The generalized Riemann hypothesis for Dirichlet L-functions

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Volume XXV

Rational flags and the second moment of Bessel periods

Abstract

We count pairs of rational flags, separate the open and closed orbital integrals and construct the real test vector near a stationary point. The resulting Poincaré series give a second moment on the marked classes. Its terminal norm constraint leads to the local correlations in the next volume.

The guide in Volume I lists all twenty-seven volumes.

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130 Rational flags at the two marked primes

130.1 The two level projections on definite automorphic forms

Consider the family $E = \mathbb{Q}(\sqrt{-d}, \mu_{q_1})$ of Corollary 11.30, and put $F = E^+$, $f = [F : \mathbb{Q}]$. Write r for the rank of the unitary group in this family. Its two distinct level primes p_0, p_1 lie in $[b^{10}, b^{11}]$ and are one modulo Q^2, where $Q = (q_0 d)^{s+6}$ is a multiple of $q_1 d$. Thus each splits completely in E, by its action on cyclotomic roots. At either prime there are f places of F, with residue field $\mathbb{F}_{p_i}$; choose one of the two places of E over each. The two residue groups are

$$H_i = \mathrm{GL}_r(\mathbb{F}_{p_i})^f, \qquad H = H_0 \times H_1.$$

For a field $\mathbb{F}_q$ and $r \geq 2$, call a monic degree-r polynomial *allowed* if it is squarefree, has nonzero constant term, and has no root in $\{a \in \mathbb{F}_q^\times : a^Q = 1\}$. Call it *primitive* if it is the minimal polynomial of a generator of $\mathbb{F}_{q^r}^\times$. Such a polynomial has degree r and is allowed. Green's parametrization attaches distinct irreducible characters to the squarefree degree-r polynomials of nonzero constant term. In this parametrization one fixes compatible generators for the character groups of the finite tori. For a polynomial with irreducible factor degrees $d_1, \dots, d_k$, take the one-box partition on each corresponding Frobenius orbit. The resulting character χ_c has degree

$$\chi_c(1) = \frac{\prod_{j=1}^{r}(q^j - 1)}{\prod_{j=1}^{k}(q^{d_j} - 1)}.$$

This is the squarefree case of Green's degree and classification theorems. [1, §8, Thms. 13–14]

In $\mathbb{C}[H_i]$ let e_i be the sum of central character idempotents for the f-tuples of allowed labels having at least one primitive member. Write $e = e_0 e_1$, viewing each factor in $\mathbb{C}[H]$. We compute the fraction δ_i that e_i discards in the regular representation and then evaluate its rank on the definite automorphic space.

Proposition 130.1 (Plancherel mass at one level prime)**.** *Let $q = p_i$, $r \geq 2$, $q \geq 3$, and let A be the sum of $\chi_c(1)^2/|\mathrm{GL}_r(\mathbb{F}_q)|$ over allowed c. Put $n = q^r - 1$. Then*

$$P = \frac{\varphi(n)}{rn} \prod_{j=1}^{r-1}(1 - q^{-j}), \qquad \delta_i = 1 - A^f + (A - P)^f.$$

In particular

$$\delta_i \leq f\left(\frac{2 + \gcd(Q, q-1)}{q} + \frac{1}{q-1}\right) + \exp\!\left(-\frac{f}{4r(1 + r\log q)}\right).$$

For every sufficiently large node of the stated family, $\delta_0 + \delta_1 \leq b^{-8}$, uniformly for its prescribed level primes.

Proof. Counting ordered bases gives $|\mathrm{GL}_r(\mathbb{F}_q)| = q^{r(r-1)/2}\prod_{j=1}^{r}(q^j - 1)$. Consequently each squarefree label has mass

$$\frac{\chi_c(1)^2}{|\mathrm{GL}_r(\mathbb{F}_q)|} = q^{-r}\frac{\prod_{j=1}^{r}(1 - q^{-j})}{\prod_{j=1}^{k}(1 - q^{-d_j})^2} \geq q^{-r}\Big(1 - \frac{1}{q-1}\Big).$$

There are $\varphi(n)/r$ primitive labels. Indeed a generator of the multiplicative group cannot lie in a smaller subfield, and its r Frobenius conjugates have the same minimal polynomial. Substitution of $k = 1, d_1 = r$ in the exact mass formula gives their total mass P.

Among the q^r monic polynomials, exactly q^{r-1} have a repeated factor. To see this, unique factorization gives $\prod_L (1+z^{\deg L}) = (1-qz^2)/(1-qz)$ for monic irreducibles L, by taking the quotient of the two Euler products for all monic polynomials at z and z^2. Each prescribed root excludes q^{r-1} polynomials. There are $1+\gcd(Q, q-1)$ excluded roots including zero. Taking the union bound and the lower mass per allowed label gives

$$1 - A \le (2 + \gcd(Q, q-1))/q + 1/(q-1).$$

This follows directly when the root-count lower bound is positive; otherwise the displayed upper bound is at least one. The ranks of distinct character idempotents add, and Plancherel masses multiply in a direct product. All labels being allowed has probability A^f; all being allowed with none primitive has probability $(A-P)^f$. These events give the asserted equality for δ_i.

For any positive integer n,

$$\frac{n}{\varphi(n)} = \prod_{p \mid n} \frac{1+p^{-1}}{1-p^{-2}} \le \zeta(2) \sum_{d \mid n} d^{-1} \le 2(1+\log n).$$

Also $\prod_{j=1}^{r-1}(1-q^{-j}) \ge 1-1/(q-1) \ge 1/2$. Thus $P \ge 1/[4r(1+r\log q)]$. Using $1-A^f \le f(1-A)$ and $(A-P)^f \le e^{-fP}$ proves the inequality.

We verify both terms on the specified fields. Since E contains $\mathbb{Q}(\mu_{2^s})$ and lies in $\mathbb{Q}(\mu_{q_1 d})$, Corollary 11.30 gives

$$x^{11}/4 \le 2^{s-2} \le f \ll_{q_0,d} x^{11}\log(2x).$$

Here $2^s = m^{22}$, $m \ge \sqrt{x}$. Also $s = O(\log x)$, $r \le 3x^5$, $(\log x)^6 \le \log b \le x^{1/100}$, and $10\log b \le \log q \le 11\log b$. Thus Q grows at most polynomially in x for fixed q_0, d, whereas b exceeds every such polynomial eventually. The first term is at most b^{-9} for large x. For the second, $r(1+r\log q)\log b \ll x^{10+2/100}$; hence $f \ge 36r(1+r\log q)\log b$ for all sufficiently large x. Each $\delta_i \le 2b^{-9}$, proving the claim. □

Theorem 130.2 (Dimension and integral index on the definite endpoint)**.** *Let the unitary group of rank r over F be definite at all real places. Take the standard compact level at the two split primes and any fixed compact open level away from them. Let K impose principal reduction at both p_0, p_1. For algebraic automorphic forms V of level K and any fixed finite-dimensional archimedean weight, one has*

$$\dim(1-e_i)V = \delta_i \dim V, \qquad \dim(1-e)V \le (\delta_0+\delta_1)\dim V \le b^{-8}\dim V$$

at sufficiently large nodes. The idempotent e lies in $|H|^{-1}\mathbb{Z}[H]$. If Λ_ℓ is any H-stable full lattice in a characteristic-zero model of V over a finite extension of $\mathbb{Q}_\ell$, put

$$I_\ell = e\Lambda_\ell/(\Lambda_\ell \cap eV_\ell), \qquad d_\ell = \textit{ramification index of the coefficient field.}$$

For arbitrary such coefficient fields and lattices at each prime,

$$\sum_\ell \frac{\operatorname{length}_{\mathcal{O}_\ell} I_\ell}{d_\ell}\log \ell \le b^{-8}\dim V\, \log|H|, \qquad \log|H| \ll_{q_0,d} x^{21}\log(2x)\log b.$$

Proof. Let $K^{(i)}$ restore the full compact group at p_i while retaining principal reduction at the other prime. Reduction of invertible integral matrices is onto, by lifting their entries and using the determinant. Hence $K^{(i)}/K = H_i$. Its right action on the finite level-K class set is free. Indeed, for a finite adelic representative g, the group of rational points lying in $gK^{(i)}g^{-1}$ is finite. The diagonal rational points are closed and discrete in the full adelic group, and both

the indicated finite-adelic set and the entire archimedean group are compact. At a place over p_{1-i}, an element of this finite group is conjugate into $1 + p_{1-i}M_r(\mathbb{Z}_{p_{1-i}})$. This group is torsion free for odd $p = p_{1-i}$. In fact, if $C = 1 + p^a B$ with $a \geq 1$ and B nonzero modulo p, the first nonzero term of $C^u - 1$ has valuation a when $p \nmid u$, and that of $C^p - 1$ has valuation $a + 1$ by the binomial formula. Iteration rules out every finite order. Thus the rational element is the identity at one completion, and hence is the identity. If a lift in $K^{(i)}$ fixes the double coset of g, this trivial intersection puts the lift in K, proving freeness.

The level-K forms have the full archimedean weight space as fiber over each double coset, since its stabilizer is also trivial. The finite class set here is the usual class set of positive-definite Hermitian lattices at a compact open level. On a free H_i-orbit identify each fiber with the fiber over one base point by translation. The sections over this orbit then form copies of the regular representation of H_i. The coefficient at one of a group algebra projection is its normalized trace on that representation. This proves the single-prime rank formula. Since e_0, e_1 are commuting orthogonal projections (choose an invariant inner product), the kernel of their product is the sum of their kernels. Proposition 130.1 gives the joint bound.

For an irreducible character χ the coefficient of the central idempotent at h is $\chi(1)\overline{\chi(h)}/|H|$, with H here the whole product and χ a tensor product character. Its numerator is an algebraic integer. The indicated set of characters is stable under every automorphism of the character-value cyclotomic field. Indeed Green's formula takes a Galois power on the roots of unity to the same power on the torus character labels. This permutes Frobenius orbits and preserves their length, primitive order, and the property of being a degree-one root of order dividing Q. Thus summing the numerators gives rational algebraic integers, as asserted. In particular both $|H|e$ and $|H|(1 - e)$ preserve every H-stable lattice.

The map $\lambda \mapsto e\lambda$ identifies I_ℓ with the quotient of Λ_ℓ by the sum of its intersections with the two eigenspaces. Indeed it is onto, and $e\lambda$ lies in the lattice precisely when both components of λ do. Using $1 - e$ instead gives the same quotient. It follows that I_ℓ is killed by $|H|$ and can be generated by at most $\dim(1 - e)V$ elements, since the image of the lattice under $1 - e$ is free of that rank over the coefficient valuation ring. The elementary divisor theorem bounds its normalized length by $\dim(1 - e)V \, v_\ell(|H|)$. Sum and use integer factorization. Finally, $\log |H| \leq f r^2 \log(p_0 p_1)$ by counting all matrices, and the estimates for f, r, p_i above give the asserted order. □

The rank computation uses the other prime's level each time it recovers a regular representation. It therefore applies to the definite space with both markings on the same field as Corollary 11.30. Estimating the norm of a given projected Poincaré series further requires its mass in these isotypic components. Projection to the global subspace of theta lifts also introduces the congruences among automorphic representations.

130.2 The finite Bessel character at the two level primes

We compute the character average that occurs at the split places in the Bessel period of the shorter construction. Its Fourier transform counts vectors for which the first powers of a matrix are linearly independent. This count controls the effect of the finite level projector on the Bessel character itself.

Let V be an r-dimensional space over $k = \mathbb{F}_q$. Fix $a \geq 1$ and $c = r - 2a \geq 1$, and choose ordered blocks of sizes $1, \dots, 1, c, 1, \dots, 1$, with a lines on each side. Let $\mathfrak{p}$ be the block upper triangular algebra, $\mathfrak{u}$ its strictly block upper triangular ideal, $U = 1 + \mathfrak{u}$, and $G = \mathrm{GL}(V)$. Fix a nontrivial additive character ϑ of k. Choose D with blocks immediately below the diagonal as follows. On consecutive lines in each arm the map is the identity in the specified coordinates.

The map from the last left line to the middle space is a vector w, and the map from the middle space to the first right line is a functional u. Suppose $\gamma = u(w) \neq 0$. Set its other blocks to zero. Nonzero coefficients along either arm can be put into this form by rescaling the line bases, which preserves nonvanishing of the middle pairing. Define

$$\psi(1 + X) = \vartheta(\mathrm{tr}(DX)) \quad (X \in \mathfrak{u}).$$

It is a group character since $\mathrm{tr}(DXY) = 0$ for $X, Y \in \mathfrak{u}$. Extend it by zero to G, and let $\psi^{\#}(h) = |G|^{-1} \sum_{g \in G} \psi(g^{-1}hg)$.

Theorem 130.3 (The Fourier transform of the finite Bessel character)**.** *Set $F(X) = \psi^{\#}(1+X)$ on matrices where $1 + X$ is invertible and set it to zero elsewhere. Use the transform*

$$\widehat{F}(A) = \sum_{X \in \mathrm{End}(V)} F(X)\overline{\vartheta(\mathrm{tr}(AX))}.$$

Write k_A for the fraction of vectors $v \in V$ for which $v, Av, \dots, A^{2a}v$ are independent. Then

$$\widehat{F}(A) = k_A / B_{q,r,a}, \qquad B_{q,r,a} = \prod_{j=0}^{2a} (1 - q^{j-r}), \qquad \sum_{\substack{h \in G \\ h \neq 1}} |\psi^{\#}(h)|^2 \leq B_{q,r,a}^{-1} - 1.$$

Proof. Before averaging over conjugates, the function on matrices is $\vartheta(\mathrm{tr}(DX))1_{\mathfrak{u}}(X)$. Each $1 + X$ in its support is invertible. The trace pairing identifies the annihilator of $\mathfrak{u}$ with $\mathfrak{p}$: testing one entry of a strict upper block tests the transposed lower block. Orthogonality of additive characters therefore gives transform $|\mathfrak{u}|1_{D+\mathfrak{p}}$. Fourier transformation commutes with conjugation. Hence $\widehat{F}(A)$ is $|\mathfrak{u}|$ times the probability that the matrix of A in a uniformly chosen basis belongs to $D + \mathfrak{p}$.

We count that probability. Record from a basis its first a columns $v_1, \dots, v_a$ and its last a coordinate functionals $\lambda_1, \dots, \lambda_a$, starting with the last functional and proceeding backwards. Write L_j for the span of the first j columns, R_j for the span of the first j of these functionals, and $W = \bigcap_i \ker \lambda_i$. Then $L_a \subset W$ and $C = W/L_a$ has dimension c. The basis identifies the original middle space with C, and thereby gives $w' \in C$ and $u' \in C^*$ with $u'(w') = \gamma$. Every tuple of independent frames with these annihilations and every such pair in C has the same number of basis lifts. Indeed one may choose a basis of the kernel of u', complement it by w', lift a resulting middle basis to W, and complete it by lifts of the basis dual to the λ_i. Equivalently G acts transitively on these data and the map from bases is equivariant.

The condition on the blocks of A in this basis says exactly

$$\begin{aligned} Av_j - v_{j+1} \in L_j, \qquad & \lambda_j A - \lambda_{j+1} \in R_j \quad (1 \leq j < a), \\ Av_a \in W, \qquad & [Av_a] = w' \text{ in } C, \\ (\lambda_a A)|_W = u' \circ (W \longrightarrow C). & \end{aligned}$$

For example the first line fixes one subdiagonal entry and kills all entries below it in the corresponding column or to its left in the corresponding row. The two last conditions fix the two blocks adjacent to C and kill the remaining blocks below them. Other entries belong to $\mathfrak{p}$.

Put $v = v_1, \lambda = \lambda_1$. The first line makes v_j a monic polynomial of degree $j - 1$ in A applied to v. It gives the analogous monic polynomial on the right of λ for λ_j. The equations $\lambda_i(v_j) = 0$ successively give $\lambda A^h v = 0$ through $h = 2a-2$, since every such h equals $i+j-2$ for $1 \leq i, j \leq a$. The condition $Av_a \in W$ gives the next vanishing. Finally $\lambda_a A^2 v_a = u'(w') = \gamma$ gives

$$\lambda A^h v = 0 \quad (0 \leq h < 2a), \qquad \lambda A^{2a} v = \gamma.$$

Conversely these moment equations force independence of $v, Av, \dots, A^{2a}v$, because a relation ending at an earlier power would express $A^{2a}v$ in terms of earlier powers. The same argument works for $\lambda, \lambda A, \dots, \lambda A^{2a}$. Choose v_{j+1} freely in $Av_j + L_j$ and choose λ_{j+1} freely in $\lambda_j A + R_j$. The moment equations give all required annihilations, put Av_a in W, and make $\lambda_a A$ vanish on L_a. Its induced functional on C pairs with $[Av_a]$ to γ, so u', w' are determined. Each solution (v, λ) gives exactly $q^{a(a-1)}$ tuples, by multiplying the two sets of choices of q^j at each step.

For each of the $q^r k_A$ independent starting vectors there are exactly q^{r-2a-1} functionals satisfying the $2a+1$ moment equations. For the other vectors there are none. Thus $\widehat{F}(A)$ is a constant times k_A, with the constant independent of A. Its average over all matrices is $F(0) = 1$, by character orthogonality. For uniform independent A, v, the probability that $v \neq 0$ is $1 - q^{-r}$. Given j independent successive vectors and the previous values of A, the next value of A on the last of them is uniform on V: a linear map prescribed on the first $j-1$ independent vectors extends in equally many ways for each image of the j-th. Its chance of missing their span is $1 - q^{j-r}$. This computes the average of k_A as $B_{q,r,a}$, proving the transform formula. Since $0 \leq k_A \leq 1$, the average squared transform is at most $B_{q,r,a}^{-1}$. Finite Fourier Plancherel gives the sum of $|F(X)|^2$ as that average. Removing $X = 0$ subtracts exactly one and proves the estimate. □

Corollary 130.4 (Mass of the projected finite Bessel character)**.** *On the fields and levels of Proposition 130.1, put $a = m/2 - 1$, $c = r - 2a$, and $K_x = 8f(p_0^{-c} + p_1^{-c})$. Let Ψ be the product of the reduced Bessel characters extended by zero on H. For all sufficiently large x,*

$$\sum_{s \in H,\ s \neq 1} |\Psi^{\#}(s)|^2 \leq K_x.$$

In the representation induced to H from the character of its Bessel unipotent subgroup, the fraction of the dimension discarded by e is

$$\delta + O(\sqrt{\delta(1-\delta)K_x}), \qquad \delta = 1 - (1-\delta_0)(1-\delta_1).$$

The implied constant is one. The same error bound with δ_i holds for the rank of $1 - e_i$. In particular all these fractions are $O(b^{-9})$.

Proof. We check the middle pairing, since its nonvanishing is used in the moment calculation. At a marked prime the split algebra is $\mathbb{Q}_p \times \mathbb{Q}_p$ with involution interchanging factors. Write the hyperbolic lines in the order $e_1, \dots, e_a$, the middle space, $e_a^*, \dots, e_1^*$. If g is the first matrix, the second equals $J^{-1}(g^{-1})^t J$, where J pairs the opposite lines and has a symmetric, unimodular middle block T. In the lattice basis the distinguished middle vector z satisfies $z^t T z = -N$, where $N > p_0, p_1$ is the separate prime level. Indeed the middle basis replaces one unit vector x_j of the form $\mathrm{diag}(1, \dots, 1, -N)$ by $x_j/\overline{\pi} + z/N$, with $\pi\overline{\pi} = N$. Its pair with z is -1, its norm is zero, and the two-by-two middle block has determinant -1. The reduction in the first matrix is the entire strictly block upper unipotent group, since entries can be lifted freely over $\mathbb{Z}_p$ and the second matrix is then determined.

The additive character reads the consecutive entries along $e_1, \dots, e_a, z$ in both components, with a unit coefficient in each. In fact the two completions are $\mathbb{Q}_p$, the different is a unit there, and division by $p_0 p_1$ has valuation -1 in each completion. In the first matrix this gives nonzero coefficients along the left arm and the last row evaluated at z. In the second matrix the inverse negates each first off-block entry, since every higher power of $g - 1$ has zero in that block. Transposition and J therefore give the reversed arm in the first matrix and, at its central end, the functional $-z^t T$. The two central blocks have pairing a nonzero scalar times $z^t T z = -N \bmod p$. This verifies precisely the hypothesis of Theorem 130.3 for the

actual reduced character. Reductions in all factors are independent: take the specified integral unipotent lift at each split place and the identity away from the marked primes. Thus the extended adelic character on its residue image is the stated product.

Conjugation averages and sums of squared absolute values factor over the product. The full sum is at most $\prod_{i=0}^{1} B_{p_i,r,a}^{-f}$, and the identity contributes one. Since $p_i \ge 3$, summing $-\log(1-y) \le 2y$ over $y = p_i^{j-r}$, $0 \le j \le 2a$, bounds the log of the product by $4f(p_0^{-c} + p_1^{-c})$. This quantity tends to zero by the scale bounds of Proposition 130.1, since $c = r - m + 2 \ge x^5 - 2\sqrt{x}$. Using $e^v - 1 \le 2v$ for small $v \ge 0$ gives K_x.

The induced character divided by its degree is exactly $\Psi^{\#}$, by the character formula for induction from a subgroup. Let $d = 1 - e = \sum_s d_s s$. Its identity coefficient is δ by the product formula for the regular trace. Since $d = d^* = d^2$, $\sum_{s \ne 1} |d_s|^2 = \delta - \delta^2$, by taking the coefficient at one in $d^* d$. The normalized trace of d on the induced space is $\sum_s d_s \Psi^{\#}(s)$. Its identity term is δ and Cauchy–Schwarz bounds the rest by $\sqrt{\delta(1-\delta) K_x}$. Taking $d = 1 - e_i$ gives the single-prime assertion. Finally $\delta \le 4b^{-9}$ by the proof of Proposition 130.1, and K_x is smaller than any fixed negative power of b on this family. □

These bounds and the odd-cocycle estimate can be used on the same entire interval in this construction. More precisely, fix q_0, d, take $V_0 \le x \le X = V_0^{3/2}$, and put

$$s_* = \lceil 22 \log_2(2\sqrt{X}) \rceil, \quad N_* = 3 q_0 d \, l_0(s_*) 2^{s_*}, \qquad t_* = \lceil 15 X^5 \rceil,$$

where $l_0(s)$ is the least power of three at least $27s$. For every possible tower exponent $1 \le s \le s_*$, put $E_s = \mathbb{Q}(\sqrt{-d}, \mu_{3 q_0 l_0(s) 2^s})$. Then one integer

$$J_I = \mathcal{J}_{N_*, t_*} \prod_{1 \le s \le s_*} h_{E_s}$$

controls the odd-cocycle lengths of Theorem 11.29, including weight zero, on every field used in the finite Bessel comparison above. Indeed $2^s = m^{22} \le (2\sqrt{X})^{22}$ at every node, and $l_0(s) \mid l_0(s_*)$ because both are powers of three. Hence all $E_s \subseteq \mathbb{Q}(\mu_{N_*})$ and all needed weights lie below t_*. Each $\log h_{E_s} \ll N_* \log(2N_*)$ by the class-number estimate in Corollary 11.30. Since $s_* = O(\log(2X))$ and $N_* \ll_{q_0,d} X^{11} \log(2X)$, the theorem gives $\log J_I \ll_{q_0,d} X^{54} \log^5(2X)$. The finite projector and its character estimate at a node use that node's own two places and field E_s. All the preceding bounds for them hold throughout the interval when V_0 is sufficiently large.

This evaluates the finite character in the Bessel term of the period. Extending it to the whole Poincaré Gram matrix requires bounds for rational elements outside the Bessel subgroup in the unfolded integral.

130.3 Rational orbital cancellation and the two frame filtrations

Retain the shorter construction and its levels from Corollary 130.4. Work at a single node x of the interval there. Write $F = E^+$, $a = m/2 - 1$, and $G = \mathrm{U}(h)$ over F. Use a Hermitian form conjugate linear in its first variable. The space over E is

$$V = L \oplus (X_0 \perp Ez) \oplus L^\vee, \qquad L = \langle e_1, \dots, e_a \rangle, \qquad h(z,z) = -N.$$

The first and last summands are opposite totally isotropic spaces, X_0 is totally positive definite, and $N > 0$. Put $n = a + 1$, $f_i = e_i$ for $i \le a$, $f_n = z$, and $T_i = \langle f_1, \dots, f_i \rangle$, with $T_0 = 0$ and $T = T_n$. Let U be the unipotent radical of the stabilizer of the flag $T_1, \dots, T_a$, and set $S = U \rtimes \mathrm{U}(X_0)$, with $\mathrm{U}(X_0)$ acting trivially on the other summands. Write $\sigma(u)$ for the sum

of the entries in positions $(i, i+1)$ in the action on the frame, $1 \le i \le a$. In the final entry one applies the last row on L to the vector z. This is additive on U, since a product of two strictly upper block matrices has zero in every first off-diagonal block. Put $\alpha = -1/(M\delta_E)$, with δ_E the chosen different generator and $M = p_0 p_1$. The finite character is the standard finite adelic additive character of E applied to $\alpha\sigma(u)$.

Lemma 130.5 (A simultaneous endomorphism for two rational characters)**.** *For $\gamma \in G(F)$, set $H = U \cap \gamma U \gamma^{-1}$. Suppose the finite character takes equal values at u and $\gamma^{-1} u \gamma$ for every $u \in H(F)$. There is an E-linear endomorphism A of V such that, for $1 \le i < n$,*

$$Af_i - f_{i+1} \in T_i, \qquad A\gamma f_i - \gamma f_{i+1} \in \gamma T_i.$$

Proof. Let $\mathfrak{n} = \operatorname{Lie} U$ over F. Matrix logarithm and exponential are inverse finite polynomials on strict block unipotents and nilpotents in characteristic zero. They commute with conjugation. For the Hermitian adjoint $u^* = u^{-1}$ amounts to $(\log u)^* = -\log u$. Thus the logarithms of $H(F)$ are exactly the intersection of $\mathfrak{n}$ and $\gamma \mathfrak{n} \gamma^{-1}$ on F-points. Also $\sigma(\exp X) = \sigma(X)$, using the same entries on the Lie algebra. On rational arguments the finite character is the inverse of its archimedean factor, the exponential of the absolute trace times $2\pi i$, up to the common choice of sign. Consequently

$$\lambda(X) = \operatorname{Tr}_{E/F}(\alpha\sigma(X))$$

and its conjugate by γ, given by $X \mapsto \lambda(\gamma^{-1} X \gamma)$, agree on the intersection. Indeed if their difference were $d \neq 0$ at X, evaluating at $X/(2[F : \mathbb{Q}]d)$ would give absolute trace $1/2$.

The two functionals now define one F-linear functional on the sum of their domains. Extend it to the unitary Lie algebra by extending a basis, and extend scalars to E. This last Lie algebra becomes $\operatorname{End}_E(V)$. For instance each endomorphism is the sum of a skew-adjoint endomorphism and an imaginary scalar times a skew-adjoint endomorphism, by taking its skew-adjoint and self-adjoint parts. The scalar extension of $\mathfrak{n}$ is the entire strict upper block space: the adjoint preserves that space in the ordering with reversed dual lines, so the same decomposition applies. Represent the resulting functional by $\operatorname{tr}(B\,\cdot)$ using the nondegenerate ordinary matrix trace pairing.

On a skew-adjoint matrix, λ is the sum of $\alpha\sigma(X)$ and its conjugate. After scalar extension the conjugate term uses only the opposite arm: conjugation on the entries of a skew-adjoint matrix is obtained by negative transposition and the Hermitian Gram matrix, which interchanges the two outer flags and preserves the middle block. Thus it has zero coefficient on each entry from a later block into a left line. Testing an arbitrary entry of this kind in row i specifies precisely the part of Bf_i below that line. The only prescribed nonzero part is αf_{i+1}, including αz for the last left line. This gives $Bf_i - \alpha f_{i+1} \in T_i$. Applying the identical argument in the γ-translated basis gives the second relation for B. Take $A = \alpha^{-1} B$. □

Theorem 130.6 (Ranks and minors of agreeing rational frames)**.** *Write $C = (h(f_i, \gamma f_l))_{i,l \le n}$, let D_i be its leading minor of order i, set $D_0 = 1$, and put $\Delta = D_n \overline{D_n} - N^2 D_a \overline{D_a}$. If $T \cap \gamma T = 0$, then Δ is totally positive and $D_n \neq 0$. Otherwise $\Delta = 0$. In the latter case, under the character agreement of Lemma 130.5, there is a unique $0 \le j \le a$ with*

$$\dim_E(T \cap \gamma T) = n - j, \qquad \dim_E(L \cap \gamma L) = a - j, \qquad D_j \neq 0.$$

There is one $\zeta \in E^\times$, $\zeta\bar{\zeta} = 1$, such that for each $j < i \le n$ a common vector in $T_i \cap \gamma T_i$ has top coefficient ζ in the first frame and 1 in the second. None of the lower levels $i \le j$ has a nonzero common vector.

Proof. The joint Gram matrix of the $2n$ columns has off-diagonal blocks C, C^*. Both diagonal blocks are zero apart from their last diagonal entry $-N$. Its determinant is $(-1)^n\Delta$. Indeed the terms using neither diagonal entry give $(-1)^n D_n\overline{D_n}$, those using both give $(-1)^{n-1}N^2 D_a\overline{D_a}$, and using exactly one leaves unequal numbers of rows and columns to match across the two blocks. A common vector gives linearly dependent columns. When the columns are independent, the span is nondegenerate at each real place. To see this, $T^\perp = L \oplus X_0$ is positive semidefinite with nullspace L. A vector in the radical of the joint span is orthogonal to both frames and has norm zero, so lies in $L \cap \gamma L = 0$; scalar extension preserves this intersection. The span contains z and a totally isotropic a-space orthogonal to z. Its negative index is at least $a + 1 = n$, and the ambient negative index is exactly n. The determinant therefore has sign $(-1)^n$ at every real place. This proves the assertions about Δ and D_n.

For the remaining assertions write $W = T \cap \gamma T$. For $0 \neq w \in W$ take the larger i of its two top indices. Write u, v for its two coefficients at i, temporarily allowing one to be zero. Applying A^{n-i} takes the two expressions into the top frame level, with coefficients u, v at $z, \gamma z$, by Lemma 130.5. The norm of this one vector is both $-Nu\bar{u}$ and $-Nv\bar{v}$. It follows that both coefficients are nonzero. In particular the two filtrations induced on W coincide. In their common level W_i, the map to the top coefficient in either frame has kernel W_{i-1}. Each nonzero successive quotient is therefore one-dimensional; its pair of coefficients spans a line generated by $(\zeta, 1)$ with $\zeta\bar{\zeta} = 1$. The map A takes such a pair unchanged to the next level for $i < n$. All nonzero quotients thus occur consecutively at the top and give the same ζ. Counting the quotients in W_n and W_a proves the two dimension formulas.

We have $T = T_j \oplus W$ and $\gamma T = \gamma T_j \oplus W$. If $y \in \gamma T_j$ is orthogonal to T_j, it is also orthogonal to $W \subset \gamma T$, since $\gamma T_j \subset \gamma L$. Hence $y \in T^\perp$. Its norm is zero, so $y \in L$ by positivity. Now $y \in W \cap \gamma T_j = 0$. The pairing of the two j-spaces is nonsingular, which proves $D_j \neq 0$. □

Proposition 130.7 (Cancellation over the stabilizer of a rational term)**.** *For class representatives $h_c, h_{c'} \in \mathrm{U}(X_0)(\mathbb{A}_f)$ write $\Gamma_c = U(F) \cap h_c K(M) h_c^{-1}$, where $K(M)$ is the principal level in the integral stabilizer. Put $\Gamma = \Gamma_c \cap \gamma\Gamma_{c'}\gamma^{-1}$. If the two rational characters disagree, the integral of the Bessel orbital kernel over every fiber $\Gamma\backslash H(F \otimes_{\mathbb{Q}} \mathbb{R})$ is zero. In particular every absolutely convergent full or cut orbital integral for this term vanishes.*

Proof. Write $H_\infty = H(F \otimes_{\mathbb{Q}} \mathbb{R})$ and use a similar notation for G_∞. In the unfolding, the two principal congruence groups act on γ by left and right multiplication; its stabilizer consists of the pairs $(u, \gamma^{-1}u\gamma)$ with $u \in \Gamma$. The finite character is trivial on Γ_c. Indeed the frame is fixed pointwise by h_c, and for $u \in U$ its first off-diagonal frame entries are unchanged by conjugation by h_c. These entries are integral, and at the levels are divisible by M; division by δ_E therefore puts their trace in the conductor of the finite character. The same argument applies to c'.

The quotient $\Gamma\backslash H_\infty$ is compact. Here are details for this particular type of arithmetic subgroup. Restrict scalars to $\mathbb{Q}$ and choose a rational flag of Lie ideals with successive central, one-dimensional quotients, successively taking rational central lines in nilpotent quotients. Exponential coordinates in a basis adapted to this flag give real coordinates on H_∞; in particular each successive central direction has group law addition on its quotient. For each rational basis element X, $\exp(dX) \in \Gamma$ for a sufficiently divisible positive integer d. This follows from the finite exponential polynomial: outside finitely many primes its coefficients and both class changes are integral, and at the exceptional primes it tends to the identity as the local divisibility of d increases. Reducing one coordinate at a time, starting at the outermost quotient, gives representatives in a product of compact intervals in exponential product coordinates. The subgroup is discrete, since its rational matrix entries lie in fixed fractional integer lattices at all finite places. This proves compactness and gives a finite invariant quotient measure.

For either Bessel function W, its defining equivariance on the left by the real unipotent is multiplication by the archimedean character χ_∞. For the full vector this is the character in the Bessel integral (translate its unipotent integration variable); the cut vector is multiplied by a function constant on left U_∞-cosets. On a rational unipotent the character is the inverse of its finite factor, by triviality of the adelic additive character on E. It follows that $K_\gamma(y) = W(y) \otimes W(\gamma^{-1}y)^*$ transforms under $h \in H_\infty$ by the character $\eta(h) = \chi_\infty(h)/\chi_\infty(\gamma^{-1}h\gamma)$. It is trivial on Γ and nontrivial by rational disagreement. For fixed y, right invariance of measure on $\Gamma\backslash H_\infty$ shows that the integral of $K_\gamma(hy)$ there is unchanged by multiplication by any $\eta(h_0)$. Choosing $\eta(h_0) \neq 1$ proves the vanishing. Finally integrate the fiber integral on $H_\infty\backslash G_\infty$ using quotient Haar measure. Both groups are unimodular. This gives zero with compact cutoffs on the outer quotient, and for every absolutely convergent orbital integral. □

Corollary 130.8 (Elimination of the isotropic parabolic on a level slice)**.** *Suppose $h_c^{-1}\gamma h_{c'}$ is integral at all finite places and that, in the adapted basis at one of the split marks, its first diagonal entry reduces to* 1*. Every rational term preserving L either belongs to $S(F)$ or has the stabilizer cancellation of Proposition 130.7. In particular this holds on either slice obtained by requiring the residue to be in the Bessel image above p_i.*

Proof. For an agreeing term with $\gamma L = L$ the preceding theorem has $j = 0$, since $a \geq 1$. Starting at $i = 1$, its common vectors with coefficient pair $(\zeta, 1)$ prove inductively that $\gamma T_i = T_i$ and $\gamma f_i - \zeta f_i \in T_{i-1}$, also at $i = n$. The unitary element $\zeta^{-1}\gamma$ lies in the flag parabolic. In its Levi quotient the diagonal on the left lines is one, and the middle unitary transformation fixes z; hence it preserves $z^\perp = X_0$ inside the middle space. Removing that middle action leaves the unipotent radical. Thus $\gamma \in \zeta S(F)$.

Both class representatives fix e_1. Its line is a primitive lattice summand, so integrality at every finite place implies that the eigenvalue ζ is integral there. Since $\zeta\bar{\zeta} = 1$, all complex conjugates of ζ have absolute value one. Such an algebraic integer is a root of unity: the monic polynomials of its positive powers have bounded degree and bounded integer coefficients, so two of the powers coincide. At the selected component $E_w = \mathbb{Q}_{p_i}$, its residue equals the indicated first diagonal entry. The group $1 + p_i\mathbb{Z}_{p_i}$ is torsion-free for odd p_i, by the convergent injective p_i-adic logarithm there. Therefore $\zeta = 1$ and $\gamma \in S(F)$. All disagreeing terms have zero fiber integral by the proposition. Every element of the Bessel image has first diagonal entry one, as required. □

On these slices the agreeing terms outside $S(F)$ therefore have either $1 \leq j \leq a$ or $\Delta \neq 0$. Quantitative control requires their multiplicities and the sizes of their integrals.

130.4 Divisors of the frame discriminant at the marked primes

Continue with the two frames of Theorem 130.6. We work with pairs of classes in the actual level. Write $f = [F : \mathbb{Q}]$ in exponents, and write $\mathcal{F} = (f_1, \dots, f_n)$ for the frame matrix. A rational term γ occurs between c, c' only if $g = h_c^{-1}\gamma h_{c'}$ carries the standard finite lattice to itself. In these coordinates the two integral frames are $\mathcal{F}, g\mathcal{F}$. Indeed both class representatives fix the original frame, and the change to the lattice basis preserves its columns. Their Hermitian Gram matrices are therefore the rational Gram matrices of the preceding subsection. The lattice form is integral, including at N: the altered middle basis there has an integral two-by-two block with diagonal $0, -N$ and off-diagonal entries of norm one. In particular C has entries in $\mathcal{O}_E$ and $\Delta \in \mathcal{O}_F$.

Theorem 130.9 (Divisibility by the indices of the two frames)**.** *Suppose $\Delta \neq 0$. At a split place $v \mid p_0p_1$ of F, let $w, \bar{w}$ be the places of E over v. Form the matrix $B_w = (\mathcal{F}, g\mathcal{F})_w$ in the*

integral lattice at w. Let ℓ_w be the length of the saturation of its column lattice modulo that lattice, and let d_w be the dimension of the intersection of the two reduced frame spans. With normalized integral valuations one has

$$\operatorname{ord}_v \Delta \;\geq\; \ell_w + \ell_{\bar{w}} \;\geq\; d_w + d_{\bar{w}}.$$

In particular on the slice where the reduction belongs to the Bessel image at all places above p_i,

$$p_i^{2nf} \mid \mathrm{N}_{F/\mathbb{Q}}(\Delta), \qquad \left|\mathrm{N}_{F/\mathbb{Q}}(\Delta)\right|^{-s} \leq p_i^{-2nfs} \quad (s \geq 0).$$

If both slices apply, the two divisors multiply. On the same slice, for any rational cell, $p_i^j \mathcal{O}_E$ contains D_j for $0 \leq j \leq a$.

Proof. The two frames each have independent reduction: the first consists of primitive basis columns and the second is carried there by a lattice automorphism. Their joint reduction thus has rank $2n - d_w$. The columns are independent in characteristic zero by Theorem 130.6. Smith normal form over the local discrete valuation ring shows that the least valuation of any maximal minor of B_w is ℓ_w, the sum of its elementary divisor exponents. Exactly d_w of those exponents are positive. Write J_w for the integral matrix pairing the two components of the ambient lattice. The joint Gram over w is $B_{\bar{w}}^{t} J_w B_w$. Expanding its determinant twice by Cauchy–Binet shows that each summand contains a maximal minor of each B and an integral minor of J_w. Their determinant is $(-1)^n \Delta$, which proves the inequalities. This argument retains nontrivial elementary divisors at arbitrary powers of the mark.

If the reduction is in the Bessel image then it preserves the whole frame span in both components, so that $d_w = d_{\bar{w}} = n$. There are f places of F over the completely split prime p_i, each of residue degree one. Taking the field norm gives the divisibility assertion; total positivity makes the inverse power estimate immediate. For the last assertion each entry of the leading j-square Gram block vanishes modulo every w over p_i: a vector in the first isotropic arm pairs to zero with the whole frame. Expand the determinant at each such place. □

Proposition 130.10 (The positive vector measured by the discriminant)**.** *Put $u = \operatorname{adj}(C) e_n^{\mathrm{std}}$, where e_n^{std} is the last coordinate vector in E^n. Let $t = \gamma \mathcal{F} u$, and let q be the X_0-component of t in the rational orthogonal middle decomposition. Then*

$$N h(q, q) = \Delta.$$

At every marked component where the reduction preserves the frame span, the vector q belongs to the p_i^{a+1}-multiple of the local positive lattice of class c, provided the reduction preserves the span at its paired component as well.

Proof. The identity $C u = D_n e_n^{\mathrm{std}}$ gives $t \in L^{\perp}$ and $h(z, t) = D_n$. Thus $t = l + q - (D_n/N) z$, where $l \in L$. On the other hand the last coordinate of u is D_a, so $h(t, t) = -N D_a \overline{D_a}$ by unitarity. Equating the two evaluations gives the formula at every embedding. In the marked lattice basis q is obtained by projecting $g \mathcal{F} u$ to its positive part, then carrying it back by the class representative. The projection is integral there: the only middle change of basis has denominators dividing N, a unit at the marks. The components of u are signed a-minors of the first a rows of C, whose entries vanish modulo p_i when the frame span is preserved in the specified components. Hence p_i^a divides u. Each positive projection of a column of $g\mathcal{F}$ vanishes modulo p_i also. Multiplication yields the claimed extra factor of p_i. □

We next allow the two reduced frame spans to share only one vector. In this situation there is a further divisor that can be retained after removing the elementary divisors of both isotropic arms. For a local lattice D, the saturation of a submodule is its fraction-field span intersected with D.

Theorem 130.11 (The two terminal columns at a meeting place)**.** *Assume $\Delta \neq 0$. At each marked component w form $B_{0,w}$ from the two isotropic arms, omitting the two last columns of the frames. Write $S_{0,w}$ for the saturation of its columns and let $\ell_{0,w}$ be their index length in $S_{0,w}$. In the free quotient by $S_{0,w}$, let b, b' be the images of the two terminal columns. The ideal generated by all determinants*

$$\det \begin{pmatrix} t_1(b) & t_1(b') \\ t_2(b) & t_2(b') \end{pmatrix}$$

as t_1, t_2 range over integral linear forms on the quotient has valuation $\ell_w - \ell_{0,w}$. Suppose the two finite characters agree on the intersection of their conjugate reduced unipotent groups. Then

$$\ell_w - \ell_{0,w} \geq 1 \qquad (d_w > 0).$$

Consequently the product of the norms of all meeting components divides the absolute finite index given by any such determinant chosen locally at every finite place, if the forms and columns are integral everywhere and the determinant is an idele.

Proof. Choose bases adapted to $S_{0,w}$ and its direct complement. The exterior product of the $2a$ isotropic columns is a uniformizer to the power $\ell_{0,w}$ times a primitive generator of the top exterior power of $S_{0,w}$, up to a unit. Wedging the two terminal columns into it uses only their images in the free quotient. Thus the minimum valuation of the maximal minors of B_w is $\ell_{0,w}$ plus the minimum valuation of a two-row minor of (b, b'). Arbitrary pairs of forms generate the same ideal as these minors, by determinant expansion. This proves the equality of valuations.

For the remaining assertion work over $\mathbb{F}_{p_i}$ at w, writing s for the local matrix. In this component the unipotent is $1+\mathfrak{n}$, with $\mathfrak{n}$ the full strict block upper algebra. Its intersection with its conjugate is $1+(\mathfrak{n} \cap s\mathfrak{n}s^{-1})$. As in the proof of Corollary 130.4, the two-component character in this single matrix realization is the additive exponential of a linear form on $X = u - 1$; its consecutive coefficients on the left frame are the same nonzero scalar. This description holds with $w, \bar{w}$ interchanged, using the other component of the additive character on the left. A character equality on the global reduced intersection gives one on each factor by taking identity components at all other places. Over the prime field the additive exponential is injective as a map to the p_i-th roots of unity. The two linear forms therefore agree on the intersection. Extend them to one functional on the full matrix algebra and represent it by ordinary matrix trace. Dividing its representing matrix by the left coefficient gives a matrix A with

$$Af_i \equiv f_{i+1} \pmod{T_i}, \qquad Asf_i \equiv sf_{i+1} \pmod{sT_i} \quad (i < n)$$

over the residue field. The prescribed lower blocks give these formulas just as in Lemma 130.5. This finite-field proof uses $u-1$ and holds independently of denominators in a matrix logarithm.

If the spans meet, write a nonzero common vector in the two frames. Let i be the largest index at which at least one coefficient is nonzero. Apply A^{n-i}. The two coefficients at the terminal positions are now the former coefficients at i, so they are not both zero. The resulting equality, modulo the span of both isotropic arms, is a nontrivial relation between the terminal columns. It remains a relation modulo the reduction of the larger saturated lattice $S_{0,w}$. Hence every two-row minor there vanishes. The index of an integral finite idele is the product of residue norms to its local valuations. Taking one factor from every meeting component proves the last assertion. □

The two estimates apply at each x on the interval in Corollary 130.4, with its own E, M and class lattices. They constrain an individual term and its terminal determinant. The second moment involves products of two amplitudes summed over all pairs in a residue class.

131 Open and closed orbital integrals

131.1 The exact real measure over a pair of frames

At a real place normalize z to have norm -1. Write $n = a{+}1$, $k = \dim X_0 \ge 1$ and $r = 2a{+}k{+}1$. In coordinates (u, x, t, v) of sizes $a, k, 1, a$ the Hermitian form has matrix J with quadratic form $u^*v + v^*u + x^*x - |t|^2$. Use the matching order of the dual lines here. Let $\mathcal{F} = (e_1, \dots, e_a, z)$ and $G = \mathrm{U}(J)$. For $g \in G$ put $Z(g) = \mathcal{F}^* J g \mathcal{F}$. For any n-square complex matrix Z set

$$\delta(Z) = |\det Z|^2 - |\det Z_{[1,a],[1,a]}|^2, \qquad \Omega = \{Z : \delta(Z) > 0\}.$$

Write $S_j = 2\pi^j/(j-1)!$ for the area of the unit sphere in $\mathbb{C}^j$ and $V_j = \prod_{l=1}^{j} S_l$. Lebesgue measure on complex entries is $d\Re\, d\Im$. On Hermitian matrices use the diagonal real and the strict upper complex entries, and on skew-Hermitian matrices use the imaginary parts of the diagonal and the same strict upper entries.

We specify Haar measures. On G use the invariant measure that at the identity is $1/V_r$ times Lebesgue measure in the skew-Hermitian tangent coordinates $J\,dg$. This is the compact-dual normalization: for $J = 1$ it gives probability measure on $\mathrm{U}(r)$, as verified below. Write P for the subgroup fixing every column of $\mathcal{F}$. Its elements are the transformations

$$(u, x, t, v) \longmapsto \big(u - B^*hx + (K - B^*B/2)v,\ hx + Bv,\ t,\ v\big),$$

where $h \in \mathrm{U}(k)$, $B \in \mathbb{C}^{k\times a}$ and $K^* = -K$. The formula follows by pairing the proposed images of the positive and dual columns with the fixed frame and with one another. Direct substitution also verifies the isometry. Give P the measure $ds = dB\,dK\,dh$ with dh probability Haar. This is left and right invariant: multiplication translates the central K variable by an expression in B, and the linear actions on B are unitary or the identity.

Theorem 131.1 (Disintegration on the open set of frame products)**.** *There are measurable representatives $g_Z \in G$ with $Z(g_Z) = Z$ for $Z \in \Omega$. For any nonnegative measurable function φ on G,*

$$\int_{Z(g)\in\Omega} \varphi(g)\, dg = \frac{V_k S_k}{V_r} \int_{\Omega} \delta(Z)^{k-1} \int_P \int_P \varphi(s' g_Z s)\, ds\, ds'\, dZ.$$

Both sides may be infinite. The inner integral is independent of the representative of the indicated frame products.

Proof. We first compute directly on the set of frames. For a complex r-by-n matrix Y of independent columns, the map $Y \mapsto Y^*JY$ is a submersion, even at a singular Gram value. In fact Y^*J is surjective and one can solve $Y^*J\,dY = H/2$ for any Hermitian variation H. Let $d\mu$ be the fiber density at $Y^*JY = -e_n e_n^*$, given in submersion coordinates by dividing Lebesgue measure dY by the specified measure on Hermitian matrices. Such coordinates define the same density on overlaps by the change of variables formula. It is invariant under G, since left multiplication preserves both measures and the Gram value.

Write $Z = \mathcal{F}^*JY$; it consists of the a dual rows of Y and the negative of its z-row. If Z is invertible pass to $Y' = YZ^{-1}$, with dual rows $(I_a, 0)$ and z-row $(0, -1)$. Let (η, ξ) be the last row of Z^{-1}, let (A, p) be the top rows of Y', and (B, x) its X_0-rows. Here η is a row and x a single column. The Gram equations for Y' are precisely

$$\begin{aligned} A + A^* + B^*B &= -\eta^*\eta, \\ p + B^*x &= -\eta^*\xi, \\ x^*x &= 1 - |\xi|^2. \end{aligned}$$

Indeed the required Gram is the negative outer product of the last row of Z^{-1}. The right side of the last equation is positive on Ω, since $\xi = \det Z_{[1,a],[1,a]} / \det Z$.

There are $a + k$ free complex rows before imposing the Gram equations. Their change from Y' to Y has real Jacobian $|\det Z|^{2(a+k)}$ with Z held fixed. On Hermitian matrices the change of prescribed Gram has Jacobian $|\det Z|^{2n}$. For a diagonal matrix with positive singular values s_i, the diagonal factors are s_i^2 and the complex off-diagonal factors contribute $(s_i s_j)^2$; unitary factors preserve the Hermitian Lebesgue measure. Dividing the two Jacobians yields $|\det Z|^{2(k-1)}$. Solving for $A + A^*$ leaves its skew part $K = (A - A^*)/2$ with factor 2^{-a} from the real diagonal variables. Each off-diagonal pair has Jacobian one, as one checks on the sum and half-difference of A_{ij} and $\overline{A_{ji}}$. Solving for p has Jacobian one. Polar coordinates for x with squared radius $1 - |\xi|^2$ give $S_k(1 - |\xi|^2)^{k-1}/2$ times uniform probability on the unit sphere. These calculations give the frame density on Ω in the coordinates $Z, B, K, \omega = x/\|x\|$:

$$d\mu = 2^{-n} S_k \, \delta(Z)^{k-1} \, dZ \, dB \, dK \, d\sigma(\omega).$$

For a fixed Z, take $B = K = 0$ and a fixed unit ω_0 to construct $Y_Z = Y'Z$ by the displayed equations. This is a frame of the required Gram, with independent columns since $\mathcal{F}^* J Y_Z = Z$. Such a frame is the image of $\mathcal{F}$ by an isometry: inside the orthogonal complement of its negative line choose duals to the a isotropic columns, then subtract one half of their mutual Gram matrix times the isotropic columns to make the duals isotropic. Complete the hyperbolic pairs with an orthonormal basis of their positive complement inside that orthogonal complement. These choices can be made measurably by using the first successful basis minors and Gram–Schmidt. They yield $g_Z \mathcal{F} = Y_Z$. Applying the formula for P to Y_Z gives exactly the free B, K and $\omega = h\omega_0$ above. The image of dh is sphere probability by rotation invariance. Consequently, in the frame density the integral over these three coordinates is the integral over the left copy of P.

It remains to compute the factor for lifting frames to G. In the space of complex square matrices let $d\nu_r$ be the fiber density of $g^* J g = J$ by the same Gram construction. It is invariant, and at the identity its density is 2^{-r} in the specified tangent coordinates: the derivative is the Hermitian part $J \, dg + (J \, dg)^*$, with unit Jacobian off the diagonal and a factor two on each real diagonal. Multiplication by J on the ambient entries preserves Lebesgue measure. Thus $dg = 2^r d\nu_r / V_r$. For the positive identity form, one can integrate its square Gram fiber by adding orthonormal columns successively. Orthogonality fixes complex coordinates with unit Jacobian; the remaining unit length equation in complex dimension j contributes $S_j/2$ by polar coordinates. The mass of that fiber is $V_r/2^r$, which verifies the stated normalization.

Disintegrate the square Gram fiber first in the columns of the frame and then in the remaining columns. Its conditional fiber over $\mathcal{F}$ can be computed there, since translation by any isometry has complex determinant of absolute value one. The equations pairing the remaining columns with $\mathcal{F}$ fix their dual rows and z-rows with unit Jacobian. For the a dual columns the remaining rows are $K - B^*B/2, B$, with factor 2^{-a} from their zero Gram as before. For the k positive columns, pairing with these duals fixes their top rows as $-B^*h$ with unit Jacobian; their remaining Gram equation is $h^*h = I_k$. Its density is $V_k \, dh / 2^k$ by the compact calculation. The ordering of these equations only permutes real or complex coordinates, so the full factor along the right copy of P in $d\nu_r$ is $2^{-a-k} V_k \, ds$. These successive changes of variables may first be made in local submersion charts for continuous compactly supported functions and then extended to nonnegative measurable functions by uniqueness of the resulting measures. Combining them multiplies the frame density by $2^{r-a-k} V_k / V_r$. Since $r - a - k = n$, this proves the formula. The same coordinates show that two representatives differ by left and right factors in P; Haar invariance proves the last assertion. □

Corollary 131.2 (The density at a rational term on the marked slices)**.** *For a rational term with $\Delta \neq 0$, write Z_v for the frame products at each normalized real place of the shorter construction. Set $\kappa_v = |M\Phi_v(\delta_E)|/(2\pi)$ and $b_{j,v} = \kappa_v^{2(n-j)}/N$ for $1 \le j \le n$. The coefficient of the two Haar integrals in the product of the open-cell densities at $(Z_v)_v$ is*

$$\left(\frac{V_k S_k}{V_r}\right)^f \left(\prod_{v,j} b_{j,v}\right)^{2(k-1)} \mathrm{N}_{F/\mathbb{Q}}(\Delta)^{k-1}.$$

For terms on the p_i-slice, its reciprocal is bounded above by

$$\left(\frac{V_r}{V_k S_k}\right)^f \left(\prod_{v,j} b_{j,v}\right)^{-2(k-1)} p_i^{-2nf(k-1)}.$$

Proof. The real frame is obtained from the rational frame by scaling the last vector by $1/\sqrt{N}$ and the lengths along consecutive left lines by successive additional factors κ_v. In the group coordinates this is the diagonal substitution making the middle norm -1 and the simple-root character have unit coefficient in the real wave. Indeed the archimedean coefficient in the wave $\exp(s - \bar{s})$ has absolute value $2\pi/|M\Phi_v(\delta_E)| = \kappa_v^{-1}$; the ratios of adjacent frame scales are κ_v. Phases of basis vectors cancel in absolute determinants. Thus each leading Gram determinant of order l has its absolute value multiplied by $\prod_{j \le l} b_{j,v}$. Since $b_{n,v} = 1/N$, subtraction of their squared absolute values at orders n and a gives $\delta(Z_v) = (\prod_j b_{j,v})^2\, \Phi_v(\Delta) > 0$. Multiply the densities of Theorem 131.1 and use Theorem 130.9 for the reciprocal bound. □

The density describes integration in the continuous Gram variable. Estimating a period at a specified rational value requires control of the function being integrated there.

131.2 Pointwise restriction at a prescribed real Gram matrix

We pass from integration in Z to evaluation at a single Z. Keep G, P, Ω, n, k, r and the Haar measures of Theorem 131.1. Write S for the real Bessel subgroup from Section 130.3. Its action on the ordered frame has image the group N_n of complex upper unitriangular matrices and kernel P. Indeed the upper unitriangular changes among the first a lines extend by their inverse adjoints on the dual lines. The possible changes $z \mapsto z + l$, $l \in L$, are all realized by the unipotent radical: for a middle-space column map D from the dual lines into $\mathbb{C}z$, the middle coordinates change by Dv, and the top line coordinates by $-D^* J_{\mathrm{mid}}(x,t) - D^* J_{\mathrm{mid}} Dv/2$. This preserves the form and the linear map $D \mapsto -D^* J_{\mathrm{mid}} z$ ranges over L. We use Lebesgue measure on the strict upper entries of N_n, which is Haar in these triangular coordinates. Together with ds on P this specifies Haar on S; equivalently it is Haar on its unipotent radical times probability Haar on $\mathrm{U}(k)$. All quotient measures below use these choices.

Put $c_G = V_k S_k / V_r$ and let $d = 2n^2$, the real dimension of the Gram variable. Use operator norm for complex matrices. For $Z_0 \in \Omega$ write $\xi_0 = (Z_0^{-1})_{nn}$. A real number P_0 will satisfy

$$P_0 \ge \max\{8r,\ \|Z_0\|,\ \|Z_0^{-1}\|,\ (1 - |\xi_0|^2)^{-1/2}\}.$$

Use the complex entry coordinates on matrices, regarded as d real coordinates. Let Q be the cube centered at Z_0 with coordinate half-width $\epsilon = P_0^{-16}/4$.

Lemma 131.3 (A section with quantitative control in a Gram cube)**.** *On Q one can choose the representatives g_Z of Theorem 131.1 real analytically, fixing a positive space of dimension $k-1$ pointwise. On Q, $\delta(Z)^{k-1} \ge \delta(Z_0)^{k-1}/2$. Both g_Z and g_Z^{-1} have analytic continuations of operator norm at most P_0^{32} to the polydisc of radius P_0^{-16} in the complexified real coordinates around Z_0.*

Proof. Use $Y = Y_Z$ constructed in Theorem 131.1, with the sphere direction a fixed positive coordinate vector. This takes values in the $2n$-space consisting of the two isotropic arms, z, and this one positive direction. Write $Y_a = (Y_1, \dots, Y_a)$ and put

$$U = \mathcal{F}(Z^*)^{-1}, \qquad w = Z^{-1}e_n, \qquad D = U_{[1,a]} + \tfrac{1}{2} Y_a w_{[1,a]} w_{[1,a]}^*,$$

where the subscript on U selects its first a columns. Since $Y^* J U = I$ and $U^* J U = -ww^*$, the columns of D are isotropic duals to Y_a and are orthogonal to Y_n. Set $t = U_n + Y_n$. This is orthogonal to every column of Y and has norm for J equal to $1 - |\xi|^2$, where $\xi = (Z^{-1})_{nn}$. Subtracting $Y_a D^* J t$ makes t orthogonal to D and preserves its norm. Send the source frame to Y, the duals to D, and the distinguished positive unit vector to $(t - Y_a D^* J t)/(1 - |\xi|^2)^{1/2}$. Extend by the identity on the other $k - 1$ positive directions. The stated Gram relations give an isometry and specify g_Z.

For the analytic estimates write $p = P_0$. Extend both Z and its entrywise conjugate using the same complexified real coordinates, with opposite signs on the imaginary part. Perturbations of either matrix have operator norm at most $2np^{-16}$. The inverse perturbation series bounds the new inverse norms by $2p$, and their changes by $4np^2 p^{-16}$. In particular the change in $1 - \xi\bar{\xi}$, relative to its central value, is at most $12np^4 p^{-16}$: at the center $|\xi_0| < 1$ and $1 - |\xi_0|^2 \geq p^{-2}$. Its analytic square root is thus single valued and its reciprocal has norm at most $2p$. On the real cube the variation of $\log|\det Z / \det Z_0|$ is at most $4n^2 p\, p^{-16}$, by the convergent matrix logarithm of $I + Z_0^{-1}(Z - Z_0)$. In $\log(\delta(Z)/\delta(Z_0))$ this term occurs twice, plus the logarithm of the relative change in $1 - |\xi|^2$. Multiplication by $k - 1 \leq p$ gives absolute value less than $\log 2$, since $p \geq 8r$. This proves the lower bound.

For clarity, on the complex polydisc the free blocks of YZ^{-1} are given by the previous Gram equations with $B = K = 0$. The row η has length at most $2p$, as does its continued conjugate. The formulas there give $\|Y\| \leq p^7$. The displayed formulas here then give $\|U\| \leq p^2$, $\|D\| \leq p^{12}$, $\|t\| \leq p^8$, and $\|Y_a D^* J t\| \leq p^{27}$. With $p \geq 8r$ these bounds give $\|g_Z\| \leq p^{32}$, also for the continued conjugate matrix. The analytic continuation of g_Z^{-1} is the continuation of $J g_Z^* J$ and satisfies the same bound. □

Lemma 131.4 (The quotient measure in the Gram cube)**.** *For every nonnegative measurable left S-invariant function v and every measurable $P' \subset P$,*

$$\int_Q \int_{P'} v(g_Z s)\, ds\, dZ \leq \frac{2}{c_G\, \delta(Z_0)^{k-1}} \int_{S\backslash G} v(g)\, d\dot{g}.$$

Proof. Theorem 131.1, disintegrated on the left by P, gives the measure on $P\backslash G$ over the open set as the image of $c_G \delta(Z)^{k-1} dZ\, ds$ under $(Z, s) \mapsto P g_Z s$. To verify this for a compactly supported function on the quotient one can multiply its lift by a nonnegative function of left-orbit integral one, using local sections and a partition of unity. The left integral in the theorem removes that factor. Monotone exhaustion gives the general measure identity. Use Lemma 131.3 and enlarge P' to P. It suffices to show that the part with $Z \in Q$ of each $P\backslash S$-fiber has measure at most one.

On a fiber meeting the cube choose a representative g_1 with $Z_1 = Z(g_1) \in Q$. For $h \in S$, $Z(hg_1) = ((h^{-1})|_T)^* Z_1$, since h preserves the form. Write $A = ((h^{-1})|_T)^*$, a lower unitriangular matrix. If $AZ_1 \in Q$, then $\|A - I\| \leq \|AZ_1 - Z_1\|\, \|Z_1^{-1}\| \leq 8n\epsilon P_0 < 1/4$. Haar on $P\backslash S$ is Lebesgue measure in the entries of A: inversion on a unipotent triangular group preserves Haar, and adjoint exchanges upper and lower coordinates with unit absolute Jacobian. Each available complex entry is in a disc of area less than one. The volume of their product is at most one. Integrating over $S\backslash G$ proves the lemma. □

Let $X_1, \dots, X_{r^2}$ be a real Hilbert–Schmidt orthonormal basis of $\operatorname{Lie} G$, with the real inner product $\Re \operatorname{tr}(X^{*}Y)$. Let $R_X W(g)$ be the derivative of $W(g\exp(tX))$ at zero, and $R_{\mathbf{i}}$ the successive derivatives in a word $\mathbf{i}$. Suppose W is a smooth, finite-dimensional Hilbert-space-valued function, with $W(hg) = \tau(h)W(g)$ for a unitary representation τ of S. For an integrable compactly supported scalar ψ set $T_\psi W(g) = \int_G \psi(y) W(gy)\, dy$. Right derivatives of W and the corresponding convolutions have the same left equivariance, so their squared norms descend to $S\backslash G$.

Theorem 131.5 (A restriction estimate for localized Bessel convolutions)**.** *Put $p = P_0$, $P' = \{s \in P : \|s\| \le p^{20}\}$, and take $\phi \in L^2(G)$ supported on $\|y\| \le p^{20}$. For every word of length at most d, suppose*

$$\|T_\psi(R_{\mathbf{i}} W)\|_{L^2(S\backslash G)} \le A p^{|\mathbf{i}|} \|\psi\|_{L^2(G)}$$

for all ψ supported on the same ball. Then

$$\int_{P'} \|T_\phi W(g_{Z_0} s)\|^2 \, ds \le \frac{p^{1000 n^2}}{c_G\, \delta(Z_0)^{k-1}} A^2 \|\phi\|_{L^2(G)}^2.$$

There are no differentiability assumptions on the scalar test function ϕ.

Proof. The ball in G is compact: a unitary matrix for J satisfies $g^{-1} = Jg^{*}J$, and G is closed in matrix space. The same holds for P'. In particular ϕ has finite L^1-norm. Differentiation in Z under its integral is valid by local uniform boundedness of the derivatives of W on compact sets.

We bound a mixed derivative in any $l \le d$ distinct real coordinates Z_j. Write $B_j(Z) = g_Z^{-1} \partial_j g_Z$. Differentiating $W(g_Z s y)$ uses the Lie element $\operatorname{Ad}(y^{-1}) \operatorname{Ad}(s^{-1}) B_j(Z)$ on its right. We describe the bounds after iteration, including the coefficients that depend on y. Express the adjoint maps and B_j in the fixed orthonormal basis. Each matrix entry of $\operatorname{Ad}(y^{-1})/p^{40}$ and of $\operatorname{Ad}(s^{-1})/p^{40}$ has modulus at most one. This follows from $\|g^{-1} X g\|_{\mathrm{HS}} \le \|g\|^2 \|X\|_{\mathrm{HS}}$ on the indicated balls. In each differentiated integral absorb the entries depending on y, after division by p^{40}, into ϕ. The resulting weights are ϕ times products of such entries; each has L^2-norm at most $\|\phi\|_2$ and is independent of s, Z.

More explicitly the mixed derivative is a sum of terms $c(Z,s)\, T_\psi(R_{\mathbf{i}} W)(g_Z s)$, with $|\mathbf{i}| \le l$. The sum of the suprema of $|c|$ on $Q \times P'$ is at most $p^{180 l}$. Here are estimates establishing this bound. On the polydisc of radius $3p^{-16}/4$, Cauchy's inequality and Lemma 131.3 give $\|B_j\| \le 4p^{80}$; each Lie coefficient, extended complex linearly from the real algebra, is bounded by p^{84}. For the latter assertion one may choose the orthonormal basis as J times the usual orthonormal skew-Hermitian basis. Its coordinates complexify to trace pairings of Hilbert–Schmidt norm one; any other real orthonormal basis is related by an orthogonal change of coordinates. A derivative of an existing coefficient costs at most $4(d+1)p^{16} \le p^{20}$, by shrinking the polydisc radius by $p^{-16}/(4(d+1))$ at each step. When the derivative hits W, the expansion uses at most r^6 summands, from the three Lie indices, of size at most p^{84+80} times the old coefficient. The factors depending on s stay in the coefficient and depend on no Gram variables. Thus the sum of suprema increases by at most $p^{20} + r^6 p^{164} \le p^{180}$ per derivative. The final polydisc still contains Q. This proves the indicated expansion and estimate by induction, starting from coefficient one.

Take the $L^2(Q \times P')$-norm. For each term, Lemma 131.4, applied to the squared norm of its convolution, bounds that norm after removal of c by

$$\left(\frac{2}{c_G \delta(Z_0)^{k-1}} \right)^{1/2} A p^{l} \|\phi\|_2.$$

The triangle inequality gives the same expression times p^{180l} for the whole mixed derivative.

Finally, on an interval of length 2ϵ, evaluation at its center is bounded in norm by $(2\epsilon)^{-1}\int\|u(t)\|\,dt+\int\|u'(t)\|\,dt$, by the fundamental theorem of calculus. Apply this successively in all d coordinates with values in $L^2(P')$; equivalently one may iterate the averaging and signed integration operators of the one-variable identity. Cauchy–Schwarz bounds the result by $(2\epsilon)^{-d/2}$ times the sum, over coordinate subsets, of $(2\epsilon)^l$ times the corresponding mixed derivative norm. Since $2\epsilon=p^{-16}/2$, $l\le d=2n^2$ and $p\ge 8r$, the scalar factor after squaring is at most

$$2(2\epsilon)^{-d}\big(2^d p^{181d}\big)^2\le p^{1000n^2}.$$

This proves the claim. □

The estimate evaluates a real convolution at the prescribed Gram value, whereas Theorem 131.1 integrates over Gram values. Corollary 131.2 supplies the product of its inverse densities at the actual rational terms. The remaining factor A concerns the specific archimedean Bessel vector and its discrete-series norm.

131.3 The formal-degree bound for the real Bessel average

We evaluate the quotient-norm constant in Theorem 131.5 for the real function that enters the shorter construction. All measures are those of Theorem 131.1 and Section 131.2. In this subsection K is a maximal compact subgroup of $G=\mathrm{U}(a+k,a+1)$. Let π be its discrete series of Harish-Chandra parameter λ. Write d_π for its formal degree with respect to dg. Inner products are conjugate linear in the first variable. Let w be its unit lowest-K-type coefficient vector, $x\in G$, and $b=\pi(x)w$. Let τ be the unitary representation of S on the finite-dimensional value space V_τ, given by the simple-root character on the unipotent radical and the chosen type on $\mathrm{U}(k)$. For $v\in V_\tau$ and a K-finite vector u of π, set

$$c_u(g)=\langle b,\pi(g)u\rangle,\qquad W_u(g)=\int_S c_u(sg)\tau(s)^{-1}v\,ds.$$

In particular $c_w(g)=\langle w,\pi(x^{-1}g)w\rangle$; the freedom to choose the real point x is part of the Bessel average.

Proposition 131.6 (Orthogonality after averaging a discrete-series coefficient)**.** *Suppose $|\lambda_i-\lambda_j|>r-1$ for every noncompact root e_i-e_j. Then W_u exists, is smooth and bounded, and belongs in quotient norm to $L^1\cap L^2(S\backslash G)$. It transforms on the left by τ. For some real $m\ge 0$ one has*

$$m=\int_S\langle b,\pi(s)b\rangle\,\langle v,\tau(s)^{-1}v\rangle\,ds,$$
$$\langle W_u,W_{u'}\rangle_{S\backslash G}=\frac{m}{d_\pi}\langle u,u'\rangle.$$

In particular $m=d_\pi\|W_w\|_2^2$. For every compactly supported $\psi\in L^2(G)$,

$$\|T_\psi W_u\|_{L^2(S\backslash G)}\le\frac{\|W_w\|_2}{\sqrt{d_\pi}}\,\|u\|\,\|\psi\|_2.$$

Proof. We use the discrete-series integrability criterion in [2]. For root α the threshold is $\frac14\sum_\beta|\langle\alpha,\beta\rangle|$ in the same metric as the pairing with λ. In type A_{r-1} take roots of squared length two. The two roots $\pm\alpha$ contribute four; $4(r-2)$ further roots each contribute one. This gives $r-1$. The criterion for the semisimple derived group applies here since multiplication by the compact center is a finite covering and the absolute values of coefficients are invariant in that

central direction. Every K-finite coefficient is consequently in $L^1(G)$, including its derivatives: applying any Lie algebra word to a K-finite vector leaves it K-finite by covariance under the finite dimensional adjoint representation. Fixed left or right translations have the same property of integrability of all derivatives. For example a left derivative after left translation is the translation of the derivative conjugated by its group element. This permits $b = \pi(x)w$ at arbitrary x.

Here is the point that allows integration on the noncompact subgroup S. If c is such a coefficient or translate, then $\sup_g \int_S |c(sg)|ds < \infty$. Take a small coordinate box transversal to S in G/S at the identity and a smooth section $t \mapsto z(t)$ with $z(0) = 1$. Evaluation at zero of $c(z(t)sg)$ is bounded by a fixed sum of integrals of absolute mixed derivatives on this box, by the iterated one-dimensional formula used in Theorem 131.5. These derivatives are linear combinations with bounded coefficients of left-translation derivatives of c at $z(t)sg$. The coordinates $z(t)s$ disintegrate Haar as $j(t)\,dt\,ds$ with smooth positive j, bounded below on the closed box. Right translation by g preserves Haar. Thus integrating this point bound in s is bounded by a finite sum of global $L^1(G)$-norms, independent of g. It applies to the derivatives also. Differentiation of W_u along right translations now gives $W_{d\pi(X)u}$; for example the integrated Taylor remainder is bounded by a constant times the square of the increment using the same bound on the second derivative. The argument iterates. It proves smoothness and boundedness; changing the integration variable gives the equivariance. The quotient L^1-norm is at most $\|v\|\|c_u\|_1$ by disintegration. Boundedness gives L^2.

Expand the inner product of the two averages, write the second integration variable in S as hs, and unfold $sg = t$ over G. Absolute interchange follows from

$$\int_G |c_u(t)| \int_S |c_{u'}(ht)|\,dh\,dt < \infty$$

by the uniform restriction estimate just proved. Schur orthogonality, in the normalization defining d_π, gives

$$\int_G \overline{c_u(t)}\, c_{u'}(ht)\,dt = d_\pi^{-1} \langle b, \pi(h)b\rangle\, \langle u, u'\rangle.$$

The value-space inner product is $\langle v, \tau(h)^{-1}v\rangle$. This proves the formulas and the nonnegativity assertion by taking $u = u' = w$. The same computation holds for finite sums of translates of K-finite vectors, since the coefficient bounds used for Fubini still apply. Hence $u \mapsto W_u$ extends by density to a scaled isometry of norm $\|W_w\|_2$ (the zero map if $m = 0$), and intertwines π with right translation. Consequently $T_\psi W_u$ is the image of $\int \psi(y)\pi(y)u\,dy$. Testing this vector against a unit vector and using Cauchy–Schwarz and Schur orthogonality gives norm at most $\|u\|\|\psi\|_2/\sqrt{d_\pi}$. □

Lemma 131.7 (Derivatives of the lowest compact type)**.** *Suppose* $\max_i |\lambda_i| \le L$*, and let the real Lie basis be as in Theorem 131.5. For any word of length* $s \le d = 2n^2$*,*

$$\|d\pi(X_{i_s}) \cdots d\pi(X_{i_1})w\| \le \big[\, 4\sqrt{r}\,(L + 3r + d + 1)\,\big]^s.$$

Proof. Conjugate the form to a diagonal sign matrix by a unitary change of basis, preserving the Hilbert–Schmidt metric. In the Cartan decomposition choose orthonormal bases of the compact and noncompact parts. For $\Re\,\mathrm{tr}(XY)$ the corresponding Casimir is $\Omega_G = \sum_{\mathfrak{p}} X^2 - \sum_{\mathfrak{k}} X^2$; write $C_K = -\sum_{\mathfrak{k}} X^2$. Differentiated unitarity on smooth K-finite vectors gives

$$\sum_i \|d\pi(X_i)u\|^2 = \langle u, (2C_K - \Omega_G)u\rangle.$$

The sum is unchanged under real orthogonal change of basis. The Harish-Chandra parameter gives the usual Casimir scalar $\|\lambda\|^2 - \|\rho\|^2$, with roots of squared length two. On a compact

highest weight μ, C_K has scalar $\|\mu+\rho_c\|^2-\|\rho_c\|^2$. These formulas use the trace form, including the compact center. Equivalently, in the complex matrix basis the quadratic element is $\sum_{ij} E_{ij}E_{ji}$; commuting opposite root vectors into triangular order adds 2ρ to its linear term, which gives the indicated shifted-square formula. The compact calculation uses the same root pairs within the compact blocks.

The lowest compact type has highest weight $\lambda+\rho_n-\rho_c$, by the discrete-series lowest-type theorem [3]; each coordinate differs from the corresponding λ_i by at most r. All weights of this compact type have coordinates bounded in absolute value by $L+r$: within either unitary block the weights lie in the convex hull of the permutations of its highest weight, by the highest-weight theorem. Lie algebra application is a K-map from the tensor product with the adjoint representation. Each weight of that representation is zero or a root, and changes a coordinate by at most one. After t applications every possible compact highest weight has coordinates bounded by $L+r+t$. Since those of ρ,ρ_c are at most r, inserting these bounds in the displayed Casimir identity bounds each next derivative by $4\sqrt{r}(L+3r+t+1)\|u\|$, also for a sum of compact types by their orthogonality. Iteration proves the lemma. □

Theorem 131.8 (Pointwise real estimate with formal degree)**.** *Take the real Bessel average $W=W_w$ above, or a scalar multiple, with a parameter satisfying the gap in Proposition 131.6. With the Gram and support conditions of Theorem 131.5, choose also $P_0\ge 4\sqrt{r}(L+3r+d+1)$. Then*

$$\int_{P'}\|T_\phi W(g_{Z_0}s)\|^2\,ds\le\frac{P_0^{1000n^2}}{c_G\delta(Z_0)^{k-1}}\,\frac{\|W\|_2^2}{d_\pi}\,\|\phi\|_2^2.$$

For W scaled to squared quotient norm $\dim V_\tau$, the numerator of the formal-degree factor is $\dim V_\tau$.

Proof. Apply Proposition 131.6 with u a word in $d\pi(X_i)$ applied to w. Its Bessel average is the corresponding right derivative of W. The derivative lemma gives the hypothesis of Theorem 131.5 with $A=\|W\|_2/\sqrt{d_\pi}$, including the case $W=0$. The construction uses the doubled parameter 2λ; its separation for the undoubled entries is at least $500r^3$, which exceeds $r-1$. The test and the Gram matrix use the normalizations of the preceding theorem. Rescaling multiplies both squared norms by the same factor. □

This estimate holds for every translate x. The identity for m in Proposition 131.6 identifies the positive local period that must be bounded below to choose and control a nonzero normalized vector in a prescribed range of x.

131.4 One open-cell pairing at all infinite places

We estimate the real integral associated with a single rational term of nonzero discriminant. The estimate keeps the cut error as an integral with its original weight. We begin at one real place, retaining G,S,P,g_Z,c_G from the preceding subsections. A point y lies in the frame box of radius ρ if $\|y^{-1}f_j\|\le\rho$ for all normalized frame vectors. Set $\ell(y)=\inf_{h\in S}(\|hy\|+1)$. For $Z\in\Omega$ with $|Z_{ij}|\le\rho^2$ and $\rho\ge 8r$, put

$$H=\frac{\rho^{9n}}{\delta(Z)},\qquad T=H^{140n}.$$

Lemma 131.9 (Bounding a relative position by the two frames)**.** *If y is in the frame box and $\ell(g_Zy)\le\rho$, then $\|y\|\le T$ for the section of Lemma 131.3.*

Proof. We have $\|Z\| \le n\rho^2 \le \rho^3$, $\delta(Z) \le |\det Z|^2 \le \rho^{6n}$, and $H \ge \rho^{3n}$. The adjugate bound yields $\|Z^{-1}\| \le \rho^{3(n-1)}/\sqrt{\delta(Z)}$. Also $(1-|(Z^{-1})_{nn}|^2)^{-1/2} = |\det Z|/\sqrt{\delta(Z)}$. All the parameters needed for Lemma 131.3 are at most H^2, so $\|g_Z\| = \|g_Z^{-1}\| \le H^{64}$. Work in its fixed-coordinate $2n$-space V_*, spanned by the isotropic and dual arms, z and one positive middle coordinate. On this space form the square matrix $D = (\mathcal{F}, g_Z^{-1}\mathcal{F})$. Its joint Gram has off-diagonal block Z^*, so the determinant calculation of Theorem 130.6 at $N = 1$ gives $|\det D| = \sqrt{\delta(Z)}$. We have $\|D\| \le H^{65}$, whence $\|D^{-1}\| \le H^{131n}$ by the singular value product formula.

Choose $h \in S$ with $\|hg_Z y\| \le 2\rho$, and write $h^{-1}\mathcal{F} = \mathcal{F}B$ with B upper unitriangular. The two matrices $y^{-1}\mathcal{F}$ and $y^{-1}g_Z^{-1}\mathcal{F}B$ have norms at most $\sqrt{n}\rho$ and 2ρ, respectively. Their cross Gram is Z^*B, so $\|B\| \le 2\sqrt{n}\rho^2\|Z^{-1}\| \le H^4$. Since $\det B = 1$, $\|B^{-1}\| \le H^{4n}$. It follows that the restriction of $y^{-1}D$ has norm at most H^{5n} and $\|y^{-1}|_{V_*}\| \le H^{136n}$. The negative eigenspace of J is contained in V_*. Writing P_- for its orthogonal projection, any $A \in G$ satisfies $AA^* = J + 2AP_-A^*$ because $AJA^* = J$. This bounds $\|y^{-1}\|^2$ by $1 + 2H^{272n}$, and proves the claim. □

Lemma 131.10 (A set of shifts in the real frame stabilizer)**.** *Let $T \ge 2$, $p \ge 64r^3(2T)^8$, and $C_T = \{g : \|g\| \le T\}$. There is a compact inversion-invariant set $B_p \subset P$ of positive finite measure, with $\|s\| \le p^7$ on B_p, such that $\mathcal{A} = B_pC_T$ satisfies, for every W transforming on the left by a unitary representation of S,*

$$\frac{1}{\mathrm{vol}_P(B_p)} \int_{\mathcal{A}} \|W(y)\|^2 \, dy \le 2(2T^2)^{n(n-1)} \|W\|^2_{S\backslash G}.$$

Proof. In the coordinates of Theorem 131.1, take all $h \in \mathrm{U}(k)$ and balls $\|B\|_{\mathrm{HS}} \le p^3$, $\|K\|_{\mathrm{HS}} \le p^6$. The matrix formula gives the norm bound; inversion takes B to $-h^{-1}B$ and K to $-K$. To bound the integral by quotient disintegration, it suffices to bound the measure of $\{s \in S : sy \in \mathcal{A}\}$. On any left S-orbit meeting $\mathcal{A}$, choose a representative $y_0 \in C_T$. Any s with $sy_0 \in B_pC_T$ is in $B_p(S \cap C_{T^2})$, because $\|y_0^{-1}\| = \|y_0\| \le T$.

For $s_0 \in S \cap C_{T^2}$, both its frame matrix $A \in N_n$ and A^{-1} have norm at most T^2. A lift $l(A) \in S$ of norm at most $(2T^2+2)^3$ is obtained by extending the leading a-block of A diagonally as in Section 131.2 and following it by the shear that adds the top part of its last column to z. The diagonal factor has norm at most $1 + T^2$; the shear has norm at most $(1 + \|A\|)^2 + 1$ by its displayed action there. The element $s_0 l(A)^{-1}$ is in P; its B-coordinate is a matrix block and its K-coordinate is the skew part of a block. Each has Hilbert–Schmidt norm at most $\sqrt{r}\, T^2(2T^2 + 2)^3 \le p$.

Multiplying B_p on the right by any such element of P increases the two radii by at most p and $p + p^4$. Indeed the central cross term in the product is the skew part of a product of the two B-coordinates, up to unitary factors. Its Hilbert–Schmidt norm is at most the product of their norms. The ratio of the enlarged P-volume to the original is thus at most $(1 + p^{-2})^{2ka}(1 + 2p^{-2})^{a^2} \le 2$ for the stated p. Integrate this bound over A using the Haar quotient $P\backslash S = N_n$. Each of its $n(n-1)/2$ complex coordinates has modulus at most T^2, so their volume is at most $(2T^2)^{n(n-1)}$. This bounds the fiber measure and proves the lemma. □

We now allow all f real places. Use one copy of these groups and measures for each place v, and product Haar on G_∞. Let W_v be any scalar multiple of the translated discrete-series Bessel average of Theorem 131.8 and $W = \bigotimes_v W_v$. Let $W_v^\flat$ multiply W_v by a bounded left S-invariant cutoff supported on $\ell(y) \le \rho_v$, and $W^\flat = \bigotimes_v W_v^\flat$. For $\gamma = (\gamma_v)_v$ define

$$K_\gamma(y) = W(\gamma y) \otimes W(y)^*, \qquad K_\gamma^\flat(y) = W^\flat(\gamma y) \otimes W^\flat(y)^*.$$

Write $Z_v = Z(\gamma_v)$ and $H_v = \rho_v^{9n}/\delta(Z_v)$.

Theorem 131.11 (A cut pairing at a rational Gram value)**.** *Suppose $n \ge 32$, $Z_v \in \Omega$, $\rho_v \ge 8r$, and $|(Z_v)_{ij}| \le \rho_v^2$. At each place assume the weight factor $4\sqrt{r}(L_v + 3r + 2n^2 + 1) \le H_v^2$. Let ϕ be a measurable scalar function on G_∞, bounded by one in absolute value and invariant under left translation by $\prod_v P$. Suppose that whenever $\phi(y)K_\gamma^\flat(y)$ is nonzero, each y_v is in the frame box of radius ρ_v. Then this cut integrand is supported in a compact set and*

$$\Big\| \int_{G_\infty} \phi(y) K_\gamma^\flat(y)\, dy \Big\|_{\mathrm{HS}} \le \left(\prod_v \frac{H_v^{10^5 n^4}\, \|W_v\|_{S\backslash G}^4}{d_{\pi_v}\, c_G\, \delta(Z_v)^{k-1}} \right)^{1/2} + \int_{G_\infty} |\phi(y)|\, \|K_\gamma(y) - K_\gamma^\flat(y)\|_{\mathrm{HS}}\, dy.$$

The inequality is valid also for a nonfactorizable weight ϕ.

Proof. First work locally. By Theorem 131.1 we may write $\gamma = s_L g_Z s_R$ with $s_L, s_R \in P$. Use $T = H^{140n}$, $p = H^{2000n}$; the bounds in Lemma 131.9 and $H \ge \rho^{3n}$ verify the hypotheses for the chart, formal degree and Lemma 131.10. Choose s in B_p with its normalized Haar restriction and set $t(s) = s_R^{-1} s^{-1}$. Any z in the frame box with $\ell(\gamma t(s) z) \le \rho$ has $s^{-1}z$ in the frame box, as s fixes the frame; also $\ell(g_Z s^{-1} z) = \ell(\gamma t(s) z)$. Thus $s^{-1}z \in C_T$ by Lemma 131.9, so $z \in \mathcal{A} = B_p C_T$. In particular, at all places together the nonzero cut integrand at $t(s)z$ has $z \in \prod_v \mathcal{A}_v$, since each $t(s)_v$ fixes the frame and the cutoff enforces the length condition. Taking any fixed shifts proves the compact support assertion.

For the full average consider the local map

$$\psi \longmapsto \Big(s \longmapsto \int_{\mathcal{A}} \psi(z) W(g_Z s^{-1} z)\, dz \Big)$$

from $L^2(\mathcal{A})$ to $L^2(B_p; V_\tau)$ with unnormalized measure there. Its squared operator norm is at most $p^{1000n^2} \|W\|_{S\backslash G}^2 / (d_\pi c_G \delta(Z)^{k-1})$ by Theorem 131.8. Indeed $\mathcal{A}$ has norm at most $p^7 T \le p^{20}$, B_p lies in the allowed ball of shifts, and inversion preserves its measure. Apply the tensor product of these operators to the scalar functions

$$\phi(z)\, \overline{\Big(\bigotimes_v W_v(z_v) \Big)_j} \quad \text{on } \prod_v \mathcal{A}_v$$

for each value-space coordinate j. The operator bound multiplies: this follows first on Hilbert tensor products from the positive-operator bound for each factor, and then on product L^2 by density of finite sums of rectangle functions. Thus it holds for the indicated general ϕ. Sum the squared output norms in j. Left equivariance under s_L and $t(s)$ only inserts unitaries on the two value spaces. Since $\phi(t(s)z) = \phi(z)$, the resulting quantity is the integral over the shifts of

$$\Big\| \int_{\prod_v \mathcal{A}_v} \phi(t(s)z) K_\gamma(t(s)z)\, dz \Big\|_{\mathrm{HS}}^2 .$$

On the input side the sum of squared norms is at most the product of $\int_{\mathcal{A}_v} \|W_v(z_v)\|^2\, dz_v$. Divide by the product of the shift volumes and apply Lemma 131.10. At each place the scalar loss, apart from $\|W_v\|_2^4 / (d_{\pi_v} c_G \delta(Z_v)^{k-1})$, is bounded by $2(2T^2)^{n(n-1)} p^{1000n^2} \le H^{10^5 n^4}$: with the chosen T, p its exponent to base H is at most $2{,}001{,}000\, n^3$, and $n \ge 32$.

For every shift the cut integral over $\prod_v \mathcal{A}_v$, after the indicated translation, is the original full-domain cut integral by its support and Haar invariance. Its difference from the corresponding integral of the full kernel is bounded by the displayed tail on G_∞. Integrate this triangle inequality against the probability measure of the shifts and use Cauchy–Schwarz and the preceding second-moment estimate. □

For the shorter construction take $\rho_v = b^{37n}$. Its cut vanishes past length b^{6n+2}, and the prescribed large-node scales have $n \geq 32$ and $r + L_v + 1 \leq b^4$; these bounds imply the numerical hypotheses above. For a near rational term the two frames at a common point are both in the box, so Cauchy–Schwarz there gives $|(Z_v)_{ij}| \leq \rho_v^2$, since J has operator norm one. The adjoint kernel $W(y) \otimes W(\gamma^{-1}y)^*$ is covered by replacing γ with γ^{-1} and conjugating ϕ. The two Gram matrices are adjoints and have the same δ. Corollary 131.2 evaluates the product of the density denominators on these very terms. To bound the global second moment one must also sum the rational orbits, their stabilizer weights, and the cut errors.

131.5 Summing over rational frame products and one normal vector

We return to the rational Hermitian space and to the integral class lattices of Section 130.4. In this subsection P is the rational group fixing every column of $\mathcal{F}$, and U_0 is the kernel of $P \to \mathrm{U}(X_0)$ on the positive middle space. Thus $P = U_0 \rtimes \mathrm{U}(X_0)$. Write $D = \mathrm{N}_{F/\mathbb{Q}}(\Delta)$ when $\Delta \neq 0$. It is a positive integer for any integral class position. We describe which rational orbits are counted by keeping the cross Gram matrix C and the first positive normal $q = q_\gamma$ of Proposition 130.10.

Proposition 131.12 (Rational frame-fixer orbits on the open cell)**.** *Each realized cross Gram matrix C with $\Delta \neq 0$ determines a single $P(F)$-double orbit. The orthogonal complement $Y_\gamma = (T + \gamma T)^\perp$ is totally positive of dimension $k-1$, and the simultaneous pointwise stabilizer of the two frames is $\mathrm{U}(Y_\gamma)$. Its projection to $\mathrm{U}(X_0)$ is the full stabilizer of q_γ. The orbits under $U_0(F)$ on the left and $P(F)$ on the right inside this double orbit are therefore distinguished exactly by q_γ.*

Proof. The joint Gram is nonsingular, with negative index n at every real embedding, by Theorem 130.6. These are all the negative directions in the ambient space. Thus its orthogonal complement is totally positive of dimension $r - 2n = k - 1$. Matching the two frames for terms of the same C gives an isometry of their joint spans. We recall an extension argument over E/F. Choose an orthogonal basis of the common abstract nondegenerate span with nonzero diagonal entries, by Hermitian diagonalization. Suppose two corresponding next vectors u, v have the same nonzero norm and are both orthogonal to the vectors already matched. For $y = u - v$ and $\beta = h(y, u) \neq 0$, the map $t \mapsto t - yh(y, t)/\beta$ preserves the form since $\beta + \bar{\beta} = h(y, y)$. It sends u to v and fixes the preceding vectors. If $\beta = 0$, then $h(u + v, u) = 2h(u, u) \neq 0$; apply the same formula with target $-v$, and then negate on the nondegenerate line Ev. Induction gives the desired rational extension. Since it matches both frames, it fixes $\mathcal{F}$ and matches the two images of $\mathcal{F}$. This proves the double-orbit assertion.

An element fixing both frames is the identity on their span, and is exactly an arbitrary unitary transformation on its orthogonal complement. The projection of Y_γ to X_0 is an isometry: $Y_\gamma \subset T^\perp = L \oplus X_0$ and its form is positive definite. Its image is orthogonal to q_γ, by pairing with $t = l + q_\gamma - (D_n/N)z \in \gamma T$ in Proposition 130.10. Since $h(q_\gamma, q_\gamma) = \Delta/N \neq 0$, dimension forces the image to be the whole hyperplane $q_\gamma^\perp$. Every unitary transformation of this hyperplane extends, through Y_γ, to the simultaneous stabilizer just described. Its action on X_0 fixes q_γ: it fixes t and z, and the middle projection on $L^\perp/L$ intertwines this action. Its image is therefore exactly $\mathrm{U}(q_\gamma^\perp)$, fixing q_γ pointwise.

Right multiplication by $P(F)$ fixes the frame image. On left multiplication by $p \in P(F)$, C stays fixed and q_γ changes to its image by the projection of p. The kernel is $U_0(F)$. Quotienting the left copy of $P(F)$ by this kernel and by the simultaneous stabilizer gives the stated classification. □

Let Λ_c be the integral class lattice in V, and write Q_c for its projection to X_0 in the rational decomposition. It is a full $\mathcal{O}_E$-lattice there. The two hyperbolic arms are primitive direct lattice summands, also after the class change. Removing their coordinates does not affect Q_c.

Lemma 131.13 (Counting integral positive normals without a loss at N)**.** *Fix C and c. Let $\mathfrak{a}$ be the ideal generated by the coordinates of* $\operatorname{adj}(C)e_n^{\mathrm{std}}$*. For an integral term on the p_i-slice, $q_\gamma \in p_i\mathfrak{a}Q_c$. More generally, if the normals being counted belong to $\mathfrak{b}Q_c$ for a nonzero integral ideal $\mathfrak{b}$, their number, when nonzero, is at most*

$$9^{kf}\big(D/\mathrm{N}_{E/\mathbb{Q}}\mathfrak{b}\big)^k.$$

On the p_i-slice one may take $\mathrm{N}_{E/\mathbb{Q}}\mathfrak{b} \ge p_i^{2nf}$.

Proof. For each middle lattice vector $x+\alpha z$ one has $N\alpha \in \mathcal{O}_E$, by pairing with z. For two such vectors the difference between $Nh(x,x')$ and $Nh(x+\alpha z, x'+\alpha' z)$ is $\overline{N\alpha}\,N\alpha'$. Hence $Nh(Q_c,Q_c) \subseteq \mathcal{O}_E$. In particular if $0 \neq u \in \mathfrak{b}Q_c$, then the totally positive $Nh(u,u)$ is in the F-ideal $\mathrm{N}_{E/F}\mathfrak{b}$. Indeed $\overline{\mathfrak{b}}\mathfrak{b}$ is the extension of this ideal, as one checks at split, inert and ramified primes. Thus $\mathrm{N}_{F/\mathbb{Q}}(Nh(u,u)) \ge \mathrm{N}_{E/\mathbb{Q}}\mathfrak{b}$.

Each projected column of $\gamma\mathcal{F}$ is in Q_c, and is in p_iQ_c at the places above p_i on the indicated slice, by the frame-span preservation there. Hence $q_\gamma \in p_i\mathfrak{a}Q_c$. The first a rows of C vanish modulo every component at p_i; their a-minors generate $\mathfrak{a}$. Thus $p_i\mathfrak{a} \subseteq p_i^n\mathcal{O}_E$.

Set $R = D^{1/(2f)}$ and $m = (\mathrm{N}_{E/\mathbb{Q}}\mathfrak{b})^{1/(2f)}$. At embedding v, use the Euclidean norm from Nh and rescale it by $R/\sqrt{\Phi_v(\Delta)}$. The scale factors have product one. By the first paragraph, distinct points of $\mathfrak{b}Q_c$ are at distance at least m in the maximum of the rescaled local norms: their squared norms have product at least m^{2f}. Each of the proposed normals has local radius exactly R at every place. If a normal exists then $R \ge m$, by comparison with zero. Disjoint open balls of radius $m/2$ around these normals lie in the ball of radius $R+m/2$. Volume in this real $2kf$-dimensional normed space bounds the number by $(1+2R/m)^{2kf} \le (3R/m)^{2kf}$, proving the claim. □

For fixed archimedean vectors write $t = 10^5n^4$, $\rho = b^{37n}$, $A_0 = \prod_{v,j} b_{j,v}$ with the scales of Corollary 131.2, and

$$B_x = \rho^{9nft}A_0^{-2(t+k-1)}\prod_v \frac{\|W_v\|_2^4}{d_{\pi_v}c_G}.$$

By that corollary the square of the first term on the right of Theorem 131.11 is exactly $B_xD^{-(t+k-1)}$.

Theorem 131.14 (A sum over the first normal on either marked slice)**.** *Fix c,c' and a marked slice p_i. Sum $B_xD^{-(t+k-1)}$ once for each distinct pair (C,q_γ) from a near integral open-cell term on this slice. With $\rho = b^{37n}$, the sum is at most*

$$B_x(1+2N\rho^2)^{2fn^2}\,9^{kf}\,p_i^{-2nfk}.$$

If the entire residue matrix is fixed at both levels, the first parenthesis can be replaced by $1+2N\rho^2/M$. For sufficiently large nodes of the shorter construction the first bound is at most B_xb^{-18nkf}.

Proof. A near term has normalized Gram entries of modulus at most ρ^2. Before the rescaling at v, the (j,l)-entry therefore has modulus at most

$$R_{jl,v} = N\rho^2\kappa_v^{-(2n-j-l)}.$$

The geometric mean over v is at most $N\rho^2$: the different generator has absolute norm $|\mathrm{disc}(E)|$, so $(\prod_v \kappa_v)^{1/f} = M|\mathrm{disc}(E)|^{1/(2f)}/(2\pi) \geq 1$. To count possible integral entries of C rescale each complex plane by $(\prod_v R_{jl,v})^{1/f}/R_{jl,v}$. The factors have product one. Distinct algebraic integers are separated by at least one in the rescaled maximum norm, since the absolute field norm of their difference is a nonzero integer. The same disjoint-ball argument as above bounds the count by $(1+2N\rho^2)^{2f}$. When the residue is fixed modulo both levels, all possible entries in the same position are congruent modulo M: in class coordinates they are the entries of $\mathcal{F}^* J g \mathcal{F}$ in the same integral Gram model. The separation is then at least M. Multiply these bounds over the n^2 positions.

For any chosen C, Lemma 131.13 counts its first normals by $9^{kf} D^k p_i^{-2nfk}$. Multiplying by the summand leaves $D^{-(t-1)} \leq 1$. This proves both bounds. For the last estimate the separate prime is chosen with $N \leq b^{60n}$, the level satisfies $p_i \geq b^{10}$, and $b \geq 9$. Thus the Gram count costs at most b^{270fn^3}. The family has $n = m/2 \leq \sqrt{x}$ and $k = r - 2n + 1 \geq x^5 - 2\sqrt{x}$, so $k \geq 270n^2$ eventually. The Gram count and 9^{kf} then cost at most b^{2nkf}, giving the assertion. □

For clarity, the finer rational orbit data can be described without counting them. On a fixed C, retain the second normal $q_{\gamma^{-1}}$ and the isometry between the two normal hyperplanes induced by Y_γ: send the positive projection of $t \in Y_\gamma$ to that of $\gamma^{-1}t$. These two projections are onto their respective hyperplanes by Proposition 131.12. If both actions are restricted to $U_0(F)$, the remaining data are exactly the two normals and this isometry. Indeed the stabilizer in the two copies of $\mathrm{U}(X_0)$ of the two normals and the isometry is the graph of $\mathrm{U}(Y_\gamma)$; each of its elements lifts to the unitary on the full space acting as the identity on the joint frame span. This is exactly the image of the stabilizer of γ. Passing further to principal unipotent congruence groups can split these orbits again. Theorem 131.14 counts its specified pairs once; the second moment weights those finer lifts.

131.6 The discriminant of two integral terms with a common link

We prove the arithmetic discriminant gain for two open-cell terms in the same level position whose positive links agree. Keep the original class lattices $\Lambda = \Lambda_c$ and $\Lambda' = \Lambda_{c'}$ in the indefinite space. Write $R = \mathcal{O}_E$, $\Lambda_X = \Lambda \cap X_0$ and $\Lambda_L = \Lambda \cap L = \bigoplus_{i=1}^a Re_i$. Use primes on subscripts for the corresponding sublattices of Λ'. A term γ carries Λ' isometrically to Λ. Let $H_\gamma = q_\gamma^\perp \subset X_0$ and $j_\gamma : H_\gamma \to X_0$ be the isometry induced by the two positive projections of $Y_\gamma = (T + \gamma T)^\perp$ in Proposition 131.12.

For $x \in H_\gamma$ its unique lift to Y_γ has the form $t_\gamma(x) = x + l_\gamma(x)$ with $l_\gamma(x) \in L$. Its inverse image under γ has positive component $j_\gamma x$ and isotropic component in L. The integral lattice of the term is

$$L_\gamma = \{x \in \Lambda_X \cap H_\gamma : l_\gamma(x) \in M\Lambda_L\}.$$

For a positive Hermitian lattice A of rank e define $d(A)$ as its real covolume at the complex embeddings, divided by $(2^{-f}\sqrt{|\mathrm{disc}(E)|})^e$. Use Euclidean volumes for its positive form at each embedding. This gives $d(R) = 1$ for the usual form, and passage to an index i sublattice of full rank multiplies d by i. Equivalently, in any rational basis with Gram G_0 and determinant ideal $\mathfrak{d}$ of the lattice coordinates, one has $d(A) = \mathrm{N}_{E/\mathbb{Q}}(\mathfrak{d})\mathrm{N}_{F/\mathbb{Q}}(\det G_0)$. Indeed this holds in the basis lattice by the real Jacobian at the complex embeddings, and the lattice index gives the general formula after clearing denominators. It agrees with the square root of the index of the local Gram determinant.

Theorem 131.15 (An exact discriminant formula for the level lift lattice)**.** *For an integral term with $\Delta_\gamma \neq 0$, define the finite index*

$$I_\gamma = [R^n : \{(h(\gamma f_j, y))_{j=1}^n : y \in M\Lambda_L + \Lambda_X\}].$$

Then the lattice L_γ has rank $k-1$ and

$$d(L_\gamma) = \frac{M^{2af}\,\mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma)}{I_\gamma}.$$

Proof. Let P here denote the n-by-a matrix $P_{ji} = h(\gamma f_j, e_i)$, and put $Bx = (h(\gamma f_j, x))_j$. All these values are integral on the lattices in question. The matrix P is the first a columns of C^* and has rank a, since C is invertible. Set $u = \mathrm{adj}(C)e_n^{\mathrm{std}}$ and let $\mathfrak{a} = (u_1, \dots, u_n)$. The functional $u^* : R^n \to \overline{\mathfrak{a}}$ is onto. Its kernel is $Y = R^n \cap \mathrm{span}_E(P)$, since $Cu = D_n e_n^{\mathrm{std}}$. The a-minors of P generate $\overline{\mathfrak{a}}$. Smith normal form at each prime gives $[Y : PR^a] = \mathrm{N}_{E/\mathbb{Q}}\mathfrak{a}$. Moreover $u^*Bx = h(q_\gamma, x)$ by the construction of the normal. Write $J = h(q_\gamma, \Lambda_X)$, a nonzero integral ideal contained in $\overline{\mathfrak{a}}$.

Set $A = MPR^a + B\Lambda_X$. It has rank n because $u^*A = J \neq 0$. For $x \in \Lambda_X \cap H_\gamma$ one has $Bx \in Y$, and $l_\gamma(x)$ has coordinates solving $Pl = -Bx$. Therefore $x \mapsto Bx \bmod MPR^a$ identifies $(\Lambda_X \cap H_\gamma)/L_\gamma$ with $(A \cap Y)/MPR^a$. Surjectivity follows because the B-part of any element in $A \cap Y$ is annihilated by u^*. The exact sequences for u^* give

$$[R^n : A] = [Y : Y \cap A]\,[\overline{\mathfrak{a}} : J],$$
$$[\Lambda_X \cap H_\gamma : L_\gamma] = \frac{M^{2af}\,\mathrm{N}_{E/\mathbb{Q}}J}{I_\gamma}.$$

This also proves the asserted rank.

It remains to compute the discriminant of the saturated hyperplane lattice. Project Λ_X orthogonally to Eq_γ. At embedding v, in a unit vector coordinate on this line, its projected coordinates are the embeddings of J divided by $\sqrt{\Phi_v h(q_\gamma, q_\gamma)}$. Covolume in an exact sequence of real Euclidean lattices is the product of the kernel covolume and the projected covolume; this follows by lifting a basis of the image and wedging with a basis of the kernel. Hence

$$d(\Lambda_X \cap H_\gamma) = d(\Lambda_X)\,\frac{\mathrm{N}_{F/\mathbb{Q}}h(q_\gamma, q_\gamma)}{\mathrm{N}_{E/\mathbb{Q}}J}.$$

Finally $d(\Lambda_X) = N^f$ in this construction. Before the class change all middle basis vectors except the distinguished one are ordinary. For that coordinate the integral middle lattice uses $x_*/\bar{\pi} + z/N,\ z$, where the finite idele $\pi\bar{\pi} = N$. A combination has zero z-part precisely when the coefficient of the first vector is divisible by N, so the pure positive coordinate lattice there is $\pi\widehat{R}$. Its ideal norm is N^f by $\pi\bar{\pi} = N$. A finite unitary class change in X_0 has determinant of absolute ideal norm one, since its conjugate times itself is one. Thus it preserves this discriminant. Inserting $h(q_\gamma, q_\gamma) = \Delta_\gamma/N$ cancels the factors of N. Multiplying the last two index formulas finishes the proof. □

Let

$$t_\gamma = [\,\Lambda \cap (L + \gamma L) : \Lambda_L + \gamma\Lambda'_L\,].$$

For a fixed residue class write $z(s)$ for the product of the norms of all marked components where the two reduced full frame spans meet. If character agreement has only been checked above one rational mark, one may form the product just over the components there. In particular on the p_i-Bessel slice the factor $z_i = p_i^{2f}$ is always available.

Lemma 131.16 (A surviving factor after both isotropic arms are removed)**.** *For an agreeing finite residue class, $t_\gamma z(s)$ divides I_γ. On a p_i-Bessel slice $t_\gamma z_i$ divides I_γ even without the agreement at the other mark.*

Proof. The lattice Λ is Hermitian unimodular. Its hyperbolic blocks are unimodular, the changed two-dimensional middle Gram has determinant -1, and the remaining block is the identity. The frame lattice $\Lambda \cap T$ is generated by the columns of $\mathcal{F}$. Its orthogonal lattice is $K = \Lambda \cap T^{\perp} = \Lambda_L + \Lambda_X$. By the perfect pairing and saturation of K, the dual of K identifies with $\Lambda/(\Lambda \cap T)$ via the Hermitian pairing. This is a local statement about free modules at a prime and its conjugate: an integral linear functional on a saturated direct summand extends over the whole ambient lattice. Its kernel in the pairing is the annihilator of K, whose rational span is T.

Enlarging $M\Lambda_L + \Lambda_X$ to K gives an index dividing I_γ. In the indicated perfect dual bases the evaluation map on K is the conjugate transpose of the columns of $\gamma\mathcal{F}$ in $\Lambda/(\Lambda \cap T)$. The index of its image in R^n is therefore the product of residue norms to the index lengths ℓ_w of the full joint frame columns at all finite primes. Indeed append the primitive first frame in adapted lattice coordinates; saturation of the combined columns is saturation of the projected columns in the quotient. Hermitian transpose exchanges conjugate places, which preserves the product of norms. Likewise t_γ is the product using the lengths $\ell_{0,w}$ of the two isotropic arms. At every place $\ell_w - \ell_{0,w} \geq 0$, by wedging in the two terminal columns after saturation as in Theorem 130.11; this part of that argument works over any local discrete valuation ring. At each agreeing meeting component the difference is at least one by that theorem. On a Bessel slice the terminal congruence $gz \equiv z \bmod L$ in both components gives the required dependence after saturation directly. Multiplication yields the divisibility assertions. □

Theorem 131.17 (The common-link discriminant gain for integral pairs)**.** *Suppose γ, η are open-cell terms in the same integral class position and with the same residue at both marked levels. Suppose their positive link graphs agree: $H_\gamma = H_\eta$ and $j_\gamma = j_\eta$ there. Put $L_{12} = L_\gamma \cap L_\eta$. Then*

$$\big(z\, d(L_{12})\big)^2 \leq M^{4af}\, \mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma)\, \mathrm{N}_{F/\mathbb{Q}}(\Delta_\eta)$$

for $z = z(s)$ under finite character agreement, or $z = z_i$ on a p_i-Bessel slice. In particular the bound applies to open terms with the same full link: imposing zero z-coordinate in both middle components of the full link gives precisely the positive graph above.

Proof. We first show $[L_\gamma : L_{12}] \mid t_\eta$. For $x \in L_\gamma$, put $x' = j_\gamma x = j_\eta x$, and write

$$t_\gamma = x + l_\gamma, \quad \gamma^{-1} t_\gamma = x' + l'_\gamma, \qquad t_\eta = x + l_\eta, \quad \eta^{-1} t_\eta = x' + l'_\eta.$$

The subscripts refer to the unique rational lifts. The given finite residue equality implies $(1 - \eta\gamma^{-1})\Lambda \subset M\Lambda$: at all primes over M it is the congruence in the common class basis, and at the other primes M is a unit. Hence

$$u(x) = \frac{t_\gamma - \eta\gamma^{-1} t_\gamma}{M} = \frac{l_\gamma - l_\eta}{M} - \eta\, \frac{l'_\gamma - l'_\eta}{M} \;\in\; \Lambda \cap (L + \eta L).$$

The two summands in $L + \eta L$ are uniquely determined, since $T \cap \eta T = 0$. The class of $u(x)$ modulo $\Lambda_L + \eta\Lambda'_L$ vanishes precisely when $l_\eta \in M\Lambda_L$. In one direction use uniqueness and $l_\gamma \in M\Lambda_L$; in the other, after removing the integral first summand from $u(x)$, the remaining integral vector belongs to $\Lambda \cap \eta L = \eta\Lambda'_L$. This yields the required injection of the quotient and the divisibility. Interchanging the terms gives $[L_\eta : L_{12}] \mid t_\gamma$.

By Theorem 131.15 the two resulting bounds are

$$d(L_{12}) \leq t_\eta\, M^{2af} \mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma)/I_\gamma,$$
$$d(L_{12}) \leq t_\gamma\, M^{2af} \mathrm{N}_{F/\mathbb{Q}}(\Delta_\eta)/I_\eta.$$

Multiply and use Lemma 131.16 on each term. The two isotropic torsion indices cancel, leaving z^{-2}. □

This comparison uses the same lattices as the open-cell integral and the same two levels. When the two positive graphs differ, their common lattice has smaller rank. Bounding and counting those pairs, as well as the terms of zero frame discriminant, is still required for the full second moment.

131.7 The discriminant gain when two open-cell links differ

We now allow distinct positive link graphs. We work first with all middle coordinates and impose the two terminal equations only after comparing the links. Write $R = \mathcal{O}_E$, use the discriminant d of Section 131.6, and put $A = \mathcal{F}_{[1,a]}$, the first a frame columns. For an open-cell term γ set

$$P_\gamma = (A, \gamma A), \quad Z_\gamma = (z, \gamma z), \quad V_\gamma = (L + \gamma L)^\perp.$$

The columns of P_γ are independent and $\dim_E V_\gamma = k + 1$. Every $t \in V_\gamma$ has coordinates $t = l + x + uz$ and $\gamma^{-1}t = l' + x' + u'z$, with $l, l' \in L$, $x, x' \in X_0$. On $D = X_0 \oplus E$ use the positive form $h_D((x, c), (y, d)) = h(x, y) + N\bar{c}d$. Then (x, u') and (x', u) have equal norm, by unitarity of γ. The first coordinate pair determines t: if it is zero then the other pair vanishes by positivity and $t \in L \cap \gamma L = 0$. Dimension now shows that the pairs are related by a unitary map $J_\gamma : D \longrightarrow D$, sending (x, u') to (x', u). Its graph is the full link; its restriction with both last coordinates zero is the positive graph of Proposition 131.12.

Take $\Lambda = \Lambda_c$, $\Lambda' = \Lambda_{c'}$ as before. Let $K^0_\gamma = \Lambda \cap V_\gamma$ and let K^+_γ be the sublattice imposing $l \in M\Lambda_L$. We identify both lattices with their first pair of coordinates in D. Let $\ell_\gamma(t) = (h(z, t), h(\gamma z, t)) = (-Nu, -Nu')$, and set

$$Q_\gamma = |\mathrm{N}_{E/\mathbb{Q}} D_n(\gamma)|, \quad D_\gamma = \mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma), \qquad G_\gamma = M^{2af} \max\{Q_\gamma / N^f, D_\gamma\}.$$

Lemma 131.18 (The full link and its two terminal linear forms)**.** *For each open integral term,*

$$d(K^0_\gamma) = \frac{Q_\gamma}{N^{2f} t_\gamma}, \qquad d(K^+_\gamma) \le \frac{M^{2af} Q_\gamma}{N^{2f} t_\gamma}.$$

At an embedding of F in $\mathbb{R}$, the product of the squares of the two nonzero singular values of ℓ_γ, using h_D and the usual metric on $\mathbb{C}^2$, is $N^2\Delta_\gamma / |D_n(\gamma)|^2$ at that embedding. Each one of the two component functionals has norm $\sqrt{N}$.

Proof. Write H for the ambient Hermitian matrix in rational coordinates, $P = P_\gamma$, $Z = Z_\gamma$, $E_0 = P^*H$ and $E_1 = Z^*H$. On $\ker E_0 = V_\gamma$ the positive graph metric is exactly the restriction of

$$H' = H + N^{-1} E_1^* E_1.$$

Indeed the indefinite norm is $h(x, x) - N|u|^2$ and the added term is $N(|u|^2 + |u'|^2)$. At an archimedean embedding form $B_0 = \left(\begin{smallmatrix} H' & E_0^* \\ E_0 & 0 \end{smallmatrix}\right)$. Its absolute determinant is $|\det H|\, |D_n|^2 / N^2$. To see this, start with the Hermitian block matrix with first row of blocks H, E_0^*, E_1^*, the two lower diagonal blocks 0 and $-NI_2$, and zero between the two lower blocks. Eliminating $-NI_2$ gives $N^2 \det B_0$. Eliminating H gives, up to sign, $\det H$ times the determinant of the joint frame Gram with N added at the two terminal diagonal entries. Its two diagonal frame blocks are now zero and its off-diagonal blocks are C, C^*. Its absolute determinant is $|D_n|^2$.

Here is the lattice consequence of this bordered calculation. The map E_0 sends Λ to a full lattice in R^{2a} of index t_γ, since Λ is Hermitian unimodular and evaluation on the columns of P has the elementary divisors of those columns at the conjugate primes. Choose a rational basis of $\ker E_0$ and then lifts of the standard basis of the target. Call the resulting basis matrix T_0. In this basis $E_0 T_0 = (0, I_{2a})$; expansion along the border gives $\det(H'|_{\ker E_0}) = |\det T_0|^2 |\det B_0|$,

where the left side denotes the determinant in the chosen kernel basis. In these coordinates the determinant ideal of Λ is the product of those of K_γ^0 and $E_0\Lambda$, by the short exact sequence, or by choosing local bases. The basis change contributes the inverse ideal of $\det T_0$. The formula for d in the preceding subsection therefore gives $d(K_\gamma^0) = Q_\gamma/(N^{2f}t_\gamma)$: the factor consisting of the norm of the original determinant ideal of Λ times $|\mathrm{N}_F \det H|$ is one by unimodularity. Imposing the congruence on l has index at most $|\Lambda_L/M\Lambda_L| = M^{2af}$.

For the singular values border B_0 a second time by E_1 on the original ambient coordinates and zero elsewhere; give this new border zero diagonal block. Its absolute determinant is $|\det H|\Delta_\gamma$, because the term $N^{-1}E_1^*E_1$ can be removed by row operations, and the remaining determinant after eliminating H is the joint Gram determinant of the two frames. Divide by $|\det B_0|$. In coordinates $E_0T_0 = (0, I)$ the upper ambient part of B_0^{-1} is the inverse kernel metric on the kernel and zero on the remaining coordinates, as is seen by solving its bordered linear equations. Thus this ratio is the determinant of the Gram of the two functionals on V_γ in the dual positive metric, proving the formula. Finally each functional is N times the last coordinate of a unitary identification with D, whose last coordinate line has squared length N. □

Proposition 131.19 (Congruence on the quotient by the common graph)**.** *For open terms γ, η in one class position with the same residue at both levels, put $I = \ker(J_\gamma - J_\eta)$ in D, $c = \operatorname{codim}_D I$ and $T_i = K_i^+ \cap I$ for $i = \gamma, \eta$. Then*

$$d(T_\gamma)d(T_\eta) \le (2/M)^{2cf} d(K_\gamma^+)d(K_\eta^+).$$

Proof. On the two source spaces consider the cross pairing

$$B(u, v) = h_D(u, v) - h_D(J_\gamma u, J_\eta v).$$

Its left and right kernels are both I, since the two maps are unitary and defined on all of D. Its operator norm is at most two. On $K_\gamma^+ \times K_\eta^+$ it is integral and divisible by M. Indeed if $t_1, t_2 \in \Lambda$ are the two lifts, expansion of the swapped terminal coordinates gives $B(u, v) = h(t_1, t_2) - h(\gamma^{-1}t_1, \eta^{-1}t_2)$. Now $\gamma^{-1}t_1, \gamma^{-1}t_2 \in \Lambda'$, and $(\gamma^{-1} - \eta^{-1})t_2 \in M\Lambda'$ by the shared residue. This proves the congruence.

Let B_i denote the orthogonal projection of K_i^+ to $I^\perp$ for h_D. Rationality of the projection makes these full lattices in the c-dimensional quotient. There B is nondegenerate and $B(B_\gamma, B_\eta) \subset MR$. Taking the determinant of the pairing gives

$$M^{2cf} \le 2^{2cf} d(B_\gamma)d(B_\eta).$$

Explicitly, in rational bases the determinant, times the conjugate determinant ideal of the first lattice and the determinant ideal of the second, generates an ideal contained in M^cR, by taking exterior powers at every finite prime. At an infinite place the absolute determinant of the pairing is at most 2^c times the square root of the product of the two positive Gram determinants. Take the absolute field norm and use the formula for d. Finally $d(K_i^+) = d(T_i)d(B_i)$ by orthogonal projection. The empty determinant is one if $c = 0$. □

Theorem 131.20 (The discriminant of any pair of open terms in one block)**.** *In the setting of Proposition 131.19, let $I_X = \{u \in I : u \in X_0,\ J_\gamma u \in X_0\}$. The common integral lattice of the two positive links is $L_{12} = I_X \cap L_\gamma \cap L_\eta$. Then*

$$d(L_{12})^2 \le (2/M)^{2cf} G_\gamma G_\eta.$$

For $c = 0$ the finer bound of Theorem 131.17 also applies with its meeting-place factor. All common dimensions in this assertion are allowed, including zero.

Proof. The two terminal maps ℓ_i coincide on I, since the input and output coordinates there agree. Their common kernel in I is I_X. Write $T_{i,X} = T_i \cap I_X$ and s for the rank of this terminal map on I. It has integral values on T_i by its definition on the integral lifts. We give bounds $d(T_{i,X}) \le A_i d(T_i)$.

If $s = 0$, take $A_i = 1$. If $s = 1$, choose a nonzero one of the two component functionals on I, so its kernel there is I_X. Its image lattice is a nonzero ideal inside R. Its real Jacobian on the orthogonal line to its kernel at each place is at most N by Lemma 131.18. Thus one can take $A_i = N^f$. If $s = 2$, use the two components together. Their image lattice in R^2 has discriminant at least one for the standard positive metric. The squared exterior norm on a subspace is bounded by that on the entire space, and Lemma 131.18 gives $A_i = N^{2f} D_i/Q_i$. In both positive ranks the covolume step is the same: the discriminant of the projected lattice orthogonal to the kernel is the image discriminant divided by the product of the real Jacobians on that complement. Divide $d(T_i)$ by the projected discriminant to get the kernel discriminant. Together with Lemma 131.18 all three cases give $A_i d(K_i^+) \le G_i/t_i$.

The lattice $T_{i,X}$ is $L_i \cap I_X$ in the positive coordinates: both terminal components vanish and its lift is integral with the indicated level condition on the isotropic part. We claim $[T_{\gamma,X} : L_{12}] \mid t_\eta$, and the reverse bound with the terms exchanged. For $u \in T_{\gamma,X}$ the positive images $j_\gamma u, j_\eta u$ coincide. Therefore the class of $(1-\eta\gamma^{-1})t_\gamma(u)/M$ in $(\Lambda \cap (L+\eta L))/(\Lambda_L + \eta\Lambda'_L)$ is defined, with kernel exactly L_{12} by the decomposition in the proof of Theorem 131.17. This argument uses agreement of images at u, which holds for every $u \in I_X$. Multiplying the resulting index inequalities and applying Proposition 131.19 gives

$$d(L_{12})^2 \le t_\gamma t_\eta A_\gamma A_\eta (2/M)^{2cf} d(K_\gamma^+) d(K_\eta^+) \le (2/M)^{2cf} G_\gamma G_\eta.$$

At $c = 0$ the two full unitary maps are equal and their positive graphs agree. □

The factors in this estimate come from congruence on the actual two class lattices. It covers each pair with nonzero frame discriminants. To apply it in a second moment one still needs the multiplicities of the integral pairs and estimates for the other cell ranks.

131.8 The pair discriminant at every rational cell rank

Keep the Hermitian space, the two class lattices and the level $M = p_0 p_1$ of the preceding subsection. We treat the terms for which the two frames have a common vector. Their rational lifts can have a common isotropic summand. We first remove that summand over $R = \mathcal{O}_E$.

Suppose that the characters agree on the rational stabilizer. For γ with $\Delta_\gamma = 0$, let j be its rank in Theorem 130.6, put $U = L \cap \gamma L$, and write $U_R = U \cap \Lambda$. Let $A_j = (e_1, \dots, e_j)$. A bar on a vector in $U^\perp$ denotes its image in $U^\perp/U$. Define

$$t_\gamma = [\, \Lambda \cap (L + \gamma L) : \Lambda_L + \gamma\Lambda'_L \,],$$
$$i_\gamma = [\, \Lambda_L/U_R : \bar{A}_j R^j \,], \qquad i'_\gamma = [\, \gamma\Lambda'_L/U_R : \overline{\gamma A_j} R^j \,].$$

The columns in each denominator span its numerator over E. In fact the common flag from Theorem 130.6 gives $L = \langle A_j \rangle \oplus U$ and $\gamma L = \langle \gamma A_j \rangle \oplus U$. The index i_γ can equally be computed from the first j isotropic rows of the frame product matrix inside the lattice generated by all its isotropic rows. Indeed evaluation of $l \in \Lambda_L$ against the second frame annihilates U_R, because $U \subset \gamma L$. Its restriction to A_j has rank j by $D_j \neq 0$, so its kernel is exactly U_R. Evaluation against the first frame proves the corresponding description of i'_γ. All indices here are absolute orders of finite quotients.

Lemma 131.21 (The two terminal vectors in the isotropic quotient)**.** *The lattice $\bar{\Lambda} = (\Lambda \cap U^\perp)/U_R$ is Hermitian unimodular. In $\bar{V} = U^\perp/U$ the subspaces $\bar{L}$ and $\overline{\gamma L}$ pair perfectly over E.*

Their sum has an orthogonal complement W of dimension $k+1$. If $d_* = \det C_{\{1,\dots,j,n\},\{1,\dots,j,n\}}$, then

$$d_* = -N\zeta D_j,$$

where ζ is the scalar for this term in Theorem 130.6. We include $j = 0$ by $D_0 = 1$. The two terminal linear forms $t \mapsto h(z,t), h(\gamma z, t)$ on W have joint rank one.

Proof. The module U_R is saturated in Λ. Locally it is a direct summand. Since Λ is self-dual, evaluation on U_R is onto its conjugate dual. Choose a local basis of U_R and dual lifts. Their joint Gram matrix has blocks $\left(\begin{smallmatrix} 0 & I \\ I & B \end{smallmatrix}\right)$, whose determinant is a unit. The summand they generate splits off orthogonally by integral projection. Its contribution to $U_R^{\perp}/U_R$ is zero, and the remaining summand is self-dual. At a split place take bases and dual lifts in the paired components of the local ring $R \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$. The same block matrix is invertible there. This argument applies also above two. Local self-duality gives the assertion about $\bar{\Lambda}$.

In the displayed bases of the two arms the cross pairing has determinant $D_j \neq 0$. The arms have zero intersection and are each isotropic. Together they span a nondegenerate $2j$-space in the $(2j+k+1)$-space $\bar{V}$. Both terminal vectors are orthogonal to U. Subtract their orthogonal projections onto the sum of the two arms. For $\bar{z}$ the subtracted vector belongs to $\bar{L}$, since $\bar{z} \perp \bar{L}$ and the two arms pair perfectly. For $\overline{\gamma z}$ it belongs to $\overline{\gamma L}$. The resulting vectors $z_0, z_1 \in W$ both have norm $-N$. The common vector at the terminal level of the two flags gives $\gamma z - \zeta z \in L + \gamma L$, whence $z_1 = \zeta z_0$. Changing each terminal vector by its indicated arm is a row or column operation of determinant one on the subframe matrix. Its remaining terminal entry is $h(z_0, z_1) = -N\zeta$. This proves the formula and shows that the two linear forms on W are proportional and nonzero. □

For any rank use the positive space $D = X_0 \oplus E$, with squared norm $h(x,x) + N\bar{u}u$. Representatives of a class of W may be taken in $(L+\gamma L)^{\perp}$. As in the open case write

$$t = l + x + uz, \qquad \gamma^{-1}t = l' + x' + u'z.$$

Changing t by a vector of U leaves both middle components fixed. The pairs (x, u') and (x', u) have equal positive norm by unitarity. If one vanishes, both vanish and $t \in U$. Since $\dim_E W = \dim_E D$, the first pair identifies W with D, and the second differs from it by a unitary map J_γ. Write K_γ^0 for the image in D of $\Lambda \cap (L+\gamma L)^{\perp}$. Write K_γ^+ for the image of those representatives that satisfy $l \in M\Lambda_L$. The lattice L_γ of the positive link is obtained by imposing $u = u' = 0$ on K_γ^+.

Proposition 131.22 (The determinant of the integral link in lower rank)**.** *For $0 \le j \le a$ in the preceding notation,*

$$d(K_\gamma^0) = \frac{|\mathrm{N}_{E/\mathbb{Q}} d_*|}{N^{2f} t_\gamma i_\gamma i'_\gamma}, \qquad [K_\gamma^0 : K_\gamma^+] \le M^{2jf}.$$

Each terminal functional has norm $\sqrt{N}$ for the positive metric at every real place.

Proof. In $\bar{V}$ take $P = (\bar{A}_j, \overline{\gamma A_j})$, $Z = (\bar{z}, \overline{\gamma z})$, and write H for the Hermitian matrix. The lattice K_γ^0 identifies with $\bar{\Lambda} \cap \ker(P^* H)$. Indeed any integral lift of such a class is already orthogonal to both arms. The positive metric on this kernel is the restriction of $H + N^{-1}(Z^*H)^*(Z^*H)$, as is seen by substituting the two expressions for t. The bordered determinant with border P^*H therefore has absolute value $|\det H|\,|d_*|^2/N^2$ at each embedding. Explicitly, adjoin the two rows Z^*H with diagonal block $-NI_2$. Eliminating that block contributes N^2. Eliminating

H instead gives $\det H$ times, up to sign, the Gram matrix of the two subframes with both terminal diagonal entries changed to zero. The latter determinant has absolute value $|d_*|^2$.

The image of P^*H on $\bar{\Lambda}$ has index $t_\gamma i_\gamma i'_\gamma$ in R^{2j}. By self-duality, local elementary divisors of the evaluation map are those of the columns of P at the conjugate prime. Their product of absolute norms is the index of PR^{2j} in its saturation. Saturating the sum of the two full arm lattices costs precisely t_γ, since U_R lies in both arms and a lift of a class in their rational sum belongs to $L+\gamma L$. Inside the sum the projected arms are disjoint. Their separate indices therefore multiply to $i_\gamma i'_\gamma$. Apply the basis-and-image calculation of Lemma 131.18 to this border. The ambient determinant ideal times the absolute norm of the ambient Gram determinant is one by unimodularity. This proves the first identity, including the empty border when $j = 0$.

In the class lattice the top isotropic coordinates are an integral direct summand, so for an integral representative the component l belongs to Λ_L. Assign its class modulo $M\Lambda_L + U_R$ to the element of K^0_γ. This is independent of the representative and has kernel K^+_γ. The target is the reduction modulo M of the rank j projective module Λ_L/U_R. Its order is M^{2jf}, by local freeness. Finally the functionals have values $-Nu, -Nu'$. Each is N times the last coordinate of a unitary identification with D, so has squared norm N. □

For an open term write $j = n = a + 1$, $d_* = D_n$, $i_\gamma = i'_\gamma = 1$. For a term of any cell rank put

$$G_\gamma = M^{2\min(j,a)f} \max\left\{ \frac{|\mathrm{N}_{E/\mathbb{Q}} d_*|}{N^f i_\gamma i'_\gamma}, \mathrm{N}_{F/\mathbb{Q}} \Delta_\gamma \right\}.$$

The second quantity in the maximum vanishes at every lower rank.

Theorem 131.23 (The common lattice at the two split levels)**.** *Take two integral terms γ, η between the same class lattices, with the same residue at both levels, each satisfying the rational frame conclusion of Theorem 130.6. Set $I = \ker(J_\gamma - J_\eta)$, $c^+ = \operatorname{codim}_D I$ and*

$$I_X = \{ y \in I : y \in X_0,\ J_\gamma y \in X_0 \}, \qquad L_{12} = I_X \cap L_\gamma \cap L_\eta.$$

Then L_{12} has full rank in I_X, and

$$d(L_{12})^2 \le (2/M)^{2c^+ f} G_\gamma G_\eta.$$

Under finite character agreement, if $c^+ = 0$ and one term is open, the left side can be multiplied by $z(s)^2$, the square of the product of the norms of the meeting places. In all cases

$$k - \dim_E I_X \ \le\ c^+ + \min\left\{ \mathbf{1}_{\Delta_\gamma \neq 0}, \mathbf{1}_{\Delta_\eta \neq 0} \right\}.$$

Proof. The comparison of full links in Proposition 131.19 applies with the quotient representatives just constructed. Indeed its cross pairing is $h(t_1, t_2) - h(\gamma^{-1} t_1, \eta^{-1} t_2)$. Its expression in middle coordinates is independent of the representatives, and choosing integral ones puts its values in MR by the two residue congruences. On the quotient by I this pairing is nondegenerate and has operator norm at most two. Thus for $T_i = K^+_i \cap I$ one has the same determinant inequality

$$d(T_\gamma) d(T_\eta) \le (2/M)^{2c^+ f} d(K^+_\gamma) d(K^+_\eta).$$

On I the two terms have the same terminal forms. If these forms have joint rank s, then $\dim I_X = k+1-c^+-s$. On all of D their rank is one for a lower-rank term by Lemma 131.21 and two for an open term by Lemma 131.18. This proves the dimension bound. Put $T_{i,X} = T_i \cap I_X$. If either term has lower rank, then $s \le 1$. At rank zero $T_{i,X} = T_i$. At rank one choose a nonzero

component on I. Its image on T_i is a nonzero integral ideal by evaluation on integral lifts. Its real Jacobian on the line transverse to the kernel is at most N at every embedding. The orthogonal projection formula for covolume therefore yields $d(T_{i,X}) \le N^f d(T_i)$. Multiplying the preceding inequality by the costs of these two cuts gives

$$d(T_{\gamma,X})d(T_{\eta,X}) \le (2/M)^{2c^+ f}\frac{G_\gamma G_\eta}{t_\gamma t_\eta}.$$

Here Proposition 131.22 supplies the factor for each lower-rank term. An open term in a mixed pair uses the first estimate of Lemma 131.18, so its minor term in the maximum suffices. If both terms are open, the same inequality, allowing $s = 2$, was proved in Theorem 131.20 before the final passage to the common integral lattice.

It remains to justify that passage when the lifts are nonunique. For $x \in T_{\gamma,X}$, choose a level lift $t_\gamma = x + l_\gamma$, $l_\gamma \in M\Lambda_L$. Choose any rational η-lift $t_\eta = x + l_\eta$ to the same positive graph point over I_X. The two inverse images have the same positive coordinate x', with respective top components $l'_\gamma, l'_\eta \in L$. Then

$$v = \frac{(1-\eta\gamma^{-1})t_\gamma}{M} = \frac{l_\gamma - l_\eta}{M} - \eta\frac{l'_\gamma - l'_\eta}{M} \in \Lambda \cap (L + \eta L).$$

The numerator is divisible by M on Λ because the two reductions agree. Any two level lifts for γ differ by a vector in $M(\Lambda \cap L \cap \gamma L)$. Consequently the class of v modulo $\Lambda_L + \eta\Lambda'_L$ is well defined and additive.

Suppose it is zero and choose $a_0 \in \Lambda_L, b_0 \in \Lambda'_L$ with $v = a_0 + \eta b_0$. The difference $d_0 = (l_\gamma - l_\eta)/M - a_0$ lies in $L \cap \eta L$ by the displayed expression. Replacing t_η by $t_\eta + Md_0$ gives a lift with top component $l_\gamma - Ma_0 \in M\Lambda_L$. It is integral, since $x \in \Lambda \cap X_0$ for the chosen γ-lift. Conversely, if an η-level lift exists, its use in the formula makes the first summand integral. The remaining summand belongs to $\Lambda \cap \eta L = \eta\Lambda'_L$. The kernel of the class map is therefore precisely L_{12}. This proves $[T_{\gamma,X} : L_{12}] \mid t_\eta$, and interchanging the terms gives $[T_{\eta,X} : L_{12}] \mid t_\gamma$. These finite indices prove the rank assertion. Multiplication cancels the two torsion indices and gives the asserted inequality.

Finally $c^+ = 0$ makes the two maps J equal everywhere. Their joint terminal ranks coincide, so if either term is open both are open. Their positive graphs agree, and Theorem 131.17 gives the stated meeting factor. □

The theorem proves the arithmetic pair estimate used in `New/Spanning/RemainingTerms/Gain.lean` on the class lattices of the shorter construction. Summing the orbital integrals also requires the number and the analytic weights of pairs with a given common lattice.

131.9 The weighted count of lattices in the marked classes

The common lattice in Theorem 131.23 lies in $\Lambda_X = \Lambda_c \cap X_0$. Its discriminant enters the period estimate with weight $d(L)^{-k}$. We sum this weight over every lattice of a prescribed corank in the actual source classes. The prime N in this calculation is the norm of the negative terminal vector with sign reversed. It is distinct from the two primes dividing M.

Put $R = \mathcal{O}_E$, $f = [E : \mathbb{Q}]/2$, and $\rho_E = |\operatorname{disc} E|^{1/(2f)}$. We first work in an arbitrary positive Hermitian unimodular lattice $\mathcal{U}$ of rank k over R. For an E-subspace V put $d_V = d(\mathcal{U} \cap V)$, with the normalized covolume of Section 131.6.

Lemma 131.24 (The sum over rational subspaces). *For $0 \le \ell \le k$ and $k \ge 1$,*

$$\sum_{\operatorname{codim}_E V = \ell} d_V^{-k-1} \le (25fk\rho_E)^{f(k+1)\ell}.$$

Proof. For a lattice P in its rational span, write $P^{\#}$ for its Hermitian dual there. The determinant-ideal formula for normalized covolume gives $d(P^{\#}) = 1/d(P)$. Orthogonal projection of $\mathcal{U}$ onto $V^{\perp}$ has dual $\mathcal{U}^{\#} \cap V^{\perp} = \mathcal{U} \cap V^{\perp}$. Indeed pairing a vector of $V^{\perp}$ with a projected vector gives its original pairing. The projection is rational and its image is a lattice, so taking its double dual recovers it. Multiplicativity of covolumes on the kernel and the orthogonal projection now gives $d_V = d_{V^{\perp}}$. Also $d_V \geq 1$, because the integral lattice in V is contained in its dual. For $\ell = 0$ the asserted sum is one.

Use the Euclidean norm $\|y\|^2 = \sum_{v|\infty} h_v(y,y)$ on all the complex embeddings in the CM type. Put $P = \mathcal{U} \cap V^{\perp}$ and $m = 2f\ell > 0$. Its real covolume is $(2^{-f}\rho_E^f)^{\ell} d_V$. Let $\lambda_1, \dots, \lambda_m$ be its Euclidean successive minima. We record the ball case of Minkowski's inequality: $\omega_m \prod_i \lambda_i \leq 2^m \operatorname{covol}(P)$, where ω_m is the unit ball volume. Choose successively a shortest lattice vector outside the real span of the ones already chosen, and orthonormalize their flag. The lengths at these choices are the successive minima, since i independent vectors of length at most λ_i cannot lie in a space of dimension $i-1$. In these orthonormal coordinates dilate axis i of the unit ball by λ_i. A nonzero lattice vector whose last nonzero coordinate is i has length at least λ_i, while a vector in the open ellipsoid with that last coordinate has length strictly less than λ_i. The half-sized open ellipsoid thus has disjoint translates by P. Comparing volume with a fundamental domain proves the inequality.[1]

Next choose $x_1, \dots, x_\ell$ in P by successively minimizing length outside their previous E-span. The real span at all embeddings of an E-space of dimension $i-1$ has dimension $2f(i-1)$. Hence $\|x_i\| \leq \lambda_{2f(i-1)+1}$. The unit ball contains a cube of side $2/\sqrt{m}$. Multiplying the minima in groups of $2f$ therefore gives

$$\prod_{i=1}^{\ell} \|x_i\|^{2f} \leq m^{m/2} (2^{-f}\rho_E^f)^{\ell} d_V = (f\ell\rho_E)^{f\ell} d_V.$$

The tuple determines V by taking its orthogonal complement.

For nonzero $y \in \mathcal{U}$, the absolute norm to $\mathbb{Q}$ from F of the totally positive algebraic integer $h(y,y)$ is at least one. The arithmetic-geometric mean inequality gives $\|y\|^2 \geq f$. Distinct lattice points are thus at least one apart. In dimension $D = 2fk$ the number in a ball of radius $t \geq 1$ is at most $(2t+1)^D$, by packing balls of radius $1/2$. For $p = 2f(k+1)$, integrate pt^{-p-1} against the count of nonzero vectors of norm at most t. This gives

$$\sum_{0 \neq y \in \mathcal{U}} \|y\|^{-p} \leq 3^{2fk}(k+1) \leq 25^{f(k+1)}.$$

Choose one of the preceding tuples for each V. Raising the product inequality to $k+1$ bounds d_V^{-k-1} by $(f\ell\rho_E)^{f(k+1)\ell} \prod_i \|x_i\|^{-p}$. Sum over all tuples of nonzero lattice vectors and use $\ell \leq k$. □

At a node in the shorter construction $N \equiv 1 \bmod q_1 d$, so N splits completely in $E \subseteq \mathbb{Q}(\mu_{q_1 d})$. At its f pairs of split places the distinguished middle coordinate imposes one row congruence at one member of each pair. More explicitly, in the middle basis $x_*/\bar{\pi} + z/N$, z of Section 131.6, a pure X_0-vector with distinguished coordinate v_* has those two coefficients $\bar{\pi} v_*$ and $-v_*/\pi$, where $\pi\bar{\pi} = N$. Their integrality is equivalent to $v_* \in \pi R$ locally. The other pure coordinates are integral. For a class of a definite isometry g, this condition is read in g^{-1} coordinates.

Proposition 131.25 (The average over the terminal rows)**.** *Let $\mathcal{C}$ be the finite set of the definite classes at the source level. Forget the level of a class and retain its underlying unimodular*

[1] For the successive-minima inequality for general symmetric convex bodies, see R. P. Bambah, A. Woods and H. Zassenhaus, Journal of the Australian Mathematical Society **5** (1965), 453–462.

lattice $\mathcal{U}$, up to isometry. To average an isometry-invariant sum on the classes over this $\mathcal{U}$, one can average uniformly over $(N^k-1)^f$ tuples of nonzero residue covectors when the summand depends only on their common kernel in $\mathcal{U}$. This common kernel is the marked lattice Λ_X. For a fixed rank $r=k-\ell$ sublattice L of $\mathcal{U}$, put $P=\mathcal{U}\cap EL$, $i=[P:L]$. The fraction of row tuples for which L belongs to the kernel is at most

$$\frac{iN^{f\ell}}{(N^k-1)^f}.$$

Proof. Fix a finite adelic representative of the coarse class and let K be its standard integral level in lattice coordinates. The finer classes over it are the orbits on K/K_{src} under the finite automorphism group of $\mathcal{U}$ in these coordinates. This follows by writing two representatives as gk, gk' in the double quotient and cancelling g. Finiteness of the automorphism group follows from discreteness on a real Euclidean lattice and compactness of its isometry group. It acts freely on these cosets. In fact stabilizing a finer coset gives a finite-order integral matrix congruent to the identity at the odd split prime p_0, where the source level is principal. The torsion calculation in Theorem 130.2 applies: if $1+p^aB$ has a nonzero first residue $B \bmod p$, prime-to-p powers preserve its depth and the p-th power increases the depth by exactly one. Such a matrix has infinite order. Thus every orbit has the same size and a uniform average over finer classes equals the coset average for an invariant sum.

The image of K in the split component at each chosen place over N is $\mathrm{GL}_k(\mathbb{F}_N)$: lift an invertible residue matrix integrally and use its inverse transpose in the conjugate component. These choices are independent at different places. Under $k\in K$ the distinguished row becomes the distinguished row of k^{-1} in the fixed coarse coordinates. The subgroup K_{src} fixes this row pointwise modulo the chosen prime. Indeed its embedded elements preserve the middle lattice and fix z. For an integral X_0-matrix A, the coefficient along z in the image of a middle basis vector x_j with $j\neq *$ is $-A_{*j}/\pi$. In the image of $x_*/\bar{\pi}+z/N$ the coefficient is $(1-A_{**})/N$. Hence on the chosen component the required row is the original unit row modulo π. Preservation at the conjugate place is compatible with the inverse transpose. The coordinate description before the proposition identifies the pure lattice with the row kernel. Transitivity on nonzero rows and equal fiber sizes now give the uniform row average.

At a chosen place choose a local basis of $\mathcal{U}$ beginning with a basis of the saturation P. By local elementary divisors take L to be generated there by $\varpi^{a_j}e_j$, $1\le j\le r$. A residue covector annihilating L must have zero j-th entry if $a_j=0$. Allowing also the zero covector bounds its number by $N^{\ell+\#\{j:a_j>0\}}$. The product of $N^{\#\{j:a_j>0\}}$ over the chosen places is at most the global index i, by $\#\{j:a_j>0\}\le\sum_j a_j$. Division by the total number of rows proves the bound. □

Theorem 131.26 (The corank sum on the period classes)**.** *Write $n=a+1$, $H=b^{n^3f}$, $\tau=b^{kf/4}$. At every sufficiently large node of the shorter construction on $V_0\le x\le V_0^{3/2}$, and for every $0\le\ell\le k$,*

$$\sum_{c\in\mathcal{C}}\sum_{\substack{L\subseteq\Lambda_c\cap X_0\\ \mathrm{rank}_R\,L=k-\ell,\ d(L)\le H}} d(L)^{-k}\le\frac{|\mathcal{C}|}{N^{kf}}\tau^{\ell+1}.$$

Here L ranges over all R-lattices of the indicated rank in their rational spans, with the zero lattice of discriminant one at rank zero.

Proof. First fix a coarse lattice and average as in Proposition 131.25. Group the lattices L by $V=EL$, and write $d=d_V$ and $i=[\mathcal{U}\cap V:L]$. Then $d(L)=di$, $d,i\ge1$. Put $\mu=2a+4$,

$r = k - \ell$. When $di \le H$ one has

$$i(di)^{-k} \le H^{\mu+1} d^{-k-1} i^{-r-\mu}.$$

Indeed after division the right side is at least $d^{\mu} i^{\ell}$ times the left side.

For a rank r projective R-lattice P the sum of $[P : L]^{-s}$ over its full-rank sublattices equals $\prod_{j=0}^{r-1} \zeta_E(s-j)$ for $s > r$. Locally at a prime of norm q, project onto the last coordinate and fix the intersection with the first $r-1$ coordinates. If that intersection has index t and the last image is $\varpi^m R_w$, there are t choices for the first coordinates of a lift of ϖ^m, and the total index is tq^m. The local series is thus $(1-q^{-s})^{-1}$ times the rank $r-1$ series at $s-1$. Every finite index sublattice is determined by these choices at its finitely many exceptional primes. Conversely their intersection inside P realizes the choices, by the Chinese remainder theorem. This proves the formula including nonfree P. At $s = r+\mu$ it is at most 2^r for our fields. Indeed $\zeta_E(t) \le \zeta(t)^{2f}$ by the Euler factors, and $\log \zeta(t) \le 2^{-t}(1 + 2/(t-1))$ for $t > 1$, by summing the tail by integration. For $t = \mu + 1 = 2a + 5$ and $(a+1)2f \le 2^{2a}$ these inequalities give $\log \zeta_E(t) < \log 2$.

Combining these bounds with Lemma 131.24 and $(N^k - 1)^{-f} \le 2^f N^{-kf}$ bounds the conditional mean over each coarse class by

$$N^{-kf} 2^{f+k} H^{2a+5} \{ N^f (25 f k \rho_E)^{f(k+1)} \}^{\ell}.$$

All terms summed are nonnegative, so finite partial sums justify the changes of order. The bound holds with the same constant for every coarse class, hence for the unweighted mean on $\mathcal{C}$.

We verify the sizes on the indicated interval. Write $m = 2n$ for the number of lifts and r_0 for the target rank. Here $\sqrt{x} \le m \le 2\sqrt{x}$, $x^5 \le r_0 \le 3x^5$, $k = r_0 + 1 - m$, and $2^s = m^{22}$. The cyclotomic field $\mathbb{Q}(\mu_{2^s}) \subseteq E$ gives $f \ge 2^{s-2}$. The inclusion $E \subseteq \mathbb{Q}(\mu_{q_1 d})$ and $q_1 d \ll_{q_0, d} x^{11} \log(2x)$ also give $2f, \rho_E \le q_1 d$. For the latter inequality the cyclotomic discriminant formula bounds its root discriminant by its conductor, and the relative discriminant formula can only lower the root discriminant in a subfield. Consequently $k \le f$, $(a+1)2f \le 2^{2a}$, and $k \ge 1000(2n)^9$ for large x. The construction imposes $x^{10^6} \le x_1$, $x_1^{10^6} \le b$ and $\log N \le 30 m \log b$. Thus $(200 f k \rho_E)^{1000} \le b$, $N \le b^{60n}$ and $b \ge 1000$ throughout the interval for large V_0. We obtain $2^{f+k} \le b^f$ and

$$1 + n^3(2a+5) \le k/4, \qquad 60n + (k+1)/1000 \le k/4.$$

The first inequality absorbs the factor $2^{f+k} H^{2a+5}$ in τ. The second absorbs the factor inside braces in τ. Multiplication by $|\mathcal{C}|$ proves the claim. □

The sum includes every common lattice of the preceding theorem. To pass from lattices to orbital pairs one must also count the distinct rational terms over a fixed lattice and residue label. Multiple such terms can have the same lattice.

131.10 Rational frames over a lattice and the weighted pair count

We count the multiplicity with which a lattice in Theorem 131.26 can occur. Fix the first class c and write $\Lambda = \Lambda_c$. For the image of the frame under a term from c' write its blocks in the original rational coordinates as

$$\gamma \mathcal{F} = \begin{pmatrix} A_\gamma \\ X_\gamma \\ v_\gamma \\ B_\gamma \end{pmatrix}.$$

The four blocks have rows along $L, X_0, Ez, L^\vee$, respectively. The entries of v_γ are the scalar coefficients along z. We order the last block by the duals of the basis of the first block. The cross products $C = \mathcal{F}^* h \gamma \mathcal{F}$ specify both $B_\gamma = C_{[1,a],*}$ and $v_\gamma = -C_{n,*}/N$. Each column of the frame is integral in Λ, and its top and bottom coordinates are integral. Two frames with the same C have their positive blocks differing by columns of Λ_X, since the terminal and bottom parts cancel and the top part can be subtracted integrally.

Let B_{res} be the image of the integral finite Bessel group in the product of the two level groups. Choose one representative for each coset sB_{res} and write a residue $s = s_0 t$ with $t \in B_{\mathrm{res}}$; call t its label. A term is counted once per double coset under the two rational pointwise unipotent frame fixers of principal level. We count cosets admitting a near representative. At both y and $\gamma^{-1} y$, near means that the points lie in the reduced frame domain and the Bessel length at every real place is at most b^{6n+2}. We take the rank j of a surviving term as in Theorem 130.6, with $j = n$ in the open case. Write $\delta = D_j \neq 0$ and form the n-by-n matrix Y from the first j rows of C followed by the remaining rows of the identity. Then $\det Y = \delta$.

Lemma 131.27 (Bounded invariants of near frames)**.** *At sufficiently large nodes of the interval of Theorem 131.26 there are at most $b^{270 n^3 f}$ possible matrices C for near integral terms. For fixed j, C, at most $b^{502 n^4 f}$ arrays occur as the top coordinates of the columns $\gamma \mathcal{F} \operatorname{adj}(Y) e_i$, $j < i \leq n$. These columns belong to T, and are unchanged by every pointwise frame fixer on the left. Moreover the gauge of Theorem 131.23 satisfies $G_\gamma \leq H = b^{n^3 f}$.*

Proof. Use the normalized frame at a real place, with scale factors $|s_h| = \kappa_v^{n-h}/\sqrt{N}$ from Corollary 131.2. We have $1 \leq \kappa_v \leq b^{23}$ for all these places. Indeed the chosen different generator, up to a factor of absolute value one, is $\sqrt{-d}\, \Phi_{q_1}'(\zeta_{q_1})$, where here $\Phi_j(T)$ denotes a cyclotomic polynomial. If $u = \operatorname{rad}(q_1)$, then u is fixed and $\Phi_{q_1}(T) = \Phi_u(T^{q_1/u})$. This identity follows by comparing the orders of the roots and the degrees. The derivative at a primitive root therefore has magnitude $(q_1/u)|\Phi_u'(\xi)|$ for some primitive u-th root ξ. These finitely many derivative values are nonzero. The bounds on q_1, b, M in the preceding subsection now give $2 \leq |\Phi_v(\delta_E)| \leq b$ and the stated bounds on $\kappa_v = M|\Phi_v(\delta_E)|/(2\pi)$.

For clarity, the reduced domain is defined by taking the strict upper coefficients of the Gram–Schmidt triangular matrix in rational frame coordinates in a set of residue representatives modulo MR in $\mathbb{C}^f$. One can take all representatives to have maximum modulus at most b^{26}. Choose a rational basis from the spanning integral elements $\zeta_{q_1}^i, \sqrt{-d}\, \zeta_{q_1}^i$. A half-open parallelepiped for M times their integer span has maximum modulus at most $2 f M \max(1, \sqrt{d}) \leq b^{26}$. For each coset of MR choose the lexicographically first of its finitely many points there. This gives a measurable set by comparison among finitely many translates.

At a reduced point the normalized triangular coefficient at (i, l), $i < l$, is the rational one times s_l/s_i, of modulus ratio at most one. The diagonal coefficients are one. Let F_y denote the matrix of normalized columns $y^{-1} f_h$. For a point in the cut choose a Bessel element u_0 with $\|u_0 y\| \leq b^{7n}$, using the infimum defining the Bessel length and $n \geq 2$. Its action on the frame is upper unitriangular. Column operations of that kind preserve the successive orthogonalized columns of F_y. Their lengths are bounded by $\|(u_0 y)^{-1}\| = \|u_0 y\|$, by orthogonalization after the change of frame. Hence each column of F_y has length at most $n b^{26} b^{7n} \leq b^{37n}$. This applies to both boxed points at the same place. Their cross products are bounded by b^{74n} since the normalized Hermitian matrix has operator norm one. Removing the two scale factors costs at most $N \leq b^{60n}$, so $|\Phi_v C_{il}| \leq b^{134n}$. Integral tuples of h entries of maximum modulus R_0 at all embeddings number at most $(1 + 2R_0)^{2fh}$. In fact the product formula gives maximum distance at least one between distinct tuples, and balls of radius $1/2$ pack inside the ball of radius $R_0 + 1/2$ in the maximum norm. With $h = n^2$ this proves the first count.

For $j < n$, the kernel of evaluation against A_j on γT equals $T \cap \gamma T$. It contains that space because $A_j \subset L$, and it has dimension $n - j$ by $D_j \neq 0$. This puts each indicated column in T. Its top coordinates are integral since Y is integral. To bound them, the unitriangular matrix U of u_0 on the normalized frame satisfies $\|U\| \le b^{7n}\|F_y\| \le b^{44n+1}$. Here the normalized fixed frame consists of Euclidean orthonormal basis vectors and $u_0 y F_y$ has coefficient matrix U. Since $\det U = 1$, for all w

$$\|w\| \le b^{46n^2}\|F_y w\|.$$

Indeed $\|Uw\| \le b^{7n}\|F_y w\|$ and $\|U^{-1}\| \le \|U\|^{n-1}$. All entries of $\operatorname{adj} Y$ and $\det Y$ have modulus at most b^{135n^2}, by determinant expansion and $n \le b$. Express the shared column both in the fixed and in the second frame at y. Its second-frame coefficients before scaling are entries of $\operatorname{adj} Y$. The ratio of any two frame scales is at most b^{23n}. The preceding coefficient inequality and the column lengths at $\gamma^{-1}y$ bound every fixed rational coefficient by

$$b^{46n^2}\, n b^{37n+23n+135n^2} \le b^{250n^2}.$$

Packing applies to the at most an top entries, proving the second count. For $j = n$ the array is empty.

For the gauge, $0 \le \Phi_v(\Delta) \le |\Phi_v(D_n)|^2$. In the lower ranks Lemma 131.21 gives $|\Phi_v(d_*)| = N|\Phi_v(D_j)|$. In the open case $d_* = D_n$. All finite row indices are at least one. Since $M \le b^{22}$, the bound for either term in the gauge is $G_\gamma \le b^{(44a+60n+270n^2)f} \le b^{n^3 f}$ for sufficiently large n. □

Proposition 131.28 (Multiplicity over a fixed positive lattice)**.** *Fix c, t and a corank ℓ lattice $L_0 \subseteq \Lambda_X$, $d(L_0) \le H$. The number of near terms (c', γ) of these ranks and this label with $L_0 \subseteq L_\gamma$ is at most*

$$b^{2000n^4(\ell+1)f}.$$

Proof. First, equality of frames up to a left level move identifies terms of one label. Indeed, after the move, the relative rational isometry fixes the first frame pointwise. Write it as uh in $P = U_0 \rtimes \mathrm{U}(X_0)$ as in Proposition 131.12. The relative finite matrix after the two class changes fixes the same frame in lattice coordinates. Its residue thus lies in B_{res}; the two residues belong to the same coset $s_0 B_{\mathrm{res}}$. Equal labels imply equality of the residues. The relative finite matrix is integral and congruent to one at M. On the middle quotient of the integral lattice $(\Lambda \cap L^\perp)/\Lambda_L$ it induces the class translate of $h \oplus 1_{Ez}$, still integral and congruent to one. Extending this middle action by the identity on the two integral arms shows that the class translate of h belongs to the definite source level, the preimage of the principal level under this extension. Thus the two source classes c' coincide. With their representatives equal, h is an automorphism at that level and is the identity by Proposition 131.25. The relative rational element is consequently a pointwise unipotent level move on the right. This proves the identification in the original double cosets.

Fix j, C and the shared array of Lemma 131.27. Write $B = C_{[1,a],*}$. We give all the level moves used below. In coordinates (p, y_0, q) on the two isotropic arms and the middle block, with middle Gram H_0, one such move is

$$(p, y_0, q) \longmapsto (p - D^* H_0 y_0 - \epsilon D^* H_0 D q,\ y_0 + Dq,\ q).$$

Here $D = M(\eta_1, \dots, \eta_a)$ with columns in $M\Lambda_X$ and $\epsilon = -\zeta_3 \in R$, so that $\epsilon + \bar{\epsilon} = 1$. Substitution in the Hermitian form verifies unitarity. The move fixes L and z pointwise. It preserves Λ and is one modulo M, since any η_i pairs integrally with the middle lattice. Its inverse is integral by triangularity. We may also use the central move $p \mapsto p + Sq$ with $S = -S^*$ in $M\ \mathrm{Mat}_a(R)$.

For each term choose an R-linear lift $x \mapsto x + Mw(x)$, $x \in L_0$, orthogonal to the second frame, with $w(x) \in \Lambda_L = R^a$. Such lifts exist by the definition of L_γ; they can be chosen linearly because the surjection of the module of level lifts onto L_0 splits over the projective module L_0. Each row of w is a functional on L_0. The submodule of functionals $x \mapsto h(\eta, x)$, $\eta \in L_0$, has index $d(L_0)^2$, by Hermitian duality in its rational span. Grouping the lifts by these classes costs at most $d(L_0)^{2a}$. Within one group the first move makes w equal to a fixed representative, because it changes its rows by $-h(\eta_i, x)$. For the resulting frames,

$$X_\gamma^* h_X x = -MB^* w(x) \qquad (x \in L_0),$$

where h_X in matrix expressions denotes the positive Gram matrix. Their positive columns therefore differ from the fixed frame's positive columns by vectors of $P_0 = \Lambda_X \cap (EL_0)^\perp$, a lattice of rank ℓ. Record these differences modulo $M\delta P_0$ in each column. There are at most $|\mathrm{N}_{E/\mathbb{Q}}(M\delta)|^{\ell n}$ records.

Within one record put $V = X_\gamma - X_{\gamma'}$. All columns of VY^{-1}/M belong to Λ_X, by the adjugate formula and the congruence. If $j < n$, its columns $i > j$ vanish because the corresponding combinations of the two frames lie in T. If $j = n$, only the last column needs attention. The vector $\gamma \mathcal{F} C^{-1} e_n$ has bottom coordinates zero and z-coordinate $-1/N$. It belongs to γT, where the form is negative semidefinite at every embedding. Thus its X_0-component has norm at most $1/\sqrt{N}$ at each embedding in the positive Hermitian space. The same holds for the second term. The last column of VC^{-1} has norm at most $2/\sqrt{N} < 1$ everywhere. It is in Λ_X. A nonzero such vector has a totally positive integral norm with product at least one, so this column also vanishes. These arguments apply after the preceding moves, since the moves fix the first frame. Multiplying VY^{-1} by Y now gives $V = DB$, with the columns of D in $M\Lambda_X$: use the first $\min(a, j)$ columns and fill the others with zeros. The first type of move, with the appropriate sign, thus makes the entire positive block equal throughout this record.

Only the top coordinates can still differ. Record their residues modulo $M\delta\bar{\delta}$ entry by entry. This costs at most $|\mathrm{N}_{E/\mathbb{Q}}(M\delta\bar{\delta})|^{an}$. Let Ξ be the top difference of two frames with one record. The equal Gram matrices of the frames give $B^*\Xi = -\Xi^* B$. Moreover Ξ vanishes on $\ker B$. For $j < n$ this kernel is spanned by $Y^{-1} e_i$ for $i > j$, since B has rank j by Theorem 130.6 and the first j rows are independent. The shared top entries are the same and have been fixed through all moves. For $j = n$, B has rank a, so B^* is injective and the Gram identity gives $\Xi \ker B = 0$.

Let R_0 be Y^{-1} times the n-by-a rectangular diagonal matrix with ones only in the first $\min(a, j)$ positions. Then $BR_0B = B$: the first $\min(a, j)$ rows are preserved and span the row space. Put $P = BR_0$, $T_0 = \Xi R_0$, so $T_0 B = \Xi$, since $1 - R_0 B$ has image in $\ker B$. The matrix

$$S = T_0 - T_0^* - P^* T_0$$

is skew-Hermitian since $P^* T_0 = R_0^* B^* \Xi R_0$ is. Multiplying the Gram identity by R_0^* on the left gives $T_0^* B = -P^* \Xi$, hence $SB = \Xi$. Each entry of R_0 is integral after multiplication by δ. The congruence on Ξ therefore puts S in $M \operatorname{Mat}_a(R)$. The central move aligns the two frames, proving they represent one term. This argument also covers $j = 0$, when the shared columns are all the columns.

There are at most $n + 1$ choices of j. The two norms of moduli used above are each at most $b^{600 n^2 f}$ by $|\Phi_v \delta| \le b^{135 n^2}$ and $M \le b^{22}$. Multiplying the two counts in Lemma 131.27, the lift count, and the two residue counts bounds the base-b logarithm, divided by f, by

$$2 + 270 n^3 + 502 n^4 + 2 a n^3 + 600 n^2 (\ell n + a n) \le 2000 n^4 (\ell + 1).$$

The first term absorbs $n + 1$. All the choices were made inside a fixed c, t, L_0, as required. $\square$

Theorem 131.29 (The sum over near pairs in agreeing residue blocks)**.** *On the interval and fields of Theorem 131.26, sum over ordered pairs of near surviving terms in the same class position (c,c') and with the same agreeing residue at both levels. At corank $\ell = k - \dim I_X$ the sum satisfies*

$$\sum_{c,c'} \sum_{\text{such pairs of corank } \ell} d(L_{12})^{-k} \le |B_{\mathrm{res}}| \, \frac{|\mathcal{C}|}{N^{kf}} \, \tau^{\ell+2}.$$

Every such pair satisfies $d(L_{12}) \le H$ and $\ell \le n^3$.

Proof. The integral positive lattice L_{12} has $d(L_{12}) \ge 1$. By Theorem 131.23 and Lemma 131.27,

$$1 \le d(L_{12}) \le H(2/M)^{c^+ f}.$$

Since $M/2 \ge b^{19}$, this gives $c^+ \le n^3/19$. The dimension assertion of Theorem 131.23 then gives $\ell \le c^+ + 1 \le n^3$. Assign to each ordered pair its first class, the common residue label and the actual lattice $L_0 = L_{12}$. Its two terms both contain this lattice. For each (c, t, L_0) their number is bounded by the square of the bound in Proposition 131.28. This remains an upper bound when the second class is required to be the same for both. Since $k \ge 1000(2n)^9$ and $\ell \le n^3$, the square is at most $b^{kf/4} = \tau$. Sum the weight $d(L_0)^{-k}$ using Theorem 131.26 and take the $|B_{\mathrm{res}}|$ labels. This proves the claim. □

The discriminant estimate and the weighted count now apply to the same pairs in the unfolded period. The absolute sizes of their orbital integrals and the contribution of the discarded cut tails still enter the second moment.

131.11 Projected frame volumes and the integral over positive shifts

Let V denote the complex Hermitian space at one infinite place of the shorter construction, in the normalized coordinates of Corollary 131.2. Its form is $h(v,w) = v^* H w$, where $H = H^* = H^{-1}$ exchanges the two isotropic arms, is 1 on the k-dimensional positive summand X_0, and is -1 on the unit terminal vector z. Put $F = (e_1, \dots, e_a, z)$, $T = \operatorname{im} F$, $L = \langle e_1, \dots, e_a \rangle$, $n = a + 1$. For an isometry y, let $\Omega = y \ker(H + 1)$. It is a negative space of dimension n. For another isometry g, put $Z = F^* H g F$. The ratio of projected frame volumes is

$$\nu_g(y) = \frac{|\det(\Omega_b^* H g F)|}{|\det(\Omega_b^* H F)|},$$

where Ω_b denotes any basis matrix of Ω. A basis change multiplies both volumes by the same positive factor. Both determinants are nonzero. Indeed $T^\perp = L \oplus X_0$ is positive semidefinite, as is $gT^\perp$, whereas Ω is negative definite.

We use the flags of Theorem 130.6. Let j be the rank of a surviving term, $i = \min(a, j)$, $p = i + 1$, $F_i = (e_1, \dots, e_i, z)$, and $B = F_i^* H g F_i$. For $j = n$ this is the full invertible Gram matrix. For $j \le a$ invertibility follows from Lemma 131.21 and the nonzero leading minor. Write X for projection on the positive middle summand.

Lemma 131.30 (A determinant bound in a negative plane)**.** *There are a matrix ω with p columns in Ω and an invertible lower triangular matrix $D = F_i^* H \omega$, depending only on y, Z, i, such that $\omega^* H \omega = -I_p$. Put $\omega' = g F_i B^{-1} D$, $V_g = X(\omega' - \omega)$ and let w be the last column of V_g. Then*

$$|\det B| \det(I_k + V_g V_g^*) \le 2^p |\det D|^2 \nu_g(y).$$

Suppose $\|y^{-1} f\| \le \rho$ for each column f of F and the Bessel length of y is at most θ. For $d = D_{pp}$ we then have

$$|\det D| \le \rho^p, \qquad \frac{|d|^2}{1 + \|w\|^2} \le \theta^{2(j-i)}.$$

Proof. The map $F^{*}H : \Omega \to \mathbb{C}^{n}$ is an isomorphism by the nondegeneracy just proved. Take the preimages ω_{0} of the columns of $F^{*}HgF_{i}B^{-1}$. Their pairings with F_{i} form the identity. Apply Gram–Schmidt in the positive metric $-h|_{\Omega}$, beginning with the last column. Each new column is a combination of that column and later ones, with a positive real diagonal coefficient. Thus $\omega = \omega_{0}D$ has the stated triangular properties and depends continuously on y for fixed Z. Continuity follows also by using the basis $y\Omega_{-}$ for a fixed basis Ω_{-} of $\ker(H+1)$ in the initial linear system.

Put $U = L \cap gL$. In the open case $U = 0$ since Z is invertible. In the lower ranks the common triangular flag gives $a - i$ independent vectors in U, one at each index from $i+1$ to a. The leading i-minor implies $\dim U \le a - i$ by pairing L with gL. The columns of ω are orthogonal to U. Indeed $F^{*}H\omega = F^{*}H\omega'$, and the vectors of $U \subset T$ pair to zero with gF_{i}. In both frames replace the omitted columns $e_{i+1}, \dots, e_{a}$ and their images by the same triangular common vectors. Each replacement has determinant of modulus one, since the two leading coefficients at every replaced index have modulus one. Extend ω to a basis of Ω. In the two pairing matrices the common columns have a zero block in the first p rows, and they have identical entries in the other rows. Expansion along these columns cancels their common determinant, which is nonzero by invertibility of the full matrices. Hence

$$\nu_{g}(y) = \frac{|\det(\omega^{*}HgF_{i})|}{|\det D|}.$$

This also covers the full frame, where the omitted block is empty.

The columns of $\omega' - \omega$ belong to $T^{\perp}$. Their Gram matrix is $V_{g}^{*}V_{g}$, because L is the radical of $T^{\perp}$. The form restricted to gT is negative semidefinite. On writing $K = -\omega^{*}H\omega'$, expansion therefore gives

$$K + K^{*} = I + V_{g}^{*}V_{g} - \omega'^{*}H\omega' \succeq I + V_{g}^{*}V_{g}.$$

If $P = (K + K^{*})/2$, then $P^{-1/2}KP^{-1/2} = I + \mathrm{i}S$ with $S = S^{*}$. Each eigenvalue of $I + \mathrm{i}S$ has modulus at least one, so $|\det K| \ge \det P$. Monotonicity of determinant on positive definite matrices follows by conjugating the larger matrix by the inverse square root of the smaller one. Also $\det(I + V_{g}^{*}V_{g}) = \det(I + V_{g}V_{g}^{*})$ by the singular values. Substituting $|\det K| = |\det D|^{2}\nu_{g}(y)/|\det B|$ proves the determinant bound.

The columns of $y^{-1}\omega$ are Euclidean orthonormal in $\ker(H+1)$. Each row of D has length at most the Euclidean length of the corresponding $y^{-1}f$, by the projection inequality. Hadamard's inequality gives the first bound on D. If $j \le a$, take a common terminal vector with coefficients $\zeta, 1$ along z, gz, respectively, where $|\zeta| = 1$. Subtract the common vectors of indices $a, \dots, i+1$, in that order, to kill the coefficients along $ge_{a}, \dots, ge_{i+1}$. This gives a vector $w_{0} \in T \cap g\,\mathrm{im}\,F_{i}$ with the same terminal coefficients. Now $F_{i}^{*}Hw_{0} = -\zeta e_{p}^{\mathrm{std}}$, so the last column of ω' is $-(d/\zeta)w_{0}$ by invertibility of B. It has squared Hermitian norm $-|d|^{2}$ and is orthogonal to the corresponding column of $\omega' - \omega$. Subtracting the latter column gives the last column of ω and hence $-1 = -|d|^{2} + \|w\|^{2}$.

If $j = n$, triangularity gives $\omega_{p} \perp L$, where ω_{p} denotes the last column. The Bessel length is one plus the infimum of $\|uy\|$ over the Bessel unipotent group, so some such u has $\|uy\| \le \theta$. As $u^{-1}z - z \in L$, Cauchy–Schwarz after applying y^{-1} gives $|d| = |h(u^{-1}z, \omega_{p})| \le \|(uy)^{-1}z\| \le \theta$. Here $\|(uy)^{-1}\| = \|uy\|$ since H is a unitary involution. This proves the second bound in every rank. □

Proposition 131.31 (The integral of the real majorant)**.** *Fix a representative g_{0}, its Z, j, and a point y in the preceding bounds. For $A \in \mathbb{C}^{k\times i}$ consider all surviving g with this Z, j and*

$$X(gF_{i}) = X(g_{0}F_{i}) + AB_{[1,i],*}.$$

There is a single nonnegative smooth function $\varphi_y(A)$ majorizing $\nu_g(y)^{-(k+i)}$ for every such g, with

$$\int_{\mathbb{C}^{k\times i}} \varphi_y(A)\,dA \le \frac{(2\rho^2)^{pi}(2^p\theta^{2(j-i)})^k}{|\det B|^{k+i}} \qquad (k\ge 7).$$

Here dA is the ordinary real Lebesgue measure in complex orthonormal coordinates. The majorant is measurable in y.

Proof. Write $D=\left(\begin{smallmatrix} D_1 & 0 \\ * & d\end{smallmatrix}\right)$. The shift in $X(\omega')$ is $AB_{[1,i],*}B^{-1}D=(AD_1,0)$. Thus $V_g=(C_1+AD_1,w)$ with C_1,w independent of A,g. Put $G=I_k+ww^*$, $s=k+i$. The lemma gives the stated domination with

$$\varphi_y(A)=\left(\frac{2^p|\det D|^2}{|\det B|}\right)^s \det\!\big(G+(C_1+AD_1)(C_1+AD_1)^*\big)^{-s}.$$

All its ingredients depend continuously on y for fixed frame products and fixed g_0. Changing variables to $W=G^{-1/2}(C_1+AD_1)$ gives an integral of the determinant factor equal to

$$|\det D_1|^{-2k}(\det G)^{-k}\pi^{ki}\prod_{t=0}^{i-1}\frac{t!}{(k+t)!}.$$

Indeed the real Jacobian for left multiplication by $G^{1/2}$ is $(\det G)^i$, and right multiplication by D_1^{-1} contributes $|\det D_1|^{-2k}$. To integrate the remaining factor, for any positive P use $\det(P+vv^*)=\det P(1+v^*P^{-1}v)$. At exponent $s'>k$ one column integrates to $\pi^k\Gamma(s'-k)/\Gamma(s')$ times $(\det P)^{1-s'}$. This follows by writing the negative power as its gamma integral and integrating the Gaussian on $\mathbb{C}^k$; nonnegativity justifies the exchange. Starting at $s'=k+i$ and repeating for all columns yields the displayed product. For $k=7$, $\pi^k/k!<1$, and the ratio decreases thereafter because $\pi/(k+1)<1$. Each $\pi^k t!/(k+t)!\le 1$. Now $\det G=1+\|w\|^2$ and $|\det D|=|\det D_1|\,|d|$. The two bounds of the lemma give the claimed integral after collecting the i-th and k-th powers. For $i=0$ the integral over the zero space is evaluation and the empty product is one. □

Theorem 131.32 (Mean over a translate of the marked column lattice)**.** *Take the fields, marked classes and interval of Theorem 131.26. At every sufficiently large node put $\rho=b^{37n}$, $\theta=b^{6n+2}$, $\tau=b^{kf/4}$. Fix a near surviving term of rank $j\ge 1$, put $e=j-i$, and set*

$$P_j=\prod_{v:E^+\hookrightarrow\mathbb{R}}|\Phi_v(D_j)|=|\mathrm{N}_{E/\mathbb{Q}}D_j|^{1/2}.$$

Choose one embedding Φ_v of E over each real place v. Use the positive columns $Q\in X^i_{0,\infty}$ in the original rational basis. A shift by MQ times the first i isotropic rows of the rational cross-product matrix induces a normalized shift $A_v=Q_vS_v$ at place v. Let $\Psi_y(Q)=\prod_v\varphi_{y_v}(Q_vS_v)$. For $\mathcal{A}=(\Lambda_c\cap X_0)^i$, any shift t, and any fundamental parallelepiped $\mathcal{P}$ for $\mathcal{A}$,

$$\frac{1}{\operatorname{vol}\mathcal{P}}\int_{\mathcal{P}}\sum_{\lambda\in\mathcal{A}}\Psi_y(Q+t+\lambda)\,dQ \;\le\; \frac{\tau}{c_E^{ki}N^{fi}}\left[\left(\frac{b^{13}}{M^2}\right)^{if}\frac{N^{ef}}{P_j}\right]^k, \qquad c_E=2^{-f}\sqrt{|\mathrm{disc}(E)|}.$$

The bound holds uniformly for the reduced points y in the Bessel length cut. The measure on each $X_{0,v}$ is Euclidean volume for its positive form.

Proof. The bounds on y_v required for the proposition hold at every place by the proof of Lemma 131.27. Write $s_{h,v}$ for the frame scale of column h, of modulus $\kappa_v^{n-h}/\sqrt{N}$. Put $r_v=$

$\prod_{h=1}^{i}\kappa_v^{n-h}$. Row h of the normalized cross products acquires a factor $\bar{s}_{h,v}$, and each column acquires its own scale. The latter scales also multiply the shifted image columns. Thus

$$S_v=\operatorname{diag}_{h\le i}(M/\bar{s}_{h,v}),\qquad |\det S_v|=\frac{(M\sqrt{N})^i}{r_v},\qquad |\det B_v|=\frac{r_v^2}{N^j}|\Phi_v(D_j)|.$$

For the last equality, the unscaled subframe has determinant of modulus $N|\Phi_v(D_j)|$ in closed rank and $|\Phi_v(D_j)|$ in open rank. The product of the squared subframe scales, including the terminal scale, is r_v^2/N^{i+1}.

The integral in the Q_v coordinates is multiplied by $|\det S_v|^{-2k}$. The factor r_v^{2k} cancels exactly and $r_v^{-2i}\le 1$, since $\kappa_v\ge 1$. By the proposition the resulting bound at v is at most

$$\left[\frac{(2\rho^2)^pN^j}{|\Phi_v(D_j)|}\right]^i\left[\frac{b^{13i}N^e}{M^{2i}|\Phi_v(D_j)|}\right]^k.$$

Indeed $2^p\theta^{2e}\le b^{13i}$. In closed rank use $i\ge 1$ and $b\ge 4$. In open rank $n=i+1$, and $2(6n+2)+(i+1)/2\le 13i$ when $i\ge 33$, as holds at these nodes. Since D_j is a nonzero algebraic integer, $P_j\ge 1$. The product of the first brackets to the power i is at most τ, since $2\rho^2\le b^{75n}$, $N\le b^{60n}$, and $(75np+60nj)i\le 135n^3\le k/4$.

Tonelli's theorem and translation of each summand identify the integral of the periodization over $\mathcal{P}$ with the integral of Ψ_y on the whole real vector space. The translates tile it up to null sets for every t. By the covolume formula of Section 131.6 and $d(\Lambda_c\cap X_0)=N^f$, the volume of $\mathcal{P}$ is $c_E^{ki}N^{fi}$. Dividing the product of the local bounds by this volume proves the assertion. □

To pass from this translation mean to an orbital contribution one must also count the integral frames with the same positive shift and integrate their cutoffs against the Bessel functions.

131.12 The orbital Schur bound and the contribution of the closed cells

Return to the rational frame $\mathcal{F}$ and the fields and interval of Theorem 131.32. Write U for the Bessel unipotent group and U^0 for the subgroup that fixes $\mathcal{F}$ pointwise. For a class c let Γ_c be the rational elements of U acting integrally on Λ_c and as the identity modulo M. Write $\Gamma_c^0=\Gamma_c\cap U^0$. The full unipotent covolume

$$v_U=\operatorname{vol}(\Gamma_c\backslash U_\infty)$$

is independent of c. Indeed the finite lattice in U of class c is conjugate to the one at the identity by its representative in $\mathrm{U}(X_0)(\mathbb{A}_{F,f})$. The determinant of this conjugation on the independent positive coordinates of the Lie algebra, over $F=E^+$, is a power of $\mathrm{N}_{E/F}\det_E h_c=1$; the remaining graded coordinates are fixed. Exponential coordinates identify U with its Lie algebra as a variety. Thus the finite rational approximation and product measure argument in Lemma 92.23 applies with this U and this principal level. It expresses the displayed volume as the volume of the adelic unipotent quotient divided by the volume of the finite open lattice. Both volumes are positive and finite.

We describe the cutoffs used in the orbital integral. The image of Γ_c on $\mathcal{F}$ is exactly the upper unitriangular group with entries above the diagonal in MR, where $R=\mathcal{O}_E$. The congruence gives the inclusion, since the two isotropic arms are split summands. Every elementary entry between two isotropic vectors lifts by the elementary matrix on L and its inverse adjoint on the dual arm. To lift an entry from the terminal vector z to e_h, choose a vector t of the middle lattice with $h(t,z)$ any specified element of R. Indeed z is part of a basis locally in the marked

lattice, so evaluation on it maps the integral dual onto R; the middle lattice is Hermitian unimodular. Set $\xi = -\zeta_3 \in R$, so $\xi + \bar{\xi} = 1$. The map

$$v \longmapsto v + M\{e_h h(t,v) - t h(e_h,v)\} - M^2 \xi h(t,t) e_h h(e_h,v)$$

is the required element of Γ_c. In the pairing of two transformed vectors the terms linear in M cancel. The square of the first-order term contributes $M^2 h(t,t)\overline{h(e_h,v)}h(e_h,w)$, canceled by the two last terms since $\xi + \bar{\xi} = 1$. Higher terms vanish by $e_h \perp t, e_h$. The map and its isometric inverse preserve the lattice and the level. Every required triangular matrix is a product of the elementary ones by elimination.

Let $R_{\mathrm{red}}(y)$ be the reduced-set indicator from Lemma 131.27. In rational frame coordinates the upper unitriangular Gram–Schmidt coefficient matrix of $y^{-1}\mathcal{F}$ changes by right multiplication by the inverse of the frame matrix of u when y is replaced by uy, $u \in \Gamma_c$. A unique matrix in the image just described puts its strict upper entries into the chosen representatives modulo MR. Reduce in increasing order of the superdiagonal, using elementary column additions. Each addition affects only its chosen entry and larger superdiagonals, and at the first superdiagonal of a difference the two choices would differ by MR. This proves uniqueness. Choose a measurable nonnegative p_c whose translates under Γ_c^0 sum to one. For example, a discrete subgroup acting on a Lie group by left multiplication has a Borel fundamental set: cover its quotient by countably many open sets admitting local sections and restrict the sections to the successive disjoint differences. Set

$$\beta_c = p_c R_{\mathrm{red}}, \qquad \beta_c^{\square} = \beta_c \, \mathbf{1}_{\{\ell(y_v) \le \theta \ \forall v\}}.$$

Summing first over Γ_c^0 in each coset gives $\sum_{u \in \Gamma_c} \beta_c(uy) = 1$. In particular the argument below applies to the class cutoffs for any choice of this partition on the pointwise fixer.

Let $W^\flat = \bigotimes_v W_v^\flat$ be the cut Bessel functions, with cut of absolute value at most one supported at length at most $\theta = b^{6n+2}$. Use the squared local quotient norm bound $\|W_v^\flat\|^2_{U_v \backslash G_v} \le d_v$, where d_v is the dimension of the value space. This is the normalization of Theorem 131.8: the uncut average has squared norm d_v when scaled there, and cutting decreases it. The quotient by U_v gives the same norm as the quotient by the Bessel group since the remaining compact group has Haar mass one and acts unitarily. The zero function also satisfies the bound. Write $d_B = \prod_v d_v$.

For a rational term γ in a fixed block (c, s, c'), let $\mathcal{O}_\gamma = \Gamma_c^0 \gamma \Gamma_{c'}^0$ as a set of rational isometries. The sum of the absolute cut amplitudes on this orbit is bounded by the nonnegative integral

$$A_\gamma = \sum_{\delta \in \mathcal{O}_\gamma} \int_{G_\infty} \beta_c(y) \beta_{c'}(\delta^{-1} y) \|W^\flat(y)\| \, \|W^\flat(\delta^{-1} y)\| \, dy.$$

The product of the two norms is exactly the Hilbert–Schmidt norm of the corresponding simple tensor. The finite residue and finite character have already been fixed in this expression.

Lemma 131.33 (Positive shifts and the multiplicity at each shift)**.** *Let $j \ge 1$ be the rank of γ, $i = \min(a, j)$, and let $H_i \subset H_{\mathrm{all}}$ be the R-span of the first i isotropic rows inside the span of all isotropic rows of $C = \mathcal{F}^* h \gamma \mathcal{F}$. Put $i_\gamma = [H_{\mathrm{all}} : H_i]$, as in Lemma 131.21, with $i_\gamma = 1$ in open rank. The orbit breaks up into at most i_γ^k sets. In each one choose a base term δ_0. The changes in the positive frame block there are $M\lambda C_{[1,i],*}$ with unique parameters $\lambda \in (\Lambda_c \cap X_0)^i$.*

Fix a reduced point y in the length cut and $\widehat{y} = uy$, $u \in U_\infty^0$. For a fixed parameter in one of these sets, the sum of $\beta_{c'}^{\square}(\delta^{-1}\widehat{y})$ over its terms is at most $\tau = b^{kf/4}$.

Proof. Right multiplication by $\Gamma_{c'}^0$ fixes the image frame. A left pointwise unipotent adds to its positive block a multiple of its bottom block $C_{[1,a],*}$. In fact the coefficient of its h-th row is

the positive component of $(u-1)e_h^\vee$, where $e_h^\vee$ is the dual isotropic basis vector. For $u \in \Gamma_c^0$ this component belongs to $M\Lambda_X$, $\Lambda_X = \Lambda_c \cap X_0$: the bottom and terminal changes are zero since the frame is fixed, and the integral top component can be subtracted. Thus all the changes belong to $M(\Lambda_X \otimes_R H_{\mathrm{all}})$ in the positive block. The submodule formed with H_i has index i_γ^k. This can be checked prime by prime, where Λ_X is free of rank k; both tensor products embed in X_0^n by extending to E, using local freeness. The first i rows are independent by the nonzero leading minor (in open rank they are rows of an invertible matrix). They are a basis of H_i and give the unique parameter relative to any base representative of the coset. All other blocks except the top are now fixed, since C fixes the terminal and bottom blocks.

For a nonzero cutoff the normalized columns $\widehat{y}_v^{-1}\delta F_v$ have lengths at most $\rho = b^{37n}$. Compare any two contributing terms of this fiber. Their rational frame difference is only in the top block and has entries $t_{hj} \in R$, since the top arm splits off integrally. Moreover $\widehat{y}_v^{-1}F_v = y_v^{-1}F_v$, and the bound in the proof of Lemma 131.27 gives $\|y_v^{-1}F_v w\| \geq b^{-46n^2}\|w\|$. A difference of normalized columns of index j' has coefficient $\Phi_v(t_{hj'})s_{j',v}/s_{h,v}$ on this fixed frame. Hence $|\Phi_v(t_{hj'})| \leq 2\rho\, b^{46n^2+23n} \leq b^{47n^2}$ at every embedding for sufficiently large n. Relative to one contributing frame, the packing bound for integral entries of Lemma 131.27 bounds the number of full frames in the fiber by $(1+2b^{47n^2})^{2fan} \leq b^{96n^4 f} \leq \tau$. An empty fiber contributes zero.

Terms with identical full frame differ on the right by an element of $\Gamma_{c'}^0$. Indeed their quotient fixes $\mathcal{F}$ and is principal on $\Lambda_{c'}$, since their residues coincide. Its action on the positive middle quotient preserves $\Lambda_{c'} \cap X_0$ and is the identity modulo M there: apply the quotient isometry to a positive vector, then subtract its top component as above. The definite automorphism group is finite, and the principal subgroup at either odd level prime is torsion free by the argument in Proposition 131.25. The positive middle action is therefore trivial, leaving exactly the pointwise unipotent fixer. The sum of the cutoffs for this full frame is bounded by the sum of the translates of $p_{c'}$ under $\Gamma_{c'}^0$, which is one. Multiplication by the frame count proves the claim. □

Theorem 131.34 (Schur's inequality on a rational orbit)**.** *For a near surviving orbit of rank $j \geq 1$ put $e = j - i$, $\mathcal{V}_i = c_E^{ki} N^{fi}$, and let P_j be as in Theorem 131.32. Then*

$$\frac{A_\gamma}{v_U d_B} \leq \frac{\tau^2}{\mathcal{V}_i}\left[\left(\frac{b^{13}}{M^2}\right)^{if} \frac{N^{ef}\sqrt{i_\gamma i'_\gamma}}{P_j}\right]^k.$$

Here i'_γ is the row index for γ^{-1}.

Proof. Write $s_0 = k + i$ and define

$$F_\gamma(y) = \sum_{\delta \in \mathcal{O}_\gamma} \beta_{c'}^{\square}(\delta^{-1}y) \prod_v \nu_{\delta_v}(y_v)^{-s_0}.$$

It is measurable, nonnegative and left Γ_c^0-invariant: left multiplication reindexes the orbit and fixes the original frame. At a reduced y in the cut, fix $u \in U_\infty^0$. For $Q \in X_{0,\infty}^i$, choose a real point $T(Q) \in U_\infty^0$ whose inverse adds $Q_v S_v$ times the first i bottom rows to the positive block at v. Such a continuous choice follows from the explicit matrix before Theorem 131.1: take $h = 1, K = 0$ and the negative of $Q_v S_v$, padded by zero columns, for B. At $uT(Q)y$ the volume ratio for δ equals the one at y for $T(Q)^{-1}u^{-1}\delta$, since $u, T(Q)$ fix the first frame. The Gram matrix is unchanged and all the common vectors of the flags retain their coefficients. In any one of the sets of Lemma 131.33, take $u^{-1}\delta_0$ as the real base frame. The difference of its positive block from that of $T(Q)^{-1}u^{-1}\delta$, with the latter minus the former, is $(Q_v + \Phi_v\lambda)S_v$ times the first i Gram rows. Indeed the shift coming from u^{-1} is common to the two frames as their bottom rows are identical.

The majorants of Proposition 131.31 can thus be taken the same throughout the fiber over λ. Lemma 131.33 bounds the remaining cutoff sum there by τ, also at $uT(Q)y$. Summing over all λ, applying Theorem 131.32, and then summing over the at most i_γ^k sets yields

$$\frac{1}{\mathcal{V}_i}\int_{\mathcal{P}} F_\gamma(uT(Q)y)\,dQ \le C_\gamma, \qquad C_\gamma = \frac{\tau^2}{\mathcal{V}_i}\left[\left(\frac{b^{13}}{M^2}\right)^{if}\frac{N^{ef}i_\gamma}{P_j}\right]^k.$$

The proof of the translation mean uses only the Gram value, the flags, and the real base frame. It therefore applies to the base $u^{-1}\delta_0$ even when u is irrational.

Integrate first on U_∞^0-orbits in G_∞ using Haar disintegration. On such an orbit the reduced indicator, the length cut and $\|W^\flat\|^2$ are constant. The remaining cutoff p_c integrates an invariant function by the quotient measure on $\Gamma_c^0\backslash U_\infty^0$. Right multiplication by each $T(Q)$ preserves this measure, since a unipotent group is unimodular. Averaging this identity over $Q\in\mathcal{P}$ and using the displayed bound therefore gives

$$\int_{G_\infty}\beta_c^{\square}(y)\|W^\flat(y)\|^2F_\gamma(y)\,dy \le C_\gamma\int_{G_\infty}\beta_c(y)\|W^\flat(y)\|^2\,dy \le C_\gamma v_U d_B.$$

For the last inequality integrate first over U_∞. The Γ_c-translates of β_c sum to one; its integral on each such orbit is v_U. The remaining quotient norm factors over the places. All the changes of order apply to nonnegative functions by Tonelli's theorem.

Apply Cauchy–Schwarz on the product of counting measure on the orbit and Haar measure on G_∞. In one squared factor insert $\prod_v \nu_{\delta_v}(y_v)^{-s_0}$, and in the other insert its reciprocal. Use the measure with density $\beta_c^{\square}(y)\beta_{c'}^{\square}(\delta^{-1}y)$; the cut on both vectors permits these boxed cutoffs in A_γ. The first squared factor is the integral just estimated. For the other change variables by $y\mapsto\delta^{-1}y$ and invert the orbit. The projected volume ratio is then inverted by its determinant definition. The inverse has Gram matrix C^*, the same P_j, j, and row index i'_γ. Thus $A_\gamma^2 \le (v_U d_B)^2 C_\gamma C_{\gamma^{-1}}$. Taking the square root proves the formula. □

Corollary 131.35 (The squared contribution of all positive closed ranks)**.** *In each agreeing residue block (c,s,c'), sum A_γ once per surviving orbit with $1\le j\le a$, and denote that sum by $A_{\mathrm{cl}}(c,s,c')$. At all sufficiently large nodes,*

$$\sum_{c,c',\,s\ \text{agreeing}}\left(\frac{A_{\mathrm{cl}}(c,s,c')}{v_U}\right)^2 \le \tau^{-40}|B_{\mathrm{res}}|\,|\mathcal{C}|\,d_B.$$

This controls also the squared norms of the signed sums on these cells, by the triangle inequality.

Proof. Only near orbits can contribute, since both frames belong to the reduced length cut wherever the integrand is positive. The common lattice and gauge of Theorem 131.23 do not change under independent level moves on either frame: left moves preserve the positive component of every lift with zero bottom component, changing its top component by an integral multiple of M; right moves have the identical property on the second lift. Thus the pair estimate applies to representatives of these orbits. Alternatively choose the near representatives when first expanding the square. For $j\le a$ we have $|\mathrm{N}_{E/\mathbb{Q}}d_*| = N^{2f}P_j^2$ and $\Delta_\gamma = 0$. The bracket of the theorem becomes

$$\left(\frac{b^{13}}{M}\right)^{if}\frac{N^{f/2}}{\sqrt{G_\gamma}}.$$

Moreover $c_E \ge 1$: the relative discriminant formula for $E/\mathbb{Q}(\mathrm{i})$ gives $|\mathrm{disc}(E)| \ge 4^f$. In particular $\mathcal{V}_i \ge 1$. For two closed terms of ranks $i, i' \ge 1$, the positive corank ℓ satisfies $\ell \le c^+$

in Theorem 131.23. Its estimate therefore gives

$$\frac{A_\gamma A_\eta}{(v_U d_B)^2} \le \tau^4\, b^{-14kf} N^{kf} (2/M)^{\ell kf} d(L_{12})^{-k},$$

using $M \ge b^{20}$. By Theorem 131.29 the sum over all ordered pairs is at most

$$|B_{\mathrm{res}}|\,|\mathcal{C}|\,d_B^2\,\tau^6 b^{-14kf} \sum_{\ell \ge 0} [\tau (2/M)^{kf}]^\ell$$

after division by v_U^2. The series is at most two for large nodes. Finally $d_B \le b^{2kf}$. To see this on the actual fields, the chosen diagram for the source at v has only columns of heights one and k; the number of single boxes is $t_v = xb$ at the designated places and zero elsewhere. Removing columns of full height removes only a determinant twist. The remaining symmetric power has a basis of degree-t_v monomials on k variables, so its dimension is at most $(t_v+1)^k \le b^{2k}$, since b exceeds every fixed power of x. Multiply over v. As $b^{-14kf} = \tau^{-56}$ and $d_B \le \tau^8$, the asserted inequality follows from $2\tau^{-42} \le \tau^{-40}$. □

The rank-zero terms, the open terms for which the discriminant exceeds the minor bound, and the removal of the Bessel length cut still enter the full second moment.

131.13 Compact rotation of the terminal vector in the open cell

In open rank the last column of the real Schur estimate lies in a complex unit ball. We integrate its contribution in the direction of the positive middle space. This integration takes place after the arithmetic unipotent average, so it requires no compact rotation of the integral lattice.

Lemma 131.36 (The angular integral of the terminal column)**.** *At one real place in open rank use the notation of Lemma 131.30. Write $d = D_{nn}$, $v_0 = gFB^{-1}e_n^{\mathrm{std}}$, $q = Xv_0$ and $x = X\omega_n/d$. Then*

$$t^2 := \|q\|^2 = \frac{\Phi_v(\Delta_\gamma)}{|\Phi_v(D_n)|^2} \le 1, \qquad s^2 := \|x\|^2 = 1 - |d|^{-2} < 1.$$

As the point y varies to hy, $h \in \mathrm{U}(X_0)$, the quantity D is fixed, x varies to hx, and the factor $|d|^2/(1+\|w\|^2)$ in the integral of the majorant becomes $(1 + t^2 - 2\Re\langle q, hx\rangle)^{-1}$. For probability Haar dh and

$$c = \tfrac{1}{2}(1 + t^2 + \sqrt{(1+t^2)^2 - 4t^2 s^2}), \qquad r_0 = ts/c,$$

its k-th power has the exact mean

$$\int_{\mathrm{U}(X_0)} (1 + t^2 - 2\Re\langle q, hx\rangle)^{-k}\, dh = \frac{c^{-k}}{1 - r_0^2} \le |d|^2.$$

Proof. In open rank $B = F^* H g F$. Thus $F^* H v_0 = e_n^{\mathrm{std}}$. The bottom components of v_0 vanish and its normalized terminal component is -1. Its squared Hermitian norm can also be computed in the gF frame. It is $-|(B^{-1})_{nn}|^2$, since only the last column of that frame has nonzero norm. The cofactor formula and the scale of the terminal vector give

$$|(B^{-1})_{nn}| = \frac{N|\Phi_v(D_a)|}{|\Phi_v(D_n)|}.$$

Equating the two norms gives the formula for t^2. The subframe is the whole frame, and $F^* H \omega_n = d e_n^{\mathrm{std}}$. The same computation using $h(\omega_n, \omega_n) = -1$ gives the formula for s^2. The last column of ω' is dv_0, hence $w = d(q - x)$. Using $1 - s^2 = |d|^{-2}$ now gives the indicated

denominator. The negative basis construction commutes with h, as h preserves both the metric and the ordered frame. This proves the transformation law for D, x.

For $st > 0$, the unit vector in the direction of hx is uniform on the sphere in $\mathbb{R}^{2k}$. Indeed the push-forward of probability Haar is the unique rotation-invariant probability under the transitive unitary action, and normalized Euclidean surface measure has this invariance. Choosing the axis in the direction of q makes the denominator equal to $c(1 + r_0^2 - 2r_0u_1)$, where u_1 is the real first coordinate on this sphere. The identities $c(1 + r_0^2) = 1 + t^2$, $cr_0 = ts$, and $c \geq 1$ follow from the quadratic formula and $t, s \leq 1$. In particular $r_0 \leq s < 1$.

For completeness, at dimension $m = 2k > 2$ the mean of $\|z-u\|^{2-m}$ over surface probability on the unit sphere is one for $\|z\| < 1$. It is a radial harmonic function there, by differentiation on compact subballs, and its value at zero is one. The radial harmonic differential equation and regularity at the origin force it to be constant. Differentiate at $z = r_0e_1$ to obtain $\int (r_0 - u_1)\|r_0e_1 - u\|^{-m}\, du = 0$. On multiplying $\|r_0e_1 - u\|^{-m}$ by $1 + r_0^2 - 2r_0u_1$ and integrating, the result is one. Consequently its mean without this multiplier is $(1 - r_0^2)^{-1}$. This proves the formula. For $st = 0$ the denominator is constant and gives the same value directly. Finally $c \geq 1$, $r_0 \leq s$ give the bound. □

Theorem 131.37 (A compact average in the orbital integral)**.** *For every near surviving orbit of rank $j \geq 1$ on the interval of Theorem 131.34 one has*

$$\frac{A_\gamma}{v_U d_B} \leq \frac{\tau^2}{\mathcal{V}_i} \left[\frac{2^{(i+1)f}}{M^{2if}} \frac{N^{(j-i)f} \sqrt{i_\gamma i'_\gamma}}{P_j} \right]^k .$$

Proof. In the exact integral in Proposition 131.31 retain the factor $R_v(y)^k = (|d|^2/(1+\|w\|^2))^k$. The factors $\pi^{ki} \prod_{t=0}^{i-1} t!/(k+t)!$ are at most one, and $|\det D|^{2i} \leq \rho^{2pi}$. After the change of coordinates in Theorem 131.32 the local bound before estimating R_v is

$$\left[\frac{(2\rho^2)^p N^j}{|\Phi_v(D_j)|} \right]^i \left[\frac{2^p N^{j-i}}{M^{2i} |\Phi_v(D_j)|} \right]^k R_v(y)^k .$$

For $j \leq a$ one has $R_v = 1$ by the common terminal vector. In open rank $q = X(gFB^{-1}e_n^{\mathrm{std}})$ is independent of the orbit representative and of the shifts in the positive columns used in Lemma 131.33. In fact left multiplication by a pointwise unipotent preserves the positive component of any vector with zero bottom component, and right multiplication fixes the input frame. The matrix B is unchanged. Also D and x are constant on U_∞^0-orbits of y, by the same equivariance of the triangular basis and the zero bottom component of ω_n. Thus at a fixed y the factor $\prod_v R_v(y_v)^k$ is common to every fiber in the forward Schur integral.

To justify averaging this factor, first perform the integration over $\Gamma_c^0 \backslash U_\infty^0$ exactly as in Theorem 131.34. The right translations by $T(Q)$ give the same bound with the displayed local factors. After this step the remaining measure on $U_\infty^0 \backslash G_\infty$ is quotient Haar times

$$\mathrm{vol}(\Gamma_c^0 \backslash U_\infty^0)\, R_{\mathrm{red}}(y)\, \mathbf{1}_{\mathrm{cut}}(y)\, \|W^\flat(y)\|^2 .$$

Indeed integration of $p_c(uy)$ over u gives that covolume for each y, by its partition identity. This covolume is finite by the same rational unipotent argument used for v_U. The displayed density is invariant under left $\prod_v \mathrm{U}(X_{0,v})$: the compact group fixes the frame, preserves the length, and acts unitarily on the values of the Bessel function. It preserves the quotient Haar measure since it normalizes U_∞^0 and preserves its Haar measure. For the latter assertion its modulus is a continuous homomorphism from a compact group to the positive reals, hence is

one. We may therefore replace the factor $\prod_v R_v(y_v)^k$ in the integral by its compact mean. Tonelli gives the product of the local means.

In open rank the lemma bounds each mean by $|d|^2 \le \theta^2$ on the cut. The product of the first displayed brackets raised to i, times θ^{2f}, is at most $b^{(135n^3+12n+4)f} \le \tau$ at these nodes, since $P_j \ge 1$. The same estimate applies in closed rank with the terminal factor one. The cutoff multiplicity costs another τ, the coset count gives i_γ^k, and the translation mean divides by $\mathcal{V}_i$. The integral of the remaining squared Bessel norm against β_c is at most $v_U d_B$. This proves that the forward squared factor divided by $v_U d_B$ is at most

$$\frac{\tau^2}{\mathcal{V}_i}\left[\frac{2^{pf}N^{(j-i)f}i_\gamma}{M^{2if}P_j}\right]^k.$$

Apply the same result to the inverse orbit and take the geometric mean as in Theorem 131.34. □

Corollary 131.38 (Open terms with bounded product of normal lengths)**.** *In each agreeing block sum A_γ over open surviving orbits for which the normalized real normal vectors satisfy*

$$\prod_v \|q_v\| = \frac{\sqrt{\mathrm{N}_{F/\mathbb{Q}}\Delta_\gamma}}{P_n} \le \frac{(M/b)^{af}}{N^{f/2}}.$$

Call this sum $A_*(c,s,c')$. Then

$$\sum_{c,c',\,s\ \text{agreeing}} \left(\frac{A_*(c,s,c')}{v_U}\right)^2 \le \tau^{-3a}|B_{\mathrm{res}}|\,|\mathcal{C}|\,d_B.$$

Proof. For an open orbit $i=a$, $i_\gamma = i'_\gamma = 1$. Write $G_0 = M^{2af}P_n^2/N^f$. Then the gauge in Theorem 131.23 satisfies

$$G_\gamma/G_0 = \max\{1, N^f\,\mathrm{N}_{F/\mathbb{Q}}\Delta_\gamma/P_n^2\}.$$

Our condition gives $\sqrt{G_\gamma/G_0} \le (M/b)^{af}$. By the theorem, $\mathcal{V}_a \ge 1$, and $2^{a+1} \le b^{a/4}$ for $b \ge 2^8$, $a \ge 1$, it follows that

$$\frac{A_\gamma}{v_U d_B} \le \tau^2\left[b^{-3af/4}\frac{N^{f/2}}{\sqrt{G_\gamma}}\right]^k.$$

Two open terms in the same block can have positive corank $\ell = c^+ + 1$; the estimate of Theorem 131.23 always gives $c^+ \ge \max(0,\ell-1)$. Applying that theorem to the product of these bounds leaves the factor $(2/M)^{\max(0,\ell-1)kf}d(L_{12})^{-k}$. The factor N^{kf} cancels upon summing by Theorem 131.29. Thus the normalized second moment is at most

$$|B_{\mathrm{res}}|\,|\mathcal{C}|\,d_B^2\,\tau^{6-6a}\sum_{\ell\ge 0}\tau^\ell(2/M)^{\max(0,\ell-1)kf}.$$

The sum is $1+\tau/(1-\tau(2/M)^{kf}) \le 3\tau$. Since $d_B \le \tau^8$ by Corollary 131.35, the result follows from $3\tau^{15-6a} \le \tau^{-3a}$. This inequality holds at all sufficiently large nodes. □

For larger products of normal lengths the compact mean still gives Theorem 131.37. The common-lattice estimate then involves the larger discriminant term in G_γ; bounding those blocks on the scale of the full period requires further information.

132 Stationary real coefficients and truncation

132.1 A stationary position at every real place

Fix a node $V_0 \le x \le V_0^{3/2}$ and a real place of the field $F = E^+$ used in Theorem 131.37. We construct a translate for the local period m of Proposition 131.6. Use normalized real coordinates as in Corollary 131.2, so the last vector z has Hermitian norm -1 and the character on the frame is $\exp(\sigma - \bar{\sigma})$, where σ is the sum of its entries immediately above the diagonal. Write $a = n-1$, $r = k+2n-1$. The basis of the normalized space consists of a isotropic vectors f_j, the positive middle space X_0 of dimension k, the line of z, and the dual vectors $f_j^\vee$. It is Euclidean orthonormal. The Hermitian form has matrix H with $H^2 = 1$ and $\langle f_i, H f_j^\vee \rangle = \delta_{ij}$.

Proposition 132.1 (The compact weights at the given node)**.** *Order the coordinates by decreasing Harish-Chandra parameter. The weight operator L of the highest vector in the lowest compact type is zero on the first $k-1$ positive coordinates. On the remaining $2n$ coordinates the signs are $+,-,\dots,+,-$, and the weights are distinct integers $\mu_0 > \cdots > \mu_{2n-1}$ in $[-3B, -B]$. Here $B = \kappa + r^8$, where κ is the weight at this place. The compact value representation has weight operator $L' = \operatorname{diag}(0,\dots,0,t)$ after a permutation, where $t = xb$ at the designated a-places and $t = 0$ elsewhere. For all sufficiently large nodes, $D_0 := 3B + t \le b^2$.*

Proof. The undoubled row parameter has entries $k/2 - 1 - i$ for $0 \le i < k-1$. At $i = k-1+j$ its entry is $-\kappa - 2(j+1)r^8$, $0 \le j < 2n$, by the shifted weight formula $\kappa_j = \kappa + (2j+1)r^8$. The first block has positive sign and the last block alternates starting with a positive sign. The prescribed scales have $r = O(x^5)$, $2n \le 2\sqrt{x}$, $\log b \ge (\log x)^6$, and $\log x_1 \le 10^{-6} \log b$. The weight κ is either b or xb/x_1, with $x_1 \mid b$. Thus $2(2n)r^8 + r = o(\kappa)$, and the row is in decreasing order. Also $\kappa \le xb$, $t \le xb$, whence $D_0 \le b^2$.

For signs ϵ_i and row entries λ_i, the formula $L = \lambda + \rho_{\mathrm{n}} - \rho_{\mathrm{c}}$ recalled in Lemma 131.7 gives twice the difference between λ_i and the compact weight at position i as

$$\sum_{j>i} \epsilon_i \epsilon_j - \sum_{j<i} \epsilon_i \epsilon_j .$$

Indeed a compact root contributes with the opposite sign to a noncompact root. The alternating block has sign sum zero, so the difference on the first block is $(k - 2 - 2i)/2 = \lambda_i$. On the last block the correction has absolute value at most $(r-1)/2$. Its entries are integers since r is odd. The gap $2r^8$ gives strict decrease, the first entry is at most $-\kappa - r^8$, and the last is at least $-\kappa - 4nr^8 - (r-1)/2 \ge -3B$. Finally the diagram defining the compact Bessel value space counts columns strictly below height k with last weight zero. Its only columns at those heights have height one, with the indicated multiplicity t. Thus it is the symmetric power with the stated highest weight. □

Theorem 132.2 (A stationary flag with a uniform length bound)**.** *Write the defining space in sign coordinates and let p be a Euclidean unitary identification of its sign form with H in the normalized space. There are independent vectors $v_0, \dots, v_a$, a positive isometric embedding $I : \mathbb{C}^k \to \mathbb{C}^r$, and $g \in \mathrm{U}(H)$ such that*

$$gpv_j = f_{j+1}\ (0 \le j < a), \qquad gpv_a = z, \qquad gpI = \text{the prescribed inclusion of } X_0,$$
$$Lv_j = v_{j+1}\ (j < a), \qquad LI - IL' \text{ has image in } \mathbb{C}v_0, \qquad \inf_{s \in S} (1 + \|sg\|_{\mathrm{op}}) \le b^{5n}.$$

Here S is the subgroup acting by upper unitriangular matrices on the ordered frame $(f_1, \dots, f_a, z)$.

Proof. Set $P(T)=\prod_{j=0}^{2n-1}(T-\mu_j)$, $Q(T)=T-t$. For a polynomial A define $[A]$ to be zero on the first $k-1$ coordinates and to have tail coordinates

$$[A]_j=\frac{A(\mu_j)}{\sqrt{|P'(\mu_j)(\mu_j-t)|}}.$$

The sign of $P'(\mu_j)$ is $(-1)^j$, and $\mu_j<t$. Thus the Hermitian product of two such vectors is

$$-\sum_j \frac{\overline{A(\mu_j)}\,C(\mu_j)}{P'(\mu_j)(\mu_j-t)}.$$

Put $v_h=[T^hQ]$ for $0\le h\le 2a$ and $u=\sqrt{P(t)}[1]$. Lagrange interpolation shows that $\sum_j R(\mu_j)/P'(\mu_j)$ is the coefficient of T^{2n-1} in R whenever $\deg R\le 2n-1$. Apply this to $R=T^{h+l}Q$. For $h+l<2a$ the Hermitian product of v_h,v_l vanishes, and for $h+l=2a$ it is -1. Applying the formula to T^h gives $\langle u,v_h\rangle_\epsilon=0$ for $h\le 2a$. Partial fractions give $\sum_j[P'(\mu_j)(t-\mu_j)]^{-1}=1/P(t)$, so u has Hermitian norm one. The first $a+1$ vectors v_h are independent since a polynomial of degree at most a vanishing at all $2n$ nodes is zero.

Let I take the first $k-1$ standard vectors to themselves and the last one to u. Multiplication by L acts as multiplication by T on $[A]$. In particular $Lu=tu+\sqrt{P(t)}v_0$, which proves the weight assertions. The Gram matrices now agree on the indicated frames and middle spaces. To extend their identification to an isometry, choose vectors dual to the independent isotropic arms in the orthogonal complement of $I(\mathbb{C}^k)\oplus\mathbb{C}v_a$. This complement is nondegenerate. Subtract from the dual vectors the arms times half the Hermitian Gram matrix of the duals. Their pairings with the arms stay equal to the identity and their mutual pairings become zero. The resulting bases give g with the required exact images.

We bound the length using a representative of Sg. Since the nodes and t are separated integers, $|P'(\mu_j)|\ge 1$ and $1\le t-\mu_j\le D_0$. Thus $|(v_h)_j|\le D_0^{h+1/2}$ and $|u_j|\le D_0^n$. Put $K_0=\sqrt{r}\,D_0^{2n}$. Apply Euclidean Gram–Schmidt without rescaling to $pv_0,\dots,pv_a$ and call the resulting columns b_h. Their lengths are at most K_0. The Hermitian product of b_h with pv_{2a-h} is still -1, since all earlier columns pair to zero with it. Consequently $\|b_h\|\ge K_0^{-1}$. For $h<a$ their isotropic duals $Hb_h/\|b_h\|^2$ have lengths at most K_0. The arms and these duals are pairwise Euclidean orthogonal, by Gram–Schmidt, $H^2=1$, and isotropy of the span of the arms. Their joint span is H-stable. The terminal vector b_a lies in its Euclidean orthogonal complement, by Gram–Schmidt and Hermitian orthogonality to the arms. Its Hermitian norm is -1.

The complement has signature $(k,1)$. Write $b_a=b_+ + b_-$ in the positive and negative eigenspaces of H. A unitary identification of those eigenspaces followed by the real two-dimensional boost with diagonal entries $\|b_-\|$ and off-diagonal entries $\|b_+\|$ sends the standard negative unit vector to b_a. Choose the directions of the two components as axes; if $b_+=0$ only a unitary identification is needed. Since $\|b_-\|^2-\|b_+\|^2=1$, this is an isometry of norm at most $2K_0$, extended by the identity in the remaining positive directions. Combining it with the map of the standard arms and duals to the orthogonal columns just constructed gives an isometry g_1^{-1} of norm at most $2K_0$. Its inverse has the same norm since H is a Hermitian involution. The frames $g^{-1}(f_1,\dots,f_a,z)$ and $g_1^{-1}(f_1,\dots,f_a,z)$ differ by an upper unitriangular change of columns. Hence $g_1g^{-1}\in S$. Finally $1+2K_0\le b^{5n}$ by $D_0\le b^2$ and the stated growth of the ranks. This proves the uniform assertion. □

Corollary 132.3 (An exact trace identity for the period phase)**.** *For $s\in S$, set $Y=p^*g^{-1}sgp-1$, and $Y'=h-1$, where h is the induced action on X_0. Then*

$$\operatorname{tr}Y=\operatorname{tr}Y',\qquad \operatorname{tr}(LY)-\operatorname{tr}(L'Y')=\sigma(s)-\overline{\sigma(s)}.$$

In particular the integrand for m in Proposition 131.6, at the translate g and the indicated unit highest weight vectors, has vanishing first derivative on S at the identity.

Proof. Use the three-block Witt basis $A, W, A^\vee$, with A the arms $v_0, \dots, v_{a-1}$, $W = I(\mathbb{C}^k) \oplus \mathbb{C}v_a$, and the isotropic duals constructed in the theorem. The middle form is $H_W = \mathrm{diag}(1_k, -1)$. Both Y and its Hermitian adjoint $Y^\dagger$, which equals $p^* g^{-1} s^{-1} g p - 1$, are upper block triangular. Indeed the isometries preserve A and $A^\perp$. The action on the middle quotient fixes the terminal vector, and therefore restricts to h on its orthogonal complement. The AA blocks of $Y, Y^\dagger$ are strictly upper triangular and the middle block of Y is $\mathrm{diag}(Y', 0)$. The last diagonal block of Y is the conjugate transpose of the first block of $Y^\dagger$. This gives the ordinary trace.

For the weighted trace, L_{AA} is the shift $v_j \mapsto v_{j+1}$ up to the last arm, whose image is $v_a \in W$. The only entry of L_{WA} is this last map. Also $L_{A^\vee A} = 0$ and the compression of L to the I-part of W is L'. Thus the contractions $\mathrm{tr}(L_{AA} Y_{AA})$ and $\mathrm{tr}(L_{WA} Y_{AW})$ add to $\sigma(s)$, including the terminal simple root. Since $L = L^\dagger$, the corresponding contractions on the dual blocks are the complex conjugates of the same contractions with $Y^\dagger$. Explicitly, $L_{A^\vee A^\vee} = L_{AA}^*$, $L_{A^\vee W} = L_{WA}^* H_W$, and $Y_{WA^\vee} = H_W^{-1} ((Y^\dagger)_{AW})^*$. They give $\overline{\sigma(s^{-1})}$. All remaining contractions vanish except the middle trace $\mathrm{tr}(L'Y')$. The entries immediately above the diagonal add under multiplication of upper unitriangular matrices, so $\sigma(s^{-1}) = -\sigma(s)$.

For the derivative, the normalized matrix coefficient of a torus weight vector of weight L has derivative $Z \mapsto \mathrm{tr}(LZ)$ on the unitary Lie algebra. On the diagonal torus this follows by differentiating its character. The complex-linear extension vanishes on off-diagonal matrix units by torus conjugation, since the coefficient is invariant under that conjugation. The same argument applies to L'. Differentiating the trace identity now cancels both coefficient derivatives against the derivative of the inverse simple-root character. □

The point and the trace identity hold at every place with the same upper bound on the length. To normalize the global test vector one still needs a lower bound for the integral m; this requires controlling the coefficient away from its stationary region.

132.2 Localization and the positive contribution near the stationary point

At the translate of Theorem 132.2 write $G_s = p^* g^{-1} s g p$, $h_s = \mathrm{pr}_{X_0} s$ and $J_s = G_s \oplus h_s$. Put $Y = G_s - 1$, $Y' = h_s - 1$, and let C_s be the matrix retaining the entries of G_s between coordinates of opposite signs and zero elsewhere. Let $\varphi(s) = \sigma(s) - \overline{\sigma(s)}$, and let

$$\Theta(s) = \langle w, \pi(g^{-1} s g) w \rangle \overline{\langle e, \tau_0(h_s) e \rangle} \exp(-\varphi(s)).$$

Here τ_0 is the compact value representation and w, e are the unit highest vectors used above. The unsigned norm $\| \cdot \|_2$ on matrices in this subsection is the Hilbert–Schmidt norm. The integral of Θ on S is m in Proposition 131.6.

Lemma 132.4 (The phase in exponential coordinates). *If $\|J_s - 1\|_{\mathrm{op}} < 1$, then*

$$\mathrm{tr}(L \log G_s) - \mathrm{tr}(L' \log h_s) = \varphi(s)$$

for the power-series logarithms at the identity.

Proof. Let $D(A, A') = \mathrm{tr}(LA) - \mathrm{tr}(L'A')$. By Corollary 132.3 and additivity of φ, $D(G_s^j - 1, h_s^j - 1) = j\varphi(s)$ for each integer $j \geq 0$. For every $l \geq 2$, expand $(G_s - 1)^l, (h_s - 1)^l$ by the binomial theorem. The coefficients sum to zero, so one may replace G_s^j, h_s^j by $G_s^j - 1, h_s^j - 1$. The l-th finite difference of the linear function $j\varphi(s)$ vanishes. This gives $D(Y^l, (Y')^l) = 0$. Termwise application of D to the two convergent series proves the formula. □

Theorem 132.5 (Localization modulo the common compact subgroup)**.** *The common first $k-1$ positive coordinates define a subgroup $K_c = \mathrm{U}(k-1)$ of S acting by the same unitary matrix on those coordinates in both factors of J. The function Θ is invariant under left translation by K_c. If $\|C_s\|_2 = c \le 1/6$, there is $u \in K_c$ with*

$$\|G_{us} - 1\|_2^2 \le 13c^2, \qquad \|h_{us} - 1\|_2^2 \le 11c^2.$$

In particular K_c consists precisely of the elements of S whose first component is in the maximal compact subgroup of the sign model.

Proof. An element of $\mathrm{U}(k-1)$ on the common coordinates extended by the identity on their complement fixes every v_h and has the stated action on I, so it defines a member of S. It fixes w: the highest weight is zero on the common block, and each adjacent compact root there has highest weight difference zero. Its root $\mathrm{SU}(2)$ therefore fixes the highest vector by the weight string formula. These root groups and the diagonal torus generate the block. It fixes e in the symmetric power, and σ vanishes there since the frame is fixed pointwise. This gives the asserted invariance.

Let P_c project to the common coordinates and put $P_o = 1 - P_c$, $\nu_i = -L_{ii}$. For each row of the sign-unitary matrix G_s, preservation of the sign form gives the identity

$$2\Re Y_{ii} = 2\sum_j |(C_s)_{ij}|^2 - \sum_j |Y_{ij}|^2.$$

In fact $\sum_j \epsilon_i\epsilon_j |(G_s)_{ij}|^2 = 1$; expanding the diagonal entry $1 + Y_{ii}$ gives the identity. The same formula holds for columns. By the exact phase identity, $\Re\,\mathrm{tr}(\nu Y) = -\Re\,\mathrm{tr}(L'Y') = t(1 - \Re(h_s)_{kk}) \ge 0$. Multiply the row identities by ν_i and sum. Since $B \le \nu_i \le 3B$ off the common block and $\nu_i = 0$ on it, the row and column estimates give

$$\|P_o Y\|_2^2 \le 6c^2, \qquad \|Y P_o\|_2^2 \le 6c^2.$$

Also the unweighted row identities summed for both factors and the ordinary trace identity give $\|Y'\|_2^2 = \|Y\|_2^2 - 2c^2$.

Write $A = (G_s)_{cc}$. From the cc-block of $G_s^* \epsilon G_s = \epsilon$ we have $\|A^*A - 1\|_2 \le \|P_o Y P_c\|_2^2 \le 6c^2$, using $\|D^*\epsilon D\|_2 \le \|D\|_2^2$. Let $A = U|A|$ be its polar decomposition. The bound on c implies invertibility, and each singular value satisfies $|s_j - 1| \le |s_j^2 - 1|$. Thus $\||A| - 1\|_2 \le 6c^2$. Left translate by the member of K_c acting by U^*. The new cc-block is $|A|$ and the noncompact part has unchanged norm. The preceding row and column estimates apply at the new point. Bounding the entries outside cc by the sum of those two estimates gives a squared norm at most $12c^2 + 36c^4 \le 13c^2$. The unweighted trace identity gives the second bound. Taking $c = 0$ shows that the indicated element is the inverse of s, since $s \mapsto G_s$ is faithful. □

Theorem 132.6 (Positive mass in a neighborhood of the identity)**.** *Put $d_* = r + k$, $\beta = b^3$, $\eta = (200\beta^5)^{-1}$. With Haar measure on the actual Bessel subgroup let*

$$\Omega_\eta = \{s : \|J_s - 1\|_{\mathrm{op}} \le \eta\}, \qquad \mathcal{B}_\beta = \{s : \|J_s\|_{\mathrm{op}} \le 1 + \beta\}.$$

At every real place, for all sufficiently large nodes,

$$\Re\Theta(s) \ge \tfrac{1}{2} \quad (s \in \Omega_\eta), \qquad \int_{\Omega_\eta} \Theta(s)\, ds \ge \tfrac{1}{2}\beta^{-16 d_*^2}\,\mathrm{vol}(\mathcal{B}_\beta) > 0.$$

On the product of the f real groups the integral on $\prod_v \Omega_{\eta,v}$ is at least $2^{-f}\beta^{-16 f d_*^2}\prod_v \mathrm{vol}(\mathcal{B}_{\beta,v})$.

Proof. For $s \in \Omega_\eta$, $Z = \log G_s$ has $\|Z\|_{\mathrm{op}} \le 2\eta$ and $\|Z\|_2 \le 2\sqrt{r}\,\eta$. It is in the real Lie algebra: conjugation of the logarithm by the sign involution after taking adjoints gives the logarithm of G_s^{-1}, which equals $-Z$ on this neighborhood. In an orthonormal real Lie basis the sum of the absolute values of its coordinates is at most $r\|Z\|_2$. Lemma 131.7 at length one and the unitary path $\exp(uZ)$, $0 \le u \le 1$, give

$$|\langle w, \pi(g^{-1} s g) w\rangle - 1| \le 8r^2(\max_i |\lambda_i| + 3r + 2n^2 + 1)\eta.$$

Here the sign basis is carried to the normalized group by p, which preserves the matrix norms. On the symmetric power the compact coefficient is $(h_s)^t_{kk}$, so its difference from one has modulus at most $t\eta$, also for $t = 0$. Finally

$$|\exp(-\varphi(s)) - 1| \le |\varphi(s)| \le (3rB + t)\eta$$

by the trace identity and the fact that φ is imaginary. Each coefficient has modulus at most one. Their product differs from one by at most the sum of these three errors. The sum is less than $1/2$ for all sufficiently large nodes since $r = O(x^5)$, $n \le \sqrt{x}$, $3B + t \le b^2$, $\max_i |\lambda_i| \le 3xb$, and $\log b \ge (\log x)^6$.

We compare the two volumes by packing in complex matrices of size d_* with the operator norm. The graph $s \mapsto J_s$ is proper: its first projection is the conjugate of the inclusion of the closed subgroup S. For $R = 1 + \beta$, take a maximal η/R-separated set J_{s_i} in the image of $\mathcal{B}_\beta$. Disjoint ambient balls of radius $\eta/(2R)$ about them lie in a ball of radius $R + \eta/(2R)$. Comparing Lebesgue volumes in real dimension $2d_*^2$ bounds their number by $(1 + 2R^2/\eta)^{2d_*^2} \le \beta^{16 d_*^2}$ for $\beta \ge 10^4$. Every J_s in that image is within η/R of some J_{s_i}. Since a sign-unitary matrix and its inverse have equal operator norm, $\|J_s J_{s_i}^{-1} - 1\|_{\mathrm{op}} \le \eta$. The right translates $\Omega_\eta s_i$ cover $\mathcal{B}_\beta$. Both sets have finite volume, and each contains a neighborhood of the identity. This proves the asserted comparison of volumes.

The group S is unimodular: its unipotent radical is unimodular and its compact factor preserves Haar on the radical. The set Ω_η is inversion invariant by the sign adjoint identity for $J_s - 1$. The two unitary coefficients and the character take conjugate values at inverses, so the integral on Ω_η is real. The lower bound follows by integrating the first assertion, and the product bound follows by Fubini on the product of these compact sets. □

In the full period the complementary region can contribute with either sign. The localization theorem describes where the integrand can have small noncompact displacement; estimating its phase and decay there and beyond the compact neighborhood is needed to bound the complete integral.

132.3 Decay of the period away from the maximal compact subgroup

We estimate the part of the actual local integral on which the displacement C_s of Theorem 132.5 is large. Put $D = (r + k)^2$, $T = D \log \beta$ and $\delta_* = (T r^{1/4})^{-1}$, with $\beta = b^3$ as before. Use coordinates in decreasing order of λ. Write $\alpha = \lambda + \rho$, $\mu = \lambda + \rho - 2\rho_c$, and $K = \mathrm{U}(p) \times \mathrm{U}(q)$ for the maximal compact subgroup of the sign unitary group. For an integer weight v and an invertible matrix A put

$$m_v(A) = (\det A)^{v_{\min}} \prod_{v_{\min} < j \le v_{\max}} \Delta_{\{i : v_i \ge j\}}(A),$$

where Δ_B denotes the principal minor on B and the empty minor is one. We first fix precisely the coefficient formula used in the estimate.

Proposition 132.7 (The compact integral in matrix coordinates)**.** *Denote the highest-vector coefficient of π in sign coordinates by c_λ. Write $y = k_0 z$ by polar decomposition, $k_0 \in K$, $z > 0$, and normalize Haar on K to have mass one. Then*

$$c_\lambda(y) = \frac{1}{M_\mu} \int_K \frac{m_\mu(k_0 u)\overline{m_\mu(u)}}{m_{-\alpha}(u^* z u)}\, du, \qquad M_\mu = \int_K |m_\mu(u)|^2 du = (\dim V_\mu)^{-1}.$$

The germ at the identity of c_λ on the complex matrix group is

$$\mathcal{F}(Z) = \frac{1}{M_\mu} \int_K \frac{|m_\mu(u)|^2}{m_{-\alpha}(u^* Z u)}\, du.$$

In this germ $\mathcal{F}(b_- Z b_+) = \mathcal{F}(Z)$ for block diagonal unitriangular b_-, b_+ that are lower and upper triangular, respectively, in the chosen order.

Proof. Use the compact-Cartan discrete-series realization by cohomological induction from the Borel for the positive roots of λ. The torus character has weight $\lambda - \rho$, an integral weight here, and the good-range inequalities are $\lambda_i - \lambda_j > 0$ for $i < j$. The bottom compact type has weight μ, as in Lemma 131.7. On positive Hermitian matrices the Flensted-Jensen integral for this compact type is the average of the conjugates of its highest-line projection times the inducing character multiplied by the determinant character of the nilradical. In Cartan polar coordinates its value at $k_0 z$ acquires the left factor $\sigma_\mu(k_0)$. These are the compact-Cartan case of the bottom-layer coefficient theorem and its duality formula. We use B. Sun, *Matrix coefficients of cohomologically induced representations*, Compos. Math. 143 (2007), Theorems 2.1, 6.3 and formula (26).

Here are the characters and projections in that formula. The determinant character adds 2ρ to the inducing weight, giving α. For a positive Hermitian matrix $A = u^* Z u$, take its unique factorization $A = n_0 d n_0^*$ with n_0 upper unitriangular and d positive diagonal. It follows from successive Schur complements starting at the last entry. Products of the last entries of d are exactly the trailing principal minors of A. Since α is strictly decreasing, $m_{-\alpha}(A) = \prod_i d_i^{-\alpha_i}$. Thus the torus factor is $1/m_{-\alpha}(A)$. The compact highest coefficient is $m_\mu(u)$: in either sign block take the tensor product of the exterior highest vectors for its columns and the determinant power of its last weight. The individual coefficients are their principal minors, and the product of unit highest vectors generates the component of highest weight μ. Taking the matrix entry of the conjugated highest-line projection, and replacing the integration variable by its inverse, gives the indicated numerator. Schur orthogonality gives $M_\mu = 1/\dim V_\mu$. In the polar decomposition the positive factor and the unitary factor belong to the sign group: the involution $y \mapsto \epsilon y^{-*} \epsilon$ fixes y and preserves both parts, so uniqueness gives the assertion.

For the local formula consider the operator-valued integral before taking the highest entry. It extends holomorphically to $\|Z - 1\|_{\mathrm{op}} < 1$, since every compressed $u^* Z u$ is then invertible and the compact integral is locally uniformly holomorphic. On positive matrices the duality formula gives covariance under $Z \mapsto a Z a^*$ for $a \in K_{\mathbb{C}}$, with factors $\sigma_\mu(a), \sigma_\mu(a)^*$ on the two sides. Consequently its germ is covariant under independent left and right $K_{\mathbb{C}}$-multipliers. Indeed both sides are holomorphic in the left multiplier, the right multiplier and the argument; the conditions that the right multiplier be the adjoint of the left and that the argument be Hermitian give a totally real set of full real dimension. Uniqueness of the Taylor expansion gives the identity of germs. At a nearby real point $y = k_0 z$ this covariance gives precisely the left factor in the polar formula, so the asserted scalar germ is the one for c_λ. Upper compact unitriangular matrices fix the highest vector. Lower ones fix its dual covector on the right of that covector. The two multipliers therefore leave its matrix entry unchanged. The identity extends along connected parameter paths on which the three arguments of the covariance identity stay in the ball of holomorphy. □

Lemma 132.8 (Decay from the opposite-sign parameter gaps)**.** *Take $H_0 = r^8$ and set $A_y = (y + \epsilon y \epsilon)/2$, the block diagonal part of y. For the parameters of Proposition 132.1,*

$$|c_\lambda(y)|^2 |\det A_y|^{H_0} \le 1, \qquad |c_\lambda(y)|^4 \le (1 + \|y - A_y\|_2^2)^{-H_0}, \qquad |c_\lambda(y)| \le (2/\|y\|_{\mathrm{op}})^{H_0}.$$

Proof. First suppose $y = z > 0$. Then $\epsilon z \epsilon = z^{-1}$, $\det z = 1$, and $A_z = (z + z^{-1})/2$ is positive. Jacobi's complementary-minor identity gives $\Delta_B(z) = \Delta_{B^c}(z)$, since z^{-1} is the conjugate of z by a diagonal sign matrix. Set $S_j = \{i : \alpha_i \ge j\}$. All coordinates in $D_j = S_{j-H_0} \setminus S_j$ have the same sign whenever that set is nonempty. Indeed opposite signs have λ-gap at least H_0: consecutive tail entries are $2r^8$ apart and the first block is separated from the negative tail by $\kappa - O(r) > r^8$. Adding the decreasing coordinates of ρ increases the gaps.

For disjoint B, C the positive principal minors satisfy $\Delta_{B\cup C}(Z) \le \Delta_B(Z)\Delta_C(Z)$ by the Schur complement formula and monotonicity of positive determinants. On D_j the compressions of A_z and z coincide. Applying the inequality first to A_z and then to the complementary minor of z gives

$$\frac{\Delta_{S_{j-H_0}}(A_z)}{\Delta_{S_j}(A_z)} \le \Delta_{D_j}(z) = \Delta_{D_j^c}(z) \le \Delta_{S_j}(z)\, \Delta_{S_{j-H_0}^c}(z).$$

Multiply over all integers j. All but finitely many ratios are one. The numerator on the left shifts the denominator by H_0 positions, giving $(\det A_z)^{H_0}$. On the right, complements have the same minor, and $\det z = 1$. The product is $m_\alpha(z)^2 = m_{-\alpha}(z)^2$, again by complementary minors.

Conjugation by $u \in K$ preserves $\det A_z$. For $y = k_0 z$ it also gives $|\det A_y| = \det A_z$, since $A_y = k_0 A_z$. The preceding inequality thus bounds the denominator in Proposition 132.7 below by $|\det A_y|^{H_0/2}$ at every integration point. Cauchy–Schwarz and invariance of Haar bound the integral of the absolute numerator by M_μ. This proves the first inequality.

Put $C = y - A_y$. The diagonal blocks of $y^* \epsilon y = \epsilon$ give $A_y^* A_y = 1 + C^* C$. Hence $|\det A_y|^2 \ge 1 + \operatorname{tr}(C^* C)$ by the nonnegative eigenvalues of $C^* C$. Finally the eigenvalues of z occur in reciprocal pairs, and the largest is $\|y\|_{\mathrm{op}}$. Each contributes $(s + s^{-1})/2 \ge 1$ to $\det A_z$, with the largest and its reciprocal together contributing at least $(\|y\|_{\mathrm{op}}/2)^2$ when $\|y\|_{\mathrm{op}} > 1$. For norm one the bound is immediate. These facts give the last two estimates. □

Theorem 132.9 (Absolute integral outside the compact tube)**.** *At every real place of the same fields and at every sufficiently large node in the prescribed interval,*

$$\int_{\|C_s\|_2 \ge \delta_*} |\Theta(s)|\, ds \le \beta^{-100D} \operatorname{vol}(\mathcal{B}_\beta).$$

Proof. On this set the compact coefficient and the character have modulus at most one. The lemma gives $|\Theta| \le \exp(-H_0 \delta_*^2/8)$, since $\log(1 + \delta_*^2) \ge \delta_*^2/2$. Here $H_0 \delta_*^2 / T = r^{15/2}/T^3 \to \infty$ uniformly: $r \in [x^5, 3x^5]$, $r + k \le 2r$, and $\log \beta = 3 \log b \le 3x^{1/100}$. The uniform upper bound is thus at most e^{-300T}. By taking the geometric mean with the global bound from the lemma and unitarity, we get on that set

$$|\Theta(s)| \le e^{-150T} \min\{1, (2/\|J_s\|_{\mathrm{op}})^{H_0/2}\}.$$

The operator norm of J_s equals that of G_s, since $\|h_s\|_{\mathrm{op}} = 1 \le \|G_s\|_{\mathrm{op}}$.

The integral of this minimum is at most $2 \operatorname{vol}(\mathcal{B}_\beta)$. Indeed the portion inside $R = 1 + \beta$ costs at most that ball's volume. For $U \ge 1$, the packing argument of Theorem 132.6 with separation $1/U$ covers $\{\|J_s\|_{\mathrm{op}} \le U\}$ by at most $(1 + 2U^2)^{2D}$ translates of $\{\|J_s - 1\|_{\mathrm{op}} \le 1\}$, which is contained in $\mathcal{B}_\beta$. On the shell $2^j R < \|J_s\|_{\mathrm{op}} \le 2^{j+1} R$, $j \ge 0$, the cost divided by $\operatorname{vol}(\mathcal{B}_\beta)$ is at most

$$12^{2D} R^{4D} (2/R)^{H_0/2}\, 2^{j(4D - H_0/2)}.$$

The first factor, including the power of $2/R$, is at most $1/2$ eventually since $H_0/D \to \infty$ and $R \to \infty$. The common ratio is also at most $1/2$. Summing proves the asserted bound, using $2e^{-150T} \le \beta^{-100D}$. □

This tail is smaller than the positive integral on Ω_η in Theorem 132.6. To compare the entire period with that positive integral, it remains to bound cancellation in the tube $\|C_s\|_2 < \delta_*$. Theorem 132.5 places its integrand within $4\delta_*$ of identity coordinates after compact translation.

132.4 A lower bound for the full real Bessel period

We prove that the signed integral in the tube of Theorem 132.9 retains a fixed fraction of the mass estimated in Theorem 132.6. The estimates below have absolute constants independent of both the dimension and the number of minors in a weight. For a projection P, $\det_P A$ denotes the determinant of PAP acting on its range, and $[P, A] = PA - AP$.

Lemma 132.10 (Expansion of a compressed product of exponentials)**.** *Let P be an orthogonal projection and let $\|A_j\|_2 \le d \le 1/40$ for $1 \le j \le 3$. Set $u_0 = \max_j \|[P, A_j]\|_2$, $F_P(A, B) = \operatorname{tr}(PA(1-P)BP)$. Then the logarithm at the identity satisfies*

$$\log \det_P(e^{A_1}e^{A_2}e^{A_3}) = \operatorname{tr}(P\textstyle\sum_j A_j) + \frac{1}{2}\sum_j F_P(A_j, A_j) + \sum_{j<l} F_P(A_j, A_l) + E,$$
$$|E| \le 16000\, d\, u_0^2.$$

Proof. Introduce a scalar t in all three exponents and expand the trace of the logarithm on $P\mathbb{C}^r$. The compression differs from its identity by at most $e^{3|t|d} - 1$ in operator norm. Its logarithmic series converges for $|t| \le 1$. Expand the exponentials and the logarithm in powers of t. The sum of absolute scalar coefficients of words of total degree l before inserting projections is bounded by the coefficient of z^l in $-\log(2 - e^{3z})$. This follows by replacing the coefficients of the logarithm by their absolute values: the majorizing series for its argument minus the identity is $e^{3z} - 1$. The coefficients are nonnegative and the series has value less than one at $z = 1/10$; hence this bound is at most 10^l.

Insert $1 = P + (1 - P)$ at each unseparated junction between letters. There are at most 2^l resulting paths per word. Those staying in the range of P give exactly the trace logarithm of the product of the three compressed exponentials, namely of $e^{PA_1P}e^{PA_2P}e^{PA_3P}$ on that range. Its determinant is the exponential of the sum of their traces. Their contribution is thus precisely the linear term. Every other path crosses the boundary of P at least twice, since it begins and ends in its range. The two crossing factors have Hilbert–Schmidt norm at most u_0; all other factors have operator norm at most d. The absolute trace is bounded by $u_0^2 d^{l-2}$ by the Hilbert–Schmidt Cauchy–Schwarz inequality. In degree two the crossing terms are exactly the two sums displayed in the statement, from a square in one exponential or from two exponentials in their indicated order. The sum in degrees $l \ge 3$ is bounded by $\sum_{l \ge 3} 20^l d^{l-2} u_0^2 \le 16000 d u_0^2$. This also gives absolute convergence of the remainders. □

Proposition 132.11 (A Gaussian estimate in the shrinking neighborhood)**.** *Use the notation of Proposition 132.7. For an integer weight v put $Q_v(A) = \sum_{i,j} |v_i - v_j||A_{ij}|^2$. Suppose Z is in the sign unitary Lie algebra and $\|Z\|_2 \le 8\delta_*$. For $q = Q_\alpha(Z)/4$ and all sufficiently large nodes,*

$$\left| c_\lambda(e^Z) e^{-\operatorname{tr}(\mu Z)} - e^{-q} \right| \le \tfrac{1}{20} e^{-q} + e^{-200T}.$$

Here μ in a trace denotes the diagonal matrix of that weight. If Z' is skew-Hermitian on the compact value space of dimension k in its defining representation, with $\|Z'\|_2 \le 8\delta_*$, the corresponding estimate is

$$\left| ((e^{Z'})_{kk})^t e^{-tZ'_{kk}} - e^{-q'} \right| \le \tfrac{1}{20} e^{-q'} + e^{-200T}, \qquad q' = \tfrac{t}{2} \sum_{j \ne k} |Z'_{kj}|^2.$$

Proof. Put $d = 8\delta_*$, which tends to zero with $dT \to 0$. Let U, V be the strict lower and strict upper parts of the compact (equal-sign) blocks of Z. For $(A_1, A_2, A_3) = (-U, Z, -V)$ all norms are at most d. Write $Y = e^{-U} e^{Z} e^{-V}$. Proposition 132.7 permits using $\mathcal{F}(Y)$ for $c_\lambda(e^Z)$. In fact all partial products with the exponents scaled between zero and one differ from the identity by at most $e^{3d} - 1 < 1$ in operator norm, so the germ identity extends to the matrices used here.

Fix a trailing coordinate projection P in decreasing order of λ and $u \in K$. Put $R = uPu^*$, $s_0 = \|[P, Z]\|_2$, $h_0 = \|[P, u]\|_2 = \|R - P\|_2$. Since taking either triangular mask can only delete entries from $[P, Z]$, one has $\|[P, A_j]\|_2 \le s_0$ and $\|[R, A_j]\|_2 \le s_0 + 2dh_0$. Apply Lemma 132.10 with projection R. The first term is $\operatorname{tr}(R \operatorname{diag} Z)$: the sum of the three arguments retains just the diagonal and the opposite-sign entries, and R is block diagonal by sign. Its difference from $\operatorname{tr}(PZ)$ is at most dh_0^2, since $\sum_i |R_{ii} - P_{ii}| = 2 \operatorname{tr}(P - PR) = h_0^2$.

The quadratic term may be evaluated at P with error at most $C dh_0(s_0 + dh_0)$ for an absolute C. Indeed the crossing blocks $RA_j(1 - R)$ and $PA_j(1 - P)$ differ in Hilbert–Schmidt norm by at most $2dh_0$, by changing one projection at a time. The reverse crossing blocks have the same bound. Their individual norms are at most $s_0 + 2dh_0$ at R and s_0 at P. Expanding each of the six trace products gives the estimate.

For $i \in P$, $j \notin P$, necessarily $i > j$. If the signs differ, only $F_P(Z, Z)/2$ contributes this pair, giving $Z_{ij} Z_{ji}/2 = |Z_{ij}|^2/2$. If the signs agree, the two square terms from U, V vanish on this pair. The four remaining coefficients of $Z_{ij} Z_{ji}$ are $1/2, -1, 1, -1$, in order from the square in Z and the three mixed terms. Here $Z_{ji} = -\overline{Z_{ij}}$, so the result is again $|Z_{ij}|^2/2$. Summing gives $s_0^2/4$, since Z_{ij}, Z_{ji} have the same modulus. The lemma and the preceding estimates yield, with an absolute constant,

$$\log \det_P(u^* Y u) = \operatorname{tr}(PZ) + s_0^2/4 + O\big(d(s_0^2 + h_0^2)\big).$$

For the weight $-\alpha$ all the projections in its definition are trailing. For any weight v, summing $\|[P, A]\|_2^2$ over its level projections gives $Q_v(A)$: the number of levels separating i from j is $|v_i - v_j|$. Also the sum of the projections is $\operatorname{diag}(v) - v_{\min} 1$. The determinant factor restores the constant since $\det Y = e^{\operatorname{tr} Z}$. It follows that

$$\frac{1}{m_{-\alpha}(u^* Y u)} = \exp\{\operatorname{tr}(\mu Z) - q + E(u)\}, \qquad |E(u)| \le C_1 d\{q + Q_\mu(u) + r^2\}$$

with absolute C_1. To pass from α to μ, the difference $2\rho_c$ has entries of modulus at most r. Thus the difference of the linear traces is at most $r^2 d$ and $Q_\alpha(u) \le Q_\mu(u) + 2r^2$ by unitarity of u.

The weight $m(u) = |m_\mu(u)|^2$ satisfies $0 \le m(u) \le e^{-Q_\mu(u)/2}$. For one of its projections, the squared compressed determinant is the determinant of $1 - Pu^*(1 - P)uP$ on the range of P. Its eigenvalues lie in $[0, 1]$, so it is at most $\exp(-\|(1 - P)uP\|_2^2)$. Unitarity gives $2\|(1 - P)uP\|_2^2 = \|[P, u]\|_2^2$. Multiply over levels, using $|\det u| = 1$. By the compact Weyl dimension formula $M_\mu^{-1} = \dim V_\mu \le (3B + r)^{r(r-1)/2} \le e^T$: every factor is the ratio of a highest-weight difference plus a root height to that height, at least one in the denominator, and all differences are at most $3B$.

After removing $e^{\operatorname{tr}(\mu Z)}$ the error from e^{-q} is consequently bounded by the integral of $m(u) e^{-q} |e^{E(u)} - 1| / M_\mu$. Where $q + Q_\mu(u) \le 1000T$, the bound on E is $o(1)$ uniformly since $r^2 \le T$ and $dT \to 0$. This portion costs at most $e^{-q}/20$ eventually. On the complement $|E| \le (q + Q_\mu(u))/4$ eventually. The unnormalized integrand is at most $2 \exp(-(q + Q_\mu(u))/4)$. Integration against probability Haar and division by M_μ costs at most $2e^{-249T} \le e^{-200T}$. This proves the first assertion.

For the compact value coefficient take the last coordinate projection and the single exponential $e^{Z'}$ in Lemma 132.10. Its linear term, after multiplication by t, is $t Z'_{kk}$, and its

quadratic term is $-q'$ by skew-adjointness. The error has modulus at most $C_2 dq'$, because $\|[P,Z']\|_2^2 = 2\sum_{j\neq k}|Z'_{kj}|^2$. If $q' \le 1000T$ the error in the exponential is $o(1)$ relative to $e^{-q'}$. Otherwise its absolute contribution is at most $2e^{-q'/2} \le e^{-200T}$. This includes $t=0$, when the coefficient is constantly one. □

Theorem 132.12 (Uniform lower bound for the complete real period)**.** *At the translate g of Theorem 132.2 and at each real place of the same fields, for all sufficiently large nodes $V_0 \le x \le V_0^{3/2}$, the complete integral of the Bessel coefficient satisfies*

$$m = \int_S \Theta(s)\,ds = d_\pi \|W_w\|^2_{L^2(S\backslash G)} \;\ge\; \tfrac14 \beta^{-16D}\,\mathrm{vol}(\mathcal{B}_\beta) > 0.$$

In particular the product of these full local periods is at least $4^{-f}\beta^{-16fD}\prod_v \mathrm{vol}(\mathcal{B}_{\beta,v})$.

Proof. If $\|C_s\|_2 < \delta_*$, Theorem 132.5 gives $u \in K_c$ with the same integrand and both components of $J_{us}-1$ of Hilbert–Schmidt norm at most $4\delta_*$. Their logarithms Z, Z' have Hilbert–Schmidt norms at most $8\delta_*$ by the logarithmic series and submultiplicativity. Their compact weight traces are imaginary. By Lemma 132.4 the two linear phases in Proposition 132.11 cancel the character in Θ. Its value is therefore the product of the first normalized coefficient in that proposition with the conjugate of the second.

Put $A = e^{-q}$, $A' = e^{-q'}$, $e_0 = e^{-200T}$. If the two factors are within $A/20 + e_0$ and $A'/20 + e_0$ of A, A', expanding their product bounds its distance to AA' by $(2/20 + 1/400)AA' + 4e_0$, since $A, A', e_0 \le 1$. In particular $\Re\Theta(s) \ge -4e_0$ throughout the tube. The tube lies in $\mathcal{B}_\beta$: the compact translation is unitary in both factors, so it preserves $\|J_s\|_{\mathrm{op}}$, and $\|J_{us}\|_{\mathrm{op}} \le 1 + 4\delta_*$.

The integral is real and equals the squared-norm expression by Proposition 131.6. Integrate its real part first on Ω_η using Theorem 132.6. On the rest of the tube the negative contribution is at most $4e^{-200T}\,\mathrm{vol}(\mathcal{B}_\beta)$. The remaining set contributes at worst the negative of the absolute integral in Theorem 132.9. Consequently

$$\frac{m}{\mathrm{vol}(\mathcal{B}_\beta)} \ge \tfrac12 e^{-16T} - 4e^{-200T} - e^{-100T} \ge \tfrac14 e^{-16T}$$

eventually. Every constant above is independent of the place, and the uniform growth bounds used are those of the given nodes. Multiplication gives the final assertion. □

The real test vector can now be multiplied by a positive scalar to give it squared quotient norm equal to the dimension of its value space. Passage to an integral projected theta period still requires the global comparison.

132.5 Normalized test vectors and their truncation

Continue with the real translates constructed in Theorem 132.2. Haar on the compact factor of $S = U \rtimes \mathrm{U}(k)$ is probability Haar, so an S-invariant scalar function has the same integral on $U\backslash G$ and $S\backslash G$. All vector norms below descend in this way.

Lemma 132.13 (Integrals of an operator norm bound)**.** *At one real place let g be its stationary translate and let $\phi(t) = (2/\|t\|_{\mathrm{op}})^{9r^2}$ on G. Write $V = \mathrm{vol}(\mathcal{B}_\beta)$ for the S-volume in Theorem 132.12. Uniformly for all $y \in G$, at all sufficiently large nodes,*

$$\int_S \phi(g^{-1}sy)\,ds \le b^{r^2}V, \qquad d_\pi \int_G \phi(t)\,dt \le b^{38r^2}.$$

Proof. Put $P = 9r^2$ and $R \geq 1$. In the part of the matrix coset $g^{-1}Sy$ of operator norm at most R, take a maximal $1/R$-separated set. As in Theorem 132.6 its cardinality is at most $(1+2R^2)^{2r^2}$. If $A = g^{-1}sy$ is within $1/R$ of $A_i = g^{-1}s_i y$ in this set, then

$$\|g^{-1}ss_i^{-1}g - 1\| = \|(A - A_i)A_i^{-1}\| \leq 1.$$

Here and below in this proof undecorated matrix norms are operator norms. The corresponding ss_i^{-1} belongs to $\mathcal{B}_\beta$, since its first component has operator norm at most two and its second is unitary. Thus the S-volume with $\|g^{-1}sy\| \leq R$ is at most $(1+2R^2)^{2r^2}V$ by right invariance. For $2^j \leq \|g^{-1}sy\| < 2^{j+1}$, $j \geq 0$, multiply the volume bound by 2^{P-jP}. The sum is at most

$$C_r V, \qquad C_r = 2^{P+1}12^{2r^2} \leq b^{r^2}$$

eventually, since the common ratio is 2^{4r^2-P}. This is uniform in y.

Taking the whole group in the packing argument gives $\int_G \phi \leq C_r \operatorname{vol}_G\{\|t-1\| \leq 1\}$: the relative points AA_i^{-1} now lie in that set. Cover this last set, of norm at most two, by right translates of the ball $\{\|t-1\| \leq \eta\}$ using separation $\eta/2$. The number required is at most $(1+8/\eta)^{2r^2} \leq \beta^{12r^2}$, since $\eta = (200\beta^5)^{-1}$ and $\beta \to \infty$. The coefficient derivative estimate in the proof of Theorem 132.6 applies to every t on this small ball. In particular $|c_\lambda(t)| \geq 1/2$, with the coefficient transported by the sign frame. By Schur orthogonality $d_\pi \int_G |c_\lambda(t)|^2\,dt = 1$. It follows that d_π times the small ball's volume is at most four. The claim follows by $C_r \leq b^{r^2}$ and $\beta = b^3$. □

Theorem 132.14 (A normalized local vector and its cut)**.** *At each real place put $d_v = \dim V_{\tau_v}$, $\theta = b^{6n+2}$, $j = 3r^3 f$, and $l_b(y) = \max\{1, \ell(y)/b^{6n}\}$. There is a positive scalar multiple F_v of the stationary Bessel average and a continuous cut $F_v^\flat$ such that*

$$\|F_v\|_{U\backslash G}^2 = d_v, \qquad \|F_v^\flat\|_{U\backslash G}^2 \leq d_v, \qquad \|F_v - F_v^\flat\|_{U\backslash G}^2 \leq d_v b^{-2r^3 f}.$$

They agree for $\ell \leq \theta/2$, and $F_v^\flat$ vanishes for $\ell \geq \theta$. Both have the prescribed S-equivariance. There is a bounded, measurable, S-invariant nonnegative function A_v such that

$$l_b(y)^j \max\{\|F_v(y)\|, \|F_v^\flat(y)\|\} \leq A_v(y), \qquad (\sup A_v)\int_{U\backslash G} A_v \leq b^{400r^2} d_v.$$

The full vector satisfies the reproducing identity

$$F_v(y) = d_\pi \int_G \overline{c_\lambda(h)} F_v(yh)\,dh.$$

Proof. Write m for the full integral in Theorem 132.12. By that theorem $c = (d_v d_\pi/m)^{1/2}$ is finite and positive, and $F_v = cW_w$ has the asserted norm. Let $F_v^\flat = \zeta(\ell(y))F_v(y)$, where ζ equals one up to $\theta/2$, is linear from one to zero on $[\theta/2, \theta]$, and is zero thereafter. The length is continuous: for $t \in G$, $\ell(yt) \leq \ell(y)\|t\|$ and $\ell(y) \leq \ell(yt)\|t^{-1}\|$. These follow by taking the infima and using $\|t\|, \|t^{-1}\| \geq 1$. They give continuity as $t \to 1$, and the length is left S-invariant by definition. Equivariance, continuity, cut support and the cut norm bound follow.

For $h = g^{-1}sy$, the length bound on g gives $\ell(y) = \ell(gh) \leq b^{5n}\|h\|$. Hence $\ell(y)/b^{6n} \leq \|h\|/2$ for large nodes. The gap $H_0 = r^8$ of Lemma 132.8 exceeds $9r^2 + j$, uniformly, since $f \ll_{q_0,d} x^{11}\log(2x)$ and $r \geq x^5$. It follows that

$$l_b(y)^j |c_\lambda(h)| \leq (2/\|h\|)^{9r^2}.$$

When $l_b(y) > 1$ one has $\|h\| \geq 2$ and uses coefficient decay with exponent $9r^2 + j$. When $l_b(y) = 1$ use decay if $\|h\| \geq 2$, and unitarity if $\|h\| \leq 2$.

Take $A_v(y) = c\int_S \phi(g^{-1}sy)\,ds$. Integration of the last estimate, using the unitarity of the action on the values, gives the pointwise bound for F_v and its cut. Tonelli gives measurability and disintegration over the quotient gives $\int_{U\backslash G} A_v = c\int_G \phi$; right invariance on S gives its left invariance as a function of y. By Lemma 132.13 and the lower bound for m,

$$(\sup A_v)\int A_v \le c^2 b^{r^2} V \int_G \phi \le 4 d_v\, \beta^{16(r+k)^2} b^{39r^2} \le b^{400r^2} d_v.$$

Outside $\ell \le \theta/2$ we have $l_b(y) \ge b$. There $\|F_v - F_v^\flat\| \le b^{-j} A_v(y)$; inside, the difference vanishes. Integrating its square gives $d_v b^{-6r^3 f + 400r^2} \le d_v b^{-2r^3 f}$ for all sufficiently large nodes.

Finally $d_\pi \int_G \overline{c_\lambda(h)}\pi(h)\,dh$ is the orthogonal projection onto the line of w: Schur orthogonality gives the matrix entries of this operator against arbitrary vectors, and the coefficient is integrable by Proposition 131.6. The map $u \mapsto W_u$ in that proposition intertwines π with right translation and is bounded into the quotient L^2-space. Applying it to the projection identity on w proves reproduction in L^2. The convolution is continuous by boundedness of F_v, integrability of its kernel, and dominated convergence. The quotient Haar measure has full support, so the continuous representatives agree everywhere. □

Corollary 132.15 (The cut at all real places of the given fields)**.** *With $d_B = \prod_v d_v$, $F = \bigotimes_v F_v$ and $F^\flat = \bigotimes_v F_v^\flat$, one has*

$$\|F\|^2_{U_\infty\backslash G_\infty} = d_B, \qquad \|F^\flat\|^2_{U_\infty\backslash G_\infty} \le d_B, \qquad \|F - F^\flat\|_{U_\infty\backslash G_\infty} \le f b^{-r^3 f}\sqrt{d_B}.$$

These specific nonzero full local tests and their cuts can be used in Theorems 131.11 and 131.37.

Proof. Product Haar and Tonelli multiply the squared quotient norms. In the difference of the tensors replace one factor at a time. In each of the f terms the changed factor contributes at most $b^{-r^3 f}\sqrt{d_v}$ and all the others at most $\sqrt{d_{v'}}$. The triangle inequality proves the estimate. The integrability, equivariance, support, choice of coefficient and local norm required for the cited estimates are those just verified. The bounds, including the exponent containing f, hold at the nodes of Theorem 132.12. □

132.6 The truncation error after summation over rational translates

Use the full tensor F, its cut $F^\flat$, and the functions A_v from Theorem 132.14. Fix a source class c and a residue s at the two marked primes. Denote by $\mathcal{T}(c, s, c')$ the rational isometries γ for which $h_c^{-1}\gamma h_{c'}$ preserves the finite lattice and reduces to s. Throughout this subsection write

$$L(y) = \prod_{v\mid\infty} l_b(y_v), \qquad J = r^3 f, \qquad d_0 = 2nrf, \qquad \rho = b^{37n}.$$

The function β_c is the reduced partition from the proof preceding Lemma 131.33. Thus it is nonnegative, is supported on reduced frames, and its Γ_c-translates sum to one.

Lemma 132.16 (Counting all translates at a prescribed length)**.** *For $u \ge 1$, uniformly in c, s, y, at every sufficiently large node one has*

$$\sum_{c'} \sum_{\substack{\gamma\in\mathcal{T}(c,s,c')\\ L(\gamma^{-1}y)\le u}} \beta_{c'}(\gamma^{-1}y) \;\le\; b^{6n^2 d_0} u^{2d_0}.$$

Consequently the sum of $\beta_{c'}(\gamma^{-1}y)L(\gamma^{-1}y)^{-J}$ over all these classes and all their translates is at most $2b^{12n^3 rf}$.

Proof. Write e_j for the rational frame columns, including the terminal column $e_n = z_0$ of norm $-N$. Write $\widehat{e}_{j,v} = s_{j,v}\Phi_v(e_j)$ for their unit Euclidean normalizations in the real model of Corollary 131.2. The transformed frame we count has columns $y_v^{-1}\gamma_v\widehat{e}_{j,v}$. At $x = \gamma^{-1}y$ in the indicated support, $\ell(x_v) \le b^{6n}u$. Choose $w_v \in S_v$ with $\|w_v x_v\| \le 2b^{6n}u$. In the normalized model these unitary isometries of the indefinite form satisfy $\|g^{-1}\| = \|g\|$. The successive orthogonalized columns of $x_v^{-1}\widehat{e}_{j,v}$ are unchanged by the unitriangular column operations induced by w_v^{-1}. Their lengths are therefore at most $2b^{6n}u$. All strict upper coefficients recovering the original columns are bounded by b^{26} on the reduced set, as shown in Lemma 131.27. Every transformed column has length at most $2nb^{26+6n}u \le \rho u$.

Two distinct transformed frames have distance at least $(N\rho u)^{-1}$ in the maximum of the column norms over the places. To prove this, take two translates γ_1, γ_2, possibly from different right classes, set $x = \gamma_1^{-1}y$, $\delta = \gamma_1^{-1}\gamma_2$, and assume the distance is strictly smaller. For each pair i, j the entry $h(e_i, (\delta - 1)e_j)$ belongs to $\mathcal{O}_E$. Indeed the two finite class lattices have the same frame, and δ takes the second lattice isometrically onto the first, where the form is integral. At v the normalized pairing has modulus at most

$$\|x_v^{-1}\widehat{e}_{i,v}\|\,\|x_v^{-1}(\delta_v - 1)\widehat{e}_{j,v}\| < 1/N,$$

since the defining Hermitian matrix has operator norm one. Restoring the two scale factors costs at most N. Thus the algebraic integer has absolute value less than one at every embedding, and its norm shows that it is zero.

It follows that $(\delta - 1)e_j \in T^{\perp} = L_0 \oplus X_0$, where here L_0 denotes the isotropic arm spanned by $e_1, \dots, e_a$. Its Hermitian norm is zero: expand $h(\delta e_j, \delta e_j) = h(e_j, e_j)$ and use its orthogonality to e_j. The definiteness of X_0 then puts the difference in L_0. Write it as $\sum_{h \le a} t_{hj} e_h$; each t_{hj} is integral because the arm is a direct lattice summand. In any nonzero column choose the largest h with $t_{hj} \neq 0$. Multiplication by w_v, upper unitriangular on this arm, leaves that top coefficient unchanged. In normalized coordinates its modulus is $|s_{j,v}/s_{h,v}|\,|\Phi_v(t_{hj})|$. On the other hand

$$\|w_v(\delta_v - 1)\widehat{e}_{j,v}\| \le \|w_v x_v\|\,\|x_v^{-1}(\delta_v - 1)\widehat{e}_{j,v}\| < \frac{2b^{6n}}{N\rho}.$$

Both scale ratios are at most b^{23n}. Thus all conjugates of t_{hj} have modulus less than one, a contradiction. This proves the separation.

The ambient real dimension of the tuple of columns is d_0. Balls of radius $(2N\rho u)^{-1}$ about distinct frames have disjoint interiors in the maximum norm. Comparing their volumes with the concentric enlargement of the ball of radius ρu bounds their number by $(1 + 2N\rho^2u^2)^{d_0}$. This is at most $(b^{6n^2}u^2)^{d_0}$, since $N \le b^{60n}$ and n tends to infinity. The first paragraph of the proof of Proposition 131.28 identifies the multiplicity at a single frame: equality of the residue implies that the relative map induces a principal-level isometry on the definite middle quotient. Hence the right classes agree, and their definite principal-level automorphism is trivial. The relative maps then lie in $\Gamma^0_{c'}$. The weights in the fiber add to at most one, being distinct terms in the partition by $\Gamma_{c'}$. This proves the first assertion without a restriction on the left point y. For the second use $L^{-J} = \int_L^{\infty} Ju^{-J-1}\,du$ and Tonelli's theorem. The extra factor is $J/(J - 2d_0) \le 2$ at large nodes. □

Theorem 132.17 (Absolute truncation error in a full rational row)**.** *Write $K_\gamma(y) = F(y)F(\gamma^{-1}y)^*$ for the matrix outer product, and $K_\gamma^{\flat}$ for its cut counterpart. Both kernels are absolutely integrable after summation over every row (c, s) with weights $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$. With the same weights put*

$$E_{csc'} = \sum_{\gamma \in \mathcal{T}(c,s,c')} \int_{G_\infty} \beta_c(y)\beta_{c'}(\gamma^{-1}y)\|K_\gamma(y) - K_\gamma^{\flat}(y)\|_{\mathrm{HS}}\,dy.$$

On the fields and interval of Theorem 132.12, for every sufficiently large node,

$$\sum_{c'} E_{csc'} \le \tfrac{1}{2} v_U b^{-k^3 f}.$$

Proof. Put $A(y) = \prod_v A_v(y_v)$, $a_* = \prod_v \sup A_v$. At $z = \gamma^{-1} y$ each kernel norm is at most $A(y) a_* L(y)^{-3J} L(z)^{-3J}$ by Theorem 132.14. The cut matrix is the full matrix multiplied by the product of all the scalar cutoffs at y, z, a real number in $[0,1]$. If the difference is nonzero, some local length exceeds $b^{6n+2}/2$. Thus $L(y)L(z) \ge b$ for large b, and

$$\| K_\gamma(y) - K_\gamma^{\flat}(y) \|_{\mathrm{HS}} \le b^{-2J} A(y) a_* L(z)^{-J}.$$

For the full and cut kernels separately the same upper bound holds with the factor b^{-2J} omitted. The last factor is summable against the right weights by Lemma 132.16. For the left weight one has

$$\int_{G_\infty} \beta_c(y) A(y)\, dy = v_U \prod_v \int_{U_v \backslash G_v} A_v.$$

Indeed at fixed y integration of $\beta_c(uy)$ over U_∞ unfolds to the integral of one over $\Gamma_c \backslash U_\infty$, of volume v_U. The A_v are left U_v-invariant. Tonelli's theorem and Theorem 132.14 consequently bound the sum of the error norms by

$$2 v_U d_B\, b^{(-2r^3 + 12 n^3 r + 400 r^2) f}.$$

This also proves the asserted absolute convergence, by deleting the gain b^{-2J}. Finally $d_B \le b^{2kf}$, $k \le r$, and $12n^3 r + 400 r^2 + 2k = o(r^3)$ on this interval because $n \le \sqrt{x}$ and $r \ge x^5$. The stated bound follows. The estimates also hold for arbitrary subsets of rational terms by nonnegativity. □

Corollary 132.18 (The error in the residue second moment)**.** *Uniformly on the same interval the array of differences between full and cut block sums, normalized by v_U, has squared Hilbert–Schmidt norm at most*

$$v_U^{-2} \sum_{c,s,c'} E_{csc'}^2 \le |\mathcal{C}| b^{-k^3 f} \le \tau^{-40} |B_{\mathrm{res}}|\, |\mathcal{C}| d_B.$$

The bound persists when each block is restricted to a subset of its rational translates or multiplied termwise by scalars of modulus at most one.

Proof. For each c, s, the sum of the squares of the nonnegative errors is at most the square of their sum. At a split place of F above p_i the number of possible residue matrices is at most $p_i^{r^2}$; there are f such places above each mark. Therefore there are at most $M^{f r^2} \le b^{22 f r^2}$ residues. Apply Theorem 132.17 and use $k^3 \ge 22 r^2$ for all sufficiently large nodes, since $k = r - 2n + 1$. The last inequality follows from $\tau = b^{kf/4}$ and $|B_{\mathrm{res}}|, d_B \ge 1$. The triangle inequality gives the assertion for signed matrix sums. It bounds their difference as an array and hence also the absolute difference of their Hilbert–Schmidt norms. □

133 Poincaré series and their marked second moment

133.1 The rank-zero contribution at the two marks

Write $H = \mathrm{U}(X_0)$, $S = U \rtimes H$, K for the finite integral stabilizer, and $K(M)$ for its principal subgroup. The H-level is $K_H(M) = H(\mathbb{A}_{F,f}) \cap K(M)$, with H extended by the identity on the arms and the terminal line. The finite set $\mathcal{C} = H(F) \backslash H(\mathbb{A}_{F,f}) / K_H(M)$ has representatives h_c.

Let B_{res} be the reduction of $S(\mathbb{A}_{F,f}) \cap K$. At the infinite places let $V = \bigotimes_v V_{\tau_v}$, $\dim V = d_B$, and let σ_∞ denote the unitary action of S_∞ in the equivariance of F. Its restriction to U_∞ is the prescribed scalar character. Its restriction to $\prod_v \mathrm{U}(k)$ is the exterior tensor product of the irreducible local value representations. The cut has the same equivariance.

Lemma 133.1 (Rational lifts of a Bessel residue)**.** *For $t \in B_{\mathrm{res}}$ and each c, there is exactly one class $c \cdot t$ with an element of $S(F)$ over $(c, t, c \cdot t)$. For any such lift $\gamma_{c,t}$, all the lifts there form the right coset $\gamma_{c,t}\Gamma_{c\cdot t}$. The rule is a right action on $\mathcal{C}$, and*

$$\gamma_{c,t}\Gamma_{c\cdot t}\gamma_{c,t}^{-1} = \Gamma_c.$$

Proof. Lift t to $k_t \in S(\mathbb{A}_{F,f}) \cap K$, and factor $h_c k_t = uh$, with $u \in U(\mathbb{A}_{F,f})$ and $h \in H(\mathbb{A}_{F,f})$. Write $h = \eta h_d a_0$, with $\eta \in H(F)$, $a_0 \in K_H(M)$. Choose $x \in U(F)$ such that $h^{-1}u^{-1}xh \in K(M)$. For this choice one only needs density of $U(F)$ in the finite adeles of U. Indeed in characteristic zero the finite logarithm and exponential polynomials are inverse rational-coordinate maps between this unipotent matrix group and its nilpotent Lie algebra. They give homeomorphisms at the finite adeles, being integral outside finitely many primes in a rational basis. The diagonal copy of F is dense in $\mathbb{A}_{F,f}$: multiply the given coordinates and congruences by a common rational integer to make them integral, and use the Chinese remainder theorem in $\mathcal{O}_F$, also imposing divisibility at any additional primes of that integer. This gives the required approximation in every Lie algebra coordinate. Now $x\eta$ is a lift since

$$k_t^{-1}h_c^{-1}x\eta h_d = (h^{-1}u^{-1}xh)a_0^{-1} \in K(M).$$

If γ_1, γ_2 are lifts for the same c, t with right classes d_1, d_2, write $\gamma_1^{-1}\gamma_2 = u'\eta'$ in $S(F)$. Its class translate is integral and principal, by equality of residues. It acts on the middle quotient of the lattice in the isotropic flag by $h_{d_1}^{-1}\eta' h_{d_2}$ extended by one on the terminal line. This induced map preserves the middle lattice and is principal there. Its extension by the identity on the two direct integral arms belongs to $K(M)$. Thus $h_{d_1}^{-1}\eta' h_{d_2} \in K_H(M)$. The two classes are equal and $\eta' = 1$ by freeness of the definite level in Proposition 131.25. The relative element then belongs to Γ_{d_1}. Conversely every such right multiple is a lift. Products and inverses of lifts have the product and inverse residues by cancellation of the intermediate class representatives. This proves the right action assertion. Conjugation by a lift takes the rational unipotent to itself. At the finite places it takes one indicated principal condition to the other because $K(M)$ is normal in K. Applying this also to its inverse proves the displayed equality. □

For a set $T \subset G(F)$, write $\mathcal{A}_T(s)_{cd}$ for the sum, over $\gamma \in T \cap \mathcal{T}(c, s, d)$, of the integrals of K_γ against the two class weights, divided by v_U. Regard these matrices as blocks on $\mathcal{H}_X = \bigoplus_{c\in\mathcal{C}} V$. Every such sum converges absolutely by Theorem 132.17. Write $R(t) = \mathcal{A}_{S(F)}(t)$.

Theorem 133.2 (The exact matrix of the Bessel terms)**.** *For $t \in B_{\mathrm{res}}$ the only nonzero block in row c of $R(t)$ is*

$$R(t)_{c,c\cdot t} = \sigma_\infty(\gamma_{c,t}).$$

These operators form a unitary representation of B_{res} on $\mathcal{H}_X$. For t outside B_{res} one has $R(t) = 0$. In particular, with $D_X = |\mathcal{C}|d_B$,

$$R(1) = I_{\mathcal{H}_X}, \qquad \sum_t \|R(t)\|_{\mathrm{HS}}^2 = |B_{\mathrm{res}}|D_X.$$

Proof. A Bessel term between any two classes has finite translate in $S(\mathbb{A}_{F,f})$, hence has residue in B_{res}. Fix a lift $\gamma = \gamma_{c,t}$, $d = c \cdot t$. The real character equals one on Γ_d: its finite factor there equals one by the conjugation and conductor computation in Proposition 130.7, and the adelic

character is trivial on rational points. Thus F is left Γ_d-invariant. At each y, all the matrices in the right coset equal $F(y)F(y)^*\sigma_\infty(\gamma)$. Their right weights sum to one by the partition for Γ_d. It remains to evaluate

$$v_U^{-1}\int_{G_\infty}\beta_c(y)F(y)F(y)^*\,dy=\int_{U_\infty\backslash G_\infty}F(y)F(y)^*\,d\dot{y}.$$

The equality follows by disintegration and the left U_∞-invariance of the matrix. Its trace is d_B by Corollary 132.15. It commutes with the product of compact middle groups: left translation by any of their elements conjugates the integrand by its value representation and preserves quotient measure. Indeed conjugation on U_∞ preserves Haar measure, since a compact group has trivial modulus. Successive applications of Schur's lemma to the irreducible tensor factors show that this integral is scalar. Its trace makes it the identity.

The displayed block is independent of the lift, as the factor on Γ_d is the identity. The product of the lifts for (c,t) and $(c\cdot t,t')$ gives the block at tt'. Hence $R(t)R(t')=R(tt')$, and $R(1)=I$. Each operator permutes classes and acts unitarily on each component. Its squared Hilbert–Schmidt norm is D_X; summing gives the formula. □

Corollary 133.3 (The full contribution of the isotropic parabolic)**.** *Let $P_L(F)=\{\gamma\in G(F):\gamma L=L\}$. For $\xi\in\mu(E)$ write $\bar{\xi}_{\mathrm{res}}$ for its scalar reduction at the marks and $\omega_\infty(\xi)$ for its unitary central character on the full real vector. Then*

$$\mathcal{A}_{P_L(F)}(s)=\sum_{\xi\in\mu(E)}\omega_\infty(\xi)R(\bar{\xi}_{\mathrm{res}}^{-1}s).$$

The supports of these summands are disjoint. The squared norm summed over all residues is $|\mu(E)|\,|B_{\mathrm{res}}|D_X$. On either marked slice, where s agrees with a Bessel residue above the specified prime, the parabolic matrix is exactly $R(s)$; its squared norm summed on that slice is $|B_{\mathrm{res}}|D_X$.

Proof. Group the sum into double cosets under Γ_c,Γ_d. This is legitimate by absolute convergence. For one coset first sum the right weights over all right multiples, giving one. Then summing the left weights unfolds the integral to $(\Gamma_c\cap\gamma\Gamma_d\gamma^{-1})\backslash G_\infty$. In detail the remaining left translates are indexed by the cosets of that intersection in Γ_c; after translation of the variable their weights add to a partition for the intersection. The kernels are equivariant under this change because the two lattice characters are trivial. For disagreeing characters the orbital integral therefore vanishes by Proposition 130.7.

An agreeing γ preserving L lies in $\xi S(F)$ for a unique $\xi\in\mu(E)$. Indeed the equal coefficient ratio on the common flag of Theorem 130.6 is the eigenvalue on its first line. The proof of Corollary 130.8 gives the conclusion before imposing its congruence on that eigenvalue. Conversely $\xi S(F)$ normalizes the Bessel unipotent and preserves its character, so every one of its elements agrees. The scalar ξ commutes with y, and its right action on the real test is the central character of the tensor of the discrete series, by the intertwining in Proposition 131.6. Thus it can be taken out of the adjoint factor in the outer product with multiplier $\omega_\infty(\xi)$. For these cosets the right weights again sum to one as in the theorem, even though the argument of each weight has also been translated by the scalar. The residue is multiplied by $\bar{\xi}_{\mathrm{res}}$. This proves the stated matrix identity.

The first diagonal entry of an element of B_{res} is one. At any one chosen component above either mark, the roots of unity of E inject on reduction, since the field there is $\mathbb{Q}_{p_i}$ and $1+p_i\mathbb{Z}_{p_i}$ is torsion-free for odd p_i. Thus distinct scalars give disjoint residue cosets. Only the identity scalar survives on the indicated slice, which contains all of B_{res}. The norm formulas now follow from the theorem. □

Proposition 133.4 (Transport of the full and cut block sums)**.** *If T is stable under left and right $S(F)$, then $\mathcal{A}_T(st) = \mathcal{A}_T(s)R(t)$ for every residue s and every $t \in B_{\mathrm{res}}$. The same identity holds with $\mathcal{A}_T$ formed from cut vectors and with $R(t)$ as in Theorem 133.2.*

Proof. For fixed d take $\delta = \gamma_{d,t}$ and $e = d \cdot t$. Right multiplication by δ identifies the terms over (c, s, d) with those over (c, st, e), including their membership in T. It also identifies their right unipotent cosets by Lemma 133.1. In either sum the right weights on a coset add to one at fixed y; every kernel in that coset is equal because the lattice character is trivial. Passing to right cosets and integration is justified by Theorem 132.17. Equivariance gives $K_{\gamma\delta}(y) = K_\gamma(y)\sigma_\infty(\delta)$, also for the cut. The only block entering the product on the right in column e is $R(t)_{de}$, which proves the identity. This evaluates the parabolic part of the marked second moment. The open rational terms in the same residues still enter the integral comparison. □

133.2 Summing the signed open orbits on a marked slice

Put $D_{\mathrm{id}} = |B_{\mathrm{res}}| D_X$, $n_p = |B_{\mathrm{res}}|\,|\mathcal{C}| N^{-kf}$ and $\Omega_i = \{s : s_i = t_i \text{ for some } t \in B_{\mathrm{res}}\}$. Write $X_\gamma = \mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma)$ for an open term. This is a positive integer. We shall retain the factor in the archimedean bound which is independent of the rational term. With the measures of Theorem 131.1 define

$$A = 10^5 n^4, \quad e = A + k - 1, \quad \Xi = \prod_{v,j} |s_{j,v}|^2, \quad d_\pi = \prod_v d_{\pi_v}, \quad c_G = V_k S_k / V_r,$$

$$C_\infty^2 = \frac{d_B^2 \rho^{9nAf}}{v_U^2 d_\pi c_G^f \Xi^{2e}}, \qquad D_{\mathrm{op}} = n_p M^{2akf} C_\infty^2.$$

Here $\rho = b^{37n}$ and $\tau = b^{kf/4}$, as before.

Lemma 133.5 (Elimination of nonagreeing residues)**.** *Let T be stable under left and right $S(F)$. Both its full and cut block matrices vanish unless the finite unipotent characters agree at the residue. Explicitly, for integral finite unipotents of residues u, u' with $us = su'$, their finite character values must coincide for the block at s to be nonzero.*

Proof. The adjoint of the array at s is the array for T^{-1} at s^{-1}: invert each rational term and change variables by $y = \gamma z$, interchanging the left and right weights. Absolute convergence justifies this computation. Applying Proposition 133.4 to T^{-1} and taking adjoints gives the left transport formula $\mathcal{A}_T(ts) = R(t)\mathcal{A}_T(s)$. Both formulas hold with cut or full kernels. For a finite integral unipotent, Lemma 133.1 in each row permits a unipotent rational lift to the same class: take $h = h_c$, $d = c, \eta = 1$ in its proof. Its archimedean character is the inverse of the finite character of the chosen residue. Indeed class conjugation fixes the simple-root sum, and on principal unipotents the finite character is one by Proposition 130.7. Hence $R(u)$ is that one scalar on the common source. The equalities for $us = su'$ give the result. One may take the integral unipotents supported at just one place, so agreement holds at each mark where a block survives. □

Proposition 133.6 (One signed integral per open orbit)**.** *In a fixed block take a union of Γ_c^0-left and Γ_d-right orbits of open terms. Choose one representative of each such orbit, choosing a near one whenever possible. Its normalized cut block is a sum of integrals*

$$v_U^{-1} \int_{G_\infty} R_{\mathrm{red}}(y) K_\gamma^\flat(y)\, dy.$$

Only near representatives need be kept. The norm of the integral for each retained representative is at most $C_\infty X_\gamma^{-e/2} + \epsilon_\gamma$, where the nonnegative errors, summed in any block, are at most E_{csd}/v_U of Theorem 132.17.

Proof. First sum over right cosets as in Proposition 133.4. The action of Γ_c^0 on these right cosets is free. In fact if $l = \gamma u \gamma^{-1}$ with l fixing the first frame and $u \in \Gamma_d$, write B for the upper unitriangular action of u on the second frame in its own basis. The cross Gram satisfies $C = CB$, by invariance under l. Since C is invertible, $B = 1$. Now l fixes both frames pointwise. Their joint span is nondegenerate and contains all negative directions in the sense that its negative index equals that of the ambient space (Theorem 130.6). Its pointwise stabilizer is compact at every real place. A unipotent element of a compact unitary group is the identity by diagonalizability. This proves freeness.

For $l \in \Gamma_c^0$, $K_{l\gamma}(ly) = K_\gamma(y)$. The sum of $\beta_c(ly)$ over this free left orbit is $R_{\mathrm{red}}(y)$, because the reduced indicator is invariant under the pointwise frame fixer and p_c is a partition for Γ_c^0. Applying these changes also to the nonnegative integrals of the error norms gives precisely the same unfolding, so the sum of their reduced-weight versions is at most E_{csd}. This proves absolute convergence for the regrouping of both matrix integrals.

If a cut integrand with this reduced weight is nonzero, both lengths are in the cut at every place. A right level move reduces $\gamma^{-1}y$ by the partition in Section 131.12; it preserves the length. Consequently the orbit contains a near representative. The reduced-weight integral is independent of the representative by the preceding unfolding.

For a near choice R_{red} is invariant under the product of the real pointwise frame fixers. Every reduced point of cut length is in the box of radius ρ by Lemma 131.27. A witnessing pair of boxed frames gives $|Z_{ij,v}| \le \rho^2$. The hypotheses of Theorem 131.11 thus apply with precisely this weight, using its adjoint-kernel version. Each $\|F_v\|_2^4 = d_v^2$, and $\delta(Z_v) = (\prod_j |s_{j,v}|^2)^2 \Phi_v(\Delta_\gamma)$ by Corollary 131.2. Substitution and division by v_U give $C_\infty X_\gamma^{-e/2}$; the other term of that theorem is the integral of the error norm just unfolded. □

Theorem 133.7 (The remaining open contribution on a marked slice)**.** *Let $T_>$ consist of open rational terms with $\sqrt{X_\gamma}/P_n > (M/b)^{af}/N^{f/2}$. For $i = 0, 1$, the normalized full array on Ω_i satisfies*

$$\left(\sum_{s \in \Omega_i} \|\mathcal{A}_{T_>}(s)\|_{\mathrm{HS}}^2 \right)^{1/2} \le (b^3/M)^{kf/2} \sqrt{D_{\mathrm{op}}} + 2\tau^{-20} \sqrt{D_{\mathrm{id}}}.$$

In particular the full array of all rational terms satisfies

$$\left(\sum_{s \in \Omega_i} \|\mathcal{A}_{G(F)}(s) - R(s)\|_{\mathrm{HS}}^2 \right)^{1/2} \le (b^3/M)^{kf/2} \sqrt{D_{\mathrm{op}}} + (4\tau^{-20} + \tau^{-3a/2}) \sqrt{D_{\mathrm{id}}}.$$

Proof. The two leading determinants of orders a, n are unchanged by left and right multiplication by Bessel elements: on the frame these perform upper unitriangular operations preserving every initial segment. Thus $T_>$ is stable. Remove rational terms with disagreeing characters, whose whole orbital integrals vanish by Proposition 130.7 and the unfolding in Corollary 133.3. This removal also preserves stability, as Bessel elements normalize the unipotent and preserve its character. By Lemma 133.5 only agreeing residues need be summed.

For the cut array at those residues apply Proposition 133.6. The error array has norm at most $\tau^{-20}\sqrt{D_{\mathrm{id}}}$ by Corollary 132.18. For the main majorant use $X_\gamma \ge 1$, $e \ge k$. The

inequality defining $T_{>}$ gives $X_\gamma > P_n^2/N^f$, so the gauge in Theorem 131.23 is $G_\gamma = M^{2af}X_\gamma$. The individual main bound is thus at most $C_\infty M^{akf}G_\gamma^{-k/2}$.

Expand the square of the sum of these bounds in each block. Each selected representative determines a distinct near orbit for the smaller left and right pointwise subgroups: equality there would imply equality in the larger orbit used in the proposition. Thus the ordered pairs form a subset of the pairs counted in Theorem 131.29. For a pair of positive corank $\ell = k - \dim I_X$ (allowing $\ell = 0$), put $c^+ = \operatorname{codim}_D \ker(J_\gamma - J_\eta)$. When $c^+ \geq 1$, the common-lattice estimate gives

$$(G_\gamma G_\eta)^{-k/2} \leq (2/M)^{\max(1,\ell-1)kf}\, d(L_{12})^{-k}.$$

If $c^+ = 0$, then $\ell \leq 1$ and the full links agree. Theorem 131.17 gives the stronger factor p_i^{-2kf} multiplying $d(L_{12})^{-k}$, since the reduced full frames meet in each of the two components at every place above the slice prime. Summing the two bounds by Theorem 131.29 and dividing the result by D_{op} gives at most

$$(\tau^2 + \tau^3)p_i^{-2kf} + \sum_{\ell \geq 0} \tau^{\ell+2}(2/M)^{\max(1,\ell-1)kf}.$$

The series is at most $4\tau^4(2/M)^{kf}$: the first two terms have the same gain, and from $\ell = 2$ onward the ratio is $\tau(2/M)^{kf} < 1/2$. The sizes of the two primes give $p_i^2 \geq M/b$. The entire display is then at most $6\tau^4(b/M)^{kf} \leq (b^3/M)^{kf}$. Taking square roots and adding the error proves the cut bound. A further $\tau^{-20}\sqrt{D_{\mathrm{id}}}$ suffices for passage to full kernels by Corollary 132.18.

On the slice, Corollary 133.3 gives the entire parabolic block. For all positive closed ranks together and the complementary open range, the two cut estimates are Corollaries 131.35 and 131.38, respectively. Their union costs at most one further cut error, by Corollary 132.18. Adding their norms gives the second inequality. □

133.3 The marked second moment and the identity block

We compare D_{op} directly with the rank-zero mass at the two prescribed primes. In this comparison the covolume of the positive coordinates of the unipotent group and the factorials in the real formal degree must both be retained. Write w_v for the weight denoted by κ in Proposition 132.1, so that w_v is distinct in notation from the frame ratio κ_v. Put $w = (\prod_v w_v)^{1/f}$, $m = 2n = 2a + 2$, $\rho_E = |\mathrm{disc}(E)|^{1/(2f)}$ and $c_E = 2^{-f}\sqrt{|\mathrm{disc}(E)|} = (\rho_E/2)^f$.

Lemma 133.8 (The terminal prime and the real formal degree)**.** *For the terminal level of the shorter construction one can use a prime $N \equiv 1 \bmod Q^2$, $Q = (q_0 d)^{s+6}$, with*

$$N_- \leq N \leq N_- \exp(\sqrt{\log b}), \qquad N_- = w^m(M\rho_E/(2\pi))^{m-1}b^{-15/2}.$$

There is an algebraic integer $\pi_N \in E$ with $\pi_N\bar{\pi}_N = N$. For every such choice at the same nodes,

$$d_\pi \;\geq\; \frac{w^{fm(k-1)}}{\left(\prod_{j=k-1}^{r-1} j!\right)^f}.$$

Proof. Here $1 \leq w_v \leq b$, since $w_v = b$ or xb/x_1 with $x_1 \geq x$. We have $\log \rho_E = O_{q_0,d}(\log(2x))$ and $b^{20} \leq M \leq b^{22}$. Thus for large x, $m \log b \leq \log N_- \leq 24m \log b$. The specified modulus Q^2 is bounded by a fixed power of x, since $s \log 2 = 22 \log m$. It is consequently at most a fixed power of $\log N_-$, uniformly on the interval, since $m \geq \sqrt{x}$. Apply the classical Siegel–Walfisz theorem for rational primes in arithmetic progressions to $Y = N_-$ and $2Y$, in the class $1 \bmod Q^2$. In the range $q \leq (\log Y)^C$ it gives $\vartheta(Y; q, 1) = Y/\varphi(q) + O_{C,B}(Y/(\log Y)^B)$ for every fixed B, where ϑ sums $\log p$ over primes. Choosing $B > C + 2$ gives a strictly positive

difference. Since $2 \le \exp(\sqrt{\log b})$, this produces a prime in the prescribed window. It splits in E, since $q_1 d \mid Q$ and $E \subseteq \mathbb{Q}(\mu_{q_1 d})$. Also $4 \mid Q$ and $\mathbb{Q}(i) \subset E$. The prime N splits in the Euclidean ring $\mathbb{Z}[i]$, so one of the two conjugate prime ideals is generated by an element of norm N. This gives the required π_N. The construction retains p_0, p_1, the fields, the weights and the interval, and $\log N \le 30m \log b$ as used above.

For the real assertion use the formal degree formula with compact-dual Haar measure in Theorem 94.9. Its denominator is $\prod_{j=1}^{r-1} j!$. The first $k-1$ Harish–Chandra entries in Proposition 132.1 are consecutive, contributing $\prod_{j=1}^{k-2} j!$ to the numerator. The other m entries have mutual gaps at least one. Each of the $m(k-1)$ cross gaps is at least w_v: the largest entry of the last block is $-w_v - 2r^8$ and the smallest entry of the first is $1 - k/2$. Multiplication of the resulting lower bounds proves the assertion. □

Lemma 133.9 (A lower bound for the unipotent covolume)**.** *In the normalized coordinates on the same group the Haar measures used above satisfy*

$$v_U \;\ge\; b^{-100n^3 f}\, \Xi^{-k} M^{2akf} c_E^{ak} N^{-kf}.$$

Proof. Take the successive quotients given by the frame action $U \to N_n$ (upper unitriangular matrices), then the positive block B on its kernel U^0, then the skew-Hermitian center. On N_n Haar is Lebesgue measure on the strict upper entries and on U^0 it is $dB\, dK$ in the coordinates of Theorem 131.1. These are the quotient measures in Lemma 131.10. At each step the covolume multiplies by that of the image lattice. Indeed, a measurable fundamental set in the image can be lifted and multiplied by a fundamental set for the intersection with the kernel. Uniqueness of the image class and then the kernel class gives a fundamental set, and the quotient integration formula gives the product of volumes. The image lattices here are discrete by the integral conditions in the coordinates below. They are cocompact since the original rational unipotent lattice is cocompact, by the proof of Proposition 130.7.

In rational frame coordinates the action on N_n has image in the strict upper entries of $N_n(M\mathcal{O}_E)$, that is, unitriangular matrices with those entries in $M\mathcal{O}_E$. A fundamental set for this latter lattice is given by reducing entries modulo $M\mathcal{O}_E$ in order of increasing superdiagonal. Its volume is $(M^{2f} c_E)^{n(n-1)/2} \ge 1$. In normalized coordinates each complex entry scales at v by $s_{j,v}/s_{i,v}$, of modulus at least b^{-23n}. The contribution of this quotient is therefore at least $b^{-23n^3 f}$.

For the projection $U^0 \to B$, let a rational dual isotropic basis vector be denoted by $e_j^\vee$. The positive part of $(u-1)e_j^\vee$ lies in $M\Lambda_X$ for every $u \in \Gamma_c^0$, as in Lemma 131.33. In fact the terminal and bottom changes are zero by the fixed frame; after subtracting the integral top summand, the principal congruence puts the positive change in that lattice. The normalized dual vector at v is $\Phi_v(e_j^\vee)/\bar{s}_{j,v}$, since its pairing with the scaled arm vector has value one. The positive space has its ordinary unitary metric. Each of the a columns of B therefore lies in the Minkowski lattice $M\Lambda_X$ scaled by $1/\bar{s}_{j,v}$ in its k complex coordinates at v. The product lattice containing this image has covolume

$$M^{2akf} (c_E^k N^f)^a \prod_{v,\, j \le a} |s_{j,v}|^{-2k} = M^{2akf} c_E^{ak} N^{af} \Xi^{-k} N^{-kf}.$$

Here $d(\Lambda_X) = N^f$ in the discriminant convention of Section 131.6, and the last frame vector has scale $N^{-1/2}$. The image contributes at least this covolume, without needing surjectivity onto the containing lattice.

In the central kernel the map adds Ke_j in the first arm to each dual vector, with rational entries in $\mathcal{O}_E$. Take off-diagonal upper complex coordinates and imaginary diagonal coordinates as in the real Haar formula. Before scaling, any nonzero difference in this lattice has a

coordinate of modulus at least one at some embedding by the algebraic norm. In the real and imaginary coordinates it thus has maximum modulus at least $1/\sqrt{2}$. The translates of a real cube of side $1/2$ have disjoint interiors, giving covolume at least $2^{-a^2 f}$. Normalization scales a complex entry by $1/(s_{i,v}\bar{s}_{j,v})$ and a real diagonal coordinate by $|s_{i,v}|^{-2}$; in either case the modulus is at least b^{-46n}. Thus the center contributes at least $2^{-a^2 f} b^{-46na^2 f}$. Multiplying the three lower bounds and using $N^{af} \geq 1$ proves the lemma. □

Theorem 133.10 (The second moment on both prescribed slices)**.** *At all sufficiently large nodes of this family,*

$$D_{\mathrm{op}} \leq (M/b^6)^{kf} D_{\mathrm{id}}, \qquad \sum_{s \in \Omega_i} \|\mathcal{A}_{G(F)}(s) - R(s)\|_{\mathrm{HS}}^2 \leq b^{-12} D_{\mathrm{id}} \quad (i = 0, 1).$$

Here $\mathcal{A}_{G(F)}$ is the full signed array, with every rational rank and the untruncated test vectors.

Proof. We first track the quantities whose logarithms are of order akf. Dropping such factors before taking ratios would lose the comparison. Since $|\log \Xi| \leq 60n^2 f \log b$, $\rho = b^{37n}$, $A = 10^5 n^4$, and $n^6 = o(k)$, the quantity $b^{200n^3 f} \rho^{9nAf} \Xi^{-2(A-1)}$ is at most $b^{o(k)f}$. All the little-o estimates here are uniform as the lower endpoint tends to infinity; $n \leq \sqrt{x}$, $k \asymp x^5$ and $\log b \geq (\log x)^6$ suffice. Using Lemma 133.9 in the definition of D_{op} and $n_p/D_{\mathrm{id}} = N^{-kf}/d_B$ gives

$$\frac{D_{\mathrm{op}}}{D_{\mathrm{id}}} \leq b^{o(k)f} \frac{d_B}{d_\pi c_G^f} \frac{N^{kf}}{M^{2akf} c_E^{2ak}}.$$

The identity $c_G^{-1} = \prod_{l=k+1}^{r} S_l/S_k$, with $S_l = 2\pi^l/(l-1)!$, is essential for the factorials. Put $h = r - k = 2a + 1 = m - 1$. Multiplying this identity by the factorials in Lemma 133.8 gives

$$\frac{1}{d_\pi c_G^f} \leq w^{-fm(k-1)} \{2^{h-1} \pi^{(h-1)k + h(h+1)/2} ((k-1)!)^2\}^f.$$

Indeed all $j!$ with $k \leq j < r$ cancel, and the numerator $(k-1)!$ from $1/S_k$ is multiplied by the first surviving factorial of the degree bound. Also

$$N^{kf} \leq w^{mkf} (M\rho_E/(2\pi))^{hkf} b^{-15kf/2} \exp(kf\sqrt{\log b}).$$

Use $c_E^{2ak} = (\rho_E/2)^{2akf}$ and $h - 1 = 2a$. All powers in these three displays combine to give

$$\frac{D_{\mathrm{op}}}{D_{\mathrm{id}}} \leq b^{-15kf/2 + o(k)f} M^{kf} d_B \left\{ w^m ((k-1)!)^2 2^{h-1-k} \pi^{h(h+1)/2 - k} \rho_E^k \right\}^f.$$

The bracket is at most $b^{o(k)}$. In detail, $m \log w \leq m \log b$, $\log((k-1)!) \leq k \log k$, $\log \rho_E = O_{q_0,d}(\log x)$, and the remaining positive exponents are $O(n^2)$. The exponential of $kf\sqrt{\log b}$ has been absorbed in $b^{o(k)f}$. For the symmetric-power value space at each place the number of degree-t_v monomials is at most $(t_v + 1)^k$, with $t_v \leq xb$. Thus $d_B \leq b^{(1+o(1))kf}$, using $\log x = o(\log b)$. The upper bound is $(Mb^{-13/2 + o(1)})^{kf}$, which proves the first inequality with room to spare.

Insert it into Theorem 133.7. After division by $\sqrt{D_{\mathrm{id}}}$ the resulting norm is at most $b^{-3kf/2} + 4\tau^{-20} + \tau^{-3a/2}$. Since $\tau = b^{kf/4}$ and kf, a tend to infinity, this is at most b^{-6} on the stated interval. Squaring gives the second inequality. □

Corollary 133.11 (The full block at the identity residue)**.** *The Hermitian operator $A_1 = \mathcal{A}_{G(F)}(1)$ on the D_X-dimensional source satisfies*

$$\|A_1 - I\|_{\mathrm{HS}}^2 \leq b^{-12} D_X.$$

At most $4b^{-12} D_X$ of its eigenvalues, counted with multiplicity, lie outside $[1/2, 3/2]$.

Proof. For $t \in B_{\mathrm{res}}$, $\mathcal{A}_{G(F)}(t) - R(t) = (A_1 - I)R(t)$ by Proposition 133.4. Right multiplication by this unitary preserves the Hilbert–Schmidt norm. Both slices contain B_{res}. Sum there and divide the bound of Theorem 133.10 by $|B_{\mathrm{res}}|$. Inversion of the rational term and the Haar change of variable in Lemma 133.5 give $A_1^* = A_1$. The spectral theorem expresses the squared norm as the sum of $|\lambda - 1|^2$; each indicated eigenvalue costs at least $1/4$. These conclusions concern the complex Poincaré pairing. The comparison of integral projected theta columns in the main argument requires further estimates. □

133.4 Poincaré series and projection at a marked prime

We realize the block matrices in a single automorphic Hilbert space, with the Haar measures and terminal prime just used. In this subsection write $\mathbb{A} = \mathbb{A}_F$, $[G] = G(F)\backslash G(\mathbb{A})$, $J = K(M)$ and $\mathcal{G} = K/J = \mathcal{G}_0 \times \mathcal{G}_1$, where $\mathcal{G}_i = \mathrm{GL}_r(\mathbb{F}_{p_i})^f$. The finite level is unrestricted within K at the other places. Each residue matrix at a mark lifts to an integral isometry: lift it invertibly on the chosen component and use the inverse-transpose action on its paired component in dual lattice bases. This gives the indicated equality for $\mathcal{G}$. On $L^2([G])^J$ write $T(s)$ for right translation by a lift of s. Put $j_M = \mathrm{vol}(J)$. Use product Haar on the adelic group and its quotient measure for counting measure on $G(F)$. Use inner products conjugate linear in the first variable.

Theorem 133.12 (The Petersson products of the full series)**.** *There is a linear map $\mathcal{P} : \mathcal{H}_X \to L^2([G])^J$ such that for every $s \in \mathcal{G}$*

$$\mathcal{P}^* T(s) \mathcal{P} = \mathcal{A}_{G(F)}(s).$$

Its series are fixed by the archimedean convolution with $\prod_v d_{\pi_v} c_{\lambda_v}$ in the convention below. The number of singular values of $\mathcal{P}$ outside $[1/\sqrt{2}, \sqrt{3/2}]$, including the zeros on its source, is at most $4b^{-12} D_X$.

Proof. Use the product orthonormal basis of the value space, with coordinates F_i of the full Bessel function. For the source basis vector at class c, coordinate i, use the measurable seed

$$\Phi_{c,i}(y, g_f) = (j_M v_U)^{-1/2}\, \mathbf{1}_{h_c J}(g_f)\, \beta_c(y) \overline{F_i(y)}.$$

Define $\mathcal{P}_{c,i}(g) = \sum_{\alpha \in G(F)} \Phi_{c,i}(\alpha g)$. We first justify the construction, including its inner products. Replace both seeds by their absolute values and initially allow the sums to be infinite. Unfolding the product of their periodizations, with the second translated by s, is justified by Tonelli's theorem. More explicitly, sum over all ordered translates of a measurable fundamental set for $G(F)$ on the left. With the first translate absorbed into the variable, the relative translate in the second seed is γ^{-1}. One obtains the integral over the adelic group of the first seed times the sum of the second seeds at $\gamma^{-1} g k_s$, where k_s lifts s. The first support requires $g_f \in h_c J$. The second support then either holds throughout this coset or is empty. The condition is $h_c^{-1} \gamma h_d \in k_s J$, since k_s normalizes J. Thus the finite integration contributes j_M, and the resulting nonnegative real sum is bounded by the absolute row estimate in Theorem 132.17 divided by v_U. In particular applying this to a seed paired with itself at $s = 1$ shows that the periodization of its absolute value lies in $L^2([G])$. The series is absolutely convergent almost everywhere; its sum is left rational invariant and right J-invariant. There are finitely many coordinates.

Apply the same change of variables to the complex integrals, now justified by the absolute bound. The two real coordinate factors are $F_i(y)\overline{F_j(\gamma^{-1} y)}$; the remaining weights are $\beta_c(y)\beta_d(\gamma^{-1} y)$. The resulting matrix entry is exactly $\mathcal{A}_{G(F)}(s)_{cd,ij}$. This proves the stated identity on the whole source.

For the real convolution group the series first by $\Gamma_c\backslash G(F)$. Along each left coset the finite support is fixed, the F_i are invariant under Γ_c, and the class weights sum to one. The grouped seeds therefore use $\overline{F_i}$ with no class weight. They satisfy $\int C(h)W(gh)\,dh = W(g)$ with $C = \prod_v d_{\pi_v} c_{\lambda_v}$, by conjugating the identity in Theorem 132.14. Here h acts only at infinity. The kernel belongs to $L^1(G_\infty)$. The grouped absolute series also equals the periodization of the absolute seed. Right translating that L^2 function and integrating it against $|C(h)|\,dh$ gives an L^2 majorant by the integral triangle inequality and unitarity. It is finite almost everywhere, permitting termwise convolution there. This proves the real assertion; the use of conjugate coordinates in the seed accounts for the conjugation convention in the matrix identity. Finally the squares of the singular values are the eigenvalues of $\mathcal{P}^*\mathcal{P} = A_1$; apply Corollary 133.11. □

For an element $x = \sum_s x(s)s$ of the finite group algebra put $R(x) = \sum_s x(s)R(s)$; the matrices $R(s)$ vanish off B_{res}. Write $T(x) = \sum_s x(s)T(s)$.

Theorem 133.13 (Compression of a projection at one mark)**.** *For either i, every orthogonal group-algebra projection $x = x^* = x^2 \in \mathbb{C}[\mathcal{G}_i]$ satisfies*

$$0 \le R(x) \le I, \qquad \|\mathcal{P}^*T(x)\mathcal{P} - R(x)\|_{\mathrm{HS}} \le b^{-6}\sqrt{D_X}.$$

Proof. Realize the unitary representation induced from $R|_{B_{\mathrm{res}}}$ as functions $f : \mathcal{G} \to \mathcal{H}_X$ satisfying $f(tg) = R(t)f(g)$, with squared norm the sum over left B_{res}-cosets of the squared value norms. The isometry Iv supported on B_{res} has value $R(t)v$ at $t \in B_{\mathrm{res}}$. Compression of right translation by s to this copy gives $R(s)$: its value at the identity is $(Iv)(s)$. Thus $R(x)$ is the compression of an orthogonal projection, and is a positive contraction.

Let $j = 1-i$. On $\mathcal{G}$ form the matrix-valued function $D(s) = \mathbf{1}_{\Omega_j}(s)(\mathcal{A}_{G(F)}(s) - R(s))$. It has squared norm at most $b^{-12}D_{\mathrm{id}}$ for counting measure, by Theorem 133.10. Right convolution $D \mapsto [s \mapsto \sum_g x(g)D(sg)]$ is an orthogonal projection on this Hilbert space, since the right translations are unitary and form a representation. If $t \in B_{\mathrm{res}}$ and $g \in \mathcal{G}_i$, then $tg \in \Omega_j$. Left transport from the proof of Lemma 133.5, and the same formula for R, give

$$\sum_g x(g)D(tg) = R(t)\big(\mathcal{P}^*T(x)\mathcal{P} - R(x)\big).$$

Take squared norms, sum over all $t \in B_{\mathrm{res}}$, and bound by the squared norm of the entire convolved function. Since $R(t)$ is unitary, division by $|B_{\mathrm{res}}|$ proves the estimate. □

Corollary 133.14 (The cuspidal projection of the series)**.** *Take the two specified central projections e_i of Section 130.1, put $e = e_0e_1$, and set $\alpha_i = D_X^{-1}\operatorname{tr}R(1-e_i)$. The image of $T(e)\mathcal{P}$ lies in the cuspidal L^2-space with the real convolution of Theorem 133.12. Its dimension satisfies*

$$\operatorname{rank}(T(e)\mathcal{P}) \;\ge\; D_X\left(\frac{\max(0,\,1-\alpha_0-\alpha_1-3b^{-6})}{1+b^{-6}}\right)^2.$$

The α_i are finite numbers in $[0,1]$, explicitly the traces of the compressed finite-group projections.

Proof. Every finite type selected by e_i has a primitive factor at some split place above p_i. This factor is cuspidal for $\mathrm{GL}_r(\mathbb{F}_{p_i})$: it is the character of a regular Frobenius orbit of length r with the one-box partition in Green's parametrization [1, §8]. In particular averaging it over the unipotent radical of any proper finite parabolic gives zero. For a proper F-parabolic with radical N_P, consider the constant term at an adelic g. At the indicated split place the conjugated parabolic by g_v^{-1} is conjugate to a standard parabolic by an integral matrix:

intersect its flag with the standard lattice and extend bases along the resulting flag of direct summands over the valuation ring. The integral points in its unipotent radical reduce onto the corresponding finite radical by lifting the free entries. Averaging the form on the right over this compact subgroup kills it. Its constant term at g is unchanged by that average, since the subgroup becomes a subgroup of $N_P(\mathbb{A})$ on the left of g, and the measure on $N_P(F)\backslash N_P(\mathbb{A})$ is right invariant. Thus the constant term vanishes. For L^2 representatives these identities hold almost everywhere: the unipotent quotient is compact by successive additive quotients, and integration on its fibers gives locally integrable constant terms by Fubini. Only finitely many reduced flag subgroups occur, so the averaging identities can be taken simultaneously outside a null set. Sum over the finitely many selected types. The finite projection commutes with real convolution.

Let $D_e = \mathcal{P}^{*}T(e)\mathcal{P}$. The commuting projections give $0 \le I - T(e) \le T(1-e_0) + T(1-e_1)$. The trace of A_1 is at least $(1-b^{-6})D_X$, by Corollary 133.11 and Cauchy–Schwarz for the matrix trace. The same trace inequality in Theorem 133.13 bounds each of the two losses by $(\alpha_i + b^{-6})D_X$. This proves the lower bound for $\operatorname{tr} D_e$ used in the numerator. Also $0 \le D_e \le A_1$, so the minimax characterization of eigenvalues gives $\|D_e\|_{\mathrm{HS}} \le \|A_1\|_{\mathrm{HS}} \le (1+b^{-6})\sqrt{D_X}$. If $D_e \neq 0$, apply $(\operatorname{tr} D_e)^2 \le \operatorname{rank}(D_e)\|D_e\|_{\mathrm{HS}}^2$ to its nonzero eigenvalues. For $D_e = 0$ the nonnegative trace lower bound is zero. The kernel of D_e is the kernel of $T(e)\mathcal{P}$, which proves the asserted rank bound. Evaluating the finite traces on this induced representation is necessary to obtain a numerical lower bound for the prescribed cuspidal projection. □

133.5 A finite Fourier transform and the projected cuspidal rank

We evaluate the loss from the two finite projections by counting cyclic vectors of a matrix over a finite field. Let V_0 have dimension $r = k + 2a + 1$ over $\mathbb{F}_q$, $a \ge 1$, and put $h = 2a+1$. Order a basis in blocks as

$$e_1, \dots, e_a, \quad W = \langle z \rangle \oplus X, \quad u_a, \dots, u_1, \qquad \dim X = k.$$

Let P be the upper block parabolic, U its unipotent radical, and $\mathfrak{p}, \mathfrak{u}$ the corresponding matrix spaces, with $\mathfrak{u}$ strictly upper in blocks. Define C to advance one step in the string $e_1, \dots, e_a, z, u_a, \dots, u_1$, to kill u_1, and to vanish on X. Fix a nontrivial additive character ψ of $\mathbb{F}_q$. On U the function $\chi(1+D) = \psi(\operatorname{tr}(CD))$ is a character: the trace reads only entries between neighboring blocks, and vanishes on $\mathfrak{u}^2$. Extend it by zero on $\mathrm{GL}(V_0) \setminus U$, and denote its average under conjugation by $\chi^{\natural}$.

Lemma 133.15 (Counting bases with prescribed successive images)**.** *For an endomorphism Y of V_0 set*

$$n_h(Y) = \#\{x \in V_0 : x, Yx, \dots, Y^{h-1}x \text{ are linearly independent}\}.$$

Then

$$\#\{g \in \mathrm{GL}(V_0) : g^{-1}Yg \in C + \mathfrak{p}\} = |U|\,|\mathrm{GL}_k(\mathbb{F}_q)|\,q^{k} n_h(Y).$$

Proof. Write η_1 for the coordinate functional on the last line u_1. A basis counted on the left gives $x = ge_1,\ \eta = \eta_1 g^{-1}$ satisfying

$$\eta Y^{j} x = 0 \quad (0 \le j < 2a), \qquad \eta Y^{2a} x = 1.$$

Indeed in that basis the only blocks below the diagonal are the specified maps to the immediate next block. The leading components of $Y^j x$ for $j < a$ are the successive top vectors, and the leading middle component at $j = a$ is z. Dually ηY^j reads the successive bottom lines, and

at $j = a$ its middle component is z^*, the functional killing X with $z^*(z) = 1$. Pairing these components proves the equations: for $j < 2a$ write $j = l + t$ with $l, t \le a$ and at least one strictly less than a; at $j = 2a$ only $z^*(z)$ survives.

Conversely, the equations determine the partial flag and the simple-block identifications of every possible basis above x, η. Put $x_j = Y^j x,\ \eta_j = \eta Y^j$ and form

$$E_j = \langle x_0, \dots, x_{j-1} \rangle, \qquad D_j = \bigcap_{t<j} \ker \eta_t \qquad (0 \le j \le a).$$

The first h powers on x are independent. In fact a dependence at an earlier power would put every later power, including $Y^{2a}x$, in the span of the earlier ones, contradicting its pairing with η. The same argument with the transpose gives independence for the first h functionals. The flag is $E_1 \subset \cdots \subset E_a \subset D_a \subset \cdots \subset D_1 \subset V_0$. On its top one-dimensional quotients prescribe x_{j-1}, and on its bottom quotients D_{j-1}/D_j prescribe the covector η_{j-1}. The middle quotient D_a/E_a contains the class of x_a, and η_a induces a functional there: these assertions use the zero moments up to $2a - 1$. Their pairing is one. Identify the class of x_a with z and the induced functional with z^*. A basis respecting these data exists by first choosing graded bases and then lifting them. It gives the required lower blocks: $Yx_j = x_{j+1}$ prescribes the maps out of the top lines, including the map into the middle. The identities $\eta_j Y = \eta_{j+1}$ prescribe the maps into the bottom lines, including the map out of the middle. For example $YD_j \subset D_{j-1}$ for $1 \le j \le a$, and the coordinate on the next bottom quotient is read by η_j on D_j. These inclusions and the corresponding top inclusions force every lower block further than one step away to vanish.

For fixed x, η the changes of such a basis are arbitrary upper unipotent block changes and arbitrary changes of basis in the k-dimensional kernel in the middle graded space. The singleton graded bases are fixed, and a middle automorphism fixing both a vector and its covector of pairing one is exactly the general linear group of that kernel. There are $|U|\,|\mathrm{GL}_k(\mathbb{F}_q)|$ choices. Each independent h-tuple of powers permits precisely $q^{r-h} = q^k$ extensions of the specified functional values to V_0. This proves the count. □

Theorem 133.16 (The conjugacy average at a split place)**.** *With $v_j(q) = \prod_{l=1}^{j}(1 - q^{-l})$, one has*

$$1 \le \sum_{t \in \mathrm{GL}(V_0)} |\chi^{\natural}(t)|^2 \le v_k(q)/v_r(q).$$

Proof. On the additive matrix algebra use the Fourier transform

$$\widehat{f}(Y) = q^{-r^2/2} \sum_{D} f(D)\psi(-\operatorname{tr}(YD)).$$

It preserves the counting L^2 norm by orthogonality of the additive characters, since the matrix trace pairing is nondegenerate. Take $f(D) = \mathbf{1}_{\mathfrak{u}}(D)\psi(\operatorname{tr}(CD))$. Its transform is $q^{-r^2/2}|U|\mathbf{1}_{C+\mathfrak{p}}$, since the trace annihilator of the strictly upper block matrices is the upper block parabolic algebra. Fourier transform commutes with averaging conjugates by the invariance of trace. Applying Lemma 133.15, the transform of $f^{\natural}$ therefore equals

$$q^{-r^2/2} \frac{v_k(q)}{v_r(q)}\, q^{-r} n_h(Y).$$

Indeed $|U| = q^{a(2k+2a+1)}$, $|\mathrm{GL}_l(\mathbb{F}_q)| = q^{l^2} v_l(q)$, and $2a(2k + 2a + 1) + k^2 + k - r^2 = -r$.

For uniform independent Y, x, the probability that the first h powers starting at x are independent is $\prod_{j=0}^{h-1}(1 - q^{j-r}) = v_r(q)/v_k(q)$. First exclude $x = 0$; after each independent list, condition on the vectors already exposed and the images of all but the last one. A uniform

linear map with these prescribed images sends the last independent vector uniformly to V_0. Excluding the current span gives the next factor. Thus $q^{-r}n_h(Y) \in [0,1]$ has mean $v_r(q)/v_k(q)$ over Y, which also bounds its second moment. Plancherel's identity now gives the upper bound. Finally $f^\natural(D)$ is supported where $1+D$ is unipotent and agrees there with $\chi^\natural(1+D)$. This identifies the two counting norms, and $\chi^\natural(1)=1$ gives the lower bound. □

Proposition 133.17 (The trace lost at an actual marked prime)**.** *For the prescribed central projection at p_i let δ_i be its discarded Plancherel mass as in Proposition 130.1, and let α_i be the lost trace in Corollary 133.14. Then on the same interval*

$$|\alpha_i - \delta_i| \le 4\sqrt{f\delta_i}\, p_i^{-(k+1)/2} \qquad (i=0,1)$$

for all sufficiently large nodes.

Proof. For $t \in \mathcal{G}_i$, so $t=1$ at the other mark, $D_X^{-1}\operatorname{tr} R(t)$ is the scalar unipotent character extended by zero off the finite unipotent image. To see this, a nonzero diagonal class block gives a rational Bessel lift $u\eta$ from that class to itself. Its middle component η preserves the integral middle quotient and its positive lattice Λ_X, since u acts trivially on the middle quotient and η fixes the terminal line. The automorphism group of that positive lattice is finite. At the other odd split mark η is principal on the middle quotient, so is the identity by the torsion argument in Proposition 131.25. Thus the lift is in $U(F)$ and t is in the finite unipotent image. On that image the scalar on every class is the same, by Lemma 133.5. If a residue is outside B_{res} its entire matrix is zero. This proves the trace identity.

At the mark the image of U is the full product of the block upper unipotent radicals just used. On a chosen split component lift any strictly upper block entries and use the unitary inverse transpose on its paired component; the integral form is perfect there and $p_i \nmid N$. Off the mark one can use the identity. In coordinates for the chosen component the character reads the entries just above the diagonal along the left arm, ending at z. The conjugate component supplies the opposite arm in reverse order, starting at z: its matrix is obtained from the inverse transpose by the integral Gram identification. On adjacent blocks the inverse of $1+D$ has entries $-D$, since D^2 skips a block. Thus the two middle coefficients are a vector on the terminal line and its Hermitian dual, with nonzero pairing modulo the mark since its norm is $-N$. Each simple coefficient is nonzero: the conductor has precisely one factor p_i from M and the different is a unit at this split place. Choose the middle basis starting with this vector and continuing in the kernel of its dual. Diagonal rescaling along the single string makes all its coefficients one, successively, including the nonzero terminal pairing. Hence up to conjugation the character on this factor is the one of Theorem 133.16, with a nontrivial additive character. This argument applies also to the inverse character dictated by the real convention.

Let $b_i^\natural$ be the conjugacy average of the normalized trace on $\mathcal{G}_i$. The character and its average factor at its f places. As $b_i^\natural(1)=1$, the theorem gives, with $q=p_i$ and counting norm,

$$\|b_i^\natural - \mathbf{1}_{\{1\}}\|_2^2 \le (v_k(q)/v_r(q))^f - 1 \le 16 f q^{-(k+1)}.$$

Indeed $-\log(1-q^{-l}) \le 2q^{-l}$; the tail sum begins at $l=k+1$. Its logarithm after multiplication by f is at most $4fq^{-(k+1)} = o(1)$. Exponentiation gives the displayed inequality.

Write $x = 1 - e_i$. Its coefficients are constant on conjugacy classes, and $\sum_t |x(t)|^2 = x(1) = \delta_i$, by taking the identity coefficient of $x^*x = x$ and the normalized trace in the regular representation. By the trace identity, $\alpha_i = \sum_t x(t) b_i^\natural(t)$. Subtract the identity coefficient and apply Cauchy–Schwarz to get the claimed comparison. □

Theorem 133.18 (Rank after both cuspidal projections)**.** *On the fields and at all sufficiently large nodes of Theorem 133.10, let $\mathcal{P}_e = T(e_0e_1)\mathcal{P}$. Its Petersson matrix and the dimension of its cuspidal image satisfy*

$$\|\mathcal{P}_e^*\mathcal{P}_e - I\|_{\mathrm{HS}} \le 2b^{-4}\sqrt{D_X}, \qquad \operatorname{rank}\mathcal{P}_e \ge (1-4b^{-8})D_X.$$

Proof. Proposition 130.1 gives $\delta_0+\delta_1 \le b^{-8}$, and the error at either mark in Proposition 133.17 is at most b^{-20} for large nodes, since $p_i \ge b^{10}$, f is polynomial in x, and $k \to \infty$. Each $R(1-e_i)$ is a positive contraction with trace $\alpha_i D_X$. Its squared Hilbert–Schmidt norm is at most that trace. By Theorem 133.13,

$$\|\mathcal{P}^*T(1-e_i)\mathcal{P}\|_{\mathrm{HS}} \le (b^{-6}+\sqrt{\alpha_i})\sqrt{D_X}.$$

The positive difference $\mathcal{P}^*\mathcal{P}-\mathcal{P}_e^*\mathcal{P}_e$ is bounded above by the sum of these two positive operators. The Hilbert–Schmidt norm is monotone on positive operators by eigenvalue monotonicity. Add $\|\mathcal{P}^*\mathcal{P}-I\|_{\mathrm{HS}}$ from Corollary 133.11. The resulting factor is at most $3b^{-6}+\sqrt{2(b^{-8}+2b^{-20})} \le 2b^{-4}$ for large b. Each zero eigenvalue of the projected Petersson matrix contributes one to its squared distance from the identity. The bound for the rank follows. The image is cuspidal by Corollary 133.14. This is a dimension estimate relative to the source; the span of its translates in the target and the integral comparison for theta lifts still require control. □

133.6 The second moment over all residues and the span of translates

Let $\mathcal{G} = K/K(M)$, $B = B_{\mathrm{res}}$, and let $\mathcal{P}_e = T(e_0e_1)\mathcal{P}$ be the projected Poincaré map of Theorem 133.18. We use counting measure on $\mathcal{G}$. Recall that $z(s)$ is the product of the norms of the marked components where the two reduced full frames meet, with empty product one.

Proposition 133.19 (A bound summed over every finite residue)**.** *On the fields and intervals of Theorem 133.10, at all sufficiently large nodes one has*

$$\sum_{s\in\mathcal{G}} \|\mathcal{P}^*T(s)\mathcal{P}\|_{\mathrm{HS}}^2 \le (M/b^5)^{kf}\,|B|D_X.$$

More precisely, on the residues with $z(s) \ge Z$, $Z \ge 1$, the contribution from the open terms $T_>$ of Theorem 133.7 has counting Hilbert–Schmidt norm at most

$$\{(\tau^2+\tau^3)Z^{-k} + 4\tau^4(2/M)^{kf}\}^{1/2}\sqrt{D_{\mathrm{op}}} + 2\tau^{-20}\sqrt{D_{\mathrm{id}}}.$$

Proof. Theorem 133.12 identifies the left side of the first display with the moment of the full signed array. The set $T_>$ is invariant under the rational Bessel group on both sides by invariance of its two defining leading determinants. Lemma 133.5 therefore eliminates every residue with disagreeing characters before the square is expanded. In each remaining block, unfold it by Proposition 133.6. After removing the cut error, the norm is bounded by the sum of $C_\infty M^{akf}G_\gamma^{-k/2}$ over its retained near representatives. This follows from $e \ge k$, $X_\gamma \ge 1$, and $G_\gamma = M^{2af}X_\gamma$ in the indicated open range, as in Theorem 133.7. The passage to the cut and the replacement of the cut integral by its majorant each cost at most $\tau^{-20}\sqrt{D_{\mathrm{id}}}$ in counting norm by Corollary 132.18. That corollary already sums over all residues.

For the square of the majorant expand in ordered pairs within each block. Distinct chosen representatives in that block belong to distinct orbits for the smaller pointwise frame fixers. The pairs are therefore among those counted by Theorem 131.29. If their full links agree, their positive corank ℓ is zero or one, and Theorem 131.17 bounds the pair weight $(G_\gamma G_\eta)^{-k/2}$ by $Z^{-k}d(L_{12})^{-k}$. If the links differ, Theorem 131.23 bounds it by $(2/M)^{\max(1,\ell-1)kf}d(L_{12})^{-k}$.

Summing the first bound gives $(\tau^2+\tau^3)Z^{-k}$ in units of D_{op}. In the second, the terms for $\ell=0,1,2$ share the gain $(2/M)^{kf}$, and from $\ell=2$ on the ratio is $\tau(2/M)^{kf}<1/2$. Their sum is at most $4\tau^4(2/M)^{kf}$. This proves the restricted estimate.

Take $Z=1$. Its expression in braces is at most $4\tau^3$ for large nodes. The parabolic terms have norm $\sqrt{|\mu(E)|D_{\mathrm{id}}}$ by Corollary 133.3. The positive closed terms and the complementary open range have cut norms $\tau^{-20}\sqrt{D_{\mathrm{id}}}$ and $\tau^{-3a/2}\sqrt{D_{\mathrm{id}}}$ as upper bounds by Corollaries 131.35 and 131.38. One application of Corollary 132.18 to their union bounds its passage back to the full kernels. Thus, divided by $\sqrt{D_{\mathrm{id}}}$, the norm of the entire array is at most

$$2\tau^{3/2}(M/b^6)^{kf/2}+\sqrt{|\mu(E)|}+4\tau^{-20}+\tau^{-3a/2},$$

using Theorem 133.10. The square of the first term is $4(M/b^{21/4})^{kf}$. The roots of unity inject into $\mathbb{F}_{p_0}^{\times}$ as in Corollary 133.3, so their number is at most b^{11}. Since $M\ge b^{20}$ and $kf\to\infty$, the whole display is at most $(M/b^5)^{kf/2}$ for large nodes. This gives the first bound. □

Define a real number determined by the two finite levels by

$$L=\frac{|\mathcal{G}|}{|B|(M/b^5)^{kf}}.$$

It grows with the nodes. Indeed, at a split component of residue size q the finite image of the Bessel group has order $q^{a(2k+2a+1)}|\mathrm{GL}_k(\mathbb{F}_q)|$. The upper block unipotents lift as in Proposition 133.17. The middle isometry fixes the terminal vector and its dual and acts arbitrarily on their common k-dimensional complement. Integral lifts exist because the terminal norm is a unit there. On the paired component the action is determined by the perfect pairing. These choices are independent at distinct finite places. Consequently

$$L=b^{5kf}M^{[(2a+1)k+(2a+1)(a+1)]f}\prod_{i=0}^{1}\left(\frac{v_r(p_i)}{v_k(p_i)}\right)^{f}.$$

Here v_j is the product in Theorem 133.16. The final product tends to one, since the negative of its logarithm is at most $4f\sum_i p_i^{-k-1}$. In particular $L\ge 1$ at every sufficiently large node.

Theorem 133.20 (Dimension obtained by finite translates)**.** *The cuspidal subspace generated by all $T(s)\mathcal{P}_e\mathcal{H}_X$, $s\in\mathcal{G}$, has dimension at least $(1-2b^{-4})^2LD_X$. Moreover, for every integer $J_0\ge 1$, at most J_0 translates generate a space of dimension at least*

$$\frac{(1-2b^{-4})^2D_X}{L^{-1}+(1+b^{-6})^2/J_0}.$$

Proof. Put $Z=\mathcal{P}\mathcal{P}^{*}$ and $A=|\mathcal{G}|^{-1}\sum_s T(s)ZT(s)^{*}$. All traces in the proof are of finite-rank operators. Expanding the square and counting ordered pairs of residues with each prescribed relative translate gives

$$\operatorname{tr}A^2=|\mathcal{G}|^{-1}\sum_s\operatorname{tr}(ZT(s)ZT(s)^{*})=|\mathcal{G}|^{-1}\sum_s\|\mathcal{P}^{*}T(s)\mathcal{P}\|_{\mathrm{HS}}^2\le D_X/L.$$

Write $E=T(e_0e_1)$, $Z_e=EZE$, $A_e=EAE$. The projection E commutes with each $T(s)$ by centrality. Consequently A_e is the same average with Z_e in place of Z. Compression cannot increase Hilbert–Schmidt norm, and

$$\operatorname{tr}A_e=\operatorname{tr}Z_e=\operatorname{tr}(\mathcal{P}_e^{*}\mathcal{P}_e)\ge(1-2b^{-4})D_X.$$

For the last step pair the difference from the identity in Theorem 133.18 with the identity matrix by Cauchy–Schwarz. The range of A_e is precisely the required span: its kernel is the

intersection of the orthogonal complements of the finitely many images, by positivity of every summand. The inequality $(\operatorname{tr} A_e)^2 \le \operatorname{rank}(A_e)\operatorname{tr} A_e^2$, applied to its eigenvalues, proves the first assertion.

For the second choose J_0 independent uniform residues and let A' be the empirical average of their conjugates of Z_e. Its trace is still $\operatorname{tr} Z_e$. Independence in the off-diagonal terms gives

$$\mathbb{E}\big[\operatorname{tr}((A')^2)\big] = (1 - J_0^{-1})\operatorname{tr}(A_e^2) + J_0^{-1}\operatorname{tr}(Z_e^2).$$

We have $\|Z_e\|_{\mathrm{HS}} \le \|Z\|_{\mathrm{HS}} = \|\mathcal{P}^*\mathcal{P}\|_{\mathrm{HS}} \le (1 + b^{-6})\sqrt{D_X}$ by Corollary 133.11. Some tuple has trace of its squared average at most the displayed mean. Apply the same rank inequality to this tuple. Repetitions do not affect its span. □

Corollary 133.21 (A translate modulo a prescribed subspace)**.** *For every subspace W of finite dimension d in the target, some $s \in \mathcal{G}$ has image modulo W of dimension at least*

$$D_X \left(\frac{\max\{0, 1 - 2b^{-4} - \sqrt{d/(LD_X)}\}}{1 + b^{-6}} \right)^2 .$$

Proof. If Q is the orthogonal projection onto W, the average trace lost to W is $\operatorname{tr}(QA_e)$, which is at most $\sqrt{d}\|A_e\|_{\mathrm{HS}} \le \sqrt{dD_X/L}$. Choose a residue with loss at most this average. Compress its conjugate of Z_e to $W^{\perp}$. The trace of the compression is the original trace less this loss and is nonnegative. Its Hilbert–Schmidt norm is at most $(1+b^{-6})\sqrt{D_X}$. The trace–rank inequality gives the assertion. Its scale is the explicit lower dimension LD_X. Passing to a bound in terms of a prescribed fraction of the full target requires a sharper comparison with the automorphic mass. □

134 Spanning by translates and the terminal norm sphere

134.1 The good source dimension and the number of translates

Use the shorter interval, the marked group $\mathcal{G} = K/J$ on Y_0, and the map $P = T(e_0e_1)\mathcal{P}$ of Theorem 133.18. Write $d_X = \dim \mathcal{H}_X$ for its domain dimension and $d = \dim H_{Y_0} = D_x$ for the common good source of the determinant. These are different spaces. The finite-level operators $T(g)$ preserve H_{Y_0} and act unitarily. Throughout this subsection $b = b_x$. The terminal level auxiliary prime N may be chosen as in Lemma 133.8. Its choice does not enter the definition of d.

Proposition 134.1 (Multiplicity on the common good source)**.** *At every sufficiently distant node one has $d > 0$. On the positive definite rank-s space I_s, let $\mathcal{C}_s$ be the set of maximal-level lattice classes and Γ_L the automorphism group of class L. Let V_λ be the compact coefficient representation of the common Harish–Chandra parameter, with unit highest vector and $d_\lambda = \dim V_\lambda$. Put $\operatorname{mass}_s = \sum_{L\in\mathcal{C}_s} |\Gamma_L|^{-1}$. Then*

$$d = |\mathcal{G}| \sum_{L\in\mathcal{C}_s} \frac{1}{|\Gamma_L|} \sum_{\gamma\in\Gamma_L} \operatorname{tr}_{V_\lambda}(\gamma)\, e(\bar\gamma) \;\le\; |\mathcal{G}|\, d_\lambda \operatorname{mass}_s . \tag{134.1}$$

Here $e = e_0e_1$ is expressed in the group algebra with counting convolution, and $\bar\gamma$ denotes reduction in the local frames of L. Use the coefficient representation in the left automorphy rule for V_λ.

Proof. The rank bound of Theorem 133.18 is at least $(1 - 4b^{-8})d_X$. Its domain dimension is $|\mathcal{C}|d_B$ by Theorem 133.2, with the identity class present and a nonzero compact type. The image

is in the fixed real discrete-series line and the principal finite level by Theorem 133.12, and has already been projected by e. The parameters at the node agree: the line in Proposition 132.1 has the $k-1$ consecutive entries and the m shifted entries of Theorem 73.8 on Y_0. Thus P takes values in that theorem's H_{Y_0}, and the positive rank gives $d>0$.

For the dimension calculation realize this good space at the same finite places on I_s. The packet argument in Theorem 73.8 applies with the signs all positive. Indeed the unimodular lattices are locally isometric by its finite frame calculation, the same two finite supercuspidal components give a simple generic parameter, and the all-positive assignment of the same regular entries is the compact member of its real packet. Keeping the finite parts and making that assignment gives multiplicity one by the same simple-parameter multiplicity formula, in both directions. Projection to the specified real line leaves the finite multiplicity unchanged. On the definite group this line projection identifies scalar forms with V_λ-valued functions on the finite classes, by compact Schur orthogonality.

At a maximal-level class L these functions at principal level are the invariants in $V_\lambda\otimes\mathbb{C}[\mathcal{G}]$ for Γ_L acting by its coefficient representation and by left translation on residue arguments. Project onto the invariants by averaging this action. It commutes with the right action $R(e)$. For the convention $R(t)f(g)=f(gt)$, the trace of left translation by h followed by $R(e)$ on the regular space is $\sum_{g\in\mathcal{G}}e(g^{-1}hg)=|\mathcal{G}|e(h)$. The last equality uses centrality. Taking the trace of the product of the two commuting projectors gives the equality in (134.1). For the upper bound discard e. No nonidentity $\gamma\in\Gamma_L$ reduces to the identity: the principal subgroup at an odd original mark is torsion free by the proof of Lemma 84.13. Left translation on $\mathcal{G}$ has trace zero for every other group element. Thus the whole principal-level dimension is exactly $|\mathcal{G}|d_\lambda\,\mathrm{mass}_s$. Projection can only decrease dimension. □

Recall the explicit number

$$L=\frac{|\mathcal{G}|}{|B_{\mathrm{res}}|(M/b^5)^{kf}}\ge 1$$

of Section 133.6. A stronger moment estimate can be used in the following argument by replacing L with any number $L_*\ge 1$ for which the positive operator

$$A=|\mathcal{G}|^{-1}\sum_{g\in\mathcal{G}}T(g)PP^{*}T(g)^{*}$$

satisfies $\mathrm{tr}(A^2)\le d_X/L_*$. For the actual columns $L_*=L$ is available: Proposition 133.19 gives this bound before the projection by e, by expanding the trace as in Theorem 133.20. Compressing that average by the orthogonal projector $T(e)$ gives A, since e is central, and cannot increase its Hilbert–Schmidt norm.

Theorem 134.2 (Independent input columns from few translates)**.** *For every positive integer j, the sum of at most j spaces of the form $T(g)P\mathcal{H}_X$ inside H_{Y_0} has dimension at least*

$$L_*d_X\bigl(1-e^{-j/(2L_*)}-8b^{-4}\bigr).\tag{134.2}$$

In particular with the explicit L there is a finite set $\mathcal{T}\subseteq\mathcal{G}$ with

$$|\mathcal{T}|d_X\le 5d\log(1+2b^2),\qquad \dim\sum_{g\in\mathcal{T}}T(g)P\mathcal{H}_X\ge(1-2b^{-2})Ld_X.$$

Independent vectors counted by this bound can be selected from the translated images of any spanning family of $\mathcal{H}_X$. They may be used as the first n columns of Theorem 84.14.

Proof. Set $\epsilon = 2b^{-4}$ and $N_* = L_* d_X$. Theorem 133.18 gives $\|P^*P - I\|_{\mathrm{HS}} \le \epsilon\sqrt{d_X}$. Consequently $\operatorname{tr} A = \operatorname{tr} P^*P \ge (1-\epsilon)d_X$. If the currently selected span has dimension v and orthogonal projector Q, then

$$\operatorname{tr}(QA) \le \sqrt{v \operatorname{tr}(A^2)} \le d_X\sqrt{v/N_*}.$$

By averaging choose g for which, with $P_g = T(g)P$, the positive operator $B = P_g^*(1-Q)P_g$ has trace at least $\operatorname{tr} A - \operatorname{tr}(QA)$. The rank of B is the exact dimension gained by adjoining the columns of P_g, because it is the Gram operator of $(1-Q)P_g$. If Q_B is the orthogonal projector onto its range in the domain, $0 \le B \le P^*P$ gives

$$\operatorname{tr} B \le \operatorname{tr}(Q_B P^*P) \le \operatorname{rank} B + \|Q_B\|_{\mathrm{HS}}\|P^*P - I\|_{\mathrm{HS}} \le \operatorname{rank} B + \epsilon d_X.$$

This last estimate uses the near-isometry on the source. It retains a linear gain in the number of missing columns.

For $v \le N_*$ use $1 - \sqrt{v/N_*} \ge (1 - v/N_*)/2$. The new dimension is at least $v + (N_* - v)/(2L_*) - 2\epsilon d_X$. Writing $s_j = (N_* - v_j)_+$ for the shortfall after j choices, monotonicity of the span therefore gives

$$s_{j+1} \le (1 - 1/(2L_*))s_j + 2\epsilon d_X.$$

The inequality also holds when $v_j > N_*$, since then the next shortfall is zero. Starting with $s_0 = N_*$, summation of the geometric progression bounds s_j by $N_*(e^{-j/(2L_*)} + 4\epsilon)$, proving the first assertion.

For $L_* = L$ choose $j = \lceil 2L\log(b^2)\rceil$; then $e^{-j/(2L)} \le b^{-2}$ and $8b^{-4} \le b^{-2}$ eventually. Also $d \ge (1-\epsilon)^2 L d_X$ by $(\operatorname{tr} A)^2 \le d \operatorname{tr}(A^2)$. It follows that $j d_X \le 5d\log(1+2b^2)$ for all sufficiently large nodes, using $L \ge 1$. Repetitions of translates may be discarded. The images of the indicated spanning family span the same sum, so elementary basis selection gives as many independent columns as its dimension. □

The unconditional rank above is measured in units of $L d_X$. For example, a bound $\operatorname{tr}(A^2) \le (1 + b^{-2})d_X^2/d$ allows one to take $L_* = d/((1+b^{-2})d_X)$ whenever this is at least one; (134.2) then has the scale of the full good source. The estimate in Proposition 133.19 uses absolute majorants on the open orbits. Obtaining this precision for the second moment on all residues is a remaining issue in applying the terminal covolume inequality.

134.2 The contribution of equal links to the full second moment

Let $b = b_x$ on the shorter interval and use the terminal prime of Lemma 133.8. In this subsection $r = k + 2a + 1$ is the rank called s in Proposition 134.1. Set $h = 2a + 1 = m - 1$, write $\mathfrak{m}_j$ for the mass of the standard positive genus in rank j, and put

$$\mathcal{M} = \frac{d_X^2}{d_\lambda \mathfrak{m}_r}, \qquad D_{\mathrm{id}} = |B_{\mathrm{res}}| d_X, \qquad \rho_E = |\operatorname{disc} E|^{1/(2f)}.$$

These normalizations use counting measure on the marked residue group. Indeed, averaging the translates of $\mathcal{P}\mathcal{P}^*$ divides the squared moment of $\mathcal{P}^* T(s) \mathcal{P}$ by $|\mathcal{G}|$, while $d \le |\mathcal{G}| d_\lambda \mathfrak{m}_r$ by Proposition 134.1.

Lemma 134.3 (The normalization on the shorter interval)**.** *At every sufficiently large node,*

$$\frac{\mathcal{M}}{D_{\mathrm{id}}} = \frac{d_B (N^k - 1)^f \mathfrak{m}_k}{M^{fa(2k+h)} d_\lambda \mathfrak{m}_r}, \tag{134.3}$$

$$\log \frac{\mathcal{M}}{D_{\mathrm{id}}} = kf \log(M b^{-15/2}) + \log d_B + f \log((k-1)!) + o(kf \log b). \tag{134.4}$$

The error is uniform on the interval. In particular

$$(M/b^8)^{kf} D_{\mathrm{id}} \le \mathcal{M} \le (M/b^6)^{kf} D_{\mathrm{id}}, \qquad D_{\mathrm{op}} \le b^{o(1)kf} \mathcal{M},$$

where D_{op} is the majorant in Section 133.2.

Proof. At the two original marks the $H = \mathrm{U}(X_0)$ level fixes a full frame. Write $\bar{H}$ for the product of the groups $\mathrm{GL}_k(\mathbb{F}_{p_i})$ at their split places. At each of the f places of F over N, the level in the chosen component of E fixes one nonzero covector. To check the exact index, use the middle basis in the proof of Proposition 131.25. At a component where π divides $N = \pi\bar{\pi}$, $\bar{\pi}$ is a unit. Preservation of the lattice by a middle matrix and its inverse requires integrality and the distinguished row congruence $e_*^t A = e_*^t \bmod \pi$. Conversely these conditions make all the coefficients integral: the possible denominators in the z-coordinate are those of A_{*j}/π for $j \ne *$ and $(1 - A_{**})/N$. The other coordinates have unit denominators. The same row condition holds for A^{-1}, and the inverse transpose preserves the dual lattice in the paired component. Reduction of GL_k acts transitively on nonzero rows. At the other primes the middle basis differs from the standard one by an integral invertible change, apart from the two prescribed principal levels. It follows that a standard class L acquires $|\bar{H}|(N^k - 1)^f$ structures before quotient by its automorphisms. Their action is free by the odd principal level argument of Proposition 131.25. Therefore

$$d_X = d_B |\bar{H}| (N^k - 1)^f \mathfrak{m}_k.$$

At a split mark the unipotent radical is the upper block radical with block sizes $1, \dots, 1, k+1, 1, \dots, 1$, with a singletons on either side. Its strict off-block entries number $(r^2 - (k+1)^2 - 2a)/2 = a(2k+h)$. The residual middle subgroup is GL_k on the complement of the terminal line. All these entries lift integrally, as in Section 133.6. Thus $|B_{\mathrm{res}}| = |\bar{H}| M^{fa(2k+h)}$. Substitution gives (134.3).

We evaluate the part of the mass that is of order $kf \log b$. Here E/F is unramified at the finite places. Indeed E contains $\mathbb{Q}(i)$ and the imaginary quadratic field of odd discriminant $-d_K$; inertia at any finite prime fixes at least one of them and complex conjugation fixes neither. Hence $|\operatorname{disc} F| = \rho_E^f$. Lemma 93.7 gives $\mathfrak{m}_{i+1}/\mathfrak{m}_i = 2^{-f} |L_F(-i, \eta^{i+1})|$ with η the quadratic character of E/F. For $j = i + 1$ its abelian factors are the f primitive Dirichlet characters of

parity j in the character group of E, taken with their primitive Euler products. The conductors for even parity have product $|\operatorname{disc} F|$ by the conductor-discriminant formula applied to F. For odd parity they have product $|\operatorname{disc} E|/|\operatorname{disc} F| = |\operatorname{disc} F|$ by the same formula for E. For a primitive factor of conductor q, the functional equation at $1-j$ gives the absolute value

$$2(j-1)!\,(2\pi)^{-j} q^{j-1/2} |L(j,\bar\chi)|.$$

Both $|L(j,\bar\chi)|$ and its reciprocal are at most $\zeta(j)$ by the Euler product. Since $\log\zeta(j) \le \sum_{n\ge 2} n^{-j} \ll 2^{-j}$ for $j \ge 3$, the factor 2^f cancels and summation gives

$$\log \frac{\mathfrak m_r}{\mathfrak m_k} = f \sum_{i=k}^{r-1} \Big(\log i! + (i+\tfrac12)\log \rho_E - (i+1)\log(2\pi) \Big) + O(f 2^{-k}). \tag{134.5}$$

Let w_v be the local weight of Lemma 133.8, $w_{\min} = \min_v w_v$, and $w = (\prod_v w_v)^{1/f}$. The first $k-1$ entries of the compact Harish–Chandra parameter are $k/2-1-i$, $0 \le i \le k-2$; the last m entries are $-w_v - 2jr^8$, $1 \le j \le m$, by Proposition 132.1. In the Weyl degree formula the first block cancels the factors $1!, \dots, (k-2)!$. Each mixed gap divided by w_v lies between one and $1 + (2mr^8 + k)/w_{\min}$. The product of the gaps internal to the last block is $(2r^8)^{m(m-1)/2} \prod_{j=1}^{m-1} j!$. Consequently

$$\log d_\lambda = fm(k-1)\log w - f \sum_{i=k-1}^{r-1} \log i! + \Delta, \qquad 0 \le \Delta \ll f\left(m^2 \log(2r) + \frac{mk(2mr^8+k)}{w_{\min}} \right).$$

Here $w_{\min} \ge b^{1-10^{-6}}$, $m = O(\sqrt{x})$, $k \asymp x^5$, and $\log b \ge (\log x)^6$, so that $\Delta = o(kf \log b)$.

Write $N = w^m (M\rho_E/(2\pi))^h b^{-15/2} e^{\theta}$, $0 \le \theta \le \sqrt{\log b}$. Insert the last two formulas in (134.3) and use $\log(N^k - 1) = k \log N + O(N^{-k})$. The factorial sums leave $f \log((k-1)!)$. The coefficient of $\log M$, before splitting off its leading term, is $f(hk - a(2k+h)) = f(k - ah)$. The remaining terms after the three displayed terms of (134.4) are

$$f\Big(m \log w - ah \log M - \tfrac12 h^2 \log \rho_E + \tfrac12 h(h+1)\log(2\pi) + k\theta \Big) - \Delta + O\big(f(2^{-k} + N^{-k})\big).$$

Each has size $o(kf \log b)$, since $m^2 = o(k)$, $w \le b$, $M \le b^{22}$, and $\log \rho_E = O_{q_0, d_K}(\log x)$. This proves (134.4). The symmetric powers at infinity give $1 \le d_B \le (xb+1)^{kf}$, and $\log((k-1)!) \le k \log k = o(k \log b)$, which give the stated two-sided inequalities. Finally the proof of Theorem 133.10, before bounding d_B, gives

$$D_{\mathrm{op}} / D_{\mathrm{id}} \le (M b^{-15/2})^{kf} d_B \exp(o(kf \log b)).$$

Indeed the logarithms of the remaining factors there are $O(f(m \log w + k \log k + k \log \rho_E + h^2)) + o(kf \log b)$. Comparing with (134.4), retaining the same d_B in both expressions, proves the final bound. □

We now retain the signs of the orbital integrals. In each block (c, s, c') restrict to the large open range $T_>$ of Theorem 133.7, with agreeing characters. Use the cut orbit representatives of Proposition 133.6 and let V_γ be the actual signed matrix integral there, including v_U^{-1}. Choose one of the permitted near representatives for each orbit. Its full link is the unitary map J_γ on $X_0 \oplus E$ defined in Section 131.7. For the product $z(s)$ of the norms of the meeting marked components, set

$$\mathcal{E} = \sum_{\substack{c,s,c' \\ z(s) < b^f}} \sum_{J} \Big\| \sum_{\gamma:\, J_\gamma = J} V_\gamma \Big\|_{\mathrm{HS}}^2 . \tag{134.6}$$

The inner sums are in a single block. The comparison below holds for every such choice of near representatives.

Theorem 134.4 (Removal of the pairs with different full links)**.** *Let $\mathcal{P}$ be the unprojected Poincaré map on these same fields, levels and intervals. At all sufficiently large nodes,*

$$\left| \left(\sum_{s \in \mathcal{G}} \|\mathcal{P}^* T(s) \mathcal{P}\|_{\mathrm{HS}}^2 \right)^{1/2} - \sqrt{\mathcal{E}} \right| \le b^{-kf/10} \sqrt{\mathcal{M}}.$$

In particular an upper estimate $\mathcal{E} \le (1 + \frac{1}{2} b^{-2}) \mathcal{M}$ suffices for $\operatorname{tr}(A^2) \le (1 + b^{-2}) d_X^2 / d$ in Theorem 134.2.

Proof. Theorem 133.12 identifies the full Gram array with the sum of the orbital integrals. On all residues combined, the parabolic contribution has norm $\sqrt{|\mu(E)| D_{\mathrm{id}}}$. The positive closed terms and the remaining part of the open range have cut norms bounded by $(\tau^{-20} + \tau^{-3a/2}) \sqrt{D_{\mathrm{id}}}$, where $\tau = b^{kf/4}$, by Corollaries 131.35 and 131.38. Their passage to full kernels costs at most $\tau^{-20} \sqrt{D_{\mathrm{id}}}$ by Corollary 132.18. Since $|\mu(E)| \le b^{11}$ and $\mathcal{M} / D_{\mathrm{id}} \ge b^{12kf}$, these terms together have norm $O(b^{-5kf}) \sqrt{\mathcal{M}}$.

For the full open array on $z(s) \ge b^f$, use Proposition 133.19 with $Z = b^f$. The squared coefficient of D_{op} in its majorant is

$$(\tau^2 + \tau^3) b^{-kf} + 4 \tau^4 (2/M)^{kf} \le b^{(-1/4 + o(1)) kf}.$$

The other term in its norm bound is $2 \tau^{-20} \sqrt{D_{\mathrm{id}}}$. By Lemma 134.3 the whole norm is at most $b^{(-1/8 + o(1)) kf} \sqrt{\mathcal{M}}$. On the complementary residues the error in using the cut orbit integrals costs at most $\tau^{-20} \sqrt{D_{\mathrm{id}}}$. The triangle inequality now reduces the assertion to comparing the norm of the signed sum of these V_γ with $\sqrt{\mathcal{E}}$.

Write $B_\gamma = C_\infty M^{akf} G_\gamma^{-k/2}$ for the individual majorant of Proposition 133.19. There $\|V_\gamma\|_{\mathrm{HS}} \le B_\gamma + \epsilon_\gamma$, and the counting norm of the blockwise sums of $\epsilon_\gamma \ge 0$ is at most $\tau^{-20} \sqrt{D_{\mathrm{id}}}$. Project each V_γ radially onto the ball of radius B_γ in its matrix space, obtaining W_γ. This changes its normed distance by at most ϵ_γ. The error in the sum within each block is bounded by the sum of these errors. The same bound on the total counting norm holds after refining blocks by J, since the sum of squares of partial sums of nonnegative errors is at most the square of their full sum.

Expand the squared norms for the W_γ before and after refining blocks by J. Their difference is a sum of inner products over ordered pairs with $J_\gamma \ne J_{\gamma'}$. Its absolute value is bounded by the sum of $B_\gamma B_{\gamma'}$ over those pairs. In the proof of Proposition 133.19 this part of the ordered-pair bound, even on all residues, is

$$D_{\mathrm{op}} \sum_{\ell \ge 0} \tau^{\ell+2} (2/M)^{\max(1, \ell - 1) kf} \le 4 \tau^4 (2/M)^{kf} D_{\mathrm{op}}.$$

Indeed different links give positive codimension in Theorem 131.23; for $\ell \ge 2$ the successive majorants have ratio $\tau (2/M)^{kf} < 1/2$. Thus the squared-norm difference is at most $b^{(-19 + o(1)) kf} \mathcal{M}$. For nonnegative u, v one has $|\sqrt{u} - \sqrt{v}| \le \sqrt{|u - v|}$. Applying this to the two squared norms and then restoring the V_γ on both sides proves the assertion, because $1/8 > 1/10$ and all error bounds are uniform.

Under the stated hypothesis on $\mathcal{E}$, squaring gives a full Gram moment at most $(1 + b^{-2}) \mathcal{M}$ for large nodes. The average of $\mathcal{P} \mathcal{P}^*$ has squared Hilbert–Schmidt norm equal to that moment divided by $|\mathcal{G}|$, by expanding the two group averages and using unitarity. Its compression by the central good projector is A, so its Hilbert–Schmidt norm dominates that of A. The inequality $d \le |\mathcal{G}| d_\lambda \mathfrak{m}_r$ gives the required conclusion. □

The estimate of $\mathcal{E}$ with the displayed leading constant is still required. Its summands retain interference between distinct rational orbits with the same unitary map J.

134.3 Nonzero leading minors at a marked place

Keep $\mathcal{M}$, the shorter interval and the unprojected map $\mathcal{P}$ of Section 134.2. In the residue space at a place above p_0, the two components give the natural action on a vector space of dimension $r = k + 2a + 1$ and its contragredient. The pairings identify the frame in the second component with $n = a + 1$ independent linear forms on the first. Call a residue s open at w if for every $1 \le l \le a$ the pairing matrix of the first l frame forms against the s_w-images of the first l frame vectors has nonzero determinant. Let $\mathcal{O}$ be the set of residues open at some w above p_0. The first a columns are the isotropic arm, and the extra column is the terminal vector. The nonvanishing refers only to the arm.

Lemma 134.5 (A weighted count of singular minors)**.** *Let a finite group H act on $\mathbb{F}_q^r$ by a representation π. Suppose $\beta \ge 0$ on H and $\langle u_h, u_j\rangle = \beta(h^{-1}j)$ for vectors in a complex Hilbert space. Write $\mu = \sum_{h\in H}\beta(h)$. Take independent l-tuples of vectors and forms, given by embeddings $B : \mathbb{F}_q^l \longrightarrow \mathbb{F}_q^r$ and $C : \mathbb{F}_q^l \longrightarrow (\mathbb{F}_q^r)^*$. Suppose that each stabilizer of a nonzero Bv under π, or of a nonzero Cy under the contragredient, has β-mass at most Y. Then*

$$q \sum_{\det(C^t\pi(h)B)=0} \beta(h) \le 2\mu + q^{l+r/2}Y.$$

The columns of C here are written in dual coordinates.

Proof. Take a nontrivial additive character ψ of $\mathbb{F}_q$. In the counting space on $V = \mathbb{F}_q^r$ use the point masses e_z and the Fourier vectors $c_x(v) = q^{-r/2}\overline{\psi(x(v))}$. Each family is orthonormal. For a nonzero x in the image of C and z in the image of B, form

$$U_x = \sum_{h\in H} c_{\pi(h)^{-t}x} \otimes u_h, \qquad V_z = \sum_{j\in H} e_{\pi(j)z} \otimes u_j.$$

Orthogonality gives $\|V_z\|^2 = |H| \sum_{s:\pi(s)z=z} \beta(s)$ by putting $s = h^{-1}j$. The same computation for the dual action gives $\|U_x\|^2 = |H| \sum_{s:\pi(s)^{-t}x=x} \beta(s)$. With the first argument of the inner product conjugate-linear,

$$\langle U_x, V_z\rangle = |H| q^{-r/2} \sum_{s\in H} \psi(x(\pi(s)z))\beta(s).$$

Cauchy–Schwarz bounds the absolute value of the character sum by $q^{r/2}Y$.

Put $Z_s = C^t\pi(s)B$. The cardinality of its left kernel is $q^{-l}\sum_{y,v}\psi(y^tZ_sv)$, by summing over v first. It is at least q when Z_s is singular. In the weighted sum over s, the pairs with $y = 0$ or $v = 0$ contribute at most 2μ after division by q^l. Each other pair contributes at most $q^{r/2}Y$ before that division. There are at most q^{2l} pairs. This proves the claim. □

A choice Ξ consists of a subset of the f places above p_0 and, at each chosen place, one nonzero vector in the span of the full frame in either component. Let $t = |\Xi|$ denote the number of chosen places and Γ_Ξ the subgroup fixing these vectors. Fixing a vector in the second component means fixing its corresponding form by the contragredient in the first. Define

$$\beta(s) = \|\mathcal{P}^*T(s)\mathcal{P}\|_{\mathrm{HS}}^2, \quad \mu(H) = \sum_{s\in H}\beta(s), \quad B_0 = b^{6k}.$$

For a choice Ξ, write $\mathcal{E}_\Xi$ for the sum in (134.6) with residue range replaced by

$$s \in \Gamma_\Xi \cap \mathcal{O}, \qquad z(s) < b^{f+6t}.$$

It uses the signed matrix integrals and exactly the same partition by full link within each block as in that equation.

Theorem 134.6 (Restriction to residues with nonzero leading minors)**.** *Uniformly for sufficiently large nodes on the shorter interval,*

$$(1-2b^{-4})^2\mathcal{M} \le \mu(\mathcal{G}), \tag{134.7}$$

$$(1-3a/p_0)\,\mu(\mathcal{G}) \le \left(\max_{\Xi:\,|\Xi|<f/4} B_0^{|\Xi|/2}\sqrt{\mathcal{E}_\Xi} + b^{-kf/10}\sqrt{\mathcal{M}}\right)^2. \tag{134.8}$$

Every choice with $t \ge f/4$ satisfies $B_0^t\mu(\Gamma_\Xi) \le b^{-kf/5}\mathcal{M}$. In particular, on the right of (134.8) it suffices to prove $B_0^t\mathcal{E}_\Xi \le (1+\frac{1}{2}b^{-2})\mathcal{M}$ for $t < f/4$ to obtain the full-source moment estimate in Theorem 134.2.

Proof. For $\mathcal{P}\mathcal{P}^* = S$, the positive operators $u_h = T(h)ST(h)^*$ are Hilbert–Schmidt. Cyclicity of trace gives $\langle u_h, u_j\rangle_{\mathrm{HS}} = \beta(h^{-1}j)$. This proves the positivity hypothesis of Lemma 134.5 on every subgroup. The average of the u_h has squared norm $\mu(\mathcal{G})/|\mathcal{G}|$. Its compression by the central good projector is A of Theorem 134.2. The trace of A is at least $(1-2b^{-4})d_X$ by the near-isometry proved there. Since A acts on a space of dimension d, its squared norm is at least $(1-2b^{-4})^2 d_X^2/d$. The mass upper bound for d in Proposition 134.1 now gives (134.7).

We first track the error estimates under restriction to Γ_Ξ. Write $Z = b^{f+6t}$. In units of $\mathcal{M}$ the term from D_{op} in the squared majorant of Proposition 133.19 on $z(s) \ge Z$, multiplied by B_0^t, is at most

$$b^{o(1)kf}\left(2b^{6kt+3kf/4}Z^{-k} + 4b^{6kt+kf}(2/M)^{kf}\right) \le b^{(-1/4+o(1))kf}.$$

We used $t \le f$, $M \ge b^{20}$, and Lemma 134.3. The parabolic, closed, complementary open and cutoff errors in the proof of Theorem 134.4 have combined norm $O(b^{-5kf})\sqrt{\mathcal{M}}$ before the weight. Their norm after multiplication by $B_0^{t/2}$ is $O(b^{-2kf})\sqrt{\mathcal{M}}$. The bounds persist upon restriction to any subset of residues. In comparing the cut sums with their refinement by J, the absolute cross terms between different links after radial projection were bounded by $b^{(-19+o(1))kf}\mathcal{M}$. Restriction and multiplication by B_0^t leave at most $b^{(-13+o(1))kf}\mathcal{M}$. The errors from the radial projections in both counting spaces remain within the cutoff bound. Applying the same square-root inequality as in Theorem 134.4 therefore gives

$$\left|\sqrt{B_0^t\,\mu(\Gamma_\Xi\cap\mathcal{O})} - \sqrt{B_0^t\,\mathcal{E}_\Xi}\right| \le b^{-kf/10}\sqrt{\mathcal{M}}.$$

Only estimates of absolute errors were restricted here. No positivity assertion about a signed orbital integral is needed.

If $t \ge f/4$, every $s \in \Gamma_\Xi$ satisfies $z(s) \ge p_0^t$: at each selected component the fixed vector itself witnesses a nonzero intersection of the two frames. Apply the same bound to all these residues using $Z = p_0^t$. Since $p_0 \ge b^{10}$, the exponent $6kt + 3kf/4 - 10kt$ is at most $-kf/4$. The terms independent of Z obey the preceding estimates. After taking square roots and adding the other orbital terms we obtain $B_0^{t/2}\sqrt{\mu(\Gamma_\Xi)} \le b^{-kf/10}\sqrt{\mathcal{M}}$, as claimed.

Choose Ξ to maximize $B_0^t\mu(\Gamma_\Xi)$ among all choices. They form a finite set, including the empty choice of value $\mu(\mathcal{G})$. The lower bound (134.7) and the result for $t \ge f/4$ force $t < f/4$ at every maximizer. Take an unselected place w above p_0. For every nonzero vector in either frame span at w, adding its stabilizer to the restrictions gives mass at most $\mu(\Gamma_\Xi)/B_0$, by maximality.

Apply Lemma 134.5 on Γ_Ξ to the first l vectors and forms at w, for $1 \le l \le a$. For all such l,

$$p_0^{l+r/2} \le b^{11(a+r/2)} \le b^{6k} = B_0,$$

since $p_0 \le b^{11}$, $r = k + 2a + 1$, and $a = o(k)$. Each singular-minor locus therefore carries at most $3\mu(\Gamma_\Xi)/p_0$. A residue outside $\mathcal{O}$ has at least one of these minors zero at this particular w. Summing bounds its mass by $3a\mu(\Gamma_\Xi)/p_0$. Multiplying the bound for its complement by B_0^t, using maximality, and applying the weighted norm comparison proves (134.8). Finally $3a/p_0 \le b^{-9}$ eventually, and $b^{-kf/10} = o(b^{-2})$. The asserted sufficient estimate thus gives $\mu(\mathcal{G}) \le (1 + b^{-2})\mathcal{M}$, which implies the stated estimate for A by compression. □

For every surviving block there is a place where all these minors are nonzero. The corresponding rational leading minors are consequently nonzero in E: reduction of their integral matrix gives the displayed residue pairing since the two class representatives belong to the middle group and fix the frame. This supplies the domain on which successive elimination of the frame coordinates is possible. The value of the signed sum on that domain still enters $\mathcal{E}_\Xi$.

134.4 Factorization of a signed sum with a fixed link

On the rational space write $Y = L \oplus W \oplus L^{\vee}$, where L has ordered basis $e_1, \dots, e_a$, $L^{\vee}$ has dual isotropic basis $e_i^{\vee}$ with $h(e_i, e_j^{\vee}) = \delta_{ij}$, and $W = X_0 \oplus Ez$ with $h(z,z) = -N$. The Hermitian form is linear in its second argument. The group U is the unipotent radical of the stabilizer of the complete flag in L. Extend each $\sigma \in \mathrm{U}(W)(F)$ by the identity on the arms. For $\mathbf{p} = (p_1, \dots, p_a) \in (E^{\times})^a$, define $w_{\mathbf{p}} e_i = p_i e_i^{\vee}$, $w_{\mathbf{p}} e_i^{\vee} = \bar{p}_i^{-1} e_i$ and $w_{\mathbf{p}}|_W = 1$. Write $D_j(g)$ for the leading minors of the pairings of the frame $(e_1, \dots, e_a, z)$ with its image under g. Put $D_0 = 1$.

Lemma 134.7 (The unipotent coordinates and the middle isometry)**.** *If $D_1(g), \dots, D_a(g) \neq 0$, there is a unique expression*

$$g = u\, w_{\mathbf{p}} \sigma v^{-1}, \qquad u, v \in U(F), \qquad \sigma \in \mathrm{U}(W)(F).$$

One has $p_i = D_i(g)/D_{i-1}(g)$, and the full link of g determines σ and is determined by σ. Write $\sigma z = \omega + \alpha z$, with $\omega \in X_0$, and $t = h(z, \sigma z)$. Then

$$D_n(g) = D_a(g) t, \qquad \Delta_g = D_a(g)\overline{D_a(g)}(t\bar{t} - N^2), \qquad N h(\omega, \omega) = t\bar{t} - N^2. \tag{134.9}$$

For $\Delta_g \neq 0$, one has $|\Phi_v(t)| > N$ at every real place. Membership in $T_>$ then depends only on t.

Proof. The ordinary upper unitriangular group on L lifts into U by its inverse adjoint on the dual arm. Acting on the left and right thus permits lower unitriangular row operations and upper unitriangular column operations on the a-by-a cross Gram matrix. Successive elimination makes it diagonal: the first pivot is its upper left entry, and after clearing its row and column the leading minors of the smaller block are the next leading minors divided by that pivot. Induction gives the claimed ratios. All leading minors are invariant under these operations. More generally they are invariant under left and right U, including the minor with the last column z, because U acts on the entire ordered frame by upper unitriangular matrices.

After diagonalization set $q_i = g e_i / p_i$. Then $h(e_j, q_i) = \delta_{ji}$ and $h(q_i, q_j) = 0$. There is an element of U taking $e_i^{\vee}$ to q_i. Indeed fix the e_i and send $y \in W$ to $y - \sum_i e_i h(q_i, y)$. These images have their original pairings with one another and with L, and are orthogonal to the q_i. The asserted map therefore preserves the whole Gram matrix and induces the identity on each graded block of the parabolic. Inverting it on the left now gives $g e_i = p_i e_i^{\vee}$. Consequently $w_{\mathbf{p}}^{-1} g$ fixes L pointwise and induces an isometry σ on $L^{\perp}/L = W$. After removing σ it lies in U, which proves existence.

To identify the middle isometry intrinsically, consider the relation between the two images of y and $g^{-1}y$ in $L^{\perp}/L$, for $y \in (L + gL)^{\perp}$. Multiplication by U on either side preserves this relation: change y by the left factor and use that both unipotents act trivially on the middle quotient. For $g = w_{\mathbf{p}}\sigma$ the relation consists of the pairs $(y, \sigma^{-1}y)$, $y \in W$. Interchanging the last scalar coordinates between the two copies of W is exactly the passage from this relation to the graph of the full link in Section 131.7. It is an invertible coordinate permutation, so equality of links is equivalent to equality of σ. For fixed $\mathbf{p}, \sigma$, uniqueness of the two unipotents follows from $U \cap (w_{\mathbf{p}}\sigma) U (w_{\mathbf{p}}\sigma)^{-1} = 1$. In fact σ normalizes U, and interchanging both arms takes the strictly upper block group to the strictly lower one when the dual arm is ordered in reverse.

Calculation at $w_{\mathbf{p}}\sigma$ gives $D_n = D_a t$ and $t = -N\alpha$. Isometry gives $h(\omega, \omega) - N\alpha\bar{\alpha} = -N$, which proves (134.9). Nonzero ω has strictly positive norm at every real place. Finally $P_n =$

$\prod_v |\Phi_v(D_n)|$ in Corollary 131.38, so that

$$\frac{\sqrt{\mathrm{N}_{F/\mathbb{Q}}\Delta_g}}{P_n} = \prod_v \left(1 - \frac{N^2}{|\Phi_v(t)|^2}\right)^{1/2}.$$

The right side, and hence its comparison with $(M/b)^{af}/N^{f/2}$, is independent of the p_i. □

Let J_w denote the local level at a finite place w of F, and take $k_{s,w}$ to be the component of a lift of the marked residue s. For a class c put $U_{c,w} = U(F_w) \cap h_{c,w} J_w h_{c,w}^{-1}$. Use Haar measure $du_{c,w}$ on $U(F_w)$ giving $U_{c,w}$ volume one. Write $\chi_\infty \prod_{w<\infty} \chi_w$ for the adelic Bessel character on U, trivial on $U(F)$. With $g = w_{\mathbf{p}}\sigma$, define

$$K_w(c, s, c'; g) = \int_{U(F_w)^2} \mathbf{1}_{h_{c,w} k_{s,w} J_w h_{c',w}^{-1}}(ugv^{-1}) \overline{\chi_w(u)}\, \chi_w(v)\, du_{c,w}\, dv_{c',w},$$
$$I_v(g) = \int_{G(F_v)} F_v^\flat(y) F_v^\flat(g^{-1}y)^*\, dy \qquad (v \mid \infty).$$

The cut functions $F_v^\flat$, their product and the real Haar measures are those of Proposition 133.6. The finite expression uses counting measure on left cosets of the two local level groups. It can have complex values.

Theorem 134.8 (Local factors for an actual open block)**.** *Fix (c, s, c') occurring in $\mathcal{E}_\Xi$ of Theorem 134.6. A link J_* occurring in that block has a unique middle isometry σ. Its signed sum in* (134.6) *equals*

$$\sum_{\gamma:\, J_\gamma = J_*} V_\gamma = \frac{1}{v_U} \sum_{\mathbf{p} \in \mathcal{Q}^a} \left(\prod_{w<\infty} K_w(c, s, c'; w_{\mathbf{p}}\sigma)\right) \bigotimes_{v\mid\infty} I_v(w_{\mathbf{p}}\sigma). \tag{134.10}$$

Here $\mathcal{Q} \subset E^\times$ is a finite set common to the blocks at the node. Each finite-place integral is a finite sum, and almost all factors equal one. Formula (134.10) *applies when t has the range in Lemma 134.7 defining $T_>$; the sum restricted to $T_>$ is zero for any other link. Both class representatives and the original residue enter the local factors.*

Proof. Every term over a residue in $\mathcal{O}$ has the required nonzero leading minors by Theorem 134.6. We first record the finiteness facts for the factorization. The construction in Lemma 134.7 gives inverse coordinates on the open set of matrices with all these minors invertible. Gaussian elimination divides only by nonzero pivots. The remaining changes of basis use the fixed nonsingular middle form, isometric inverse, and the pivots and their conjugates. In particular the inverse coordinates are continuous over each completion on that open set, also in a split algebra by applying the same operations in its two paired components. Where the fixed form and bases are unimodular and the pivots are local units, integral input with integral inverse gives integral unipotent factors and integral middle isometry with integral inverse. This follows step by step: the row and column additions, the vectors q_i, the correcting map on W, and the normalized Weyl map are all integral there, as are their inverses by the unitriangular or isometry formulas.

For fixed $g = w_{\mathbf{p}}\sigma$ its intersection $U(F_w)gU(F_w)$ with any compact set has relatively compact unipotent coordinates. Indeed the leading minors of orders $1, \dots, a$ are fixed nonzero constants on this orbit. The closed locus fixing those minors in the compact set is contained in the domain of the continuous inverse just described. Its image is compact. Applied to the compact finite-place coset in the definition of K_w, this bounds both local unipotents. The condition in that integral is unchanged by left $U_{c,w}$ on u and left $U_{c',w}$ on v. The local characters on

these subgroups are trivial by the level and conductor computation in Proposition 130.7. Thus K_w is a sum over a finite set of pairs of cosets, each of measure one. At all but finitely many places the level is standard, the class representatives are integral, g and its inverse are integral and the pivots are units. The preceding integral coordinate calculation gives exactly the two integral unipotent subgroups as the preimage of the integral level. This gives $K_w = 1$ there.

At a real place the cut functions are supported on $U(F_v)C_v$ for some compact C_v. In fact their length bound is a bound on the infimum of $1 + \|hy\|$ over the Bessel group, whose quotient by $U(F_v)$ is compact. Bounded matrices in the unitary group lie in a compact set because their inverses are obtained by adjunction with the fixed form. If both real factors at y contribute, write $y = uc_1$, $g^{-1}y = vc_2$, with $c_i \in C_v$. Then $u^{-1}gv = c_1c_2^{-1}$. The compact-set argument of the preceding paragraph bounds u, v, so y lies in a compact set as well. The local cut functions are bounded there. The real integral converges absolutely and the product integral factors by Fubini.

At the node only finitely many pivot tuples can contribute to (134.10). For a contributing term all D_j are integral, since after the two class changes the isometry preserves the integral lattice, the frame is integral, and the class changes fix it. At each embedding the D_j are bounded by their maxima on $C_vC_v^{-1}$, by the same invariance under U. These are finitely many fixed bounds at the node. The image of $\mathcal{O}_E$ in its Minkowski space is a lattice. Thus we may take a finite set $\mathcal{D}$ comprising 1 and all nonzero integers of E satisfying the bounds for at least one of these minor orders at every embedding (enlarging to the maximum bound for all orders is harmless). Take $\mathcal{Q} = \{d/d' : d, d' \in \mathcal{D}\}$. Any pivot of an orbit contributing to the finite and real integrals then lies in $\mathcal{Q}$.

It remains to unfold with the phases and measures. Write $U_c = \prod_{w<\infty} U_{c,w}$ and $\Gamma_c = U(F) \cap U_c$. The diagonal rational unipotent group is dense in $U(\mathbb{A}_{F,f})$: exponential and logarithm are inverse polynomial maps with rational coefficients, integral outside finitely many primes, and additive rational points are dense in the finite adeles by approximation. Consequently $\Gamma_c \backslash U(F) \longrightarrow U_c \backslash U(\mathbb{A}_{F,f})$ is a bijection. Every left coset of the compact open subgroup is open and contains a rational point. Two rational representatives differ on the left by an element of U_c precisely when they differ by an element of Γ_c. The same assertions apply to c'.

By uniqueness in Lemma 134.7, the rational terms of this double coset are parametrized without multiplicity by the pairs $(u, v) \in U(F)^2$. Fix their two left level cosets. For $\lambda \in \Gamma_c, \lambda' \in \Gamma_{c'}$, the term is $\lambda u g v^{-1} \lambda'^{-1}$. In the integral with the two class weights of Section 133.1 substitute $y = \lambda u y_0$. The second argument becomes $\lambda' v g^{-1} y_0$. Unipotent equivariance of the cut function and triviality on the two rational level subgroups leave the scalar $\chi_\infty(u)\overline{\chi_\infty(v)}$. Summing the two class weights independently gives one in each variable, by their partition property. This unfolding is valid in absolute value as well: the weights are nonnegative, and after summing them the integrand is bounded by the product of the two cut norms on the compact overlap just proved. Thus the contribution of the fixed level cosets is the scalar times $v_U^{-1} \bigotimes_v I_v(g)$.

Finally $\chi_\infty(u) = \prod_w \overline{\chi_w(u)}$ for rational u, and the conjugate relation holds for v. The two coset bijections identify the admissible pairs exactly with the local condition $ugv^{-1} \in h_{c,w} k_{s,w} J_w h_{c',w}^{-1}$ at every w. Outside a finite set there is just the integral pair of cosets, so their weighted count is the product of the K_w. Summing over the pivots gives the claimed formula. This regrouping agrees with the cut integrals V_γ because Proposition 133.6 unfolds the same sum of terms against the same two class weights. The full link and the range $T_>$ are invariant under every unipotent move, as shown in Lemma 134.7. □

In the energy $\mathcal{E}_\Xi$ the squared norm is taken after summation over $\mathbf{p}$. Thus the finite character sums with different pivots occur together with the corresponding real matrix integrals. An

estimate of this quadratic expression with the leading constant in Theorem 134.6 remains necessary for this argument.

134.5 The class sum on the terminal norm sphere

Let $H = \mathrm{U}(X_0)$ and $H_W = \mathrm{U}(W)$ in the notation of Section 134.4. Thus H is the stabilizer of z in H_W. Put $Q(x) = h(x, x)$ on X_0. Fix a choice Ξ of marked vectors and write $\Omega_\Xi = \{s \in \Gamma_\Xi \cap \mathcal{O} : z(s) < b^{f+6|\Xi|}\}$. We compute the sum over the two definite classes and the middle isometry in $\mathcal{E}_\Xi$. The level in $H(\mathbb{A}_{F,f})$ is $K_H = H(\mathbb{A}_{F,f}) \cap J$. In integrals over that group give K_H volume one.

Let $\mathcal{Q}$ be the finite set of pivot values constructed in Theorem 134.8. It also contains $D_n/D_a = t$ for any contributing term. Keep the $t \in \mathcal{Q}$ for which

$$\beta_t = (t\bar{t} - N^2)/N$$

is totally positive, is represented by Q over F, and satisfies the $T_>$ condition of Lemma 134.7 through its value of t. Denote this finite set by $\mathcal{T}$. For each t choose one rational $\sigma_t \in H_W(F)$ with $h(z, \sigma_t z) = t$. To obtain one, take $Q(\omega) = \beta_t$ and extend the isometry of nondegenerate lines $z \longmapsto \omega - (t/N)z$ by Witt's extension theorem. For each $\mathbf{p} \in \mathcal{Q}^a$ use the matrix

$$A_{\mathbf{p}}(t) = \bigotimes_{v \mid \infty} I_v(w_{\mathbf{p}} \sigma_t)$$

in the Hilbert–Schmidt space of the coefficient representation.

For $x \in H_W(\mathbb{A}_{F,f})$ write $K(s; w_{\mathbf{p}} x)$ for the product of the finite integrals in Section 134.4 with both class representatives equal to one, using the standard U-level for both measures. Define the nonnegative function

$$B_{\Xi,t}(x) = \sum_{s \in \Omega_\Xi} \left\| \sum_{\mathbf{p} \in \mathcal{Q}^a} K(s; w_{\mathbf{p}} x) A_{\mathbf{p}}(t) \right\|_{\mathrm{HS}}^2 .$$

On the finite adelic sphere $Q(\omega) = \beta_t$, set

$$\Psi_{\Xi,t}(\omega) = \int_{H(\mathbb{A}_{F,f})} B_{\Xi,t}(xh)\, dh, \qquad xz = \omega - (t/N)z. \tag{134.11}$$

The integral uses one fiber of the map $x \mapsto xz$. Two choices of x differ on the right by $H(\mathbb{A}_{F,f})$, so its value is independent of that choice.

Proposition 134.9 (Unfolding the right definite class)**.** *The test $\Psi_{\Xi,t}$ is nonnegative, has compact support on the finite adelic sphere, and is invariant under K_H. The identity for the actual energy is*

$$v_U^2 \mathcal{E}_\Xi = \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \sum_{\substack{\omega \in X_0(F) \\ Q(\omega) = \beta_t}} \Psi_{\Xi,t}(h_c^{-1} \omega). \tag{134.12}$$

Proof. If $\sigma, \sigma' \in H_W(F_v)$, $v \mid \infty$, have the same last entry t, the X_0-parts of σz and $\sigma' z$ have the same positive length by (134.9). An element $h_1 \in H(F_v)$ takes the first part to the second. Then $(\sigma')^{-1} h_1 \sigma$ fixes z. Thus $\sigma' = h_1 \sigma h_2^{-1}$ for some $h_2 \in H(F_v)$. The arm interchange $w_{\mathbf{p}}$ commutes with H. Changing the real integration variable by h_1 shows that all the matrices $I_v(w_{\mathbf{p}}\sigma)$ are carried to $I_v(w_{\mathbf{p}}\sigma')$ by the same two unitary factors on the coefficient spaces, by the left equivariance of $F_v^\flat$. Consequently any linear combination of the real tensor matrices has the same squared norm when σ is replaced by σ_t, with its finite coefficients held fixed.

In the finite integrals conjugate the two unipotent variables by h_c^{-1} and $h_{c'}^{-1}$, respectively. These middle elements preserve the Bessel character. The chosen Haar measures go to the standard ones since the two normalizing open subgroups go to $U \cap J$. The resulting factor is

$K(s; w_{\mathbf{p}} h_c^{-1}\sigma h_{c'})$. This expression as a function of $h_c^{-1}\sigma h_{c'}$ is invariant on both sides by K_H. Commuting a level element across the appropriate unipotent variable preserves its character and its normalizing subgroup, and $jk_sJ = k_sJ = k_sJj$ for $j \in J$. Residues with disagreeing characters give zero even for a fixed pivot tuple. Indeed, if u_0, v_0 are integral finite unipotents with $u_0 k_s J v_0^{-1} = k_s J$, replacing the integration variables by $u_0 u, v_0 v$ preserves the coset condition and multiplies the character factor by $\overline{\chi_f(u_0)}\chi_f(v_0)$. Unequal character values therefore force the integral to vanish.

For fixed pivots, nonzero finite integral forces the middle coordinate into a compact set at every place by the inverse coordinate calculation of Theorem 134.8. At almost all places this set is the standard integral middle group. On a compact neighborhood of a middle element the unipotent variables in the integral can be confined to a common compact set, by the same inverse coordinate argument applied to the integral coset and fixed pivots. The characteristic function of that coset is locally constant. Its integral is therefore a locally constant function there, as one also sees by covering the compact integration set by finitely many cosets. At all but finitely many places it is just the indicator of the standard integral middle group. Hence $B_{\Xi,t}$ has compact support and is locally constant. Its fiber integral in (134.11) is measurable, supported in the image of that compact set, and invariant by K_H on ω. The fibers over the stated adelic sphere exist. Locally this is Witt extension. At almost every finite place the prescribed vector and z are integral of the same unit norm and an integral isometry suffices. Over an unramified quadratic valuation ring each such line splits off a unimodular complement. To see that unimodular Hermitian lattices of the same rank there are isometric, choose a vector of unit norm by reduction, rescale by surjectivity of the norm on units, split off its line and repeat on the complement. At a split place a primitive vector and its paired covector with unit pairing likewise split off a direct summand and its dual. These arguments give an integral extension at every place required for the restricted product.

Now insert (134.10) and group by $\sigma z = \omega - (t/N)z$. For fixed rational ω, t all choices of σ are $\sigma_0 h$, $h \in H(F)$, for any one choice σ_0; every such choice gives one link. The pairs (h, c') give each right coset $hh_{c'}K_H$ in $H(\mathbb{A}_{F,f})$ exactly once. Existence is the definition of the definite classes, and uniqueness uses $H(F) \cap h_{c'} K_H h_{c'}^{-1} = 1$ as in Proposition 131.25. Each coset has volume one. Their sum is exactly the integral in (134.11) with $x = h_c^{-1}\sigma_0$. Terms with no admissible finite representative give zero. The resulting sums and integrals are of nonnegative squared norms, so the regrouping is legitimate. This proves (134.12). □

Normalize additive measure at a finite place w by giving $\mathcal{O}_{F_w}$ and $\mathcal{O}_{E,w}^k$ volume one. On the smooth fiber $Q(\omega) = \beta \neq 0$, let $\nu_{\beta,w}$ be the measure $d\omega/dQ$ in these additive measures. Equivalently, one can integrate a locally constant compactly supported test over $Q \in \beta + \mathfrak{p}_w^j$, divide by the measure of $\mathfrak{p}_w^j$ and pass to the limit. In local coordinates with Q as one coordinate this formula is the ordinary change of variables.

Choose a finite set S of places of F containing both marks, the divisors of N, and the places where any of the pivots or entries in $\mathcal{Q}$ is a nonunit. Include the finitely many exceptions to the standard integral bases and levels of Theorem 134.8. On the sphere, write $\Psi_{\Xi,t,S}$ for (134.11) using only the places of S in its finite integrals, product measures and fiber. Write $\delta_w(\beta) = \nu_{\beta,w}(\mathcal{O}_{E,w}^k)$ and $c_E = 2^{-f}\sqrt{|\operatorname{disc} E|}$.

Theorem 134.10 (The mean on a prescribed terminal sphere)**.** *For every Ξ one has*

$$v_U^2 \mathcal{E}_\Xi = |\mathcal{C}| \, \frac{\sqrt{|\operatorname{disc} F|}}{c_E^k} \left(\frac{\pi^k}{(k-1)!\, N^{k-1}} \right)^f \sum_{t \in \mathcal{T}} \prod_{v \mid \infty} \left(|\Phi_v(t)|^2 - N^2 \right)^{k-1} \mathcal{D}_{\Xi,t}, \tag{134.13}$$

$$\mathcal{D}_{\Xi,t} = \left(\prod_{w \notin S} \delta_w(\beta_t) \right) \int_{Q(\omega_S) = \beta_t} \Psi_{\Xi,t,S}(\omega_S) \prod_{w \in S} d\nu_{\beta_t, w}.$$

The product over $w \notin S$ differs from one by $O(f 2^{1-k})$, uniformly in t, S, Ξ. In particular its error is $o(b^{-6})$ on the same shorter interval.

Proof. For $w \notin S$ the local factor in each term of $B_{\Xi,t}$ is the indicator of the integral middle group, independent of $\mathbf{p}$. It is real and idempotent. Its fiber integral equals one for integral ω_w and zero otherwise. Indeed $\omega_w - (t/N)z$ is integral exactly then, and has the unit norm $-N$. The integral extension of the isometry of lines proved above gives an integral point of the fiber. The rest of the integral points form one coset of the integral H-level, of volume one. Hence outside S the test on the sphere is exactly the product of the standard lattice indicators. This also shows β_t is integral there, by its formula and the choice of S.

We give the mean formula with its measure constant. Fix $t \in \mathcal{T}$, put $\beta = \beta_t$, choose $x_0 \in X_0(F)$ of norm β, and put $H_0 = \mathrm{U}(x_0^\perp)$. All rational vectors of this norm are in the orbit of x_0 by Witt extension. Integrate $\sum_{Q(x)=\beta} \Psi(h_f^{-1} x)$ over $H(F) \backslash H(\mathbb{A}_F)$ with probability Haar measure, with $\Psi = \Psi_{\Xi,t}$. Its integral is the class sum divided by $|\mathcal{C}|$. The integrand is invariant under the full compact group at infinity and K_H at finite places, and the finite-level stabilizers are trivial, so that each class has equal measure. Unfolding the rational orbit gives the integral of $\Psi(h_f^{-1} x_0)$ on $H_0(\mathbb{A}_F) \backslash H(\mathbb{A}_F)$ in quotient Tamagawa measure, multiplied by $\tau(H_0)/\tau(H)$. One sees the factor by first taking the quotient by $H_0(F)$ and then integrating the stabilizer fiber of volume $\tau(H_0)$. Both unitary groups have Tamagawa number two, as both Hermitian dimensions $k-1, k$ are positive and both forms are definite at every real place. See Gan, Hanke and Yu, Section 4, equation (4.3).

To identify the quotient measure, over F take the top differential on the smooth sphere obtained by dividing a constant volume form on X_0 by dQ. It is H-invariant because the determinant of a unitary isometry has E/F-norm one. The quotient of invariant rational top forms on H and H_0 is proportional to this form by a scalar in $F^\times$, since the tangent quotient at x_0 is the tangent space of the sphere and the action is transitive. The absolute values of the scalar have product one. In the Tamagawa normalization the character-module convergence factors of H and H_0, including the global leading L-value, cancel. Indeed over an algebraic closure these groups are general linear groups, whose characters are generated by determinant, the restriction of determinant is determinant, and conjugation acts on that generator by inversion in both cases. The discriminant normalization for an F-group of dimension d' is $|\operatorname{disc} F|^{-d'/2}$ when rational-coordinate finite measures give the base integer rings volume one. The dimension difference here is $2k-1$. Hence the quotient is exactly the fiber measure obtained by disintegrating the globally normalized additive measures on E^k and F by Q. This follows also place by place from the quotient of top forms, since their underlying additive dimensions differ by $2k-1$. Local Witt extension and the integral extension at almost all places identify the full adelic sphere with the quotient being integrated.

For our finite conventions and ordinary Lebesgue measures on $\mathbb{C}^f$ and $\mathbb{R}^f$, the covolumes of the rational additive lattices in their adeles are respectively c_E and $\sqrt{|\operatorname{disc} F|}$. This follows by taking the product of the finite integer rings and a fundamental parallelotope of the Minkowski lattice. Thus disintegration of the additive measures of covolume one multiplies the product

of local fiber measures by $\sqrt{|\operatorname{disc} F|}/c_E^k$. At an infinite place the total fiber measure at $\beta_v > 0$ is $\pi^k \beta_v^{k-1}/(k-1)!$, obtained by differentiating the real $2k$-dimensional volume $\pi^k u^k/k!$ of $\{\|x\|^2 \le u\}$ at $u = \beta_v$. Use $\beta_v = (|\Phi_v(t)|^2 - N^2)/N$ and the factorization outside S. Inserting the resulting mean for every t in (134.12) yields (134.13).

It remains to estimate the standard local densities. At $w \notin S$ put $q = |\mathcal{O}_{F_w}/\mathfrak{p}_w|$ and $\eta = 1$ in the split case, $\eta = -1$ in the unramified field case. For a nontrivial additive character ψ of $\mathbb{F}_q$ and any $d \neq 0$, the sum of $\psi(d\,x\bar{x})$ over the quadratic residue algebra is ηq. In the split case sum first over the second coordinate in $x\bar{x} = uv$. In the field case every nonzero residue norm has $q+1$ preimages and zero has one. Character orthogonality therefore gives

$$\#\{x : Q(x) = c \bmod \mathfrak{p}_w\} = q^{2k-1} + \eta^k q^{k-1} \begin{cases} -1, & c \neq 0, \\ q-1, & c = 0. \end{cases}$$

At each nonzero residue vector the differential of Q is a nonzero linear form over $\mathbb{F}_q$ by the perfect residue trace pairing. Such a solution has precisely q^{2k-1} lifts at each successive digit for any prescribed target lift, including in residue characteristic two. For unit β this proves $\delta_w = 1 - \eta^k q^{-k}$. For positive valuation the primitive solutions contribute $1 + \eta^k (q-1) q^{-k} - q^{1-2k}$. When the valuation is at least two, writing a nonprimitive vector as ϖy adds $q^{2-2k}\delta_w(\beta/\varpi^2)$ by additive scaling. There is no such contribution at valuation one. This recursion, ending at valuation zero or one, gives $|\delta_w(\beta) - 1| \le 4q^{1-k}$ for $k \ge 3$ by summing the geometric series with ratio q^{2-2k}. There are at most f places over any rational prime, hence $\sum_{w \notin S} q_w^{1-k} \ll f 2^{1-k}$ as k tends to infinity. The product converges absolutely and has the asserted error, since $f \ll_{q_0, d_K} x^{12}$, $k \asymp x^5$, and $\log b \le x^{1/100}$. $\square$

The integral in $\mathcal{D}_{\Xi,t}$ still contains the correlations between the finite character sums for different pivots. The remaining upper estimate concerns this finite-place integral together with the matrices $A_{\mathbf{p}}(t)$ at the infinite places.

134.6 Simultaneous pivot congruences at the two original marks

Let $\mathcal{W}$ consist of the $2f$ places of F above the two split primes p_0, p_1. At $w \in \mathcal{W}$ the residue algebra of E has two components, each $\mathbb{F}_{q_w}$, where $q_w = p_i$ if $w \mid p_i$. In the first component the residue group acts on its natural space of dimension r. On the other it acts by the contragredient under the perfect pairing. Say that s has *unit pivots at* w if the leading frame Gram minors of orders $1, \dots, a$ are nonzero in both components. This condition can be tested on a lift to the integral lattice group and depends only on its residue.

Permit a choice Ξ as in Section 134.3 to use places from either of the two primes, with one nonzero vector in either frame component at each chosen place. Let $t = |\Xi|$. Define

$$\Omega_{\Xi}^{\sharp} = \{s \in \Gamma_{\Xi} : \ z(s) < b^{f+6t}, \ s \text{ has unit pivots at every } w \notin \Xi \text{ in } \mathcal{W}\}.$$

For $t < f/4$, let $\mathcal{E}_{\Xi}^{\sharp}$ be the signed sum of squares in (134.6) with this residue mask. There is an unselected place over p_0, so every residue in this mask is in $\mathcal{O}$. Both (134.10) and the sphere formula (134.13) hold with this mask. Indeed their proofs fix a set of residues in the open cell and sum the squared norms there. The level K_H, the two class sums and the terminal sphere measure are the same.

Theorem 134.11 (Unit pivots outside the selected places)**.** *Put $B_0 = b^{6k}$ and $q_* = \min(p_0, p_1)$. At all sufficiently large nodes of the shorter interval the actual second moment $\mu(\mathcal{G}) = \sum_s \|\mathcal{P}^* T(s) \mathcal{P}\|_{\mathrm{HS}}^2$ satisfies*

$$(1 - 12af/q_*)\, \mu(\mathcal{G}) \ \le \ \left(\max_{\Xi:\, |\Xi| < f/4} B_0^{|\Xi|/2} \sqrt{\mathcal{E}_{\Xi}^{\sharp}} + b^{-kf/10} \sqrt{\mathcal{M}} \right)^2 .$$

In this bound $12af/q_* \le b^{-9}$ for all sufficiently large nodes. Every maximizing choice for $B_0^{|\Xi|}\mu(\Gamma_{\Xi})$ among choices at both marks has fewer than $f/4$ selected places.

Proof. By (134.4), $M \ge b^{20}$, $d_B \ge 1$, and $(k-1)! \ge 1$, the identity scale satisfies the sharper lower bound

$$\mathcal{M}/D_{\mathrm{id}} \ge b^{(25/2 - o(1))kf}.$$

This margin allows the stabilizer weights at up to $2f$ places. In the argument of Theorem 134.4, the combined parabolic, closed, complementary open and cutoff norms are bounded by

$$\big(\sqrt{|\mu(E)|} + O(\tau^{-20} + \tau^{-3a/2})\big)\sqrt{D_{\mathrm{id}}}.$$

After multiplying by $B_0^{t/2}$ and dividing by $\sqrt{\mathcal{M}}$, the result is at most $b^{(-1/4 + o(1))kf}$, since $|\mu(E)| \le b^{11}$ and $3kt \le 6kf$. The absolute contribution of cross terms with different links after radial projection in that proof was bounded by $b^{(-19+o(1))kf}\mathcal{M}$ in the squared norm. Its weighted bound for $t \le 2f$ is at most $b^{(-7+o(1))kf}\mathcal{M}$. Both statements hold on every residue subset.

Every $s \in \Gamma_{\Xi}$ has $z(s) \ge q_*^t$ since a nonzero fixed frame vector witnesses an intersection at that component. Use Proposition 133.19 with $Z = q_*^t$. In the squared majorant the term containing Z^{-k}, in units of $\mathcal{M}$ and with weight B_0^t, is bounded by $b^{o(1)kf} 2 b^{3kf/4 + 6kt} q_*^{-kt}$. For $t \ge f/4$ its exponent to base b is at most $(-1/4 + o(1))kf$, using $q_* \ge b^{10}$. The other term, $4B_0^t \tau^4 (2/M)^{kf} D_{\mathrm{op}}/\mathcal{M}$, is at most $b^{(-7+o(1))kf}$. Taking square roots and adding the other orbital contributions gives $B_0^t \mu(\Gamma_{\Xi}) \le b^{-kf/5}\mathcal{M}$ for all such choices. The empty choice has value at least $(1 - 2b^{-4})^2 \mathcal{M}$ by (134.7). Consequently every maximizer has $t < f/4$.

Fix a maximizer. For each unused place and each of its two components, the stabilizer of every nonzero frame vector there has mass inside Γ_{Ξ} at most $\mu(\Gamma_{\Xi})/B_0$. Adding that one

restriction to Ξ proves this directly. To test nonvanishing of a minor at the first component apply Lemma 134.5 to the natural representation and the primal and paired dual frame there. At the other component apply it to the contragredient representation, interchanging the two frames. Its dual is the natural representation, so the same stabilizer bound applies on both sides in each case. At both marks $q_w^{l+r/2} \le b^{6k}$ for $1 \le l \le a$, by the numerical bounds in Theorem 134.6. A single singular minor carries at most $3\mu(\Gamma_\Xi)/q_*$. Summing over at most $2f$ places, two components, and a orders loses at most $12af\mu(\Gamma_\Xi)/q_*$. The stated upper bound for this fraction follows from $q_* \ge b^{10}$, $f \ll_{q_0,d_K} x^{12}$, and $\log b \ge (\log x)^6$.

On the residues with unit pivots at all unused places remove $z(s) \ge b^{f+6t}$. In the preceding squared majorant the term depending on $Z = b^{f+6t}$ is at most $b^{(-1/4+o(1))kf}\mathcal{M}$, now for every t. The bound for different links and the cutoff bounds above continue to hold under this restriction. The comparison of square roots proved in Theorem 134.4 thus has error at most $b^{-kf/10}\sqrt{\mathcal{M}}$ in the weighted norm. Apply it to the subset of Γ_Ξ with unit pivots at every unused place. Its weighted squared mass is at least $(1 - 12af/q_*)B_0^t\mu(\Gamma_\Xi)$, before the large-$z$ part is removed, and maximality makes this at least $(1 - 12af/q_*)\mu(\mathcal{G})$. This proves the result. □

Proposition 134.12 (Evaluation of a marked factor with unit pivots)**.** *Use the finite factor of Theorem 134.8 after removing the two class representatives as in Proposition 134.9. Fix an unselected w and $s \in \Omega_\Xi^\sharp$. Over the paired residue algebra factor s uniquely as*

$$s = \bar{u}_s w_{\bar{\mathbf{p}}_s} \bar{x}_s \bar{v}_s^{-1}.$$

The bars in this display mean reduction. Here $\bar{u}_s, \bar{v}_s$ are unipotents, $\bar{x}_s$ is a middle isometry and the entries of $\bar{\mathbf{p}}_s$ are units in both components. The local factor vanishes unless $\mathbf{p}$ consists of local units, x is integral, and

$$\mathrm{red}_w(\mathbf{p}) = \bar{\mathbf{p}}_s, \qquad \mathrm{red}_w(x) = \bar{x}_s.$$

When these conditions hold, the local Haar measures of Section 134.4 give

$$K_w(s; w_{\mathbf{p}} x) = \overline{\chi_w(\bar{u}_s)}\, \chi_w(\bar{v}_s).$$

Here the character on a residue unipotent is evaluated on any integral lift. In particular, let $\mathfrak{L}_\Xi$ be the product of both prime ideals of E over every unused $w \in \mathcal{W}$. If two rational pivot tuples contribute to the same block at these places, then for every coordinate i

$$p_i - p_i' \in \mathfrak{l}\mathcal{O}_{E,\mathfrak{l}} \quad \text{for every } \mathfrak{l} \mid \mathfrak{L}_\Xi.$$

For $|\Xi| < f/4$, $\mathrm{N}\mathfrak{L}_\Xi \ge b^{35f}$. If $p_i \ne p_i'$ and $(p_i - p_i') = \mathfrak{n}/\mathfrak{d}$ in coprime integral ideals, then $\mathfrak{L}_\Xi \mid \mathfrak{n}$ and

$$\prod_{v \mid \infty} |\Phi_v(p_i - p_i')|^2 \;\ge\; b^{35f}/\mathrm{N}\mathfrak{d}.$$

Proof. At w the old level is principal of depth one, the middle basis is integral, and N is a unit. The elimination in Lemma 134.7 works over the product of the two residue fields when each leading minor is a unit. Row operations, the dual isotropic completion, and the upper and lower triangular uniqueness use only inversions of these units and the nonsingular middle Gram matrix. Over the local ring the same elimination shows that an integral lift of s has integral factors with the specified reductions. In particular, if $uw_{\mathbf{p}}xv^{-1}$ reduces to s, the minors force each rational pivot to be a local unit, and uniqueness over the local algebra identifies u, x, v with the factors obtained by integral elimination. Conversely integral factors of the specified reductions give residue s. Each residual unipotent lifts because the radical at a split place is

the block upper unipotent group in one component, with the paired component determined by inverse adjoint. The possible u form one coset of $U(F_w)\cap J_w$, as do the possible v. Each has volume one, and the character is constant on each. This proves the local formula.

The ratios of minors determining $\bar{\mathbf{p}}_s$ are fixed by s. Any two contributing rational tuples therefore have the indicated congruences in both components, also in the original class notation because the class changes fix the frame. Each unused w contributes $q_w^2 \ge b^{20}$ to the absolute ideal norm. There are $2f-|\Xi| > 7f/4$ such places. A nonzero difference is integral at all their primes and its numerator ideal contains each as a factor. Finally the absolute field norm of the difference equals both the product of squared complex absolute values and $\mathrm{N}\mathfrak{n}/\mathrm{N}\mathfrak{d}$. This gives the last inequality. □

Corollary 134.13 (The integrated factor at an unused mark)**.** *For $|\Xi| < f/4$ enlarge the mask $\Omega_{\Xi}^{\sharp}$ by dropping the inequality on $z(s)$, while retaining all the fixed vectors and all the unit pivot conditions. The resulting energy bounds $\mathcal{E}_{\Xi}^{\sharp}$ from above. In the sphere formula for this enlarged energy take an unused $w\in\mathcal{W}$, and write $q=q_w$ and $A_w=\mathcal{O}_E\otimes_{\mathcal{O}_F}\mathcal{O}_{F_w}$. For two pivot tuples $\mathbf{p},\mathbf{p}'$ and terminal entry θ the entire local factor in the expanded quadratic expression is zero unless*

$$\theta\in A_w,\qquad \mathbf{p},\mathbf{p}'\in (A_w^{\times})^{a},\qquad \mathbf{p}\equiv\mathbf{p}' \pmod{\mathfrak{p}_w A_w}.$$

When these conditions hold its value is

$$q^{\,2a(2k+2a+1)}\,|\mathrm{GL}_k(\mathbb{F}_q)|\,\delta_w\big((\theta\bar{\theta}-N^2)/N\big).$$

The density is evaluated by the recursion in Theorem 134.10 with split sign $+1$.

Proof. For a fixed link block the energy is a squared norm. Adding residues therefore gives the stated upper bound and does not alter any sum within a block. The enlarged mask is a product of local conditions. In (134.11) expand the finite squared sum in the two pivot indices and sum the residues at w before integrating the corresponding fiber. This order is legitimate because all pivot and residue sums are finite and each test at a finite place is bounded on compact support.

At an integral middle matrix x the two local integrals can both be nonzero for a given residue only when the pivot tuples have the required unit reductions, by Proposition 134.12. In that case the product of one value with the conjugate of the other is one. Once $\bar{\mathbf{p}}$ and the reduction of x are fixed, the residue s_w is specified uniquely by the two freely chosen unipotents $\bar{u}_s,\bar{v}_s$. The local residue sum is thus $|\bar{U}_w|^2=q^{2a(2k+2a+1)}$ times the indicator of an integral middle matrix. For a nonintegral middle matrix every term is zero.

For fixed ω_w the fiber of interest consists of the middle isometries carrying z to $\omega_w-(\theta/N)z$. Here N is a unit and the middle lattice is the standard orthogonal sum. This fiber has an integral point exactly when θ,ω_w are integral. Indeed the image vector has the same unit norm as z, so one can extend the isometry of integral lines as in Proposition 134.9. All integral points of the fiber then differ on the right by the maximal integral subgroup of $H(F_w)$. Its volume is $|\mathrm{GL}_k(\mathbb{F}_q)|$ because its principal kernel has volume one in the measure of (134.11). Finally the measure of integral ω_w on its prescribed norm sphere is precisely $\delta_w((\theta\bar{\theta}-N^2)/N)$. Multiplication gives the formula. □

The remaining terms now involve integral congruences between the rational pivots at every unused mark. Their denominators at the other places and their real orbital integrals still enter the required leading-constant estimate.

135 References

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