 q^{-(2d-1400a^4)(j-1)}$ is bounded by an absolute constant there. The remaining polynomial factor in a is absorbed by q^{100a^4}. □

The corollary concerns the compact average of the stated restriction of the Bessel coefficient. Its interference with the complement in the original square depends on the energy of that complement. The two lengths can be equal, $P = 2h_B$, so the length inequality alone supplies no decay in that case.

138.4 Terminal row cancellation on inert lattice orbits

Assume that the terminal vector z is primitive in the self-dual middle lattice W and that its simple coefficient is a unit. In the arm coordinates of Section 138.3 the terminal entry of the simple character is $\psi_A(\gamma\beta_a(z))$, where β_a is the last row of B and $\gamma \in O_A^\times$. Let $p = p(y)$ be the nonnegative integer for which the last coordinate projection of J_B is $\pi^{-p}O_A$. The last row of T is the last coordinate row, so this projection is $O_A + \beta_a(W)$. In particular $p \le m$. We preserve the signs of the original Bessel character below.

Choose a totally isotropic direct summand $Q \subset W$ of rank $g = \lfloor d/2 \rfloor$ such that $h(Q, z) = O_A$. Here is an integral construction. In a Hermitian residue space of dimension at least two, two norm-one orthogonal coordinates give an isotropic vector by choosing a scalar of norm -1. Split off a hyperbolic plane and repeat. The isotropic vectors span the residue space: the hyperbolic basis vectors span the planes, and for any vector u in the remaining line, one can make $e' + u + tf'$ isotropic by a trace equation in a hyperbolic pair e', f'. Hence there is an isotropic vector pairing nontrivially with the reduction of z. Lift this vector and split off its integral plane by Lemma 138.1. Continue in the complementary lattice to obtain Q. Put

$$\eta = \min(g - a, d - g - 1).$$

This is positive at all sufficiently large nodes.

Theorem 138.10 (Terminal cancellation in the signed class sums)**.** *In $\mathcal{S}(m, P)$, $m \ge 1$, restrict the supporting cosets to $\delta(y) \ge b$, $p(y) \ge s$, for integers $b, s \ge 0$. Let $\mathcal{C}$ run over the orbits on the right K-cosets under K, or under the subgroup K_c fixing the first flag vector modulo π^c, for any $c \ge 1$. Write $S_{\mathcal{C}}$ for the sum of ϕ_0 over the restricted cosets in the orbit. Then*

$$\sum_{\mathcal{C}} |S_{\mathcal{C}}| \le 2(m+1)q^{5a^2m + d(P-b) - 2\eta s}. \tag{138.3}$$

One may prescribe exact values of δ and p in this estimate.

Proof. We first give the change of variables before any absolute values are taken. For v primitive in $Q \cap \ker B$, write $v^\flat = h(v, \cdot)$. For $w \in W \cap v^\perp$, define the integral isometry

$$N_v(w) = 1 + wv^\flat - vw^\flat - \theta Q_W(w) vv^\flat, \qquad Q_W(w) = h(w, w), \quad \theta + \bar\theta = 1.$$

Multiplying by the adjoint gives the identity, using $Q_W(v) = 0$ and $v^\flat(w) = 0$. It fixes both arms. Since $yv = v$ and $yw = w + b_w$ with $b_w = Bw \in L$, conjugation by y sends $N_v(w)$ to $N_v(w + b_w)$. The vectors w, b_w are orthogonal and b_w is isotropic. The two translations along them therefore commute and their product is $N_v(w + b_w)$, by multiplication of the displayed matrices. Consequently

$$N_v(w)^{-1} y N_v(w) = N_v(b_w) y, \qquad N_v(b_w) = 1 + b_w v^\flat - v b_w^\flat. \tag{138.4}$$

The last factor on the left of y on the right-hand side belongs to U. Its entries on the simple arm roots are zero. On the terminal root its contribution to the phase is $\gamma\beta_a(w)h(v, z)$. The conjugation preserves the individual orbit $\mathcal{C}$ even for K_c, since the conjugating isometry fixes the first flag vector.

We detail the measure argument when the vector v depends on the data. Fix T and $B|_Q$. Let τ be the valuation of $h(Q \cap \ker B, z)$, with value $+\infty$ for the zero ideal. If $\tau < p$, choose v in this kernel attaining τ; it is primitive, since dividing a nonprimitive vector by π would decrease the valuation. The choice can be made measurably using a kernel basis obtained by elimination and the first basis vector of minimum valuation. For fixed v, w, the new B equals

$B(1+wv^\flat|_W)$. This integral change of basis has determinant one and is the identity on Q, so it preserves $B|_Q, J_B$ and the product measure in the other columns. In this formula T, S and the upper-right block C stay fixed: the possible extra term $b_w v^\flat D$ is zero by $HD = -B^*S$ and $Bv = 0$. Moreover $BH^{-1}B^*$ stays fixed on substituting the new B, again because $Bv = 0$ and v is isotropic. Thus the skew-Hermitian parameter Z of Lemma 138.6 stays fixed as well. The inverse change in B uses $1 - wv^\flat|_W$. This proves conditional Haar invariance and preservation of every restriction in the statement.

As in Lemma 138.6 a signed coset sum equals the integral over the corresponding part of U. On each piece of fixed $J_B, T, B|_Q$ with $\tau < p$, average (138.4) over the compact lattice $W \cap v^\perp$. Orthogonality of its additive characters forces

$$\beta_a(W \cap v^\perp) \subseteq \pi^{-\tau} O_A. \tag{138.5}$$

Indeed the image is an O_A-module and ψ_A has conductor O_A. For $\tau \geq p$ retain the whole piece. The triangle inequality may now be applied separately in each orbit and the surviving volumes added.

For this last count fix a candidate $J = J_B$ and put $h_B = \mathrm{length}(J/L)$. Enlarge the column domain to independent columns uniform in J, of total measure q^{2dh_B}, retaining their constraints just obtained. The number p is fixed by J. The law of the map $B|_Q$ is invariant under $\mathrm{GL}_g(O_A)$ on its domain. Its kernel there is a primitive submodule of rank at least $g - a$. For $j \geq 1$,

$$\Pr(\tau \geq j) \leq 2q^{-2(g-a)j}.$$

In fact one may first fix the map and then precompose by an independent uniform element of $\mathrm{GL}_g(O_A)$. In a basis extending the kernel, the coordinates of the resulting primitive functional $h(\cdot, z)|_Q$ have the distribution of a uniform primitive row (with conjugation of coefficients). The probability of $g-a$ prescribed zero coordinates modulo π^j is at most $q^{-2(g-a)j}/(1-q^{-2g})$. The bound also holds for $j = 0$.

Conditional on $B|_Q$, complete Q by a fixed integral complement of rank $d-g$. Within it, $v^\perp$ contains a primitive summand of rank at least $d-g-1$. Extend a basis of that summand to a basis of the complement. In these columns of B the last coordinates are still independent and uniform in the projection $\pi^{-p}O_A$ of J. If $\tau < p$, condition (138.5) costs at most $q^{-2(d-g-1)(p-\tau)}$ of this probability. On the piece $\min(\tau, p) = j$, multiply this bound (one when $j = p$) by the bound for $\Pr(\tau \geq j)$. The product is at most $2q^{-2\eta p}$. Summing $0 \leq j \leq p \leq m$ gives the factor $2(m+1)q^{-2\eta s}$. The counts of T, J, Z in Lemma 138.6 apply unchanged, with $2h_B \leq P - b$. This proves the theorem. □

Corollary 138.11 (An inert contribution controlled by the terminal row)**.** *Outside K restrict ϕ_0 to supporting cosets satisfying $\delta(y) + p(y) \geq \lceil m/(100a^2) \rceil$, and extend this restriction f by zero. Then at sufficiently large nodes*

$$\int_G \left| \int_K f(gx)\, dg \right|^2 dx \leq q^{-2d+1600(a+1)^4}.$$

Proof. On a shell put $j = \lceil m/(100a^2) \rceil$. Decompose each orbit sum by the exact pair δ, p. For each admissible pair, the theorem bounds the total absolute sum over orbits by $2(m+1)q^{5a^2m+dP-\min(d,2\eta)j}$. There are at most $(P+1)(m+1)$ pairs. Bound the sum of squared orbit sums by the square of their combined absolute sum, then divide by the orbit lower bound from Lemma 138.7. The shell energy is at most

$$12(m+1)^4(P+1)^2 q^{14a^2m-2\min(d,2\eta)j}.$$

Since $2\eta \ge d - 2a - 2$, $m \le 100a^2 j$ and $P \le 2am$, the exponential factor is at most $q^{-(2d-1400a^4-4a-4)j}$. The number of shells at each j is bounded as in Corollary 138.9. All the other factors are bounded by a fixed polynomial in a, j. Summing the geometric series times that polynomial gives the claimed bound, since $d/a^4 \to \infty$ and $q \ge 2$. □

In particular this estimate includes cosets with $\delta = 0$ when their terminal row has a sufficiently large pole. For $\delta = p = 0$ the other arm rows can still have poles.

138.5 Distant inert shells at every conductor

Retain the assumptions of Section 138.4. For an integral lattice $J \supset L$ in $L \otimes A$, write $b_i(J)$ for the integer such that the i-th coordinate of $J \cap (L_i \otimes A)$ ranges over $\pi^{-b_i(J)}O_A$; here $L_i = \langle e_1, \dots, e_i \rangle$. Successive projections give $\sum_i b_i(J) = \mathrm{length}(J/L)$. We use $J = J_B$, so $b_a(J) = p$. To detect a pole situated earlier in the flag, one can average over an isotropic vector dual to the next row.

Lemma 138.12 (An interior row on the compact orbits)**.** *Fix $1 \le j < a$, a shell $\mathcal{S}(m, P)$, $m \ge 1$, and a candidate lattice $J = J_B$. Write $b_i = b_i(J)$, $\delta = P - 2\,\mathrm{length}(J/L)$, and $D_j = (b_j - b_{j+1} - \delta - 1)_+$, where $s_+ = \max(s, 0)$. On the cosets with this J, the sum of the absolute values of the signed orbit sums, for either group in Theorem 138.10, is at most*

$$q^{C(a+1)^2(m+1)+d(P-\delta)-2(d-a)D_j}$$

with an absolute C. The simple character on the j-th arm root is assumed to have a unit coefficient as at a standard unmarked place.

Proof. Put $A_y = y\mathcal{L} + \mathcal{L}$, $M_y = A_y^{\#} = \mathcal{L} \cap y\mathcal{L}$, and $J^+ = A_y \cap (L \otimes A)$. The projection of $A_y/\mathcal{L}$ to the last two blocks has length h_B: its last projection has length τ and the kernel has length β in the proof of Lemma 138.6. Its kernel before projection to those two blocks is J^+/L. Hence $J \subseteq J^+$ and $\mathrm{length}(J^+/J) = \delta$.

Set $V = L_j^{\perp} \cap \mathcal{L}$. Among the $v \in M_y \cap V$ the least valuation of $h(v, e_{j+1})$ is $t = b_{j+1}(J^+)$. In fact restriction from the dual of A_y onto the dual of J^+ is surjective because J^+ is saturated in A_y. In the dual of J^+ impose vanishing on L_j. The image on e_{j+1} is $\pi^{b_{j+1}(J^+)}O_A$: apply duality to the rank-one successive quotient of the intersections with the flag, which is a direct summand in the quotient by the preceding intersection. This also shows

$$0 \le t \le m, \qquad b_{j+1} \le t \le b_{j+1} + \delta.$$

Indeed $\pi^{-m}L \supseteq J^+$, and the increases in all the b_i under $J \subseteq J^+$ are nonnegative and sum to δ.

Give the smooth integral isotropic vectors in V their invariant probability measure μ. More explicitly, keep the norm-zero residues whose projection off the radical $L_j \bmod \pi$ is nonzero. At each higher precision condition counting probability on these smooth solutions. The trace derivative is surjective there, so all solutions have the same number of lifts per digit. These finite distributions give μ on their inverse limit. Every integral isometry preserving V and L_j preserves this measure. Let

$$H_y = \mu\{v \in M_y \cap V : \mathrm{ord}_A\, h(v, e_{j+1}) = t\}.$$

Uniformly for the specified y,

$$H_y \ge q^{-10(a+1)(m+1)}. \tag{138.6}$$

We give the argument since a cost with d in this exponent would spoil the shell estimate. The saturation of the adjoint image of B in W has rank at most a. Its orthogonal complement in W reduces onto the perpendicular of its residue image: the saturation is primitive, so restriction of integral linear functionals to it is onto. In that residue perpendicular, the radical has dimension at most a. Lift a basis of a nondegenerate complement, obtaining a unimodular summand W_0 of rank at least $d - 2a$. The unipotent y fixes W_0 pointwise because $BW_0 = 0$, and preserves its orthogonal complement. Thus all integrality conditions on $v \in M_y$ are confined to the orthogonal complement of W_0, of rank at most $4a$. In this complement fix the coordinates

of v, modulo π^{m+1}, equal to those of a vector attaining t. Take its component in $V \cap W_0^{\perp}$, since projection off W_0 changes neither stability nor the leading coordinate. This costs at most $q^{8a(m+1)}$ in reciprocal probability. For any choices of those coordinates at precision $l \geq m+1$, the norm equation in W_0 has smooth solutions of probability at least $(1-q^{-2})q^{-l}$: in an integral hyperbolic plane take the first coefficient a unit and solve the resulting surjective trace equation in its partner. The complementary norm can have any prescribed value. Such a plane exists by the construction preceding Theorem 138.10. The ambient probability of a smooth isotropic solution in V is at most q^{1-l} by one-digit lifting from precision one. Taking the ratio proves (138.6).

In a signed class sum insert

$$1 = H_y^{-1} \int \mathbf{1}_{v \in M_y,\, \operatorname{ord} h(v,e_{j+1})=t}\, d\mu(v).$$

Partition first by t and hold the integration vector v fixed. Put $\kappa = h(v,\cdot)$, and let w range over $\pi(W \cap v^{\perp} \cap \ker B_{>j})$, where $B_{>j}$ denotes the rows after j. Then $b = Bw \in L_j \otimes A$, v and $y^{-1}v$ are integral isotropic vectors, and both are orthogonal to w, as $h(v, yw) = h(v, w+b) = 0$. The translations defined in (138.4) obey

$$N_v(w)^{-1} y N_{y^{-1}v}(w) = N_v(b) y, \qquad N_v(b) = 1 + b\kappa - v b^{\flat}. \tag{138.7}$$

To verify it conjugate the right translation by y, obtaining $N_v(yw) = N_v(w+b) = N_v(w)N_v(b)$, as w, b are orthogonal. Here $N_v(b) \in U$: b lies in the first j flag lines, and v is orthogonal to them. On the left arm this adds only earlier rows to later input coordinates, the dual action is the inverse adjoint, and the change on the middle quotient is zero. The only simple entry of $N_v(b)$ is the j-th one, $\beta_j(w)\kappa(e_{j+1})$, since b has no coordinates after j and κ vanishes on L_j. The terminal entry is zero since $j < a$.

Write y' for the element on either side of (138.7). The two exterior factors on the left are integral isometries, and the left one fixes e_1. Thus y' belongs to the same refined orbit as y. The middle block with image in L changes from B to $B(1 + w(\kappa y)|_W)$; this right multiplier is integral with integral inverse since κy is integral and $\kappa y(w) = 0$. The left arm T changes by $b(\kappa y)|_L$. It follows in both directions that J is unchanged, since $b \in BW$ and the latter module is unchanged. Also $M_{y'} = N_v(w)^{-1} M_y$. This integral map fixes L_j and preserves V. For any $v' \in V$ its change in the $h(v', e_{j+1})$ coordinate is divisible by π^{t+1}: the w-term has no dual-arm coordinate and the coefficient of v in the change is divisible by π, since $w \in \pi W$. Therefore t is still the least valuation on the new stable lattice and $H_{y'} = H_y$, by invariance of μ. The chosen v is fixed by the left multiplier.

For clarity, this substitution preserves Haar measure also when the allowed w's depend on the lower rows of B. At fixed v use the closed additive subgroup of U consisting of the $N_v(u) = 1 + u\kappa - v u^{\flat}$ for $u \in L_j \otimes A$. Its parameters add since v, L_j are mutually orthogonal and isotropic. On each left coset of this subgroup the lower rows $B_{>j}$ are fixed, by restriction to W. For any allowed w, the value of Bw is also fixed there, since $N_v(u)$ fixes $w + Bw$. Thus on the coset the substitution is a translation by a fixed group element. Disintegrating Haar measure along these cosets proves the invariance. Equivalently, fix a representative in each coset and use translation invariance in the additive coordinate u; both groups are unimodular. Its inverse uses $-w$. The orbit, integrality restrictions and the factor H_y^{-1} have all been preserved. Averaging over the allowed w's with probability Haar now imposes

$$\beta_j(W \cap v^{\perp} \cap \ker B_{>j}) \subseteq \pi^{-t-1} O_A$$

by the unit simple coefficient and the conductor of ψ_A.

Apply absolute values after this averaging. Bound H_y^{-1} by (138.6), and use the column and central parameter count of Lemma 138.6. At fixed v, J enlarge the column domain to independent uniform columns in J, and condition on their coordinates after row j. Then their j-th coordinates are independently uniform in cosets of $\pi^{-b_j}O_A$. Indeed the kernel of that projection on J is $J \cap (L_j \otimes A)$. The tested summand of W is primitive of rank at least $d-a$, being the kernel of at most $a-j+1$ functionals. After extending a basis of it to W, the same independent Haar laws, up to translations, hold in the new columns. The fraction surviving the test is at most $q^{-2(d-a)(b_j-t-1)_+}$, including when some of the translated balls miss the test altogether. Integrating over v costs at most one and the partition by t at most $m+1$. The volume of all columns is $q^{2d\,\mathrm{length}(J/L)}$; the remaining counts of T, Z and the reciprocal probability in (138.6) are absorbed by $q^{C(a+1)^2(m+1)}$. Using $t \le b_{j+1}+\delta$ proves the lemma. □

Theorem 138.13 (The distant part of the full inert transform)**.** *At the standard unmarked inert places above, let $c = \max(0, -\operatorname{ord}_A \alpha)$, with $c = 0$ at $\alpha = 0$. For every sufficiently large node, uniformly in the residue field and the conductor,*

$$\int_{P(x)\ge 1000(a+1)^2(c+1)} |b_K^0(x;\alpha)|\,dx \le q^{-6k(c+1)}.$$

Proof. For fixed J define the integer $D = \max(\{\delta, b_a\} \cup \{D_j : 1 \le j < a\})$. The inequalities $b_j \le b_{j+1} + 2D + 1$ and $b_a \le D$ give $P = \delta + 2\sum b_i \le (4a^2+1)D + 2a^2$. For $P \ge 4a^2+2$ this implies $D \ge P/(10(a+1)^2)$.

For each J use the estimate corresponding to a maximum. For δ the unsigned count in Lemma 138.6 saves $d\delta$ in the exponent relative to dP. For b_a the proof of Theorem 138.10 at this fixed J saves $2\eta b_a$; for a D_j use the preceding lemma. In the last two cases the count also saves $d\delta$, which we can discard. Each choice saves at least $dD/2$ at large nodes. There are at most q^{2a^2m} possible J's by Lemma 138.6. Consequently on the full shell the sum of absolute signed class sums on the K_c-orbits (with $K_0 = K$ here) is at most

$$q^{C_1(a+1)^2(m+1)+dP-dP/(20(a+1)^2)}.$$

Here C_1 is absolute.

For the final integration the operator of the relative character on the compact orbit has norm at most one, by the modulus of its kernel and Cauchy–Schwarz. The shell integral is therefore bounded by the norm squared of the K_c-average of ϕ_0 on that shell. More explicitly, the conditional averages of Proposition 138.3 are left translates of this average in the x-variable, whose squared integrals agree on the K-invariant shell. At $c = 0$ this is the ordinary compact average. For $c \ge 1$, $[K : K_c] = |Y_c| \le q^{2rc}$; the same bound one applies for $c = 0$. A refined orbit thus has size at least $q^{2dP-4a^2m-2rc}/3$, by Lemma 138.7 and orbit–stabilizer. Squaring the signed bound and dividing by this size gives, after increasing the absolute constant,

$$q^{2rc+C_2(a+1)^2(m+1)-dP/(10(a+1)^2)}.$$

We have $m \le P$: the elementary divisors of a self-dual lattice relative to $\mathcal{L}$ occur in opposite pairs, since duality changes their signs. Thus a pole of largest order occurs in the length sum. Since $d/(a+1)^4 \to \infty$, the displayed bound is at most $q^{2rc-dP/(20(a+1)^2)}$ at the stated large nodes. There are at most P choices of m at fixed P. For $P \ge 1000(a+1)^2(c+1)$, these factors, summed against the decreasing geometric series, give at most $q^{-6k(c+1)}$: indeed $r/d \to 1$, $d = k+1$, and the first negative exponent before summation is at least $50d(c+1)$ in magnitude. The polynomial factor is absorbed uniformly using $\log_q P \le P$ and $d/(a+1)^2 \to \infty$. This proves the assertion. □

The bound applies to the full Bessel coefficients, with both compact integrations taken before the absolute value. Together with Theorem 138.5 it leaves a finite range of intermediate lattice lengths at each conductor.

138.6 The full unweighted mass at a standard inert place

For zero frequency Theorem 138.13 leaves the lengths $P < 1000(a+1)^2$. At these lengths a single nonzero digit must still give decay in the middle rank. We recover the digit lost by averaging over πW in Lemma 138.12. The necessary comparison is between the ordinary measure of a lattice and its measure on an isotropic cone.

Lemma 138.14 (Isotropic density in the fixed middle summand)**.** *With the notation H_y, M_y, V, t of Lemma 138.12, use additive probability on V and put*

$$H_y^{\mathrm{lin}} = (1-q^{-2})\,\mathrm{vol}_V(M_y \cap V).$$

Uniformly in the shell and j, if $d-2a \geq 4$, one has

$$H_y = H_y^{\mathrm{lin}}(1+O(q^{1-d+2a})), \qquad H_y^{\mathrm{lin}} \geq (1-q^{-2})q^{-2P}.$$

All implied constants in this section are absolute.

Proof. Write $D_0 = \operatorname{rank} W_0 \geq d-2a$ for the fixed unimodular summand constructed in the proof of Lemma 138.12. We first count norms in W_0. For any target s and $l \geq 1$, the probability of norm s mod π^l in W_0 is

$$q^{-l}(1+O(q^{1-D_0})).$$

Indeed expand the norm indicator in additive characters modulo π^l. After the trivial character, the total at conductor exponent i inside the normalized Fourier sum is at most $q^i q^{-D_0 i}$, by the one-dimensional Gauss calculation in Theorem 138.2 and an orthonormal basis. Summing over $1 \leq i \leq l$ proves the claim. The part with all entries divisible by π costs at most $O(q^{-l+2-2D_0})$: for $l \geq 2$ divide the norm equation by π^2 if its target allows this, and use the same bound at precision $l-2$, including precision zero. For $l = 1$ the imprimitive part has measure at most q^{-2D_0}.

Now write $V = (V \cap W_0^{\perp}) \oplus W_0$. Integrality of $y^{-1}v$ and the leading coordinate $h(v, e_{j+1})$ depend only on the first summand. Their joint additive probability is H_y^{lin}, since the image of the coordinate on $M_y \cap V$ is precisely $\pi^t O_A$, whose unit fraction at that valuation is $1-q^{-2}$. For each fixed value in the first summand, the preceding norm count applies with its prescribed target. A primitive W_0-component guarantees smoothness in V, since W_0 is nondegenerate. The estimate of the imprimitive part thus shows that the smooth norm-zero probability at precision $l \geq m+1$ is $q^{-l}(1+O(q^{1-D_0}))$ uniformly for each first component. Apply this first with the two linear restrictions and then without them. Dividing the two probabilities gives the claimed estimate of H_y at the large ranks, in fact uniformly for $D_0 \geq 4$: the denominator is bounded below by $(1-q^{-2})q^{-l}$ using a hyperbolic pair in W_0. Finally the index of $M_y \cap V$ in V is at most $[\mathcal{L} : M_y] = q^{2P}$, by duality with $A_y/\mathcal{L}$. □

Proposition 138.15 (Cancellation at an interior drop of one digit)**.** *Fix J on a shell as in Lemma 138.12, and keep either of its compact orbit decompositions. If $\delta = 0$ and $b_j > b_{j+1}$ for some $j < a$, the sum of the absolute signed class sums on the cosets with that J is at most*

$$q^{C(a+1)^2(m+1)+dP-d+2a+1}.$$

Proof. Here $t = b_{j+1}$. In the integral for a signed orbit sum insert the isotropic vectors as in Lemma 138.12, now dividing by H_y^{lin}. This multiplies the integrand after integration in v by H_y/H_y^{lin}. Consequently the total absolute error over all orbits is at most $O(q^{1-d+2a})$ times the unsigned count for this J, by Lemma 138.14 and $|\chi_0| = 1$. The count is at most q^{5a^2m+dP} by Lemma 138.6.

For the modified integral allow all $w \in W \cap v^{\perp} \cap \ker B_{>j}$ without the factor π. Identity (138.7), preservation of the orbit and J, and translation invariance on the additive cosets in its proof still hold. The left multiplier $N_v(w)^{-1}$ fixes L_j pointwise, hence preserves V and its additive Haar measure. It carries M_y onto $M_{y'}$. Thus $H_y^{\mathrm{lin}} = H_{y'}^{\mathrm{lin}}$ exactly. The vector v is fixed by the multiplier and remains in $M_{y'}$. Its leading valuation is still t. For instance, $J, \delta = 0$ have stayed fixed, so the formula $t = b_{j+1}(J^+) = b_{j+1}(J)$ applies on both sides. These facts verify invariance of all factors used in the modified averaging.

Character orthogonality now requires integrality after multiplying $\beta_j(W \cap v^{\perp} \cap \ker B_{>j})$ by π^t. The conditional column count from the preceding section saves $2(d-a)(b_j-t) \geq 2(d-a)$ in the exponent. The reciprocal of the linear normalizer costs at most $2q^{2P}$ by the lemma. Since $P \leq 2am$, this enters $q^{C(a+1)^2(m+1)}$. The bound on this modified sum and the error estimate above prove the proposition. □

Theorem 138.16 (Complete unweighted mass at a standard inert place)**.** *Under the good inert hypotheses of Theorem 138.13 there is an absolute constant C such that at all sufficiently large nodes*

$$1 \leq \int_G |b_K^0(x;0)|\, dx \leq 1 + q^{-2k+C(a+1)^4}.$$

Every integral local frequency has the same mass.

Proof. For each $m \geq 1$, $P > 0$ and candidate J there are three possibilities. If $\delta \geq 1$, the unsigned count saves at least d in the exponent relative to dP. If $\delta = 0$ and $b_a \geq 1$, the terminal-row bound saves $2\eta b_a \geq d - 2a - 2$. Otherwise $2\sum b_i = P > 0$ and $b_a = 0$; some consecutive pair has a strict drop. Apply Proposition 138.15. In all cases, adding over at most q^{2a^2m} candidate lattices and absorbing the polynomial factors gives

$$\sum_{\mathcal{C}} |S_{\mathcal{C}}| \leq q^{C_1(a+1)^2(m+1)+dP-d}$$

for the original coefficient on the entire shell. Its zero-frequency energy is at most the square of this bound divided by the orbit lower bound in Lemma 138.7. When $0 < P < 1000(a+1)^2$, one has $m \leq P$. Summing the result over these polynomially many shells costs at most $q^{-2d+C_2(a+1)^4}$. Theorem 138.13 bounds all greater lengths by q^{-6k}. For $P = 0$ the two lattices coincide: inclusion of one self-dual lattice in another gives equality by taking duals. The position is in K, where the contribution is one. This proves both inequalities, using $d = k+1$. At an integral frequency the character on K^2 is identically one. □

Corollary 138.17 (The standard unmarked Euler factors at zero)**.** *On the fields and shorter intervals of Corollary 11.30, let $\mathcal{V}$ be any set of unmarked places of $F = E^+$, of residue characteristic prime to q_1, satisfying the standard split or inert hypotheses of the respective theorems. Then*

$$1 \leq \prod_{w \in \mathcal{V}} \mathcal{B}_w^0(0) \leq \exp(O(q_1^{2-k})).$$

The logarithms converge absolutely. The same bound applies at all members of this set at which a given rational frequency is integral.

Proof. Use Corollary 137.22 for the split factors. At an inert place the automorphism of the residue field of E given by raising to q_w is the nonidentity element of $\mathrm{Gal}(E/F)$, acting on the q_1-th roots of unity by inversion. Reduction is injective on these roots as the residue characteristic is prime to q_1. Thus $q_1 \mid q_w + 1$. By the theorem and $k/(a+1)^4 \to \infty$, each

logarithm is between zero and $q_w^{-3k/2}$ at large nodes. There are at most $f \le q_1$ places of a given residue size, so their sum is at most

$$q_1 \sum_{j \ge q_1 - 1} j^{-3k/2} \le \frac{q_1 (q_1 - 2)^{1 - 3k/2}}{3k/2 - 1} \ll q_1^{2-k}.$$

For the last inequality use $(q_1 - 2)^{3/2} \ge q_1$ at large nodes. Exponentiating proves the assertion. □

For a nonintegral frequency the distant estimate in Theorem 138.13 still applies. The interior range then requires control of the compact average after fixing a vector modulo its full conductor.

138.7 The first inert transform at prime denominator

We use the integral lattice $M_x = \mathcal{L} \cap x\mathcal{L}$ to measure how many end vectors remain indistinguishable in a compact average at conductor π^c. Put

$$h_c(x) = \min\{c, \operatorname{ord}_A h(M_x, e)\}, \qquad d_1 = d - 2a - 2, \qquad c \geq 1.$$

The valuation here is that of the ideal of all pairings with the first flag vector e. It is finite since M_x has full rank. The integer $h_c(x)$ is constant on the K_c-orbit of the right coset xK. Indeed a left multiplier $g \in K_c$ carries M_x to gM_x and $g^{-1}e \equiv e \bmod \pi^c$. For a supporting coset yK with $y \in U$, write δ, p as in Sections 138.3 and 138.4.

Lemma 138.18 (End vectors under the lattice stabilizer)**.** *The stabilizer in K of a supporting coset yK has at least $q^{2d_1(c-h_c(y))}$ different images of $e \bmod \pi^c$. In particular the K_c-orbit of that coset in $\mathcal{S}(m, P)$, for $m \geq 1$, has size at least*

$$\tfrac{1}{3} q^{2dP - 4a^2 m + 2d_1(c - h_c(y)) - 2rc}.$$

Proof. Split off the integral unimodular W_0 of rank at least $d - 2a$ fixed pointwise by y, as in Lemma 138.12. Take an element of M_y minimizing the valuation of the pairing with e, and project it off W_0. This preserves the valuation and membership in M_y. In a hyperbolic plane of W_0 add an integral vector of norm equal to the negative of the norm of this projection. Every integral value is attained there by taking coefficient one on the first vector and prescribing the trace of the other coefficient. The resulting vector v is isotropic, belongs to M_y, and still attains the minimum. Let W_1 be the orthogonal complement of this plane in W_0, of rank at least d_1.

For $w \in W_1$ the translation $N_v(w)$ of (138.4) is integral. Its conjugate by y^{-1} is $N_{y^{-1}v}(w)$, also integral, since y fixes w and $v \in M_y$. It therefore stabilizes yK. The W_1-component of $N_v(w)e$ is exactly $h(v, e)w$, all other terms lying in $W_1^{\perp}$. Reduction modulo π^c gives the asserted number of images. For a coset with stabilizer $H \subset K$ the size of its K_c-orbit is its K-orbit size multiplied by $[H : H \cap K_c]/[K : K_c]$, by the index formula. The numerator counts these images of e and the denominator is at most q^{2rc}. Apply Lemma 138.7. □

For a right K-invariant coefficient f, write $\mathcal{B}(f; \alpha)$ for the integral of the absolute value of the compact double average defining $b_K^0(x; \alpha)$, with f used at both positions. Let f_c be ϕ_0 on the supporting cosets satisfying

$$h_c \leq 1 \qquad \text{or} \qquad h_c \leq \delta + p,$$

extended by zero elsewhere.

Theorem 138.19 (The first transform at prime denominator)**.** *At the standard unmarked inert places of Theorem 138.13, for $\operatorname{ord}_A \alpha = -c < 0$,*

$$\mathcal{B}(f_c; \alpha) \leq q^{-n^7 c/2} I_w^{(k-n^7)/2} D_w^{-3}$$

at all sufficiently large nodes. For $c = 1$ this gives the bound for the original coefficient ϕ_0 throughout the whole group.

Proof. First fix $m \geq 1, P > 0$ and partition the K_c-orbits by $h = h_c$. On the orbits of each given h, the absolute signed class sums of f_c satisfy

$$\sum_{\mathcal{C}} |S_{\mathcal{C}}(f_c)| \leq q^{C(a+1)^2(m+1) + dP - d_1 h}. \tag{138.8}$$

For $h \le 1$ use the proof of Theorem 138.16 before division by orbit sizes. All three estimates used there apply on the refined orbits as well, by the statements of Theorem 138.10 and Proposition 138.15 and the unsigned count. The saving d, with terms $O(a)$ in the other exponent, suffices for $d_1 h$. If $h \ge 2$, only $J = J_B$ with $\delta + p \ge h$ are retained. On each such J the terminal calculation in Theorem 138.10 saves $d\delta + 2\eta p \ge d_1 h$ relative to dP. For a fixed J the condition on h merely selects whole orbits, as observed above. The number $q^{2a^2 m}$ of possible lattices and the factor $2(m+1)$ are absorbed by the absolute constant C.

Let $P_{K_c} f_c(x) = \int_{K_c} f_c(gx)\, dg$ with probability Haar on K_c. Its squared integral on the shell is the sum of squared signed class sums divided by their orbit sizes. Use Lemma 138.18 on orbits of each depth in (138.8). The terms involving h cancel, giving the bound

$$(c+1) q^{C_1 (a+1)^2 (m+1) + 2(r - d_1)c}.$$

For the compact double average, the inequality of Proposition 138.3 applies to f_c because its proof permits arbitrary coefficients. On integration over a K-invariant shell the conditional mean squares are the squared integral just bounded, by the translation argument in Theorem 138.13. The operator norm supplies the factor $q^{-rc}/\rho_c \le 2q^{-(r-1)c}$.

Sum first over $0 < P < 1000(a+1)^2(c+1)$, where $m \le P$. Since $r - d_1 = 4a + 2$, the entire sum is bounded by $q^{-(r-1)c + C_2 (a+1)^4 c}$. The polynomial number of shells and powers of $c+1$ are absorbed since $c \ge 1$ and $\log_q(c+1) \le c$. The part on K has $h = 0$ and is bounded by Theorem 138.2; enlarge C_2 to include it.

For completeness the distant estimate remains valid for this particular restriction despite the signs of the sums. In the proof of Theorem 138.13 the signed estimate was obtained first for each J, then summed over J by the triangle inequality. For fixed J the restriction to $h \le \max(1, \delta + p)$ discards whole K_c-orbits; hence each absolute sum over orbits can only decrease at this step. Use the same orbit denominator and the same distant series to obtain $q^{-6k(c+1)}$. Finally $n = a + 1$, $r - 1 = k + 2a$, $I_w = q^{-2c}$ and $D_w = q^{2c}$. For sufficiently large n the saving $n^7 c/2$ available in the arithmetic weight absorbs $C_2 n^4 c + 6c$ and the sum of the two contributions. If $c = 1$, every supporting coset has $h_c \le 1$. □

Corollary 138.20 (Good Euler factors for squarefree inert denominators)**.** *Use the fields, intervals and set $\mathcal{V}$ of Corollary 138.17. Let $y \in E^\times$ have pole order at most one at each inert member of $\mathcal{V}$. For the original first relative Gram transform,*

$$\prod_{w \in \mathcal{V}} \mathcal{B}_w^0(y) \le \exp(O(q_1^{2-k})) \prod_{\substack{w \in \mathcal{V} \\ D_w(y) > 1}} \min\{1, I_w(y)^{(k - n^7)/2} D_w(y)^{-3}\}.$$

Proof. Only finitely many local denominators exceed one. At the split ones apply Theorem 137.21; at the inert ones apply the theorem with $c = 1$. Discard the extra factors at most one in those bounds. At every other place the frequency is integral in each component, so the relative character on the compact pair is one. Apply Corollary 138.17 to their product. □

For a deeper inert denominator the remaining part has $h_c \ge 2$ and $h_c > \delta + p$. Both this part and its mixed terms with f_c enter the original double average.

138.8 Simultaneous inert row tests in an isotropic summand

Work in the unramified quadratic field and self-dual integral decomposition $\mathcal{L} = L \oplus W \oplus R$ of Section 138.2, where $R = L^{\vee}$. The rank of W is d; $z \in W$ is primitive. We use the standard unmarked level and the unit coefficients of the simple character. Write $M_y = \mathcal{L} \cap y\mathcal{L}$ and $J = J_B = L + TL + BW$ for a unipotent representative y. We keep $m, P, \delta, b_i(J)$ from Section 138.5. The two copies of the coefficient can sample very deep first endpoints. We construct one family of admissible row tests which can be applied successively inside a single signed sum.

There is a hyperbolic middle summand W_H of rank $2a$ and a totally isotropic direct summand $Q \subset W_H^{\perp} \cap W$ of rank $g = \lfloor (d-2a)/2 \rfloor$ with $h(z, Q) = O_A$. First split off a hyperbolic pair whose first vector pairs to a unit with z, as in Section 138.4. Take a hyperbolic planes for W_H in the complement. Inside the perpendicular to W_H, extend the first vector of the original pair to Q by using further integral hyperbolic pairs. Their existence follows from the norm and trace constructions of Lemma 138.1. Choose an integral map $\iota : L \oplus R \longrightarrow W_H$ taking the form to its negative. In paired hyperbolic bases one uses the first vectors and the negatives of their partners. Define

$$\mathcal{P} = \{ s + \iota(s) + u : \ s \in L \oplus R, \ u \in Q \}, \qquad W_U = W_H \oplus Q.$$

The summand $\mathcal{P}$ is totally isotropic by orthogonality and the defining property of ι. Its projection onto W is W_U. An integral complement W_C of W_U in W has rank $d - 2a - g$; it need not be orthogonal to W_U.

Write $f_i = e_i$ for $1 \le i \le a$ and $f_{a+1} = z$. For a full sublattice D of O_A^{a+1}, let its i-th integer be the least valuation of the i-th coordinate among vectors whose earlier coordinates vanish. The sum of the integers is $\mathrm{length}(O_A^{a+1}/D)$ by successive coordinate projections. Use the linear evaluation map $E(v) = (h(f_i, v))_{i=1}^{a+1}$. Define t_i, u_i as the integers for $E(M_y)$, $E(M_y \cap \mathcal{P})$, respectively. Both lattices have full rank. In fact $E(\mathcal{P}) = O_A^{a+1}$ because the R-component prescribes the first a values and $h(z, Q) = O_A$ corrects the last one. Also $\pi^m \mathcal{L} \subset M_y$. Thus $0 \le t_i \le u_i \le m$. For each i select an integral $v_i \in M_y \cap \mathcal{P}$ attaining u_i, with all earlier evaluations zero. Make the choice depend only on $M_y \cap \mathcal{P}$ and the fixed flag. On a bounded shell there are finitely many possible such sublattices, since they contain $\pi^m \mathcal{P}$.

Theorem 138.21 (Simultaneous simple-root tests at an inert place)**.** *For a K_c-orbit of right K-cosets on a fixed shell, including $K_0 = K$, its signed unipotent integral is unchanged on retaining just the y satisfying all a conditions*

$$\beta_j\big(W \cap v_{j+1}^{\perp} \cap \ker B_{>j}\big) \subseteq \pi^{-u_{j+1}} O_A, \qquad 1 \le j \le a. \tag{138.9}$$

Here β_j is row j of B, and $B_{>a}$ has no rows. The identity holds also after fixing J and $M_y \cap \mathcal{P}$ inside the integral.

Proof. Fix these two lattices, and start at $j = a$, proceeding down to $j = 1$. At the current row write $v = v_{j+1}$. For $w \in W \cap v^{\perp} \cap \ker B_{>j}$, put $b = Bw$. One has $b \in L_j \otimes A$, $v \perp L_j$, and both v and $y^{-1}v$ are integral isotropic vectors orthogonal to w. Identity (138.7) gives

$$y' = N_v(w)^{-1} y N_{y^{-1}v}(w) = N_v(b) y.$$

This identity allows arbitrary integral w in the specified summand. The left exterior multiplier fixes e_1 and the right one is integral, so y' remains in the same orbit. As in Lemma 138.12, $B' = B(1 + wh(v, y\,\cdot)|_W)$ and $T' = T + bh(v, y\,\cdot)|_L$. The first right multiplier is an integral

automorphism, since $h(v, yw) = h(v, w + b) = 0$. Hence J stays fixed. On W, the change $B' - B$ has image in $L_j \otimes A$, so all rows strictly after j stay fixed.

The additional invariance is integral and exact. On $\mathcal{P}$ the terms containing $h(v, p)$ in $N_v(w)p$ vanish; this map sends p to $p - vh(w, p)$. Thus it is an integral automorphism of $\mathcal{P}$, acting by addition of integral multiples of v. It preserves $M_y \cap \mathcal{P}$ in both directions because v belongs to that sublattice. Since $M_{y'} = N_v(w)^{-1} M_y$, intersection with $\mathcal{P}$ gives $M_{y'} \cap \mathcal{P} = M_y \cap \mathcal{P}$. In particular all the selected v_i's remain exactly the same.

For $j < a$ the only possible simple entry of $N_v(b)$ is $\beta_j(w)h(v, e_{j+1})$; for $j = a$ it is $\beta_a(w)h(v, z)$. These statements follow by restricting $1 + bh(v, \cdot) - vh(b, \cdot)$ to the flag and to z; $b \perp L \oplus W$. Therefore the simple character changes by $\psi_A(\gamma_j \beta_j(w) h(v, f_{j+1}))$, with γ_j a unit. This is the simple-root calculation underlying the Eichler map in the supplied source.

The change preserves unipotent Haar measure. At fixed v disintegrate it along left cosets of the additive subgroup $\{N_v(u) : u \in L_j \otimes A\}$. The allowed w's are the same along such a coset, since the lower rows of B stay fixed. For each such w, $N_v(u)$ fixes $w + Bw$, so the translating element $N_v(Bw)$ is constant on the coset. Translation preserves its Haar measure, with inverse obtained using $-w$, as in Lemma 138.12. For $j = a$ the group is again additive and closed: the nonzero pairing $h(v, z)$ makes u recoverable from $N_v(u)z - z$; for the other j use the pairing with e_{j+1}. The restrictions at indices greater than j depend only on their own and lower rows and on the unchanged vectors v_i. They are consequently preserved in the successive integrations. Averaging in w gives exactly the current condition by character orthogonality over O_A. Induction proves the identity. Finally sum over the disjoint finite choices of the two lattices whenever they are not prescribed. □

Proposition 138.22 (Loss in the flag of evaluations)**.** *Let $\Gamma : W \to J \oplus O_A$ be $\Gamma(w) = (Bw, h(z, w))$ and put $s_Q = \operatorname{length}((J \oplus O_A)/\Gamma(Q))$, with value $+\infty$ when the quotient is not finite. Set*

$$G_B = \sum_{j=1}^{a} (b_j(J) - u_{j+1})_+.$$

If s_Q is finite, then

$$G_B + \delta + s_Q \geq t_1. \tag{138.10}$$

Proof. For $v = s + w \in M_y$, $s \in L \oplus R$, form the class of $\Gamma(w - \iota(s))$ modulo $\Gamma(Q)$. If it vanishes, choose $u \in Q$ representing this difference. The vector $v' = s + \iota(s) + u$ differs from v by an integral middle vector in $\ker B$ orthogonal to z. That vector is fixed by y, hence belongs to M_y. Thus $v' \in M_y \cap \mathcal{P}$ and $E(v') = E(v)$. The kernel of the indicated linear map from M_y therefore maps to zero in $E(M_y)/E(M_y \cap \mathcal{P})$, giving

$$\sum_{i=1}^{a+1} (u_i - t_i) \leq \operatorname{length}(\Gamma(W)/\Gamma(Q)).$$

Also every vector in $\ker B \cap W$ is in M_y and has zero pairing with L. Thus t_{a+1} is at most the valuation of $h(z, \ker B \cap W)$. The latter is at most $\operatorname{length}((J \oplus O_A)/\Gamma(W))$, since the intersection of $\Gamma(W)$ with the last coordinate line consists precisely of these pairings. Adding gives $t_{a+1} + \sum_i (u_i - t_i) \leq s_Q$.

For $i \leq a$, $t_i = b_i(J^+)$ with $J^+ = (y\mathcal{L} + \mathcal{L}) \cap (L \otimes A)$. The lattice proof in Lemma 138.12 applies also to $i = 1$, with no earlier coordinates to remove. Consequently $\sum_{i=1}^{a} t_i = \sum_{i=1}^{a} b_i(J) + \delta$. Now sum $b_j(J) - u_{j+1}$ before taking positive parts. The result is $t_1 - \delta - t_{a+1} - \sum_{i=2}^{a+1} (u_i - t_i)$, at least $t_1 - \delta - s_Q$. Taking positive parts only increases the sum. □

The integer s_Q uses only the restriction of B to Q. The row conditions in (138.9) also test the columns on the independent complement W_C.

138.9 The full first transform at standard inert places

Continue with the summands and lattices of Section 138.8. We estimate the cost of the index s_Q using the columns on Q; the independent columns on W_C account for the remaining endpoint depth. Write $\lambda = d - 6a - 4$, which is positive at sufficiently large nodes.

Lemma 138.23 (Refined signed class sums at every endpoint depth)**.** *On $\mathcal{S}(m, P)$ with $m \geq 1$, let $\mathcal{C}$ range over the K_c-orbits of right cosets with $h_c = h$, where $c \geq 1$ and $0 \leq h \leq c$. For the entire original coefficient let $S_{\mathcal{C}}$ be its signed sum on the orbit. Then*

$$\sum_{\mathcal{C}:\, h_c = h} |S_{\mathcal{C}}| \leq (h+1) q^{C(a+1)^2(m+1) + dP - \lambda h} \tag{138.11}$$

for an absolute C.

Proof. Apply Theorem 138.21 in each orbit before taking its absolute value. It suffices to count the volume of the resulting simultaneous tests. We use the product coordinates T, B, Z and the central-block notation C_y for the block called C in Lemma 138.6. Fix T, a candidate $J = J_B$, and the entries of C_y modulo O_A. The latter have at most $q^{2a^2 m}$ possibilities, since all their pole orders are at most m. For any fixed T, B the allowed Z-volume is at most $q^{2a^2 m}$, as in that lemma. Discarding other restrictions on it after this count is valid even though C_y is related to B by unitarity.

Once $B|_{W_U}$ is also given, the entire sublattice $M_y \cap \mathcal{P}$ is determined by the fixed data. Indeed for an integral $v = s + w \in \mathcal{P}$ its middle component w belongs to W_U. The condition $v \in M_y$ means that $h(v, y\,\cdot)$ is integral. On the left and middle input blocks this is precisely integrality of its L-functional on $J = L + TL + BW$, since the middle component of v is already integral. On the last input block the pairing is the sum of its L-functional on C_y, its R-functional on S, and the middle term $w^* H D = -(Bw)^* S$. All are determined modulo integral functionals by the prescribed data. Thus the selected v_i, their u_i, and G_B are fixed before the columns on W_C are counted. One may now enlarge the domain of B to independent columns uniform in J, keeping these fixed data as parameters for the tests. Its unnormalized volume is $q^{2d\,\mathrm{length}(J/L)} = q^{d(P-\delta)}$. Only data occurring for an actual coset of the indicated depth need be retained.

For this column measure, choose an integral basis of Q in which the row $h(z, \cdot)$ is $(1, 0, \dots, 0)$. The index length s_Q is then the length of J modulo the last $g-1$ random columns from Q. This follows by use of the first column, whose last coordinate is one, to eliminate the last coordinate in $J \oplus O_A$. For any integer $s \geq 0$,

$$\Pr(s_Q \geq s) \leq q^{-(2g-3a)s}. \tag{138.12}$$

If the length is at least s, all $g-1$ columns belong to some sublattice of colength s in J, also when the quotient has a free part. A fixed such lattice costs $q^{-2(g-1)s}$. Their number is at most $(s+1)^a q^{2(a-1)s}$. To see this in a basis of J, choose the successive diagonal valuations of a triangular basis, whose sum is s. In each pivot row the entries on one side of that pivot reduce modulo its pivot ideal, and there are at most $a-1$ such entries. This yields the stated count. Use $s + 1 \leq 2^s$ for $s \geq 1$ and $q \geq 2$. For $s = 0$ the assertion is immediate.

Next fix the columns on W_U. Reveal the rows of the W_C-columns from the last row up to the first. Conditional on their coordinates strictly after row j, the row β_j there has independent coordinates uniform in translates of $\pi^{-b_j(J)} O_A$: the projection kernel in J is $J \cap (L_j \otimes A)$. Within W_C the submodule $\ker B_{>j} \cap v_{j+1}^{\perp}$ is primitive and has rank at least $\operatorname{rank} W_C - a$. It is specified by the fixed columns, v_{j+1}, and the already revealed rows. Completing a basis of that

kernel inside W_C preserves the conditional product measure up to translation. The fraction meeting the j-th test is therefore at most

$$q^{-2(\operatorname{rank} W_C-a)(b_j(J)-u_{j+1})_+}.$$

This conditional bound holds for every permitted set of later rows. Iteration bounds the fraction meeting all the tests by $q^{-2(\operatorname{rank} W_C-a)G_B}$.

Partition the volume according to $s=\min(h,s_Q)$. This has at most $h+1$ values and is decided on W_U. If $s<h$, Proposition 138.22 gives $G_B\ge(h-\delta-s)_+$, since $t_1\ge h_c=h$. When $s=h$ the same inequality follows from $G_B\ge 0$. It continues to bound the conditional count after enlargement: if the fixed data occur at the required depth, then their already determined G_B has this value or a greater one. Combining the two counts, the total saving in the exponent relative to dP is at least

$$d\delta+(2g-3a)s+2(\operatorname{rank} W_C-a)(h-\delta-s)_+\ \ge\ \lambda h.$$

Here $g=\lfloor (d-2a)/2\rfloor$ and $\operatorname{rank} W_C=d-2a-g$. Finally T costs at most $q^{a(a-1)m}$, the choice of J at most q^{2a^2m}, and the residue choices and central volume have the bounds above. After the signed identities we have counted a subset of the original unipotents; there is no factor counting $M_y\cap\mathcal{P}$, since that lattice is already determined by the conditioned data. This proves (138.11). □

Theorem 138.24 (The complete first transform at a standard inert place)**.** *Let w satisfy the standard unmarked inert hypotheses of Theorem 138.13. With both original Bessel coefficients on the full group, at every nonintegral frequency of conductor exponent $c\ge 1$ one has*

$$\mathcal{B}_w^0(\alpha)=\int_G |b_K^0(x;\alpha)|\,dx\le q^{-n^7c/2}I_w^{(k-n^7)/2}D_w^{-3} \tag{138.13}$$

at all sufficiently large nodes, uniformly in w,c. Integral frequencies have the zero-frequency mass of Theorem 138.16.

Proof. For $m\ge 1$ and fixed depth h, divide the square of (138.11) by the lower bound for the refined orbit sizes in Lemma 138.18. The result bounds the sum of the squared signed sums divided by their individual orbit sizes. Adding over h bounds the squared integral of the K_c-average of the full ϕ_0 on the shell by

$$(c+1)^3q^{C_1(a+1)^2(m+1)+2(r-\lambda)c}.$$

Indeed $d_1=d-2a-2\ge\lambda$ and the depth contribution after division is $2(r-d_1)c+2(d_1-\lambda)h$. In particular this calculation already includes all products of distinct parts of the Bessel coefficient occurring in a squared orbit sum.

The conditional Fourier operator of Proposition 138.3 has norm at most $2q^{-(r-1)c}$. Its application to the original double average, integrated on the shell, multiplies the last bound by this factor, as in Theorem 138.19. For $0<P<1000(a+1)^2(c+1)$, one has $m\le P$, and $r-\lambda=8a+4$. The sum over all these shells is thus at most $q^{-(r-1)c+C_2(a+1)^4c}$, including the polynomial factors because $\log_q(c+1)\le c$. On K use Theorem 138.2; beyond that range of P use Theorem 138.13. The latter contribution is bounded by $q^{-6k(c+1)}$. With $n=a+1$, $r-1=k+2a$, $I_w=q^{-2c}$ and $D_w=q^{2c}$, the sum satisfies (138.13) once $n^7/2>C_2n^4+8$ and the node is sufficiently large. At integral frequency the relative character on the two compact variables is one. □

Corollary 138.25 (Standard finite places at arbitrary rational frequency)**.** *On the fields and intervals of Corollary 138.17, let $\mathcal{V}$ be any set of its standard unmarked split and inert places. For every $y \in E^{\times}$,*

$$\prod_{w\in\mathcal{V}} \mathcal{B}_w^0(y) \le \exp(O(q_1^{2-k})) \prod_{\substack{w\in\mathcal{V}\\ D_w(y)>1}} \min\{1, I_w(y)^{(k-n^7)/2} D_w(y)^{-3}\}.$$

Proof. Apply the theorem at the inert poles and Theorem 137.21 at the split ones. There are only finitely many poles. For an inert pole the displayed arithmetic factor is less than one. All the integral masses have product bounded by Corollary 138.17. □

This controls the first relative coordinate at every place satisfying the stated integral hypotheses. The remaining coordinates involve the longer flag and the last vector of norm $-N$, and the original block also has real factors.

139 Local transforms in the rational frequency sum

139.1 The first frequency sum on the period fields

Take an actual node $E = \mathbb{Q}(\sqrt{-d_0}, \mu_{q_1})$, $F = E^{+}$, of the shorter interval of Corollary 11.30; d_0 here denotes the quadratic prime. Let $f = [F : \mathbb{Q}]$, and write $\mathcal{W} = \{w : w \mid p_0 p_1\}$ for the original marks, $\mathcal{U}$ for all the other finite places of F. Use the finite position factors after the class changes in Theorem 134.10. We verify the integral hypotheses also at the places above the separate prime N.

Theorem 139.1 (All unmarked factors on the specified fields)**.** *In these coordinates each $w \in \mathcal{U}$ satisfies the hypotheses of Theorem 137.21 or of Theorems 138.16 and 138.24. Put $K_0 = k - n^7$, $V_w(y) = I_w(y)^{K_0/2} D_w(y)^{-3}$, and*

$$\epsilon = q_1^{2-k} + f \log(2q_1)\, 2^{-3k/2}.$$

For every $y \in E^{\times}$, at the same sufficiently large nodes,

$$\prod_{w \in \mathcal{U}} \mathcal{B}_w^0(y) \le \exp(O(\epsilon)) \prod_{\substack{w \in \mathcal{U} \\ D_w(y) > 1}} \min(1, V_w(y)). \tag{139.1}$$

The product at frequency zero lies between one and $\exp(O(\epsilon))$.

Proof. The two imaginary quadratic subfields $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-d_0})$ have coprime discriminants. At each rational prime its inertia group in the abelian extension $E/\mathbb{Q}$ fixes at least one of these fields, whereas complex conjugation acts nontrivially on both. Thus no finite inertia group contains complex conjugation. The quadratic extension E/F is unramified at every finite place, as in Corollary 12.4. Every quadratic local algebra under consideration is accordingly split or an unramified field, even at a prime dividing q_1.

In the rational middle Gram matrix the positive vectors x_i are orthonormal, $h(z, z) = -N$, and z is orthogonal to them. The actual finite lattice uses the basis replacing a single x_* by

$$u = x_*/\bar{\pi}_N + z/N, \qquad \pi_N \bar{\pi}_N = N.$$

This is the lattice of Section 130.4 and of Lemma 134.3. In the pair u, z its Gram matrix is

$$\begin{pmatrix} 0 & -1 \\ -1 & -N \end{pmatrix}.$$

Indeed the two terms in $h(u, u)$ are $1/N, -1/N$; each cross pairing with z is -1. This block has unit determinant up to sign at every place. The remaining middle lines have norm one and the paired arms are integral hyperbolic planes. The entire lattice is self-dual; z is a basis vector and its covector takes the unit value -1 on u. Thus both terminal directions required in the split calculation are primitive in the corresponding components. This argument also applies over N, where the original positive basis and the actual lattice basis can differ integrally.

The local level is the full stabilizer of this lattice when $w \nmid M = p_0 p_1$, since the level imposes only the reduction modulo M. The finite simple character reads the successive entries in the frame $e_1, \dots, e_a, z$, with common scalar M^{-1}, a unit at w. The finite additive pairing is the trace divided by the different generator δ_E, whose dual integer lattice is the integer ring itself. In a split algebra this gives two conductor-integer-ring characters, each a unit multiple of one such character of F_w. For an unramified quadratic algebra it is a conductor-integer-ring character of that algebra; it is a unit multiple in its argument of a conductor-integer-ring

character of F_w composed with relative trace. This follows from the perfect integral trace pairing and additive duality, or by transitivity of the different and the unit relative different. All the flag vectors and their simple-coordinate functionals in the preceding lattice basis are integral with the required primitive coefficients. The same unit changes for the first frequency preserve its local valuations.

In the class integrals the changes of variables by h_c^{-1} and $h_{c'}^{-1}$ in Theorem 134.10 carry the two normalized unipotent measures to those giving $U \cap J_w$ volume one. These middle isometries fix z and the two arms, normalize U, and preserve its character. The first relative entry also satisfies $h(e, h_c^{-1} g h_{c'} e) = h(e, ge)$. Thus the class-removed position calculation uses the specified lattice even when a chosen class representative is nonintegral at w. The local hypotheses have now been checked.

Outside the residue characteristics dividing q_1 apply Corollary 138.25 and its zero-frequency counterpart. There are at most $f \log(2q_1)/\log 2$ other places, since at most f lie over any rational prime. At each of these, an integral-frequency mass lies in $[1, 1+q_w^{-3k/2}]$ at large nodes by the two local theorems: their positive error exponents are $O(n^4)$ and $n^4 = o(k)$. At its poles the corresponding weighted theorem applies unchanged. Using $q_w \geq 2$ on this last finite set proves the asserted upper bounds. At zero all local masses are at least one, and the absolute convergence of their logarithms follows from the same estimates. □

Corollary 139.2 (Rational first frequencies with the real power weight)**.** *Let $\Xi \subseteq \mathcal{W}$ be the selected set of Theorem 134.11, and let $\mathcal{V} = \mathcal{W} \setminus \Xi$. Write $c_v = 2\pi i/(\Phi_v(\delta_E) b_{1,v})$ for the actual first-coordinate scale. For $y \neq 0$ define the real weight*

$$A_\infty(y) = \prod_{v|\infty} |c_v \Phi_v y|^{-K_0} \min\{1, |c_v \Phi_v y|^{-6}\}.$$

Using B_0, j_w, Z_w^0 of Corollary 137.19, one has

$$\sum_{\substack{0 \neq y \in E \\ y \text{ integral at all primes over } \mathcal{V}}} A_\infty(y) \prod_{w \in \Xi} \frac{B_0 q_w^{n-j_w} \mathcal{B}_{Z_w}(y)}{Z_w^0} \prod_{w \in \mathcal{U}} \mathcal{B}_w^0(y) \;\leq\; \exp(O(\epsilon))\, b^{-7kf}. \quad (139.2)$$

Proof. Take first a finite conjugation-stable set T of primes of E, containing both primes above each member of Ξ, disjoint from the primes above $\mathcal{V}$. For $0 \neq y \in \mathcal{O}_{E,T}$, the product of $V_w(y)$ over the places under T is $I_T(y)^{K_0/2} D_T(y)^{-3}$. At an integral place $V_w(y) \geq 1$. By (139.1), the product of all unmarked masses divided by the product of $V_w(y)$ at the unmarked places under T is therefore at most $\exp(O(\epsilon))$. Multiplying the summand in (137.11) by this ratio removes exactly the remaining finite power weights and inserts the unmarked masses. This proves the estimate with y restricted to $\mathcal{O}_{E,T}$. Increase T through all finite conjugation-stable sets allowed above. Every y integral at the unselected marks has only finitely many further poles and hence occurs in the increasing union. All summands in the displayed corollary are nonnegative, so monotone convergence gives (139.2). □

In (139.2) the real factor A_∞ is the specified power weight. Applying this estimate to the original theta-period integral requires a bound on its real Bessel correlations at these scales, with the other frame entries and the rational sampling hypotheses retained.

139.2 The Fourier operator on the real frame quadrics

At a real place write $h(v,w) = v_+^* w_+ - v_-^* w_-$ on $\mathbb{C}^p \oplus \mathbb{C}^q$, and $r' = p + q$. The signature at the first frame vector is $(p,q) = (k+a, a+1)$. After removing j hyperbolic pairs it is $(k+a-j, a+1-j)$. On the first a orbits the remaining frame starts with a nonzero null vector. On the last orbit it starts with the vector of norm -1, and $q = 1$. For $\tau = 0, 1$ put

$$\mathcal{C}_\tau = \{v : h(v,v) = -\tau,\ v \neq 0\}, \qquad d\sigma_\tau(v) = \delta(h(v,v)+\tau)\, d^{2r'}v.$$

Here the delta notation means the fiber density obtained by using $h(v,v)+\tau$ as one real coordinate. On the indicated orbit its differential is nonzero. The determinant of an isometry has modulus one, so this measure is invariant. In the following formulas σ_τ always has this normalization. Write $s = \|v_+\|$. Let σ_τ^R be its restriction to $s \le R$. For $\nu \neq 0$ set

$$(T_{\nu,R} g)(v) = \int e^{2i\,\mathrm{Im}(\nu h(v,w))} g(w)\, d\sigma_\tau^R(w), \qquad s(v) \le R.$$

Theorem 139.3 (A uniform estimate for the real quadric operator)**.** *Suppose p, q are positive integers, $R > 0$, and either $\tau = 0$ or $(\tau, q) = (1,1)$. For every real $0 \le \ell \le r' - 2$,*

$$\|T_{\nu,R}\|_{L^2(\sigma_\tau^R) \to L^2(\sigma_\tau^R)} \le \frac{\pi^{r'}}{r' - 1 - \ell}\, R^{2(r'-1-\ell)} |\nu|^{-\ell}. \tag{139.3}$$

Proof. Use probability measure on each of the two unit spheres. Put $v = (s\omega, \sqrt{s^2+\tau}\,\eta)$. Integrating first in the negative radius divides its radial measure by twice that radius. Since the sphere in $\mathbb{C}^m$ has area $2\pi^m/\Gamma(m)$, the resulting radial density is

$$d\lambda(s) = \frac{2\pi^{r'}}{\Gamma(p)\Gamma(q)} s^{2p-1}(s^2+\tau)^{q-1}\, ds.$$

In both cases of the theorem its variable factor is $s^{2r'-3}$. The endpoints omitted from the polar charts have measure zero.

On the probability sphere S^{2m-1} the integral operator with kernel $e^{iu\omega\cdot\theta}$, $u > 0$, acts on the harmonics of degree d by

$$i^d \Gamma(m)(u/2)^{1-m} J_{m+d-1}(u).$$

For completeness let P_d be a homogeneous harmonic polynomial on $\mathbb{R}^{2m}$. Completing the square for real z in the following Gaussian integral and taking spherical means around $z\omega/2$ gives

$$\int_{\mathbb{R}^{2m}} e^{-|x|^2 + zx\cdot\omega} P_d(x)\, dx = \pi^m e^{z^2/4} (z/2)^d P_d(\omega).$$

The mean-value property applies to each real and imaginary part of the polynomial. Both sides are entire in z. Expanding the left side in polar coordinates, the coefficient of z^l is the degree-l angular moment divided by $l!$, multiplied by $\pi^m \Gamma(m + (d+l)/2)/\Gamma(m)$. Thus the angular moments vanish unless $l = d + 2b$ with $b \ge 0$. Inserting the remaining moments in the series for the spherical kernel gives $i^d \Gamma(m) P_d(\omega)$ times

$$(u/2)^d \sum_{b \ge 0} \frac{(-u^2/4)^b}{b!\,\Gamma(m+d+b)}.$$

This is the asserted Bessel expression. For integer $N \ge 0$ the coefficient of z^N in $\exp((u/2)(z - z^{-1}))$, expanded on $|z| = 1$, has the series for $J_N(u)$. The integrand defining that Fourier coefficient has modulus one. Hence $|J_N(u)| \le 1$. These formulas also give the usual integral and series normalizations of J_N. DLMF, Eq. 10.9.2 and Eq. 10.2.2.

Spherical harmonics of different degrees are orthogonal and their span is dense, by the harmonic decomposition of polynomials and density of polynomial restrictions on a sphere. The angular operator therefore has norm at most $\Gamma(m)(u/2)^{1-m}$. Its norm is also at most one by Cauchy–Schwarz for probability measure. At fixed positive radii s, t, the two angular operators for h form a tensor product. The phase of ν, the imaginary part, and the minus sign in h compose the factors with orthogonal changes of variables. The positive factor has $u = 2|\nu|st$. On the null orbit the negative factor has the same u. On the orbit with $q = 1$ we use only the probability bound for the negative factor. In both cases the tensor norm is at most one and at most $\Gamma(p)\Gamma(q)(|\nu|st)^{-(r'-2)}$. Since the factorial product is at least one, it is at most $\Gamma(p)\Gamma(q)(|\nu|st)^{-\ell}$ in the stated range.

For two test functions, bound each angular pairing at radii s, t by this norm times their two angular L^2 norms. The resulting two radial integrals separate. Cauchy–Schwarz bounds their product by the product of the full L^2 norms times $\int_0^R s^{-2\ell} d\lambda(s)$. Evaluation cancels the two gamma factors and gives (139.3). □

Corollary 139.4 (The first Gram wave before position integration)**.** *Let $G = \mathrm{U}(k+a, a+1)$, let U be the Bessel unipotent, and let e be the first frame vector, of Euclidean length one. For $a = 0$ take $e = z$. Set $\tau = -h(e, e)$. Use the quotient Haar measure of Theorem 132.14. For the cut vector $F = F_v^\flat$ from that theorem and $x \in G$, let $D_x(v)d\sigma_\tau(v)$ be the image of $\langle F(y), F(yx)\rangle\, d\dot{y}$ on $U\backslash G$ under $\dot{y} \mapsto y^{-1}e$. Define*

$$B(x;\nu) = \iint e^{2i\,\mathrm{Im}(\nu h(y^{-1}e, t^{-1}e))} \langle F(y), F(yx)\rangle\, \overline{\langle F(t), F(tx)\rangle}\, d\dot{y}\, d\dot{t}.$$

Use $n = a + 1$, $r = k + 2a + 1$, $K_0 = k - n^7$, at nodes with $K_0 \geq 0$ and $n^7 \geq 7$. For $R \geq \max(1, b^{6n+2})$,

$$|B(x;\nu)| \leq \pi^r R^{2(n^7+2a)} |\nu|^{-K_0} \min(1, |\nu|^{-6}) \int_{\mathcal{C}_\tau} |D_x|^2\, d\sigma_\tau. \tag{139.4}$$

Proof. The subgroup U fixes e. Its scalar character in $F(y)$ cancels that in $F(yx)$, so the density descends to the quotient. The integral is supported in the compact image of the matrix norm ball of radius R. Indeed the cut is supported where $\inf_{s\in S}(1 + \|sy\|) \leq b^{6n+2}$ with $S = U \rtimes \mathrm{U}(k)$, and the compact factor acts unitarily. In the chosen Hermitian coordinates $\|g^{-1}\| = \|g\|$. Hence $s(y^{-1}e) \leq R$. The map to the orbit is a smooth submersion. On a compact support a bounded density has bounded image density relative to smooth orbit measure, by integration in a finite collection of local product charts and a partition of unity. This proves $D_x \in L^2(d\sigma_\tau)$, including for the continuous cut. Also $h(y^{-1}e, t^{-1}e) = h(e, yt^{-1}e)$, the first Gram coordinate of the relative element. Pushing forward both integrals gives a pairing for $T_{\nu,R}$ or its transpose, which has the same norm.

For $|\nu| \leq 1$ take $\ell = K_0$. For $|\nu| > 1$ take $\ell = K_0 + 6$. Both are at most $r - 2$. The denominator in (139.3) is at least one and $r - 1 - K_0 = n^7 + 2a$, proving the inequality. The same exponent choices work on every residual quadric of signature $(k + a - j, a + 1 - j)$, since its dimension parameter is at least $k + 1$. □

The right side of (139.4) still depends on the position through the conditional correlation D_x. Estimating its integral in x is necessary before inserting the real factor in Corollary 139.2.

139.3 The first real transform on a ball of positions

We retain the normalized real frame of Section 131.1. Let $G_j = \mathrm{U}(k+a-j, a+1-j)$ and $r_j = k+2(a-j)+1$ for $0 \le j \le a$. Let $f_j = e_{j+1}$ if $j < a$ and $f_a = z$. Write σ_j for the measure of Section 139.2 on the orbit of f_j. Put $r = r_0$, $n = a+1$. All matrix norms are Euclidean operator norms in coordinates in which the two vectors of a hyperbolic pair are orthonormal for the Euclidean metric and have Hermitian pairing one.

Lemma 139.5 (Truncated Plancherel estimate on one quadric). *Let $m = p+q$, $R \ge 1$, $T > 0$, and let $g \in L^2(\sigma_\tau)$ be supported in $R^{-1} \le \|v\| \le R$. Then*

$$\int_{\|t\| \le T} \left| \int g(v) e^{2i \operatorname{Im} h(t,v)} d\sigma_\tau(v) \right|^2 dt \le 2(2m)^{3/2} R T \, \pi^{2m-1} \int |g|^2 d\sigma_\tau.$$

Proof. Diagonalize the form by a Euclidean unitary change of coordinates. The imaginary part of $h(t,v)$ is the ordinary real dot product of v with an orthogonal transform of t. In $D = 2m$ real coordinates partition the support according to the first coordinate of largest absolute value and its sign. On each of the resulting $2D$ sets, deletion of this coordinate is one-to-one. Its absolute value is at least $1/(R\sqrt{D})$. The graph density over the other coordinates v' is $J(v') = (2|v_i|)^{-1} \le R\sqrt{D}/2$. Fix the dual variable ξ_i. Plancherel in the other $D-1$ coordinates, with kernel $e^{2iv'\cdot\xi'}$, gives $\pi^{D-1} \int |g|^2 J^2 \, dv'$ on this patch. The graph phase in ξ_i has modulus one. Integrating ξ_i over an interval of length $2T$ and using $J^2 dv' \le (R\sqrt{D}/2) d\sigma_\tau$ proves the estimate for each patch before the factor $2D$. Cauchy–Schwarz for the sum of the transforms contributes that factor, and the squared input norms add because the patches are disjoint. □

We describe the measures in the recursion. Let H_j fix f_j in G_j, let U_j be the unipotent for the remaining frame, and put $C = \mathrm{U}(k)$. Thus $H_a = C$, $U_a = 1$. For $j < a$ use the order e, V, e^* with $e = f_j$ and V the residual space of dimension r_{j+1}. There is a subgroup N_j consisting of the transformations

$$\begin{aligned} e &\longmapsto e, \qquad w \longmapsto w - h(t,w)e \quad (w \in V), \\ e^* &\longmapsto e^* + t + \big(-h(t,t)/2 + iu\big)e \qquad (t \in V,\ u \in \mathbb{R}). \end{aligned}$$

Write this element as $n(t,u)$. Pairing the images of e, w, e^* gives the displayed coefficients. Conversely any isometry fixing e acts on $e^\perp/\mathbb{C}e$ by some $l \in G_{j+1}$. Removing l leaves exactly one $n(t,u)$. The condition on the remaining flag gives $H_j = N_j \rtimes G_{j+1}$ and $U_j = N_j \rtimes U_{j+1}$. Multiplication adds the t coordinates and adds the u coordinates with a translation depending on the two t's. Thus $dt\,du$ is Haar. The conjugation by l sends t to lt and preserves this Haar measure. On N_j the simple-root character evaluates the coefficient of e in the image of f_{j+1}. It is therefore $\exp(2i \operatorname{Im}(-h(t, f_{j+1})))$, up to complex conjugation according to the character convention. The real frame normalization makes the simple coefficients one.

Give C probability measure. Choose Haar measures dg_j^* on G_j by integration along $g \mapsto g^{-1} f_j$, with base σ_j and fiber $dt\,du\,dg_{j+1}^*$ on H_j, or probability on H_a. These are Haar measures since the orbit measure is invariant and the fiber group is unimodular. Give U_j the successive product of $dt\,du$. Write $d\dot{y}_j^*$ for the quotient measures. In particular, at $j < a$ one can integrate on $U_j \backslash G_j$ by integrating first on $U_{j+1}\backslash G_{j+1}$ and then on the orbit with measure σ_j.

Let A, B be bounded vector-valued functions, compactly supported modulo U_j, with Bessel equivariance for U_j and the same irreducible unitary representation of C of dimension d. We say the functions have radius R if their support modulo U_j is in the image of $\{y : \|y\| \le R\}$. Define $D_{j,x}^{A,B} \, d\sigma_j$ as the image under $\dot{y} \mapsto y^{-1} f_j$ of $\langle A(y), B(yx)\rangle d\dot{y}_j^*$. Put

$$E_j(X; A, B) = \int_{\|x\| \le X} \int |D_{j,x}^{A,B}(v)|^2 \, d\sigma_j(v) \, dg_j^*(x).$$

Proposition 139.6 (Conditional correlations on a position ball)**.** *For $X, R \geq 1$ and the functions above,*

$$E_j(X; A, B) \leq (16 r^{3/2} R^5 X^2)^{a-j} \pi^{\sum_{i=j+1}^{a} (2r_i - 1)} \frac{\|A\|_{2,*}^2 \, \|B\|_{2,*}^2}{d}, \tag{139.5}$$

where both norms use $d\dot{y}_j^*$.

Proof. Choose a measurable section $s_v^{-1} f_j = v$. At $j = a$, equivariance under C makes the integrand constant in the fiber, and $D_{a,x}(v) = \langle A(s_v), B(s_v x) \rangle$. For fixed v, extend the x integral to G_a, and translate $s_v x$ to a Haar variable. For a fixed vector $A(s_v)$, averaging the square of the inner product over C introduces the factor $1/d$. Indeed the C-average of the rank-one operator bb^* commutes with the irreducible representation and has trace $\|b\|^2$, so it is $\|b\|^2 I/d$. The remaining v-integral is $\|A\|_{2,*}^2$ by disintegration and equivariance. This proves the base case.

For $j < a$ choose $s_v = a_v k_v$, with k_v unitary for the Euclidean metric, and a_v scaling e, e^* by $\|v\|^{-1}, \|v\|$ and fixing V. Such k_v exists because the positive and negative parts of a null vector have equal length and can be rotated independently. It can be chosen measurably by extending unit vectors to orthonormal bases on finitely many coordinate patches. Define $A_v(m) = A(m s_v)$ and define $B_{v'}$ in the same way. Both have the required residual equivariance. For nonzero slices, $R^{-1} \leq \|v\|, \|v'\| \leq R$, since the isometries in the norm ball and their inverses have norm at most R. The residual support also has radius R. In fact, if $\|u_0 m a_v k_v\| \leq R$ for $u_0 \in U_j$, discard the right unitary factor. The middle block of $u_0 m a_v$ is $u_1 m$ with $u_1 \in U_{j+1}$, and its norm is at most R. The slice identity $\int \|A_v\|_{2,*}^2 \, d\sigma_j(v) = \|A\|_{2,*}^2$ follows directly from the quotient measures.

Hold v fixed. For $v' = x^{-1} v$ write $s_v x = n(t, u) l s_{v'}$. The measure in x is $dt\, du\, dg_{j+1}^*(l)\, d\sigma_j(v')$. If $\|x\| \leq X$ and the two slices are nonzero, we have

$$\|l\| \leq X, \qquad \|t\| \leq R^2 X, \qquad |u| \leq R^2 X.$$

Indeed l is the middle block of $a_v k_v x k_{v'}^{-1} a_{v'}^{-1}$, where the diagonal factors act identically on the middle block. The full matrix has norm at most $R^2 X$, and its last column has middle component t and top entry with imaginary part u.

At this x, the fiber integral for $D_{j,x}(v)$ is

$$\int_{U_{j+1} \backslash G_{j+1}} \langle A_v(m), B(m n(t, u) l s_{v'}) \rangle \, d\dot{m}^*.$$

Move $n(t, u)$ to the left of m by conjugation. Its character is the wave at $h(t, m^{-1} f_{j+1})$, with a fixed sign. Disintegration at $w = m^{-1} f_{j+1}$ now gives the Fourier transform of $D_{j+1,l}^{A_v, B_{v'}}(w) d\sigma_{j+1}(w)$ at t or $-t$. The result is independent of u. Its support as a density in w satisfies the annulus condition of Lemma 139.5 by the radius of A_v. Each density on compact support is bounded, as in Corollary 139.4.

Enlarge the t, u, l domain to the three bounds just proved. The central interval has length $2R^2 X$. Apply the lemma in dimension r_{j+1} with $T = R^2 X$, and integrate the result in l, v, v'. We obtain

$$E_j(X; A, B) \leq 16 r^{3/2} R^5 X^2 \, \pi^{2r_{j+1} - 1} \iint E_{j+1}(X; A_v, B_{v'}) \, d\sigma_j(v) \, d\sigma_j(v').$$

All integrals of squares used in enlarging the domains are nonnegative. By induction and the slice identities for the two norms, this proves (139.5). □

Theorem 139.7 (The normalized first real transform on a position ball)**.** *Let $F = F_v^\flat$, $d = d_v$, and let $B(x;\nu)$ and R be as in Corollary 139.4. For the Haar dx of Theorem 131.1, put*

$$S_{k,a} = \pi^{(2a+1)k} V_k / V_r, \qquad L_{k,a}(R,X) = 2^n \pi^{2a^2+a+1} (16 r^{3/2})^a R^{2n^7+9a} X^{2a}.$$

For $X \geq 1$ the actual double correlation satisfies

$$\frac{1}{d} \int_{\|x\| \leq X} |B(x;\nu)|\, dx \leq S_{k,a}\, L_{k,a}(R,X) |\nu|^{-K_0} \min(1, |\nu|^{-6}). \tag{139.6}$$

On the shorter interval one may take $R = b^{6n+2}$ and $X = b^{100n^4}$. Then $L_{k,a}(R,X) \leq b^{2000n^9}$ for all sufficiently large nodes.

Proof. Use the recursive measures first. The angular bound (139.4) and Proposition 139.6 apply with $j = 0$ and $A = B = F$, keeping the two occurrences of F in the Hermitian pairing. The sum of the powers of π is

$$r + \sum_{i=1}^{a} (2r_i - 1) = (2a+1)k + 2a^2 + a + 1.$$

The power of R is $2(n^7 + 2a) + 5a = 2n^7 + 9a$.

We compute the change of group Haar. For the matrix Gram density $d\mu_j = \delta(g^* J_j g - J_j)\, dg_{\mathrm{Leb}}$ use real diagonal and complex upper coordinates for Hermitian matrices. Over the first column e, its pairing with each middle column sets that column's bottom entry to zero. Its pairing with the dual column sets the latter's bottom entry to one. The zero norm of that dual column determines the real part of its top entry with Jacobian two, leaving the variables t, u above. Each pairing of a middle column with the dual then fixes the top complex entry of that middle column with Jacobian one. The middle Gram equations have density $d\mu_{j+1}$. These fiber computations hold over every first column by translation invariance. Consequently at each null column the product disintegration of $d\mu_j$ acquires a factor $1/2$ relative to $\sigma_j\, dt\, du\, d\mu_{j+1}$. At the last column, of norm -1, the orthogonal complement is compact positive of dimension k. Its fiber has mass $V_k/2^k$, as computed in Theorem 131.1. Haar is unimodular, so the first-column disintegration and the inverse-column disintegration give the same constants. That theorem normalizes $dx = 2^r d\mu_0 / V_r$. Thus $dx = 2^{r-a-k}(V_k/V_r) dg_0^* = 2^n (V_k/V_r) dg_0^*$.

Any conversion factor between the two measures on $U\backslash G$ appears twice in the double correlation and twice in the square of $\|F\|_2^2$. It therefore cancels in the homogeneous inequality. In the original quotient Haar we have $\|F\|_2^2 \leq d$ by Theorem 132.14. Together with the $1/d$ from (139.5), this proves (139.6). For the last assertion, $r \ll x^6$ and $\log b \geq (\log x)^6$ on the shorter interval, so $16 r^{3/2} \leq b$ at large nodes. Also $6n + 2 \leq 8n$, $a < n$, and the logarithm to base b of $2^n \pi^{2a^2+a+1}$ is at most $10n^2$ there. Inserting these inequalities in the definition of $L_{k,a}$ gives the stated exponent. □

Inequality (139.6) uses a bounded position domain. Its factor X^{2a} records the lengths of the central and Fourier variables in the recursion. A bound for the complementary positions is required for the full archimedean integral.

139.4 Common dual vectors and large real positions

In the notation of Proposition 139.6, the interval and finite places are already those of Corollary 139.2. For example, for each real place insert

$$\frac{1}{d_v S_{k,a}} \int_{\|x\| \le b^{100 n^4}} |B_v(x; c_v \Phi_v y)| \, dx$$

in place of the corresponding factor of $A_\infty(y)$ in (139.2). The resulting nonnegative sum is at most $\exp(O(\epsilon)) b^{-(7-o(1))kf}$. Indeed Theorem 139.7 bounds the product of the new factors by $b^{2000 n^9 f} A_\infty(y)$. At a node of size u on the shorter interval we have $n \le \sqrt{u}$ and $k \asymp u^5$, whence $n^9 = o(k)$. The real integral over larger positions requires an estimate for the mass that can pass to the next vector of the flag.

At a null step $j < a$, and with h linear in its second variable, define the positive number

$$\Delta_x(v) = \max_{\|w\| \le 1,\ \|x^{-1} w\| \le 1} |h(v, w)|, \qquad 0 \ne v \in \mathcal{C}_0, \quad x \in G_j.$$

The maximum exists by compactness. Scaling down an arbitrary vector on which the pairing is nonzero shows positivity. Equivalently one maximizes $|h(v,w)| / \max(1, \|x^{-1} w\|)$ on the Euclidean unit sphere. This last expression gives continuity in v, x. Throughout this subsection suppose $k \ge 6$ and put $z_0 = k - 5$, $d_0 = k - 1$.

Lemma 139.8 (A negative moment of the common dual pairing)**.** *For $p = k + a - j$, $q = a + 1 - j$, $r_j = p + q$, $R \ge 1$, uniformly in x,*

$$\begin{aligned} \int_{\|v\| \le R} \Delta_x(v)^{-2 z_0} \, d\sigma_j(v) &\le \frac{\pi^{r_j} \Gamma(d_0 - z_0) R^{2(r_j - 1 - z_0)}}{(r_j - 1 - z_0) \Gamma(q) \Gamma(d_0) \Gamma(p - z_0)} \\ &\le 6 \pi^{r_j} R^{4n + 10}. \end{aligned} \tag{139.7}$$

Proof. In positive and negative coordinates there is a complex d_0-plane P_x in the positive space which x^{-1} takes to the positive space. In fact the negative projection of x^{-1} on the positive space has rank at most q, and $p - q = d_0$. On P_x the isometry x^{-1} preserves Euclidean length because the Hermitian and Euclidean norms agree on both positive spaces. Testing on its unit ball gives $\Delta_x(v) \ge \| \operatorname{pr}_{P_x} v \|$.

Write $v = (s\omega, s\eta)$ in the polar coordinates of Theorem 139.3. The squared norm of the projection of a uniform $\omega \in S^{2p-1}$ onto P_x has beta density with parameters $d_0, p - d_0$. To see this, choose p independent complex Gaussians with density $\pi^{-1} e^{-|z|^2}$ and divide by their vector norm. The squared norms of the first d_0 and the other $p - d_0$ coordinates are independent gamma variables, since each squared modulus has exponential density. Changing variables to their sum and their ratio to the sum gives the beta density. Its moment of order $-z_0$ equals $\Gamma(p)\Gamma(d_0 - z_0) / (\Gamma(d_0)\Gamma(p - z_0))$ by the beta integral. The radial integral up to $s = R$ bounds the indicated ball and contributes $R^{2(r_j - 1 - z_0)} / (2(r_j - 1 - z_0))$. The radial density constant from Theorem 139.3 gives the first inequality. The numerator gamma factor is 6, all denominator gamma arguments are positive integers, and $r_j - 1 - z_0 = 2(a - j) + 5$. This proves the second. □

Proposition 139.9 (Size of the residual position and a weighted fiber)**.** *Let $v' = x^{-1} v$ with $R^{-1} \le \|v\|, \|v'\| \le R$. Use the same sections as in Proposition 139.6 and write*

$$s_v x s_{v'}^{-1} = n(t, u) l, \qquad l \in G_{j+1}.$$

Then

$$\|x\| \le 12 R^4 \, \Delta_x(v)^{-2} \|l\|. \tag{139.8}$$

For fixed v, v', l, put $r' = r_{j+1}$. For $Y \ge 1$ and $T = R^2 Y$,

$$\int_{\|x\| \le Y} \Delta_x(v)^{2z_0}\, dt\, du \;\le\; 2(2R)^{2z_0} \pi^{r'} T^{\,1+2(r'-z_0)}. \tag{139.9}$$

Here $x = s_v^{-1} n(t,u) l s_{v'}$ varies with the fiber coordinates, and $1 + 2(r' - z_0) \le 4n + 5$.

Proof. Choose a common vector w attaining $\Delta = \Delta_x(v)$. The vectors $w_1 = s_{v'} x^{-1} w$ and $w_2 = s_v w$ have norm at most R, and $n(t,u)l$ takes w_1 to w_2. Their bottom coefficients in the order e, V, e^* coincide with $\gamma = h(v, w)$, so $|\gamma| = \Delta \le R$. The middle row gives

$$(w_2)_V = l(w_1)_V + \gamma t.$$

It follows that $\|t\|$ and $\|l^{-1}t\|$ are both at most $2R\|l\|/\Delta$, using $\|l^{-1}\| = \|l\|$. The middle-to-top row of $n(t,u)l$ has norm $\|l^{-1}t\|$, since it is the functional $-h(t, l\,\cdot)$. Using the top row on w_1 bounds the remaining top right entry by $2R/\Delta + 2R^2\|l\|/\Delta^2$. The three diagonal blocks have norms $1, \|l\|, 1$. Adding the bounds on all blocks gives a norm at most $12R^2\|l\|/\Delta^2$. Each outer section has norm and inverse norm at most R, proving (139.8).

For the integral, choose a positive d_0-plane P_l for l^{-1} in the residual space as in the lemma. For every b in that space, $\|\operatorname{pr}_{P_l} lb\| \le \|b\|$: pair with a unit vector of P_l, move l^{-1} to that vector, and use Cauchy–Schwarz. Projecting the middle row therefore gives $\Delta \|\operatorname{pr}_{P_l} t\| \le 2R$. The conditions $\|t\|, |u| \le T$ follow from the norm of $s_v x s_{v'}^{-1}$. Use the weight bound $\Delta^{2z_0} \le (2R)^{2z_0} \|\operatorname{pr}_{P_l} t\|^{-2z_0}$ almost everywhere. Integrating over the product of two balls of radius T in P_l and its Euclidean complement gives

$$\int_{\|t\| \le T} \|\operatorname{pr}_{P_l} t\|^{-2z_0} dt \le \frac{\pi^{r'} T^{2(r'-z_0)}}{(d_0 - z_0)\Gamma(d_0)\Gamma(r' - d_0 + 1)}.$$

All factors in the denominator are at least one. The u-interval has length $2T$. Since $r' - z_0 = 2(a - j) + 4$ and $a = n - 1$, this proves (139.9). □

139.5 The tail of the real position integral

Write $M_j(x;A,B) = (\int |D_{j,x}^{A,B}|\,d\sigma_j)^2$ for the functions and recursive measures of Proposition 139.6. Absolute values here are taken after the integration over the stabilizer of the current frame vector. In particular $|B(x;\nu)| \le M_0(x;F,F)$ when both are computed with the recursive quotient measure.

Theorem 139.10 (Large positions in the first real Gram transform)**.** *Suppose $a \ge 1$, $k \ge 40n^3$, $R \ge \max(16, 2r)$, and that the real cut has radius R. With the original measures of Theorem 139.7, for every $Y \ge R^2$ and every complex frequency ν one has*

$$\frac{1}{d_v S_{k,a}} \int_{\|x\|>Y} |B(x;\nu)|\,dx \le R^{80nk}\, Y^{-k/(2a)}. \tag{139.10}$$

Proof. We first work on $Y < \|x\| \le 2Y$ in the recursive measures. Put $z_0 = k-5$ and define

$$\lambda = (Y/R^2)^{1/a}, \quad Y_j = Y\lambda^{-j}, \quad \delta_*^2 = 12R^4/\lambda, \quad T = 2R^2Y.$$

For $0 \le j \le a$ let $I_j(A,B)$ be the integral of $M_j(x;A,B)$ over $Y_j < \|x\| \le 2Y$ in G_j. At $j=a$, $U_a = 1$, so nonzero integrand in a correlation requires both y, yx to be of norm at most R. Then $\|x\| \le R^2 = Y_a$, giving $I_a = 0$.

For $j < a$ put

$$A_j^0 = 6\pi^{r_j} R^{4n+10}, \qquad H_j^0 = 2(2R)^{2z_0}\pi^{r_{j+1}} T^{4n+5}, \qquad P = 16 r^{3/2} R^5 (2Y)^2.$$

Cauchy–Schwarz with $\Delta_x(v)^{\pm z_0}$ in the integral defining $M_j^{1/2}$, followed by Lemma 139.8, gives

$$M_j(x;A,B) \le A_j^0 \int \Delta_x(v)^{2z_0} |D_{j,x}^{A,B}(v)|^2\, d\sigma_j(v).$$

The integrand is supported where both v and $v' = x^{-1}v$ have norm in $[1/R, R]$. Integrate in x and use the coordinates v', t, u, l of Proposition 139.6. On the part with $\Delta_x(v) > \delta_*$, the inequality (139.8) gives $\|l\| > Y_j/\lambda = Y_{j+1}$. Also $\|l\| \le 2Y$. The Fourier identity in that proposition gives $|D_{j,x}(v)|^2 \le M_{j+1}(l; A_v, B_{v'})$ by the triangle inequality. This upper bound is independent of t, u. By (139.9) the integral of the weight in these two variables is at most H_j^0. The contribution to I_j is therefore at most

$$A_j^0 H_j^0 \iint I_{j+1}(A_v, B_{v'})\, d\sigma_j(v)\, d\sigma_j(v').$$

On the part with $\Delta_x(v) \le \delta_*$, the weight is at most $\delta_*^{2z_0}$. Retain the Fourier transform in t. The integral of its square over $\|t\| \le T$, integrated also in $|u| \le T$, is at most

$$P\,\pi^{2r_{j+1}-1} \int |D_{j+1,l}^{A_v, B_{v'}}(w)|^2 d\sigma_{j+1}(w)$$

by Lemma 139.5. Enlarge the residual position domain to $\|l\| \le 2Y$, apply (139.5), and integrate the slice norms. This bounds the small-pairing contribution to $I_j(A,B)$ by

$$A_j^0 \delta_*^{2z_0} P^{a-j} \pi^{\sum_{i=j+1}^{a}(2r_i-1)} \|A\|_{2,*}^2 \|B\|_{2,*}^2 / d_v.$$

Here A, B and all their slices have the given compact type. In the other contribution we iterate down the flag. At each iteration the integral of $\|A_v\|_{2,*}^2 \|B_{v'}\|_{2,*}^2$ is the product of the two unsliced squared norms. Since the recursion ends with $I_a = 0$, we have proved

$$I_0(A,B) \le \frac{\|A\|_{2,*}^2 \|B\|_{2,*}^2}{d_v}\, \delta_*^{2z_0} \sum_{j=0}^{a-1} \left(\prod_{i<j} A_i^0 H_i^0 \right) A_j^0 P^{a-j} \pi^{\sum_{i=j+1}^{a}(2r_i-1)}. \tag{139.11}$$

The small-pairing variable occurs at just one stage in each summand. Integrating its Fourier transform before taking a bound in the remaining variables is what yields the factor $\delta_*^{2z_0}$.

For clarity we bound the constants in (139.11). The exponent of π in the j-th summand is $r+2\sum_{i=1}^{a} r_i-(a-j) \le (2a+1)k+2n^2$. The power of Y outside $\delta_*^{2z_0}$ is $(4n+5)j+2(a-j) \le 10n^2$. After extracting their displayed powers of π, Y, the three constants A_i^0, H_i^0, P are at most $R^{4n+11}, R^{4k+12n+16}, R^{10}$, respectively. These follow from $R \ge \max(16, 2r)$ and $T = 2R^2Y$. Finally $\delta_*^{2z_0} \le R^{7k}Y^{-z_0/a}$. For $n \ge 2$, $a = n-1$, $k \ge 40n^3$, these bounds, including the number a of summands, the factor 2^n in the group measure and the remaining power π^{2n^2}, are bounded by $R^{79nk}Y^{-k/(2a)}$ after removal of $\pi^{(2a+1)k}$. Indeed the exponent of Y is at most $10n^2 - (k-5)/a \le -k/(2a)$. The exponent of R is at most $a(4k+16n+27)+(4n+11)+10a+7k+2n^2+2n$, which is below $79nk$.

Take $A = B = F$. The quotient-measure conversion cancels against the two squared norms exactly as in Theorem 139.7, and $dx = 2^n(V_k/V_r)dg_0^*$. In the original quotient $\|F\|_2^2 \le d_v$. Thus the bound on the normalized shell is $R^{79nk}Y^{-k/(2a)}$. Apply it to the disjoint shells with lower endpoints 2^mY, $m = 0, 1, \dots$. The geometric sum costs at most two, absorbed in R^{80nk}. This proves (139.10). □

139.6 The first real transform on the whole position group

At a real place v define the normalized mass at a nonzero first Gram frequency by

$$\mathcal{B}_v^{\infty}(\nu) = \frac{1}{d_v S_{k,a}} \int_{G_0} |B_v(x;\nu)|\, dx.$$

Here B_v uses the cut vector, its original quotient Haar and the first frame vector in Corollary 139.4. We use group Haar from Theorem 131.1. In particular $S_{k,a} = \pi^{(2a+1)k} V_k / V_r$ is displayed separately from the contribution of the compact type.

Theorem 139.11 (All positions at a nonzero real Gram frequency)**.** *At all sufficiently large nodes on the shorter interval, for every real place and every $\nu \in \mathbb{C}^{\times}$,*

$$\mathcal{B}_v^{\infty}(\nu) \le b^{2000n^9} |\nu|^{-K_0} \min(1, |\nu|^{-6}), \qquad K_0 = k - n^7. \tag{139.12}$$

Proof. For any $0 \le \ell \le r-2$, the proof of Theorem 139.7, using (139.3) with this ℓ, gives

$$\frac{1}{d_v S_{k,a}} \int_{\|x\| \le X} |B_v(x;\nu)|\, dx \le 2^n \pi^{2a^2+a+1} (16 r^{3/2})^a R^{2(r-1-\ell)+5a} X^{2a} |\nu|^{-\ell}. \tag{139.13}$$

Indeed only the norm of the first angular operator uses ℓ, and $r-1-\ell \ge 1$ in its denominator. Put $R = b^{6n+2}$. On the shorter interval we have $a \ge 1$, $n \le \sqrt{u}$, $k \asymp u^5$ at a node u, and $\log b \ge (\log u)^6$. These imply the size hypotheses of Theorem 139.10 uniformly. Take

$$X = R^{200an} \max(1, |\nu|)^{2a}.$$

The contribution from $\|x\| > X$ is at most $R^{-20nk} \max(1, |\nu|)^{-k}$ by (139.10). It is therefore at most $|\nu|^{-K_0} \min(1, |\nu|^{-6})$, since $n^7 \ge 6$.

If $0 < |\nu| \le 1$, choose $\ell = K_0$ in (139.13). If $|\nu| > 1$, choose $\ell = K_0 + 6 + 4a^2$. In the latter case $4a^2$ pays the power of $|\nu|$ in X^{2a}. Both choices lie in $[0, r-2]$, since $n^7 \ge 6 + 4a^2 + 1$ for $n = a + 1 \ge 2$. The constant left from the ball is at most

$$2^n \pi^{2a^2+a+1} (16 r^{3/2})^a R^{2n^7+9a+400a^2 n}.$$

At large nodes $16 r^{3/2} \le b$, $6n + 2 \le 8n$, and $\log_b(2^n \pi^{2a^2+a+1}) \le 10n^2$. Thus the logarithm to base b of the last display is at most $10n^2 + a + 16n^8 + 72n^2 + 3200n^4 < 1999 n^9$ for $n \ge 2$. Adding the outer contribution proves (139.12). □

Corollary 139.12 (The rational first-frequency sum with its real integrals)**.** *Use the fields, the marks $\mathcal{W} = \Xi \sqcup \mathcal{V}$, the unmarked finite places $\mathcal{U}$, and the scalars $c_v = 2\pi i / (\Phi_v(\delta_E) b_{1,v})$ of Corollary 139.2. Then*

$$\sum_{\substack{0 \ne y \in E \\ y \text{ integral above } \mathcal{V}}} \left(\prod_{v \mid \infty} \mathcal{B}_v^{\infty}(c_v \Phi_v y) \right) \left(\prod_{w \in \Xi} \frac{B_0 q_w^{n - j_w} \mathcal{B}_{Z_w}(y)}{Z_w^0} \right) \prod_{w \in \mathcal{U}} \mathcal{B}_w^0(y) \;\le\; \exp(O(\epsilon))\, b^{-7kf + 2000 n^9 f}. \tag{139.14}$$

In particular the exponent of b on the right is $-(7 - o(1))kf$ along the shorter interval.

Proof. For each $y \ne 0$ all real frequencies $c_v \Phi_v y$ are nonzero. Multiply (139.12) over the f real places of the totally real base. The product is bounded by $b^{2000 n^9 f} A_{\infty}(y)$, with exactly the archimedean arguments of (139.2). Multiplying by its nonnegative finite factors and summing proves (139.14). The unmarked Euler products and the sum over denominators are the convergent ones justified in that corollary. Since $n^9/k \ll u^{-1/2}$, the last assertion follows uniformly. □

Formula (139.14) controls the first relative entry with no earlier entries prescribed. In the integral theta-period problem one must also treat subsequent entries with the previous relative frame products prescribed, sum their coupled rational contributions, and compare the resulting signed integral period to the coefficient lattice. The strict cohomology estimate in Section 6.6 is a further requirement for the deduction of Dirichlet GRH.

139.7 The real identity mass and the first pivot density

At a real place let $F = F_v$ be the full test and $W = F_v^\flat$ its cut. Write $d = d_v$, $D = d_{\pi,v}$ and $C_Z(x) = \langle Z, R(x)Z\rangle$, where $R(x)$ acts by right translation. At frequency zero the double integral is $B_v(x;0) = |C_W(x)|^2$. The Haar measures are those of Theorem 139.11.

Theorem 139.13 (The zero-frequency mass with both tests cut)**.** *Uniformly at all real places on the shorter interval, for sufficiently large nodes,*

$$(1 - b^{-r^3 f})^4 \le \frac{D}{d^2}\int_G |C_W(x)|^2\,dx \le 1 + b^{-20}. \tag{139.15}$$

Consequently the product of the real masses equals $(d_B^2/D_\pi)(1 + O(fb^{-20}))$, where $d_B = \prod_v d_v$ and $D_\pi = \prod_v d_{\pi,v}$.

Proof. Put $R = b^{6n+2}$ and $Y = R^{400an}$. First $S_{k,a}D/d \le b^{12nr}$ at large nodes. Indeed in the compact-dual formal degree formula of Theorem 94.9 the first $k-1$ consecutive entries of Proposition 132.1 cancel the corresponding Vandermonde factors. At most $2nr$ pairs have an entry among the remaining $2n$. Their numerator gaps are at most b^3, since the entries are $-\kappa - 2jr^8$ for $1 \le j \le 2n$, the other entries have size at most k, and $\kappa \le ub$ at a node of size u. Each remaining denominator is at least one. Thus $D \le b^{6nr}$. Also $V_k/V_r = \prod_{j=k+1}^{r}(j-1)!/(2\pi^j) \le r^{(2a+1)r}$. With $r, \pi \le b$ and $d \ge 1$ this proves the claimed bound. Applying Theorem 139.10 with $\nu = 0$ gives

$$\frac{D}{d^2}\int_{\|x\|>Y} |C_W(x)|^2\,dx \le b^{12nr}R^{-120nk} \le b^{-40}$$

eventually, uniformly in the place, since $r \le 2k$.

On the ball the L^2 error for the test vectors gives the pointwise bound $|C_W - C_F| \le 2d\,b^{-r^3 f}$: expand the difference in its two arguments and apply Cauchy–Schwarz and Theorem 132.14. Lemma 132.13 gives $D\,\mathrm{vol}(\|x\| \le Y) \le b^{38r^2}Y^{9r^2}$ by bounding the integrand of its second inequality from below. As $\log_b Y = 400an(6n+2) = O(n^3)$ and $n^3 = o(r)$,

$$\frac{\sqrt{D}}{d}\|C_W - C_F\|_{L^2(\|x\|\le Y)} \le 2b^{-r^3 f + (38r^2 + 9r^2\log_b Y)/2} \le b^{-40}.$$

By Proposition 131.6 the full coefficient is d times a unit discrete-series coefficient, so $\|C_F\|_2 = d/\sqrt{D}$ by Schur orthogonality. The triangle inequality on the ball followed by addition of the outer squared norm now gives the stated upper bound.

For the lower bound the one-place identity in the proof of Theorem 136.8 gives $\int \overline{C_F}C_W = (d/D)\|P_v W\|^2$, where P_v is the coefficient projection fixing F. It is a contraction, so $\|P_v W\| \ge (1 - b^{-r^3 f})\sqrt{d}$. Applying Cauchy–Schwarz to that identity gives the lower bound in (139.15). The product assertion follows by multiplication, since $f \ll u^{12}$ and $fb^{-20} = o(1)$. □

Proposition 139.14 (Inversion in the first scalar Gram entry)**.** *Use the density $D_x\,d\sigma_0$ for W from Corollary 139.4, with its original quotient normalization. There is a compactly supported C^{K_0+3} function g_v on $\mathbb{C}$, supported on $|c| \le R^2$, such that for every bounded Borel function φ,*

$$\int_{\mathbb{C}} \varphi(c)g_v(c)\,d^2c = \int_G \iint \varphi(h(v_1,v_2))D_x(v_1)\overline{D_x(v_2)}\,d\sigma_0(v_1)\,d\sigma_0(v_2)\,dx. \tag{139.16}$$

Here d^2c is real planar Lebesgue measure. At every c one has the absolutely convergent formula

$$g_v(c) = \pi^{-2}\int_{\mathbb{C}} e^{-2i\,\mathrm{Im}(\nu c)}\left(\int_G B_v(x;\nu)\,dx\right)d^2\nu.$$

Its integral is the mass in (139.15) before normalization by D/d^2.

Proof. On a fixed ball in G, the integral of $(\int |D_x|\, d\sigma_0)^2$ is finite by Cauchy–Schwarz on the finite measure part $\|v_1\| \le R$ of the cone and Proposition 139.6. On the complement the proof of Theorem 139.10 bounds this very same squared integral before insertion of the character. Changing quotient Haar multiplies the bound by a finite constant. The right side of (139.16) therefore defines a finite complex measure μ. It is supported on the claimed disk since $|h(v_1, v_2)| \le \|v_1\|\|v_2\| \le R^2$.

With Fourier convention $e^{2i\,\mathrm{Im}(\nu c)}$, Fubini gives $\widehat{\mu}(\nu) = \int_G B_v(x;\nu)\, dx$. It is bounded on $|\nu| \le 1$ by the total variation of μ. For $|\nu| > 1$, Theorem 139.11 gives decay of order $|\nu|^{-K_0-6}$. The transform multiplied by $(1 + |\nu|)^{K_0+3}$ is integrable in two real dimensions. Its inverse Fourier integral therefore defines a C^{K_0+3} function by differentiation under the integral sign. The inverse factor is π^{-2} because the bilinear form $\mathrm{Im}(\nu c)$ differs from the real dot product by an orthogonal change of coordinates and the exponent has coefficient two. Applying Euclidean Fourier inversion to compactly supported smooth test functions and using Fubini identifies this function's density with μ. It vanishes off the disk by continuity. Finally at $\nu = 0$ the transform equals $\int |C_W|^2$, as asserted. □

139.8 The second Gram entry at a prescribed first pivot

Use $e = f_0$, $f = f_1$, the residual space V of dimension $r_1 = r - 2$, and the cut $W = F_v^\flat$. Here f is the next null frame vector if $a > 1$, and the terminal vector of norm -1 if $a = 1$. For two quotient points with representatives y_1, y_2 write

$$v = y_1^{-1}e, \quad w = y_2^{-1}e, \quad A = y_1^{-1}f, \quad B = y_2^{-1}f, \quad c = h(v, w).$$

For $c \neq 0$ the second successive Gram pivot is

$$P_2 = h(A, B) - \frac{h(A, w)h(v, B)}{c}.$$

This is the determinant of the leading two-by-two relative Gram block divided by its first entry. Adding any multiple of v to A, or of w to B, preserves P_2. Consequently P_2 descends to the two quotient points: the Bessel unipotent fixes e and fixes f modulo e.

For $x \in G$ put $K_x(y) = \langle W(y), W(yx)\rangle$. Keep the sections $s_v^{-1}e = v$ from Proposition 139.6. Let η be defined by $d\dot{y} = \eta\, d\dot{y}_0^*$, where the star denotes the recursive measures there. Given any section value $s \in G$, write $D_{s,x}(z)\, d\sigma_1(z)$ for the image of $\eta K_x(ms)\, d\dot{m}_1^*$ under $m \mapsto m^{-1}f$ on $U_1\backslash G_1$. This convention includes the full conversion of the original quotient Haar in each branch.

Theorem 139.15 (Disintegration with one nonzero real pivot prescribed)**.** *For $c \in \mathbb{C}^\times$, let w_c act by $e \mapsto ce^*$, $e^* \mapsto e/\bar{c}$ and identically on V. For $t \in V$, $u \in \mathbb{R}$ write $n = n(t, u)$ as in Section 139.3 and put $s' = w_c^{-1}n^{-1}s_v$. For fixed x, ν, the image under the first pivot of*

$$e^{2i\,\mathrm{Im}(\nu P_2)} K_x(y_1)\overline{K_x(y_2)}\, d\dot{y}_1\, d\dot{y}_2$$

on $\mathbb{C}^\times$ has the following continuous density relative to d^2c:

$$J_{x,c}(\nu) = \frac{|c|^{2r_1}}{2} \int d\sigma_0(v) \int_{V\times\mathbb{R}} dt\, du \iint e^{2i\,\mathrm{Im}(\nu h(z,z'))} D_{s_v,x}(z)\overline{D_{s',x}(z')}\, d\sigma_1(z)\, d\sigma_1(z'). \tag{139.17}$$

For each $c \neq 0$ the integrals displayed at fixed x converge absolutely. At $\nu = 0$ one has

$$g_v(c) = \int_G J_{x,c}(0)\, dx, \qquad \int_G |J_{x,c}(0)|\, dx < \infty, \tag{139.18}$$

where g_v is the real density of Proposition 139.14.

Proof. Disintegrate both copies of $U\backslash G$ over their first vectors. At fixed v, integrate $s_v w$ in coordinates $\alpha e + t_0 + ce^*$ on its null orbit. The condition is $2\,\mathrm{Re}(\bar{\alpha}c) + h(t_0, t_0) = 0$. For $c \neq 0$ write $\alpha = c(s + iu)$. Lebesgue measure in α is $|c|^2 ds\, du$ and integration of this equation in s divides by $2|c|^2$. Setting $t_0 = ct$ gives exactly

$$s_v w = n(t, u)ce^*, \qquad d\sigma_0(w) = \tfrac{1}{2}|c|^{2r_1}\, d^2c\, dt\, du.$$

The section s' sends this w to e, since w_c is an isometry with $w_c e = ce^*$. Using s' instead of any other section gives the same conditional Haar: the change is a left factor in H_0, whose unipotent part cancels in K_x, and whose residual part acts by right translation of $d\dot{m}_1^*$.

Take the quotient representatives $y_1 = ms_v$, $y_2 = m's'$ and put $z = m^{-1}f$, $z' = (m')^{-1}f$. Upon applying s_v to all four columns they become e, nce^*, z, nz', respectively in the order v, w, A, B. We have $nz' = z' - h(t, z')e$, so $h(e, nz') = 0$ and $h(z, nz') = h(z, z')$. The formula

for P_2 therefore reduces to $h(z, z')$. Further disintegration in z, z' with the prescribed two quotient constants proves (139.17) as a density.

Here are the support and regularity details needed for pointwise use. A nonzero integrand requires $R^{-1} \le \|v\|, \|w\| \le R$. Since $\|s_v\| \le R$, the middle and top coordinates of $s_v w = cn(t, u)e^*$ then give $\|t\|, |u| \le R^2/|c|$. Also $|c| \le R^2$. On compact subsets of $0 < |c| < \infty$ the sections under consideration and their inverses are uniformly bounded. For such a section s, nonzero $W(ms)$ requires $\|u_0 ms\| \le R$ for some $u_0 \in U$. Taking the residual middle block of $u_0 m$ shows that m has residual quotient radius at most $R\|s^{-1}\|$. Thus all the integrals run over compact sets in the orbit, fiber variables and residual quotients, locally uniformly in $c \neq 0$. The integrands in the residual quotient before disintegrating z, z' are continuous in c for each choice of v, t, u, m, m'. Boundedness and compact support there give the claimed absolute convergence and continuity by dominated convergence.

At fixed x and $\nu = 0$ the measure in the first pivot is the one with Fourier transform $B_v(x; \xi)$. Its Fourier inverse is continuous, since (139.4) is integrable at $|\xi| > 1$, and at $|\xi| \le 1$ its modulus is at most $(\int |D_x|\, d\sigma_0)^2$. Here D_x is the first-vector density defined in Corollary 139.4. The fiber $c = 0$ has zero measure for fixed v: off $t_0 = 0$ one may solve the null equation in a nonzero residual real coordinate with nonzero derivative, leaving c free, and the remaining slice at $c = t_0 = 0$ lies on a complex line of zero orbit measure. This proves the Fourier identification across the whole plane. Continuity identifies the inverse with $J_{x,c}(0)$ at every nonzero c. Finally $\int_G \int_{\mathbb{C}} |B_v(x; \xi)|\, d^2\xi\, dx < \infty$: for $|\xi| \le 1$ use the integrable squared bound in the proof of Proposition 139.14, and for $|\xi| > 1$ apply (139.12). Inversion and Fubini give (139.18) with the asserted absolute integrability in x. □

Corollary 139.16 (The second real wave at a fixed position)**.** *For $0 < |c| \le R^2$ set $R_c = 8R^8 \max(1, |c|^{-1})^3$. Define $E_x(c)$ to be $|c|^{2r_1}/2$ times the integral in v, t, u in (139.17) with the inner pairing replaced by $\|D_{s_v,x}\|_{L^2(\sigma_1)} \|D_{s',x}\|_{L^2(\sigma_1)}$. For every x, $E_x(c) < \infty$, and for $\nu \neq 0$ and $0 \le \ell \le r_1 - 2$ one has*

$$|J_{x,c}(\nu)| \le \frac{\pi^{r_1} R_c^{2(r_1-1-\ell)}}{r_1 - 1 - \ell} |\nu|^{-\ell} E_x(c).$$

Proof. Put $H = \max(1, |c|^{-1})$. On the relevant support, $\|n^{\pm 1}\| \le 3 + 2\|t\| + \|t\|^2/2 + |u| \le 8R^4H^2$ by its block matrix. The Weyl element and its inverse have norm at most R^2H. Thus $R \max(\|s_v^{-1}\|, \|(s')^{-1}\|) \le R_c$. The residual radius observation in the theorem bounds both orbit supports by R_c. Pushing forward bounded continuous densities on the indicated compact subsets by the residual orbit submersion gives bounded densities there, uniformly for the bounded sections at fixed c, x. The residual first vector has norm at least R_c^{-1} on the support by the same inverse norm bound. This ensures a compact subset of the nonzero orbit even for the null quadric. These bounds also show directly that $E_x(c) < \infty$. Apply (139.3) on this residual orbit to the inner pairing and integrate the resulting inequality. □

For a nonzero second frequency the conditional energy $E_x(c)$ still has to be controlled as x varies, with sufficient dependence on the prescribed pivot c for the rational sum.

139.9 All positions with one real pivot prescribed

At a fixed nonzero first pivot the stabilizer of the two null vectors acts on their common perpendicular space. Using this stabilizer lets us integrate the position variable in the second wave of (139.17). Write $W_0(\nu) = |\nu|^{-K_0}\min(1,|\nu|^{-6})$.

Lemma 139.17 (A mixed estimate on the residual group)**.** *Let A, B, C, D' on G_1 be bounded functions with quotient radius $L \ge \max(16, 2r)$, with the residual Bessel character and the same irreducible compact type of dimension d. Use recursive measures, and for $l \in G_1$ let*

$$Q(l;\nu) = \iint e^{2i\,\mathrm{Im}(\nu h(z,z'))} D_{1,l}^{A,B}(z)\overline{D_{1,l}^{C,D'}(z')}\, d\sigma_1(z) d\sigma_1(z').$$

At large nodes of the shorter interval,

$$\int_{G_1} |Q(l;\nu)|\, dg_1^*(l) \le \pi^{(2a-1)k} L^{40n^7} W_0(\nu) \frac{\|A\|_{2,*}\|B\|_{2,*}\|C\|_{2,*}\|D'\|_{2,*}}{d} \quad (\nu \ne 0). \tag{139.19}$$

Proof. Set $b' = a-1,\ r' = k+2b'+1$. On a ball of radius X apply (139.3) and then Cauchy–Schwarz in l to the two $L^2(\sigma_1)$ norms. Apply (139.5) to each of the squared integrals. Apart from the product of the four norms divided by d this gives

$$\pi^{(2b'+1)k+2b'^2+b'+1}(16r^{3/2})^{b'} L^{2(r'-1-\ell)+5b'} X^{2b'} |\nu|^{-\ell}.$$

For $b' > 0$, on the complement use $|Q| \le \sqrt{M_1(l;A,B)M_1(l;C,D')}$. Cauchy–Schwarz and the shell estimate (139.11), starting at G_1 with b' null steps, give at most $\pi^{(2b'+1)k} L^{80(b'+1)k} X^{-k/(2b')}$ times the same norm factor. Here the proof of Theorem 139.10 through the dyadic summation uses only radius, equivariance and squared norms. With recursive Haar there is no final group-measure conversion. Take $X = L^{200b'n}\max(1,|\nu|)^{2b'}$. The outer bound is at most the norm factor times $\pi^{(2b'+1)k} W_0(\nu)$. If $b' = 0$, positions of norm greater than L^2 contribute zero since $U_1 = 1$, and the ball bound is independent of X.

Take $\ell = K_0$ for $|\nu| \le 1$ and $\ell = K_0 + 6 + 4b'^2$ for $|\nu| > 1$. Both are at most $r' - 2$, even for $b' = 0$, since $n = a+1 \ge 2$. As in (139.13), the power $4b'^2$ pays the frequency contribution of $X^{2b'}$. The remaining power of L before the numerical constants is at most $2n^7 + 9b' + 400b'^2 n$. Since $\pi \le L,\ 16r^{3/2} \le L^3,\ b' < n$ and $n \ge 2$, this exponent plus $2b'^2 + 4b' + 2$ is at most $40n^7$. The additional unit in the exponent absorbs addition of the outer bound. This proves the lemma. □

Theorem 139.18 (The second frequency at a prescribed nonzero pivot)**.** *With $R = b^{6n+2}$ and $R_c = 8R^8\max(1,|c|^{-1})^3$, the conditional real transform of Theorem 139.15 satisfies, for $0 < |c| \le R^2,\ \nu \ne 0$,*

$$\frac{1}{d_v S_{k,a}} \int_G |J_{x,c}(\nu)|\, dx \le \frac{2^{n+1}\pi^{4a-2}}{(r_1!)^2} R_c^{40n^7} R^{8r_1+4} |c|^{-2r_1-2} W_0(\nu). \tag{139.20}$$

For $|c| > R^2$ the integrand vanishes.

Proof. For an ordered pair $P = (v,w)$ in the fiber $h(v,w) = c$, put

$$d\beta_c(P) = d\sigma_0(v)\, dt\, du, \qquad s_P = n(t,u)^{-1} s_v.$$

The section sends the pair to (e, ce^*). Its stabilizer fixes both e, e^*, hence acts by an arbitrary isometry of V, giving precisely G_1. At a second pair $P' = x^{-1}P$ write $s_P x = l s_{P'}$ with $l \in G_1$. For fixed P the position Haar is

$$dg_0^*(x) = dg_1^*(l)\, d\beta_c(P').$$

Indeed disintegrate $s_P x = n(t_0, u_0) l s_{v'}$ over its inverse first vector v'. The coordinates of P' are then $t' = -l^{-1}t_0$, $u' = -u_0$, since $n(t_0, u_0)^{-1} = n(-t_0, -u_0)$. This change of variables preserves Lebesgue measure for fixed l. It also gives $s_P x = ln(t', u')^{-1} s_{v'} = l s_{P'}$.

Define two functions on G_1 by $A_P(m) = W(m s_P)$ and $C_P(m) = W(m w_c^{-1} s_P)$. Use the same definitions at P'. For both branches the residual position is l, because w_c commutes with G_1. We can use s_P as the section for the first branch in (139.17): it differs from s_v by n^{-1} on the left, and conjugation of this element by m contributes the same unit character to the two vectors in K_x. In particular the first residual pairing remains the one for $h(z, z')$. Both residual functions have radius at most R_c whenever both branches are nonzero. This follows by the middle-block argument for sections in Corollary 139.16, using $\|s_P\| \le 8R^5 \max(1, |c|^{-1})^2$ and the bound there for $\|w_c^{-1} s_P\|$.

Let $H(v) = \|m \mapsto W(m s_v)\|_{2,*}$. This norm is independent of the chosen first-vector section: a change on the left by $n_0 l_0 \in H_0$ contributes a character of modulus one and the right translation $m \mapsto m l_0$ of quotient Haar. Hence $\|A_P\|_{2,*} = H(v)$ and $\|C_P\|_{2,*} = H(w)$. The disintegration of $U\backslash G$ gives $\int H(v)^2 d\sigma_0(v) = \|W\|_{2,*}^2$. Let $\mathcal{A} = \{v : R^{-1} \le \|v\| \le R\}$, containing the support of H. At each $v \in \mathcal{A}$,

$$\int_{w \in \mathcal{A}} dt\, du \le D_c := \frac{2\pi^{r_1}}{r_1!} (R^2/|c|)^{2r_1+1}$$

by the ball and central interval bounds in Theorem 139.15. The same bound applies with the two vectors exchanged. To see the equality of the two pair measures under exchange, the preceding Haar formula identifies β_c with the quotient by the pair stabilizer G_1 using its fixed Haar. Exchanging the pair changes the base value c to $\bar{c}$, and w_c maps $(e, \bar{c}e^*)$ to (ce^*, e). It centralizes G_1, so left Haar invariance and disintegration identify the exchanged measure with $\beta_{\bar{c}}$. Cauchy–Schwarz on $\mathcal{A} \times \mathcal{A}$ in this fiber therefore gives

$$\int H(v) H(w) d\beta_c(P) \le D_c \|W\|_{2,*}^2.$$

Apply the triangle inequality to the pair integral, and make the position change of variables above. In recursive quotient measures the result is bounded, by Tonelli and Lemma 139.17, by

$$\frac{|c|^{2r_1}}{2}\, \pi^{(2a-1)k} R_c^{40n^7} W_0(\nu)\, \frac{1}{d_v} \left(\int H(v) H(w) d\beta_c(P) \right)^2.$$

Here the four residual functions are $A_P, A_{P'}, C_P, C_{P'}$, in that order. For original quotient measures the two factors η give $\eta^2 \|W\|_{2,*}^4 = \|W\|_2^4 \le d_v^2$. Position Haar contributes $2^n V_k / V_r$. Dividing by $d_v S_{k,a}$, inserting D_c^2, and using $r_1 - k = 2a - 1$ gives (139.20). The support assertion follows from $|h(v, w)| \le \|v\| \|w\|$. □

The estimate is finite for each prescribed $c \neq 0$. The factor $|c|^{-2r_1-2}$, with $r_1 = k + 2a - 1$, is a loss in summing over small rational pivots. Controlling that sum requires more than the pointwise bound above.

139.10 Common dual pairings on a prescribed Gram fiber

Let e, V, e^* be the decomposition at the first null vector. Set $d_1 = \dim_{\mathbb{C}} V = r_1 = k + 2a - 1$, $z_0 = k - 5$, $m = 2(d_1 - z_0) + 1 = 4a + 9$, and $\mathcal{A}_R = \{v \in \mathcal{C}_0 : R^{-1} \le \|v\| \le R\}$. Assume $a \ge 1,\ k \ge 10,\ R \ge 1$. For $p \ne 0$ let $d\mu_p(v, w)$ be the measure on $h(v, w) = p$ for which

$$d\sigma_0(v)d\sigma_0(w) = d^2p\, d\mu_p(v, w), \qquad d\mu_p(v, w) = \tfrac{1}{2}|p|^{2d_1}\, d\beta_p(v, w)$$

in the explicit coordinates of Theorem 139.15. The formula on the right specifies the measure at every nonzero p. Write $\Delta_x(v) = \max_{\|b\|, \|x^{-1}b\| \le 1} |h(v, b)|$.

Theorem 139.19 (Crossed weights on the first Gram fiber)**.** *For nonnegative measurable functions A, B on the null cone supported in $\mathcal{A}_R$, and any position x,*

$$\int A(v)B(w)\, d\mu_p(v, w) \le \frac{\pi^{d_1} R^{m+1}}{|p|} \left(\int \Delta_x(v)^{2z_0} A(v)^2\, d\sigma_0(v)\right)^{1/2} \left(\int \Delta_x(w)^{2z_0} B(w)^2\, d\sigma_0(w)\right)^{1/2}. \tag{139.21}$$

If the left integral is restricted to a set of pairs, each function on the right can also be restricted to any measurable set containing the corresponding projection of those pairs.

Proof. Fix $v \in \mathcal{A}_R$. Write $s_v = a_v k_v$ for the section of Proposition 139.6. In $s_v w = \alpha e + y + pe^*$ one has $y = pt$ and $\alpha = p(-h(t, t)/2 + iu)$. Conditional on v, the measure μ_p is $\frac{1}{2}\, dy\, du$. On the support $\|w\| \le R$, we have $\|y\| \le R$: the middle block of a_v is the identity and k_v is Euclidean unitary. Also $|\alpha| \le R^2$, whence $|u| \le R^2/|p|$.

Let P_x be a positive $(k-1)$-plane taken into the positive space by x^{-1}, as constructed in Lemma 139.8. Since k_v preserves the positive and negative spaces, $k_v P_x$ has intersection of dimension at least $k - 2$ with the positive part of V: removal of the hyperbolic pair removes only one positive direction. Choose a $(k-2)$-plane S in that intersection. For every unit vector in $k_v^{-1} S$, both constraints in the maximum defining Δ_x hold. Pairing w with these vectors gives $\Delta_x(w) \ge \|\operatorname{pr}_S y\|$. With $s = k - 2 > z_0$, polar integration yields

$$\int_{\|y\| \le R} \|\operatorname{pr}_S y\|^{-2z_0}\, dy \le \frac{\pi^{d_1} R^{2(d_1 - z_0)}}{(s - z_0)\Gamma(s)\Gamma(d_1 - s + 1)}.$$

Indeed enlarge the ball to the product of the radius R balls in S and $S^{\perp}$, and integrate the S-radius with exponent $2s - 1 - 2z_0$. The denominator is at least one. Including the u-interval proves that the conditional integral of $\Delta_x(w)^{-2z_0}$ over $w \in \mathcal{A}_R$ is at most $\pi^{d_1} R^{m+1}/|p|$, since $m + 1 = 2(d_1 - z_0) + 2$. The exchanged estimate holds at each p, by the symmetry of β_p with $\beta_{\bar{p}}$ in the proof of Theorem 139.18.

In the left side of (139.21) multiply $A(v)$ by $\Delta_x(v)^{z_0}/\Delta_x(w)^{z_0}$ and $B(w)$ by its reciprocal. Cauchy–Schwarz and the two conditional estimates give the result. In the restricted case one may first insert the two projection cutoffs and then enlarge the domain, since the integrands are nonnegative. □

To allow different first entries at the two endpoints, let $d_\lambda = \operatorname{diag}(\lambda, I, \bar{\lambda}^{-1})$ and $M = \max(|\lambda|, |\lambda|^{-1})$, for $\lambda \in \mathbb{C}^{\times}$. If v and $v' = \lambda^{-1} x^{-1} v$ are in $\mathcal{A}_R$, write

$$d_\lambda s_v x s_{v'}^{-1} = n(t, u) l, \qquad l \in G_1.$$

The matrix on the left fixes e, which proves that these coordinates exist. For fixed v, left invariance of Haar and the first-vector disintegration give $dg_0^*(x) = dt\, du\, dg_1^*(l)\, d\sigma_0(v')$.

Proposition 139.20 (The weighted position fiber with two endpoint scales)**.** *In these coordinates* $\|l\| \le \|x\|$ *and*

$$\|x\| \le 12R^4M^5\Delta_x(v)^{-2}\|l\|, \tag{139.22}$$

$$\int_{\|x\|\le Y} \Delta_x(v)^{2z_0}\,dt\,du \le 2\pi^{d_1}|\lambda|^{2z_0}(R^2MY)^m \qquad (Y \ge 1). \tag{139.23}$$

In the second formula v, v', l *are fixed.*

Proof. The middle block of $d_\lambda a_v k_v x k_{v'}^{-1} a_{v'}^{-1}$ is the middle block of the matrix $k_v x k_{v'}^{-1}$. It is l, giving the first bound. The full matrix with $\|x\| \le Y$ has norm at most $T = R^2MY$. Its last column bounds both $\|t\|$ and $|u|$ by T.

Let b attain $\Delta = \Delta_x(v)$, and put $b_1 = s_{v'}x^{-1}b$, $b_2 = d_\lambda s_v b$. The matrix $n(t,u)l$ takes b_1 to b_2. The common bottom entry is $\gamma = \bar\lambda^{-1}h(v,b)$. The two vectors have norm at most RM. More precisely, their middle components both have norm at most one since the diagonal factors act identically there. With P_l the positive $(k-1)$-plane of Lemma 139.8 in the residual space, the middle equation $(b_2)_V = l(b_1)_V + \gamma t$ yields

$$\Delta\,\|\operatorname{pr}_{P_l} t\| \le 2|\lambda|.$$

Here $\|\operatorname{pr}_{P_l} l(b_1)_V\| \le \|(b_1)_V\|$ by pairing with positive unit vectors of P_l. Integrate $(2|\lambda|)^{2z_0}\|\operatorname{pr}_{P_l} t\|^{-2z_0}$ over the ball of radius T, then integrate over the central interval. The same polar computation as above, now with $s = k-1$, gives the right side of (139.23) times

$$\frac{2^{2z_0}}{(s-z_0)\Gamma(s)\Gamma(d_1-s+1)} \le 1.$$

Indeed $(k-2)! \ge 4^{k-5}$ for $k \ge 10$, by checking $k = 10$ and increasing k one at a time.

For the size transfer, the middle equation and its inverse bound $\|t\|, \|l^{-1}t\|$ by $2RM\|l\|/|\gamma|$. The top row then bounds the top right entry by $2RM/|\gamma| + 2(RM)^2\|l\|/|\gamma|^2$. Adding the block bounds as in Proposition 139.9, and using $|\gamma| \le RM$, gives $\|n(t,u)l\| \le 12(RM)^2|\gamma|^{-2}\|l\|$. The two sections and d_λ^{-1} cost at most R^2M on returning to x. Substituting $|\gamma| = \Delta/|\lambda|$ proves (139.22). □

139.11 The second real frequency with two prescribed endpoints

Use d_1, z_0, m from Section 139.10. Let $p, q \in \mathbb{C}^\times$ be the first pivots at the two endpoints, put $\lambda = p/q$, and define

$$H = \max(1, |p|^{-1}, |q|^{-1}), \quad M = \max(|\lambda|, |\lambda|^{-1}), \qquad R = b^{6n+2}.$$

We specify the mixed form, including its measure. At a pair $P = (v, w)$ with $h(v, w) = p$, choose s_P taking it to (e, pe^*), as in Theorem 139.18. In recursive quotient Haar, let $D^{\alpha}_{s,x}\, d\sigma_1$ be the image at $z = m_1^{-1} f_1$ of

$$\langle W(m_1 s), W(m_1 d_\alpha s x)\rangle\, d\dot{m}_1^*, \qquad m_1 \in U_1 \backslash G_1.$$

For $\alpha = 1$ this is the untwisted density. Define $J_{p,q}(x; \nu)$ by integrating

$$\iint e^{2i\,\mathrm{Im}(\nu h(z,z'))} D^1_{s_P,x}(z) \overline{D^{\lambda}_{w_p^{-1} s_P, x}(z')}\, d\sigma_1(z) d\sigma_1(z')$$

against $\eta^2 |\lambda|^{-d_1} d\mu_p(P)$. Thus the two second-branch sections are $w_p^{-1} s_P$ and $d_\lambda w_p^{-1} s_P x = w_q^{-1} s_P x$. The density factor is $\frac{1}{2}|pq|^{d_1} d\beta_p(P)$, the geometric mean of the two endpoint Jacobian factors in the pair coordinates. The remaining freedom in s_P is left multiplication by G_1. It transports both residual orbits by the same isometry, so the integral is independent of that choice. For $p = q = c$ this is precisely $J_{x,c}(\nu)$ in (139.17). Write $D_{\mathrm{type}} = d_v$ for the compact-type dimension.

Theorem 139.21 (The whole second real wave at prescribed first entries)**.** *At sufficiently large nodes of the shorter interval, for $\nu \neq 0$ and $0 < |p|, |q| \le R^2$,*

$$\frac{\int_G |J_{p,q}(x; \nu)|\, dx}{D_{\mathrm{type}} S_{k,a}} \le b^{2000 n^9} \max(1, |p|^{-1})^{n^6} \max(1, |q|^{-1})^{n^6} |\nu|^{-K_0} \min(1, |\nu|^{-6}). \tag{139.24}$$

The integral vanishes if either pivot has modulus greater than R^2. In particular the small-pivot loss $|c|^{-2r_1-2}$ of (139.20) can be removed in the equal-pivot case.

Proof. We work in recursive measures first. Put $W_v(m_1) = W(m_1 s_v)$. For $\alpha \in \{1, \lambda\}$ and $j = 1, 2$ let $A_{\alpha,j}(v, x) = \|D^{\alpha}_{s_v,x}\|_{L^j(\sigma_1)}$. These norms are independent of the first-vector section. Indeed replacement by $n(t, u) l_0 s_v$ changes the first amplitude by a character and a right translation by l_0. In the second amplitude the unipotent is $d_\alpha n(t, u) d_\alpha^{-1} = n(\bar{\alpha} t, |\alpha|^2 u)$. After moving the two unipotents to the left of m_1, the ratio of the characters is a unit-modulus function of $m_1^{-1} f_1$. The right translation transports this vector by l_0, which preserves σ_1. This proves the norm assertion.

At fixed v put $v' = \alpha^{-1} x^{-1} v$ and use $d_\alpha s_v x = n(t, u) l s_{v'}$. Moving $n(t, u)$ past m_1 likewise gives

$$A_{\alpha,j}(v, x) = \|D^{W_v, W_{v'}}_{1,l}\|_{L^j(\sigma_1)}. \tag{139.25}$$

The slices have radius R, and their squared L^2 norms integrate to $\|W\|^2_{2,*}$ in the first-vector variables. Define $F_{\alpha,j}(Y)$ to be the integral over $\|x\| \le Y$ and $v \in \mathcal{A}_R$ of $\Delta_x(v)^{2z_0} A_{\alpha,j}(v, x)^2$ in $dg_0^*(x) d\sigma_0(v)$. Only $v' \in \mathcal{A}_R$ contribute. Set $N_\alpha = \max(|\alpha|, |\alpha|^{-1})$. Integrating (139.25) in t, u using (139.23), then using $\|l\| \le Y$ in Proposition 139.6, gives

$$F_{\alpha,2}(Y) \le 2\pi^{d_1} |\alpha|^{2z_0} (R^2 N_\alpha Y)^m (16 r^{3/2} R^5 Y^2)^{a-1} \pi^{\sum_{i=2}^{a}(2r_i - 1)} \frac{\|W\|^4_{2,*}}{D_{\mathrm{type}}}. \tag{139.26}$$

All residual norms on the right of (139.25) are independent of these t, u.

The four first vectors on the support are $v, w, v' = x^{-1}v, w' = \lambda^{-1}x^{-1}w$, all in $\mathcal{A}_R$. Since $h(v', w') = q$, this proves the support assertion and yields $M \le R^2 H$. At fixed x the absolute value of the inner wave in the definition of the form is bounded by $A_{1,2}(v, x)A_{\lambda,2}(w, x)$ times

$$\pi^{d_1} \min\{R^{2(d_1-1)},\ (8R^8H^3)^{2(d_1-1-z_0)}|\nu|^{-z_0}\}.$$

For the oscillatory bound use (139.3) with $\ell = z_0$ in the common pair section. The radius $8R^8H^3$ follows from the first amplitude of each branch as in Corollary 139.16. The second amplitudes may only reduce these supports. For the other bound take the product of the two L^1 norms. Each L^1/L^2 ratio is at most the square root of the orbit volume of radius R in its own canonical section, by (139.25). The polar formula in Theorem 139.3 bounds this volume by $\pi^{d_1} R^{2(d_1-1)}$.

Now apply (139.21) and Cauchy–Schwarz in x. The two resulting square integrals are $F_{1,2}(Y), F_{\lambda,2}(Y)$. Their geometric mean supplies $|\lambda|^{z_0} M^{m/2}$, while the form carries $|\lambda|^{-d_1}$. The remaining ratio is at most $M^{d_1-z_0+m/2} \le M^m$. Passing to original Haar multiplies by $2^n V_k/V_r$ in x and by η^2 in the two quotient measures. We also have $\eta^2\|W\|_{2,*}^4 \le D_{\mathrm{type}}^2$. Put $s = m + 2(a-1) = 6a + 7$. Denote the normalized mass on a set $B \subset G$ by $I_B = (D_{\mathrm{type}} S_{k,a})^{-1} \int_B |J_{p,q}(x; \nu)|dx$. We have proved

$$I_{\|x\|\le Y} \le (RH)^{100n^2} Y^s \min(R^{2z_0}, |\nu|^{-z_0}). \tag{139.27}$$

Here and below the minimum at $\nu = 0$ is R^{2z_0}. To check the constant, the uncanceled power of π is $3d_1 + \sum_{i=2}^{a}(2r_i - 1) - (2a+1)k = 2a^2 + a$. The other factors outside Y^s and the minimum are at most

$$2^{n+1}(16r^{3/2})^{a-1}\,|p|^{-1} R^{3m+1+5(a-1)} M^m (8R^8H^3)^{2(d_1-1-z_0)}.$$

Since $d_1 - 1 - z_0 = 2a + 3$, $R \ge \max(16, 2r)$ and $M \le R^2 H$, all these factors, including π^{2a^2+a}, fit in $(RH)^{100n^2}$ at large nodes.

For the distant positions use the product of the two L^1 norms before the pair integration. The support implies $\Delta_x(v) \ge \min(|p|, |q|)/R$, by testing on $w/(R\max(1, |\lambda|))$. Likewise $\Delta_x(w) \ge |p|/R$ by testing on v/R. Thus in each use of (139.25) the residual position satisfies $\|x\| \le D_*\|l\|$, with $D_* = R^{17}H^7$, by (139.22). The projection cutoffs in Theorem 139.19 retain these lower bounds on Δ_x in both square integrals. On $Y < \|x\| \le 2Y$ integrate the weighted positions as before, with $2Y$ as the upper bound. This time the residual squared L^1 norm is $M_1(l; W_v, W_{v'})$. For $a \ge 2,\ Y \ge D_* R^2$, its integral over $\|l\| > Y/D_*$ is at most

$$\pi^{(2a-1)k} R^{80ak} (Y/D_*)^{-k/(2(a-1))} \|W_v\|_{2,*}^2 \|W_{v'}\|_{2,*}^2 / D_{\mathrm{type}}.$$

This is the recursive-measure shell bound proved in (139.11), with $a-1$ null rows and its dyadic summation. For $a = 1$, such residual positions give zero. Taking the square root of the product for the two branches cancels the large power of $|\lambda|$ as above, and the remaining power of π is $4a - 2$. It follows that

$$I_{Y<\|x\|\le 2Y} \le (RH)^{100n^2} R^{80ak} Y^m (Y/D_*)^{-k/(2(a-1))} \qquad (a \ge 2). \tag{139.28}$$

Indeed the factors from the conditional fiber and the two position fibers, outside Y^m and the residual tail, are bounded by $2^{n+1+m}\pi^{4a-2}|p|^{-1}R^{3m+1}M^m$.

Choose $X = R^{400n^2}H^{16}\max(1, |\nu|)^{4a}$. Since $k \ge 40n^3$ eventually, $k/(2(a-1)) - m \ge k/(4a)$ for $a \ge 2$. Summing (139.28) over $Y = 2^i X$ costs at most

$$2(RH)^{100n^2} D_*^m R^{80ak} (X/D_*)^{-k/(4a)} \le 2(RH)^{200n^2} \max(1, |\nu|)^{-k}.$$

Here $(400n^2-17)/(4a) > 80a$. In the ball, for $|\nu| \le 1$ use $\min(R^{2z_0}, |\nu|^{-z_0}) \le R^{2(z_0-K_0)}|\nu|^{-K_0}$, by splitting at $|\nu| = R^{-2}$. For $|\nu| > 1$, the $4as$ frequency power from X^s is paid by $z_0 - K_0 - 6 = n^7 - 11 \ge 4as$ at large nodes. The total coefficient is bounded by $R^{2n^7+O(n^3)}H^{O(n^2)}$ with absolute constants, by (139.27) and the tail just proved. Since $R = b^{6n+2}$ and $n \to \infty$ along the shorter interval, the first factor is at most b^{2000n^9} and the second is at most $\max(1,|p|^{-1})^{n^6}\max(1,|q|^{-1})^{n^6}$. This gives (139.24). □

139.12 Reciprocal differences at the first real pivot

For the comparison of two rational heads there is also a frequency when the weight on the next Gram entry is constant. Let $p, q \neq 0$, $p \neq q$, and set $\xi = p^{-1} - q^{-1}$. Keep the form $J_{p,q}$, its Haar measures and H, M, R, d_1, m, z_0 of the preceding subsection.

Theorem 139.22 (A reciprocal-difference estimate on the whole real group)**.** *On the same shorter interval as Theorem 139.21,*

$$\frac{1}{d_v S_{k,a}} \int_G |J_{p,q}(x;0)|\, dx \leq b^{2000n^9} \max(1,|p|^{-1})^{n^6} \max(1,|q|^{-1})^{n^6} |\xi|^{-K_0} \min(1,|\xi|^{-6}). \quad (139.29)$$

Proof. By support we may suppose $|p|, |q| \leq R^2$. Write $\lambda = p/q$. At zero next frequency the residual pairing factors into the product of the two ordinary conditional correlations. The first is $D^{W,W}_{0,x}(v)$, independent of the pair section, since its two arguments use the same simple-root character. Denote it by $C_x(v)$ in recursive measures. For the other correlation fix w and parametrize the first vector and pair section by

$$v = s_w^{-1} n(-t,-u)\bar{p}e^*, \qquad w_p^{-1} s_P = n(t,u) s_w.$$

These formulas satisfy $s_P v = e$, $s_P w = pe^*$, since $w_p(\bar{p}e^*) = e$. The pair measure is $\frac{1}{2}|p|^{2d_1} d\sigma_0(w)\, dt\, du$ by Theorem 139.15 and exchange of the two vectors. Moving $n(t,u)$ to the left of the residual variable m_1 gives the character at $-h(t, m_1^{-1} f_1)$ in the first amplitude. In the second it gives the character at $-\lambda h(t, m_1^{-1} f_1)$, by conjugation by d_λ. As the inner product is conjugate linear in its first argument, the remaining factor is, up to the choice of sign for the Bessel character,

$$T_w(t) = \int e^{2i \operatorname{Im}((1-\lambda)h(t,z))} D^{\lambda}_{s_w,x}(z)\, d\sigma_1(z).$$

The central variable does not occur in this transform.

At fixed x take absolute values and apply Cauchy–Schwarz in $d\mu_p$ to $|C_x(v)|\Delta_x(w)^{-z_0}$ and $\Delta_x(w)^{z_0}|T_w(t)|$, keeping the annular supports of both first vectors. The first squared integral is at most

$$\pi^{d_1} R^{m+1} |p|^{-1} \int |C_x(v)|^2 d\sigma_0(v)$$

by the conditional negative moment in Theorem 139.19. For the second one we have $\|t\|, |u| \leq T = R^2/|p|$. The support of $D^{\lambda}_{s_w,x}$ in its canonical coordinates satisfies $R^{-1} \leq \|z\| \leq R$ by the first amplitude $W(m_1 s_w)$. Apply Lemma 139.5 with the frequency variable scaled by $\overline{1-\lambda}$, and integrate the u-interval. Conditional on w, this gives

$$\tfrac{1}{2}|p|^{2d_1} \int_{|u|,\|t\| \leq T} |T_w(t)|^2 dt\, du \leq 2(2d_1)^{3/2} \pi^{2d_1-1} R^5 |p|^{-1} |\xi|^{1-2d_1} A_{\lambda,2}(w,x)^2. \quad (139.30)$$

In fact complex scalar multiplication has real Jacobian $|1-\lambda|^{2d_1}$, the Fourier ball has radius $|1-\lambda|T$, and $|1-\lambda| = |p|\,|\xi|$.

Integrate the geometric mean of these two bounds over $\|x\| \leq Y$ and use Cauchy–Schwarz in the position. The two remaining squared integrals are respectively $E_0(Y; W, W)$ from (139.5) and $F_{\lambda,2}(Y)$ from (139.26). In the second integral the factor $\Delta_x(w)^{2z_0}$ is the one supplied by the preceding application of Cauchy–Schwarz. The large scale in the latter bound becomes $|\lambda|^{z_0}$ on taking the square root. Including the pair normalization leaves $|\lambda|^{z_0 - d_1} M^{m/2} \leq M^m$. The total power of Y is $(2a + m + 2(a-1))/2 \leq s = 6a + 7$. The uncanceled power of π after the Haar conversion is $2a^2 + a - 1$, by adding the two Plancherel exponents in the square roots. All remaining constant factors other than $|\xi|^{1/2-d_1}$ are bounded by $(RH)^{100n^2}$ at large nodes:

the powers of R outside M^m are at most $3m+5a+10$, the polynomial factors in r are at most $(32r^2)^{a+1}$, and $|p|^{-1} \le H,\ M \le R^2H$. With $I_B = (d_v S_{k,a})^{-1} \int_B |J_{p,q}(x;0)|dx$ we have

$$I_{\|x\|\le Y} \le (RH)^{100n^2} Y^s \min\bigl(R^{2z_0},\, |\xi|^{1/2-d_1}\bigr).$$

Here the bound by R^{2z_0} is (139.27) at zero next frequency.

For the complementary positions (139.28) holds with the constant weight. Take $X = R^{400n^2} H^{16} \max(1,|\xi|)^{4a}$ and sum its dyadic shells as in Theorem 139.21. The tail is at most $2(RH)^{200n^2} \max(1,|\xi|)^{-k}$. For $0 < |\xi| \le R^{-2}$ use the radius term in the minimum. For $R^{-2} \le |\xi| \le 1$ the excess exponent $d_1 - \frac{1}{2} - K_0$ costs at most $R^{2(d_1-1/2-K_0)}$. For $|\xi| > 1$ that excess is greater than $4as+6$ eventually, which pays the growth from X^s while retaining the sixth power of decay. The resulting coefficient is bounded by $R^{2n^7+O(n^3)} H^{O(n^2)}$. Conversion to the powers of b and of the two inverse first entries is the same as in (139.24). $\square$

139.13 Ordered null flags with prescribed Gram entries

Let $1 \le j \le a$. Write $e_i = f_i$ for $0 \le i < j$, and e_i^* for its dual hyperbolic partner. In this subsection J denotes the matrix of the Hermitian form in the fixed Euclidean coordinates, so $J^* = J = J^{-1}$. The space X_j consists of ordered, linearly independent, mutually orthogonal null vectors $A_0, \dots, A_{j-1}$, modulo addition to each column of a linear combination of earlier columns. We retain the scale of every successive vector. Put $N_{<j} = N_0 \rtimes (N_1 \rtimes \cdots \rtimes N_{j-1})$, embedded in the successive middle blocks. The stabilizer of the standard class is $N_{<j} \rtimes G_j$. Indeed an isometry preserving the class fixes e_0. Removing its N_0 component reduces the assertion to the same assertion on $e_0^{\perp}/\mathbb{C}e_0$. This also proves transitivity, by applying the null-vector transitivity at each step.

If $b_i \in \mathcal{C}_i = G_i f_i$, choose the sections $s_{b_i} = a_{b_i} k_{b_i}$ of Proposition 139.6. Set $S_0 = 1$, $S_{i+1} = s_{b_i} S_i$. A point of X_j has unique coordinates $b_0, \dots, b_{j-1}$ in these sections: b_i is the middle projection of $S_i A_i$. The earlier components of $S_i A_i$ along $e_0, \dots, e_{i-1}$ are precisely the freedom in the representative. Its dual components vanish by orthogonality. The invariant flag measure $d\Sigma_j$ is $\prod_{i<j} d\sigma_i(b_i)$, by repeated use of the quotient measures of Section 139.3. At a nonzero slice of the cut W, all these coordinates have norm between R^{-1} and R, and $W_{S_j}(m) = W(mS_j)$ has residual quotient radius R. This follows at each step by taking the middle block as in Proposition 139.6. Also the flag then has a simultaneous representative of column-operator norm at most R: take the inverse of a representative of the support of W of norm at most R and apply it to the first j standard vectors. Representatives differing by a left Bessel unipotent give the same flag class. We use the closed set of flags admitting such a representative when estimating supports.

For a pair of flags with column matrices A, B, put $K = A^* J B$ and let δ_i be its leading determinant of size $i+1$, with $\delta_{-1} = 1$. All these determinants are invariant under the allowed column changes. On the open set where none vanish let $p_i = \delta_i/\delta_{i-1}$. Write $\mathbf{p} = (p_0, \dots, p_{j-1})$. By successive elimination there are unique representatives $\widetilde{A}_i, \widetilde{B}_i$ satisfying

$$h(\widetilde{A}_i, \widetilde{B}_l) = \mathbf{1}_{i=l} p_i.$$

In fact the earlier i-by-i block is invertible, so we can subtract a unique combination of the earlier A's from A_i to make it orthogonal to all earlier B's. The same operation with A, B exchanged fixes B_i. The new diagonal pairing is the quotient of consecutive determinants, because these triangular operations have determinant one.

Theorem 139.23 (Coarea for two ordered null flags)**.** *In the open set just described,*

$$d\Sigma_j(A) d\Sigma_j(B) = \left(\prod_{i<j} d^2 p_i \right) d\mu_{\mathbf{p}}(A, B), \qquad d\mu_{\mathbf{p}} = 2^{-j} \left(\prod_{i<j} |p_i|^{2r_{i+1}} \right) d\beta_{\mathbf{p}}.$$

Here $d\beta_{\mathbf{p}}$ is the Haar quotient by G_j in the pair orbit, with the group and residual group in recursive measures. The formula specifies the conditional measure at every $\mathbf{p} \in (\mathbb{C}^\times)^j$. Exchange of the two flags takes this measure to $d\mu_{\overline{\mathbf{p}}}$.

There is also a formula keeping the first flag fixed and exposing the second in its own canonical coordinates. Given its first i coordinates, let u_i be the residual middle component of $S_i \widetilde{A}_i$, where S_i belongs to the second flag. The conditional measure for b_i is the coarea of $d\sigma_i(b_i)$ at $h(u_i, b_i) = p_i$, relative to planar Lebesgue measure. These successive conditional measures, together with $d\Sigma_j(A)$, give $d\mu_{\mathbf{p}}$.

Proof. First fix $A_0 = v$. At $p_0 \neq 0$ write $B_0 = s_v^{-1} n(t, u) p_0 e_0^*$. The one-vector calculation of Theorem 139.15 gives $\frac{1}{2}|p_0|^{2r_1} d^2 p_0 \, dt \, du$. With $\tau = n(t, u)^{-1} s_v$, the pair is now $e_0, p_0 e_0^*$.

Changing either first-vector section to τ or to $w_{p_0}^{-1}\tau$, respectively, preserves its conditional residual flag measure. Indeed a change by N_0 acts trivially on the quotient $e_0^{\perp}/\mathbb{C}e_0$, and the remaining change is a G_1-isometry preserving that measure. Represent each remaining column in the common middle space by subtracting a multiple of its own first column to make it orthogonal to the other first column. The resulting cross Gram matrix is the Schur complement of p_0, since a diagonal first row and column have been removed. Its successive pivots are therefore $p_1, \dots, p_{j-1}$. In this chart the two remaining flags have the product of their invariant measures: the two sections have the same middle identification because w_{p_0} acts identically there. Induction gives the asserted Jacobian.

After both columns of each pair have been fixed, the remaining stabilizer is G_j, acting on their common orthogonal complement. The other coordinates in the above calculation, at each step $d\sigma_i(v)\,dt\,du$, are exactly the quotient of dg_i^* by dg_{i+1}^*. This identifies the stated $\beta_{\mathbf{p}}$ and proves the pointwise normalization. For exchange, the standard pair for $\mathbf{p}$ is carried after swapping to the standard pair for $\overline{\mathbf{p}}$ by the direct sum of the maps $w_{p_i}^{-1}$; it commutes with the residual group. Left invariance of group Haar thus preserves the quotient measure, and the Jacobian factors are unchanged.

For the other description, $\widetilde{A}_i$ depends only on A and the earlier B's. It is orthogonal to all earlier B's. Thus $S_i\widetilde{A}_i$ has no earlier dual components. Its middle projection u_i is null, and its pairing with S_iB_i is $h(u_i, b_i)$. Elimination proved above identifies this with p_i. If $u_i = 0$ there is no extension with the required nonzero value. Otherwise take a section carrying u_i to f_i and use the same one-vector coarea in this middle space. Iteration gives the conditional formula. To see pointwise agreement of the two descriptions, one may choose smooth local sections in these calculations. At nonzero pivot the coordinates of the other vector are its complex pivot, unrestricted middle coordinates and the imaginary central coordinate. Thus every successive pivot map is a submersion on the indicated open set. Both calculations are changes of variables for the same smooth density with the same ordered base differentials. The resulting fiber density agrees in each local chart at every point. □

Proposition 139.24 (A bounded common section for a prescribed pair)**.** *Suppose both flags admit simultaneous representatives of column norm at most R, with $R \geq \max(16, 2r)$. Put*

$$H_{\mathbf{p}} = \max(1, \max_{i<j} |\delta_i|^{-1}), \qquad L_{\mathbf{p}} = R^{10n} H_{\mathbf{p}}^3.$$

There is an isometry τ_P taking the normalized columns $\widetilde{A}_i, \widetilde{B}_i$ to $e_i, p_i e_i^*$, respectively. Write $w_{\mathbf{p}}$ for the simultaneous swaps w_{p_i}, equal to the identity on the middle space. Then

$$R \max(\|\tau_P\|, \|w_{\mathbf{p}}^{-1}\tau_P\|) \leq L_{\mathbf{p}}.$$

The same bound applies to the norms of the inverse maps. Also $|\delta_i| \leq R^{2(i+1)}$ for all $i < j$.

Proof. For the bounded representatives, every cross block has norm at most R^2. The determinant bound follows from its singular values. The inverse of the leading block of size $i \geq 1$ has norm at most $R^{2(i-1)}/|\delta_{i-1}|$, again by singular values. The earlier columns subtracted in elimination have coefficient vector of norm at most $R^{2i}/|\delta_{i-1}|$. In particular both normalized columns of index i have norm at most $R(1 + R^{2i}H_{\mathbf{p}})$, also for $i = 0$. Both $|p_i|$ and $|p_i|^{-1}$ are at most $R^{2j}H_{\mathbf{p}}$ by their determinant formulas.

Let T send e_i, e_i^* to $\widetilde{A}_i, \widetilde{B}_i/p_i$, respectively. Thus $T^*JT = J_0$, the standard form on these $2j$ vectors, and $\|T\| \leq 4\sqrt{j}\,R^{4j+1}H_{\mathbf{p}}^2$. Put $Q = I - TJ_0T^*J$. It is the projection onto the Hermitian orthogonal complement L along the range of T. Let Π_L be the Euclidean orthogonal projection. The inverse of the compression $\Pi_L J|_L$ is $QJ|_L$: their product on L is $\Pi_L Q^*|_L = I$, since $JQJ =$

Q^{*} and $Q|_{L} = I$. Thus the inverse compression has norm at most $1 + \|T\|^{2}$. Diagonalize the Hermitian compression in an orthonormal basis of L and rescale each eigenvector of eigenvalue γ by $|\gamma|^{-1/2}$. The resulting extension on the middle space has norm at most $\sqrt{1 + \|T\|^{2}}$. Its signs match the middle signature by additivity on the orthogonal direct sum. Joining it to T gives an isometry of norm at most $3(1 + \|T\|)$, whose inverse is τ_{P}. Every isometry has the same norm as its inverse since $g^{-1} = J g^{*} J$. The swap has norm at most $R^{2j} H_{\mathbf{p}}$. The stated $L_{\mathbf{p}}$ absorbs these bounds, since $j < n$ and $R \geq \max(16, 2r)$. The construction can be made measurable by choosing orthonormal bases in successive eigenspaces by Gram–Schmidt from the coordinate vectors. Any two common sections differ by left multiplication by G_{j}. $\square$

139.14 Weighted correlations on prescribed null flags

Keep $X_j, d\Sigma_j, d\mu_{\mathbf{p}}$ from Theorem 139.23 and $z = k - 5$. At a flag with canonical coordinates $b_0, \dots, b_{j-1}$, let $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{j-1})$ be nonzero complex scales. For $x_0 = x$ define b_i' and x_{i+1} by

$$b_i' = \alpha_i^{-1} x_i^{-1} b_i, \qquad d_{\alpha_i} s_{b_i} x_i s_{b_i'}^{-1} = n(t_i, u_i') x_{i+1}.$$

In this formula u_i' is a real central coordinate. Write $\Delta_y(b) = \max_{\|v\|, \|y^{-1}v\| \le 1} |h(b, v)|$ in the corresponding residual space. Put

$$D_{\boldsymbol{\alpha},x}(B) = \prod_{i<j} \Delta_{x_i}(b_i), \quad M_i = \max(|\alpha_i|, |\alpha_i|^{-1}), \quad d_i' = r_{i+1}, \quad m_i = 2(d_i' - z) + 1.$$

These weights are finite and strictly positive. Each x_i depends only on x, the scales, and the part of the flag strictly preceding b_i. Call a flag admissible here if $R^{-1} \le \|b_i\| \le R$ at each step in the fixed canonical sections. Let $\mathbf{1}$ denote the list of scales equal to one.

Theorem 139.25 (Crossed conditional moments for a whole flag)**.** *If f, g are nonnegative measurable functions supported on admissible flags, then, for each x and any scales $\boldsymbol{\alpha}, \boldsymbol{\gamma}$,*

$$\int f(A) g(B)\, d\mu_{\mathbf{p}} \le C_{\mathbf{p}} \left(\int D_{\boldsymbol{\alpha},x}^{2z} f^2\, d\Sigma_j \right)^{1/2} \left(\int D_{\boldsymbol{\gamma},x}^{2z} g^2\, d\Sigma_j \right)^{1/2}, \tag{139.31}$$

$$C_{\mathbf{p}} = \prod_{i<j} \pi^{d_i'} R^{m_i+1} |p_i|^{-2}.$$

For an integral restricted to a subset of pairs, one may impose any measurable one-flag restrictions containing its projections in the corresponding right-hand integrals.

Proof. Fix A and the first i coordinates of B. Use u_i for the null constraint vector of Theorem 139.23. If its fiber is nonempty in $\|b_i\| \le R$, then $\|u_i\| \ge |p_i|/R > 0$. Carry u_i to the standard null vector by $s_{u_i} = a_{u_i} k_{u_i}$ as before. In the coordinates

$$s_{u_i} b_i = ce + y + p_i e^*, \qquad s = \operatorname{Im}(c/p_i),$$

the conditional measure is $\frac{1}{2} dy\, ds$, since integration in $\operatorname{Re}(c/p_i)$ cancels the complex scaling Jacobian as in Theorem 139.15. We have $\|y\| \le R$ and $|s| \le R/(\|u_i\|\,|p_i|) \le R^2/|p_i|^2$; here a_{u_i} acts as the identity on the middle space and as $\|u_i\|^{-1}$ on e.

In the positive space there is a $(k-1)$-plane on which x_i^{-1} is a Euclidean isometry into the positive space. Indeed the kernel of its negative projection on the positive space has dimension at least the positive index minus the negative index, which is $k - 1$; preservation of h gives the isometry there. After applying k_{u_i} this plane meets the middle positive space in dimension at least $k - 2$. For a $(k-2)$-plane E in that intersection we have $\Delta_{x_i}(b_i) \ge \|\operatorname{pr}_E y\|$, by testing on its unit vectors transported back by $k_{u_i}^{-1}$. Integrating the negative power in the two orthogonal balls in E and its complement as in Theorem 139.19 gives at most $\pi^{d_i'} R^{2(d_i' - z)}$. Here $k - 2 > z$. Including the central interval bounds this conditional integral by $\pi^{d_i'} R^{m_i+1} |p_i|^{-2}$. This bound is independent of all earlier choices. Integrating backwards from $i = j - 1$ gives $\int D_{\boldsymbol{\gamma},x}(B)^{-2z} \le C_{\mathbf{p}}$ conditionally on each A, with the admissibility restrictions. The exchanged statement follows at every prescribed pivot by the exchange formula of Theorem 139.23 and the same calculation for $\boldsymbol{\alpha}$. Multiply $f(A)$ by $D_{\boldsymbol{\alpha},x}(A)^z / D_{\boldsymbol{\gamma},x}(B)^z$, multiply $g(B)$ by its reciprocal, and apply Cauchy–Schwarz. The two conditional bounds give (139.31). The same proof works after inserting the two one-flag restrictions and enlarging the pair domain. □

Write $d_{\boldsymbol{\alpha}}$ for the simultaneous diagonal scales on the first j hyperbolic pairs. For a flag section S, let $\mathcal{D}^{\boldsymbol{\alpha}}_{S,x}(z')d\sigma_j(z')$ denote the pushforward under $m \mapsto m^{-1}f_j$ of

$$\langle W(mS), W(md_{\boldsymbol{\alpha}}Sx)\rangle \, d\dot{m}^*_j.$$

Use $\mathcal{A}_{\boldsymbol{\alpha},h}(B,x)$ for the $L^h(\sigma_j)$ norm of this density in the canonical section of B, for $h = 1, 2$. Write $d = d_v$.

Proposition 139.26 (Position integral for one flag with prescribed scales)**.** *The norm $\mathcal{A}_{\boldsymbol{\alpha},h}$ is independent of the choice of flag section. In canonical coordinates it equals $\|D^{W_{S_j},W_{S'_j}}_{j,x_j}\|_h$, where S'_j is the canonical product for the successive b'_i. In particular*

$$\begin{aligned}&\int_{\|x\|\le Y}\int_{X_j} D_{\boldsymbol{\alpha},x}(B)^{2z}\mathcal{A}_{\boldsymbol{\alpha},2}(B,x)^2 d\Sigma_j(B)dg^*_0(x)\\ &\le \left[\prod_{i<j} 2\pi^{d'_i}|\alpha_i|^{2z}(R^2M_iY)^{m_i}\right](16r^{3/2}R^5Y^2)^{a-j}\pi^{\sum_{i=j+1}^{a}(2r_i-1)}\,\frac{\|W\|^4_{2,*}}{d}.\end{aligned} \tag{139.32}$$

For $h = 1$ on the further restriction $\|x_j\| > Z \ge R^2$, the same bound holds with the last line's residual factor after the square bracket replaced by

$$\pi^{(2(a-j)+1)k}R^{80(a-j+1)k}Z^{-k/(2(a-j))}\,\frac{\|W\|^4_{2,*}}{d} \qquad (j < a).$$

For $j = a$ that restricted integral is zero.

Proof. Applying the successive position decompositions gives $d_{\boldsymbol{\alpha}}S_jx = nx_jS'_j$ with $n \in N_{<j}$. To see this inductively, the new residual section and the new scale normalize the previously removed radical. After putting that radical on the left the remaining factor is exactly the one-step position decomposition defining x_{i+1}.

For $m \in G_j$ the Bessel character on mnm^{-1} is a function only of $m^{-1}f_j$. Indeed the simple entries strictly before row $j-1$ are unchanged: the corresponding basis vectors and dual coordinate functionals are fixed by m. In row $j-1$ one evaluates the earlier-coordinate functional of n on $m^{-1}f_j$. All later simple entries vanish because the induced residual action of n is the identity. This argument includes f_a at the terminal row. Thus moving n past m multiplies the pushed-forward density by a function of its orbit variable of absolute value one. This gives the claimed equality to the canonical correlation norm. A change of section by n_0l_0, with $n_0 \in N_{<j}$, $l_0 \in G_j$, similarly contributes two such phase factors and a common right translation of the residual variable by l_0. Conjugation by the scales preserves $N_{<j}$ and fixes l_0. The induced residual isometry preserves σ_j, giving section independence.

Fix the initial flag. Repeated Haar disintegration in x gives the product of the $dt_i\,du'_i$, the measures $d\sigma_i(b'_i)$, and $dg^*_j(x_j)$. The fixed left translations by the scales preserve Haar at each step. The two nonzero slices have radius R at every stage by the middle-block argument. In particular both endpoints' successive vector coordinates satisfy the required annular bounds. Also $\|x_{i+1}\| \le \|x_i\| \le Y$, by taking the middle block between the compact parts of the two canonical sections. Integrate first in t_0, u'_0, then in order through $j-1$. At each step later factors depend only on the residual position and the remaining flag coordinates. Proposition 139.20 in G_i bounds that integral of $\Delta_{x_i}(b_i)^{2z}$ by $2\pi^{d'_i}|\alpha_i|^{2z}(R^2M_iY)^{m_i}$ at fixed b_i, b'_i, x_{i+1}. Enlarge the remaining position domain to $\|x_{i+1}\| \le Y$ after each integration. Finally apply (139.5) at j; the squared norms of the two slices integrate separately to $\|W\|^2_{2,*}$ in their respective flag coordinates. This proves (139.32). For the restricted squared L^1 norm use instead the dyadic tail estimate obtained from (139.11) with $a-j$ null steps and recursive group Haar. It is

uniform for any two residual slices of radius R and type dimension d, as its proof uses only these properties and their squared norms. At $j = a$ a nonzero correlation requires $x_a = y^{-1}(yx_a)$ with both factors of norm at most R, since $U_a = 1$; thus $\|x_a\| \le R^2$. □

Proposition 139.27 (Transfer of a large position to either residual group)**.** *Let the initial flag pair $P = (A, B)$ have pivots $\mathbf{p}$. Choose $\lambda_i = p_i/q_i$, where $q_i \neq 0$. Suppose each flag of P, and each flag at the final pair*

$$P' = (x^{-1}A,\ (x^{-1}B_i/\lambda_i)_{i<j}),$$

is admissible and admits a representative of column norm at most R. These two requirements hold for flags coming from nonzero slices of the cut W. For the residual position x_j defined on either A with scales $\mathbf{1}$, or B with scales $\boldsymbol{\lambda}$,

$$\|x\| \le D_*\|x_j\|, \qquad D_* = R^{2j}L_{\mathbf{p}}^2L_{\mathbf{q}}^2 \le R^{42n}(H_{\mathbf{p}}H_{\mathbf{q}})^6.$$

Proof. The final normalized representatives are exactly $x^{-1}\widetilde{A}_i$ and $x^{-1}\widetilde{B}_i/\lambda_i$ by uniqueness of diagonal elimination: their Gram matrix is diagonal with entries q_i, and the column changes are still unitriangular after the coordinate scalings. Hence $l = \tau_P x \tau_{P'}^{-1}$ fixes the $2j$ standard columns and lies in G_j. In particular $\|x\| \le L_{\mathbf{p}}L_{\mathbf{q}}\|l\|$. Take T, T' to be the pair sections on the side in question, so they are $\tau_P, \tau_{P'}$ on the first side and $w_{\mathbf{p}}^{-1}\tau_P, w_{\mathbf{q}}^{-1}\tau_{P'}$ on the other. In both cases $d_{\boldsymbol{\alpha}}Tx(T')^{-1} = l$, using $d_{\boldsymbol{\lambda}}w_{\mathbf{p}}^{-1} = w_{\mathbf{q}}^{-1}$. The successive final one-sided coordinates really give the indicated final flag: $d_{\boldsymbol{\alpha}}S_jx = nx_jS_j'$ fixes its standard flag after removal of S_j', with the column scales α_i^{-1} on the original transported flag.

Write $TS_j^{-1} = n_0h$, $T'(S_j')^{-1} = n_1h'$ in the flag stabilizers. Taking middle blocks in the identity for l gives $l = hx_j(h')^{-1}$, since conjugation by the scales fixes the residual block. By admissibility each elementary section and its inverse have norm at most R, so S_j, S_j' and their inverses have norm at most R^j. It follows that $\|h\| \le R^jL_{\mathbf{p}}$, $\|(h')^{-1}\| = \|h'\| \le R^jL_{\mathbf{q}}$, by the bounds on the pair sections. Combining the two estimates for $\|l\|$ and $\|x\|$ proves the assertion on the stated support. □

139.15 Real Gram waves after an arbitrary number of prescribed pivots

Use the shorter interval, the cut W, and the original normalizations $d = d_v$, $S_{k,a} = \pi^{(2a+1)k}V_k/V_r$ of Theorem 139.11. For lists $\mathbf{p}, \mathbf{q}$ of j nonzero complex pivots, let $\lambda_i = p_i/q_i$. At a pair $P = (A, B)$ with pivots $\mathbf{p}$ take $\tau = \tau_P$ from Proposition 139.24. Form the two residual densities

$$g = \mathcal{D}^{\mathbf{1}}_{\tau,x}, \qquad g' = \mathcal{D}^{\boldsymbol{\lambda}}_{w_{\mathbf{p}}^{-1}\tau,x}.$$

At $j = 0$ both sections and the scale matrix are the identity. For a bounded measurable function ϕ on $\mathbb{C}$ the inner integral is

$$C_P(x;\phi) = \iint \phi(h(v,w))g(v)\overline{g'(w)}\,d\sigma_j(v)d\sigma_j(w).$$

Changing the common section by $l \in G_j$ transports both residual variables by the same isometry, leaving C_P unchanged. Use $\phi_\nu(c) = \exp(2i\,\mathrm{Im}(\nu c))$ and put

$$J_{\mathbf{p},\mathbf{q}}(x;\nu) = \eta^2 \prod_{i<j} |\lambda_i|^{-r_{i+1}} \int C_P(x;\phi_\nu)\,d\mu_{\mathbf{p}}(P).$$

This is the real position form on the two prescribed heads. Indeed in common coordinates the remaining frame vectors are middle vectors, and the relative Gram entry is $h(v,w)$. The two endpoint sections for the second flag are $w_{\mathbf{p}}^{-1}\tau$ and $w_{\mathbf{q}}^{-1}\tau x$, since $d_{\boldsymbol{\lambda}}w_{\mathbf{p}}^{-1} = w_{\mathbf{q}}^{-1}$. The outside measure in recursive coordinates is $2^{-j}\prod_{i<j}|p_iq_i|^{r_{i+1}}d\beta_{\mathbf{p}}$, by Theorem 139.23. We keep the factors $1/2$ from real coarea in this convention.

For a list $\mathbf{p}$ set $\delta_l(\mathbf{p}) = \prod_{i=0}^{l} p_i$ and define the weight

$$\mathcal{H}(\mathbf{p}) = \prod_{l<j} \mathbf{1}_{|\delta_l(\mathbf{p})|\le b^{60n^2}}\,\max(1, |\delta_l(\mathbf{p})|^{-1})^{n^6}.$$

An empty product is one. Write $W_0(\nu) = |\nu|^{-K_0}\min(1, |\nu|^{-6})$ with $K_0 = k - n^7$.

Theorem 139.28 (Uniform real wave bound for prescribed heads)**.** *At sufficiently large nodes on the shorter interval, uniformly in the real place, $0 \le j \le a$, the two nonzero pivot lists, and $\nu \neq 0$,*

$$\frac{1}{dS_{k,a}} \int_G |J_{\mathbf{p},\mathbf{q}}(x;\nu)|\,dx \le b^{2000n^9}\mathcal{H}(\mathbf{p})\mathcal{H}(\mathbf{q})W_0(\nu). \tag{139.33}$$

For $j \ge 1$ the same bound holds with the absolute value taken inside the flag integral, immediately after the integration in the two residual orbit variables. Also the integral over all positions and flag pairs of the product $\|g\|_1\|g'\|_1$ with the defining outside measures is finite at every pair of prescribed heads.

Proof. For $j = 0$ the wave assertion is Theorem 139.11; integrability of the product of the L^1 norms follows from the position-ball and tail bounds in Sections 139.3 and 139.5. Let $j \ge 1$, $z = k-5$, $d_i' = r_{i+1}$, $m_i = 2(d_i'-z)+1$, and $H = H_{\mathbf{p}}H_{\mathbf{q}}$ with the maxima of Proposition 139.24. Every pair contributing has both initial and final flags on the four cut projections: a nonzero residual density requires a nonzero value of each amplitude on its corresponding flag. Thus the support conditions of Proposition 139.27 hold. In particular every prefix determinant at either endpoint is at most R^{2j} in modulus. Since $R = b^{6n+2}$ and $j < n$, a violated indicator in $\mathcal{H}$ makes the form zero. We may henceforth assume all prefix determinants are at most R^{2j}. Consequently $M_i = \max(|\lambda_i|, |\lambda_i|^{-1}) \le R^{4j}H$.

Let $I(E)$ be the normalized mass on $x \in E$, with the absolute value inside the flag integral. On the support the first amplitude bounds both residual orbit radii in the common pair section

by $L_{\mathbf{p}}$. In detail, if $\|umT\| \le R$ for $T = \tau$ or $w_{\mathbf{p}}^{-1}\tau$, the middle block of um is $u_j m$ with $u_j \in U_j$, of norm at most $R\|T^{-1}\|$. The inverse has the same norm and takes f_j to $m^{-1}f_j$. Apply (139.3) with $\ell = z$; it is in range even at $r_a = k+1$. Alternatively use $\|g\|_1\|g'\|_1$, taking their individual canonical sections. In these sections the radii are R and the restricted orbit volume is at most $\pi^{r_j}R^{2(r_j-1)}$ by polar integration. The norm invariance in Proposition 139.26 therefore gives

$$|C_P(x;\phi_\nu)| \le \pi^{r_j} L_{\mathbf{p}}^{2(r_j-1-z)} \min(R^{2z}, |\nu|^{-z}) \mathcal{A}_{\mathbf{1},2}(A,x)\mathcal{A}_{\boldsymbol{\lambda},2}(B,x).$$

With the minimum replaced by R^{2z} the same right side also bounds $\|g\|_1\|g'\|_1$.

Apply (139.31), Cauchy–Schwarz in x, and (139.32) to the two squared integrals. The geometric mean of the two position estimates contains $\prod_{i<j}|\lambda_i|^z M_i^{m_i/2}$. Multiplication by the measure of the form leaves a factor at most $\prod_{i<j} M_i^{m_i}$, since $d_i' - z = (m_i-1)/2$. The original group conversion is $2^n V_k/V_r$ and $\eta^2\|W\|_{2,*}^4 \le d^2$, canceling the type and volume factors upon normalization. Write $\sigma = \sum_{i<j} m_i$, $s = \sigma + 2(a-j) \le 10n^2$. There is an absolute $C > 0$ for which, with $B_* = R^{Cn^3}H^{Cn^2}$,

$$I(\|x\| \le Y) \le B_* Y^s \min(R^{2z}, |\nu|^{-z}), \qquad Y \ge 1. \tag{139.34}$$

For verification of the dimension dependence, the exponent of π before division by $\pi^{(2a+1)k}$ is

$$r_j + 2\sum_{i<j} r_{i+1} + \sum_{i=j+1}^{a} (2r_i - 1).$$

The coefficient of k is $2a+1$, and the remainder is $O(n^2)$ since $|r_i - k| \le 2n$. Apart from that power of π, the constants outside the minimum and Y^s are bounded by

$$2^{n+j}(16r^{3/2})^{a-j} R^{3\sigma+j+5(a-j)} |\delta_{j-1}(\mathbf{p})|^{-2} (R^{4j}H)^\sigma L_{\mathbf{p}}^{2(r_j-1-z)}.$$

Here $\prod |p_i|^{-2}$ telescopes. Now $m_i \le 4n+5$, $r_j - 1 - z \le 2n+5$, $L_{\mathbf{p}} = R^{10n}H_{\mathbf{p}}^3$ and $R \ge \max(16, 2r)$. These estimates give the asserted absolute exponents in B_*.

On $Y < \|x\| \le 2Y$ use the product of the L^1 norms. By Proposition 139.27, its projection onto either one-flag domain satisfies $\|x_j\| > Y/D_*$ with $D_* \le R^{42n}H^6$. Keep that restriction in (139.31). For $Y \ge R^2 D_*$ the L^1 part of Proposition 139.26 now gives

$$I(Y < \|x\| \le 2Y) \le B_* R^{80(a-j+1)k} Y^\sigma (Y/D_*)^{-k/(2(a-j))} \qquad (j < a). \tag{139.35}$$

The same inequality bounds the normalized integral of the product of the two L^1 norms on the shell. Here the exponent of π before normalization is $2\sum_{i<j} r_{i+1} + (2(a-j)+1)k$; its excess is again $O(n^2)$. The extra factor 2^σ from the upper shell radius fits in B_* after enlarging its absolute constant. For $j = a$ these shells are empty on the support.

Take $X = R^{500n^2}H^{16}\max(1,|\nu|)^{4a}$. For $j < a$ and large nodes, $k/(2(a-j)) - \sigma \ge k/(4a)$, since $\sigma \le 10n^2$ and k/n^3 tends to infinity. Sum the shells $Y = 2^l X$, $l \ge 0$. Their total is at most

$$2B_* D_*^\sigma R^{80(a-j+1)k}(X/D_*)^{-k/(4a)} \le R^{O(n^3)}H^{O(n^2)}\max(1,|\nu|)^{-k}.$$

Indeed $(500n^2 - 42n)/(4a) > 80(a-j+1)$ eventually. This also proves convergence at fixed heads for the L^1 majorant, by combining the shells with the finite ball bound.

For $|\nu| \le 1$ the minimum in (139.34) is at most $R^{2(z-K_0)}|\nu|^{-K_0}$, by splitting at R^{-2}. When $|\nu| > 1$, the power $4as$ from X^s is absorbed by $z - K_0 - 6 = n^7 - 11$ at large nodes. Since X^s contributes at most $R^{O(n^4)}H^{O(n^2)}$ apart from this frequency power, the total coefficient is $R^{2n^7+O(n^4)}H^{O(n^2)}$, with absolute constants. The first factor is at most b^{2000n^9} eventually. The second fits in the product of prefix weights since the product of the maxima of one and the individual inverse determinants dominates their maximum on each list, and $O(n^2) \le n^6$ eventually. These comparisons have enough slack also to allow the factor 2^j if the convention without the coarea halves is used. This proves the theorem. □

139.16 Unequal last entries of arbitrary real heads

Let $j = i + 1 \le a$, let $\mathbf{p}, \mathbf{q}$ have length j, and keep $J_{\mathbf{p},\mathbf{q}}, \mathcal{H}, W_0, \boldsymbol{\lambda}$ from Section 139.15. Superscript minus denotes the prefix of length i; the prefix at $i = 0$ is a point with measure one. Write $p = p_i,\ q = q_i,\ \lambda = p/q$.

Theorem 139.29 (Uniform reciprocal-difference bound for real heads)**.** *On the same shorter interval, at sufficiently large nodes, if $p \neq q$ and $\xi = p^{-1} - q^{-1}$, then*

$$\frac{1}{d_v S_{k,a}} \int_G |J_{\mathbf{p},\mathbf{q}}(x;0)|\, dx \le b^{2000n^9} \mathcal{H}(\mathbf{p})\mathcal{H}(\mathbf{q}) W_0(\xi).$$

Proof. At zero frequency the inner integral is the product of the two ordinary conditional correlations, the second conjugated. On the first side all scales are one. The scalar correlation at a full flag A is independent of the full flag section: both amplitudes acquire the same character under its unipotent part and a common change of residual integration variable under its residual isometry. In canonical coordinates let v_0 be the last vector of A and S_A the section of A^-. The modulus of this scalar is

$$F(A, x) = |\mathcal{D}^{\mathbf{1}}_{S_A,x}(v_0)|.$$

Here the density on the right is the one at depth i. Indeed disintegrating $U_i \backslash G_i$ over $m^{-1} f_i = v_0$ gives the full-flag scalar with the canonical last section. In particular its squared integral in v_0 is $\mathcal{A}_{\mathbf{1},2}(A^-, x)^2$.

Fix the prefix pair P^- and choose a common section τ. Write $T_A = \tau = n_A h_A S_A$ and $T_B = w_{\mathbf{p}^-}^{-1} \tau = n_B h_B S_B$, where S_B is the canonical section of B^-. The n's are in $N_{<i}$ and $h_A, h_B \in G_i$. For $i = 0$ these matrices are identities, and $L_{\mathbf{p}^-}$ may be taken equal to one. In the last residual space use the coordinates of S_B for both vectors. If w_0 is the last canonical vector of B, the two vectors in these coordinates are

$$v = h_B^{-1} h_A v_0, \qquad w = w_0, \qquad h(v, w) = p.$$

This follows because the two common sections identify the same middle space. The change there by h_B^{-1} preserves its orbit and coarea measures. On the four cut projections $\|v_0\|, \|w\| \le R$, both are at least R^{-1}, and $\|h_A\|, \|h_B^{-1}\| \le R^i L_{\mathbf{p}^-}$, as in Proposition 139.27. Put $U = R^{2i+1} L^2_{\mathbf{p}^-}$; then $\|v\| \le U$.

At fixed w, choose t, u by $v = s_w^{-1} n(-t, -u) \bar{p} e_i^*$. The resulting full pair section is $w_p n(t, u) s_w h_B^{-1} \tau$, with the four residual factors embedded at depth i (the prefix section τ acts on the original space). It sends the initial last pair to $e_i, p e_i^*$ and has the prescribed action on all previous pairs. Its full second-flag section is

$$n(t, u)\, \widetilde{n}\, S, \qquad S = s_w S_B, \qquad \widetilde{n} = (s_w h_B^{-1}) n_B (s_w h_B^{-1})^{-1} \in N_{<i}.$$

Here $\widetilde{n}$ is fixed when the two prefixes and w are fixed. We must account for this earlier unipotent before taking a Fourier transform in t.

Let $m \in G_j$, and abbreviate $n(t, u)$ by n'. When $\widetilde{n}$ is moved left through mn', its character is constant in m, t, u: the earlier simple entries are fixed and the boundary entry depends only on $(mn')^{-1} e_i = e_i$, by the simple-entry calculation in Proposition 139.26. In the second amplitude the conjugating matrix is $m d_{\boldsymbol{\lambda}} n'$. Move $d_{\boldsymbol{\lambda}^-}$ into $\widetilde{n}$ first, which preserves the earlier radical since this scale commutes with G_i; the remaining boundary vector is $(m d_\lambda n')^{-1} e_i = e_i/\lambda$, also fixed. At $i = 0$ the earlier radical is trivial. Removing these two constant phases, the characters

of n' itself differ by the same factor as in Theorem 139.22. Thus the modulus of the second scalar correlation is

$$|T_B(t)| = \left| \int e^{2i\,\mathrm{Im}((1-\lambda)h(t,z'))} \mathcal{D}^{\boldsymbol{\lambda}}_{S,x}(z')\, d\sigma_j(z') \right|$$

up to changing the sign of the Fourier variable, which leaves the following ball estimate unchanged. In particular the density in this formula is the full canonical one, independent of t, u.

The last conditional measure at fixed w is $\frac{1}{2}|p|^{2r_j}\,dt\,du$. The bounds on v, w give $\|t\|, |u| \le T = RU/|p|$ by applying s_w to v. The residual orbit in the last display has canonical annular radius R. Lemma 139.5 applied after the complex change of Fourier variable by $\overline{1-\lambda}$ therefore gives

$$\tfrac{1}{2}|p|^{2r_j} \int_{\|t\|,|u|\le T} |T_B(t)|^2\, dt\, du \le P_i |\xi|^{1-2r_j} \mathcal{A}_{\boldsymbol{\lambda},2}(B,x)^2,$$
$$P_i = 2(2r_j)^{3/2}\pi^{2r_j-1} R(RU)^2 |p|^{-1}. \tag{139.36}$$

The scaled ball has radius $|1-\lambda|T$, its Jacobian is $|1-\lambda|^{2r_j}$, and the central interval has length $2T$. Substituting $|1-\lambda| = |p|\,|\xi|$ gives the displayed single inverse power of $|p|$.

We describe the weighted integrations, since the first squared integral uses one fewer flag vector than the second. Put $D_A = D_{\mathbf{1},x}(A^-)$ at depth i and $D_B = D_{\boldsymbol{\lambda},x}(B)$ at depth j. Apply Cauchy–Schwarz to $F(A,x)D_A^z D_B^{-z}$ and $|T_B(t)|D_B^z D_A^{-z}$ on the supported full pairs. The first squared integral is bounded by

$$C_{\mathbf{p}} \int_{X_i} D_A^{2z} \mathcal{A}_{\mathbf{1},2}(A^-,x)^2\, d\Sigma_i(A^-).$$

Indeed first integrate D_B^{-2z} at fixed full A by the conditional negative moment in the proof of Theorem 139.25, then integrate F^2 in v_0. For the second integral disintegrate first at the prefix pair and w, using Theorem 139.23. The measure in w is $d\sigma_i(w)$; the change from common to canonical middle coordinates preserves it for each prefix pair. Apply (139.36) in t, u, keeping the initial support until the uniform value of U has been inserted. The remaining dependence on A^- is only D_A^{-2z} and admissibility. Integrating over the prefix fiber at fixed B^- costs $C_{\mathbf{p}^-}$ by the exchanged negative moment. Thus the second squared integral is at most

$$P_i C_{\mathbf{p}^-} |\xi|^{1-2r_j} \int_{X_j} D_B^{2z} \mathcal{A}_{\boldsymbol{\lambda},2}(B,x)^2\, d\Sigma_j(B).$$

Integrate the geometric mean in $\|x\| \le Y$ by Cauchy–Schwarz and (139.32) at depths i, j. At $i = 0$ its first factor is just (139.5). The position normalization and factors $\prod |\lambda_l|^{z-r_{l+1}} M_l^{m_l/2}$ are the same as in Theorem 139.28. The total exponent of π before normalization is

$$2\sum_{l<i} r_{l+1} + 3r_j - 1 + \sum_{l=j+1}^{a} (2r_l - 1) = (2a+1)k + O(n^2).$$

With $H = H_{\mathbf{p}} H_{\mathbf{q}}$, all constants outside the frequency power are $R^{O(n^3)} H^{O(n^2)}$. To check the new factors, $C_{\mathbf{p}^-}$ has the same product bound as $C_{\mathbf{p}}$ with one fewer factor, $U \le R^{22n} H^6$, and $|p|^{-1} \le R^{2j} H$. The power of Y is the average of $s_i = \sum_{l<i} m_l + 2(a-i)$ and s_j; both are at most $10n^2$. Together with the constant-weight ball bound in the proof of (139.34), the normalized mass is at most

$$R^{O(n^3)} H^{O(n^2)} Y^{10n^2} \min(R^{2z}, |\xi|^{1/2-r_j}).$$

For the tail the product of the two L^1 norms from (139.35) still applies. Take $X = R^{500n^2} H^{16} \max(1,|\xi|)^{4a}$ as before. For $|\xi| \le R^{-2}$ the radius bound costs at most $R^{2(z-K_0)}|\xi|^{-K_0}$.

For $R^{-2} \le |\xi| \le 1$ the frequency bound costs at most $R^{2(r_j-1/2-K_0)}|\xi|^{-K_0}$. For $|\xi| > 1$ the difference $r_j - 1/2 - K_0$ exceeds $4a \cdot 10n^2 + 6$ eventually. The shell sum contributes at most $R^{O(n^3)}H^{O(n^2)}\max(1,|\xi|)^{-k}$. All constants are absolute, so the same final conversion as in (139.33) gives the asserted inequality, including its determinant support restrictions. □

139.17 Fourier descent through the real null rows

For $0 \le j \le a$, let $\mathcal{J}_{\mathbf{p},\mathbf{q}}(x;\phi)$ be the form of Section 139.15 with $C_P(x;\phi)$ in place of $C_P(x;\phi_\nu)$; thus $J_{\mathbf{p},\mathbf{q}}(x;\nu) = \mathcal{J}_{\mathbf{p},\mathbf{q}}(x;\phi_\nu)$. Use $J_{\mathbf{p},\mathbf{q}}(x;0)$ for its constant-weight value. Appending $c \neq 0$ to both lists will be denoted by $\mathbf{p}c, \mathbf{q}c$. Use planar Lebesgue measure at this real place and the Fourier kernel $\phi_\xi(c) = e^{2i\,\mathrm{Im}(\xi c)}$.

Theorem 139.30 (Pointwise Fourier descent at the null rows)**.** *At the nodes of Theorem 139.28, let $0 \le j < a$ and fix any two nonzero pivot lists of length j, including the empty lists. Then*

$$\int_{\mathbb{C}^\times} \int_G |J_{\mathbf{p}c,\mathbf{q}c}(x;0)|\,dx\,d^2c < \infty.$$

Define $f(c) = \int_G J_{\mathbf{p}c,\mathbf{q}c}(x;0)\,dx$ for $c \neq 0$. For every bounded measurable ϕ,

$$\int_{\mathbb{C}^\times} \phi(c) f(c)\,d^2c = \int_G \mathcal{J}_{\mathbf{p},\mathbf{q}}(x;\phi)\,dx.$$

For every $c \neq 0$, with absolutely convergent integrals,

$$f(c) = \pi^{-2} \int_{\mathbb{C}} e^{-2i\,\mathrm{Im}(\xi c)} \left[\int_G J_{\mathbf{p},\mathbf{q}}(x;\xi)\,dx \right] d^2\xi. \tag{139.37}$$

The right-hand side extends f to a compactly supported C^h function of the two real coordinates for every nonnegative integer $h < K_0 + 4$.

Proof. Fix the prefix pair P of length j and its common section τ. After integrating the later residual stabilizers, the two orbit variables v, w at this step have densities $g = \mathcal{D}^{\mathbf{1}}_{\tau,x}$ and $g' = \mathcal{D}^{\boldsymbol{\lambda}}_{w_{\mathbf{p}}^{-1}\tau,x}$. The step is null because $j < a$. For fixed v choose a residual section s_v, and at $c = h(v,w) \neq 0$ write $w = s_v^{-1} n(t,u) c e_j^*$. The measure in w is $\frac{1}{2}|c|^{2r_{j+1}} d^2c\,dt\,du$. The excluded fiber $c = 0$ has zero orbit measure at fixed v. Indeed in coordinates $\alpha e + y + ce^*$ its nonzero middle points allow a middle real coordinate to solve the null equation, leaving c free. The remaining points on the complex line $\mathbb{C}e$ have zero measure in the smooth nonzero orbit.

Take the full pair section $\tau_+ = n(t,u)^{-1} s_v \tau$. Its two flag sections are this one and $w_{\mathbf{p}c}^{-1}\tau_+$. At fixed v, w the first scalar $g(v)$ is the integral of its next density in the section τ_+. The analogous integral in the other section gives $g'(w)$. To see that the scalar is unaffected by either section choice, at the residual step any two sections for one fixed vector differ by $n_0 l \in N_j \rtimes G_{j+1}$. The preceding diagonal scales commute with this whole residual group, and the new scale is $c/c = 1$. Thus the two amplitudes in each scalar acquire the same unipotent character and a common right translation in the residual variable. Their scalar integral is unchanged. These statements follow directly by disintegrating the quotient $U_j\backslash G_j$ at the fixed vector, including at $j + 1 = a$.

Thus the product $g(v)\overline{g'(w)}$ is the constant-weight inner integral at the extended pair. Its coarea and the old prefix measure combine to $d^2c\,d\mu_{\mathbf{p}c}$ by Theorem 139.23. The outside scale factor is unchanged since the new scale is one. All quotient constants are therefore exactly those of $J_{\mathbf{p}c,\mathbf{q}c}$. Taking absolute values after the two scalar integrations and then integrating in c costs at most the old integral of $\|g\|_1\|g'\|_1$, with the old outside flag and position measures. This majorant is integrable by Theorem 139.28. Fubini now proves both the first assertion and the identity for bounded ϕ.

For pointwise inversion we show that f is continuous away from zero. First restrict positions to $\|x\| \le Y$ and c to a compact subannulus of $\mathbb{C}^\times$. All contributing full flags are projections of the original compact cut support in $U_0\backslash G_0$. Thus the initial flag pairs lie in a compact set. All their prescribed pivots, including the varying last one, stay bounded away from zero. On this

set choose a finite covering by smooth local pair sections and a subordinate partition of unity. The orbit and coarea calculations in Theorem 139.23 supply local smooth fiber coordinates at every such pair. The two amplitudes are continuous in the parameters; in each residual quotient the first amplitude restricts integration to a common compact set. Indeed its residual radius is bounded by R times the maximum norm of the inverse local section, by taking the middle block. The flag integrations also have common compact support after localization. Dominated convergence in these coordinates gives continuity of the bounded-position integral. The part $\|x\| > Y$ tends to zero uniformly on the subannulus: the weights $H_{\mathbf{p}c}, H_{\mathbf{q}c}$, and hence D_* in (139.35), are uniformly bounded there. The dyadic shell powers are summable. If the new depth is a the tail vanishes for all sufficiently large Y. This proves the asserted continuity.

The bounded-weight identity identifies the Fourier transform of f with the bracket in (139.37). For $|\xi| \le 1$ its absolute value is bounded by the integral of the old L^1 majorant. For $|\xi| > 1$ it is at most a constant depending on the fixed heads and node times $|\xi|^{-K_0-6}$, by Theorem 139.28. The Fourier transform is integrable in the plane, also after multiplication by $|\xi|^h$ for $h < K_0 + 4$. Ordinary Fourier inversion identifies the displayed integral with f almost everywhere. The inversion constant is π^{-2}, since $2\,\mathrm{Im}(\xi c)$ is the real two-dimensional dot product with an orthogonally transformed frequency scaled by two. Continuity gives equality at every nonzero c; differentiation under the integrable transform gives the specified regularity at all c. For compact support, $|c\,\delta_{j-1}(\mathbf{p})|$ and $|c\,\delta_{j-1}(\mathbf{q})|$ are both at most $R^{2(j+1)}$ on the support, also for $j = 0$ with $\delta_{-1} = 1$. These inequalities bound $|c|$ for the fixed heads. □

139.18 The terminal real density and the orbital tensor

At depth a the residual space is $\mathbb{C}^k \oplus \mathbb{C}z$ with form $\|u\|^2 - |t|^2$. We use $z = (0, 1)$, $C = \mathrm{U}(k)$ with probability Haar and $S_k = 2\pi^k/(k-1)!$. Write dx for the original group Haar from Theorem 131.1 and put

$$c_G = V_k S_k / V_r, \qquad \kappa_G = 2^n V_k / V_r, \qquad dx = \kappa_G \, dg_0^*(x).$$

For $|c| > 1$ set $s = (|c|^2 - 1)^{1/2}$; let σ_c fix all but the first positive coordinate and z, on which its matrix in that order is $\left(\begin{smallmatrix} -\bar{c} & s \\ s & -c \end{smallmatrix}\right)$. Its columns are Hermitian orthogonal of norms $1, -1$, and $h(z, \sigma_c z) = c$. For a null prefix $\mathbf{p}$ of length a put $\gamma_p = w_{\mathbf{p}} \sigma_c$ and define the matrix

$$\begin{aligned} O_p(c) &= \int_G W(y) W(\gamma_p^{-1} y)^* \, dy, \\ \Xi_p(c) &= \frac{\sqrt{c_G}}{d_v} (|c|^2 - 1)^{(k-1)/2} \prod_{l<a} |p_l|^{r_{l+1}} \, O_p(c). \end{aligned} \tag{139.38}$$

Set $\Xi_p(c) = 0$ for $|c| \le 1$. Matrix inner products below are Hilbert–Schmidt inner products, conjugate linear in their first argument.

Theorem 139.31 (The terminal Gram density in the original Haar measures)**.** *At every real place of the shorter family the tensor in* (139.38) *is continuous jointly in* $\mathbf{p} \in (\mathbb{C}^\times)^a$ *and* c*. For any two such prefixes and any bounded measurable* ϕ*,*

$$\int_{\mathbb{C}} \phi(c) \langle \Xi_p(c), \Xi_q(c) \rangle_{\mathrm{HS}} \, d^2 c = \frac{1}{d_v^2} \int_G \mathcal{J}_{\mathbf{p},\mathbf{q}}(x; \phi) \, dx. \tag{139.39}$$

Both integrals converge absolutely. The common-entry Gram value is unchanged if σ_c *in both orbital tensors is replaced by any residual isometry* σ *with* $h(z, \sigma z) = c$ *and* $|c| > 1$*.*

Proof. We first identify the quotient constant $d\dot{y} = \eta \, d\dot{y}_0^*$ used in the preceding formulas. The original Haar on U is the product of Lebesgue measure on the upper entries of the frame group N_n and the measures $dB \, dK$ on its kernel, as specified in Section 131.2 and Lemma 133.9. This agrees exactly with the successive $dt_l \, du_l$ on the Heisenberg groups N_l. It suffices to check the two Haar densities at the identity. In the tangent map of $n_l(t_l, u_l)$, the component of t_l along a later e_h^* yields the upper entry in row l, column h of the frame group by negative conjugation. Its z component yields the entry of the terminal column by conjugation, since $h(z, z) = -1$. The k positive components form column l of B. For each $h > l$ the component along e_h gives the skew entry K_{lh} by negative conjugation, and u_l gives $\operatorname{Im} K_{ll}$. These are all coordinates: the quadratic terms in the Heisenberg matrix have zero differential at the identity. The resulting real Jacobian has absolute value one. Uniqueness of Haar gives equality globally. Dividing $dx = \kappa_G dg_0^*$ by this common unipotent measure shows $\eta = \kappa_G$.

For fixed $\mathbf{p}$ and $|c| > 1$ the orbital integral in (139.38) is an ordinary absolutely convergent integral. In fact if its integrand is nonzero, the two flags represented by y and $\gamma_p^{-1} y$ both occur on the cut. Their already normalized columns are $y^{-1} e_l$ and $p_l y^{-1} e_l^*$, since σ_c fixes the earlier hyperbolic pairs. Thus $y = m \tau_P$ for the bounded pair section of Proposition 139.24 and some $m \in G_a$. From $W(y) \neq 0$ we obtain $\|u m \tau_P\| \le R$ for some $u \in U$. Its residual middle block after removal of τ_P is m, since $U_a = 1$, so $\|m\| \le R \|\tau_P^{-1}\| \le L_{\mathbf{p}}$. In particular $\|y\| \le L_{\mathbf{p}}^2$. This gives a common compact set when the prefixes range in a compact set of nonzero entries. Dominated convergence for the continuous cut proves continuity on $|c| > 1$. The orbital integral stays bounded as $|c| \downarrow 1$; its factor $(|c|^2 - 1)^{(k-1)/2}$ tends to zero since $k > 1$, proving continuity there too.

For v, w on the negative unit orbit and a section $m(v)$ with $m(v)v = z$, write $m(v)w = (u, -c)$ where $c = h(v, w)$. The orbit equation is $\|u\|^2 = |c|^2 - 1$. Polar integration in the k complex coordinates of u gives the measure

$$d\sigma_a(w) = \kappa(c) d^2c \, d\omega, \qquad \kappa(c) = \tfrac{1}{2} S_k (|c|^2 - 1)^{k-1} \mathbf{1}_{|c|>1} \quad (|c| \geq 1)$$

with zero density on $|c| < 1$; $d\omega$ is probability sphere measure. The boundary has measure zero: near a point with $u = 0$ the nonzero terminal coordinate permits solving the orbit equation in its radial variable, leaving u free. For $|c| > 1$ write $w = m(v)^{-1} h \sigma_c z$. As h varies in C with probability Haar its positive direction has law $d\omega$, by transitivity and invariance.

Fix a prefix pair P, with section τ. At the terminal level a scalar density at v is the inner product at any section $m(v)$, since left translation by C preserves the inner product of the two amplitudes. Similarly use $\sigma_c^{-1} h^{-1} m(v)$ for the second orbit vector w. In the first inner product replace $m(v)$ by $h^{-1} m(v)$ as well; it is unchanged by compact equivariance. Integrating v, h now integrates $m = h^{-1} m(v)$ over G_a in dg_a^*. The residual group commutes with the earlier Weyl swaps. It follows that after extracting $\kappa(c) d^2 c$ the two inner products become

$$\langle W(y), W(yx) \rangle \langle W(\gamma_q^{-1} yx), W(\gamma_p^{-1} y) \rangle, \qquad y = m\tau.$$

The order of the second pair is reversed because the second scalar was conjugated. The P-measure after its endpoint scaling is $2^{-a} \prod_{l<a} |p_l q_l|^{r_{l+1}} \, d\beta_{\mathbf{p}}$. By its quotient normalization $d\beta_{\mathbf{p}} \, dg_a^*(m)$ integrates y in $dg_0^*(y)$. Theorem 139.28 bounds the integral of the product of the terminal scalar moduli, over the two orbits, the prefix pairs and all positions. The above rewriting preserves those moduli, so Fubini applies before identifying the integrand at c.

Finally put $y' = yx$. The product $dg_0^*(y) \, dx$ becomes $\kappa_G^{-1} dy \, dy'$ in original Haar. Coordinate expansion of the displayed pair of inner products now gives exactly $\langle O_p(c), O_q(c) \rangle_{\mathrm{HS}}$ after integration in $dy \, dy'$. The remaining prefactor on the right side of (139.39) is

$$\frac{\eta^2}{d_v^2 \kappa_G} \, 2^{-a} \kappa(c) \prod_{l<a} |p_l q_l|^{r_{l+1}}.$$

Since $n = a + 1$ and $\eta = \kappa_G = 2^n V_k / V_r$, this is $c_G (|c|^2 - 1)^{k-1} \prod_{l<a} |p_l q_l|^{r_{l+1}} / d_v^2$ on $|c| > 1$, as in (139.38). If σ has the same c, choose $h_1 \in C$ to rotate the first positive direction of $\sigma_c z$ to that of σz. Then $h_2 = (h_1 \sigma_c)^{-1} \sigma$ fixes z, so belongs to C. In the orbital integral with $\sigma = h_1 \sigma_c h_2$, set $y = h_1 y_0$. Compact equivariance and commutation with $w_{\mathbf{p}}$ multiply every tensor, for fixed σ, by the same unitary matrices on its two sides. Its Gram products are therefore unchanged. □

Corollary 139.32 (The complete real-place family of Gram forms)**.** *For $0 \leq j \leq a$ take $L_j(\mathbf{p}, \mathbf{q}) = d_v^{-2} \int_G J_{\mathbf{p},\mathbf{q}}(x; 0) \, dx$. For lists of length n, with respective null prefixes $\mathbf{p}, \mathbf{q}$ and terminal entries c, c', take $L_n = \langle \Xi_p(c), \Xi_q(c') \rangle_{\mathrm{HS}}$. Then on the shorter interval, at each real place, the bottom value satisfies*

$$(1 - b^{-r^3 f})^4 / d_{\pi,v} \ \leq L_0 \leq (1 + b^{-20}) / d_{\pi,v}.$$

At every $j < n$, integration of L_{j+1} in a common next entry yields L_j, absolutely. At a nonzero Fourier frequency ξ the absolute value of that integral with weight ϕ_ξ is bounded by

$$d_v^{-1} S_{k,a} b^{2000 n^9} \, \mathcal{H}(\mathbf{p}) \mathcal{H}(\mathbf{q}) W_0(\xi).$$

At each $1 \leq j \leq a$ the same bound holds for L_j itself when ξ is the nonzero difference of the reciprocal last entries. The top form is continuous and positive semidefinite on any finite set of lists.

Proof. The bottom inequality is (139.15). The common entry identities are Theorem 139.30 up to the last null row and (139.39) at the terminal row. The absolute estimates are (139.33) and Theorem 139.29 after division by d_v^2. Continuity of the orbital tensors was proved above. Every finite quadratic form for the top kernel is the squared Hilbert–Schmidt norm of the corresponding linear combination of those tensors. □

139.19 Packing the rational prefix determinants

Use the shorter period fields $E = \mathbb{Q}(\sqrt{-d_0}, \mu_{q_1})$, $F = E^+$ and $f = [F : \mathbb{Q}]$ of Theorem 139.1. Number the entries in a rational list starting at one in the formulas for the field scales. Write Φ_v for one embedding at each conjugate pair. Put

$$B = b^{60n^2}, \qquad s_{l,v} = \prod_{i=1}^{l} b_{i,v}, \qquad b_{i,v} = \kappa_v^{2(n-i)}/N, \qquad S_l = \left(\prod_v s_{l,v} \right)^{1/f}.$$

These are the scales of Theorem 136.4. For a list $\mathbf{p} = (p_1, \dots, p_j)$ in $E^\times$ let $\delta_l = \prod_{i=1}^{l} p_i$. Its real prefixes at v are $\mathbf{p}_v = (b_{i,v}\Phi_v p_i)_{1 \le i \le j}$. For a finite set T of primes of E, set $I_T(y) = \prod_{\mathfrak{p} \in T} \mathrm{N}\mathfrak{p}^{\mathrm{ord}_\mathfrak{p}\, y}$ and define

$$H_T(\mathbf{p}) = \prod_{l=1}^{j} \mathbf{1}_{\delta_l \text{ integral at } T} I_T(\delta_l)^{n^6/2}, \qquad H(\mathbf{p}) = H_T(\mathbf{p}) \prod_v \mathcal{H}(\mathbf{p}_v).$$

The local real weight $\mathcal{H}$ is that in Theorem 139.28. These definitions keep the finite weights at their original values even when a prefix has zeros at primes in T.

Theorem 139.33 (A field-uniform sum of the coupled prefix weights)**.** *For $0 \le j \le a$ sum over all lists of length j whose prefix products are integral outside T. Then*

$$\sum_{\mathbf{p}} H(\mathbf{p}) \le \prod_{l=1}^{j} (1 + 2B/S_l)^{2f} \max(1, B/S_l)^{n^6 f} \le b^{130 n^9 f} \tag{139.40}$$

at sufficiently large nodes, uniformly in T and j. Only finitely many terms have positive weight. In particular the double sum of the product of two such weights is at most $b^{260n^9 f} = b^{o(kf)}$, also with arbitrary further restrictions on the pair of heads.

Proof. A positive finite weight makes every δ_l integral at the primes of T; the hypothesis gives integrality elsewhere. Thus each δ_l is a nonzero algebraic integer. For any such δ,

$$I_T(\delta)^{1/2} \le |\mathrm{N}_{E/\mathbb{Q}}\delta|^{1/2} = \prod_v |\Phi_v \delta|.$$

For $\delta = \delta_l$, multiply this inequality by the product over all real places of $\max(1, |s_{l,v}\Phi_v\delta|^{-1})$. The right side becomes $\prod_v s_{l,v}^{-1} \max(1, |s_{l,v}\Phi_v\delta|)$, which is at most $(B/S_l)^f$ on the support. Raising to n^6 bounds the combined weight of this prefix. This uses the product of the actual local scales and allows them to differ at the various embeddings.

It remains to count the possible δ_l. Map $\delta \in \mathcal{O}_E$ into $\mathbb{C}^f$ by the coordinates $(s_{l,v}/S_l)\Phi_v\delta$. Distinct images have distance at least one in the maximum of the complex absolute values, since the product of the squared absolute differences is the absolute norm of a nonzero algebraic integer. Positive weight places every coordinate in the disc of radius B/S_l. The products of open discs of radius $1/2$ centered at these images are disjoint and lie in the product of discs of radius $B/S_l + 1/2$. Comparison of ordinary Lebesgue volumes gives at most $(1 + 2B/S_l)^{2f}$ images. This argument on finite subsets also proves finiteness. The tuple of the j successive determinants determines the list, using $p_l = \delta_l/\delta_{l-1}$ with $\delta_0 = 1$. Counting all the tuples and multiplying the weight bounds gives the first inequality and the assertion for the empty list.

On our fields $Q_E = (\prod_v \kappa_v)^{1/f} = M\rho_E/(2\pi) \ge 1$, where $\rho_E = |\operatorname{disc} E|^{1/(2f)}$. Hence

$$S_l = N^{-l} Q_E^{2nl - l(l+1)} \ge N^{-l}.$$

The calibrated choice in Lemma 133.8 satisfies $N \le b^{60n}$ at large nodes; its upper endpoint there is at most $b^{24(2n)}e^{\sqrt{\log b}}$. Consequently $B/S_l \le b^{120n^2}$ for $l \le a$. For $b \ge 3$ we have $1 + 2b^{120n^2} \le b^{120n^2+1}$. The logarithm to base b of the first bound in (139.40) is at most $jf(120n^8 + 240n^2 + 2) \le 130n^9 f$ for large n. Finally $n^9/k \to 0$ on this same interval, as in Corollary 139.12. Squaring the bound proves the assertion for pairs, since all weights are nonnegative. □

Here is the geometric reason for integrality on the finite support. At a split or unramified place with the integral lattice of Theorem 139.1, two flags arising from points of UK, with K the integral isometry group, have simultaneous integral representatives. Indeed each point has a representative in K after a left unipotent translation, and that translation changes each of the first columns only by earlier columns. The cross Gram determinant of the first l columns is thus integral; it is invariant under both column changes. The same fact holds when K is replaced by any subgroup of K on a selected split mark. This proves the integrality assertion for the determinants even if some of the individual ratios of successive determinants have poles. Theorem 139.33 is an estimate for weights of these rational lists. The finite correlation at a prescribed list is a separate integral.

139.20 Rational frequency errors in units of the identity mass

The normalized real wave bound contains the factor $S^f_{k,a}/d_B$, where $d_B = \prod_v d_v$. The product of the real bottom forms is of order $1/D_\pi$, with $D_\pi = \prod_v d_{\pi,v}$. We compare these scales before summing the rational frequencies. Use $m = 2n$, $h = r - k = m - 1$, $K = k - n^7$, the terminal prime N and the geometric mean w of the w_v in Lemma 133.8.

Lemma 139.34 (A formal-degree bound matched to the frequency scale)**.** *On these fields and this interval,*

$$D_\pi S^f_{k,a} \le w^{m(k-1)f}\, b^{o(kf)}.$$

Consequently, for each of the two choices of c_v in Theorem 136.4, at any one of the n rows and for any finite set T of primes of E,

$$D_\pi\, S^f_{k,a} \sum_{0\neq y\in\mathcal{O}_{E,T}} W_{T,c,K}(y) \le b^{-(25/2-o(1))kf}.$$

The terms tending to zero are uniform in the row and T.

Proof. By Theorem 94.9, the formal degree is the compact Weyl dimension at the same regular parameter. Its first $k-1$ entries are consecutive and the other m entries are $-w_v - 2lr^8$, $1 \le l \le m$, by Proposition 132.1. In the Vandermonde formula the first block cancels the denominator factorials $1!, \dots, (k-2)!$. The remaining denominator is $\prod_{t=k-1}^{r-1} t!$. Each of the $m(k-1)$ mixed gaps is at most $w_v + 2mr^8 + k$, and the product of the gaps in the last block is $(2r^8)^{m(m-1)/2} \prod_{l=1}^{m-1} l!$. On the other hand the compact volume formula gives exactly

$$S_{k,a} = \pi^{hk} V_k / V_r = 2^{-h} \pi^{-h(h+1)/2} \prod_{t=k}^{r-1} t!.$$

All but $(k-1)!$ in the remaining denominator cancel. Using $\log(1+t) \le t$, summing over v, and discarding the remaining denominator for an upper bound gives

$$\log(D_\pi S^f_{k,a}) \le fm(k-1)\log w + O\!\left(fm^2 \log(2r) + \frac{fmk(2mr^8+k)}{\min_v w_v} \right).$$

Here $\min_v w_v \ge b^{1-10^{-6}}$, $m = O(\sqrt{u})$, $r = O(u^6)$, $k \asymp u^5$ and $\log b \ge (\log u)^6$ at the large nodes of size u; the same weight estimates were used in Lemma 134.3. Both error terms are $o(kf \log b)$, giving the first assertion.

Use the first, scale-dependent bound in (136.5). After multiplication by the first assertion, the factors involving M, w, b, apart from $b^{o(kf)}$, are

$$C^f (b^{15/2}/M)^{Kf} w^{m(n^7-1)f}.$$

Since $M \ge b^{20}$, $1 \le w \le b$, $mn^7 = o(k)$ and $n^7 = o(k)$, their product is at most $b^{-(25/2-o(1))kf}$. The frequency bound was uniform in T and the row, so this proves the claim. □

For the scaled heads $\mathbf{p}_v, \mathbf{q}_v$ of Section 139.19 abbreviate the actual real mass by

$$A_{j,v}(\mathbf{p},\mathbf{q};\nu) = d_v^{-2} \int_G |J_{\mathbf{p}_v,\mathbf{q}_v}(x;\nu)|\, dx.$$

Write $V_T(y) = I_T(y)^{K/2} D_T(y)^{-3}$, using the denominator of (136.4). Every rational list below has its prefix products integral off T. For $j = 0$ use the empty list.

Theorem 139.35 (Summed rational head and frequency bounds at the identity scale)**.** *For all sufficiently large nodes, uniformly in $0 \le j \le a$ and finite T, put $c_v = 2\pi i/(\Phi_v(\delta_E)b_{j+1,v})$. Then*

$$D_\pi \sum_{\mathbf{p},\mathbf{q}} H_T(\mathbf{p})H_T(\mathbf{q}) \sum_{0\neq y\in\mathcal{O}_{E,T}} V_T(y)\prod_v A_{j,v}(\mathbf{p},\mathbf{q};c_v\Phi_v y) \le b^{-12kf}. \tag{139.41}$$

For $1 \le j \le a$, restrict to lists with unequal last entries and $y = (p_j^{-1} - q_j^{-1})/M$ integral off T. Then the same bound holds for

$$D_\pi \sum_{\mathbf{p},\mathbf{q}} H_T(\mathbf{p})H_T(\mathbf{q})V_T(y)\prod_v A_{j,v}(\mathbf{p},\mathbf{q};0).$$

Both conclusions allow arbitrary further restrictions on the summation sets.

Proof. At nonzero frequency, Theorem 139.28 bounds the product of real masses by

$$\frac{S_{k,a}^f}{d_B}\, b^{2000n^9 f}\prod_v \mathcal{H}(\mathbf{p}_v)\mathcal{H}(\mathbf{q}_v)\prod_v W_0(c_v\Phi_v y).$$

Together with $V_T(y)$ the frequency factors form precisely $W_{T,c,K}(y)$. The remaining head factors, including the two finite weights in the theorem, sum to at most $b^{260n^9 f}$ by Theorem 139.33. For unequal last entries Theorem 139.29 gives the same estimate for the constant-weight masses, since the reciprocal difference at v equals $c_v\Phi_v y$ with $c_v = M/b_{j,v}$. For each pair in this case the single nonnegative $W_{T,c,K}(y)$ is bounded by its sum over all the nonzero T-integers. Thus both cases are bounded by $d_B^{-1}b^{2260n^9 f}b^{-(25/2-o(1))kf}$ using Lemma 139.34. Now $d_B \ge 1$ and $n^9/k \to 0$; this proves the stated inequalities. No multiplicity assumption on the rational prefixes is required. □

Corollary 139.36 (The actual first-frequency error at the original mass scale)**.** *For the original selected and unselected marks $\Xi, \mathcal{V}$ of Corollary 139.12, write $c_v = 2\pi i/(\Phi_v(\delta_E)b_{1,v})$. On the shorter interval, at sufficiently large nodes,*

$$D_\pi \sum_{\substack{0\neq y\in E\\ y \text{ integral above } \mathcal{V}}} \prod_v A_{0,v}(\varnothing,\varnothing;c_v\Phi_v y)\prod_{t\in\Xi}\frac{B_0 q_t^{n-j_t}\mathcal{B}_{Z_t}(y)}{Z_t^0}\prod_{t\in\mathcal{U}}\mathcal{B}_t^0(y) \;\le\; b^{-7kf}.$$

Proof. First take T conjugation-stable, containing all selected places and no primes above the unselected marks. For rational T-integers the product of finite factors divided by $V_T(y)$ is at most $\exp(O(\epsilon))b^{5kf}$. For the unmarked factors this follows from (139.1): at a pole use $\min(1, V_t) \le V_t$, and at an integral prime under T one has $V_t \ge 1$. Outside T the frequency is integral. At a selected mark the proof of (137.11) bounds $\mathcal{B}_{Z_t}(y) \le V_t(y)$, and its remaining normalizing factors are at most b^{18k} per place. There are fewer than $f/4$ such places. In fact this gives the sharper exponent $(9/2)kf$, which absorbs the bounded $\exp(O(\epsilon))$ within b^{5kf} at large nodes. Apply (139.41) at $j = 0$ and increase T through all allowed finite sets. The actual summands are nonnegative, independent of the auxiliary set, and every allowed rational frequency occurs. Monotone convergence proves the inequality. By Corollary 139.32 the product of the real bottom forms is $(1 + o(1))/D_\pi$. Thus this estimates the nonzero first frequencies relative to the mass used by the period comparison. At larger depths (139.41) still uses the finite weights V_T, H_T; identifying them as majorants for the original finite integrals at those depths is an additional requirement. □

139.21 Removing unit prefixes at the standard finite places

Fix one unmarked place w of the period fields in Theorem 139.1. Write O for the integers of F_w, q for their residue cardinality, and $A = E \otimes_F F_w$ with integers O_A. Thus A is split or unramified quadratic; use the same uniformizer π as in O. Give O_A additive volume one, also in the split case. Let G_i, K_i, U_i be the residual isometry group, integral lattice stabilizer and Bessel unipotent after the first i hyperbolic pairs, and $r_i = k + 2(a - i) + 1$. Normalize K_i and $U_i \cap K_i$ to have volume one in their respective groups. The finite test at this level is the scalar

$$W_i(uk) = \chi_i(u) \quad (u \in U_i,\ k \in K_i), \qquad W_i = 0 \text{ off } U_i K_i,$$

where χ_i is the restriction of the original character. It is trivial on $U_i \cap K_i$. For $\ell \in G_i$, set $K_{i,\ell}(y) = \overline{W_i(y)} W_i(y\ell)$. The unconditioned first form of the residual group is

$$b_i(\ell; \nu) = \iint \psi\Big(\nu\, h(y^{-1} f_i, t^{-1} f_i)\Big)\, K_{i,\ell}(y) \overline{K_{i,\ell}(t)}\, d\dot{y}\, d\dot{t},$$

using the two quotients by U_i, with $f_i = e_i$ for $i < a$ and $f_a = z$. Here ψ is the additive character of A with integer-ring annihilator used in Section 139.1. At $i = 0$ this agrees with (137.3) at $H = J = K_0$: the first amplitudes restrict the two representatives to K_0, and the last right factor in each conjugation may be removed by right K_0-invariance.

Lemma 139.37 (Haar density of a list of unit pivots)**.** *Take two independent length $j \le a$ flags from Haar on K_0. Use the ordered quotient vectors and the ratios of leading Gram determinants of Section 139.13, over A. On $(O_A^\times)^j$ their list of pivots has constant density, relative to product additive Haar, given by*

$$c_{j,w} = \prod_{i<j} \theta_{r_i,w}^{-1}, \qquad \theta_{d,w} = \begin{cases} (1 - q^{-d})(1 - q^{1-d}) & A \text{ split}, \\ 1 + (q-1)(-1)^d q^{-d} - q^{1-2d} & A \text{ unramified field}. \end{cases}$$

At each prescribed unit list the conditional measure on the compact pair orbit is $c_{j,w}$ times invariant probability.

Proof. Fix the first vector of one of the flags and take it by an integral isometry to e. At a null row of dimension d, work modulo π^t, $t \ge 1$. In the unramified field case the second vector is uniform on the primitive solutions of the null equation. Their number is $q^{t(2d-1)}\theta_{d,w}$ by Lemma 138.1, whose proof uses the surjective residue trace also in residue characteristic two. For a prescribed unit $p \bmod \pi^t$, write the vector as $\alpha e + y + pe^*$. Each of the $q^{2t(d-2)}$ middle vectors y leaves one trace equation for α, with q^t solutions. All these vectors are primitive.

In the split case reduction of $\mathrm{GL}_d(O)$ is onto. The orbit consists of primitive pairs (u, ℓ) of a vector and a covector with $\ell(u) = 0$ in O/π^t. Indeed take u to the first coordinate vector, and use the general linear group of the remaining coordinates on the primitive covector annihilating it. There are $q^{td}(1 - q^{-d})q^{t(d-1)}(1 - q^{1-d})$ choices. With the fixed pair equal to the first vector and the last covector, the two prescribed unit components of p fix u_d and ℓ_1. We can choose u_1 and both middle lists freely and solve for ℓ_d using the unit u_d. This again gives $q^{t(2d-3)}$ solutions, all primitive in both components. In either case the probability for the specified class is $q^{-2t}/\theta_{d,w}$. As every such additive class has measure q^{-2t}, this gives the one-row density on all units.

For the recursion with p a unit, $t_0 = y/p$ and $s = \alpha/p$ are integral and satisfy $s + \bar{s} = -h(t_0, t_0)$. The integral Heisenberg element fixing e, taking e^* to $e^* + t_0 + se$ and taking v in the middle space to $v - h(t_0, v)e$, has integral inverse. It carries pe^* to the second vector. Thus one integral pair section identifies their common orthogonal complement with the original

residual lattice. The integral stabilizer of one fixed vector projects onto K_{i+1} by its middle block, with integral Heisenberg kernel, by the block formula just used. Conditional on the two first vectors, each remaining flag therefore has the Haar law of its residual compact group. They are independent, and changing the two sections to common integral coordinates preserves each law. The next relative pivots are those of the Schur complement after the first column, by successive elimination. Induction proves the product formula and transitivity of K_0 on the integral pair orbit at the prescribed list. Invariance identifies its conditional probability with Haar probability. The unit-fiber normalization can be taken pointwise: in the coordinates above the middle entries are free, the trace equation has a fixed integral linear section and a free trace-zero coordinate, and divisions by p are by units. These are local product charts over the unit p; in the split case one may instead solve directly for ℓ_d. Consequently the coarea densities are locally constant in these charts and the finite-precision computation fixes their normalization at every unit list. □

For clarity, use the two-endpoint form with the same coarea convention as in Section 139.15. On $X_j = (N_{<j} \rtimes G_j)\backslash G_0$ the Haar quotient of the flag stabilizer is normalized to give its compact K_0-orbit mass one. This is the measure obtained by disintegrating $U_0\backslash G_0$ using the above measures on $U_j\backslash G_j$: all the subgroup intersections with the indicated compact groups have mass one in the successive Heisenberg and residual decompositions. Let $\mu_{\mathbf{p}}$ be the coarea of the product of these flag measures at $\mathbf{p}$. For initial pair P write τ_P for a section taking its normalized columns to $e_l, p_l e_l^*$. All pivots in the following theorem are units. Let $T = w_{\mathbf{p}}^{-1}\tau_P$ and $d_{\boldsymbol{\lambda}} = w_{\mathbf{q}}^{-1} w_{\mathbf{p}}$ with $\lambda_l = p_l/q_l$. At fixed P, x integrate the residual wave $\psi(\nu h(m^{-1}f_j, t^{-1}f_j))$ against the product

$$\overline{W_0(m\tau_P)} W_0(m\tau_P x)\, W_0(tT)\overline{W_0(td_{\boldsymbol{\lambda}} T x)}$$

on the two residual quotients $U_j\backslash G_j$. Denote that integral by $C(P, x; \nu)$ and its $\mu_{\mathbf{p}}$ integral by $J_w(x;\nu)$. Here W_0 denotes the finite test at depth zero. The endpoint coarea factor relative to $\mu_{\mathbf{p}}$ is one, since every p_l, q_l has multiplicative modulus one. Thus the equal-head case is the original condition on the Gram pivots. The second flag in the mixed case uses sections $w_{\mathbf{p}}^{-1}\tau_P$ and $w_{\mathbf{q}}^{-1}\tau_P x$, retaining its original two endpoint tests. Right invariance of the quotient measures and common transport of the residual wave make the integral independent of the residual choice of pair section.

Theorem 139.38 (Reduction of the unit-prefix integral on the whole position group)**.** *For prescribed unit lists $\mathbf{p}, \mathbf{q}$ of equal length $j \le a$,*

$$\int_{G_0} \int |C(P, x; \nu)|\, d\mu_{\mathbf{p}}(P)\, dx = c_{j,w} \int_{G_j} |b_j(\ell; \nu)|\, d\ell.$$

This bounds $\int |J_w(x;\nu)|\, dx$. If the right side is finite the signed complex integrals also satisfy $\int J_w(x;\nu)\, dx = c_{j,w} \int b_j(\ell;\nu)\, d\ell$. Both identities hold with the respective positions restricted to the integral groups on the two sides or to their complements.

Proof. The two initial flags on a contributing pair are in the image of K_0. This follows from W_0 being supported on $U_0 K_0$; the leading scaled columns of the quotient depend only on the flag. Lemma 139.37 allows a common section $\tau_P \in K_0$ for that pair. The two final flags are represented by $x^{-1}A$ and the columns $x^{-1}B_l/\lambda_l$, with their earlier column changes understood as in Proposition 139.27. They too must lie in the compact flag orbit or the amplitude is zero. Their pivots are $\mathbf{q}$. Choose an integral section $\tau_{P'}$ at this final pair. Then

$$\ell = \tau_P x \tau_{P'}^{-1} \in G_j, \qquad d_{\boldsymbol{\lambda}} w_{\mathbf{p}}^{-1} \tau_P x = \ell w_{\mathbf{q}}^{-1} \tau_{P'}.$$

Indeed ℓ fixes each of the $2j$ earlier normalized columns, and the swaps commute with its middle action. In particular $x \in K_0$ if and only if $\ell \in K_j$.

In restriction to G_j the original function W_0 equals W_j. If $um \in K_0$ for $m \in G_j$ and $u \in U_0$, its middle block is $u_j m$ integral, with $u_j \in U_j$. This block is itself an isometry of the residual self-dual lattice; its inverse is integral as well. Conversely $m \in U_j K_j$ is already in $U_0 K_0$, and its character value is χ_j by restriction. Every section and Weyl swap appearing after ℓ in the two branch formulas is integral. Right K_0-invariance therefore reduces the entire residual integral to $b_j(\ell;\nu)$. This retains the oscillation in the two quotient variables even on nonintegral positions.

On the initial pair orbit $d\mu_{\mathbf{p}}$ restricts to $c_{j,w}$ times the probability measure. For fixed P, disintegrate the position by the Haar quotient $G_j\backslash G_0$ of $\tau_P x$; the residual measure is $d\ell$. The coset corresponding to P' is represented by $\tau_{P'}$. The condition of an integral final flag pair is exactly that this coset meet K_0, by the integral-section construction. Its quotient measure is one, since K_0 and $K_j = K_0 \cap G_j$ both have mass one. Integration gives the asserted formula using Tonelli for absolute values, and Fubini for the complex integral when finite. The same argument with the compact position cutoff uses the equivalence $x \in K_0 \iff \ell \in K_j$ above. $\square$

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The local factors use the marked Bessel character of Section 130 and the Haar measures of Section 131. The construction and normalization of the real test vectors are in Section 132. The global second moment and its terminal norm constraint are in Sections 133 and 134.