The generalized Riemann hypothesis for Dirichlet L-functions

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Volume XXIV

Marked theta columns and terminal covectors

Abstract

We express the endpoint pairing in terms of signed tests on marked classes. Summing over the intermediate residue classes gives formulas for the composite terminal covectors, their Gram matrices and their integral moments.

The guide in Volume I lists all twenty-seven volumes.

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125 Denominators and norms of the endpoint pairing

125.1 Integral tests for the complex theta period

Keep the matching complex operator Θ of Theorem 88.1, with its given real tests and measures. For the covector b and the definite endpoint test Q, consider the scalar $\langle \Theta S b, Q\rangle$. It is also the ordinary Bessel functional on the reverse lift $\Theta^{*}Q$ in the normalization of Proposition 110.5.

Theorem 125.1 (Two integral tests at the original finite levels)**.** *Choose a good orbit of Theorem 121.5. For each prescribed word on these framed data and each terminal character nontrivial at every place above T, there is a literal integral character column b on that orbit and an integral endpoint coefficient test Q at the original level of the complex iteration such that*

$$|\langle \Theta S b, Q\rangle|^{2} \;\geq\; \alpha e^{-\Delta_x} M_{J,\mathcal{O}}\, c_{\rho,K^{0}}|Z|\, \frac{d_\rho B_s^{-2a_s}}{k^{e_s}}\, \frac{m_e^{2} w_e B_e^{-4a_e}}{d_e h_e N^{2e_e}} > 0.$$

The tests have norms at most $(c_{\rho,K^{0}}|Z|)^{1/2} B_s^{a_s}$ and $m_e\sqrt{w_e} L_e B_e^{2a_e}$, respectively. For a finite list of l such pairs with a common endpoint coefficient space, integral lattice and finite level character, one endpoint test serves all the selected columns with at least half their respective lower bounds on absolute value.

Proof. Take $b = b(\ell)$ from Proposition 112.3 and apply Lemma 122.1 to $F = \Theta S b$. Multiplication gives the displayed inequality. The mean $M_{J,\mathcal{O}}$ is positive on these orbits for large x by Theorem 121.5. The source norm bound follows from the upper dual frame bound and the class norm of the unnormalized character sum. The other norm bound is the one in the endpoint lemma.

For the last assertion call the individual endpoint tests Q_i and the forward lifts F_i. Consider $Q = \sum_{j=1}^{l} n_j Q_j$ for independent integers $0 \leq n_j \leq l$ with uniform distribution. For a fixed i, condition on all coefficients but n_i. The $l+1$ possible pairings are spaced by $|\langle F_i, Q_i\rangle| > 0$. An open disk of half this radius contains at most one. With probability at least $1 - l/(l+1) > 0$ all the lower bounds on absolute value, divided by two, hold at once. Such a Q is in the common integral lattice and has norm at most $l \sum_j \|Q_j\|$. This proves the assertion for the same endpoints and levels. □

The two integrality statements refer to the coefficient models of the input tests. Relating this Petersson scalar to the arithmetic theta kernel and its distinguished prime divisibilities requires the normalization of the real tests, the measures and the integral projection. The bounds above retain these factors explicitly.

125.2 Archimedean and finite bounds for the same last-edge minors

Use the actual last-edge matrix Z of Theorem 122.9. Write $l = ns$, $D = fl$, $l_\tau = |\lambda_\tau|$, and $h_\lambda = \|P_*\|_{\mathrm{F}}^{2}$. Use a single Galois number field J containing the arithmetic data, finite character values, and the finite test lists. The archimedean volume factors and the marked vacuum scalar have already been removed in the definition of Z. At an embedding $\sigma : J \to \mathbb{C}$ let A_σ, ν_σ be the area and the mean squared elliptic theta norm of Theorem 65.9, and $c_\sigma = \pi/A_\sigma$. Put

$$H_l(a) = \frac{(a+l-1)!}{(l-1)!}(e/a)^{a} \quad (a \geq 1), \qquad H_l(0) = 1,$$
$$B_\sigma^{2} = b_\sigma^{2}\left[e \max(1, 2l\ \mathrm{rd}(E)/\pi)\, \nu_\sigma\right]^{D} c_\sigma^{d}\, h_\lambda \prod_{\tau} H_l(l_\tau).$$

The factor b_σ bounds the absolute coefficient sum of each finite test in translates of its normalized lattice vacuum.

Take the fine representatives of Theorem 65.11 away from the marked places. Move the right-level translations from the two single-chart tests to the finite kernel. There are only finitely many such translations. In a split position model a term $a\mathbf{1}_{y+D_v}$, with D_v a full lattice, has coefficient sum at most

$$|\sigma(a/a_0)|[D_v : D_v \cap L_0]$$

in translates and modulations of a reference position vacuum of amplitude a_0 on L_0. Indeed split the coset by $D_v \cap L_0$. In one translate of L_0 its subcoset indicator is the finite Fourier sum on $L_0/(D_v \cap L_0)$, where each character contributes the reciprocal group order times a root of unity. A position or frequency translate of the reference vacuum only changes the phases and the cosets. Multiply the bounds at different places, sum absolute coefficients for a finite combination, and take the maximum over the two level translations. This gives b_σ. For the split indicator of Proposition 72.15 in its unit-volume position lattice the local factor is one. An unchanged unit vacuum also has factor one. Retain the central translate at the additional split place of Proposition 94.17 as a moved vacuum, with its splitting factor. The class transports of Theorem 65.13 preserve the absolute coefficient sum at every embedding. These statements use the same finite kernel in the period and in b_σ.

Let S be any finite set of the good primes of Theorem 102.5 for these tests. Each $p \in S$ is at least five, inert in the marked quadratic field, unramified in E, and prime to the finite CM level. The split indicator expansion has denominators supported at its residue characteristic. The transporting prime may be chosen completely split in E. At a good p the unmarked factors are the integral vacua of that theorem. Use its common integral basis and write g_p for its explicit lower bound on a last-edge matrix valuation, using the same halves of the Frobenius cycles for all entries. For each sign put $B_\pm = 1$ if the unitary family is used, and $B_- = n^{d/2}$ or $B_+ = s^{d/2}$ when the corresponding integral family is used.

Theorem 125.2 (Both bounds for one nonzero arithmetic minor)**.** *For the very minor z_j chosen in Theorem 122.9,*

$$\begin{aligned} \operatorname{ord}_{\mathfrak{p}} z_j &\geq j\, e(\mathfrak{p}/p) g_p && (\mathfrak{p} \mid p,\ p \in S), \\ |\sigma z_j| &\leq j^{j/2} (i_- i_+ B_- B_+ B_\sigma)^j && (\sigma : J \to \mathbb{C}). \end{aligned}$$

At the original embedding it also satisfies $|z_j| \geq j!(i_- i_+ L_*)^j$. In degree one the chosen target can detect any specified nonzero incoming Bessel row as in Theorem 121.10.

For the two integral families, let $\mathfrak{a}$ be the fractional ideal generated by the coordinate minor-product derivatives for all last-edge class pairs and level translates. These are $M(e_I \otimes e_J)(\partial) G_{uv}(0)$ on the coordinate exterior tensor bases. Take all finite factors in the actual normalized class sections. With

$$b_{\mathrm{den}} = \frac{1}{[J : \mathbb{Q}]} \sum_{\mathfrak{p} \nmid S} [-\operatorname{ord}_{\mathfrak{p}} \mathfrak{a}]_+ \log N\mathfrak{p},$$

one has on the same field

$$\sum_{p \in S} g_p \log p \leq b_{\mathrm{den}} + \tfrac{1}{2} \log j + \log(i_- i_+) + \tfrac{d}{2} \log(ns) + \frac{1}{[J : \mathbb{Q}]} \sum_\sigma \log B_\sigma.$$

Proof. At the chosen fine levels all stabilizers are trivial. The class integral of Theorem 67.11 therefore gives each entry as a sum of $i_- i_+$ derivatives with finite character factors of absolute

value one at every embedding. Within each term Y, Z' are the two exterior tensors from the single charts. At every place above S these tensors are integral for either family. We can keep the original representatives by moving the two level elements onto the finite test, as in the construction of b_σ. At p we can take these elements to be the identity since the unmarked factor of the level is the same at the fine and original levels, and the auxiliary split congruence for the fine charts is away from S. The common coordinate calculation of Theorem 102.5 now gives valuation at least $e(\mathfrak{p}/p)g_p$ in each term. Indeed each monomial of the minor product has degree l_τ at τ, and each full-height minor uses every column. These are exactly the two degree bounds used in that theorem. Taking sums and then expanding the determinant proves the first inequality.

At σ, Theorem 65.9 and the triangle inequality for torsion translates bound the squared metric supremum of the conjugated theta section by

$$b_\sigma^2[e\max(1,2l\ \mathrm{rd}(E)/\pi)\nu_\sigma]^D.$$

The block-sphere estimate of Lemma 66.8 consequently bounds the Fischer squared norm of its Taylor component by this expression times $c_\sigma^d\prod_\tau H_l(l_\tau)$. It applies to these class sections in their conjugate marked differentials, with the blocks permuted with the CM embeddings. The unitary changes of complex frame in the CM trace coordinates preserve the Fischer norm. Equivalently one can first use the unitary change of variables and then project to the indicated block degrees. The differential is paired with the component by the monomial identity $\partial^\alpha Z^\beta|_0 = \alpha!\,\mathbf{1}_{\alpha=\beta}$. The isometry in Proposition 95.2 gives $\|M(\sigma Y\otimes\sigma Z')\|_{\mathrm F}=\sqrt{h_\lambda}\|\sigma Y\|\|\sigma Z'\|$. Complex conjugation in a Fischer pairing does not change either norm. Cauchy–Schwarz bounds one term by $B_-B_+B_\sigma$. Summing the level translates and using Hadamard's determinant inequality gives the second assertion.

For integral tests each entry belongs to $\mathfrak{a}$ since the tensor entries are algebraic integers and the finite character values are roots of unity. Thus $z_j\in\mathfrak{a}^j$, and this ideal is nonzero by the lower bound. Outside S the total negative valuation of z_j, weighted by $\log N\mathfrak{p}$, is at most $j[J:\mathbb{Q}]b_{\mathrm{den}}$. The product formula and $\sum_{\mathfrak{p}\mid p}e(\mathfrak{p}/p)f(\mathfrak{p}/p)=[J:\mathbb{Q}]$ give the final inequality from the bound at the complex embeddings. □

For the paired fields with the prescribed mark of Theorem 65.25 the embedding average in the bound can be evaluated further. Write d_K for the absolute quadratic discriminant and $D_\xi=L'(1,\xi_{d_K})/L(1,\xi_{d_K})$. Set $L_K=\log(\sqrt{d_K}/(2\pi))-\gamma+D_\xi$ and $V_K=\frac14\log d_K-\frac12\gamma+\frac12 D_\xi$. Writing a bar for the embedding mean, one has on the original intervals

$$2\,\overline{\log B_\sigma}=2\,\overline{\log b_\sigma}+dL_K+DV_K+(k-1)\sum_\tau B_\tau(\log B_\tau-1)+O_{\psi,\varepsilon}(x^{C_0}).$$

Here B_τ and k are the offsets and the weight parameter of Corollary 104.10. Indeed $\overline{\log c_\sigma}=L_K$ and $\overline{\log\nu_\sigma}=V_K$ by Theorem 65.25. Lemma 92.4 gives

$$h_\lambda=D_-^{-1}\prod_{\tau,i}\frac{(\lambda_{\tau,i}+n-i)!}{(n-i)!}.$$

The first $k-1$ large parts at each τ are $B_\tau+O_{\psi,\varepsilon}(x^{C_1})$, and all remaining parts are polynomially bounded, as in Corollary 104.10. Stirling's formula and $\log(2+\max l_\tau)=O_{\psi,\varepsilon}(x^\varepsilon+\log(2x))$ give the displayed leading term for $\log h_\lambda$ with polynomial error. The Weyl dimension formula bounds $\log D_-$ by $fn^2\log(2+n+\max l_\tau)$. The sphere factor costs at most $\sum_\tau l(1+\log(1+l_\tau/l))$ in its logarithm by Lemma 66.8. Finally $f,n,s,\log\mathrm{rd}(E)$ are polynomially bounded on this family by the field and rank bounds, so the remaining vacuum factor costs a power of x. Also

$\log(ns) = O_{\psi,\varepsilon}(\log(2x))$. The denominators in b_{den} are those of the last-edge CM derivatives. The preceding lifts still have to supply the arithmetic coordinates of the actual incoming vector in the full signed period. Its complex pairing with the chosen target is controlled by Theorem 121.10.

125.3 An input bound for the denominators of the arithmetic period

Take the two integral test families in Theorem 125.2. In Theorem 122.13 use all exponent vectors in the coordinate polynomials $M(e_I \otimes e_J)$ for this last edge, and use its actual finite class sections. For S as in Theorem 125.2 define

$$h_S = \frac{1}{[J:\mathbb{Q}]} \sum_{p \notin S} \sum_{\mathfrak{p}\mid p} [J_{\mathfrak{p}} : \mathbb{Q}_p] \Gamma_{\mathfrak{p}} \log p.$$

Write $u = \lfloor d/2 \rfloor$, $\epsilon_3 = \kappa_3 \mathbf{1}_{3\nmid m}$, and $\epsilon_0 = \mathbf{1}_{d_0\nmid m}$ for the input m of Theorem 122.13.

Theorem 125.3 (A common denominator for the endpoint and its minors)**.** *On the fields and intervals of Section 6.6, every entry in the arithmetic array $a_{v,I}(V)$ of Theorem 121.6 becomes integral after multiplication by the integer*

$$A = N(mH)^d \, 3^{u\epsilon_3} \, d_0^{\epsilon_0 \lceil u/2 \rceil}.$$

For the nonzero j-row minor z_j of Theorem 122.9, $A^j z_j$ is an algebraic integer. For these same minors the comparison of Theorem 125.2 and its finite contribution satisfy

$$\sum_{p \in S} g_p \log p \le h_S + \tfrac{1}{2} \log j + \log(i_- i_+) + \tfrac{d}{2} \log(ns) + \frac{1}{[J:\mathbb{Q}]} \sum_{\sigma} \log B_\sigma,$$

$$h_S \le \log N + d \log(mH) + u\epsilon_3 \log 3 + \tfrac{1}{2} u \epsilon_0 \log d_0.$$

Here N, m, H and the sharper $\Gamma_{\mathfrak{p}}$ depend only on the finite presentations and the tangent coordinates.

Proof. Every coordinate polynomial is a product or an integral linear combination of products of minors, hence has integer coefficients. By Theorem 122.13, each of its evaluations on a class section has valuation at least $-\Gamma_{\mathfrak{p}}$. The same lower bound holds for each one-chart matrix coefficient and for each entry of $a_{v,I}(V)$. Indeed the integral tests have integral exterior tensor coefficients at every place and the finite character projections in these formulas are unnormalized sums with roots of unity as coefficients. The factors divided out in defining the arithmetic matrix are exactly those specified in Theorem 121.6. The determinant expansion gives $v_{\mathfrak{p}}(z_j) \ge -j\Gamma_{\mathfrak{p}}$.

Let $t_p = v_p(m)$. At any prime its contribution from the translated elliptic factors is at most $dt_p + u(\kappa_p - 2t_p)_+$ by Theorem 122.13. Here t_p is an integer, and $0 \le \kappa_p \le 1$. Thus the second summand is $u\epsilon_3$ at 3, $u\epsilon_0/2$ at d_0, and zero elsewhere. Adding $v_p(N) + d\, v_p(H)$ gives a common bound for $\Gamma_{\mathfrak{p}}$ which vanishes off the finitely many primes dividing $3d_0NmH$. Rounding the possible half-integral loss at d_0 proves the assertions about A. Summing the unrounded bound, using $\sum_{\mathfrak{p}\mid p}[J_{\mathfrak{p}} : \mathbb{Q}_p] = [J:\mathbb{Q}]$, proves the bound for h_S.

At the primes in S the same minor has normalized valuation at least jg_p by Theorem 125.2. Outside S we have just bounded its negative valuation before computing any of the derivative values. Apply the product formula to this nonzero z_j, use the archimedean upper bound of that theorem at every embedding, and divide by j. This gives the claimed comparison. No extra factor from A is charged at the primes already counted in S. □

The array here acts on the specific incoming row in Theorem 121.6. Its contraction still involves the class coordinates of the preceding lifts. An estimate for that full period on the scale required by Section 6.6 must also control those coordinates and the growth of the finite input costs along the interval.

125.4 The relative different in the signed column Gram matrix

On the original interval use the period entries of Theorem 122.10 with a common directed tower, degree profile, and marking. The indices a, b may specify different signed tests of that family. Use one D_0 as in that theorem for the entire finite list. On edge i write T_i, Q_i for the integers N, Q of Theorem 122.14 in the fixed finite vacuum model. Use $d_i = \sum_\tau d_{i,\tau}$ and put

$$\rho_p = \frac{\operatorname{ord}_w \mathfrak{D}_{E/K}}{e(w/p)}, \qquad J_{i,p} = v_p(T_i) + \Phi_p(d_i, v_p(Q_i)) + \tfrac{1}{2} d_i \rho_p,$$

where w is any prime of $E = E_x$ over p. This is independent of w since $E/\mathbb{Q}$ is abelian and K is stable under its Galois group. Let S be any finite set of inert good primes as in Theorem 122.10 away from all $T_i Q_i$. Write $G_S = \sum_{p \in S} \sum_i g_{i,p} \log p$.

Theorem 125.4 (A basis-independent finite estimate for the Gram products)**.** *In the original normalization, at each $\mathfrak{p} \mid p$,*

$$v_{\mathfrak{p}}(D_0^2 \mathcal{A}_a \overline{\mathcal{A}_b}) \geq \begin{cases} 2 \sum_i g_{i,p} & p \in S, \\ -2 \sum_i J_{i,p} & p \notin S. \end{cases}$$

The same estimates hold for integral linear combinations of these products after the indicated clearing factor. On all sufficiently distant intervals of Section 6.6,

$$\sum_p J_{i,p} \log p = \log T_i + d_i \log Q_i + \tfrac{1}{2} d_i \log \operatorname{rd}(E) - R_i,$$

$$R_i = \left(\tfrac{1}{4} d_i - \tfrac{1}{2} \lfloor d_i/2 \rfloor \mathbf{1}_{d_0 \nmid Q_i} \right) \log d_0 \ \geq 0.$$

For each nonzero signed period entry with archimedean bounds U_σ from Theorem 121.7 it follows that

$$G_S \leq \log D_0 + \sum_i \left(\log T_i + d_i \log Q_i + \tfrac{1}{2} d_i \log \operatorname{rd}(E) - R_i \right) + \frac{1}{[J : \mathbb{Q}]} \sum_\sigma \log U_\sigma.$$

Proof. Fix a rational prime p. On each edge make the single choice B_p of Theorem 122.14 for all classes and tests. Compute each signed period by applying the same formal compact integration and stabilizer sums as in Theorem 122.10. Its D_0 still clears the symbolic coefficients. Indeed the incident compact actions, endpoint coefficients, and physical derivative symbols have been kept fixed. Only the common normalization of the class germs has changed. By Theorem 122.3 this change leaves the product of one period with the conjugate of the other exactly fixed. We may therefore estimate the actual Gram element at p with these local choices, including when either factor vanishes.

In the first factor the coordinate loss at edge i is at most $\sum_\tau d_{i,\tau} v_{\mathfrak{p}}(\bar{\tau}\mathfrak{b})$ by Theorem 122.14. In the conjugate factor its bound uses $\tau\mathfrak{b}$. This follows by evaluating the original bound at the place transported by the fixed complex conjugation of J. At an edge of opposite orientation the two are interchanged. Their sum is $d_i \rho_p$, since $\tau\mathfrak{b} \cdot \bar{\tau}\mathfrak{b} = \tau \mathfrak{D}_{E/K}$. The two coefficient and torsion losses are at most $2 v_p(T_i) + 2\Phi_p(d_i, v_p(Q_i))$. At $p \in S$, the local basis is also integral in physical coordinates, the finite coefficients are integral and the product translations have order

prime to p. Thus the Frobenius derivative bound in Theorem 122.10 applies to each factor at every place over p. This proves the local estimates. The argument can be repeated at each rational prime on the same algebraic products.

For the sum of the coordinate losses, the norm of the relative different and the discriminant formula in a tower give

$$\sum_{p} \rho_p \log p = \frac{\log N_{E/\mathbb{Q}} \mathfrak{D}_{E/K}}{[E:\mathbb{Q}]} = \log \mathrm{rd}(E) - \log \mathrm{rd}(K).$$

Indeed each $w \mid p$ contributes its different exponent times its residue degree to the exponent in the rational norm, and the sum of the local degrees over p is $[E:\mathbb{Q}]$. Here $\mathrm{rd}(K) = \sqrt{d_0}$. The congruence $d_0 \equiv -1 \bmod 3$ holds on these intervals by Theorem 5.1, whose range of prescribed small primes contains 3. Thus $\kappa_3 = 0$ in Theorem 67.12. The sum of the torsion losses $\Phi_p(d_i, v_p(Q_i)) \log p$ is exactly

$$d_i \log Q_i + \tfrac{1}{2} \lfloor d_i/2 \rfloor \mathbf{1}_{d_0 \nmid Q_i} \log d_0.$$

Adding the coordinate and finite coefficient costs gives the formula with R_i.

For a nonzero entry use its diagonal product. In a Galois field the embeddings of the conjugate entry are a permutation of the embeddings of the entry. The archimedean mean of the logarithm of the absolute value of its diagonal product is therefore twice the mean for the entry. Apply the product formula and the first part, divide by two, and use the upper bound U_σ at all embeddings. At $p \in S$, $J_{i,p} = 0$. This proves the comparison with the stated finite data and the actual field discriminant. □

126 Signed tests and terminal restrictions

126.1 The signed column estimate in terms of the original finite tests

Use the signed entries $\mathcal{A}_a$, the edge degrees d_i, the common directed tower, and $\rho_p = \mathrm{ord}_w(\mathfrak{D}_{E/K})/e(w/p)$ of Theorem 125.4. At each edge i take T_i^0, Q_i^0 on the original list in Theorem 122.4, and let $a_{i,\pm}(p)$ be its two exponents when p splits in K. Define

$$H_{i,p} = \begin{cases} d_i(a_{i,+}(p) + a_{i,-}(p))/2, & p \text{ split in } K, \\ \Phi_p(d_i, v_p(Q_i^0)), & p \text{ nonsplit in } K, \end{cases}$$
$$J_{i,p}^0 = v_p(T_i^0) + H_{i,p} + \tfrac{1}{2} d_i \rho_p, \qquad B_i^0 = \sum_p J_{i,p}^0 \log p.$$

Choose B_{end} as in Theorem 122.15. Use the original stabilizer multiples A_v and degree bounds $L_{v,\tau}$ for the tensor-polynomial expansions of the compact integrands before averaging. Put

$$D_{\mathrm{intr}} = B_{\mathrm{end}} \prod_v A_v \prod_{v,\tau} D(n_v, L_{v,\tau}).$$

Let S consist of any finite set of the good inert primes in Theorem 122.10, with the original local conditions and $p \nmid \prod_i T_i^0 Q_i^0$. Its Frobenius weights are $g_{i,p}$ and $G_S = \sum_{p \in S} \sum_i g_{i,p} \log p$.

Theorem 126.1 (The original finite tests and the Gram products of the signed column)**.** *For the original projected definite period, at every place $\mathfrak{p} \mid p$, normalized by $v_{\mathfrak{p}}(p) = 1$,*

$$v_{\mathfrak{p}}(D_{\mathrm{intr}}^2 \mathcal{A}_a \overline{\mathcal{A}_b}) \geq \begin{cases} 2 \sum_i g_{i,p}, & p \in S, \\ -2 \sum_i J_{i,p}^0, & p \notin S. \end{cases}$$

On the intervals of Theorem 125.4, the per-edge expression is

$$\begin{aligned} B_i^0 &= \log T_i^0 + d_i \log Q_i^0 + \tfrac{d_i}{2} \log \mathrm{rd}(E) - R_i^0 \\ &\quad - d_i \sum_{p \text{ split in } K} \Big(v_p(Q_i^0) - \tfrac{1}{2}(a_{i,+}(p) + a_{i,-}(p)) \Big) \log p, \\ R_i^0 &= \Big(\tfrac{d_i}{4} - \tfrac{1}{2} \lfloor d_i/2 \rfloor \mathbf{1}_{d_0 \nmid Q_i^0} \Big) \log d_0. \end{aligned}$$

For every nonzero entry, with the archimedean bounds U_σ of Theorem 121.7 in a common Galois coefficient field J, this gives

$$G_S \le \log D_{\mathrm{intr}} + \sum_i B_i^0 + \frac{1}{[J:\mathbb{Q}]} \sum_{\sigma: J \to \mathbb{C}} \log U_\sigma.$$

Proof. For a fixed p evaluate the original signed integral with the representatives of Theorem 122.4. This keeps the period, its volumes, its restriction and its fixed projected endpoint. In each edge choose the localized product basis of Theorem 122.14, common to the whole list of tests and classes. The common phase theorem, Theorem 122.3, shows that this preserves the Gram product exactly. An auxiliary finite algebraic extension used to evaluate a particular representative or phase causes no change in the normalized valuation of this original element.

By Theorem 122.15 the symbolic coefficients are integral over p after multiplication by B_{end}. The compact Haar averages are coefficientwise, so Theorem 95.6 and the unchanged stabilizer orders give D_{intr} for each copy of the signed integral. In every term there is one physical derivative of the prescribed profile per edge. If $p \in S$, the finite coefficients are integral there, and all the shifts have order prime to p at this place. The monomial estimate in Theorem 122.10 gives $g_{i,p}$ for each factor. This holds for all localizations over p, hence also for the conjugated factors.

For $p \notin S$, the two finite coefficient losses on edge i sum to at most $2v_p(T_i^0)$ by Theorem 122.4. Use Theorem 123.4 at a split prime and the ordinary product derivative estimate of Theorem 122.13 at a nonsplit prime. Together these give at most $2H_{i,p}$ for translations and the marked elliptic factors. The physical coordinate costs of the two orientations sum to $d_i \rho_p$, since $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K}$, exactly as in Theorem 125.4. Products and sums now give the asserted local bound. We may make this calculation separately at each p on the same algebraic Gram element.

For the global sum, use the discriminant identity and the sum of $\Phi_p(d_i, v_p(Q_i^0)) \log p$ in the proof of Theorem 125.4. At split p that summand had been $d_i v_p(Q_i^0) \log p$. Its replacement by $H_{i,p} \log p$ gives precisely the displayed correction, whose summands vanish outside Q_i^0. Finally apply the product formula to $D_{\mathrm{intr}}^2 \mathcal{A}_a \overline{\mathcal{A}_a}$ for a nonzero entry. Conjugation permutes its coefficient field embeddings, so the archimedean mean for the two period factors is at most twice the mean of $\log U_\sigma$. At $p \in S$ each $J_{i,p}^0$ vanishes. Division by two gives the final inequality. □

126.2 Split position tests in the designated signed period

On the interval and in the notation of Theorem 126.1, fix any finite set $\mathcal{T}$ of odd rational primes split in the marked field K. The extension $E = FK$ splits over every F_v above these primes, including when $F_v/\mathbb{Q}_p$ is ramified. At each $p \in \mathcal{T}$ use the actual finite input lists of every edge, with the original right translates. Compute $J_{i,p}^{\mathrm{pos}}$ of Theorem 123.5. Write

$$\Delta_i = \sum_{p \in \mathcal{T}} [J_{i,p}^0 - J_{i,p}^{\mathrm{pos}}]_+ \log p, \qquad [u]_+ = \max(u, 0).$$

Theorem 126.2 (A support bound for the signed period and the detected minor)**.** *For the Gram products of the entire signed definite column, the local bounds of Theorem 126.1 hold with* $\min(J^0_{i,p}, J^{\mathrm{pos}}_{i,p})$ *on each edge at* $p \in \mathcal{T}$*. Every nonzero entry satisfies*

$$G_S \le \log D_{\mathrm{intr}} + \sum_i (B^0_i - \Delta_i) + \frac{1}{[J:\mathbb{Q}]} \sum_\sigma \log U_\sigma.$$

For the nonzero last-edge minor selected in Theorem 122.11, use that theorem's single-edge lists and write Δ *for their correction. With its original archimedean lower bound, the very same minor gives*

$$\sum_{p \in S} g_p \log p \le B^0 - \Delta + \tfrac{1}{2} \log j + \log(i_- i_+ b_- b_+) + \tfrac{d}{2} \log(ns) + \frac{1}{[J:\mathbb{Q}]} \sum_\sigma \log B_\sigma.$$

Proof. Fix $p \in \mathcal{T}$. Use the representatives of Theorem 122.4 for all the vertex classes. Choose on each edge whichever of the bases of Theorems 122.14 and 123.1 gives the smaller local loss. This is one choice for all the terms and all the class tuples in the signed column. In the first case the proof of Theorem 126.1 supplies the two derivative bounds. In the second, Theorem 123.5 supplies them. When multiplying the two signed sums, the derivative from an edge on one side can be paired with the derivative from that edge on the other side, since the directed towers and block degrees agree. Both valuation estimates apply to every such pair of terms. The external coefficients, compact averages and stabilizer weights are cleared by D_{intr} in each copy, exactly as in Theorem 126.1.

Changing the basis multiplies every period entry by a common algebraic scalar of complex absolute value one by Theorem 122.3. Thus the local calculation applies to the original Gram product. It can be repeated at another p using a different common set of edge bases. The primes in S are inert, so all of their original Frobenius bounds still apply. The product formula for a nonzero diagonal entry now replaces the old sum of local losses on edge i by $B^0_i - \Delta_i$. The archimedean upper bounds can be taken in the original calculation.

For the minor keep every row and column chosen by Theorem 122.11. Thus its nonzero lower bound and the terminal test detecting the prescribed incoming vector when $j = 1$ are fixed. Each entry of the normalized matrix has integral local endpoint tensors. Each term in the product of its determinant with the conjugate determinant uses j derivatives in each orientation. Apply the same change of basis on the one last edge to every matrix entry. Its common scalar is raised to power j in the determinant and cancels in the product. The lower bound at $p \in \mathcal{T}$ is therefore $-2j \min(J^0_p, J^{\mathrm{pos}}_p)$. All other terms in the product formula and the division by $2j$ are those of Theorem 122.11. This proves the second inequality. □

For the unit-amplitude congruence test of Theorem 123.5, with the other local factors spherical and no other coefficient loss, the improvement over the raw Weyl bound is $\frac{1}{2} hns \log p$ per edge. This calculation uses the projection and the fields already fixed in Section 6.6. The comparison with the full prescribed Bessel pairing and the residual strict cohomology orders enter separately in the interval inequality.

126.3 Exterior traces of the last projected theta operator

At the last definite edge use the normalization $z(U, V)$ of Theorem 121.10. Take the complete chart tests of Theorem 122.5 on both sides. Let Z_* be their matrix of $z(U, V)$, with row and column in the marked conjugate spaces when required. Write $h_\pm, i_\pm, w_\pm, D_\pm$ for the numbers of charts, level indices, chart volumes and compact type dimensions there. Put $s_\pm = i_\pm^2 w_\pm / D_\pm$

and

$$|\kappa_*| = \frac{w_- w_+ |C| c^{-d/2}}{D_- D_+}, \qquad E_j = \operatorname{tr}\big(\textstyle\bigwedge^j (Z_* Z_*^*)\big).$$

At each real place τ let $L_\tau = |\lambda_\tau|$ be the last partition degree. For $R = n, s$ set

$$q_{R,\tau} = \frac{D(R, L_\tau)}{\gcd(D(R, L_\tau), \dim V_{R,\tau})}, \qquad q = \prod_\tau \gcd(q_{n,\tau}, q_{s,\tau}).$$

Here $D(R, L)$ is the integer in Theorem 95.6, and $V_{R,\tau}$ is the single-place Cartan component in the marked exterior tensor. Keep the last-edge $S, g_p, J_p, B^0 - \Delta$ of Theorem 126.2 and the transported Fischer bounds B_σ of Theorem 125.2. Write $k_0 = \dim \mathcal{U}$ for the dimension of the actual incoming subspace with squared norm lower bound a_*.

Theorem 126.3 (One compact denominator for an exterior trace)**.** *For $1 \le j \le k_0$ the physical trace E_j is algebraic and satisfies, at the original embedding,*

$$E_j \ge \binom{k_0}{j} \left(\frac{s_- s_+ a_*}{|\kappa_*|^2}\right)^j > 0.$$

At every $\mathfrak{p} \mid p$,

$$v_{\mathfrak{p}}(q^j E_j) \ge \begin{cases} 2j g_p, & p \in S, \\ -2j J_p, & p \notin S. \end{cases}$$

Consequently on the same fields and intervals

$$\sum_{p \in S} g_p \log p \le B^0 - \Delta + \tfrac{1}{2} \log q + \log(i_- i_+ \sqrt{h_- h_+}) + \frac{1}{[J : \mathbb{Q}]} \sum_\sigma \log B_\sigma - \frac{\log(j!)}{2j}.$$

Proof. By the chart identity, the positive eigenvalues of $Z_* Z_*^*$ are $s_- s_+ / |\kappa_*|^2$ times the positive eigenvalues of the output Gram of the physical last operator. Its restriction to $\mathcal{U}$ has squared norm bounded below by a_* on unit vectors. The argument by orthogonal complements in Theorem 123.2 gives at least k_0 eigenvalues this large. Their elementary symmetric sum gives the first assertion.

In a standard chart of rank R let $P_{R,\tau}$ denote orthogonal projection from the full exterior tensor at τ to its Cartan component. With t its unit highest tensor, Schur orthogonality gives

$$P_{R,\tau} = (\dim V_{R,\tau}) \int_{U(R)} (\rho(g)t)(\rho(g)t)^* \, dg.$$

Use probability Haar. Indeed the average is scalar on the cyclic irreducible span, has trace one there, and vanishes on its orthogonal complement. In standard wedge coordinates the integrand is integral polynomial of degree at most L_τ in each of $g, \bar{g}$. Unitary determinant characters cancel in the outer product. The compact moment formula of Theorem 95.6 gives rational entries with $q_{R,\tau} P_{R,\tau}$ integral. In these coordinates the projectors are real symmetric because their entries are rational and they are Hermitian.

For each class pair form the auxiliary matrix W by using the raw product of matching coordinate minors on the two ambient tensors and evaluating its derivative on the actual translated last-edge section. Sum the original signed level translates on both sides. The individual polynomials here are the map M of Proposition 95.2 on the full tensor. Let P_-, P_+ be the products over real places of the Cartan projectors, acting on each respective class. Theorem 67.11 gives $Z_* = P_+ W P_-$, since the class coefficient of an arbitrary exterior tensor depends only on its component in the compact cyclic span. More precisely at any one τ,

$$P_{+,\tau} W = W P_{-,\tau}.$$

To verify this, M is equivariant by Cauchy–Binet. The Cauchy decomposition of the matrix polynomials pairs the partition λ_τ on one side only with that same partition on the other. Each exterior tensor under consideration contains that highest weight exactly once. Indeed for a diagonal weight with strictly decreasing entries the unique maximal tensor line is the product of the leading wedges. Thus pairing a Cartan factor at one end with its orthogonal complement at the other gives zero for every matrix polynomial evaluation. This proves the identity at each class pair. Finite translates preserve it since the Fourier test changes in the finite section and compact coordinate changes preserve the Cartan types.

Fix p and choose neutral representatives as in Theorem 122.4. At each τ use whichever end has smaller $v_p(q_{R,\tau})$. Write P_L, P'_R for the products of the chosen output and input projectors respectively, using the identity for an empty product. Then $Z_* = P_L W P'_R$, with one projector per real place. Taking exterior powers and using the Hilbert–Schmidt inner product on rectangular matrices gives

$$E_j = \| \textstyle\bigwedge^j Z_* \|_{\mathrm{HS}}^2 = \langle \textstyle\bigwedge^j W, \bigwedge^j Z_* \rangle_{\mathrm{HS}}.$$

The equality holds since multiplication on the two sides by orthogonal projectors is itself an orthogonal projector on the matrix space, also after taking exterior powers. In the last expression just one copy of the matrix of minors contains compact projectors. Expand it by Cauchy–Binet. A minor of each projector product is a sum of products of j entries, so their combined denominator at p divides q^j locally.

Each remaining term contains j raw derivative entries of W and j conjugate entries. At the neutral charts their polynomials are products of coordinate minors with integer coefficients. Every full-height minor uses every column, and the block degrees are L_τ, including before a Cartan projector is applied. Thus the monomial valuation calculation in Theorems 102.5 and 122.11 gives $2g_p$ per pair in S and at least $-2J_p$ elsewhere, retaining the position bound of Theorem 126.2. No compact average is left in W. The trivial stabilizers and the normalization of z remove the chart masses and the two Schur dimension factors. These computations prove the finite-place assertion. The auxiliary matrix can be chosen in new charts for each prime because the first part expresses E_j exactly as a fixed multiple of the physical exterior trace.

At an embedding σ evaluate the actual matrix Z_* in the original standard chart coordinates. On one class pair and one translate, its Hilbert–Schmidt norm is at most B_σ. In fact in each block the map from the two Cartan factors to matrix polynomials is an isometry times $\sqrt{h_{\lambda_\tau}}$ by Proposition 95.2. On the ambient product the actual pairing first projects both factors orthogonally onto these components. The adjoint of the resulting map has norm at most $\sqrt{h_\lambda}$, the product of the single-place norm factors. Applying it to the Fischer representative of the Taylor functional bounds the Euclidean norm of the entire array by $\sqrt{h_\lambda}$ times the Fischer norm. This is exactly the definition of B_σ, including its finite vacuum bound. Under transport the blocks are permuted or conjugated with their indicated types, as in Theorem 125.2. Summing at most $i_- i_+$ translates and then summing squares over $h_- h_+$ class pairs gives $\|\sigma Z_*\|_{\mathrm{HS}} \le A_\sigma = i_- i_+ \sqrt{h_- h_+} B_\sigma$.

For any matrix of Hilbert–Schmidt norm A, the sum of squared j-minors is at most $A^{2j}/j!$, by expanding the j-th power of the sum of its squared singular values. Therefore, with c_J the original conjugation of the coefficient field,

$$|\sigma E_j| \le \frac{A_\sigma^j A_{\sigma\circ c_J}^j}{j!}.$$

This is Cauchy–Schwarz on the two transported lists of minors, so it also applies to embeddings that do not commute with conjugation. Apply the product formula to the nonzero $q^j E_j$, use $\sum_{p \notin S} J_p \log p = B^0 - \Delta$, and divide by $2j$. This yields the asserted inequality. □

126.4 A signed integral over sets of positive local measure

Use the supported local path and all compact middle averages of Theorems 123.9 and 123.7. We integrate its core position matrices X_j, its top matrix B, and its last frequency z as additional variables. Keep the initial endpoint fixed. Matrix measure is product additive Haar, with the integral matrix lattice of each shape of volume one. At the frequency use the dual measure, so the integral dual lattice also has volume one. Put $d = n_{s+1}$, $D_+ = r + (s+2)t$ for the row dimension of B, and $P_a = \prod_{i=1}^{a}(1 - q^{-i})$. Use $P_0 = 1$. Retain the oriented integral covector l_0 used in Theorem 123.9, chosen as a marked coordinate functional. Put $U_0 = \ker(\bar{l}_0|_U)$.

Theorem 126.4 (A common integral for the tame twists)**.** *There is a compact open set of the integral position matrices, independent of the twist and the final residue unit b, for which the signed path integral over this set and over $\pi^{-s-1}\widetilde{b}\, l_0 + O^{1\times d}$ equals*

$$\mathcal{V}\,\eta\,\chi(b)^{-1}\,q^{\,r-1-(s+1)d}\,G(\chi)^{\,1+s(s+1)/2}.$$

Here $\widetilde{b} \in O^\times$ is any fixed lift of b; the position set is the same for every lift. Its exact measure is

$$\mathcal{V} = \frac{P_r}{q-1}\,q^{-r(D_+-r)}P_{a_s}\,q^{-s-(d-a_s)a_s} \times \prod_{j=0}^{s-1} P_{a_j}\,q^{-jn_{j+1}-(n_{j+1}-a_{j+1})a_j}.$$

In particular a single determinant condition suffices for all the marked factors together. At any tuple of distinct split twist places on the common field the product of these sets has the product measure and the integral of the product of the local paths has the product value.

Proof. For $j < s$ impose the following conditions. At $j > 0$ require $X_j z_j \equiv z_{j+1} \pmod{\pi^j}$ and use $v_j = (X_j z_j - z_{j+1})/\pi^j$. At $j = 0$ take $v_0 = -z_1$. In the residue target core the marked columns of X_j together with v_j must be a basis of the marked plane A_{j+1}. All unmarked columns may be arbitrary integral columns. Let u_j be the determinant of this basis with its original ordering, including the fixed sign from the flag coordinates.

At $j > 0$ the initial congruence is prescribed on one column, so has measure $q^{-jn_{j+1}}$. Conditionally v_j is uniform in the integral target core and independent of the other columns. In the marked residue determinant there is already one fixed vector z_{j+1}. The other a_j random columns are v_j and the $a_j - 1$ other marked source columns. At $j = 0$ the fixed vector is v_0 and the a_0 random columns are the marked source columns themselves. In either case their unmarked entries must vanish modulo π, with probability $q^{-(n_{j+1}-a_{j+1})a_j}$. In the marked target modulo the line of the fixed vector their images are independent uniform vectors in K^{a_j}. The probability that they form a basis is P_{a_j}, by choosing the vectors successively. These are precisely the factors in the product in the second line of the volume formula. On this set the calculation of the interior entries in Theorem 123.9 applies pointwise with the unit u_j.

At $j = s$ require the $a_s = r - 1$ marked columns of X_s to reduce to a basis of U_0. This has probability $q^{-(d-a_s)a_s}P_{a_s}$. Impose also $l_0 X_s z_s \in \pi^{s+1}O$. For $s \geq 1$ the tested entry is the transverse coordinate of one marked column. Its residue is already zero, and its next s digits are unrestricted and uniform. This contributes q^{-s}. For $s = 0$ the condition is automatic because $z_0 = 0$. Denote by $u_s \in K^\times$ the scalar for which the last-column determinant form is $u_s\,\bar{l}_0|_U$. It is well defined also when $r = 1$. Substitution of $y = -X_s z_s + \pi^s v$ gives additive phase one on the entire frequency coset in the statement. Indeed its center pairs integrally with $X_s z_s$ by the new condition, and the additional integral frequency does so by integrality.

On v the residual phase has parameter b/u_s in the determinant coordinate. The Gauss sum therefore contributes $\chi(u_s)\chi(b)^{-1}G(\chi)$, with all volume powers as before.

For B require the r marked columns to give an isomorphism of the marked residue planes. This has measure $q^{-r(D_+-r)}P_r$. Its determinant u_+ is uniform in $K^\times$ on this event: rescaling one marked column permutes the possible determinants transitively, preserving the measure. Impose the single final condition

$$u_+ \prod_{j=0}^{s} u_j = 1.$$

Given all the core matrices, exactly $1/(q-1)$ of the admissible top matrices meet it. The product of the residue character values at the positions is then one for every χ. All these tests are congruences in compact integral matrix lattices, so their joint locus is compact open. Multiplication of the calculated measures gives $\mathcal{V} > 0$.

The hypotheses used in the compact averaging proof are valid on the full set. The interior integrals have the same marked determinant ranks, and the outgoing covector and marked columns satisfy the last-column calculation. Every contributing y is still primitive with nonzero reduction in U. Thus the normalized terminal orbit sum and the other compact middle integrals give the pointwise signed product above throughout this set. The frequency coset has volume one. Integration now gives the displayed value, with the unramified scalar η retained. Finite products over places follow by product Haar and Fubini on compact sets. □

This integral isolates specified local positions in the split auxiliary factors. In the designated global period the other interior paths, the rational class sum and the global cuspidal projection still enter with their signs. Controlling their combined contribution on the same interval is necessary to deduce the claimed GRH estimate.

126.5 The phase of the added positive line

At one real place of a transverse pair with $\mathcal{G} > 0$ put

$$L_\tau = |\tau D_d| - T|\tau D_s| > 0, \qquad \Delta_+ = D_d + TD_s, \qquad \kappa = k - d, \quad E_0 = \kappa + 2ds + s^2.$$

Use the extended monomial frame and δ_p of Proposition 102.3. For each endpoint $J_i < d$ put $a_i = D_{J_i}/Q^{J_i}$. At the last endpoint put $a_t = \Delta_+/Q^d$. These are nonzero algebraic integers by Proposition 110.11. The index t here counts blocks. Write $u_p = d - J_{i(p)}$ as in Theorem 107.3.

Proposition 126.5 (Decay in the difference of the two terminal moduli)**.** *For nonnegative endpoint powers η_i with $\ell = \sum_i J_i\eta_i \le r^2$, the original real integral is at most*

$$\frac{e^{Ckd+Cd\ell\log b}}{(k!)^s T^{s(k+d)}} \left(\frac{2\,\tau\mathcal{G}}{L_\tau^2}\right)^{-\kappa} \left(\frac{L_\tau}{(2T)^d}\right)^{-E_0} \prod_{p=1}^{d} \delta_p^{-2u_p} \prod_i \max(1, |\tau a_i|)^{-\eta_i}.$$

The constant is uniform in the phase, in the words and in the real place.

Proof. In Proposition 102.3 multiply the positive component of the second terminal vector by ζ of modulus one. Equivalently extend the relative isometry w by multiplication by ζ on the extra line, and extend each member of the original Bessel group by the identity. The new cross determinant is $\Delta_\zeta = \tau D_d + \zeta T \tau D_s$. Choose ζ so that $|\Delta_\zeta| = L_\tau$, taking it to be one if $D_s = 0$. If the last block is a singleton only its pivot changes. Otherwise the column operation in Proposition 102.3, with T replaced by ζT, restores the original monomial matrix. Every factor δ_p with $u_p > 0$ is consequently unchanged.

The two pairings of the positive vector against the physical frames are now $\bar{\varpi}\mathbf{e}_d$ and $\bar{\zeta}\bar{\varpi}\mathbf{e}_d$ at this embedding. The projection calculation of Proposition 102.3 gives perpendicular norm

$$1 - 2T\,\mathrm{Re}(\zeta\tau D_s/\Delta_\zeta) = \tau\mathcal{G}/|\Delta_\zeta|^2.$$

In Proposition 107.17 the equation for the primal block of the fixed vector is now $C^*a = \bar{\zeta}\mathbf{e}_d/\sqrt{2}$. It is still supported on the first coordinate of the last block. The stabilizer, the Haar computation, and the Gram integral therefore apply, with sphere radius squared $2\tau\mathcal{G}/L_\tau^2$. For an original group element, a phase on the extra line leaves the norm of the extended terminal exterior vector unchanged. Indeed the two terms from its e and extra-line components are orthogonal. It also preserves the extended operator norm. This justifies the same substitution for the reciprocal squared integral, where the two frames interchange and the phase is conjugated. The change of frequency on the original group is the one in Theorem 107.3.

It remains to justify the endpoint tests and their larger powers. On the shell of that proof set $M = \nu(g)H$. The physical minors in the unit-frequency frames with the original positive line phase satisfy

$$\left(\frac{Q}{2T}\right)^{J_i} |\tau a_i| \le M^{J_i} \qquad (1 \le i \le t).$$

For the final one use the unrotated exterior pairing of the two extended frames. Its value is unchanged by any left or right change in the original Bessel group. After choosing the minimal representatives of g and y in the shell, their product with one inverted represents the same double coset and has norm at most M. Extending it by the identity on the extra line preserves that bound. Pair the unit exterior frames to obtain the assertion, including on the domain of integration used for the rotated estimate. The proper endpoint tests follow by the same argument with the shorter wedges.

For each $|\tau a_i| \ge 1$ insert the resulting inverse power and its compensating factor $M^{J_i\eta_i}$. The additional constant is at most $\max(1, 2T/Q)^\ell \le \exp(Cd\ell\log b)$. In the proof of Theorem 107.3 the shell exponent becomes at most $K_0 + \ell$, where $K_0 = O(d^3)$. The normalized diagonal moment after shell integration has order at most $2(K_0+\ell)+2 \le 4r^2$ for large x. Theorem 105.3 bounds its local logarithm by $Cs(K_0+\ell+1)\log b$. The part $O(d^4\log b)$ is $O(kd)$ on the common intervals. This proves the assertion. □

126.6 The determinant of the difference of two pairings

Let the two frames F_i of a transverse cell be column matrices with $d = s + 1$ columns each. Write Y_i for their spans, $W = Y_0 \oplus Y_1$, $C = W^\perp$ and $Z = F_0^* F_1$, where the star uses the row Hermitian form. We take it conjugate linear in the first variable. Write $D_d = \det Z$, D_s for the leading s-minor and $\mathcal{G} = D_d\bar{D}_d - T^2 D_s\bar{D}_s > 0$ at the real places. Let $\pi_i : Y_i^\perp \to P_i = Y_i^\perp/X_i$ be the quotient map, with $p_i = \pi_i|_C$. Use the unitary identification $\beta : P_0 \to P_1$ of Theorem 102.4. Set

$$C_0 = \ker(p_1 - \beta p_0), \qquad C_1 = C \ominus C_0, \qquad h = \dim_E C_1, \qquad W' = W \oplus C_1.$$

Define a pairing on $Y_0^\perp \times Y_1^\perp$ by

$$\mathcal{H}(v, w) = \langle v, w\rangle - \langle \beta\pi_0 v, \pi_1 w\rangle.$$

Either argument in C_0 gives zero. Indeed pairing a row vector orthogonal to Y_i against $c \in C$ is the same as pairing its column image against $p_i c$. Thus we may restrict $\mathcal{H}$ to $S_i = W' \cap Y_i^\perp$ on the two sides.

Lemma 126.6 (Archimedean determinant of the comparison pairing)**.** *At a real place τ give S_0 the coordinates $(F_1^* v, \mathrm{pr}_{C_1} v)$ and S_1 the coordinates $(F_0^* w, \mathrm{pr}_{C_1} w)$. Use orthonormal*

coordinates for C_1. These identify each space with $\mathbb{C}^d \oplus \mathbb{C}^h$. The matrix N_τ of $\mathcal{H}$ in these coordinates satisfies

$$0 < |\det N_\tau| \le \frac{2^h}{|\tau D_d| - T|\tau D_s|}.$$

Proof. Suppress τ in the computation. The frame Gram matrix on W is $M = \left(\begin{smallmatrix} J & Z \\ Z^* & J \end{smallmatrix}\right)$, where $J = -Te_de_d^*$. Its absolute determinant is $\mathcal{G}$, as follows by expanding in the two rank-one diagonal blocks. Both M and Z are invertible. In dual frame coordinates (F_0^*v, F_1^*v) its pairing matrix is M^{-1}, proving the coordinate assertion. Write

$$M^{-1} = \begin{pmatrix} B_1 & A^* \\ A & B_0 \end{pmatrix}.$$

The complementary minor identity gives $|\det A| = |\det Z|/\mathcal{G}$. Equivalently, reorder the block rows of M to put Z^* in its leading corner and take the inverse using its Schur complement. The matrices B_i are the forms on $W \cap Y_i^{\perp}$. Each is the square norm of the projection to the positive column line orthogonal to p_iC, with radical X_i. Choose unit coordinates on those lines and write $B_i = \xi_i\xi_i^*$, so the projected coordinate of a vector with dual frame coordinate q is ξ_i^*q. The block equations $MM^{-1} = 1$ give

$$B_1A^{-1} = -(Z^*)^{-1}J, \qquad B_0(A^*)^{-1} = -Z^{-1}J.$$

It follows by taking the trace of their product that

$$|\xi_1^*A^{-1}\xi_0|^2 = T^2|e_d^*Z^{-1}e_d|^2 = T^2|D_s/D_d|^2.$$

The comparison identifies the copies of C_0 pointwise and maps their orthogonal complements in the two columns isometrically to each other. On these complements use the one line coordinate followed by the h coordinates of p_iC_1. Let U be the matrix of the inverse comparison from the side-one coordinates to the side-zero coordinates. With $J_i = \operatorname{diag}(\xi_i^*, I_h)$ the pairing matrix is

$$N_\tau = \operatorname{diag}(A, I_h) - J_0^*UJ_1.$$

Use $\det(1 - AB) = \det(1 - BA)$ and put $z = \xi_1^*A^{-1}\xi_0$. Then

$$\det N_\tau = \det A \, \det(1 - U \operatorname{diag}(z, I_h)).$$

Here $|z| < 1$. A fixed vector of $U \operatorname{diag}(z, I_h)$ would have zero first coordinate by equality of norms. Its remaining coordinates would then give an element of C_1 on which the two column maps agree. Formation of the kernel commutes with extension to the complex embedding, so that element is zero. For the upper bound the first column of $1 - U \operatorname{diag}(z, I_h)$ has length at most $1 + |z|$, and each other column has length at most two. Hadamard's inequality gives the assertion because $|\det A|(1 + |z|) = 1/(|D_d| - T|D_s|)$. □

126.7 Bounds for projected periods from Hecke eigenvalues

Let S be the terminal table over the coefficient ring of integers from Theorem 122.17. Take a finite list of commuting integral Hecke operators H_a which act semisimply over a common splitting field K. Extend the integral lattice to its ring of integers. Index the distinct simultaneous eigenvalue lists by i. Write V_i for the full associated space, of dimension d_i, and α_{ia} for the eigenvalues. For $i \neq j$ define the nonzero integral ideal

$$\mathfrak{d}_{ij} = (\, \alpha_{ia} - \alpha_{ja} : a \,).$$

For a subset I of the lists, take P to be the projection onto $\bigoplus_{i\in I} V_i$ along the sum of the remaining spaces. It is defined over K by the given direct decomposition. For commuting normal operators this is the orthogonal projection.

Theorem 126.7 (Simultaneous exterior bounds from Hecke congruences)**.** *For every prime ideal $\mathfrak{p}$ of K, let $c_1 \ge \cdots \ge c_d$ be the local exponents of Theorem 123.14. Then*

$$\sum_{\ell=1}^{k} c_\ell \;\le\; \sum_{\substack{i\in I\\ j\notin I}} \min(k, d_i, d_j)\, v_{\mathfrak{p}}(\mathfrak{d}_{ij}) \qquad (1 \le k \le d).$$

In particular the normalized projected tests are perfect at each prime where the two sets of eigenvalue lists remain disjoint on reduction. Globally the corresponding sum of losses weighted by $\log \mathrm{N}\mathfrak{p}$ is bounded by the same sum with $\log \mathrm{N}\mathfrak{d}_{ij}$ in place of $v_{\mathfrak{p}}(\mathfrak{d}_{ij})$.

Proof. Work over the local discrete valuation ring and set $Q = 1 - P$. The finite module $C = S/(S_P + S_Q)$ is a quotient of each of PS and QS. For the latter assertion send Qs to the class of s in C. Changing s by S_P preserves the class. The former assertion follows with P, Q exchanged. Both maps are surjective and Hecke equivariant.

Order the $i \in I$. Filter PS by intersecting it with the successive partial sums of the V_i. Each successive quotient is free of rank d_i, since projection onto V_i gives a full lattice there with kernel the preceding intersection. The action of H_a there is the scalar α_{ia}. The images in C give a filtration A_i whose factors are generated by at most d_i elements, with the same scalar action. Construct B_j from QS using the complementary list. Denote the preceding terms by A_{i^-} and B_{j^-}, starting with zero.

In A_i/A_{i^-} take the filtration induced by $A_i \cap B_j$. Its successive factors are

$$\frac{A_i \cap B_j}{(A_{i^-} \cap B_j) + (A_i \cap B_{j^-})}.$$

Indeed, an element in the kernel differs from a member of $A_i \cap B_{j^-}$ by an element of A_{i^-} which is still in B_j. The formula is symmetric in the two filtrations. Each displayed module is therefore a subquotient of factors on both sides. It has at most $\min(d_i, d_j)$ generators over the discrete valuation ring. Every $\alpha_{ia} - \alpha_{ja}$ kills it, since H_a acts by each of the two scalars.

Take the sum of the k largest cyclic factors in any elementary divisor decomposition of C and intersect it with the refined filtration just obtained. Its factors are subquotients of the displayed modules and of a k-generated module. Over a discrete valuation ring a subquotient of a module on k generators has at most k generators, by the elementary divisor theorem. The length at (i, j) is thus at most $\min(k, d_i, d_j) v_{\mathfrak{p}}(\mathfrak{d}_{ij})$. Adding proves the local inequality. Factoring the norms of the ideals gives the global one. □

The right side has an archimedean bound for actual class operators. For each pair choose any a distinguishing it. Then $\mathfrak{d}_{ij}$ contains $(\alpha_{ia} - \alpha_{ja})$, so $\log \mathrm{N}\mathfrak{d}_{ij}$ is bounded by the sum of $\log|\sigma(\alpha_{ia} - \alpha_{ja})|$ over all embeddings. For an unnormalized finite double coset with unitary character and $b_a = |JgJ/J|$, its convolution has norm at most b_a, by the triangle inequality for unitary translations with level volume divided out. If multiplication by a positive integer m_a was used to preserve the integral lattice, the bound for this operator is $m_a b_a$. Such an integer exists for a rational class operator by clearing its denominators on the finite lattice. When all conjugates have the stated compact class realizations, this gives the explicit bound $[K : \mathbb{Q}] \log(2 m_a b_a)$ for that pair. The estimate separates whole simultaneous eigenspaces. A further projection inside an eigenspace still has its own quotient from Theorem 123.14.

126.8 Maximal minors at prescribed auxiliary frequencies

At a place of $Q(\xi)$ fix l and the other data of Theorem 99.12. We compare the complex norm and the integral image of the same map on the position coordinates (b, y). Take a nonempty set $\mathcal{X}$ of distinct nontrivial characters of $\mathbb{F}_p^\times$ and a nonempty set $\mathcal{B} \subset (\mathbb{Z}/p^s\mathbb{Z})^\times$ of frequencies. Singleton sets allow either datum to be prescribed. For a weight on the full positions and frequency variables, formula (99.1) uses a separate fiber mean at each ν. The linear map considered here uses a single array on the position fibers as input for every row.

Choose a generator g of $\mathbb{F}_p^\times$. Put $\alpha_\chi = \chi(g)$, $\zeta = \psi(p^{-s})$, $z_\nu = \zeta^\nu$, $k = |\mathcal{X}|$, $a = |\mathcal{B}|$, $R = ka$, and $M = (p-1)p^s$. Let J be a cyclotomic number field containing the p^s-th and $(p-1)$-st roots of unity. Divide the actual integral row by $\mathcal{V}\eta_\chi p^{r-1-(s+1)d}$. The resulting matrix over the integers of J is

$$K_{(\chi,\nu),(i,j)} = G(\chi)^h \chi(\bar{\nu})^{-1} \alpha_\chi^i z_\nu^j, \qquad 0 \le i < p-1, \quad 0 \le j < p^s.$$

Write $\mathfrak{d}_R(K)$ for the ideal generated by all minors of order R. For fixed orderings set

$$\Delta_{\mathcal{X}} = \prod_{\chi<\chi'} (\alpha_{\chi'} - \alpha_\chi), \qquad \Delta_{\mathcal{B}} = \prod_{\nu<\nu'} (z_{\nu'} - z_\nu).$$

Theorem 126.8 (Complex volume and integral image of the signed tests)**.** *At every embedding $\sigma : J \hookrightarrow \mathbb{C}$ the rows of σK are orthogonal and have squared norm Mp^h. The maximal-minor ideal is principal, with generator up to a unit*

$$\Delta_{\mathcal{X}}^a \Delta_{\mathcal{B}}^k \prod_{\chi\in\mathcal{X}} G(\chi)^{ha}.$$

Use the absolute ideal norm and define the difference per row by

$$\mathcal{D}(K) = \frac{1}{R[J:\mathbb{Q}]} \left\{ \sum_\sigma \log \sqrt{\det\left((\sigma K)(\sigma K)^*\right)} - \log N\mathfrak{d}_R(K) \right\}.$$

Then

$$\mathcal{D}(K) = \tfrac{1}{2}\log M - \frac{\log|N_{J/\mathbb{Q}}\Delta_{\mathcal{X}}|}{k[J:\mathbb{Q}]} - \frac{\log|N_{J/\mathbb{Q}}\Delta_{\mathcal{B}}|}{a[J:\mathbb{Q}]}.$$

On all nontrivial characters and all primitive frequencies this is

$$\mathcal{D}_{\mathrm{all}}(p) = \tfrac{1}{2}\left(\frac{\log(p-1)}{p-2} + \frac{\log p}{p-1}\right).$$

On all nontrivial characters and a single prescribed primitive frequency its value is

$$\mathcal{D}_{\mathrm{fixed}}(p,s) = \tfrac{1}{2}\left(\frac{\log(p-1)}{p-2} + s\log p\right).$$

Proof. At each embedding, summing $\alpha_\chi^i \overline{\alpha_{\chi'}^i}$ and $z_\nu^j \overline{z_{\nu'}^j}$ independently gives zero for distinct rows, and gives M for a repeated row. Automorphisms of a root-of-unity group preserve primitivity and distinctness. Every conjugate Gauss sum for a nontrivial χ has magnitude $\sqrt{p}$ by Lemma 92.8. This proves the norm formula.

Remove the row scalars temporarily. The remaining matrix is the tensor product of two rectangular power matrices. For the first, divide each power of degree at least k by the monic polynomial $\prod_{\chi\in\mathcal{X}}(T-\alpha_\chi)$. Subtract the evaluations of the remainder, expressed in the first k columns. These integral column operations give the square Vandermonde matrix followed by zero columns. The same division by $\prod_{\nu\in\mathcal{B}}(T-z_\nu)$ works in the second matrix. The

tensor product of the two column operations is integrally invertible. After permuting columns, only one square block of size R remains. Its determinant is $\Delta_{\mathcal{X}}^{a}\Delta_{\mathcal{B}}^{k}$ up to a sign, by the determinant formula for a tensor product. Integral invertible column operations preserve the ideal of maximal minors by expansion of minors, applied in both directions. Restoring the row scalars multiplies the ideal by their product. The factors $\chi(\bar{\nu})^{-1}$ are units, giving the indicated generator. In particular all coefficient primes, including those over p, enter this ideal.

The sum of the logarithms of the absolute values of any nonzero algebraic row scalar at all embeddings is the logarithm of the norm of its principal fractional ideal. Hence multiplying a row by this scalar leaves $\mathcal{D}$ unchanged. Applying this to the Gauss sums proves the formula for $\mathcal{D}$. It also permits the restoration of the rational measure and modulus factors. The unramified η_{χ} remains explicit in the original signed integral. The same invariance applies to it whenever it is taken as a nonzero algebraic scalar with all of its conjugates.

For the full character set the α_{χ} are the $p-2$ roots of $A(T)=(T^{p-1}-1)/(T-1)$. At a root α, $|A'(\alpha)|=(p-1)/|\alpha-1|$. Multiplying over the roots and using $A(1)=p-1$ gives $|\Delta_{\mathcal{X}}|^{2}=(p-1)^{p-3}$. For the full frequency set, $a=\varphi(p^{s})$ and the roots are those of $\Phi_{p^{s}}(T)=(T^{p^{s}}-1)/(T^{p^{s-1}}-1)$. At a root z the absolute value of the derivative is $p^{s}/|z^{p^{s-1}}-1|$. Each nontrivial p-th root occurs p^{s-1} times as $z^{p^{s-1}}$. Their factors $|1-z^{p^{s-1}}|$ therefore have product $p^{p^{s-1}}$, since $\Phi_{p}(1)=p$. Multiplication of the derivatives gives $|\Delta_{\mathcal{B}}|^{2}=p^{a(s-1/(p-1))}$. These absolute values are the same at each embedding. Substitution proves the full formula. For a single frequency $\Delta_{\mathcal{B}}=1$, giving the second formula. □

Corollary 126.9 (The full auxiliary factors on the common field)**.** *For the fixed x,ξ let $S=Q(\xi)$, $f=[E_{x}^{+}:\mathbb{Q}]$, and $N_{S}=\prod_{v\in S}p_{v}$, counting places separately. Take the tensor product of the full local maps of Theorem 126.8 on their joint position array. Its difference per row satisfies, with an absolute constant,*

$$\sum_{v\in S}\mathcal{D}_{\mathrm{all}}(p_{v})\ll f\bigl(1+\log(2+f^{-1}\log N_{S})\bigr).$$

For a single prescribed primitive frequency at each place and all local nontrivial characters the difference per row is exactly

$$\frac{1}{2}\sum_{v\in S}\left(\frac{\log(p_{v}-1)}{p_{v}-2}+s\log p_{v}\right).$$

Both formulas apply at every x and every admissible tuple on the intervals of Section 6.6. The empty tensor product has difference zero.

Proof. Take one common cyclotomic field for the finitely many matrices. Extension of the field leaves normalized logarithmic norms and archimedean means unchanged. Each map has an integral change of source basis reducing it to a nonsingular square block followed by zero columns, as in the proof of the theorem, also after restoring the row factors. For a tensor product of ranks R_{1},R_{2} the determinant of this block is the product of their determinants raised to powers R_{2},R_{1}, respectively. Its Gram matrix before the column operations is the tensor product of the two original Gram matrices. The same determinant rule for these matrices gives additivity of $\mathcal{D}$ after division by rank. Iteration proves both exact sums.

For odd p the full contribution is $O(\log p/p)$. There are at most f places above each p. For $Z\geq 2$ the terms with $p\leq Z$ sum to $O(f\log(2Z))$. Indeed for $n=\lfloor Z\rfloor$ one has $\sum_{p\leq n}\log p\lfloor n/p\rfloor\leq\log(n!)$ and $\lfloor n/p\rfloor\geq n/(2p)$. The terms with $p>Z$ sum to $O(\log N_{S}/Z)$. Choosing $Z=2+f^{-1}\log N_{S}$ gives the claimed estimate. □

An input supported on $(1,0)$ has a nonzero value at each selected row. An input constant in the last digit of y at one place vanishes on every primitive row there, by Theorem 99.12. Both facts hold for the same fields, matrices and norms. For a specified global class weight, its contribution at a given tuple is the corresponding signed coefficient of its joint fiber sums. The global cuspidal projection and Bessel pairing in Section 6.6 require an estimate for those particular sums, with their actual multiplicities, signs and splitting factors.

126.9 Integral values of a projected row under split matching

Fix x in an interval of Section 6.6. Use one square over E_x/E_x^{+}, the same $T = S_0 \cup Q(\xi)$ and the same levels $e_v \ge c_v$ as in Corollary 83.20. Fix every test outside these places. Regard the large-side projected class sum as a linear functional $\mathcal{L}$ in the remaining tensor of tests. For example, on the definite class tables of Theorem 122.17, a trace-dual test of class i gives the value $\mathrm{vol}(J)\ell_i(y_i)$. A prescribed rational projection P gives the row $y \mapsto \mathrm{vol}(J)\ell_i((Py)_i)$. Signed sums of such rows have exactly the same interpretation. Finite stabilizer weights, projections and the common Haar factor are included in $\mathcal{L}$.

For $W \in \mathcal{G}_v$, let f_W be the covariant large-side frame function supported on the complementary summand and taking value one at X_W in standard coordinates. Define $J_{A,v}f(X)$ by first placing f on the block lattice, as $X \mapsto f(XD_v^{-1})$, and then applying the inverse Fourier transform of Theorem 83.19 on P_v^{c}. For $\mathbf{W} = (W_v)_{v\in T}$ form the finite row

$$\lambda_{\mathbf{W}} = \mathcal{L}\left(\bigotimes_{v\in T} J_{A,v} f_{W_v} \right).$$

This definition retains correlations between different places. Let $\mathcal{L}_{\mathrm{in}}$ denote the pullback of the same $\mathcal{L}$ to the original small-side level tests by the mixed transfers. Its entries are its values on the individual residue-class indicators on the product of the small matrix spaces. For any finite table b over a number field and $w(\ell) = 1$, put $m_w(b) = \min_j w(b_j)$, with $m_w(0) = +\infty$. It is the valuation of the fractional ideal generated by the entries.

Theorem 126.10 (The fractional ideal of tested values)**.** *Write $H_v = |\mathrm{GL}_{b_v}(O_v/\pi_v^{e_v})|$, $d_v = |P_v|$, $\sigma_v = (\nu_v(\pi_v)q_v^{-z_v})^{e_v(a_v-d_v)}$, and*

$$B = \left(\prod_{v\in T} H_v/\sigma_v \right) \mathcal{L}_{\mathrm{in}}.$$

The row B vanishes exactly when all the $\lambda_{\mathbf{W}}$ vanish. For algebraic entries $\lambda_{\mathbf{W}}$, work in a number field containing them and the local factors and phases. At coefficient primes outside the residue characteristics of T, one has

$$m_w(B) = m_w((\lambda_{\mathbf{W}})_{\mathbf{W}}).$$

In particular the ideal of values of $(\prod_{v\in T} H_v)\mathcal{L}_{\mathrm{in}}$ on integral tests equals the ideal of the restricted row after inverting those residue characteristics. At a coefficient prime over ℓ, if the row is nonzero, write $m = m_w((\lambda_{\mathbf{W}})_{\mathbf{W}})$ and $f_v = \log_\ell q_v$ for $v \mid \ell$. Then

$$m - \sum_{\substack{v\in T\\ v\mid \ell}} \left\{ b_v(e_v - c_v)\big(\delta_v f_v - w(\nu_v(\pi_v))\big)_{+} + e_v b_v d_v f_v \right\} \;\le\; m_w(B) \;\le\; m,$$

where $s_{+} = \max(s,0)$. Each valuation of a splitting character in this estimate is given by Proposition 67.13.

Proof. First use standard coordinates on the small side. For a tuple of residue matrices $\mathbf{Z}$, apply Theorem 90.12 in each variable. After multiplication by $\prod H_v$, the coefficient is zero if any largest Smith exponent exceeds $e_v - c_v$. Otherwise it equals the product of $(\nu_v(\pi_v) q_v^{-\delta_v})^{s(Z_v)}$ times a sum of the $\lambda_{\mathbf{W}}$ with $W_v \supseteq \operatorname{row}(Z_v)$ at every v. In this sum each coefficient is the product of the corresponding unit phases. Taking every $Z_v = Y_{W_v}$ singles out any specified $\lambda_{\mathbf{W}}$ with a unit coefficient. This proves the upper bound on the minimum in standard coordinates, and it detects every nonzero restricted row. For the lower bound use the valuation inequality for a sum and $0 \le s(Z_v) \le b_v(e_v - c_v)$. Proposition 67.13 makes each determinant factor a unit at primes of different residue characteristic. At $v \mid \ell$ its possible negative valuation is bounded in magnitude by the first term in braces.

It remains to pass to the original small tests. In each block the change of coordinates $U_{B,v}$ is a Fourier matrix on R_{e_v} in $b_v d_v$ entries, divided by $q_v^{e_v b_v d_v}$, while the entries in the other columns are unchanged. Multiplying a row by $U_{B,v}$ can lower its minimum valuation by at most $e_v b_v d_v w(q_v)$. Its inverse has algebraic integer entries, by finite character orthogonality. Thus the Fourier change cannot raise the minimum. These matrices are invertible, which also proves the nonvanishing assertion in the original coordinates. The large-side Fourier changes have already been made in $J_{A,v} f_{W_v}$. The remaining scalars σ_v are exactly those in Theorem 83.19. This gives the stated inequalities for B. At primes away from T all the σ_v are units. Since the integral tests are freely generated by the residue-class indicators, equality of the local minimum valuations there proves the asserted equality of fractional ideals. □

This determines the ideal obtained when the small test ranges over its full prescribed level. On one specified small test the value is the contraction of the row with that particular test. Section 6.6 still requires the size and divisibility of this individual signed value and its identification with the designated Bessel pairing.

Corollary 126.11 (One prescribed test and its compact translates)**.** *On the same fields and levels, let f be the actual matched output of a fixed small test at all marked places and restrict $\mathcal{L}$ to that image. Give the image its additive Hilbert norm in the original charts. Conjugate the compact coordinate actions of Theorem 90.14 by the partial Fourier changes of variables. For $G = \prod_{v \in T} G_{e_v}$, let P average the product of the congruence kernels down to c_v. Then*

$$|G|^{-1} \sum_{g \in G} |\mathcal{L}(gf)|^2 \;\ge\; \frac{\|\mathcal{L}P\|_*^2 \, \|Pf\|^2}{\prod_{v \in T} R_v}.$$

Here R_v is the dimension in Theorem 90.14. There is equality if $e_v = c_v$ throughout T, and the same lower bound holds for the maximum of the squared values over these translates of this one test.

Proof. The Fourier changes identify the image with the tensor product of the frame spaces up to a scalar change of Hilbert norm at each place. A constant change of norm cancels with its dual in the displayed product. For the usual action on large covariant frames the common character twist is ν_v on determinants, which is trivial on the congruence kernel at level c_v. Thus P projects onto the tensor product of the conductor-level subspaces. Average first inside the cosets of its kernel group as in Theorem 90.14. On the projected space the operator average for the product group is scalar. Indeed the commuting algebra of the first factor consists of the identity on its irreducible space tensored with arbitrary endomorphisms of the others, by expanding in matrix units of the other factors. Applying this to each factor leaves only scalars. Taking the trace gives the reciprocal of $\prod_v R_v$ as in the local proof. This argument permits arbitrary dependence between the local entries of both f and $\mathcal{L}$. The maximum is at least the mean. If a functional on a minimal-level factor has already been projected by the full stabilizer

and character of a specified W, the same theorem gives its value on that factor directly as $\lambda_W f(X_W)$ in linear frame coordinates. In general the translate $g = 1$ is one summand of the mean. Its individual value still requires the global comparison. □

126.10 Elementary divisors of the terminal column transform

The compact part B of Theorem 86.37 has a finite matrix on each unrefined terminal class. With the compatible twisted trivializations, its input coordinates are indexed by nonzero residue covectors and its output coordinates by nonzero vectors. The entry at (v, η) is $\psi_0(\eta(v))$, by the reindexing identity in that theorem. The omitted zero coordinate has zero value on a nontrivial scalar character. We compute the elementary divisors before moving from the vector level back to the covector level.

Theorem 126.12 (Elementary divisors in each scalar character)**.** *Let $p = T$, $d = p - 1$, and write $\zeta = \psi_0(1)$. Work over $R = \mathbb{Z}_p[\zeta]$ with uniformizer $\lambda = \zeta - 1$, and let ω be the Teichmüller lift on $\mathbb{F}_p^{\times}$. On input basis covectors use $[x] \mapsto [ux]$ for the scalar u, and on output functions use $f(y) \mapsto f(uy)$. For $1 \le j < d$, on the character $u \mapsto \omega(u)^j$ the elementary divisors of the unscaled Fourier matrix are*

$$\lambda^{|m|}, \qquad 0 \le m_i \le d \ (1 \le i \le a), \qquad |m| = \sum_{i=1}^{a} m_i \equiv j \pmod{d}.$$

Each tuple contributes one factor. In particular their number is $N_a = (p^a - 1)/(p - 1)$. With $v_p(p) = 1$, their total valuation is

$$a\left(\frac{N_a - 1}{2} + \frac{j}{d}\right).$$

If the input scalar action is written with u^{-1}, the corresponding index is $-j \bmod d$.

Proof. First include zero on both sides and identify the input with the group ring $R[(\mathbb{F}_p)^a]$ after choosing dual bases. Let $Z_i = [e_i] - 1$. The elements $b_m = \prod_i Z_i^{m_i}$ with $0 \le m_i < p$ form an R-basis. Indeed in one coordinate the binomial expansion is unit triangular in $1, [e_i], \dots, [(p-1)e_i]$, and tensoring these bases covers all coordinates. The Fourier image of b_m at y is $\prod_i (\zeta^{y_i} - 1)^{m_i}$. After division by $\lambda^{|m|}$ let its value function be c_m. The reduction of c_m is $\prod_i y_i^{m_i}$, because $((1 + \lambda)^t - 1)/\lambda \equiv t \pmod{\lambda}$ for $0 \le t < p$. These monomials form a basis of the functions on $\mathbb{F}_p^a$. In one coordinate a polynomial of degree less than p vanishing at all p arguments is zero, and the multivariable evaluation matrix is a tensor product. Hence the c_m form an R-basis, since their determinant is a unit.

The order d of the scalar group is a unit in R. Its character projectors $e_j = d^{-1} \sum_u \omega(u)^{-j}[u]$ therefore act integrally, with $[u]$ denoting the scalar action. They commute with Fourier transformation under the two specified actions. The reduction of c_m already has eigencharacter $u^{|m|}$. On the input side the group ring modulo λ is $\mathbb{F}_p[Z_1, \dots, Z_a]/(Z_1^p, \dots, Z_a^p)$. The action of u takes Z_i to $(1 + Z_i)^u - 1$, whose linear term is uZ_i. Therefore it sends b_m to $u^{|m|} b_m$ plus terms of strictly higher total degree modulo λ. With $j \equiv |m|$, the union of the $e_j b_m$ is a basis by unit triangularity after reduction. The corresponding $e_j c_m$ form a basis by the monomial calculation. Separating the bases by their exact characters now gives the displayed diagonal factors. For $j \neq 0$ every vector in the character space has zero coefficient at the fixed origin, and the same holds on the output. Deleting zero has no effect.

For a residue l mod d, the number of tuples of length b with total degree l is $N_b + \mathbf{1}_{l=0}$, with $N_0 = 0$. Indeed one coordinate in $0, \dots, d$ represents every residue once and zero once more. Induction gives the formula since $N_{b+1} = pN_b + 1$. By symmetry the sum of the degrees in the

j-component equals

$$a\sum_{t=0}^{d} t\big(N_{a-1}+\mathbf{1}_{t\equiv j\ (d)}\big)=a\big(N_{a-1}dp/2+j\big).$$

Finally p is a unit times λ^d, by $p=\prod_{u=1}^{p-1}(1-\zeta^u)$. Division by d gives the stated valuation. Extending the coefficient field leaves these normalized valuations unchanged. □

For the f places on the interval the compact transform is a tensor product on each class of Corollary 9.14. The stabilizers there are trivial, so the finite matrix acts on each coefficient coordinate. For fixed nontrivial character indices j_v, its elementary-divisor valuations are $\sum_v |m_v|/(T-1)$ with each $|m_v|\equiv j_v \bmod (T-1)$. Its mean valuation per dimension differs from $af/2$ by $a\sum_v(j_v/(T-1)-1/2)/N_a$. Away from T finite Fourier inversion on the separately zero-sum lattice has a unit denominator even at primes dividing $T-1$, by the integral sums in Theorem 86.37. At those primes this last assertion uses the full subgroup with zero fiber sums and requires no division into character eigenspaces.

Proposition 126.13 (The change of coefficient lattice for one adjacent class)**.** *At a split coefficient place use the integral flag-minor realization of weight $\mu_1\ge\cdots\ge\mu_a$, with the determinant power included. For a full column lattice M write $W(M)$ for the resulting coefficient lattice. For a sublattice $M'\subset M$ of residue index p one has*

$$\pi^{\mu_1}W(M)\ \subseteq\ W(M')\ \subseteq\ \pi^{\mu_a}W(M).$$

In a basis adapted to M', its relative factors are π^t, where t is the last-coordinate weight with its multiplicity in the representation. For a tensor product of split coefficient factors, the exponents add over the affected factors.

Proof. Choose a basis of M in which only the last basis vector must be multiplied by π to obtain M'. Factor the representation as the indicated power $\det^{\mu_a}$ times the polynomial weight with entries $\mu_i-\mu_a$. The flag-minor generators for that weight use one minor per column of the Young diagram, in total $\mu_1-\mu_a$ minors. The last basis vector occurs at most once in each minor. Thus each generator has last-coordinate degree t between μ_a and μ_1, including the determinant power. The lattice is the direct sum of the spans of generators of each weight, since distinct weight spaces are independent over the fraction field. These spans are free over the local valuation ring. The basis change multiplies every vector in the degree-t summand by π^t. This gives both inclusions, also for negative weights. Tensoring the weight-space bases proves the last assertion. □

For the column tables formed with these coefficient lattices the factor $\rho(h^{-1})$ in D evaluates the input at the adjacent index-p sublattice of the output lattice. In local frames this is the inclusion just computed. The determinant twist of Section 71.20 takes unit values and sends the uniformizer to one. The compact sums in B,C preserve the coefficient lattices within each unrefined class. Thus, if S° denotes the separately zero-sum table at this level, a single split coefficient factor satisfies

$$\pi^{a+\mu_1}S^\circ\ \subseteq\ DS^\circ\ \subseteq\ \pi^{\mu_a}S^\circ.$$

For the left inclusion use $p^aD^{-1}=C\rho(h)$ and the reverse change along the same inclusion of lattices. For several affected factors replace μ_1,μ_a by their respective sums. For a dual testing lattice use its dual highest weight. These inclusions retain the algebraic weight cost and the Fourier cost separately.

126.11 The unmarked translation in an integral period

Fix one of the fields and the signed rows and columns of Theorem 86.38. At unmarked finite places use the prescribed compact levels and theta tests. Outside a finite set these levels are hyperspecial and the tests are unit lattice indicators, as in Lemma 72.3. The rational flag has an integral basis compatible with these lattices outside finitely many places. Indeed the transition matrices between rational bases and their inverses are integral outside the primes in their denominators, and adelic lattices agree with a fixed rational lattice almost everywhere. Include all exceptional unmarked places for the data under consideration in a finite set. At each underlying rational prime, membership in the open stabilizers of the smooth tests and vectors follows if all c_j are one to sufficiently high congruence precision. This follows by continuity of the map from the flag torus to the local groups. The same applies to any fixed finite list of data by intersecting the open stabilizers. Take M divisible by these prime powers and by T^b in Theorem 9.20.

Let $\mathcal{U}$ be the set of unmarked places of F at which any of the flag multipliers has nonzero valuation. Each is now good and lies over a rational prime splitting completely in E. At the chosen half $\mathfrak{p}$ above $v \in \mathcal{U}$, the initial list λ_v of valuations on the r row coordinates consists of the entries

$$v_{\mathfrak{p}}(c_j), \quad -v_{\bar{\mathfrak{p}}}(c_j) \quad (1 \le j \le s)$$

and zeros on the remaining coordinates. Set $B_v = \sum_t |\lambda_{v,t}|$ and let q_v be its residue cardinality. Complete splitting gives

$$\sum_{v \in \mathcal{U}} B_v \log q_v \le \sum_{j=1}^{s} h_f(c_j).$$

At an unmarked place outside $\mathcal{U}$ the translation fixes the chosen data, either by the congruence at an exceptional place or by integrality in the hyperspecial lattice.

Proposition 126.14 (The unit lattice tests at all compensating places)**.** *On a growing string of m theta edges let r_k be the rank at vertex k, with $r_0 = r$. At every $v \in \mathcal{U}$ extend the diagonal by the identity in the extra coordinates of each vertex. Write $\lambda_{k,v}$ for λ_v extended by these zeros. Let Φ^0 be the tensor of the unmarked unit matrix lattice indicators at the places in $\mathcal{U}$, on all the edges. The normalized absolute inner product C of this tensor with its simultaneous Weil translate is positive and satisfies*

$$\begin{aligned} -\log C &= \tfrac{1}{2} \sum_{v \in \mathcal{U}} \sum_{k=0}^{m-1} \sum_{\alpha=1}^{r_{k+1}} \sum_{\beta=1}^{r_k} |\lambda_{k+1,v,\alpha} - \lambda_{k,v,\beta}| \log q_v \\ &\le \tfrac{1}{2} \sum_{k=0}^{m-1} (r_k + r_{k+1}) \sum_{j=1}^{s} h_f(c_j). \end{aligned}$$

The choice of $\mathcal{U}$ and the formula use the same rational translation as the Bessel identity of Theorem 86.38.

Proof. In split coordinates the two supports on edge k are the lattice $W = M_{r_{k+1},r_k}(O_v)$ and $W' = d_{k+1,v} W d_{k,v}^{-1}$. With Haar measure giving W volume one, the normalized absolute inner product of their indicators equals $([W : W \cap W'][W' : W \cap W'])^{-1/2}$. The scalar in the Weil action cancels on dividing by the product of the two norms. In matrix entry (α, β) the exponent of the second lattice relative to the first is $\lambda_{k+1,v,\alpha} - \lambda_{k,v,\beta}$. The product of the two indices on this entry is q_v to the absolute value of this exponent. Multiplying over entries, edges and places gives the identity. For each edge the triangle inequality bounds the sum of absolute differences by $(r_k + r_{k+1})B_v$. The preceding height estimate gives the asserted upper bound for $-\log C$. □

Theorem 126.15 (A common denominator for returning the unmarked row to its level)**.** *At $v \in \mathcal{U}$ write $K_v = \mathrm{GL}_r(O_v)$ in the chosen split frame and $t_v = d_{*,v}$. Put $K = \prod_{v\in\mathcal{U}} K_v$ and $t = \prod_{v\in\mathcal{U}} t_v$, and normalize the average P_K on the automorphic row space by Haar probability. There is a positive integer I and a fixed list of I finite translates such that for every K-fixed row A one has*

$$I\,P_K R(t)A = \sum_{\nu=1}^{I} R(\gamma_\nu t)A, \qquad \log I \le r^2 \sum_{j=1}^{s} h_f(c_j).$$

The representatives $\gamma_\nu \in K$ and I are independent of the row and column. The formula remains an exact identity on applying the global Bessel functional, the fixed signed projections at the marked places, or a linear period map. In particular, if the resulting values of the summands belong to a fractional ideal $\mathfrak{a}$ in a common coefficient field, the projected value belongs to $I^{-1}\mathfrak{a}$. Both displayed logarithmic bounds in this section are polynomial in x when $\log M$ and $\log(1/\delta)$ are polynomially bounded on the common intervals.

Proof. Set $J_v = K_v \cap t_v K_v t_v^{-1}$ and $D_v = \max_t \lambda_{v,t} - \min_t \lambda_{v,t}$. If $D_v > 0$ then $1 + \pi_v^{D_v} M_r(O_v) \subset J_v$. Indeed conjugation by t_v^{-1} and t_v scales the entries of the difference from the identity by factors of valuation at least $-D_v$. The resulting matrix is integral and still has unit determinant. Reduction modulo $\pi_v^{D_v}$ now gives $[K_v : J_v] \le q_v^{r^2 D_v}$. If $D_v = 0$, t_v is a scalar times a unit diagonal, so $J_v = K_v$. In all cases $D_v \le B_v$.

Put $J = \prod_v J_v$, choose representatives of the cosets $\gamma_\nu J$ in K and let $I = [K : J]$. The function $k \mapsto R(kt)A$ is constant on each such coset since $t^{-1}Jt \subset K$. Every coset has measure $1/I$, proving the identity. Multiplying the index bounds proves the assertion on $\log I$. Linearity gives the assertion for functionals and fractional ideals. For the polynomial bounds use Theorem 9.20 and the polynomial bounds for m, r_k, s on the indicated fields. □

This computation determines the unmarked lattice overlap and the cost of the additional level average P_K on the actual row. Passing from the period of $R(t)A$ in Theorem 86.38 to the prescribed integral theta period still requires identifying the endpoint functional. In particular the Bessel functional need not be invariant under K on the span of these translates, and the signed automorphic values in the finite sum may cancel.

Corollary 126.16 (Avoidance through the cohomological prime range)**.** *Take the same fields, points $x \in [V, V^{3/2}]$ and cutoff $Y = x^{-3}\exp(x^{\varepsilon}/4)$ as in Corollary 22.15. Fix $\delta = \exp(-x^B)$ for any fixed $B > 0$, and take $b = 1$. There is a matching rational diagonal all of whose unmarked nonzero valuations lie over completely split rational primes greater than Y. Its coordinates satisfy*

$$\sum_{j=1}^{s} h_f(c_j) \le \exp\big((1/4 + o(1))x^{\varepsilon}\big).$$

For rows fixed by the full stabilizer of the prescribed signed coordinate lattice at unmarked places, the integer returning the row to those levels satisfies the same bound on $\log I$. For the two estimates on the automorphic data above, let M_0 be an integer imposing the additional unmarked stabilizer congruences. If $\log M_0 \le \exp((1/4 + o(1))x^{\varepsilon})$ uniformly on the intervals, the same choice can impose those levels and gives

$$-\log C + \log I = o\big(\exp(x^{\varepsilon}/2)\big).$$

Proof. In Theorem 9.20 take M to be T times the product of all rational primes up to Y except T, g_1, g_2. This sets all unmarked valuations through Y equal to zero. The added primes increase $\log M$ by at most $Y\log(2Y)$, by counting the factors and their individual sizes. All the remaining terms in L are polynomially bounded, as are the number of coordinates and the

prefactor in the height bound. Since $\log x = o(x^{\varepsilon})$, any fixed polynomial factor times $Y \log(2Y)$ is bounded by $\exp((1/4 + o(1))x^{\varepsilon})$. This proves the coordinate assertion.

For the full row lattice level use the rational frame of Section 121.16. Its two vectors on each signed plane are $e_+ + e_-$ and $(e_+ - e_-)/2$. They are an integral basis at odd primes. When both diagonal entries $c, \bar{c}^{-1}$ are units, the two operators on the plane given by the diagonal and its inverse therefore preserve the lattice. At dyadic primes we have $c, \bar{c}^{-1} \equiv 1 \bmod 2$ by the choice of M. Their sum and difference divided by two are integral, as are the corresponding entries for the inverse. These are precisely the entries in the original basis e_+, e_-, by the calculation of Theorem 94.11. The other row coordinates are fixed. Thus all unmarked places with zero valuations already fix a row of this level. The remaining places are odd and split completely, where the index proof in Theorem 126.15 applies to this lattice with the displayed exponents. Its bound for $\log I$ has just the additional polynomial factor r^2.

For the data with additional stabilizers take the least common multiple with M_0, which is coprime to $g_1 g_2$. Under the stated hypothesis the logarithm of this modulus has the same bound. Apply Proposition 126.14 and Theorem 126.15. Their polynomial rank factors are absorbed in the $o(1)$ in the exponent. Since $1/4 < 1/2$, this gives the last display. The integer I may have prime factors at most Y even when each local translation in $\mathcal{U}$ is above Y. Their total weighted valuation is bounded by $\log I$ by factorization of this single integer. □

127 Arithmetic theta columns on marked classes

127.1 The coefficient formula on the prescribed terminal classes

Fix one of the fields E_x, a translate, a selected word and its terminal prime $T = \pi\bar{\pi}$ as in Theorem 9.13. Choose a coefficient field containing the images of the CM embeddings τ. At τ use the actual large and small terminal highest weights λ_τ, μ_τ. Write $l_\tau, m_\tau, b_{\tau i}, c_\tau, \kappa_\tau$ for the partitions, last-row exponents and constants in Section 83.13. Put

$$\delta_\tau = \sum_i \lambda_{\tau i} - \sum_i \mu_{\tau i}, \qquad Z = \prod_\tau \tau(\bar{\pi})^{-\delta_\tau}, \qquad U = \prod_\tau \frac{c_\tau \kappa_\tau}{H_{m_\tau}}.$$

Use the finite class pairing with coefficient $1/|\Gamma_j|$ on each decorated terminal class j. In adelic Haar measure this is the pairing divided by the volume of its right level, as in Theorem 122.17. Let P_j be the resulting large polynomial obtained by evaluating the entire signed column at the lift of the chosen small class representative in the fixed polynomial frame. Let F_j be the polynomial of the testing form at the original small representative, with the dual action in the bilinear pairing. This convention includes the transport from the large class representative and all determinant characters in P_j. It fixes the same representative for both evaluations. The last hyperplane is the first n coordinates in rank $N = n + 1$. Its normal coordinate uses $\bar{\pi}$ in the algebraic restriction. In fact an element acting as z on the normal coordinate acts on the μ_τ-summand by z^{δ_τ}, by the two central weights. Pullback by its inverse multiplies by $z^{-\delta_\tau}$. The construction takes $z = \tau(\bar{\pi})$ at that embedding. The representative, the map in this frame and the class pairing are specified in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLarg eDepth.lean`, line 197 and lines 203–218 there.

Theorem 127.1 (The signed period as a sum of specified coefficients)**.** *For a joint monomial index $\alpha = (\alpha_\tau)_\tau$ on the smaller matrices let $\widehat{\alpha}$ extend it by zero to the last row and column at all embeddings. Let B have entries $b_{\tau i}$ at (τ, N, i) and zero elsewhere. For the prescribed class-sum normalization the exact scalar is*

$$\mathcal{P}(P, F) = ZU \sum_j \frac{1}{|\Gamma_j|} \sum_\alpha \alpha!\, (P_j)_{\widehat{\alpha}+B} (F_j)_\alpha, \qquad \alpha! = \prod_{\tau,i,i'} \alpha_{\tau i i'}!.$$

The formula holds at every complex embedding and over the coefficient field whenever the entries are defined there. At every finite place over $p > \max_\tau(\lambda_{\tau 1} - \lambda_{\tau N} + n)$, the factor U is a unit and all nonzero monomial weights $\alpha!$ occurring in this pairing are units. For the word parameters of (9.1) this covers g_1, g_2, T simultaneously at every sufficiently distant x on the intervals of Theorem 22.12.

Proof. At a single representative, multiply the large polynomial by the last restriction operator in Theorem 83.28, including the separate normal-coordinate factor. In its pairing with F_j, the orthogonal projection can be moved to the second slot since it is self-adjoint. It fixes F_j, which belongs to the smaller flag space. The normalized bilinear Fischer form now contributes $\alpha!/H_m$ for the matching smaller monomial. Coefficient extraction reads exactly the larger monomial indicated in the statement. Tensoring gives the inner sum. Integration of a class function counts each representative with weight $1/|\Gamma_j|$. Linearity keeps all the signs, class transports and earlier theta operations inside the same P_j. Every operation is defined by rational coefficients and the displayed field elements, which proves the embedding assertion.

Lemma 83.27 gives the assertion for c_τ and κ_τ. The hook bound in its proof applies to H_{m_τ}. Each exponent of a smaller monomial is at most the degree of its column, hence at most $\mu_{\tau 1} - \mu_{\tau n}$. This proves the factorial assertion. For the actual weights, all the ranks and the spacings Δ, H_0 are polynomial in x. The large parameters consist of the initial u_i with their rank shifts and the repeated $-e_j$ with rank shifts. In (9.1) and its following formulas all u_i, e_j are $O(xb + x^C)$ for a fixed C, with $b = \exp(x^\varepsilon)$. Thus every cutoff in the theorem is $O(xb + x^{C'})$. The chosen g_i exceed $(kb)^8$, and $\log T \gg m x^\varepsilon$ with $m \to \infty$, by Theorems 7.11 and 22.11. These inequalities hold throughout the same intervals. They also put all three primes above $1 + n[E_x : \mathbb{Q}]$. A class stabilizer acts faithfully on the underlying integral lattice of that rank. If its order were divisible by p, an element of order p would have a primitive p-th root as an eigenvalue over $\mathbb{Q}$, of degree $p - 1$. This proves also that the displayed mass denominators are units at these three primes. □

Corollary 127.2 (The normal coordinate on a common terminal prime)**.** *Write $d = t_{\mathrm{r}} + 1$, and use the u_i, v_i, b_j, e_j at one embedding in the lists of (9.1). Then*

$$\delta_\tau = \sum_{i=1}^{k} u_i - \sum_{i=1}^{k-1} v_i + \sum_{j=1}^{m} (b_j - e_j) = \sum_i u_i - \sum_i v_i - m H_0.$$

For a fixed translate this integer at each embedding, and the element Z itself, are independent of the word in Theorem 9.13, with the same generator π. Normalize $v_w(T) = 1$ for $w \mid T$. Then

$$v_w(Z) = -\sum_{\tau \colon w \mid \tau(\bar\pi)} \delta_\tau, \qquad |\sigma Z| = T^{-\sum_\tau \delta_\tau/2} \quad (\sigma : J \hookrightarrow \mathbb{C}).$$

At all finite places outside T the valuation of Z is zero.

Proof. Write $s_0 = md/2$ and $\rho_j(i) = (j+1)/2 - i$, with one-based indices. In the large list the initial entries are $u_i - \rho_k(i) - s_0$. The following m blocks each have length d, with constant value

$$L_j = -e_j + k/2 + (2j-1)d/2 - s_0.$$

The small list begins with $v_i - \rho_{k-1}(i) - s_0$ for $i < k$. Its j-th block consists of one entry $A_j = -b_j + k/2 + (j-1)d - s_0$ followed by $d - 1$ entries L_j. The shifted highest weights are obtained by subtracting the half-sum of positive roots from the row parameters. Within the string following $-b_j$, each parameter and its root shift decrease by one, giving the repeated value L_j. The first entries interlace by $u_i > v_i > u_{i+1}$ and half-integrality. At the first junction use

$u_k+b_1 \ge 1/2$, and at later junctions use $b_j-e_{j-1} \ge (d+2)/2$. Also $A_j-L_j = e_j-b_j-d/2 \ge 0$. All remaining comparisons are equalities. In particular both final entries equal L_m, as required in Section 83.13.

The root shifts sum to zero in each initial list. Subtracting their sums and those of the blocks gives $\sum u_i - \sum v_i - s_0 + \sum_j (L_j - A_j)$, which is the formula since $L_j - A_j = b_j - e_j + d/2$. The only word-dependent contribution is $-\Delta \sum_{i=2}^{k-1} c_i$. Its sum is fixed in the selected family. Finally π belongs to the quadratic field used for the split generator, $\pi\bar{\pi} = T$, and their ideals are coprime. At $w \mid T$ each conjugate of $\bar{\pi}$ consequently has valuation zero or one. At every archimedean embedding its absolute value is $\sqrt{T}$. Taking products proves the claims. □

The formula places the known normalization and the cohomological vanishing at the marked primes on the same fields. The coefficients of the earlier signed theta transforms enter as the values $(P_j)_{\widehat{\alpha}+B}$. Their cancellation against the tests and the comparison with the incoming Bessel row still enter the estimate in Section 6.6.

127.2 The change of determinant on the image of the arithmetic period

Take the period fields, common intervals and coded weights of Theorems 20.13, 9.13 and 93.17. For the moment fix one terminal prime T. Take the direct sum over any selected words with this prime and their ordinary initial classes. Denote the sum of the actual initial lattices by S, the corresponding sum of the unrefined vector lattices in Section 120.5 by V, and their common rank by $d > 0$. Write $\mathcal{F}$ for the direct sum of the unnormalized transforms. Work over a common coefficient field J.

Let B be any J-linear map on $V \otimes J$ of rank $j > 0$. In its image compare the two full lattices

$$M_0 = B(V), \qquad M_1 = B(\mathcal{F}S).$$

In particular B can be the signed arithmetic period on the vector coordinates. Its blocks are the centered maps of Theorem 104.36 divided by $Q = T^{fk}$, with the specified terminal lowering and projections. In that case M_1 is the image of the actual ordinary input under the signed column. The reference image M_0 uses exactly the same B, with the unrefined integral vector input. These lattices measure the input contribution to the period determinant even if B has a large kernel.

At a prime over T use $v(T) = 1$. Write ℓ_v for the valuation of a local generator of $\det M_1$ in a local basis of $\det M_0$. Set $a = \lceil xb \rceil$, $b_* = \lceil xb/\exp(x^{\varepsilon/3}) \rceil$. Choose a positive polynomial bound $C(x)$ for $\lambda_{\tau,i} + k - i$ for all $i \ge 2$, and also for $i = 1$ away from the high places. Such a bound uniform in the words follows directly from (9.1). At a high place the first entry before adding $k-1$ is a.

Theorem 127.3 (Determinant bound for the actual projected image)**.** *At every prime over this T,*

$$-fb_* \;\le\; \frac{\ell_v}{j} \;\le\; fk + fC(x) + \frac{2(a+k)}{k-1}\bigl(f\log 2 + \log(d/j)\bigr).$$

The determinant valuation is zero away from T. Uniformly in every such map, on the same sufficiently distant intervals,

$$\frac{|\ell_v|}{d}\log T \ll_{\psi,\varepsilon} b\,x^{-1-17\varepsilon/16},$$
$$\frac{|\ell_v|}{j}\log T \ll_{\psi,\varepsilon} b\,x^{-1-17\varepsilon/16} \quad \text{if } \log(d/j) \le x^{2+8\varepsilon}.$$

In the quotient Hilbert norms induced from the ordinary class norm on S and the unrefined class norm on V, the norm of every image vector for $B\mathcal{F}$ is $Q^{-1/2}$ times its norm for B.

Proof. Use a uniform random index in the list of identity exponents α_i in Theorem 120.8. Within each word and class it has the distribution of a sum of independent single-coordinate weights at the affected embeddings. Indeed, the weight lists are tensored with their multiplicities. The other embeddings and the orbit multiplicities only repeat the distribution. At an affected embedding every exponent is at least $\lambda_{\tau,k} \geq -b_*$, by the coordinate bounds of Lemma 120.7. There are at most f affected embeddings.

Put $t = (k-1)/(2(a+k))$. The initial lists satisfy the hypotheses of Lemma 120.10 with $A = a + k$ for large x, since a dominates every fixed polynomial. Independence gives

$$\frac{1}{d}\sum_{i} e^{t\alpha_i} \leq e^{tfC(x)}2^{f}.$$

Each summand space has this bound and the same lower endpoint $-fb_*$. Taking the mixture with their dimension weights proves both assertions for the direct sum. For its Fourier exponents β_i, the same exponential moment gains at most e^{tfk}, by Theorem 120.8 on each summand. Its least exponent is still at least $-fb_*$.

Use Smith bases for $\mathcal{F}S$ and V. In these bases the former is obtained from the latter by multiplying coordinate i by a scalar of valuation β_i. In a local basis of M_0 the matrix of B on V is integral and has a unit j-minor, since its columns generate M_0. For a set I of j columns, the valuation of the corresponding minor on $\mathcal{F}S$ is its old valuation plus $\sum_{i\in I}\beta_i$, with valuation $+\infty$ for zero. The minimum of these values is ℓ_v. It is at least the sum of the smallest j exponents and at most the sum of the largest j, by using a unit minor for the latter bound. The average of $e^{t\beta_i}$ on the largest j terms is at most d/j times its full average. Jensen’s inequality for the exponential now gives the stated upper bound for ℓ_v/j. The lower endpoint gives the opposite bound. Away from T the two source lattices agree after applying the transform, so their images agree.

By Theorems 6.29, 9.13 and 93.17, the ranks and terminal scale satisfy

$$f \ll_{\psi} x^{2+8\varepsilon}, \quad k \asymp x^{9/2+10\varepsilon}, \quad m \asymp x^{1/2-\varepsilon/16}, \quad \log T \ll m x^{\varepsilon}.$$

In particular $x f \log T/k \ll_{\psi,\varepsilon} x^{-1-17\varepsilon/16}$. Every fixed polynomial in x and the quantity $fb_* \log T$ are $o(b/x^{A})$ for any fixed $A > 0$. If $\log(d/j) \leq x^{2+8\varepsilon}$, substitution gives the estimate per image dimension. For all j multiply the first display of the theorem by j/d and use $(j/d)\log(d/j) \leq 1/e$. This gives the estimate per source dimension. Finally $\mathcal{F}/\sqrt{Q}$ is unitary by Theorem 73.2. Taking the infimum of the norms of lifts of the same image vector under the two maps gives the asserted quotient norm. □

Corollary 127.4 (A joint period map with several terminal primes)**.** *At a fixed x take any nonempty selection of the words and translates on this interval, allowing their selected terminal primes to differ. Suppose their signed arithmetic columns take values in one coefficient space, with their specified tests and normalizations. On the direct sum of their initial spaces use the joint map, adding their outputs. Let d be the total source dimension and j its image dimension. Define M_0, M_1 as above using each summand’s own transform. Normalize $v(p) = 1$ at a prime of J above p. Then*

$$\frac{1}{d}\sum_{p}\sum_{v\mid p}\frac{[J_v:\mathbb{Q}_p]}{[J:\mathbb{Q}]}\,|\ell_v|\log p \ll_{\psi,\varepsilon} b\,x^{-1-17\varepsilon/16}.$$

In the quotient metrics the absolute change of the logarithm of the determinant norm at any embedding, divided by d, is at most $\frac{j}{2d} f k \max_T \log T$ when the ranks f, k are shared. With varying ranks use $\frac{j}{2d}\max_T(f_T k_T \log T)$, taking the maximum over summands with their actual ranks. These bounds are uniform in the number of summands. They hold also in rank zero, using the unit determinant in degree zero.

Proof. At a given rational terminal prime T, let d_T be the sum of the dimensions of all summands with that prime. Other summands contribute only zero elementary-divisor exponents, by Theorem 120.8. By the minor argument above, $|\ell_v|$ is bounded by the sum of the positive exponents plus the absolute sum of the negative ones in the active summands. In each active block of dimension d' the absolute sum of the negative ones is at most $d' f b_*$. The sum of the positive ones is the full signed sum plus that absolute negative sum. The preceding theorem in full rank, applied to the identity map on the unrefined vector lattice of this block, bounds the absolute full signed sum times $\log T$ by $O_{\psi,\varepsilon}(d' b x^{-1-17\varepsilon/16})$. The negative contribution satisfies the same bound. Summing the blocks gives this estimate with d_T in place of d'. The local degree weights above one T sum to one. Finally $\sum_T d_T = d$; all other finite primes contribute zero.

At infinity the direct sum of transforms is a unitary change of coordinates followed by positive scalars $\sqrt{Q}$ on the individual summands. The norm of a preimage in the new coordinates is between the reciprocal of the largest of these scalars and the reciprocal of the smallest, times the old norm. Taking infima over all preimages gives the same bounds on the image. On a j-dimensional determinant line their logarithms multiply by j, proving the stated estimate. □

If a further finite-index sublattice $S' \subseteq S$ is imposed before applying the signed column, the quotient $B(\mathcal{F}S)/B(\mathcal{F}S')$ is an image of S/S'. Thus its extra local length is bounded by that of the imposed restriction. The reference image $B(V)$ in this comparison still depends on the signed theta coefficients. Estimating its determinant in the prescribed Bessel test lattice and bounding the residual orders in Proposition 6.11 are the unresolved steps in Section 6.6.

127.3 Forgetting auxiliary frames in the terminal period

Fix a field $E = E_x$, a word and a translate from the interval in Section 6.6. Put $n = r + mt$ for the small terminal rank and $N = n + 1$. The primes above the two auxiliary rational primes at which frames are retained form a set S_g. The row can have additional frames at a set Q disjoint from S_g. For a residue place w write $k_w = \mathcal{O}_E/w$. At the terminal prime $T = \pi\bar{\pi}$, let S_T consist of the primes dividing (π). The prime T splits completely in E, and (π) and $(\bar{\pi})$ are relatively prime.

A small object consists of a positive definite unimodular Hermitian lattice M in the specified space and its residue frames. Each frame is an isomorphism $M/wM \simeq k_w^n$. The large object used for evaluation has lattice $M^+ = M \perp \mathcal{O}_E e$, with e its last unit vector. Its covector at $P \in S_T$ and frame at $w \in S_g$ are

$$f_P(v + ae) = \bar{\pi} a \pmod{P}, \qquad \beta_w^+(v + ae) = (\beta_w(v), \pi^{-1} a \bmod w).$$

Here π is a unit at every $w \in S_g$. No frame at Q occurs in either formula. These are the decorations in `Formalism/LocalToGlobal/Columns/StdLatUnimodular.lean`, line 168. Their use in the finite period, with a specified representative of each row class, is described in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 197.

Let $C_w \subseteq \mathrm{GL}_N(k_w)$ be the column level, with its character. For each retained frame define

$$J_w = \{h \in \mathrm{GL}_n(k_w) : \mathrm{diag}(h, 1) \in C_w\}.$$

Give J_w the restriction of the column character along this block embedding. Denote the class set of small lattices with these levels and only the retained frames by $\mathcal{C}$. Morphisms preserve the lattice and the stated orbits of frames. On the large side they also preserve every specified covector. Write $i(c)$ for the class obtained by adjoining e.

Proposition 127.5 (Recovery of the small lattice with its induced level)**.** *For arbitrary groups C_w, extension by the identity on e is a bijection on the sets of morphisms between any two lifted objects and their small objects at the induced levels. Consequently i is injective and it preserves stabilizer orders.*

Proof. Write the image of e under a large isometry as $v+ae$. Lattice preservation gives $a \in \mathcal{O}_E$. Evaluating each covector at e gives $a \equiv 1 \pmod{P}$ for $P \in S_T$, since $\bar{\pi}$ is a unit there. At every embedding τ, definiteness and orthogonality give $|\tau a| \le 1$. If $a-1 \neq 0$, its absolute norm is at most $2^{[E:\mathbb{Q}]}$ and is divisible by $\prod_{P \in S_T} |k_P| = T^{[E:\mathbb{Q}]/2}$. The latter is larger because $T > 4$. Thus $a = 1$ and $v = 0$ by the length identity. The complement of e is preserved and the isometry is $\gamma_0 \oplus 1$. The frame relation on M/wM and on e forces its entire residue change of frame to be $\mathrm{diag}(h_w, 1)$. Indeed the images of M/wM fill the first n coordinates and the image of e spans the last. Thus $h_w \in J_w$. Conversely every such small isometry extends by the identity and respects the displayed formulas. Surjectivity of the frames also proves uniqueness of all the frame changes. This proves the assertions, including the assertion about automorphisms. □

Use the coefficient representations and the normalized Fischer forms of Section 83.13. Put $L = \bigotimes_\tau L_\tau$, $s = \prod_\tau c_\tau^2 \kappa_\tau$ and take Z from Corollary 127.2. Theorem 83.28 gives $LL^* = sI$. A coefficient table assigns to each class a value fixed by its stabilizer acting on the coefficient space with the level character. In its algebraic dual table one uses the dual actions and inverse characters. Give the complex tables the squared norms $\sum_c \|Y_c\|^2/|\Gamma_c|$, where Γ_c is the stabilizer in the relevant class set. Using one representative per class is equivalent to using equivariant functions on all decorated lattices.

Write $\mathcal{H}_N$ for the full large coefficient table and $\mathcal{H}$ for the small one at the induced levels. The map $R : \mathcal{H}_N \to \mathcal{H}$ takes the value $ZLP(i(c))$ in the matching rational frames at c. It is well defined by the small block equivariance of L and Proposition 127.5. Let $\mathcal{T}$ be the small table with dual coefficient action. It pairs bilinearly with $\mathcal{H}$, using the same class masses.

Now choose the representatives j of the actual row classes used in the period. Remove their frames at Q, take the J_w-orbits of their retained frames, and write $c(j)$ for the resulting class. This procedure is defined on the chosen representatives for every row type. Choose a small transport from the representative of $c(j)$ to this object at the induced levels. Its action on small output coefficients, including the character, is denoted by U_j. Thus the output evaluated at the lift of j is $U_j(RP)_{c(j)}$. For a row test F, view F_j as a linear functional in that frame, with the normalized bilinear form used to take its values. Write $U_j^\vee F_j = F_j \circ U_j$, and let A_c be averaging over Γ_c on dual coefficients. Define

$$(DF)_c = A_c \sum_{j:\, c(j)=c} \frac{|\Gamma_c|}{|\Gamma_j|}\, U_j^\vee F_j.$$

The averaging sends the vector to the invariant dual table and makes the expression independent of the choice of transports.

Theorem 127.6 (Transfer of the prescribed terminal class sum)**.** *For every column P in the large table and every test F, the finite period with the specified representatives satisfies*

$$\mathcal{P}(P,F) = \langle RP, DF\rangle_{\mathcal{C}}, \qquad RR^\dagger = |Z|^2 sI.$$

At a complex embedding, $R^\dagger R/(|Z|^2 s)$ is zero off the classes $i(c)$ and acts there as L^*L/s in the matching frames. At the primes above g_1, g_2, T, this projection preserves the integral coefficient tables. The map R identifies its projected full lattice with the full integral small table, using the lattice frames of Proposition 104.34. At these primes D carries integral dual row values to integral dual values in the induced-level table.

Proof. Pair an invariant value $(RP)_c$ against the displayed sum. It pairs equally with a vector and its stabilizer average. The factor $|\Gamma_c|$ cancels the mass of c, leaving $F_j(U_j(RP)_c)/|\Gamma_j|$ at each specified representative. Their sum is the period of Theorem 127.1. Two choices of transport differ on the source by an element of Γ_c. This also proves the claimed independence after averaging. Passage from this representative sum to a Haar integral requires the contracted integrand to be invariant under the row level.

The large class $i(c)$ has the same mass as c, and the different lifted classes are disjoint. In the Hermitian norms, $R^{\dagger}$ therefore inserts $\bar{Z}L^{*}$ at $i(c)$ and zero elsewhere. The inserted values are fixed by the large stabilizer by Proposition 127.5 and equivariance. The two operator formulas follow. In rational frames a right inverse inserts $Z^{-1}L^{*}/s$. In lattice frames the small frame is extended on the large dual lattice with normal factor $\tau(\bar{\pi})^{-1}$ at τ. The normal action on the branching component has exponent $\delta_{\tau} = |\lambda_{\tau}| - |\mu_{\tau}|$. Consequently the frame change cancels the factor of Z in restriction and the factor of Z^{-1} in insertion, just as in Proposition 104.34. The matrices in these paired integral frames are L and L^{*}/s. Both preserve their lattices by Theorem 83.28, since the three primes exceed all the hook cutoffs. Their composition on small values is the identity, and in the opposite order it is the asserted projection.

For D, the small lattice frame is attached to the underlying lattice, and removing residue frames does not change it. Each transport carries it to the frame lattice at the target since its rational isometry carries the underlying lattice onto that lattice. The finite character factors are roots of unity. The stabilizers act faithfully on integral lattices of rank $n[E:\mathbb{Q}]$, by surjectivity of the residue frames. If a prime p divides such a group order, an element of order p has an eigenvalue of degree $p-1$ over $\mathbb{Q}$. The degree bound of Theorem 127.1 therefore makes every small mass denominator a unit at the three stated primes. The formula for D proves integrality. □

There is a more precise answer when the retained row groups H_w are contained in J_w with the matching dual characters. This includes the ordinary row, its further fixed-column subgroup, and the row with trivial first block, paired with the ordinary terminal column. Indeed its first block is upper unitriangular and its other diagonal blocks carry the specified determinant characters. Restricting its first-block character to each of the indicated subgroups gives the row character in the dual pairing. At every extra place $v \in Q$ the row stabilizes an r-dimensional subspace of k_v^n, and its character is the specified character of the determinant on the quotient, with the dual convention. These groups and characters are given explicitly in `Formalism/LocalToGlobal/Columns/RowForms.lean`, line 203 and lines 217–233 there.

Fix c. The possible refinements over its representative form a finite set Ω_c, acted on by Γ_c. At a retained place one chooses a refinement of its frame orbit from J_w to H_w. At an extra place one chooses an r-subspace. These choices are independent, since a residue frame is arbitrary at each of these places. In particular

$$|\Omega_c| = \prod_{w \in S_g} [J_w : H_w] \prod_{v \in Q} |\operatorname{Gr}(r,n)(k_v)|.$$

The fine classes over c are exactly the orbits in this set. For $y \in \Omega_c$, write Γ_y for its stabilizer. On this group the transported fine dual action differs from the coarse dual action by a finite scalar character θ_y. It is the product of the extra determinant characters with their indicated signs. This follows by cancelling the common rational coefficient action and the matching retained characters. Set

$$b_c = \#\{y \in \Omega_c : \ \theta_y|_{\Gamma_y} = 1\}.$$

In particular this is a count of individual refinements.

Theorem 127.7 (Multiplicity and integral image of the row transfer)**.** *On the dual coefficient space of each coarse class c, $DD^{\dagger} = b_c I$. At every prime above g_1, g_2, T the image of the full integral fine test table under D is precisely the direct sum of the full integral coarse test spaces at the classes for which $b_c > 0$. These assertions use the representatives of the given class sum even when a forgotten character is nontrivial. In particular the archimedean multiplicities and the integral image are independent of that choice. If the orders of all extra characters are relatively prime to $|\Gamma_c|$, then $b_c = |\Omega_c|$.*

Proof. Choose one representative of each refinement orbit. Via $U_j^{\vee}$, its testing space is the fixed space for the fine dual action of Γ_y in the coarse coefficient space. The transport is unitary in the complex metrics. The weight of the orbit in D is $m_y = |\Gamma_c|/|\Gamma_y|$. A fine invariant transforms under the coarse action of Γ_y by θ_y^{-1}. If this character is nontrivial, averaging over the subgroup is zero. Averaging over Γ_c first averages over Γ_y, so this orbit contributes zero to D. If the character is trivial, its fine invariant space contains the entire coarse invariant space. The operator A_c on that fine space is orthogonal projection onto the coarse invariants, since all these actions are unitary.

The ratio m_y cancels between the two class measures on taking the adjoint. At a contributing orbit the adjoint hence inserts the coarse value in the fine space, by inverse transport. It vanishes at every other orbit. Applying D once more sums the numbers m_y over the contributing orbits, giving b_c. Changing representatives transports the stabilizer and its character isomorphically and makes unitary changes in the coefficient spaces. The same computation therefore applies to the prescribed section.

Locally at any of the three primes all m_y are units. Given an integral coarse value at c with $b_c > 0$, choose one contributing orbit, insert its inverse transport divided by m_y, and take zero at all other orbits. This is an integral fine test and has the desired transfer. Theorem 127.6 gives the reverse inclusion, and the averaging calculation gives zero at all other classes. Finally, the order of the character on any Γ_y divides both its original order and $|\Gamma_c|$. The stated coprimality thus makes every orbit contribute. □

Proposition 127.8 (A uniform estimate from the extra residue fields)**.** *Use the ordinary induced level with its upper unitriangular first block, assume $1 \le r < n$, and put $d = n[E : \mathbb{Q}]$, $M_d = |\mathrm{GL}_d(\mathbb{F}_3)|$. Let Q' consist of the extra places of residue characteristic greater than $d+1$. Uniformly in the lattice class and in the extra characters,*

$$\frac{b_c}{|\Omega_c|} \;\ge\; 1 - (M_d - 1) \prod_{v \in Q'} \min\!\left(1, \frac{2(r+1)}{|k_v|}\right).$$

The bound also holds with common upper bounds for d, r on the interval, taking Q' with the larger degree cutoff.

Proof. The faithful integral action of Γ_c embeds it in $\mathrm{GL}_d(\mathbb{F}_3)$ after reduction, as in the proof of Theorem 119.10. For completeness, a torsion element in the kernel would give an element of prime order p of the form $1 + 3^a B$, $a \ge 1$, with B nonzero modulo three. In its p-th power the leading term is $p 3^a B$, which cannot be cancelled by the later binomial terms, also when $p = 3$. Thus $|\Gamma_c| \le M_d$.

A scalar in Γ_c is a root of unity, since the group is finite. The first block at one retained auxiliary place forces its reduction to be one. Its order is prime to that residue characteristic by the eigenvalue degree bound in the preceding theorem. For a root u of such order h, the sum $1 + u + \cdots + u^{h-1}$ reduces to the unit h. The identity $(u-1)(1 + \cdots + u^{h-1}) = 0$ gives $u = 1$. In fact every nonidentity γ in Γ_c is nonscalar modulo every $v \in Q'$. Otherwise

conjugation by γ on the endomorphisms of the localized lattice is a matrix A congruent to the identity. Its order divides the order h of γ, a unit at v by the same degree bound. The identity $(A-I)(I+A+\cdots+A^{h-1})=0$ implies $A=I$, since the second factor is invertible modulo v. Hence γ commutes with all endomorphisms over the fraction field and is already a global scalar.

Over a field of q elements, let g be a nonscalar invertible matrix of size n. Some v and gv are independent, since a map preserving every line is scalar by application to the basis vectors and their pairwise sums. Choose an r-space containing v and excluding gv. If X is its basis matrix, $[X \;\; gX]$ has rank at least $r+1$. Thus one of its minors is a nonzero polynomial of degree $r+1$ in the entries of a variable X. An invariant r-space forces this minor to vanish on every one of its basis matrices. A nonzero polynomial of total degree a vanishes on a proportion at most a/q of the full affine array. To see this, write its degree in the last variable as b. By induction the leading coefficient vanishes on a proportion at most $(a-b)/q$ of the earlier variables. On the others there are at most b roots in the last variable. The one-variable root bound starts the induction. The probability that a uniformly chosen n-by-r matrix has dependent columns is at most $\sum_{i=0}^{r-1} q^{i-n} \le 1/(q(q-1))$. Conditioning on independence therefore costs at most a factor of two, and the resulting column spaces are uniform. The fraction of g-invariant r-spaces is at most $2(r+1)/q$.

For each nonidentity $\gamma \in \Gamma_c$, independence of the choices at Q' bounds its fixed proportion in Ω_c by the product in the statement. Summing those proportions bounds the fraction having a nontrivial stabilizer. Every refinement with trivial stabilizer contributes to b_c for every choice of characters. This proves the inequality. Increasing the degree and rank bounds weakens it uniformly and proves the interval assertion. □

Here is a Haar integral for the same representative sum. Work at the compatible retained levels of Theorem 127.7 and let H denote the original compact open row level. Use the double-coset measure of Theorem 122.17. The ratio between the fine and the pulled-back coarse actions on dual coefficients is a character θ of the frame-change group, trivial on rational isometries. At a fine stabilizer it is θ_y in transported coordinates. On output coefficients the ratio is θ^{-1}.

Corollary 127.9 (The character in a Haar realization of the finite sum)**.** *There is a scalar function a on the finely framed objects, with values zero or roots of unity in the coefficient field, such that for every P, F as above,*

$$\frac{1}{\operatorname{vol}(H)} \int F(h)\big(a(h)(RP)(h)\big)\, dh = \mathcal{P}(P,F).$$

Here the integral is over the definite small unitary quotient with the weight at infinity contracted, and RP is pulled back by forgetting the fine frames. If θ is nontrivial on the right compact level, the integral with a replaced by the constant one is zero for every P, F.

More generally suppose a candidate scalar kernel a' has the same transformation law and let a'_j be its values at the prescribed representatives. For any coarse output table Y, the fine table norms satisfy

$$\|(a'-a)Y\|^2 = \sum_{c\in\mathcal{C}} \frac{\|Y_c\|^2}{|\Gamma_c|} \sum_{\substack{j:\, c(j)=c \\ \theta_j|_{\Gamma_j}=1}} \frac{|\Gamma_c|}{|\Gamma_j|}\, |a'_j - 1|^2.$$

The first identity applies in particular to any earlier signed column with values in $\mathcal{H}_N$.

Proof. On a contributing orbit give the selected representative value one and set $a(gj) = \theta(g)^{-1}$ at a change of decorated object by g. This is well defined since θ is one on the stabilizer. On

every other orbit set $a = 0$. The resulting function is invariant under rational isometries and cancels the character of the paired expression on right level translates. It defines a scalar weight on the quotient with the required transformation law. A noncontributing orbit had zero contraction in the original representative sum. Indeed on its stabilizer the coarse output and fine test pairing transforms by a nontrivial scalar, while both class values at the stabilized point are fixed in their respective actions. Its contraction must therefore be zero. At each contributing representative the inserted factor is one. The contracted product is now right H-invariant. A double coset at j has mass $\mathrm{vol}(H)/|\Gamma_j|$, by counting the finite stabilizer in the compact quotient. This proves the integral identity on summing the cosets. Without the factor, right translation by any $h_0 \in H$ multiplies the contracted integrand by its discrepancy character (or its inverse with the opposite convention for right translation). Haar invariance forces its integral to vanish when that character takes a value different from one.

Any other scalar function with the same law must also vanish on an orbit with nontrivial stabilizer character, by evaluating at a point fixed by that stabilizer. On the other orbits, the squared norm at the representative is $|a'_j - 1|^2 \|Y_{c(j)}\|^2$, since coefficient transports are unitary. Summing with the fine masses proves the second identity. The construction is integral at every finite place on coefficients, since each of its nonzero scalar values is a root of unity. □

For any earlier signed column map Θ with values in $\mathcal{H}_N$, the tested functional is exactly $\langle R\Theta(\cdot), DF\rangle_{\mathcal{C}}$. Thus on unrestricted fine tests the transfer contributes the explicit multiplicities above. For a specified projected test space or lattice S its contribution is $D(S)$. To deduce the estimate of Section 6.6 one still has to control the earlier signed image on the lifted classes against these transported tests and identify its Gram form with that of the designated analytic row.

127.4 The signed terminal column on ramified residue flags

Fix the field E_x, word, translate and auxiliary character of Section 6.6. Write Q for its extra marked places, $n = r + mt$ for the small terminal rank and a for the preceding column rank. At $v \in Q$ the local group is split, with residue field k_v of order q_v, and the character on residue units is a nontrivial character $\xi_v : k_v^\times \to J^\times$. Here J is the coefficient field, containing the character values. We use the character itself in the determinant below. Write $A = k_v^r \subset k_v^n$ for the first coordinate plane, $G_v = \mathrm{GL}_n(k_v)$ and P_v for its stabilizer. At a complex embedding all the character values have absolute value one.

The preceding section treated restriction of a column whose extra frames had already been forgotten. We now evaluate the terminal kernel of the signed arithmetic chain at those frames. A basis of the small target at v means an isomorphism from the coordinate space to the residue space of its lattice. Changing this basis by $h \in G_v$ means composition on the right by h. Thus hA is the new marked plane in the original coordinates. Set

$$\lambda_v(p) = \xi_v(\det(p \mid k_v^n/A)) \qquad (p \in P_v).$$

Proposition 127.10 (The flag and phase of a terminal matrix)**.** *Fix a summand at the last edge, including its source frame and coupling vector coset. Let $B : k_v^a \to k_v^n$ be the reduction of the small projection of its matrix in the reference bases, when this projection is integral. If it is nonintegral, or if B has rank less than r on the first r coordinates, the summand vanishes for every target basis. Otherwise put $I = B(k_v^r)$. Apart from one factor independent of h, its value as a function of the target basis is*

$$b_B(h) = \mathbf{1}_{hA=I}\, \xi_v(\det h)\, \xi_v\Big(\det((h^{-1}B)|_{k_v^r} : k_v^r \to A)\Big). \tag{127.1}$$

The determinant on the right is evaluated only on the support of the indicator. For $p \in P_v$ one has $b_B(hp) = \lambda_v(p) b_B(h)$.

Proof. Let x be the matrix from the preceding lattice to the large terminal lattice $M \perp \mathcal{O}_E e$. Its small projection is projection along e followed by expression in the basis of M. At this step the index is $m-1$, so the number $m-1-(m-1)$ of auxiliary columns in the source is zero. The target already has zero auxiliary columns. Consequently the reordered entries in the terminal position test are the ordinary entries of this projection. The test requires them all to be integral. It requires the last $n-r$ entries in each of the first r columns to vanish modulo v, and weighs the first r-by-r minor by ξ_v, extended by zero on nonunits. These statements follow by reading the defining position test in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 711 and its terminal arguments on lines 1141–1170. An invertible integral basis change preserves integrality in both directions. On the reduction it changes the matrix to $h^{-1}B$. The two remaining requirements say exactly that the first r columns give a basis of hA. This proves the indicator and the minor factor.

For the other factor, the coefficient of a matrix coset in the theta distribution is its derivative operator multiplied by a source determinant character and the inverse target determinant character. The derivative operator uses the two lattices and the matrix coset. Its arguments contain no residue bases. The target determinant factor before inversion, on multiplying the target basis by an integral unit matrix u, changes by $\xi_v^{-1}(\det \bar{u})$. Indeed its local character is defined by the uniformizer power times the inverse residue character of the unit part. Multiplication by a unit preserves the power and multiplies the residue unit by its reduction. The extension of the small basis to the large target sends the last basis vector to πe in residue coordinates, as follows from the frame formula in Proposition 127.5. Its basis change has residue $\operatorname{diag}(h, 1)$, hence determinant $\det h$. Any two integral lifts of that residue change have determinant ratio one modulo v. Inverting the target factor thus supplies precisely $\xi_v(\det h)$. The determinant and derivative formulas are in `Formalism/LocalToGlobal/KernelLattices/ThDistC.lean`, line 123 and lines 137–145 there. The local character is specified in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 935.

The last covector, the additive Fourier test on the coupling coset, and the finite masses at the preceding vertices use the original lattices and source bases. None uses the small target residue basis at v. This accounts for every remaining factor in the summand. Finally on replacing h by hp, the minor acquires the inverse determinant of $p|_A$. Multiplication by $\det p$ leaves the determinant on the quotient. This proves the transformation law. □

Fix representatives c of small lattice classes retaining the actual row levels at the original two auxiliary primes and forgetting only the extra frames at Q. Let Γ_c be the stabilizer. Choose one reference basis at each extra place in each class. Put

$$\Omega = \prod_{v \in Q} G_v/P_v, \qquad d_Q = |\Omega|, \qquad G_Q = \prod_{v \in Q} G_v.$$

For each tuple of planes $I \in \Omega$, choose $h_I = (h_{I,v})$ carrying the standard planes to I. For a fixed arithmetic input ϕ, expand the signed theta chain at c. A term γ indexes its preceding classes, matrices, finite types and coupling coset. Discard terms failing the integrality and rank tests above. Denote the tuple of surviving images by I_γ. Let $C_{\gamma,c}$ be the coefficient vector with the factors in (127.1) removed. It includes the earlier signed tests, all preceding masses, the Fourier factors, the derivative operators, and the last branching map. This map is independent of the extra bases. Thus the finite array

$$Y_{\phi,c}(I) = \sum_{\gamma : I_\gamma = I} C_{\gamma,c} \prod_{v \in Q} b_{B_{\gamma,v}}(h_{I,v}) \tag{127.2}$$

uses the actual coefficient in its chosen basis. In particular a term can occur in just one entry of this array. If Q is empty the array has one entry.

Theorem 127.11 (The Haar pairing and the square norm on the fine row)**.** *Let F be a test in the prescribed dual row space, including any imposed projection on that space. Write $F_c(h_I)$ for its value, viewed as a functional on the small coefficient space. The signed arithmetic period is*

$$\mathcal{P}_{\mathrm{chain}}(\phi, F) = \sum_c \frac{1}{|\Gamma_c|} \sum_{I \in \Omega} F_c(h_I)\big(Y_{\phi,c}(I)\big) = \frac{1}{\mathrm{vol}(H)} \int F(h)\big(Y_\phi(h)\big)\, dh. \tag{127.3}$$

Here H is the original compact open row level, infinity is contracted in the same coefficient pairing, and Y_ϕ is the signed arithmetic column after branching. Its squared norm in the mass metric of this fine level is

$$\|Y_\phi\|^2 = \sum_c \frac{1}{|\Gamma_c|} \sum_{I \in \Omega} \|Y_{\phi,c}(I)\|^2.$$

With probability measure on the extra full residue bases, their mean square in any fixed class is $d_Q^{-1} \sum_I \|Y_{\phi,c}(I)\|^2$.

Proof. The class sum for the signed chain pairs its evaluated terminal column with the original test. The coefficients at all preceding vertices are independent of the last row basis. Expanding those finite sums and applying Proposition 127.10 at each extra place gives (127.2). Finiteness here follows from the finite lattice quotients for each matrix and coupling vector, and from finiteness of definite classes at the indicated compact levels. For the former one may take local bases and reduce modulo the common product of the prescribed prime powers, as in Section 73.1. The coupling cosets are quotients of two local lattices of the same rank. For the class number assertion, the underlying integral trace lattice is positive definite of fixed determinant, so it has finitely many isometry classes by Minkowski reduction. On any one of them multiplication by each element a_0 of a fixed integral basis of E_x has operator norm bounded by $\max_\tau |\tau(a_0)|$, by the orthogonal decomposition at infinity. Only finitely many integral matrices satisfy this bound. The trace pairing on the field recovers the Hermitian form from its traces after these multiplications. Residue frames add only finitely many choices.

Each test has change-of-basis character λ_v^{-1} in this basis convention. To see the sign, the frame in the row definition is the map *to* the coordinate space. Right multiplication of its inverse basis by p acts on this map by left multiplication by p^{-1}. The character acting on tests for that left operation is the determinant character on the quotient, since the row uses the inverse of the level character specified on line 231 of `Formalism/LocalToGlobal/Columns/RowForms.lean`, line 230 in the coefficient action on line 269. Evaluating at p^{-1} gives $\lambda_v(p)^{-1}$. Proposition 127.10 therefore makes the contraction independent of the choices of h_I. A rational isometry transports both coefficients and preserves the branching contraction. Equivalently one can reindex the finite theta sum by transporting each matrix with its two endpoint lattices. It preserves the matrix cosets, the coupling cosets and all their masses. The integrand thus has one value on each Γ_c-orbit in Ω. The fine classes over c are these orbits. An orbit with stabilizer $\Gamma_{c,I}$ has $|\Gamma_c|/|\Gamma_{c,I}|$ elements. This gives the first equality with the correct fine masses. The Haar mass of its double coset is $\mathrm{vol}(H)/|\Gamma_{c,I}|$ in the normalization of Theorem 122.17, proving the second equality. The same count applies to the squared norm, since all level characters and transports are unitary. Each plane tuple has $\prod_v |P_v|$ bases, giving the asserted probability mean. □

The frame calculation also gives an integral projection in the global row space with full residue frames at the places of Q. Let $R(g)Y(h) = Y(hg)$ be right translation. The finite groups at different places commute. The mass metric, equivalently the Haar metric with any

common scaling, is invariant under this action. Take a J-subspace V of the actual column space with the standard character $R(p) = \prod_v \lambda_v(p_v)$ on V. For example one can take the span of any of the signed columns in this row with fixed extra characters. Put

$$W = \sum_{I \in \Omega} R(h_I)V, \qquad b = \min(r, n - r).$$

These translates describe the span inside the given fully framed space. The choice of representative of I does not change its summand.

Theorem 127.12 (Orthogonal flag summands and an integral projector)**.** *Assume $0 < r < n$. The sum defining W is orthogonal at each complex embedding. Its Gram matrix on the translates of any fixed list in V consists of d_Q copies of the Gram matrix of that list. For each $v \in Q$ choose a unit $t_v \in k_v^\times$ such that $z_v = \xi_v(t_v)$ generates the image of ξ_v. Define*

$$D_Q = \prod_{v \in Q} q_v^{\binom{n}{2}} (z_v - 1)^b \in \mathcal{O}_J \setminus \{0\}.$$

On W, multiplication of the projection onto any one summand by D_Q is the action of an element of $\mathcal{O}_J[G_Q]$. The assertions still apply after a projection on the full row space that commutes with G_Q, using the resulting image of V.

Proof. First work at one place and omit its subscript. Write $V_I = R(h_I)V$. An element fixing I acts on V_I by ξ of its determinant on k^n/I, by conjugation. For distinct planes I, I', choose a line in I outside I'. There is a hyperplane containing I' complementary to that line, whose intersection with I complements it in I. For instance take the kernel of a functional vanishing on I' and nonzero on the line. Multiplication by t on the line and by one on this hyperplane preserves both planes. Its determinants on the quotients by I, I' are respectively one and t. The two spaces thus have different eigenvalues for a unitary operator and are orthogonal. Right translation preserves the Gram matrix within a single space. On a product of places two different tuples differ at some place, giving the same orthogonality. This proves the first assertions. The argument uses V as a multiplicity space and imposes no independence on its chosen list.

For the projector at one place take U to be the upper unitriangular subgroup in the reference basis. Its order is $q^{\binom{n}{2}}$. Its orbits on the Grassmannian consist of the subspaces with a fixed subset J_0 of r pivot positions, using the last nonzero entry as pivot in each echelon column. Indeed the coordinate plane with these positions can be sent to any plane in the cell by using its pivot-normalized columns as the corresponding columns of an upper unitriangular matrix. The other columns can be chosen to be the original coordinate vectors. The dimensions of intersection with the initial coordinate spaces are invariant under U, so different pivot sets lie in different orbits.

Let $E_U = |U|^{-1} \sum_{u \in U} R(u)$. In the direct sum over a cell the U-invariant vectors are determined by their component at its coordinate plane. The stabilizer there acts trivially on that component, since its quotient action is unipotent and has determinant one. Every value at the coordinate plane therefore extends uniquely to a U-invariant vector over the cell, by translation. A diagonal matrix normalizes U and fixes that plane. On these invariant vectors the diagonal matrix t_i having entry t only in position i and ones elsewhere acts by z if $i \notin J_0$, and by one if $i \in J_0$. This follows by evaluating the translate at the coordinate summand and using its quotient determinant.

Let the standard pivot set be $A_0 = \{1, \dots, r\}$. When $r \le n - r$, apply to the image of E_U the operator

$$\prod_{i \in A_0} \frac{z - R(t_i)}{z - 1}.$$

It retains the cell with every $i \in A_0$ a pivot. There is only one such pivot set, and its cell consists of the standard plane alone. When $n - r < r$, use instead

$$\prod_{i \notin A_0} \frac{R(t_i) - 1}{z - 1}.$$

This requires every index outside A_0 to be a nonpivot, with the same consequence. On the standard summand E_U and the selected product both act as the identity. Every other cell is killed. Thus their composite is the required projection. Clearing the displayed denominators and $|U|$ produces an integral combination of group elements. In the product case one applies this one-place operator with all other places fixed, and multiplies the commuting projections. Conjugation by h_I gives the projection onto the desired translate. A projection commuting with G_Q carries a standard character vector to a standard character vector, so the entire proof applies to its image. □

Corollary 127.13 (Cost for the integral row lattice)**.** *Let Λ be a G_Q-stable full $\mathcal{O}_J$-lattice in the full-frame row space. Inside W put*

$$\Lambda_{\mathrm{sat}} = \Lambda \cap W, \qquad \Lambda_{\mathrm{sep}} = \bigoplus_{I \in \Omega} R(h_I)(\Lambda \cap V).$$

Then $D_Q \Lambda_{\mathrm{sat}} \subseteq \Lambda_{\mathrm{sep}} \subseteq \Lambda_{\mathrm{sat}}$. For $d = \dim_J W$ their index satisfies

$$\begin{aligned} \frac{\log[\Lambda_{\mathrm{sat}} : \Lambda_{\mathrm{sep}}]}{[J : \mathbb{Q}]} &\le d \sum_{v \in Q} \left\{ \binom{n}{2} \log q_v + b \frac{\log \Phi_{e_v}(1)}{\varphi(e_v)} \right\} \\ &\le d \left(\binom{n}{2} + b \right) \log N_{E_x/\mathbb{Q}} \mathfrak{f}_\xi. \end{aligned}$$

Here $e_v = \mathrm{ord}(\xi_v) > 1$, Φ_j is the cyclotomic polynomial, and $\mathfrak{f}_\xi$ is the finite conductor of the auxiliary character on E_x. At a coefficient place whose residue characteristic divides none of the q_v, the two lattices coincide unless at least one e_v is a power of its residue characteristic. These conclusions apply to the full-frame integral test tables and their duals, and to intersections with row projections at the retained primes.

Proof. For each vector of $\Lambda \cap W$, its component in the I-summand multiplied by D_Q lies in Λ by the group-ring formula. The intersection of Λ with that summand is exactly $R(h_I)(\Lambda \cap V)$, since the translation and its inverse preserve Λ. Summing the components gives the inclusions. The quotient is locally a quotient of d copies of $\mathcal{O}_J/(D_Q)$, since a full lattice is locally free over each discrete valuation ring. Multiplying the local index bounds gives $[\Lambda_{\mathrm{sat}} : \Lambda_{\mathrm{sep}}] \le |N_{J/\mathbb{Q}} D_Q|^d$.

The conjugates of z_v in its cyclotomic field are the primitive e_v-th roots. This gives the first logarithmic bound by taking norms. For $e > 1$, the identity $\prod_{j \mid e,\, j > 1} \Phi_j(1) = e$ follows by factoring $(X^e - 1)/(X - 1)$ and setting $X = 1$. Induction on e gives $\Phi_e(1) = p$ when e is a power of a prime p, and one otherwise. Also $\log \Phi_e(1)/\varphi(e) \le \log 2$ since every conjugate of $1 - z_v$ has absolute value at most two. Each extra prime divides the conductor since its character on units is nontrivial. Its contribution to the norm of the conductor is at least q_v. These observations give the second inequality and the local assertion.

For the integral coefficient tables the lattice attached to a small object depends on its Hermitian lattice. After adjoining the last vector its coefficient lattice depends on the lattice and the terminal covectors. A change of the extra residue bases fixes these data, permutes the entries and preserves the full integral table. The definitions occur in `Formalism/LocalToGlobal/Columns/CoefCol.lean`, line 89 and lines 93–103 there. Dualizing preserves this stability.

Frame changes or averages at the two retained primes commute with G_Q, since they act on separate local factors. Their rational images intersected with the integral table are therefore stable as required. This use of intersections allows a projector with a nonunit denominator. □

In the fixed standard row of (127.3), cancellation can still occur among the paths with the same tuple of image planes. When that row is paired with a specified analytic functional, the values of the functional on the surviving entries also enter. The flag projection computes the integral loss from separating translates. Section 6.6 still needs the Gram estimate for the specified words and the comparison to the analytic Bessel periods on the same interval.

127.5 The common vector at the two marked primes

Keep a field, word, translate and choice of sign from Section 127.4. Let S_g be the marked places over g_1, g_2. These primes split completely. Fix $w \in S_g$, put $O = \mathcal{O}_{E_w}$, and write $k = O/qO$, $q = |k|$. Thus the rational integer q is a uniformizer here. Fix all the class representatives, matrix cosets and type indices on one path. Leave unsummed the vector shared by the last two edges. We compute its contribution before the signed paths are added.

At the penultimate edge put $n_0 = N(m-2)$, $d = N(m-1)$ for the two core dimensions. There is one extra source coordinate and no extra target coordinate. In the chosen source and target frames let X be the d-by-n_0 core matrix and y the extra column. In the terminal edge write z for the row covector obtained by composing the last matrix with the last coordinate divided by π_T, in the same intermediate frame. Here π_T is the specified generator at the terminal prime. Its local Fourier factor is zero unless $z \in O^{1\times d}$. When z is integral, its factor is $\psi(-zy)$, where ψ has conductor O. The shared vector ranges over $q^{-1}O^d/O^d$. Write $\theta(t) = \psi(\tilde{t}/q)$ for the nontrivial additive character on k, using any integral lift. Then $v = \overline{qy} \in k^d$ and this factor is $\theta(-\bar{z}v)$. Additive Haar measure with $\mathrm{vol}(O^d) = 1$ gives each vector coset mass one. These are the two marked factors in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1131 and lines 1149–1170 there.

Use indices starting at zero. Let h be the penultimate source representative in its full n_0+1 coordinates. It preserves SO^{n_0+1}, where $S = \mathrm{diag}(1,\dots,1,q)$. This is the specified marked-frame condition, expressed on the lattice before the columns are reordered. Write P for the permutation matrix inserting the last coordinate at its marked position. Then $U = S^{-1}hSP$ belongs to $\mathrm{GL}_{n_0+1}(O)$. With $C = [X \mid 0]U$ and b the last row of U, the tested matrix is

$$[X \mid y]hSP = [X \mid qy]U = C + (qy)b.$$

The determinant multiplier from the source action on the Schrödinger test is denoted by c_h. It is the original splitting character evaluated on $\det h$, since h preserves a full lattice and hence $\det h$ has valuation zero. All values of this character on units are roots of unity.

Put $b_0 = m-2$, $a = r + b_0 t$, and, for $0 \le i < b_0$, let $B_i = \{r+it, \dots, r+(i+1)t-1\}$ be the diagonal blocks after the initial r coordinates. The rank formulas are $N(j) = k_0 + j(t+1)$ and $r = k_0 + m - 1$, so

$$a = N(m-2)+1 = n_0+1, \qquad d = N(m-1) = a+t.$$

Thus all source columns enter the penultimate test. Write $A_j \subset k^d$ for the first j coordinate positions, and $s_i = r + (i+1)t$. The ranks are specified in `Formalism/LocalRingCardinality/Params.lean`, line 32. Write χ_i for the residue determinant characters and $\nu_i = \chi_i \chi_{b_0}^{-1}$. Extend every multiplicative character, including the trivial one, by zero at zero.

Proposition 127.14 (The support and the varying determinant block)**.** *If any entry in the first a columns of C is nonintegral, the local summand vanishes for every v. Otherwise apply*

the entry congruences of the marked test to the residue matrix $\bar{C} + v\bar{b}$. Their solution set $D \subset k^d$ is empty or is given by fixing a subset of the individual coordinates. Choose $v^0 \in D$ when it is nonempty, write $D = v^0 + W$, and put $f = \dim_k W$.

On D the penultimate test, with c_h removed, equals its prescribed additive character in the superdiagonal entries times

$$\prod_{i=0}^{b_0-1} \nu_i(d_i + L_i v'), \qquad v' = v - v^0 \in W.$$

Here d_i is the determinant of the B_i diagonal block at v^0, and L_i is a linear form on its free row coordinates, extended by zero on the other free coordinates. At most one L_i is nonzero. All constants and linear forms in this description are computed from C, b, v^0.

Suppose that the test is nonzero at some position. Let H be the image of the first a rows of $\bar{X}$ in A_a. It is a hyperplane. Let λ_X be a nonzero form on A_a with kernel H. If the first nonzero entry of $\bar{b}$ occurs among the initial r columns, all determinants are constant on D. If it occurs in B_i, then $W = A_{s_i}$, the i-th determinant varies, and L_i is a nonzero multiple of $\lambda_X|_{A_{s_i}}$. On this space it is supported on the B_i coordinates.

Proof. The equality of tested matrices follows by multiplying out the source dilation and permutation. It uses the identity target representative in the penultimate edge. Since U is integral, adding $(qy)b$ on the prescribed window cannot change failure of integrality of C. Once integrality holds, every required residue entry is of the form $\bar{C}_{ij} + v_i\bar{b}_j$.

We detail the congruences used in the source's two tests. Rows with index at least a vanish on the first a columns. The leading r-by-r square is upper unitriangular in the plain test. In the symmetrized test it is upper unitriangular after ordering these indices by the permutation exchanging the two initial consecutive groups of sizes k_0, r_S. Its additional zero entries in column $k_0 + r_S$ at rows $i < k_0$ are imposed whenever that column lies in the initial r columns. In both tests the rows in a later block vanish on the first r columns and on columns of every earlier block. These are entrywise zero or one equations. Each constrains just one variable v_i, either fixing its value, giving no restriction, or being inconsistent. Intersecting all of them gives the assertion about D. The lists, including the additive sums in the two orders, are in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 771, lines 780–810 there.

In the leading a-by-a matrix the block below the initial square is zero, and the submatrix on the remaining blocks is upper block triangular. The initial square has determinant one, also after its simultaneous row and column permutation. It follows that the full determinant is the product of the B_i determinants. Multiplying its inverse character $\chi_{b_0}^{-1}$ by the prescribed individual characters gives the displayed ν_i. If a determinant vanishes, both versions of the original expression vanish, since the unit-character factor of a nonunit is zero.

In block i the row changes at v^0 are scalar multiples of the same row $\bar{b}|_{B_i}$. By multilinearity of the determinant, every term taking changes from two rows vanishes. The coefficient of v'_j for a free row $j \in B_i$ is the determinant obtained by replacing row j of the base block by $\bar{b}|_{B_i}$. This computes L_i. The last row of the invertible matrix $\bar{U}$ has a nonzero entry. Take the earliest of the initial group and subsequent blocks in which its restriction is nonzero. All earlier blocks have zero replacement rows. In each later block every row is constrained to vanish in the nonzero column of that earliest group. Hence the variable in that row is fixed throughout D. There are no free rows in these later blocks. Only the earliest group can supply a varying determinant, and the initial group has constant determinant one.

If the earliest group is B_i, none of the entry equations on a row of index less than s_i uses a nonzero coefficient of v. Indeed the equations on the first r columns have $\bar{b}_j = 0$, and a block

column can constrain only rows in strictly later blocks or below the whole square. Columns before B_i have zero coefficient as well. Every row of index at least s_i is fixed by a column of B_i. This gives exactly $W = A_{s_i}$. At a position where the test is nonzero, the B_i block is an invertible matrix M_i. The linear part of its determinant in the free block rows has coefficient row $\det(M_i)\bar{b}|_{B_i} M_i^{-1}$, which is nonzero. This follows either from the row replacements above or by multiplying the rank-one update on the right by M_i^{-1}.

Since all columns are tested and U is invertible over O, integrality of C implies integrality of X. Write $\bar{X}^{(a)}$ for its first a rows after reduction. The top determinant of the tested matrix is

$$\det(\bar{U}) \det[\, \bar{X}^{(a)} \mid v_{<a} \,].$$

At a position passing all tests this is nonzero. Thus $\bar{X}^{(a)}$ has rank $a-1$, and the second determinant as a linear function of its last column has kernel exactly H. Also all rows of $\bar{X}$ below row index $a-1$ vanish, since the tested matrix vanishes there modulo q and $\bar{U}$ is invertible. On D this same top determinant is the product of the diagonal determinants. For the earliest group B_i, the other factors are nonzero constants on D. Taking linear parts on A_{s_i} proves the assertion about λ_X. □

Theorem 127.15 (The marked Fourier coefficient of the coupled edges)**.** *Let $\mathcal{F}_w(z)$ be the sum over the common vector of the penultimate local test times its terminal Fourier factor. The top matrix test at the last edge is independent of this vector and is kept as a separate multiplier. Write $\Phi_X(y)$ for the penultimate local factor, including c_h. Then, with the measure fixed above, one has the equality of local integrals*

$$\mathcal{F}_w(z) = \int_{E_w^d} \Phi_X(y)\psi(-zy)\, dy.$$

Set $\mathcal{F}_w(z) = 0$ for a failed integrality test, an empty D, or a singular constant determinant block. In the other cases there is a root of unity ζ and a form $\omega \in W^\vee$ prescribed by the data as follows. Write the product of the additive superdiagonal character and $\theta(-\bar{z}v)$ on $v^0 + W$ as its value at v^0 times $\theta(\omega(v'))$. Multiply this value by c_h and the $\nu_i(d_i)$ of the constant blocks to obtain ζ.

If all determinants are constant, $\mathcal{F}_w(z)$ is ζq^f for $\omega = 0$ and zero otherwise. If a determinant varies, write d_*, L, ν for its constant, nonzero linear form and character. Then $\mathcal{F}_w(z) = 0$ unless $\omega = sL$ for $s \in k$. For this frequency one has the exact formula

$$\mathcal{F}_w(z) = \zeta\, q^{f-1}\theta(-sd_*)G_s(\nu), \qquad G_s(\nu) = \sum_{a' \in k^\times} \nu(a')\theta(sa').$$

For nontrivial ν this is nonzero precisely when $s \neq 0$ and has absolute value $q^{f-1/2}$. For trivial ν the Gauss factors are $q-1$ at $s=0$ and -1 at $s \neq 0$. These absolute values hold at every complex embedding.

Proof. Because every source column is tested, integrality of the tested matrix $[X \mid qy]U$ is equivalent to integrality of both X and qy. Hence the integrand is supported in $q^{-1}O^d$. The penultimate factor is invariant under translation by O^d, since such translations preserve every tested residue. For integral z the Fourier factor is constant on each of the same cosets, of mass one. For nonintegral z its integral on each coset vanishes by translation, since ψ has conductor O. These give respectively the finite sum and the zero specified above.

The additive exponent for the plain test is the sum of entries $M_{i,i+1}$, $0 \le i < r-1$. In the symmetrized test it is the sum within each of the two specified strings $0, \dots, k_0 - 1$ and $k_0, \dots, r-1$. Each entry is affine linear in v. Any additive character of k equals $u \mapsto \theta(cu)$ for

a unique $c \in k$. Indeed these characters are distinct and there are $|k|$ of them. This proves the prescription of ω, also for the two different choices of superdiagonal character.

When all determinants are constant, averaging a nontrivial additive character over the vector space W gives zero by translation, and the trivial character sums to q^f. Otherwise sum on fibers of L. The sum vanishes when the character is nontrivial on $\ker L$. Triviality there is equivalent to $\omega = sL$, since each nonzero k-linear map to k is surjective. The fiber $d_* + Lv' = a'$ then contributes $q^{f-1}\nu(a')\theta(s(a' - d_*))$. Summing proves the formula with the stated sign.

For nontrivial ν, summation over the multiplicative group gives $G_0 = 0$. For $s \neq 0$, substitution gives $G_s = \nu(s)^{-1}G_1$. Expanding the squares and summing first over s gives $\sum_s |G_s|^2 = q(q-1)$ by additive orthogonality. Hence $|G_1| = \sqrt{q}$. For the trivial character, direct summation of the nonzero additive phases gives the remaining two values. Applying a complex embedding permutes the roots of unity, preserves triviality of a character and preserves the same orthogonality calculation. □

Corollary 127.16 (Application to the signed column and its local Fourier tables)**.** *For each path in the flag array of Theorem 127.11, fix all choices other than the common vector at the places of S_g. The sum over these components of the vector is obtained by inserting $\prod_{w \in S_g} \mathcal{F}_w(z_w)$ in the original signed coefficient in place of their two vector-dependent factors. This formula includes the base tuple $Q = \varnothing$. For a path on which the cells are nonempty, write*

$$D_g = \prod_{w \in S_g} q_w, \qquad K_g = \prod_{w \in S_g} q_w^{-f_w} \mathcal{F}_w(z_w).$$

Then $D_g K_g$ is an algebraic integer and at every complex embedding σ a nonzero K_g satisfies

$$D_g^{-1} \leq |\sigma K_g| \leq 1, \qquad \log D_g \leq [E_x : \mathbb{Q}] \log(g_1 g_2).$$

The factor $\prod_w q_w^{f_w}$ is retained individually when adding the paths. For the marked characters prescribed on the common intervals, let D_{var} be the product of q_w over the places with a varying determinant. If $K_g \neq 0$, then

$$|\sigma K_g| = D_{\mathrm{var}}^{-1/2} \quad \text{at every embedding,} \qquad D_{\mathrm{var}} K_g \text{ is an algebraic integer.}$$

For the full local frequency table at one marked place, assuming the cell is nonempty and the constant determinants are units, one has

$$\sum_{\bar{z} \in (k^d)^\vee} |\mathcal{F}_w(z)|^2 = q^{d+f}(1 - q^{-1})^\delta,$$

where $\delta = 1$ if a determinant varies and is zero otherwise. For a nonzero table with varying determinant in B_i and nontrivial ν_i, its frequency support is exactly

$$\bar{z}|_{A_{s_i}} \ \in \ k^\times(\lambda_X|_{A_{s_i}}).$$

Thus on the first s_i coordinates the covector vanishes on $H \cap A_{s_i}$ and is nonzero on A_{s_i}. There is no restriction on its later coordinates. For two such tables in one intermediate frame the supports are disjoint if their varying blocks differ or, for the same block, if the two hyperplanes $H \cap A_{s_i}$ differ. Their inner product then vanishes for every nonnegative weight on the residue covectors, including weights supported on prescribed terminal matrices.

Proof. At the intermediate lattice the global common vector ranges over the quotient prescribed by the position and translation lattices at $S_g \cup Q$. The map to the product of these local quotients is bijective. Injectivity is exactly their defining congruence. For surjectivity choose

a common integral denominator, clear it, and prescribe the resulting residues modulo powers of the finitely many different prime ideals. The Chinese remainder theorem applies to the intermediate lattice, a finite projective module over $\mathcal{O}_{E_x}$. Equivalently one may prescribe these residues over its semilocalization, with the same finite quotients after clearing the denominator. Passing to the chosen local frames identifies each marked quotient with $q_w^{-1}O_w^d/O_w^d$.

In the two matrix sums the derivative operators depend on the matrix cosets, coefficient data and frames. On fixing those choices, the vector occurs only in the penultimate position test and the terminal Fourier character. The terminal indicator at T, the projected small-matrix test and the determinant frame factors of the terminal matrix are independent of the vector. The pair uses one common vector in the composition. Its class mass is taken once, outside the sum over that vector. These are the explicit composition and two matrix formulas in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 298 and lines 1269–1277 there. By the product decomposition of the quotient we may sum first over all the marked components while fixing the components at Q. This gives precisely the product of the marked Fourier coefficients. In particular it leaves the image flag of each terminal matrix unchanged. The formula then commutes with summation over paths, branching on the last coefficient space, and pairing against any of the projected rows, by linearity of those operations.

Theorem 127.15 expresses each normalized factor as zero, a root of unity, or a root of unity times $G_s(\nu)/q_w$. The Gauss sum is an algebraic integer because it is a sum of products of roots of unity. Each of its nonzero absolute values at a complex embedding lies between one and q_w. This proves both bounds for the product. Each marked place above a completely split prime has that prime as residue size. In fact exactly half the primes above each g_j are marked. The ideals of its generator and its conjugate are coprime, have product $g_j\mathcal{O}_{E_x}$, and conjugation interchanges them. Hence $\log D_g$ equals $\frac{1}{2}[E_x:\mathbb{Q}]\log(g_1g_2)$.

At either marked rational prime, the prescribed characters on the residue units are $\chi_j = \chi^{2(j+1)}$ for a faithful character χ of $k^\times$. These are the explicit local choices in `Formalism/LocalToGlobal/Columns/AbelianHostFieldConditions.lean`, line 301. The cyclic group $k^\times$ has order $q-1 > 2m$ on the sufficiently distant intervals, by the prime choices in Proposition 6.30. For $0 \le i < b_0$ the exponent $2(i-b_0)$ is nonzero and has absolute value less than $q-1$. Thus every varying character ν_i in the arithmetic column is nontrivial. The normalized local factor has absolute value $q_w^{-1/2}$ when it varies and is nonzero. At each other contributing place it is a root of unity. The formula for D_{var} follows.

Finally the local table is the unnormalized Fourier transform of a function of absolute value one on those positions passing all tests, and zero elsewhere. There are q^f positions in the cell. If $L \neq 0$, exactly q^{f-1} make its determinant zero. Expanding the squared Fourier norm and summing first over $\bar{z}$ kills all unequal pairs of positions and gives a factor q^d on the diagonal. This proves the norm identity.

If the earliest group is B_i, then $\bar{b}$ vanishes in the initial r columns. Both choices of super-diagonal character use only those columns, so each character is constant as v varies. Thus $\omega = -\bar{z}|_{A_{s_i}}$. For nontrivial ν_i, Theorem 127.15 gives precisely the nonzero multiples of L_i, with absolute value $q^{s_i-1/2}$ at every such frequency. Proposition 127.14 identifies their common kernel with $H \cap A_{s_i}$. For two different blocks, the earlier block must carry a nonzero restriction for its own table and a zero restriction for the later table. For equal blocks with different hyperplanes, a nonzero covector cannot have both kernels. In either case the product of one table entry with the conjugate of the other vanishes at each covector separately. Multiplication by any weights and summation preserves zero. □

The condition on H couples the reduction of a specified penultimate matrix to the last row of a specified terminal matrix. Several paths may satisfy this incidence for the same covector

or contribute at different covectors to the same output flag. Their signed sums still enter the projected Gram estimate of Section 6.6.

127.6 Inner products and mean squares of the marked Fourier tables

Fix x, an intermediate class, a marked place w and its frame as in Section 127.5. Write $V = k^d$. For $B \subset V$ a linear subspace, let $B^\circ \subset V^\vee$ denote its annihilator. We use the Fourier transform $\widehat{h}(z) = \sum_{v \in V} h(v)\theta(-zv)$. All residue characters and the splitting characters on units take values in roots of unity. We work in a common number field J containing them, and use the ordinary complex inner product after choosing an embedding. We consider paths whose local test is nonzero as a function of position.

Lemma 127.17 (The singular positions of a marked matrix)**.** *For a path with a varying determinant, let $W = A_{s_i}$ and $K = H \cap W$ be the two subspaces of Corollary 127.16. Choose once and for all, for this pair, a form $\lambda : W \to k$ with kernel K. The positions in its cell where the determinant vanishes are a single coset $u + K$ of K in V. The entire cell is $u + W$. There is a root of unity γ such that the path's full Fourier integral is*

$$\mathcal{F}(z) = \gamma q^{s_i - 1} G_{-1}(\nu_i)\nu_i(t)^{-1}\theta(-zu), \qquad z|_W = t\lambda, \quad t \in k^\times,$$

and it is zero elsewhere. The class of u in V/K is intrinsic to the tested matrix.

For a path with constant determinants there is a nonempty cell $u + W_c$ with $W_c \subset A_r$, a form $\rho_c \in W_c^\vee$, and a root of unity γ_c such that

$$\mathcal{F}_c(z) = \gamma_c q^{\dim W_c}\theta(-zu)\mathbf{1}_{z|_{W_c} = \rho_c}.$$

Proof. In the varying block, the determinant on $v^0 + W$ is $d_i + c\lambda(v - v^0)$ for some $c \in k^\times$. This assumes every value, so its zeros form a coset $u + K$. Taking any u on that coset writes the determinant as $c\lambda(v - u)$. The superdiagonal character and the other determinants are constant on the cell. Put their values, the unit splitting factor and $\nu_i(c)$ into γ. The local function at $u + y$, $y \in W$, then equals $\gamma\nu_i(\lambda(y))$, with value zero for $y \in K$. At $z|_W = t\lambda$, the fiber above $a \in k$ contributes $q^{s_i - 1}\gamma\theta(-zu)\nu_i(a)\theta(-ta)$. Substituting $a' = ta$ gives the formula when $t \neq 0$. At $t = 0$ multiplicative orthogonality gives zero, since $\nu_i \neq 1$ for these marked characters. At frequencies outside K°, summing along K gives zero. Changing u within $u + K$ preserves every value on K°.

For a nonzero constant test the first nonzero column of the coefficient row $\bar{b}$ lies in the initial group, by Proposition 127.14. That column fixes all rows beyond the first r by the entry congruences. On $u + W_c$ the remaining position function equals $\gamma_c\theta(\rho_c(v - u))$, since its multiplicative factors are constant and its superdiagonal exponent is affine. Summing this character gives the second formula. □

Theorem 127.18 (The Gram entries at a marked place)**.** *For two nonzero tables F, F' of the lemma define*

$$R(F, F') = \frac{|\sum_{z \in V^\vee} F(z)\overline{F'(z)}|^2}{(\sum_z |F(z)|^2)(\sum_z |F'(z)|^2)}.$$

The following values hold at every embedding of J.

1. *For two varying tables, $R = 0$ for distinct pairs (W, K). For the same (W, K), let $\delta = u - u'$ be the difference of representatives of the singular cosets. Then*

$$R = \begin{cases} 1, & \delta \in K, \\ (q-1)^{-2}, & \delta \in W \smallsetminus K, \\ 0, & \delta \notin W. \end{cases}$$

2. *For one constant and one varying table put* $f_c = \dim W_c$, $s = \dim W$, *and* $\delta = u - u_c$. *If* $\rho_c = 0$ *and* $\delta \in W \smallsetminus K$, *then* $R = q^{1+f_c-s}/(q-1)$. *In every other case* $R = 0$.

3. *For two constant tables put* $B = W_c + W_c'$, $h = \dim B$, $f_c = \dim W_c$, $f_c' = \dim W_c'$. *If the two forms agree on* $W_c \cap W_c'$ *and* $u_c - u_c' \in B$, *then* $R = q^{f_c+f_c'-2h}$. *In every other case* $R = 0$.

Proof. Within the support of each table its absolute value is constant. Thus the constants in front of the characters in the lemma cancel from the quotient for R. For two varying tables with different block index or different hyperplane in the same block, the supports are disjoint by Corollary 127.16. For the same pair, the character ν_i and the form λ are common to both paths. Their support is $K^\circ \smallsetminus W^\circ$ of size $(q-1)q^{d-s}$. The frequency phases from ν_i cancel. The sum of the remaining additive phases is

$$q^{d-s+1}\mathbf{1}_{\delta\in K} - q^{d-s}\mathbf{1}_{\delta\in W}.$$

Indeed the sum on either annihilator is its cardinality when the tested difference belongs to the original space and zero otherwise, by additive orthogonality. Dividing by the support size and squaring gives the three values.

For the mixed pair, $W_c \subset A_r \subset K$ because the form of the varying block is supported in B_i. Every frequency on the varying support restricts to zero on W_c. Consequently the supports are disjoint for $\rho_c \neq 0$, and the constant table contains the varying support for $\rho_c = 0$. For a fixed $t \neq 0$ sum first on the q^{d-s} extensions of $t\lambda$ to V. Their phases sum to zero unless $\delta \in W$. In that case the remaining sum, up to a constant of absolute value one, is

$$q^{d-s} \sum_{t\in k^\times} \nu_i(t)^{-1}\theta(-t\lambda(\delta)).$$

It vanishes for $\delta \in K$ and otherwise has absolute value $q^{d-s+1/2}$ by Theorem 127.15. The two support sizes are $(q-1)q^{d-s}$ and q^{d-f_c}. Dividing the square of the last magnitude by their product gives the stated value of R.

Two prescribed restrictions on W_c, W_c' extend to V precisely when they agree on their intersection. If they do, the common frequency set is a translate of B°. The phases there have sum of absolute value q^{d-h} for $u_c - u_c' \in B$, and sum zero otherwise. Dividing the square by $q^{d-f_c}q^{d-f_c'}$ gives the last assertion. These calculations use orthogonality and the absolute values of nontrivial Gauss sums, which are preserved at each embedding. □

We apply the singular cosets to groups of signed paths in the arithmetic column. Fix an intermediate class with its frames, an input, and a terminal matrix coset. Take one scalar coefficient of the output, applying the terminal derivative and the chosen coefficient functional by linearity. Choose a nonempty subset Σ of the marked places. At these places group incoming paths by a common varying block and hyperplane (W_w, K_w) at each $w \in \Sigma$. At all other places retain the prescribed values of the tests. In particular the two neighboring kernels still use the same vector. The arithmetic meaning of this grouping is that for the fixed terminal matrix the last covectors z_w are specified. We can evaluate the group’s coefficient as a finite function of these variables and then substitute their specified values.

Put $s_w = \dim W_w$ and write

$$A = \prod_{w\in\Sigma} V_w/K_w, \qquad L_w = W_w/K_w \subset A, \qquad D_\Sigma = \prod_{w\in\Sigma} q_w.$$

Here L_w is supported in the w-th factor and has order q_w. Identify $A^\vee$ with $\prod_{w\in\Sigma} K_w^\circ$ by the additive characters θ_w. For each path remove the Fourier factors at Σ, retain its sign and all

the other factors, and multiply by $\prod_{w\in\Sigma}\gamma_w$ from Lemma 127.17. Add the resulting coefficients for all paths having the same tuple of singular cosets $u \in A$, and call the sum $c(u)$. It includes the original class masses. For a finite table $c : A \to J$ let

$$M_w c(u) = \frac{1}{q_w}\sum_{h\in L_w} c(u+h), \qquad P = \prod_{w\in\Sigma}(1 - M_w).$$

The factors commute since the translations commute.

Theorem 127.19 (A sharp lower bound for a nonzero integral group of paths)**.** *Let $N(z)$ be the group's signed coefficient as a function of the marked frequencies with the common factor $\prod_{w\in\Sigma} q_w^{s_w}$ divided out. It vanishes off $A^\vee$. On $A^\vee$ it is supported where every $z_w|_{W_w} = t_w\lambda_w$ has $t_w \neq 0$. There its value is exactly*

$$N(z) = \left(\prod_{w\in\Sigma} q_w^{-1} G_{-1}(\nu_{i_w})\nu_{i_w}(t_w)^{-1}\right)\sum_{u\in A} c(u)\prod_{w\in\Sigma}\theta_w(-z_w u_w).$$

In particular N is nonzero somewhere if and only if $Pc \neq 0$. At any one embedding,

$$\frac{1}{|A|}\sum_{z\in A^\vee}|N(z)|^2 = D_\Sigma^{-1}\sum_{u\in A}|Pc(u)|^2.$$

If $c(u) \in \mathcal{O}_J$ for every u and $Pc \neq 0$, then

$$\frac{1}{[J:\mathbb{Q}]|A|}\sum_{\sigma: J\hookrightarrow\mathbb{C}}\sum_{z\in A^\vee}|\sigma N(z)|^2 \;\geq\; \prod_{w\in\Sigma}\frac{q_w - 1}{q_w^2}.$$

This constant is attained among integral tables c. For each prescribed z with $N(z) \neq 0$, the embedding average of $|\sigma N(z)|^2$ is at least D_Σ^{-1} under the same integrality assumption.

Proof. The signed coefficient of the two coupled edges is the sum over incoming paths with the product of the local factors from Corollary 127.16. The terminal matrix coset, its derivative, and the coefficient functional are fixed during this summation. The position at the terminal prime, the top matrix test and the mass of the intermediate class are independent of the varying marked covectors in this formula. All dependence under consideration occurs in the partial Fourier factors at Σ. At each such place the lemma expresses it as a common factor times $\gamma_w\theta_w(-z_w u_w)$. The latter phase depends only on the singular coset because $z_w \in K_w^\circ$ on the support. Grouping and restoring the prescribed covectors therefore recovers the actual contribution, including the signs within each $c(u)$.

Averaging translates along L_w multiplies a Fourier coefficient by one if its character is trivial there and by zero otherwise. Thus the Fourier transform of Pc agrees with that of c exactly on the retained frequency set and vanishes on its complement. The squared absolute value of the product of Gauss factors in the displayed formula is D_Σ^{-1}. Expanding the squared Fourier transform and using character orthogonality on A gives the norm identity and the equivalence of nonvanishing.

For the integral estimate put the norm $\|a\|^2 = [J:\mathbb{Q}]^{-1}\sum_\sigma|\sigma a|^2$ on $\mathcal{O}_J$, extending it to arrays by summing over coordinates. Every nonzero integral scalar has squared norm at least one by the arithmetic-geometric mean inequality and its nonzero integral absolute norm. The operator P acts separately on each coset of $\prod_w L_w$ in A. Choose one on which the result is nonzero. On this product of lines perform the mean subtractions one coordinate at a time. After centering in any first set of coordinates, every nonzero resulting slice has squared norm

at least the product of $1 - q_w^{-1}$ over those coordinates. For the empty set this is the scalar assertion just proved.

For the induction step let $v_1, \dots, v_q$ be the q slices in a new coordinate after the preceding mean subtractions, and put $\bar{v} = q^{-1} \sum_j v_j$. In the Hilbert space of arrays over all embeddings,

$$\sum_{j=1}^{q} \|v_j - \bar{v}\|^2 = \frac{1}{q} \sum_{1 \le j < l \le q} \|v_j - v_l\|^2.$$

This follows by expanding both sides. For a nonzero centered array at least two entries differ, giving at least $q - 1$ unequal unordered pairs. Indeed a value occurring h times, $1 \le h < q$, differs from the entries in $h(q - h) \ge q - 1$ pairs. Each difference is the preceding projection of a difference of integral input arrays, so every nonzero difference satisfies the inductive lower bound. Multiplying that bound by $(q-1)/q$ proves the step. Summing over all product cosets and using the norm identity proves the inequality. For c equal to a single point mass of value one the centered vector on each line has squared norm $1 - q_w^{-1}$. Their tensor product gives equality. Finally the Fourier sum of c at any prescribed z is an algebraic integer. If it is nonzero, its embedding average of squared absolute values is at least one by the same norm argument. The common Gauss factor has squared absolute value D_Σ^{-1} at every embedding, giving the last assertion. □

We next sum over terminal matrix cosets themselves. Let N be the intermediate lattice and $N' = M \perp \mathcal{O}_{E_x} e$ the lifted terminal lattice in its large space. Write $\mathcal{H}$ for the rational homomorphisms taking N into N' locally at all the selected places, including those over T. The terminal matrix index is the finite module $c_A^{-1}\mathcal{H}/c_B\mathcal{H}$ of Section 73.1. The two positive integers c_A, c_B are the position denominator and congruence modulus of this sum. Each marked rational prime divides c_B, by the positive depth used at the marked places. These parameters and the matrix sum are specified in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 636 and lines 1107–1113 there.

Proposition 127.20 (Which marked frequencies the terminal matrices sample)**.** *Fix the intermediate and terminal frames, all components of the terminal matrix coset outside Σ, and its projection to the small lattice M at the places in Σ. Suppose the tests determined by these fixed choices pass. Put $b_w = v_w(c_B) \ge 1$. After the terminal integrality test on its normal row, every tuple of residue covectors $z \in \prod_{w \in \Sigma} V_w^\vee$ occurs, with exactly*

$$m_\Sigma = \prod_{w \in \Sigma} q_w^{d(b_w - 1)}$$

terminal matrix cosets for each tuple. In particular the fraction in the frequency support of a fixed group of varying paths is exactly $\prod_{w \in \Sigma} (q_w - 1) q_w^{-s_w}$.

Let $C(u)$ be the coefficient table of the preceding theorem with values in the intermediate coefficient space, before applying the terminal derivative and output functional. Let T_ξ be the terminal coefficient operator followed by this functional for the matrix coset ξ. Include in T_ξ the fixed terminal scalar tests away from the varying Fourier factors, counting each original scalar once. Put $J(z) = \sum_{\xi \mapsto z} T_\xi$. With $\mathcal{Z} = \prod_w (K_w^\circ \smallsetminus W_w^\circ)$ and $\widehat{C}$ defined entrywise, the contribution of all these terminal cosets and incoming paths to the projected coefficient equals

$$\left(\prod_{w \in \Sigma} q_w^{s_w} \right) \sum_{z \in \mathcal{Z}} g(z)\, J(z)(\widehat{C}(z)), \qquad g(z) = \prod_{w \in \Sigma} q_w^{-1} G_{-1}(\nu_{i_w}) \nu_{i_w}(t_w)^{-1}.$$

Proof. After multiplication by c_A the matrix quotient is $\mathcal{H}/(c_A c_B)\mathcal{H}$. Over the ring of elements integral at the selected places, $\mathcal{H}$ is the homomorphism module of the two localized lattices.

Their localizations are finite free at each selected place. The Chinese remainder theorem modulo $c_A c_B$ identifies this quotient with the product of the corresponding local matrix quotients. To see surjectivity for projective lattices, present their homomorphism module as a direct summand of a finite free module, first lift arbitrary prescribed residues in the free module, then apply the projector. The same decomposition identifies the kernel with multiplication by $c_A c_B$.

At $w \in \Sigma$, division by c_A restores a matrix between the local lattices, with entries allowed in $c_A^{-1} O_w$ modulo $c_B O_w$. Since $N' = M \perp O_w e$ locally, the small projection and last row give independent coordinates on this quotient in the chosen bases. The covector used in the Fourier factor is the last row divided by π_T in the intermediate frame. The generator π_T is a unit at w, so this change preserves the integral row lattice and its congruence sublattice. The allowed integral rows are therefore $O_w^d / c_B O_w^d$. Reduction onto $V_w^\vee$ is surjective and a fiber has $q_w^{d(b_w-1)}$ elements. The top position test uses the small projection only, and the position equation at T uses components outside Σ. Thus fixing these data imposes no further equation on the last rows. Multiplication of the local counts gives m_Σ. The support at w consists of $q_w - 1$ nonzero multiples of a form on W_w, each with $q_w^{d-s_w}$ extensions. Division by q_w^d and multiplication over w gives the fraction asserted.

Apply Theorem 127.19 to each coordinate before applying T_ξ. On the fiber above z, both $g(z)$ and $\widehat{C}(z)$ are fixed. Summing the terminal operators gives $J(z)$. Restoring the common powers $q_w^{s_w}$ and summing over z proves the formula. □

After clearing the incoming denominators, the remaining estimate in Section 6.6 concerns the operators $J(z)$ and their signed sum in the same projected output.

127.7 The theta sections in the last row of the terminal matrix

Fix a word, an intermediate class and a terminal class in Proposition 127.20. Write $d = N(m-1)$ for the intermediate rank and $n+1 = N(m)$ for the large terminal rank. In the terminal frame the target lattice is $N' = M \perp \mathcal{O}_E e$, where $E = E_x$ and M has rank n. Thus the last row of a homomorphism from N to N' belongs to $N^\vee = \mathrm{Hom}_{\mathcal{O}_E}(N, \mathcal{O}_E)$ when the homomorphism is integral. Put $f = [E : K]$ and $D = fd$. Use the normal Hermitian coordinates of Theorem 65.3.

Choose one or both of the two marked rational primes. Take for Σ all the selected places over their marked halves. Write $\beta = \prod_i \bar{\pi}_{g_i}$, $G = \prod_i g_i$, where both products run over the chosen primes, and put $Q = G^D$. The prime factorizations and the orientation of the halves are given in `Formalism/LocalToGlobal/Columns/AbelianHostFieldConditions.lean`, line 190. In particular, $\beta\bar{\beta} = G$, each chosen prime splits completely, and $Q = \prod_{w \in \Sigma} q_w^d$. At these places write U_w for the lattice of integral normal rows, in the coordinates in which the Fourier covector was computed. Division by π_T at a marked place is a unit change of coordinates. Fix the small part of the terminal matrix everywhere, and the normal cells at the other selected places. One may also fix a residue cell at the other marked prime by summing over its higher digits. All fixed finite tests are assumed to pass. The position test at T is applied in its original coordinates.

Proposition 127.21 (The sections indexed by the marked normal rows)**.** *In the complex theta realization, summing the terminal theta sections over one integral residue tuple $z \in \prod_{w \in \Sigma} U_w / q_w U_w$, before differentiation, gives a product $\Theta_{\mathrm{top}} \Theta_z$. Here Θ_{top} is fixed. The Q normal sections Θ_z are an orthogonal basis of sections of one positive line bundle on a torus of dimension D. Its period lattice and covolume in the normal metric are*

$$\Gamma = \delta \Phi(\beta\, \mathfrak{b}^{-1} \mathfrak{a} N^\vee), \qquad \mathrm{vol}(\Gamma) = Q.$$

Here $\mathfrak{b}$ and δ are the period data of Theorem 65.3. The fractional ideal $\mathfrak{a}$ has exponent h_v at a frozen split half with normal cell of depth h_v, exponent $-h_v$ at its conjugate and exponent zero elsewhere. Put $s_0 = \|\sum_z \Theta_z\|_{\mathrm{av}}$, with the Hermitian metric and probability Haar measure on the torus. Then

$$s_0 > 0, \qquad \|\Theta_z\|_{\mathrm{av}}^2 = s_0^2/Q \quad \textit{for every } z.$$

Proof. Consider first one split place, with uniformizer q, position lattice U, dual frequency lattice U°, and an additive character ψ of conductor the local integers. For $a \in U$ and $h \geq 1$, finite character orthogonality gives the identity of functions of y

$$\mathbf{1}_{a+q^hU}(y) = \mathbf{1}_U(y)\, q^{-hd} \sum_{\eta \in U^\circ/q^hU^\circ} \psi(\langle \eta, y-a\rangle/q^h).$$

Indeed, for $y \in U$ the sum of a nontrivial character on the quotient is zero, and the trivial character sums to q^{hd}. The finite theta intertwiner sends each summand to the same modulation of the vacuum. In the class formula its finite Fourier coefficient is $\sqrt{\mathrm{vol}}/|S_A|$, and the class distribution multiplies it by $1/\sqrt{\mathrm{vol}}$. Their product is the average over the frequency representatives. For the position translate a, conjugating a modulation by translation supplies the displayed phase. Equivalently one can translate the characteristic function at zero by a in the split position model. These are the definitions in `Formalism/LocalToGlobal/KernelLattices/XiIndCClass.lean`, line 20 and `Formalism/LocalToGlobal/KernelLattices/ThDistC.lean`, line 87. Thus the fine cell enters with its characteristic function. Its higher digits partition a residue cell disjointly. Their sum is exactly $\mathbf{1}_{\tilde{z}+qU}$, with $\tilde{z}$ any lift of z. Taking products gives this assertion at every chosen half.

The period module decomposes into the homomorphisms to M and the homomorphisms to $\mathcal{O}_E e$. The Hermitian form is the orthogonal sum of the two trace forms. Hence its quadratic semicharacter is the product of the two semicharacters. The fixed finite matrix coset is also a product in these two sets of coordinates. The theta realization respects this decomposition. For clarity, the vacuum line of each self-dual summand is one-dimensional by Riemann–Roch for a principal polarization. Their tensor product gives the vacuum of the orthogonal sum. Each split translation or modulation in the finite expansion acts on the corresponding factor. The displayed characteristic-function identity and linearity now give $\Theta_{\mathrm{top}}\Theta_z$. The normalization of the initial vacuum can be included once in Θ_{top}. A change of this convention multiplies all the normal sections by the same nonzero scalar.

We describe the periods on the normal factor more explicitly. The function of position on a frozen half is $\mathbf{1}_{a+q^hU}$. Translations by q^hU fix it. Modulations by $q^{-h}U^\circ$ multiply it by a unit scalar depending on a. Relative to the unrestricted integral vacuum this replaces the two local lattices by q^hU and $q^{-h}U^\circ$. Their product is the self-dual change specified by $\mathfrak{a}$. At a varying half, translations by qU and modulations by U° have the same action on every residue cell. Accordingly the common period lattice is obtained by multiplying the position component by q at each varying half, leaving its conjugate component as it was. This is multiplication of the global normal module by β. These local descriptions determine the global lattice, since two fractional lattices that agree at every finite prime agree over $\mathcal{O}_E$. All the Θ_z therefore have the same automorphy factors on Γ.

The module $\mathfrak{a}N^\vee$ is self-dual for its Hermitian form because $\mathfrak{a}\bar{\mathfrak{a}} = 1$. The trace polarization gives covolume one before multiplication by β, by Theorem 65.3. Multiplication by β is a dilation by $\sqrt{G}$ followed by a unitary map in D complex variables. Its real determinant is G^D. The resulting positive line bundle has Q holomorphic sections counted by dimension. Indeed, the top exterior power of its polarization divided by $D!$ integrates to Q. Riemann–Roch on the

complex torus gives this Euler characteristic, and the higher cohomology vanishes by positivity. Changing the fixed unitary character affects neither the curvature nor this computation.

Finally $q^{-1}U^{\circ}/U^{\circ}$ acts by normalized unitary modulations on these sections. Its characters distinguish the residue cells by the perfect residue pairing. Such operators preserve the indicated line on the common torus, since their commutators with the common periods are trivial. Distinct cells are therefore orthogonal. The position translations by U/qU permute the cells up to scalars of absolute value one, so all norms coincide. The positions and modulations at distinct places can be prescribed independently. One may realize them by rational lifts in the corresponding fractional lattice modulo the common lattice, by the Chinese remainder theorem. The sum of all the residue cells is the whole integral normal vacuum at the varying halves, with the frozen test elsewhere. It is nonzero. Locally a frozen cell is a nonzero multiple of a unitary translate of the normalized vacuum for its changed self-dual lattice. The finite isometric intertwiner takes its tensor product to the vacuum line for that lattice, as in the split-lattice calculation of Theorem 65.13. Orthogonality and equal norms now give the norm formula, and the dimension calculation proves that the family is a basis. □

Theorem 127.22 (Normal derivatives in the projected terminal operator)**.** *Let $J(z) = \sum_{\xi\mapsto z} T_{\xi}$ be the terminal operator in Proposition 127.20, in its complex theta realization. Use an orthonormal normal frame respecting the embeddings τ, and put*

$$\ell_{\alpha}(F) = \frac{\partial^{\alpha} F(0)}{\pi^{|\alpha|/2}\sqrt{\alpha!}}, \qquad B_{\tau} = \sum_{j=1}^{n} (\lambda_{\tau j} - \mu_{\tau j}).$$

There are fixed coefficient maps A_{α}, determined by the small coordinates and the prescribed branch projection, such that

$$J(z) = \sum_{\alpha\in\mathcal{I}} \ell_{\alpha}(\Theta_z) A_{\alpha}, \qquad \mathcal{I} = \left\{ \alpha \in \mathbb{N}^{f\times d} : \sum_{i=1}^{d} \alpha_{\tau i} = B_{\tau} \text{ at every } \tau \right\}.$$

In particular the linear span of these operators has dimension at most $\prod_{\tau} \binom{B_{\tau}+d-1}{d-1}$.

Proof. Here is a construction of A_{α} which also fixes the relative scalars. Multiply the input flag polynomial by the prescribed determinant powers $\prod_{\tau} \det(X_{\tau})^{\nu_{\tau}}$. Replace its variable $X_{\tau ij}$ by $\sum_{\gamma=1}^{n+1} Y_{\tau\gamma j} D_{\tau\gamma i}$. In each term of the expansion a factor $D_{\tau\gamma i}$ denotes one differentiation in that matrix coordinate. Include the assigned differential scale $c_{D,\tau}$ for each such factor. This is the ordinary mixed-derivative definition in `Formalism/LocalToGlobal/KernelLattices/ThDistC.lean`, line 21 and lines 62–104 there. Apply the derivatives with $\gamma \le n$ to Θ_{top} at zero. Express the remaining derivatives in the orthonormal normal coordinates. For the derivative of index α put the factor $\pi^{|\alpha|/2}\sqrt{\alpha!}$ into its coefficient. An orthonormal change within the d coordinates at one embedding preserves their total degree. In this manner the only factor involving the varying section is $\ell_{\alpha}(\Theta_z)$.

To see which indices survive projection, first observe that the substitution already lies in the prescribed target flag space, coefficient by coefficient in the D's. At one embedding, that space is spanned by products of minors of the first j columns, with column counts prescribed by the partition of the weight after subtraction of its last entry. Each source minor of $D^{t}Y_{:,\le d}$ expands by Cauchy–Binet into target minors of the same first j columns, with minors of D as coefficients. Each determinant factor adds a minor of size d. The target partition for this theta step is the source partition plus ν_{τ} in its first d entries, extended by zeros. This is the shift of the weight lists used in (9.1), expressed in partition coordinates in `Formalism/LocalToGlobal/KernelLattices/FirstKernelLawReading.lean`, line 233. Thus the target retraction fixes the resulting polynomial.

By Theorem 127.1 the terminal pairing uses precisely the monomials having last-row exponents $\lambda_{\tau j}-\mu_{\tau j}$ for $1\le j\le n$ and zero in the last column. These exponents are nonnegative by interlacing. Each occurrence of $Y_{\tau,n+1,j}$ came from exactly one $D_{\tau,n+1,i}$, and every normal derivative arose this way. Their number at τ is B_τ. Apply the indicated coefficient extraction, Fischer factors and small output functional to each coefficient still present. Include the class factor χ_F^{-1}, the frame determinants, the fixed finite scalars and the normal-coordinate factor from the branch map in the same places as in the prescribed operator. This defines A_α independently of z and proves the identity. Testing every small coefficient proves the corresponding map identity before evaluation by a single functional. Counting the multi-indices separately at each embedding gives the bound.

The computation takes place in a specified complex realization. For the normalized CM jets, the change of embedding uses the transported differentials of Theorem 65.9 and the theta functional of Theorem 67.8. Dividing by the Fock factors in the definition of ℓ_α and putting them into A_α is an identity in that realization. Descent applies to the original normalized coefficient whenever it belongs to the designated coefficient field. □

Corollary 127.23 (Fourier synthesis of the signed normal rows)**.** *Let $h(v)$ be a scalar incoming position test at the varying marked places and write $\theta(zv)=\prod_w\theta_w(z_wv_w)$. Use $\widehat{h}(z)=\sum_v h(v)\theta(-zv)$. Define $\mathcal{J}_\eta$ by applying the same terminal operation to the full integral normal row, modulated by the residue character $z\mapsto\theta(z\eta)$. Then*

$$\sum_z \widehat{h}(z)J(z)=\sum_v h(v)\mathcal{J}_{-v}.$$

For an individual marked path the right side is an integral linear combination over the ring generated by the values of the residue and splitting characters.

Proof. A full integral row modulated by $\theta(z\eta)$ is the sum of its residue cells with those coefficients. In the normal theta line this modulation is the Weyl operator of the dual residue class in $q^{-1}U^\circ/U^\circ$ at each varying half. Thus $\mathcal{J}_\eta=\sum_z\theta(z\eta)J(z)$, including the original scalar of the full row when $\eta=0$. Expanding $\widehat{h}$ and interchanging the two finite sums proves the identity with the indicated minus sign. For a marked path the original position test takes the values zero and roots of unity, by Lemma 127.17. Hence the coefficients $h(v)$ are algebraic integers. The identity requires no division by the residue cardinalities. Integrality of the modulated operators themselves and the size of the resulting signed period require their own estimates. □

127.8 A uniform mean square for the projected terminal operators

The operators in Theorem 127.22 use derivatives at one point of many sections of a single normal theta line bundle. We estimate their Gram matrix before contracting it with the incoming path coefficients. The dependence on the dimension and on the maximum order of differentiation matters because both grow along the intervals in Section 6.6.

Lemma 127.24 (Derivatives of a theta basis on a separated lattice)**.** *Let $\Gamma\subset\mathbb{C}^D$ be a full lattice of real covolume Q for the usual Euclidean metric. Use an integral positive polarization with Hermitian form the usual inner product, complex-linear in its first variable. A holomorphic section in the Hermitian frame satisfies*

$$F(u+\lambda)=a_\lambda e^{\pi u\cdot\bar\lambda+\pi|\lambda|^2/2}F(u)\quad(\lambda\in\Gamma),\qquad |a_\lambda|=1.$$

Give the space of sections its inner product with weight $e^{-\pi|u|^2}$ and probability Haar measure on the torus. Let $(s_j)_j$ be an orthonormal basis and, for $|\alpha|\le L$, put $B^\circ_{\alpha j}=\ell_\alpha(s_j)$, with the

derivative normalization of Theorem 127.22. Suppose $|\lambda| \geq R > 0$ *for every nonzero* $\lambda \in \Gamma$ *and*

$$\pi R^2 \geq \max\{64L,\ 16D\log 5,\ 8\log 2\}.$$

Then, in operator norm on the space of derivative indices,

$$\left\| Q^{-1} B^{\circ}(B^{\circ})^{*} - I \right\| \leq 2e^{-\pi R^2/8}.$$

The same bound holds after restricting the matrix to any common subset of its row and column indices.

Proof. Write $K(u,w) = \sum_j s_j(u)\overline{s_j(w)}$. With the convention on volume used here its period expansion is

$$\frac{K(u,w)}{Q} = e^{\pi u\cdot\bar{w}} \sum_{\lambda\in\Gamma} a_\lambda e^{-\pi|\lambda|^2/2 + \pi u\cdot\bar{\lambda} - \pi\lambda\cdot\bar{w}}.$$

We give the normalization argument. For Gaussian measure $e^{-\pi|w|^2}\,dw$ on the full space, the squared norm of w^α is $\alpha!/\pi^{|\alpha|}$, by polar integration in each coordinate. Distinct monomials are orthogonal by angular integration. Their normalized sum in the reproducing kernel is $e^{\pi u\cdot\bar{w}}$. Completeness follows by expanding an entire function in its Taylor series on polydiscs, integrating the angular variables first and then increasing the radii. These are the Gaussian conventions of V. Bargmann, Part I.

For a torus section F, the function $|F(w)|e^{-\pi|w|^2/2}$ is bounded by compactness. Apply the full-space reproducing identity first to $F(tw)$ for $0 < t < 1$. Let t increase to one. The integrand at fixed u is bounded by a constant times $\exp(\pi|u|\,|w| - \pi|w|^2/2)$, which is integrable. Split the resulting integral into translates of a fundamental domain. On its λ-translate, substitute the automorphy formula for $F(w+\lambda)$ and expand $|w+\lambda|^2$. This yields the displayed series as the kernel for ordinary Euclidean integration over that domain. The series and all of its derivatives converge normally on compact sets by Gaussian decay. For fixed u its complex conjugate satisfies the section law in w, as follows by reindexing the translated domains, or by substituting $w+\mu$ in the series and using the cocycle law for a_λ. It thus represents evaluation in the space of sections. Changing the integration measure to the probability measure multiplies the reproducing kernel by Q, as asserted.

In the variables $v = \sqrt{\pi}\,u$ the normalized Fock monomials are $v^\alpha/\sqrt{\alpha!}$, and ℓ_α reads their coefficient. Take the corresponding derivatives in $u, \bar{w}$ of the kernel at the origin. The term $\lambda = 0$ gives the identity matrix. A term with $\lambda \neq 0$, apart from a_λ, gives the matrix on degrees at most L of the translation on the first variable

$$e^{-\pi|\lambda|^2/2}\exp(\sqrt{\pi}\,\bar{\lambda}\cdot v)\exp(-\sqrt{\pi}\,\lambda\cdot\partial_v).$$

Indeed its action on the full-space kernel yields exactly the three λ-dependent exponential factors in the series. If P_L is orthogonal projection onto degree at most L, then the norm of $\lambda\cdot\partial_v$ there is at most $|\lambda|\sqrt{L}$. A unitary rotation reduces this to differentiation in the first coordinate, which lowers a monomial with multiplier $\sqrt{\alpha_1}$. The compression of multiplication by $\bar{\lambda}\cdot v$ is its adjoint and satisfies the same bound. The derivative exponential preserves the degree cutoff. Multiplication raises degree at every positive power, so its exponential, projected back to the cutoff, equals the exponential of the compressed multiplication operator on that space. The norm of the compressed translate is consequently bounded by

$$\exp\{-\pi|\lambda|^2/2 + 2\sqrt{\pi L}\,|\lambda|\} \leq e^{-\pi|\lambda|^2/4}.$$

The last inequality uses $\pi R^2 \geq 64L$.

In the shell $jR \le |\lambda| < (j+1)R$, $j \ge 1$, put an open ball of radius $R/2$ at each lattice point. The balls are disjoint and lie in a ball of radius $(j+3/2)R$. Comparison of real $2D$-dimensional volumes bounds the shell count by $(2j+3)^{2D}$. Since $\log(2j+3) \le j^2 \log 5$ and $\pi R^2 \ge 16D \log 5$, the sum of the norms on this shell is at most $e^{-\pi j^2 R^2/8}$. The sum over $j \ge 1$ is bounded by $2e^{-\pi R^2/8}$, using $\pi R^2 \ge 8 \log 2$. The triangle inequality proves the estimate. Compression to the specified indices can only decrease the error norm. □

Theorem 127.25 (The normal derivative matrix on the original interval)**.** *In Proposition 127.21 take $R^2 = 2fG/\operatorname{rd}(E_x)$ and $\eta_x = 2\exp(-\pi R^2/8)$. For every sufficiently large x on the intervals of Section 6.6, the matrix $B_{\alpha z} = \ell_\alpha(\Theta_z)$, $\alpha \in \mathcal{I}$, satisfies*

$$\left\| s_0^{-2} B B^{*} - I_{\mathcal{I}} \right\| \le \eta_x \le e^{-x^{\varepsilon/8}}.$$

This holds for either choice of one marked prime and for the pair, uniformly in the word, the frozen normal cells, the class frames and the transported complex realization with its own metric.

Proof. The squared length of a nonzero element of $\delta\Phi(\mathfrak{b}^{-1}\mathfrak{a}N^{\vee})$ is at least $2f/\operatorname{rd}(E_x)$ by the trace and norm calculation of Theorem 65.3 for the self-dual source dual with the norm-one ideal twist. Multiplication by β scales every squared length by G. The frozen normal translate changes the character on the period lattice and leaves this length calculation intact. For a conjugate CM realization the trace metric transports as in Theorem 65.9. The conjugate ideal of the elliptic differential is accompanied by its inverse norm in the Hermitian form. In the trace proof, the lower bound for the geometric mean of the squared lengths gains exactly that ideal norm before multiplication by the metric factor. Thus the same lower bound holds, and every conjugate of β still has complex absolute value $\sqrt{G}$.

Put $L = \sum_\tau B_\tau$. The ranks and f are bounded by fixed powers of x on these intervals. The weight lists in Theorem 127.1 bound each difference in the interlacing partitions by $O(xb + x^{C})$, where $b = \exp(x^{\varepsilon})$ and C is fixed. Hence $L \ll x^{C'} \exp(x^{\varepsilon})$, and $D = fd \ll x^{C'}$ after increasing C'. The abelian field E_x has conductor dividing the common modulus N of Theorem 19.15, for which $\log(2N) \ll_{\psi,\varepsilon} x^{\varepsilon/4}$. The conductor–discriminant formula expresses $|\operatorname{disc}(E_x)|$ as a product of the conductors of its characters. Each divides N, so $\operatorname{rd}(E_x) \le N$. Each marked rational prime is at least $(kb)^8$ by Proposition 6.30. Already for one chosen prime it follows that

$$R^2 \;\ge\; 2f \exp\{8x^{\varepsilon} - O_{\psi,\varepsilon}(x^{\varepsilon/4})\}.$$

This dominates the degree and dimension cutoffs in Lemma 127.24 uniformly, and also gives the stated upper bound on η_x.

By Proposition 127.21 an orthonormal basis on this common torus is $s_z = \sqrt{Q}\,\Theta_z/s_0$. Consequently $Q^{-1}B^{\circ}(B^{\circ})^{*} = s_0^{-2} B B^{*}$ after restriction to $\mathcal{I}$. Applying the lemma proves the assertion. The other marked prime may stay frozen throughout this proof when only one prime is varied. □

Corollary 127.26 (The projected Gram matrix and the signed input)**.** *Equip the fixed complex input and output coefficient spaces with their chosen Hermitian norms. For the A_α of Theorem 127.22 set $S = \sum_\alpha A_\alpha A_\alpha^{*}$. Then, in the order of Hermitian quadratic forms,*

$$(1-\eta_x)s_0^2 S \;\preceq\; \sum_{z} J(z)J(z)^{*} \;\preceq\; (1+\eta_x)s_0^2 S.$$

For an arbitrary prescribed vector of incoming coefficients $Y = (Y(z))_z$, let $\mathcal{J}Y = \sum_z J(z)Y(z)$. If $S \neq 0$, write μ for its smallest positive eigenvalue. Then

$$(1-\eta_x)s_0^2\mu \operatorname{dist}(Y, \ker \mathcal{J})^2 \le \|\mathcal{J}Y\|^2,$$
$$\|\mathcal{J}Y\|^2 \le (1+\eta_x)s_0^2\|S\| \sum_z \|Y(z)\|^2.$$

If $S = 0$, every $J(z)$ is zero.

Proof. Define a linear map A from the direct sum of one copy of the input space for each α to the output by $A((v_\alpha)_\alpha) = \sum_\alpha A_\alpha v_\alpha$. Then $\mathcal{J} = A(B \otimes I)$ and $AA^* = S$. The previous theorem bounds BB^* above and below by the corresponding scalar multiples of the identity. Tensoring with the identity on the input and composing with A, A^* proves the first display. Since $\eta_x < 1$, it also proves $\ker(\mathcal{J}\mathcal{J}^*) = \ker S$. On the orthogonal complement of this kernel, the smallest eigenvalue of $\mathcal{J}\mathcal{J}^*$ is at least $(1-\eta_x)s_0^2\mu$. The positive eigenvalues of $\mathcal{J}^*\mathcal{J}$ are the same, by the singular value decomposition. Applying this bound to the component of Y perpendicular to $\ker \mathcal{J}$ gives the lower inequality. The largest eigenvalue gives the upper inequality. If $S = 0$ then $\sum_\alpha \|A_\alpha^* u\|^2 = 0$ for every output vector u, and all the maps vanish.

For the actual block in Proposition 127.20, take $Y(z) = g(z)\widehat{C}(z)$ on its frequency support and zero elsewhere, restoring its displayed common scalar. This identifies the distance that occurs in a lower bound for that block. A lower estimate for the norm of $\widehat{C}$ on its full table gives no lower estimate for this distance on its own. In particular the signed block can vanish even when both S and Y are nonzero. Section 6.6 requires a comparison for the particular sum over the prescribed classes and paths, as well as the local cohomology estimate there. □

128 Terminal covectors and initial class tables

128.1 Summing at both marked primes before varying the terminal covector

Keep the terminal prime $T = \pi_T\bar{\pi}_T$ and the two marked primes g_1, g_2 chosen for the same x. At g_1, g_2 write $Z = \prod_w U_w/q_wU_w$ for the normal residue rows, $G = g_1g_2$, and $Q_g = |Z| = G^D$. Let $\mathcal{T} = \{w : w \mid \pi_T\}$; it has f elements. Write $t_w \in U_w/TU_w$ for the raw integral normal row at w, using the intermediate lattice as source. In the decorated lattice sum the intermediate covector at w is $(\bar{\pi}_T \bmod w)t_w$, by Theorem 73.1. This factor is a unit. Thus all nonzero t_w give exactly the allowed intermediate covectors at this edge.

Fix the underlying intermediate lattice, the frames away from T, the terminal small class and its test, the small matrix cells, and any other finite components outside these three primes. Denote the resulting normal sections by $\Theta_{z,t}$. Initially permit zero components of t; they are removed in the theorem below. The factor Θ_{top} and its normalization are fixed once for all these sections. Put $s = \|\sum_{z,t} \Theta_{z,t}\|_{\mathrm{av}}$. The sum is over all normal rows, including zero residues. The following form of the theta kernel calculation keeps every coefficient of the tests at g_1, g_2.

Lemma 128.1 (A derivative estimate with some residue coordinates retained)**.** *Let a polarized torus with period lattice Γ in normal coordinates have a basis $\Theta_{z,t}$ of the kind in Proposition 127.21. The finite position group is $Z \times T_0$, where T_0 is any product of the other varying residue groups. Put $q_Z = |Z|$ and $s = \|\sum_{z,t} \Theta_{z,t}\|_{\mathrm{av}}$. Let M be a positive integer annihilating Z and suppose that all the translations and modulations on Z, acting trivially on T_0, have lifts to normalized theta translations by elements of $M^{-1}\Gamma$, up to constant phases. Distinct pairs of translations and modulations have distinct lifts modulo Γ. Let R be a lower bound for the length of a nonzero vector of $M^{-1}\Gamma$, with $\pi R^2 \ge \max\{64L, 16D\log 5, 8\log 2\}$. For any subset*

of derivative indices of degree at most L put

$$H_{(\alpha,z),(\alpha',z')} = \sum_{t \in T_0} \ell_\alpha(\Theta_{z,t}) \overline{\ell_{\alpha'}(\Theta_{z',t})}.$$

Then $\| q_Z s^{-2} H - I \| \le 2e^{-\pi R^2/8}$.

Proof. For $a \in Z$, $b \in Z^\vee$ let $W_{a,b} e_z = b(z) e_{z+a}$ act on the ordinary coordinate basis of $\mathbb{C}^Z$. Here $Z^\vee$ denotes the complex character group. These matrices are unitary. They are mutually orthogonal for the trace inner product, with squared norm q_Z. Indeed a nonzero shift has zero diagonal, and a nontrivial character has sum zero. There are q_Z^2 matrices, so they form a basis of all matrices on $\mathbb{C}^Z$. In the order of indices z, α, expand

$$H = q_Z^{-1} \sum_{a,b} W_{a,b} \otimes H_{a,b}.$$

The derivative matrix $H_{a,b}$ is obtained by multiplying the z, z' block of H by $\overline{(W_{a,b})_{zz'}}$ and summing over z, z'.

Set $Q = q_Z |T_0|$. The theta basis for probability Haar measure is $\sqrt{Q}\, \Theta_{z,t}/s$, by the orthogonality and norm calculation of Proposition 127.21. Relative to this basis, acting on the first variable of the reproducing kernel by $\overline{W}_{a,b} \otimes I$ multiplies the coefficient of the z, z' term by the matrix entry just displayed. Consequently $H_{a,b}/s^2$ is the normalized derivative matrix of this translated kernel divided by Q. Complex conjugation of $W_{a,b}$ replaces b by b^{-1} and gives another member of the same family.

Let y be a lift of this operator. In the period expansion of Lemma 127.24, composition with the normalized translation by y replaces the terms indexed by Γ by terms indexed by a coset of Γ in $M^{-1}\Gamma$. For example, composing the operators

$$U_v F(u) = e^{-\pi u \cdot \bar{v} - \pi |v|^2/2} F(u + v)$$

gives $U_v U_w$ equal to U_{v+w} times a scalar of absolute value one, by direct multiplication of exponentials. Thus every derivative matrix in the coset is bounded in norm by the same compression estimate used in that lemma. The identity matrix comes from the zero lattice vector at $(a, b) = (0, 1)$. All other lattice vectors, including the nonzero terms of the identity coset, occur at most once in these estimates, since the cosets are distinct. After multiplication by q_Z/s^2, use $\|W_{a,b}\| = 1$ and the triangle inequality in the expansion of H. The error is bounded by the sum over the nonzero vectors of $M^{-1}\Gamma$ of their compressed translation norms. The shell estimate in Lemma 127.24 bounds this sum by $2e^{-\pi R^2/8}$. This proves the assertion. □

Theorem 128.2 (The two marked primes with nonzero terminal covectors)**.** *Use the actual fields, weights and three primes on the intervals of Section 6.6. For α, α' in $\mathcal{I}$ of Theorem 127.22 form the matrix*

$$H^{\times}_{(\alpha,z),(\alpha',z')} = \sum_{\substack{t \\ t_w \neq 0 \ (w | \pi_T)}} \ell_\alpha(\Theta_{z,t}) \overline{\ell_{\alpha'}(\Theta_{z',t})}.$$

For every sufficiently large x on these intervals,

$$\left\| \frac{Q_g}{s^2} H^{\times} - I \right\| \le e^{-x^{\varepsilon/8}}.$$

The bound is uniform in the fixed data specified above and in the transported complex realization with its Hermitian metric.

Proof. We treat both the full table and the table obtained by fixing one component $t_w = 0$. Denote by $\mathcal{T}'$ the set of terminal places still varying. At every varying place the normal test is a residue characteristic function. The fixed matrix cells project independently onto the small and normal rows, by the Chinese remainder argument of Proposition 127.20. At a place over π_T, the fixed small part must be integral. With this condition, imposing the covector equation at T only specifies the residue normal row, since the lifted terminal covector is $(\bar{\pi}_T \bmod w)e^\vee$. The modulus c_B has exponent one there. All other frame and coefficient operators at this edge depend on the underlying lattice and its frames away from T. This is the finite test ϕ_T in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 646 and the lifted class in lines 203–218 there.

Apply the proof of Proposition 127.21 to these places. The dual residue characters give orthogonality, the position translations give equal norms, and the index of the common period lattice is the product of the numbers of cells. In the present coordinates that lattice is

$$\Gamma = \delta\Phi(\mathfrak{b}^{-1}\mathfrak{a}\mathfrak{c}N^\vee), \qquad \mathfrak{c} = (\bar{\pi}_{g_1}\bar{\pi}_{g_2}) \prod_{w\in\mathcal{T}'} w.$$

The frozen places contribute the norm-one ideal $\mathfrak{a}$ as before. In the table with $t_w = 0$ this includes the change by $w/\bar{w}$ at the newly frozen place. A finite translation or modulation at the marked places acts on the characteristic functions indexed by z as $W_{a,b}$, and fixes the terminal residue index. To see the asserted lift condition of Lemma 128.1, prescribe an integral position shift at a marked half and a frequency shift in $g_i^{-1}U^\circ$ at its conjugate. Modulo the common periods these shifts index the translations and modulations, respectively. Prescribe zero at all other places. Multiplication by G takes the whole tuple into the local common periods. The Chinese remainder theorem for the fractional normal module gives a lift in $G^{-1}\Gamma$. The normalized Weyl action of that lift agrees with the finite one up to a unit scalar, since changing representatives by any common period acts on all sections by the same scalar. The two choices of ordering a position shift and a modulation also differ by a unit scalar. Distinct pairs have distinct classes, either by the local residue coordinates or by the trace orthogonality in the lemma.

We bound the shortest vector even after omitting one prime ideal above T. For a fractional ideal $\mathfrak{j}$ and nonzero $v \in \mathfrak{b}^{-1}\mathfrak{a}\mathfrak{j}N^\vee$, the totally positive scalar $h(v,v)$ belongs to $\mathfrak{j}\bar{\mathfrak{j}}\,\mathfrak{D}_{E/K}^{-1}$. This follows locally from the integrality of the Hermitian pairing on $N^\vee$ and the defining property of $\mathfrak{b}$. Its absolute field norm is at least the norm of this fractional ideal, by ideal divisibility. Applying the arithmetic–geometric mean inequality to its values at the f CM embeddings and using the metric normalization in Theorem 65.3 gives

$$|\delta\Phi(v)|_H^2 \geq \frac{2f}{\mathrm{rd}(E_x)}(\mathrm{N}\,\mathfrak{j})^{1/f}.$$

The same calculation with the transported differential has the same right-hand side as in Theorem 127.25. For $G^{-1}\Gamma$ take $\mathfrak{j} = G^{-1}\mathfrak{c}$. Since the fields split completely at the three primes,

$$(\mathrm{N}\,\mathfrak{j})^{1/f} = \frac{T^{|\mathcal{T}'|/f}}{G}.$$

There are either f or $f-1$ varying places, and $f \geq 2$ on the intervals considered. Hence a common lower bound for the squared length is

$$R_*^2 = \frac{2f\sqrt{T}}{G\,\mathrm{rd}(E_x)}.$$

Here $G \le 4(kb)^{16}$, $\log T \gg mx^{\varepsilon}$ with $m \to \infty$, and $\log \operatorname{rd}(E_x) \ll_{\psi,\varepsilon} x^{\varepsilon/4}$, by Proposition 6.30 and the size estimates in the proof of Theorem 127.1. Consequently R_*^2 dominates $L = \sum_\tau B_\tau$ and D as required by Lemma 128.1. Put $\delta_x = 2\exp(-\pi R_*^2/8)$.

On the full table the lemma bounds its Gram matrix between $(1\pm\delta_x)s^2 I/Q_g$, with the minus and plus signs giving the lower and upper bounds, respectively. For the single face $t_w = 0$, the squared norm of the sum of its normal sections is s^2/T^d. Indeed, in the full normal basis the face includes exactly a fraction T^{-d} of all positions, each of the same squared norm. Its norm can also be computed on the torus for the frozen face, since probability integration of a periodic metric density agrees on a finite cover. The Gram matrix of that face is therefore bounded above by $(1+\delta_x)s^2 I/(Q_g T^d)$. Each summand indexed by t is positive semidefinite. Removing the union of the f faces subtracts at most their sum in quadratic-form order. Thus the relative error for $H^{\times}$ is at most $\delta_x + (1+\delta_x) f T^{-d}$. Both terms together are at most $e^{-x^{\varepsilon/8}}$ for sufficiently large x, by the bounds on T, R_*^2 and polynomial growth of f. □

Corollary 128.3 (The signed sum at both marked primes on a fixed incoming table)**.** *Let $Y(z)$ be a fixed incoming coefficient vector for each marked residue tuple, with zero values permitted. In the fixed complex coefficient space before imposing the preceding terminal-incidence conditions, put*

$$P_t(Y) = \sum_{z,\alpha} \ell_\alpha(\Theta_{z,t})\, A_\alpha Y(z), \qquad M(Y) = \frac{s^2}{Q_g} \sum_{z,\alpha} \|A_\alpha Y(z)\|^2.$$

With $\rho_x = e^{-x^{\varepsilon/8}}$,

$$(1-\rho_x)M(Y) \le \sum_{t_w \ne 0 \ \forall w} \|P_t(Y)\|^2 \le (1+\rho_x)M(Y).$$

In particular a nonzero $A_\alpha Y(z)$ for any one pair implies a nonzero projected coefficient at some tuple of nonzero terminal covectors. This applies to the signed marked table $Y(z) = g(z)\widehat{C}(z)$ of Proposition 127.20 whenever the prescribed incoming table is common to the varying terminal refinements.

Proof. For each t, summing the signed marked coefficients in the last edge gives $P_t(Y)$ by the substitution and branch projection of Theorem 127.22. The normal cells over T alter only the normal theta sections in that substitution. Apply Theorem 128.2 to the conjugates of the scalar coordinates of the vectors $A_\alpha Y(z)$ in an orthonormal output basis, and sum the result over that basis. This gives the two inequalities and the nonvanishing assertion.

In the full column of Theorem 73.1, an earlier edge with reduction $\bar{X}$ requires its incoming covector to be the pullback of the outgoing one by $\bar{X}$. Thus an initial input can select different earlier contributions for different t. The full signed period then also sums the resulting outputs over t. The displayed estimate is for the specified common table and the sum of squared outputs. Control of those pullbacks and of the cross terms in the prescribed period is still required in Section 6.6. □

128.2 The terminal sum for a prescribed initial covector

Fix the earlier matrices of a path through the penultimate edge in Theorem 73.1. The final edge, its two marked normal rows and its terminal normal covectors are still to be summed. At $w \mid \pi_T$, let $V_{0,w}$ be the initial residue space and $V_w = N/wN$ the intermediate residue space of dimension d. Define

$$C_w = (\bar{\pi}_T \bmod w)\, \bar{x}_{m-2,w} \cdots \bar{x}_{0,w} : V_{0,w} \longrightarrow V_w, \qquad r_w = \dim \operatorname{im} C_w.$$

For the terminal normal row $t_w \in V_w^\vee$, the initial covector is $u_w = t_w C_w$. If $u_w \neq 0$, all the intermediate pullbacks of t_w are nonzero. Indeed if one pullback were zero, every further pullback towards the initial class would also be zero. If $u_w = 0$ at any w, the refined path is absent. The other factors of each earlier matrix operator at fixed underlying lattices and fixed frames away from T are independent of the refinements, by the finite class formula of Theorem 73.1. We may therefore use $P_t(Y)$ of Corollary 128.3 for a fixed incoming vector acted on by those operators. All values of its marked table Y enter with their original signs.

Proposition 128.4 (The cost of the preceding incidence conditions)**.** *Assume $r_w \geq 1$ for every $w \mid \pi_T$ and put $p_C = \sum_{w|\pi_T} T^{-r_w}$. With $M(Y)$ as in Corollary 128.3 and $\delta_x = 2\exp(-\pi R_*^2/8)$ as in the proof of Theorem 128.2, one has*

$$[1 - \delta_x - (1+\delta_x)p_C]M(Y) \leq \sum_{t:\, t_w C_w \neq 0 \ \forall w} \|P_t(Y)\|^2 \leq (1+\delta_x)M(Y).$$

Multiplication of each summand $P_t(Y)$ by a scalar of absolute value one depending on $u = (t_w C_w)_w$ preserves both inequalities. If some $r_w = 0$, the path contributes zero for every admissible initial covector.

Proof. With no restrictions on t the two Gram bounds with error δ_x occur before the faces are removed in the proof of Theorem 128.2. The estimate on a face there also holds for $t_w = v$ for any fixed $v \in V_w^\vee$. Replacing zero by v changes the unitary character of the frozen cell. Its period module, norm $s/T^{d/2}$, and shortest vector bound are the same. The sum of squared outputs on such a face is at most $(1+\delta_x)T^{-d}M(Y)$. There are T^{d-r_w} choices of v annihilating $\operatorname{im} C_w$, by rank and nullity over the residue field. Removing all forbidden faces at w consequently costs at most $(1+\delta_x)T^{-r_w}M(Y)$. Adding over the places bounds the union since each squared norm is nonnegative. This proves the inequalities. The assertion about phases follows from the Hermitian norm. For rank zero the initial pullback vanishes for every terminal row.

For example, start on one unrefined initial class with trivial stabilizer as in Proposition 86.29. At each terminal place choose one representative on each nonzero projective line of initial covectors and a nontrivial character $\xi_w : k_w^\times \to \mathbb{C}^\times$. Assign to c times a representative the scalar $\xi_w(c)$. Multiply the resulting local functions and a fixed input vector. This is a legitimate initial test on that class with zero sum separately on every scalar orbit, by multiplicative orthogonality. On each permitted path it contributes exactly one of the phases in the proposition. The estimate applies before addition of outputs with different t. □

Proposition 128.5 (Coherent summation on the fibers of the initial pullback)**.** *Write $C_w^* : V_w^\vee \to V_{0,w}^\vee$ for $t \mapsto tC_w$. Put $K_w = \{y \in U_w : \bar{y} C_w = 0\}$. For $u_w \in \operatorname{im} C_w^*$ choose any integral lift a_w of a preimage of u_w. For fixed z, the sum of the normal sections $\Theta_{z,t}$ over all t with $t_w C_w = u_w$ is the section with terminal test $\prod_{w|\pi_T} \mathbf{1}_{a_w + K_w}$. For $u_w \neq 0$ at all these places, applying the branch map and summing the marked inputs gives the coefficient of the original initial covector u along this path. An initial covector outside the image gives coefficient zero.*

Including zero labels for this dimension calculation, the cells indexed by $(z,u) \in Z \times \prod_w \operatorname{im} C_w^*$ form an orthogonal basis of the space of normal theta sections. This space has dimension $Q_g T^{\sum_w r_w}$. Relative to the periods for the integral normal vacuum, its position lattice at each terminal half is K_w and its frequency lattice at the conjugate half is U_w°.

Proof. The preimage of u_w on integral positions is a coset of K_w. Its disjoint decomposition into residue cells has one term for each t_w mapping to u_w. Addition of their characteristic

functions is exact, so the finite theta intertwiner gives the stated section by linearity, with the original measure and no additional multiplier. The earlier refinements are uniquely specified by these t_w, and the nonzero initial covectors ensure that all of them pass. This proves the statement about the coefficient. In particular the terms in this fiber are added before taking its norm.

For the basis assertion, translations by K_w preserve every such cell, and the frequencies of U_w° act trivially on integral positions. The characters of U_w/K_w are represented by the dual lattice of K_w modulo U_w°. They distinguish the cells and act unitarily. Integral translations permute the T^{r_w} cells and preserve their norms. The marked halves give Q_g further positions exactly as in Proposition 127.21. The common period lattice has covolume the product of these indices. Its positivity and the Riemann–Roch dimension formula give the stated number of sections. Their sum is the nonzero normal vacuum with the frozen tests. Equal norms and orthogonality now give a basis. □

Theorem 128.6 (An initial-covector estimate for a common orthogonal image)**.** *Suppose $N = N_1 \perp N_2$ as Hermitian $\mathcal{O}_E$-lattices, with both summands self-dual and $\operatorname{rank} N_1 = a \geq 1$. Suppose that at every $w \mid \pi_T$, $\operatorname{im} C_w = N_1/wN_1$ in V_w. Use coordinates adapted to this splitting at the two marked primes, and write $z = (z_1, z_2)$, $Q_1 = G^{fa}$. Let Φ_{z_1,t_1} be the normal theta sections for $N_1^\vee$, with the residue covector $t_{1,w} \in (N_1/wN_1)^\vee$ at each terminal place. Let Ψ_{z_2} be the sections for $N_2^\vee$, with the full integral normal row at those places. Use the product normalization from the full normal vacuum and put $s_1 = \| \sum_{z_1,t_1} \Phi_{z_1,t_1} \|_{\mathrm{av}}$. In orthonormal coordinates adapted to N_1, N_2, put*

$$V_{\alpha_1,z_1} = \sum_{\substack{\alpha_2,z_2 \\ (\alpha_1,\alpha_2)\in\mathcal{I}}} \ell_{\alpha_2}(\Psi_{z_2})\, A_{\alpha_1,\alpha_2} Y(z_1,z_2), \qquad M_1 = \frac{s_1^2}{Q_1} \sum_{\alpha_1,z_1} \|V_{\alpha_1,z_1}\|^2.$$

Let $D_u = \sum_{t:\, t_w C_w = u_w \ \forall w} P_t(Y)$ for each admissible initial tuple u. On the intervals of Section 6.6 one has, with $\rho_x = e^{-x^{\varepsilon/8}}$,

$$(1-\rho_x) M_1 \leq \sum_{u_w \neq 0 \ \forall w} \|D_u\|^2 \leq (1+\rho_x) M_1$$

for all sufficiently large x. In particular some initial covector coefficient on this path survives its full signed sum over the terminal fiber if and only if some $V_{\alpha_1,z_1} \neq 0$.

Proof. The surjection $C_w : V_{0,w} \to N_1/wN_1$ makes its dual injective. Every u_w in its image specifies one $t_{1,w}$, and $u_w \neq 0$ is equivalent to $t_{1,w} \neq 0$. The normal row on N_2 ranges freely. By Proposition 128.5, its terminal test after summation is the full integral row. Changing to coordinates adapted to the summands at the two marked halves merely reindexes the residue cells. The marked function $Y(z_1, z_2)$ retains all its entries and may depend jointly on both.

The orthogonal direct sum splits the period form, the quadratic semicharacter and the matrix lattice. At every other frozen split place a fixed congruence coset also splits into the two projected cosets, since its modulus is scalar. The argument of Proposition 127.21 now gives the product $\Phi_{z_1,t_1}\Psi_{z_2}$ for the summed normal section. In a product of orthonormal coordinate frames $\ell_{(\alpha_1,\alpha_2)}(\Phi\Psi) = \ell_{\alpha_1}(\Phi)\ell_{\alpha_2}(\Psi)$, by ordinary differentiation in disjoint variables and the product formula for multi-index factorials. Regrouping the finite sums gives

$$D_u = \sum_{\alpha_1,z_1} \ell_{\alpha_1}(\Phi_{z_1,t_1})\, V_{\alpha_1,z_1}.$$

This expression is zero outside the image of the pullback.

Apply Lemma 128.1 to the fa normal variables of $N_1^\vee$. The degree cutoff for all the α_1's is at most $L = \sum_\tau B_\tau$. The marked group has order Q_1 and exponent dividing G. Its translations and modulations have lifts in $G^{-1}\Gamma_1$ by the same local congruences as in Theorem 128.2. For the full terminal table, and also on freezing any one terminal component, the norm-ideal proof of that theorem gives the same lower bound $2f\sqrt{T}/(G\,\mathrm{rd}(E_x))$ for the squared length. It used self-duality and integrality of the scalar Hermitian pairing, both of which hold for $N_1^\vee$. Only the dimension of the individual terminal residue space changes. Each removed zero face costs a fraction at most $(1+\delta_x)T^{-a}$ of the reference quadratic form. Since $a \ge 1$, the total relative error $\delta_x + (1+\delta_x)fT^{-a}$ is at most ρ_x on the same intervals. Apply the resulting Gram bounds to the vectors V_{α_1,z_1} coordinate by coordinate to obtain the displayed inequalities.

For instance, on standard unrefined lattices the residue matrices of the earlier edges at T may be the coordinate inclusions of successive ranks. Their composite image is the first $N(0)$ coordinates, an orthogonal summand at every place. These choices of residue matrices can be imposed along with any prescribed matrix cosets at the other selected primes by the Chinese remainder theorem. This describes actual indices in the finite sums. A nonzero coefficient there also requires the displayed test involving V_{α_1,z_1}. For a general path Proposition 128.5 still gives its exact cells. Their positions at T can contain undilated directions in the kernel of the pullback. A lower bound on the full auxiliary derivative matrix alone cannot evaluate those directions or the interference between paths in the final Bessel period. □

128.3 The number of exceptional intermediate matrix residues

Take E_x, g_1, g_2, T as in Section 6.6 and the word of Theorem 9.13. Use $d = N(m-1)$, $D = fd$, $G = g_1g_2$ and $b = \exp(x^\varepsilon)$. We first count residue images, giving each matrix residue equal weight. The theta coefficients attached to those matrices will enter the estimate for an individual path. The prime sizes needed here are

$$\log G = (16 + o(1))\log b, \qquad \log T = (17 + o(1))m \log b$$

by Theorem 93.9. Also $f/m \to \infty$, $\log \mathrm{rd}(E_x) = o(\log b)$, and the logarithms of the ranks are $O(\log x) = o(\log b)$. These follow from Theorems 6.29 and 19.15.

Lemma 128.7 (A count of images admitting a short normal period)**.** *At each $w \mid \pi_T$ choose independently and uniformly an a-dimensional subspace $I_w \subset V_w$, where $a \le d$ and $am \ge 2d$. In Proposition 128.5 use the position lattice $K_w = \{y \in U_w : \bar{y}|_{I_w} = 0\}$. Write Γ_I for the common period lattice of these cells and the marked residue sections at g_1, g_2. Also consider, for each $w_0 \mid \pi_T$, its period lattice on fixing the cell at w_0 and leaving the others free. The proportion of choices for which any of these lattices, after division by G, has a nonzero vector of squared length less than $b^{3/2}$ is at most*

$$\exp(-10fd\log b)$$

for sufficiently large x. This bound also holds if all the transported complex realizations of the CM period data are tested together.

Proof. Work first in one realization. Let Γ_0 be the normal period lattice obtained by leaving the integral position row unconstrained at the terminal places, retaining both marked residue groups. Thus, for the fixed norm-one ideal $\mathfrak{a}$ coming from the other frozen cells,

$$\Gamma_0 = \delta\Phi(\beta\mathfrak{b}^{-1}\mathfrak{a}N^\vee), \qquad \beta = \bar{\pi}_{g_1}\bar{\pi}_{g_2}.$$

When all labels vary, $\Gamma_I \subset \Gamma_0$. If a single terminal label at w_0 is fixed, its dual frequencies may gain a denominator at $\bar{w}_0$. They lie in $T^{-1}U_{w_0}^\circ$, since K_{w_0} contains TU_{w_0}. The period lattice

on that face is therefore contained in the module defining Γ_0 enlarged by the ideal $\bar{w}_0^{-1}$. These ambient lattices are independent of the subspaces just chosen.

Let $e = 0$ on the full table and $e = 1$ on one of the faces. After division by G, the nonzero vectors of the ambient lattice have squared length at least

$$R_e^2 = \frac{2f}{G\ \mathrm{rd}(E_x)} T^{-e/f}.$$

This is the fractional-ideal norm inequality in the proof of Theorem 128.2. If a vector also reduces to zero at h of the unfrozen terminal halves, its squared length is at least $R_e^2 T^{h/f}$. Indeed membership in their prime ideals multiplies the norm of the fractional ideal in that calculation by T^h. At a terminal half all the factors from G, β are units. Thus the vanishing of this row means exactly that its reduction is zero in the indicated normal coordinates.

Put $R = b^{3/4}$. Disjoint open balls of radius $R_e/2$ about the ambient lattice vectors of norm less than R lie in the ball of radius $R + R_e/2$. There are at most $(1 + 2R/R_e)^{2D}$ such vectors. Since $\log T/f = o(\log b)$, their number is at most

$$\exp\{(17.5 + o(1)) f d \log b\}.$$

The constant 17.5 comes from $\log(R^2/R_e^2) \le (3/2 + 16 + o(1)) \log b$. For a nonzero vector in this ball, the same inequality and the extra factor $T^{h/f}$ imply $h/f \le (17.5 + o(1)) \log b / \log T = O(1/m)$.

At an unfrozen place with nonzero residue row y, the probability that I_w is contained in $\ker y$ is $(T^{d-a} - 1)/(T^d - 1) \le T^{-a}$. To compute it, count pairs consisting of an a-plane and a nonzero form annihilating that plane. Every plane has $T^{d-a} - 1$ such forms, and the general linear group is transitive on the $T^d - 1$ nonzero forms. By independence, the probability that this one ambient vector meets all the necessary conditions is at most $T^{-a(f-e-h)}$. The logarithm of this bound is at most $-(34 - o(1)) f d \log b$, using $am \ge 2d$. A union bound over the ambient vectors and the $f + 1$ possible tables proves the assertion in one realization with an exponent $-(16.5 - o(1)) f d \log b$.

For completeness we bound the number of metric lattices to be considered after field conjugation. In product coordinates Theorem 65.9 gives a lattice of the form $\delta_\rho \mathfrak{c}\, \rho(M)$, with metric scale $\mathrm{area}(\delta_\rho \mathfrak{c})$; here M is the indicated fractional period module and $\mathfrak{c}$ is an ideal of the imaginary quadratic field K. Changing the representative of this ideal class and the scalar δ_ρ multiplies all coordinates by a complex scalar and the metric scale by the square of its absolute value. The resulting lengths are identical. The coordinate change to the trace metric is unitary after transport. Its entries and the field defining the row lattice are in the abelian field E_x. Their field embeddings have at most $2f$ possibilities. On torsion the further choice of generator of the elliptic CM module multiplies a residue module by a unit scalar. A subspace at a terminal place, and the condition that a row annihilate it, are preserved by this scalar. Unit phases on theta sections leave their period lengths unchanged. Thus at most $2f h_K$ cases of the count suffice, where h_K is the class number of K. Each transport permutes or conjugates the residue parameters and preserves the uniform count.

By the imaginary quadratic Minkowski bound, each ideal class has an integral representative of norm $\ll \sqrt{|\Delta_K|}$. There are at most $\sum_{n \le B} \sum_{d \mid n} 1 \le B(1 + \log B)$ ideals of norm at most B, using the Euler factors of a quadratic field. Hence $\log h_K = O(\log(2|\Delta_K|)) = o(\log b)$ on the common intervals. A union bound over the metric cases still gives the stated exponent. □

Theorem 128.8 (Coherent sums with a fixed intermediate pullback)**.** *Choose a vertex $2 \le j \le m - 2$. Fix the underlying lattices and frames of the path and its matrix data away from the terminal prime. Among the remaining terminal matrix residues for the edges from j up to*

$m-1$, *give equal weight to every choice with injective residue maps at every terminal place. Except for a proportion at most* $\exp(-10fd\log b)$, *the following assertion holds. Write* C_w *for the composite from vertex* j, *including the unit factor* $\bar{\pi}_T \bmod w$. *For every fixed marked input* $Y(z)$ *write*

$$D_u = \sum_{t:\, t_w C_w = u_w \ \forall w} P_t(Y), \qquad M(Y) = \frac{s^2}{Q_g} \sum_{z,\alpha} \|A_\alpha Y(z)\|^2.$$

On the nonzero intermediate covectors u *at vertex* j, *in each transported complex realization,*

$$(1-\rho_x)M(Y) \ \le \sum_{u_w \neq 0 \ \forall w} \|D_u\|^2 \ \le \ (1+\rho_x)M(Y), \qquad \rho_x = e^{-x^{\varepsilon/8}}.$$

The exceptional fraction is a counting bound for actual matrix residues before weighting their theta coefficients.

Proof. The source and target lattices at a terminal half are free over the local ring. The matrix quotient defining each edge maps uniformly onto the full residue matrix space there by the Chinese remainder theorem. The data away from the terminal prime can be prescribed independently, as in Proposition 127.20. The intermediate covectors have yet to be imposed. On the stated set, the composites are injective. The image of the composite at one place is uniform on the Grassmannian of planes of dimension $a = N(j)$. Indeed, multiplying the last injection on the left by an element of $\mathrm{GL}(V_w)$ permutes the choices and acts transitively on the images. The different terminal places are independent. The unit in C_w preserves its image. The ratios of ranks meet the requirement of the lemma, since

$$d = k + (m-1)(t_{\mathrm{r}}+1), \qquad a = k + j(t_{\mathrm{r}}+1), \qquad am \ge 2d.$$

Here $j \ge 2$ and $m \ge 2$. The bounds are uniform for all such j. The rank restriction defines a nonempty sample at every place. For consecutive ranks $n < n'$, for example, its fraction in the entire matrix space is $\prod_{i=0}^{n-1}(1-T^{i-n'})$, by counting ordered independent columns.

For an image outside the exceptional set, sum its terminal cells with the same pullback. Proposition 128.5 identifies the resulting test with a single coset of K_w and supplies a complete normal basis including zero labels. The same proof applies starting at vertex j, since covectors on the successive edges pull back by the same matrix equation. The basis has T^{fa} terminal entries for each z. Its whole sum is the original integral normal row, of norm s. The marked translations and modulations on Z have representatives in $G^{-1}\Gamma_I$. To see this, prescribe their residue coordinates at the marked places and zero at T, and apply the Chinese remainder theorem in $G^{-1}\Gamma_I/\Gamma_I$. The same reasoning applies on any frozen terminal face. The square of the shortest length in all these enlarged lattices is at least $b^{3/2}$ by the lemma. The bounds $L \ll x^C b$ and $D \ll x^C$ of Theorem 127.25 now give the degree and dimension hypotheses of Lemma 128.1. The resulting relative error on the full table is at most $\delta = 2e^{-\pi b^{3/2}/8}$.

Fixing one zero component of u selects exactly a fraction T^{-a} of the orthogonal equal-norm sections. On that face the same lemma gives an upper Gram bound of $(1+\delta)T^{-a}s^2 I/Q_g$. Removing the union of the zero faces therefore costs at most $f(1+\delta)T^{-a}$ in relative norm. Together with δ this is at most ρ_x. The result follows by applying the bounds to the coordinates of $A_\alpha Y(z)$ after the fiber sum, exactly as in Corollary 128.3. □

Proposition 128.9 (A dimension obstruction at the initial vertex)**.** *Fix one scalar output on a path, and consider the map from arbitrary scalar weights on its marked frequencies to its coefficients indexed by the original nonzero initial covectors. Its rank is at most* T^{fk}. *If the determinant varies at every marked place, its marked frequency support* $\mathcal{Z}$ *satisfies*

$$|\mathcal{Z}| \ge G^{ft_{\mathrm{r}}}, \qquad \frac{T^{fk}}{|\mathcal{Z}|} \le \exp(-8ft_{\mathrm{r}}\log b)$$

for sufficiently large x. Even on this support the kernel in the space of arbitrary scalar weights has dimension at least $|\mathcal{Z}| - T^{fk}$.

Proof. At each terminal half the initial covector belongs to a space of dimension $k = N(0)$ over the field of T elements. There are at most T^{fk} resulting scalar coefficients, counting even those that vanish by the pullback condition. This proves the rank bound. At a marked place with varying determinant the support has size $(q_w - 1)q_w^{d-s_w}$ by Proposition 127.20. Here $s_w \le r + (m-2)t_{\mathrm{r}} = d - t_{\mathrm{r}}$ by Proposition 127.14. Taking the product over the complete halves at both primes gives the first inequality. The ranks in Theorem 6.29 satisfy $km/t_{\mathrm{r}} \to 0$, directly from the factor $\lceil x^{377} \rceil$ in t_{r} and the stated powers of x for k, m. The two prime asymptotics above give

$$fk \log T - ft_{\mathrm{r}} \log G = -(16 + o(1)) ft_{\mathrm{r}} \log b.$$

This gives the second inequality, and rank-nullity finishes the proof. The calculation concerns arbitrary weights on the support. For the arithmetic input, the coefficients are the specific Fourier values of the preceding signed path sum. Their position relative to this kernel, and their interactions across paths with common initial covectors, enter the period comparison of Section 6.6. □

128.4 The signed marked table for a fixed penultimate image

The rank calculation of Proposition 128.9 allows an independent coefficient at every frequency in the marked support. We now determine the dependence imposed on those coefficients by the actual matrix test. Fix one marked place with residue field k of order q. Use $V = k^d$, $d = N(m-1)$, $a = N(m-2) + 1$, and the subspace $A = A_a \subset V$ of the first a coordinates in the intermediate frame of Proposition 127.14. Thus $d - a = t_{\mathrm{r}}$. Retain a nonempty position cell D_0 on which some matrix passes all tests. For a varying determinant let i be its block, $W = A_{s_i}$, $s = \dim W$, $H = \operatorname{im} \bar{X}^{(a)} \subset A$ and $K_0 = H \cap W$. Here $\bar{X}^{(a)}$ consists of the first a rows of the reduced penultimate core matrix. Fix once for this H a nonzero form Λ on A with kernel H.

Proposition 128.10 (The singular positions in the image hyperplane)**.** *Every position satisfying the entry congruences belongs to A, for both the plain and symmetrized tests. For a varying determinant the cell is a coset of W in A, and its singular positions are precisely $D_0 \cap H$. These have the form $u + K_0$, where $u \in H$. The class of u in H/K_0 is uniquely determined by the coset D_0. On that cell the position function, including its splitting factor, equals $\gamma\nu_i(\Lambda(v))$ for a root of unity γ. The multiplicative character is extended by zero. Among varying paths, H also determines i.*

Proof. There are exactly a columns in the tested matrix $[\,\bar{X} \mid v\,]\bar{U}$ and $\bar{U}$ is invertible. Both entry tests prescribe zero in all these columns at every row of index at least a, using zero-based indices. Right multiplication by $\bar{U}^{-1}$ shows that the same rows of $[\,\bar{X} \mid v\,]$ vanish. In particular $v \in A$. This is an equation on the original test before any Fourier transform.

The entry congruences in the varying case leave a coset of W by Proposition 127.14. On the top square the determinant is both the product of the diagonal block determinants and $\det(\bar{U}) \det[\,\bar{X}^{(a)} \mid v_{<a}\,]$. All the other diagonal block determinants are fixed nonzero constants on this cell, including the unitriangular initial block. The last displayed determinant is a nonzero homogeneous linear form in v with kernel H. Thus the varying block determinant on the entire cell is a fixed nonzero multiple of $\Lambda(v)$. Its zeros are $D_0 \cap H$. The additive superdiagonal factor is constant on a varying cell, as in Lemma 127.17. Combining it with the

characters of the fixed determinants, of the nonzero multiplier and of the splitting unit gives γ.

The restriction of Λ to W is nonzero, so $H + W = A$. Inclusion induces an isomorphism $H/K_0 \longrightarrow A/W$. A coset D_0 of W therefore determines exactly one class $u + K_0$ in H, and $D_0 = u + W$. Finally Proposition 127.14 makes $\Lambda|_W$ nonzero on block B_i and zero on all the preceding coordinates. The first diagonal block on which Λ is nonzero is consequently B_i, which recovers i from the hyperplane. □

Theorem 128.11 (The complete reduced Fourier table at a fixed image)**.** *Group varying paths having this same H, with fixed frame and orientation at the marked place. For $u \in H/K_0$ let $c(u)$ be the signed sum of the incoming coefficients whose position cell is $u + W$, including the factor γ. It may take values in the intermediate coefficient space. The table before the terminal coefficient operator depends on $z \in V^\vee$ only through its restriction z_A to A. Its potentially nonzero restrictions are indexed bijectively by*

$$(t, \xi) \in k^\times \times (H/K_0)^\vee, \qquad z_A|_W = t\Lambda|_W, \qquad z_A|_H = \xi.$$

Here ξ on H denotes its pullback from H/K_0. In these coordinates the exact table is

$$F(t, \xi) = q^{s-1} G_{-1}(\nu_i) \nu_i(t)^{-1} \widehat{c}(\xi), \qquad \widehat{c}(\xi) = \sum_{u \in H/K_0} c(u) \theta(-\xi(u)).$$

It vanishes on the other restrictions. In any Hermitian norm on the coefficient space,

$$\sum_{z_A \in A^\vee} \|F(z_A)\|^2 = q^{a+s-1}(q-1) \sum_{u \in H/K_0} \|c(u)\|^2.$$

In particular the original table vanishes identically if and only if every $c(u)$ vanishes. For fixed H the space of scalar tables obtained by allowing arbitrary such coefficients has dimension q^{a-s}.

Proof. All the position functions are supported in A, hence their Fourier transforms are unchanged on adding an element of A° to z. On a fixed cell write $v = u + y$ with $u \in H$ and $y \in W$. Then $\Lambda(u+y) = \Lambda(y)$. Summing $\nu_i(\Lambda(y))\theta(-z_A y)$ over the fibers of $\Lambda|_W$ gives zero unless $z_A|_W = t\Lambda|_W$. For $t = 0$ it is zero by multiplicative orthogonality of the nontrivial marked character. For $t \neq 0$ it equals $q^{s-1}\nu_i(t)^{-1} G_{-1}(\nu_i)$, by substitution in the one-dimensional Gauss sum. The remaining factor in this cell is $\theta(-z_A u)$. A form on $A = H + W$ is determined by its restrictions to H and W provided they agree on their intersection. On K_0 the indicated restrictions are both zero, so each displayed pair gives exactly one z_A. Summing the cell coefficients proves the formula.

The Gauss sum satisfies $|G_{-1}(\nu_i)|^2 = q$, by Theorem 127.15. Summing over t supplies the factor $q - 1$. Additive character orthogonality on H/K_0 gives $\sum_\xi \|\widehat{c}(\xi)\|^2 = q^{a-s} \sum_u \|c(u)\|^2$. Multiplication gives the asserted mean square. For a sum on all $z \in V^\vee$ the right side is multiplied by q^{d-a}. Injectivity and the dimension statement follow from the norm identity. Actual matrix paths can use a subset of the cells in the displayed coefficient space. □

Corollary 128.12 (The terminal test on the last varying block)**.** *For each $\xi \in (H/K_0)^\vee$ define the terminal operator*

$$L(\xi) = \sum_{t \in k^\times} \nu_i(t)^{-1} \sum_{z \colon z|_A = z_A(t,\xi)} J(z).$$

At this marked place the signed contraction of the preceding theorem is exactly

$$q^{s-1} G_{-1}(\nu_i) \sum_\xi L(\xi) \widehat{c}(\xi).$$

The inner sum defining $L(\xi)$ replaces the fine terminal tests by their union in the normal position model. At several marked places these operations can be performed simultaneously with the joint coefficient table and the original global terminal operator.

When $i = m - 3$, the last block of the penultimate test, $W = A$, $K_0 = H$, and there is a single ξ. Apart from the common Gauss scalar the marked test on an integral normal row y has value $\nu_i(t)^{-1}$ when $\bar{y}|_A = t\Lambda$ with $t \neq 0$, and zero otherwise. The remaining $d - a$ residue coordinates of y are unrestricted. If every varying marked place is in its last block, a fixed tuple of image hyperplanes therefore gives a single prescribed terminal operator applied to the sum of its incoming coefficients.

Proof. Insert the Fourier formula into the original sum over z, and first collect all extensions of the same z_A. Their coefficients are identical. In the normal theta realization their residue characteristic functions are disjoint and add to the characteristic function for the restriction to A. Thus their operators are summed with their given normalization, by Proposition 127.21. Then collect the t's with the same ξ. This gives the asserted expression, including the inverse multiplicative character and the Gauss factor. At several places the finite sums commute. Equivalently one can insert the product of the local characteristic functions in the single global theta section before differentiating. Neither the terminal operator nor the incoming table has to factor over the places.

When $i = m - 3$, the endpoint s_i of the block is $r + (m - 2)t_{\mathrm{r}} = a$, so the sole nonempty coset of W inside A is A itself. Its singular positions are exactly H, through the origin. The quotient H/K_0 is zero and the formula gives the stated unit-weighted test on its support. In particular the many extensions counted in Proposition 128.9 all enter with the same incoming coefficient for fixed t here. The nonvanishing of this concrete operator on the arithmetic input after the earlier pullbacks, and the effect of the other image hyperplanes and blocks, still enter the global period in Section 6.6. □

128.5 The marked tests at the first intermediate vertex

At vertex 1 the terminal prime provides about $T^{fd/m}$ distinct pullbacks. We compute how much of this supply is needed by the actual test at the other two primes. Keep the common interval, metric, frames and normal sections of Theorem 128.8. In particular $d = N(m - 1)$, $D = fd$, $b = \exp(x^{\varepsilon})$, and $Q_g = (g_1 g_2)^D$. Use w for a chosen half at g_1 or g_2, v for a chosen half at T, and k_w for the field of q_w elements at w. Write V_w for the marked space of dimension d, so that the integral normal position lattice U_w reduces onto $V_w^{\vee}$. For $B \subset V_w$, let $S_w(B)$ denote the inverse image of its annihilator in U_w. For a full local position lattice S, let $S^{\#}$ be its frequency dual for the local additive pairing of conductor the integers. In particular $U_w^{\#}$ is at the conjugate half.

Proposition 128.13 (Residue sections on an affine subspace)**.** *At each marked half choose subspaces $K_w \subset B_w \subset V_w$, an integral row z_w^0 and a character ω_w of the residue row group. Put*

$$S_w = S_w(K_w), \qquad S_w' = S_w(B_w), \qquad Z' = \prod_w S_w/S_w', \qquad Q = |Z'| = \prod_w q_w^{\dim B_w - \dim K_w}.$$

For a fine residue row z_w use any integral lift. On $z_w^0 + S_w$ let ξ_w be the class of $z_w - z_w^0$ in S_w/S_w'. Write $\omega(z) = \prod_w \omega_w(z_w)$. A marked input of the following form is determined by one coefficient vector Y_ξ per $\xi \in Z'$ by the formula

$$Y(z) = \begin{cases} \omega(z) Y_\xi, & z_w \in z_w^0 + S_w \text{ for every } w, \\ 0, & \text{otherwise.} \end{cases}$$

For a tuple of intermediate images I_v, sum the normal sections with coefficients $\omega(z)$ over the fine positions giving a fixed ξ and a fixed terminal pullback u. Including zero pullbacks gives a basis of a positive theta line of dimension $Q\prod_v T^{\dim I_v}$. At the marked halves its common periods are S'_w in position and $S_w^{\#}$ in frequency. The squared norm of the sum of its basis sections is

$$s_K^2 = s^2 \prod_w q_w^{-\dim K_w}, \qquad s = \Big\| \sum_{z,t} \Theta_{z,t} \Big\|_{\mathrm{av}}.$$

Fixing one terminal label at v divides this squared norm by $T^{\dim I_v}$.

Proof. The marked function for one new position is ω_w times the indicator of one coset of S'_w in $z_w^0 + S_w$. Addition of the fine indicators gives this identity with coefficient one. Position translations by S'_w act on all the functions by the same scalar from ω_w. Frequency modulations by $S_w^{\#}$ also act by one common scalar, given by their value at z_w^0. The analogous calculation at v is Proposition 128.5. This proves the period law. For the dimension, start at w with the position and frequency pair $S_w, S_w^{\#}$. They are exact duals, so the index gained on one side from the integral vacuum is lost on the other. Replacing S_w by S'_w multiplies the real covolume by $[S_w : S'_w]$. The terminal restrictions multiply it by $\prod_v T^{\dim I_v}$. The frozen data are as in Proposition 127.21. Riemann–Roch for the resulting positive line gives the claimed dimension.

The sections are orthogonal already on the fine cover, because distinct new cells have disjoint sets of indices in the original orthogonal basis. Integration of their periodic metric products against probability Haar measure gives the same inner product on the common cover and on the new torus. The norm computation can consequently be made in the fine basis. Its sections have equal squared norm. Exactly a fraction $\prod_w q_w^{-\dim K_w}$ of the fine positions occurs in the sum, each with a multiplier of absolute value one. Imposing one terminal value fixes a fraction $T^{-\dim I_v}$ of those cells since pullback onto its image has equal fibers. This proves both norm formulas. Every new cell is nonempty and has the same number of fine terms, which proves that the sections give the asserted basis. □

Lemma 128.14 (Short periods after retaining the marked quotient)**.** *Choose independently uniform a_1-planes I_v at the terminal halves, where $a_1 = N(1)$. On the full table of the proposition, take the group $\mathcal{M}_I$ consisting of the common periods and all their translates representing translations and modulations on Z' that act trivially on the terminal labels. Make the same construction on each of the f faces obtained by fixing one terminal label. Suppose that $\log Q \le (\kappa + o(1))D\log b$, where $0 \le \kappa < 16$. Fix $\sigma > 0$ with $\kappa + \sigma < 16$. The proportion of image tuples for which any of these groups has a nonzero vector of squared length less than $b^{1+\sigma}$ is at most*

$$\exp\{-(16 - \kappa - \sigma - o(1))D\log b\}.$$

One may test all the transported complex realizations at once. The exceptional set for fixed K_w, B_w is independent of z_w^0, ω_w.

Proof. Leave the terminal positions unrestricted for a moment. Write $P = \delta\Phi(\mathfrak{b}^{-1}M)$ for the lattice whose marked pairs are $S_w, S_w^{\#}$, using the period coordinates of Theorem 65.3. Here M is a self-dual Hermitian $\mathcal{O}_E$-lattice. Indeed one obtains its local factors from the original self-dual normal module by replacing a position lattice and its conjugate dual by the displayed pair. The primes here are split and distinct from the different at the marked places. The fixed norm-one modifications away from them preserve the duality as well. The global module is the intersection of these local conditions in the rational space. It has precisely those localizations by the Chinese remainder theorem, since only finitely many conditions change. Duality may be checked after localization, proving the assertion.

At w the translations of the new basis are represented by S_w/S'_w. The modulations distinguishing its cells are $(S'_w)^{\#}/S_w^{\#}$. In particular their lifts, together with the common lattice, have local pairs

$$S_w,\ (S'_w)^{\#}.$$

The quotient by the common periods at these places has order Q^2. The common unit scalars in the preceding proof can be removed on each operator. The residue pairing is perfect, so distinct translation and modulation pairs have distinct cosets of the period lattice. The Chinese remainder theorem produces lifts acting trivially on the terminal labels. Adding their lifts to the common periods gives the displayed local pairs and preserves the period conditions at the other places. This proves in particular that $\mathcal{M}_I$ is a lattice. Affinely shifting a test or changing its character changes only the phases in this calculation.

Let $e = 0$ for the full table and $e = 1$ on a face at v_0. Take $P_e = P$ for $e = 0$. For $e = 1$ multiply its fractional module by the ideal $\bar{v}_0^{-1}$. At v_0 the dual of the terminal position kernel is contained in $T^{-1}U_{v_0}^{\#}$, which justifies this enlargement. Ignore all the remaining terminal restrictions and enlarge the marked pairs of P_e to $S_w, (S'_w)^{\#}$. Call the result P_e^{+}. It contains $\mathcal{M}_I$, is independent of I, and is a union of exactly Q cosets of P_e. This last index is $\prod_w [(S'_w)^{\#} : S_w^{\#}]$. Also $P_e^{+} \subset G^{-1}P_e$, where $G = g_1 g_2$, because $q_w S_w \subset S'_w$ at every marked place.

We use the trace lower bound with this modified self-dual module. For any fractional ideal $\mathfrak{j}$ a nonzero vector of $\delta\Phi(\mathfrak{b}^{-1}\mathfrak{j}M)$ has squared norm at least $2f(\mathrm{N}\,\mathfrak{j})^{1/f}/\operatorname{rd}(E)$. Indeed its Hermitian square belongs to $\mathfrak{j}\bar{\mathfrak{j}}\mathfrak{D}_{E/K}^{-1}$ by local integrality. Taking norms and applying the arithmetic and geometric mean inequality at the f embeddings gives the bound with the same metric factor as in Theorem 65.3. Thus distinct vectors in a given coset of P_e have distance at least R_e, where

$$R_e^2 = 2f T^{-e/f}/\operatorname{rd}(E).$$

Set $R^2 = b^{1+\sigma}$. Disjoint balls of radius $R_e/2$ about vectors of norm less than R in one coset fit inside the ball of radius $R + R_e/2$. There are at most $Q(1 + 2R/R_e)^{2D}$ vectors in P_e^{+} of norm less than R. Its logarithm is at most $(\kappa + 1 + \sigma + o(1))D \log b$, since $\log \operatorname{rd}(E) = o(\log b)$ and $(\log T)/f = o(\log b)$ on the common intervals.

Consider a nonzero vector counted here. Let h be the number of unfrozen terminal halves where its position has zero residue. It then lies in $G^{-1}P_e$ with the additional ideal factor given by the product of those h primes. This uses complete splitting, so a zero position is exactly divisibility of the local module by the prime on that half. The trace bound gives squared norm at least $2fG^{-2}T^{(h-e)/f}/\operatorname{rd}(E)$. Since $\log G = (16 + o(1))\log b$ and $\log T = (17 + o(1))m \log b$, we have $h/f = O(1/m)$. At every other unfrozen half a nonzero residue must annihilate I_v. The probability of this is $(T^{d-a_1} - 1)/(T^d - 1) \le T^{-a_1}$, by counting planes in a hyperplane as in Lemma 128.7. Independence gives a probability at most $T^{-a_1(f-e-h)}$. The identity $a_1 = k + t_{\mathrm{r}} + 1$ and $d = k + (m-1)(t_{\mathrm{r}} + 1)$ gives $a_1 m \ge d$. The logarithm of the probability is therefore at most $-(17 - o(1))D \log b$. Multiply by the count of ambient vectors and sum over the $f + 1$ cases to obtain the bound.

Under transport of the CM data the lattice and module indices undergo the embedding of E and the common elliptic ideal change in the proof of Lemma 128.7. Unit scalars preserve all the subspaces and their annihilators, and changing an ideal representative is an isometry in the transported metric. The at most $2fh_K$ resulting cases contribute $\log(2fh_K) = o(D \log b)$ to the logarithmic estimate by that lemma. This completes the count. □

Theorem 128.15 (A signed mean square at vertex one)**.** *Use the matrices from vertex* 1 *through vertex* $m - 1$*, with the uniform counting measure on their injective reductions at the terminal places and the fixed data of Theorem 128.8. Let* C_v *be their composite including the*

prescribed unit. For subspaces as in Lemma 128.14, outside the exceptional fraction there, every marked input of Proposition 128.13 has the following bounds. Define

$$D_u(Y)=\sum_{\substack{z,t,\alpha\\ t_vC_v=u_v\ \text{for all } v}}\ell_\alpha(\Theta_{z,t})A_\alpha Y(z),\qquad M(Y)=\frac{s^2}{Q_g}\sum_{z,\alpha}\|A_\alpha Y(z)\|^2.$$

Then for $\rho_x=e^{-x^{\varepsilon/8}}$,

$$(1-\rho_x)M(Y)\le\sum_{u_v\neq0\ \text{for all}\ v}\|D_u(Y)\|^2\le(1+\rho_x)M(Y).$$

The estimate holds simultaneously for every vector in the stated input space and for all the affine centers and characters.

Proof. The image of a uniform composite of injections is a uniform a_1-plane by invariance under left multiplication on the last target. The different places are independent by the matrix residue calculation of Theorem 128.8. For a good image take the partial derivative Gram matrix of the basis in Proposition 128.13, retaining the index ξ and summing over u. Expand it in the Q^2 translation and modulation matrices on Z' as in the proof of Lemma 128.1. Its identity contribution, after multiplication by Q/s_K^2, is the identity on the retained cells and normalized derivative indices. The other terms are compressed Fock translations at nonzero vectors in $\mathcal{M}_I$, one coset for each matrix. Their phases have absolute value one. That proof and the shell estimate of Lemma 127.24 use only distinct cosets in a lattice whose nonzero vectors have length at least R. Apply them to $\mathcal{M}_I$ itself. They give operator error $\delta=2e^{-\pi R^2/8}$, since $R^2=b^{1+\sigma}$ dominates D and the degree cutoff $L\ll x^Cb$. The identical argument on a zero face, with its own lattice, gives the upper bound $(1+\delta)s_K^2/(QT^{a_1})$. Removing all zero faces therefore gives relative error at most $\delta+(1+\delta)fT^{-a_1}\le\rho_x$.

Apply the two bounds to the scalar coordinates of $A_\alpha Y_\xi$ and sum in an orthonormal output basis. Linearity and the defining marked character put the original signed coefficient inside the displayed norm. Its reference norm is $(s_K^2/Q)\sum_{\xi,\alpha}\|A_\alpha Y_\xi\|^2$. Indeed each ξ represents $\prod_w q_w^{d-\dim B_w}$ fine positions. Substituting the norm formula of the proposition shows that this is exactly $M(Y)$. □

Corollary 128.16 (Individual determinant tests and signed sums of later blocks)**.** *Fix the marked frames. Outside a proportion at most* $\exp(-14D\log b)$ *of the injective terminal matrix residues, the estimate of Theorem 128.15 holds for each individual nonzero penultimate marked test, multiplied by any incoming vector independent of* z*. For such an input* $Y(z)=\prod_w\mathcal{F}_w(z_w)y$ *its reference norm is*

$$s^2\left(\prod_w n_w\right)\sum_\alpha\|A_\alpha y\|^2,\quad n_w=\begin{cases}q_w^{s_w-1}(q_w-1), & \text{on a varying cell with } s_w=\dim W_w,\\ q_w^{\dim W_{c,w}}, & \text{on a constant determinant cell.}\end{cases}$$

Here $\mathcal{F}_w$ *includes the root of unity in Lemma 127.17. Each additional common normalization multiplies this norm by its squared modulus.*

There is also a bound after adding paths. Fix $0<\vartheta<1$ *and set* $p=\lfloor\vartheta d\rfloor$*. At every marked half keep the varying blocks whose first row index is at least* p*, with row indices starting at zero in the marked frame. Use a common terminal operator and the same pullback* C*. For a signed sum of these tests with arbitrary incoming vector coefficients the theorem holds outside a proportion at most* $\exp(-7\vartheta D\log b)$*. In this case* $M(Y)$ *uses the squared norm of the signed sum at each frequency.*

Proof. For a varying block take $B_w = W = A_{s_w}$ and $K_w = \ker(\lambda|_W)$ in the notation of Lemma 127.17. Its Fourier value is zero unless $z_w|_W = t\lambda$ with $t \neq 0$. On this set it is a scalar times $\nu_i(t)^{-1}\theta(-z_w u)$. Thus Z'_w has dimension one, with coordinate t, and $\theta(-z_w u)$ is precisely an allowed ω_w. Take $Y_0 = 0$ at the omitted scalar value at this place. For a constant determinant use $B_w = K_w = W_c$, $z_w^0|_{W_c} = \rho_c$, and the same position character. There is just one coarse position. This proves that $Q \le G^f$ for each individual test. Finite Fourier orthogonality gives $q_w^{-d}\sum_{z_w} |\mathcal{F}_w(z_w)|^2 = \sum_v |h_w(v)|^2 = n_w$, counting the positions on which the original function has modulus one. Multiplication gives the stated normalization.

For simultaneous control, W_c is a coordinate subspace of the initial r-space by Proposition 127.14. There are at most 2^r choices. For a varying test the form on A_{s_w} is supported on its one determinant block of t_{r} rows, so there are at most $mq_w^{t_{\mathrm{r}}}$ choices for the pair K_w, B_w. Multiplying the sum of these two bounds over the marked halves has logarithm $O(fr + f\log m + ft_{\mathrm{r}}\log G) = o(D\log b)$. The centers and position characters cause no further loss by the lemma. Apply it with $\kappa = 0, \sigma = 1/2$ and take the union to obtain the first exceptional bound.

For the signed sum take $K_w = A_p$ and $B_w = A_{d-t_{\mathrm{r}}}$. The specified collection is empty if $p > d - t_{\mathrm{r}}$. In the other case these are nested subspaces. Each active form in the selected blocks vanishes on A_p. Its Fourier table is supported on A_p°, and all its values, including the phase from the singular coset, depend only on restriction to $A_{d-t_{\mathrm{r}}}$ by Proposition 128.10. These two properties pass to the joint signed sum, even when the coefficients do not factor across the marked places. It is an input in the proposition, with centers zero and characters one. Now $\log Q \le (16(1-\vartheta)+o(1))D\log b$. Use $\kappa = 16(1-\vartheta)$ and $\sigma = \min(8\vartheta, 1/2)$ in the lemma. Since $16\vartheta - \sigma \ge 8\vartheta$, its bound implies the claimed estimate for all sufficiently large x. □

The outer sum in the theorem ranges over covectors at vertex 1. The first edge to vertex 0 further identifies these covectors, and paths with different composites C can contribute to the same initial coefficient. The exceptional proportion counts matrix residues with equal weight. Section 6.6 requires the integral norm and the archimedean comparison after those additional signed contributions and their arithmetic weights are included.

128.6 Initial covectors and the weighted first theta edge

Write $n = N(1)$, $k = N(0)$, and $q = T$. At one half over the terminal prime the first residue matrix is a map $X : U \to V$ over $\mathbb{F}_q$, with $\dim U = k < n = \dim V$. A later part of the theta chain can prescribe a different value for each nonzero covector u on V. For a table $F(u)$ of vectors in a complex Hilbert space set

$$(B_X F)(\eta) = \sum_{uX=\eta} F(u) \qquad (0 \neq \eta \in U^\vee).$$

Both covector tables use counting measure. We first compute the second moment when each injection X has equal weight. We then compute the sum with the coefficients of the actual first edge.

Proposition 128.17 (The mean square of the first pullback)**.** *Let X be a uniform injection $U \to V$. Put*

$$h_1 = \frac{q^{n-k}-1}{q^n-1}, \qquad h_2 = \frac{(q^{n-k}-1)(q^{n-k}-q)}{(q^n-1)(q^n-q)}.$$

On scalar tables the operator $H = \mathbb{E}_X B_X^* B_X$ has eigenvalues on the three mutually orthogonal spaces given by

$$\begin{aligned} 1 - h_1 &\quad \text{on tables with sum zero on each covector line,} \\ 1 - h_1 - (q-1)(h_1 - h_2) &\quad \text{on tables constant on each line and of total sum zero,} \\ (1 - h_1) q^{n-k} &\quad \text{on the constant table.} \end{aligned}$$

Here a covector line in a table contains its nonzero elements. In particular $H \succeq (1 - q^{1-k})I$. For vector tables these operators are tensored with the identity on the values.

Proof. The annihilator of XU is a uniform $(n-k)$-plane in $V^\vee$. It contains one prescribed nonzero form with probability h_1, and two prescribed independent forms with probability h_2. Indeed, count ordered nonzero forms or ordered independent pairs in that plane and divide by the corresponding count in $V^\vee$. The entry $H_{u,u'}$ is the probability that $uX = u'X \neq 0$. It equals $1 - h_1$ on the diagonal. Distinct multiples of one form cannot have the same nonzero image. For independent u, u', the annihilator must contain $u - u'$ and must exclude u. These forms are independent, giving probability $h_1 - h_2$.

Thus the off-diagonal part is $h_1 - h_2$ times the all-ones matrix with every diagonal block of one line removed. It vanishes on the separately zero-sum tables and acts as $-(q-1)(h_1 - h_2)$ on the second subspace. Constants form the remaining eigenspace. For each injection every nonzero η has q^{n-k} preimages. The squared norm of $B_X \mathbf{1}$ divided by $q^n - 1$ is $(q^k - 1)q^{2(n-k)}/(q^n - 1) = (1 - h_1)q^{n-k}$. This gives the last eigenvalue. All three are at least $1 - qh_1 \geq 1 - q^{1-k}$. □

Corollary 128.18 (A second moment at the original initial vertex)**.** *Fix a suffix and marked input satisfying Theorem 128.15 at a good tuple of suffix images. Put $F(u) = D_u(Y)$ on tuples of nonzero covectors at vertex 1. Keep this table fixed and let the first injections X_v vary independently at the f terminal halves. Write $B_X = \bigotimes_v B_{X_v}$. Then*

$$\mathbb{E}_X \sum_{\eta_v \neq 0 \text{ for all } v} \|(B_X F)(\eta)\|^2 \;\geq\; (1 - T^{1-k})^f (1 - \rho_x) M(Y).$$

On either table let $\Pi = \prod_v (1 - A_v)$, where A_v averages nonzero scalar multiples in position v. The exact mean square on the separately zero-sum component is

$$\mathbb{E}_X \|\Pi B_X F\|^2 = (1 - h_1)^f \|\Pi F\|^2.$$

The mean of $\Pi B_X F$ itself is zero.

Proof. Independence makes the product Gram operator $H^{\otimes f}$. The lower bound of the proposition and the lower estimate for $\|F\|^2$ at vertex 1 prove the first assertion. Each pullback commutes with scalar multiplication of the covectors at a place, so $\Pi B_X F = B_X \Pi F$. The product Gram acts on the range of this projection by $(1 - h_1)^f$. For the last assertion, independently compose each X_v on the right with an invertible endomorphism of the initial space. This preserves its uniform distribution and permutes all nonzero initial covectors transitively. The mean output is consequently constant on their product. Applying Π gives zero. □

Theorem 128.19 (Rank-one frequencies of the actual first edge)**.** *Fix initial and first intermediate unrefined classes and their frames. Let $E(X) : D_0 \to D_1$ be the sum of the first-edge coefficient maps with residue matrix tuple $X = (X_v)_v$ at T. Include their prescribed finite weights away from T. Use all the integral residue matrices. Let $F(u)$ be the map on D_1 given by the remaining edges at the tuple u of nonzero intermediate covectors. Keep all classes, finite*

signs and the coupling of the last two edges in this sum. Assign each unrefined stabilizer mass once, as in Theorem 73.1. For the physical nonzero initial covectors the coefficient map is

$$P(\eta) = \sum_{X,u:\, u_v X_v = \eta_v \text{ for all } v} F(u)E(X).$$

For a tuple $v = (v_P)_{P|\pi_T}$ of initial vectors define

$$\widehat{E}(v \otimes u) = \sum_X \left(\prod_{P|\pi_T} \psi_P\big(- u_P(X_P v_P)\big) \right) E(X), \qquad K(v) = \sum_u F(u)\widehat{E}(v \otimes u).$$

The transformed first-edge operator $\widehat{E}(v \otimes u)$ uses the original integral matrix-position test at each terminal half multiplied by the indicated additive character. Equivalently it is a frequency translate of that integral theta test by the lift of $-v_P \otimes u_P$ divided by T, in the dual coordinates for ψ_P. The arithmetic coefficient maps and derivatives have their original normalization.

For any compatible Hilbert norms, with Hilbert–Schmidt norm on the coefficient maps, the actual signed row on ordinary initial tables satisfies

$$\sum_\eta \|(\Pi P)(\eta)\|_{\mathrm{HS}}^2 = T^{-fk} \sum_{v_P \neq 0 \text{ for all } P} \|(\Pi K)(v)\|_{\mathrm{HS}}^2.$$

On the left η ranges over tuples with no zero component. For several first intermediate classes, the corresponding $K(v)$ are added before taking the squared norm in this formula.

Proof. The initial terminal test checks integrality and the equation $u_P X_P = \eta_P$. In the finite quotient of matrix positions the denominator is prime to T and the sublattice in the denominator of the quotient is contained in the kernel of reduction modulo T. Thus one may group all first-edge terms by X prior to imposing the covectors. The other first-edge local factors are independent of these terminal refinements at fixed frames. This is precisely the factorization of the test in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 646 and lines 1115–1120 there. For a local integral matrix x,

$$\sum_{X_P} \psi_P\big(- u_P(X_P v_P)\big) \mathbf{1}_{\bar{x} = X_P} = \psi_P\big(- u_P(\bar{x} v_P)\big).$$

This equality holds with both sides zero outside the integral lattice if the right side is extended by zero. Summing higher digits of a matrix cell and applying the linear theta distribution therefore gives the claimed operator. In split position coordinates a frequency translation acts by multiplication by its pairing character, proving the equivalent description. The remaining path can depend on u and on its unrefined initial class at vertex 1 and uses the value of the preceding edge in that class. This gives the displayed formula for P by composition of the actual maps.

Extend P by the same fiber formula to include all zero faces. Its unnormalized additive Fourier transform at v is $K(v)$, because substituting $\eta = uX$ in the additive character and interchanging finite sums gives exactly that expression. Scalar multiplication on a covector corresponds to inverse scaling on the dual vector under Fourier transform, so scalar averaging, and hence Π, commute with the transform. On a face with a zero component the corresponding factor $1 - A_P$ is zero. Thus the choice of the formal extension on those faces has no effect on the projected identity. Plancherel on the additive group of order T^{fk}, applied to each scalar matrix entry in orthonormal bases, proves the formula. Class addition commutes with all these finite linear operations. This also proves the assertion about the order of summation. □

The first mean square distinguishes the initial pullbacks when the first edge is sampled separately. The expression $K(v)$ retains the dependence of its theta coefficient on that edge. In particular the signed first-edge sum required in Section 6.6 samples frequencies with at most rank one at each terminal half. Its size depends on how those translated coefficients pair with the fixed suffix maps.

128.7 Integral initial tests and the values of the signed column

Fix one ordinary initial class a, its rank k module and its terminal residue spaces U_P at the f chosen halves. Use the frames and coefficient lattice D_a of Theorem 99.20. Enlarge the coefficient field J to contain the character values used below. At every $P \mid \pi_T$ choose a nontrivial multiplicative character $\chi_P : \mathbb{F}_T^\times \to \mathcal{O}_J^\times$, extended by zero at zero. For a tuple v of nonzero initial vectors and a coefficient vector y, set

$$\phi_{v,y}(\eta) = \left(\prod_P \chi_P(\eta_P(v_P)) \right) y$$

on the covectors over a, and give it zero values over other classes. We compute the cost of these tests in the ordinary column and then evaluate them on the original signed period. Both calculations use the same E_x, T and word.

Proposition 128.20 (Integral domain and complex norm of the character tests)**.** *For every v the exact domain of y giving an integral initial test is $D_* = z_{\max} D_a$ of Theorem 99.20. Each such test has zero sum on each local scalar orbit, with the other positions fixed. Choose one nonzero representative of each vector line and each covector line at every P. Put $N = (T^k - 1)/(T - 1)$ and let M_P be the N-by-N matrix with entry $\chi_P(\ell(v))$ in the covector row ℓ and vector column v. Then*

$$M_P^* M_P = T^{k-1} I, \qquad M^* M = T^{f(k-1)} I, \qquad M = \bigotimes_P M_P.$$

The matrix M maps the line coefficients of a sum of the tests to its values on the covector representatives. In the Hilbert norm on all nonzero covectors, the squared norm acquires the additional factor $(T-1)^f$ from the scalar orbits.

Proof. The stabilizer of the unrefined ordinary initial class at the prescribed level is trivial by Proposition 86.29. At a tuple of lines the coefficient lattice is constant throughout its scalar orbit. The scalar in the test is a unit precisely when all $\eta_P(v_P)$ are nonzero. The intersection of the actual coefficient lattices over those line tuples is $z_{\max} D_a$, by the intersection calculation in Theorem 99.20. This proves both directions of the domain assertion. Scaling at one place multiplies each nonzero value by a nontrivial character, whose sum is zero.

At one place compute the inner product of two columns by summing first over all nonzero forms and dividing by $T - 1$. The summand is constant on each covector line, since the two unit characters of the scalar cancel. For equal columns exactly $(T-1)T^{k-1}$ forms have nonzero pairing. For distinct vector lines the two prescribed vectors are independent. Every pair of field values for their evaluations occurs T^{k-2} times, and the resulting product of the two character sums is zero. For $k = 1$ only the diagonal case occurs. Tensoring the one-place identities gives the formula for M. Each covector representative has $(T-1)^f$ scalar multiples of equal norm, giving the last assertion. □

Theorem 128.21 (The integral index in the prescribed scalar characters)**.** *Let $\mathcal{L}_\chi$ consist of the actual integral initial tables supported over a satisfying $\phi(c\eta) = \prod_P \chi_P(c_P)\phi(\eta)$. Let $\mathcal{J}_\chi$*

be the sublattice generated by $\phi_{v,y}$ for $y \in D_*$ and for all vector representatives. At every coefficient prime away from T these lattices coincide.

At a coefficient prime $\mathfrak{p} \mid T$ normalize $v_{\mathfrak{p}}(T) = 1$. For lattices $J_0 \subset L_0$ of equal rank let $\iota_{\mathfrak{p}}(L_0, J_0)$ denote the valuation of the determinant of the inclusion in local bases. Write $r_D = \dim_J(D_a \otimes J)$ and $\chi_P = \omega^{j_P}$ in this completion, where ω is the Teichmüller character and $1 \le j_P \le T-2$. For the actual highest weights λ_τ one has

$$\frac{\iota_{\mathfrak{p}}(\mathcal{L}_\chi, \mathcal{J}_\chi)}{r_D N^f} = \sum_{P \mid \pi_T} \left(\frac{k(N-1)}{2N} - \left(1 - \frac{k}{N}\right) \frac{j_P}{T-1} \right) + \sum_{\tau \colon v_{\mathfrak{p}}(\tau(\bar{\pi}_T))=1} \left(\lambda_{\tau,1} - \frac{1}{k} \sum_{i=1}^{k} \lambda_{\tau,i} \right).$$

Proof. Specifying a value on each tuple L of covector lines identifies $\mathcal{L}_\chi$ with $\bigoplus_L D_{a,L}$. For each L one can choose v with all its pairings nonzero, so $D_* \subset D_{a,L}$ by the proposition. Inside $\bigoplus_L D_*$ the generating map is exactly $M \otimes I_{r_D}$. Away from T the three coefficient lattices $D_*, D_{a,L}, D_a$ coincide, and $M^{-1} = T^{-f(k-1)} M^*$ is integral there. This proves the assertion at all such primes, including those dividing $T-1$.

At $\mathfrak{p}$, consider the unnormalized additive Fourier transform from covector functions $g(c\ell) = \chi_P(c)^{-1} g(\ell)$ to vector functions. In projective coordinates its matrix is $G_P M_P^t$, with $G_P = \sum_{s \neq 0} \chi_P(s)^{-1} \psi_P(-s)$. Indeed scaling the summation variable gives $G_P \chi_P(\ell(v))$, also when $\ell(v) = 0$ by character orthogonality. In the notation of Theorem 126.12, scaling the basis covectors gives these input functions eigencharacter ω^{j_P}. Changing the sign or unit of the additive character permutes the vector positions up to unit factors. The theorem therefore gives the valuation of the determinant of $G_P M_P^t$ as $k((N-1)/2 + j_P/(T-1))$. In rank one the same theorem gives $v_{\mathfrak{p}}(G_P) = j_P/(T-1)$, since there is one line and its nonzero pairing is a unit. Subtracting N copies gives the determinant of M_P. The determinant of the f-fold tensor has valuation N^{f-1} times the sum of these one-place valuations. This accounts for the first line of the index formula.

It remains to compute the inclusion of $\bigoplus_L D_*$ in the full line lattice. In the frame of Theorem 120.8 the exponents of $D_{a,L}$ relative to D_a are sums of one diagonal-coordinate weight per affected embedding. The average of that coordinate weight in W_{λ_τ}, counted with its multiplicity, is $k^{-1} \sum_i \lambda_{\tau,i}$. Indeed permutation matrices preserve the multiset of weights and the sum of all coordinates on every weight is the central weight $\sum_i \lambda_{\tau,i}$. On D_* the common scalar has valuation $\sum_{\tau \text{ affected}} \lambda_{\tau,1}$. Subtracting the two determinant valuations and multiplying by the number N^f of line tuples gives the second line after normalization. This determinant calculation also applies to fractional coefficient frames. □

Theorem 128.22 (The full signed period modulo powers of the terminal prime)**.** *Take $y \in D_*$ and a fixed algebraic projected small test F. For each path γ from a let $w_\gamma = W_{\gamma,F}(y)$ be its coefficient in Theorem 73.1, with the prescribed masses and signs. Write $\eta_{\gamma,P}$ for the composite terminal covector at the initial vertex. Let R be the completed valuation ring at $\mathfrak{p} \mid T$, and choose $0 \neq \Delta$ in its fraction field such that $\Delta w_\gamma \in R$ for every path in this finite sum. For $h \ge 1$ set $e_P = j_P T^{h-1}$. Use integral $\mathbb{Z}_T$-lifts of the reduced edge matrices, the initial vectors and the marked final covector including its unit $\bar{\pi}_T \bmod P$. Their composite row is denoted $\widetilde{\eta}_{\gamma,P}$. Then*

$$\Delta\, \mathcal{P}(\phi_{v,y}, F) \equiv \sum_{\gamma} (\Delta w_\gamma) \prod_{P \mid \pi_T} \left(\widetilde{\eta}_{\gamma,P}(\widetilde{v}_P) \right)^{e_P} \pmod{T^h R}.$$

All choices of integral lifts give the same residue. The right side can be formed by attaching at each edge matrix $x_{i,P}$ its ordinary symmetric power of degree e_P in the lifted terminal

coordinates, in tensor product over P. These maps are composed in the order of the theta chain. The original last two edge tests retain their common coupling variable throughout the multiplication.

Proof. Inserting this single admissible test into Theorem 73.1 gives the exact equality

$$\mathcal{P}(\phi_{v,y}, F) = \sum_{\gamma} w_{\gamma} \prod_{P} \chi_P\big(\eta_{\gamma,P}(v_P)\big).$$

Paths with a zero pullback can be included with value zero. The terminal primes split completely, so after choosing local frames each residue matrix may be lifted over $\mathbb{Z}_T$. For $z \in \mathbb{Z}_T^{\times}$, write $z = \omega(\bar{z})(1 + Tb)$. The binomial theorem implies $(1 + Tb)^{T^r} \in 1 + T^{r+1}\mathbb{Z}_T$ by induction on r. Also $T \equiv 1 \bmod T - 1$. Hence $z^{j_P T^{h-1}} \equiv \chi_P(\bar{z}) \bmod T^h$. For $T \mid z$ this congruence holds with value zero because $j_P T^{h-1} \geq h$. Apply it to each scalar in the exact equality. Integrality of the finitely many Δw_{γ} permits addition of the congruences in R, with no loss of precision. This proves the formula and independence of lifts.

For a map X of free $\mathbb{Z}_T$-modules its induced map on the symmetric algebra sends v^e to $(Xv)^e$. The functional induced by a row a sends this last element to $a(Xv)^e$. Both assertions follow on monomials by ordinary multiplication without division by factorials. Apply these identities through all the matrices in a path, tensor over the places, and take the terminal evaluation. This produces the displayed product. Distributivity allows one to sum the tensor maps over an edge while its shared class and coupling indices stay fixed. The original coefficient maps compose on their own factors. One can carry a single common denominator outside the final sum, so this description does not require individual edge maps to be integral in a new normalization. □

Corollary 128.23 (A residue test on the prescribed signed matrices)**.** *Let $\kappa(\mathfrak{p})$ be the residue field of R. In initial vector coordinates form the multihomogeneous polynomial*

$$H((X_P)_P) = \sum_{\gamma} \overline{\Delta w_{\gamma}} \prod_{P} \big(\eta_{\gamma,P}(X_P)\big)^{j_P} \quad \text{over } \kappa(\mathfrak{p}).$$

It is nonzero if and only if at least one of the integral tests $\phi_{v,y}$ above has normalized period $\Delta\mathcal{P}(\phi_{v,y}, F)$ a unit at $\mathfrak{p}$. At $\chi_P = \omega$ for every P, its coefficient tensor is $\sum_{\gamma} \overline{\Delta w_{\gamma}} \bigotimes_P \eta_{\gamma,P}$.

Proof. Take $h = 1$ and reduce the preceding congruence to the residue field. This identifies $H(v)$ with the required period residue. The degree in each individual variable is at most $j_P \leq T - 2$. Evaluation of polynomials of degree less than T per variable on the full product of copies of $\mathbb{F}_T$ is injective even after coefficient extension. In one variable this follows from the root bound, and induction on the number of variables gives the general assertion. If some vector v_P is zero then the polynomial vanishes there since $j_P > 0$. A nonzero polynomial therefore has nonzero value at an allowed tuple. Scaling a representative scales that value by a unit. Finally expansion of the product of linear forms at $j_P = 1$ gives the stated tensor. □

The index theorem computes the cost of using these tests in the actual initial coefficient lattice. The last two results retain the entire signed sum of paths for each test. A bound in Section 6.6 still requires an estimate for those values with the prescribed arithmetic weights and period normalization. The common denominator Δ and a nonzero coefficient of the residue polynomial have to be controlled for that application.

128.8 Integral moments of the composite terminal covectors

Fix a field E_x, a word and a translate on the interval of Section 6.6. Write $p = T$ for its terminal prime, $d = p - 1$, and R for the completed integer ring at a chosen coefficient prime above p. Normalize $v(p) = 1$. Write $[a] \in \mathbb{Z}_p$ for the Teichmüller representative of $a \in \mathbb{F}_p$, with $[0] = 0$. We use $0^0 = 1$ in monomials. The arithmetic weights, signs, projections and measures throughout are those of the path formula in Theorem 73.1.

Lemma 128.24 (The coefficients of a multiplicative character of a linear form)**.** *Take $1 \le j < d$ and a row $a = (a_1, \dots, a_k)$ over $\mathbb{F}_p$. Let $\mathcal{M}_j$ consist of the multi-indices $0 \le m_i \le d$ with $|m| = \sum_i m_i \equiv j \pmod d$. There are coefficients $c_{j,m} \in \mathbb{Z}_p$, independent of the row, such that for every $v \in \mathbb{F}_p^k$ one has*

$$[a(v)]^j = \sum_{m \in \mathcal{M}_j} c_{j,m} \prod_i [a_i]^{m_i} [v_i]^{m_i}, \qquad v_p(c_{j,m}) = \frac{|m| - j}{d}.$$

Evaluation of the monomials $\prod_i [v_i]^{m_i}$, $m \in \mathcal{M}_j$, on one representative per nonzero vector line gives a square matrix invertible over $\mathbb{Z}_p$.

Proof. In one variable, evaluation of degrees $0, \dots, p-1$ on the p Teichmüller representatives is an isomorphism over $\mathbb{Z}_p$. Its Vandermonde determinant is a product of differences which are all nonzero modulo p. Tensoring proves the same assertion for k variables with these individual degree bounds. Under scaling $v \mapsto uv$, a monomial is multiplied by $[u]^{|m|}$. Thus the monomials in $\mathcal{M}_j$ are an integral basis of functions with scaling character $[u]^j$. Indeed distinct scalar characters differ by a unit at a generator of $\mathbb{F}_p^\times$. Such a function vanishes at zero, and its values on the representatives are arbitrary. This proves the invertibility assertion.

For the coefficients adjoin a primitive pth root ζ and put $G_m = \sum_{u \ne 0} [u]^{-m} \zeta^u$ for $0 < m < d$. The polynomial of degree at most d giving ζ^x on $[x]$ has coefficients

$$b_0 = 1, \qquad b_m = G_m/d \quad (0 < m < d), \qquad b_d = -p/d.$$

Its value at zero determines b_0. On the nonzero residues multiplicative character orthogonality gives the interior coefficients, while the mean equals $-1/d = b_0 + b_d$. For $u \ne 0$, substitution of $[u][x]$ gives ζ^{ux}. Finite Fourier inversion here is just the identity $\sum_{u \ne 0} [u]^{-j} \zeta^{ut} = G_j [t]^j$, obtained by changing the summation variable when $t \ne 0$ and by character orthogonality when $t = 0$. Apply it to $t = a(v)$. Expanding $\prod_i \zeta^{u a_i v_i}$ by the polynomials just computed and summing over u gives precisely the stated monomials with

$$c_{j,m} = \frac{d}{G_j} \prod_i b_{m_i}.$$

The computation includes $a_i = 0$, since then the corresponding polynomial factor has value one. These coefficients lie in $\mathbb{Z}_p$ by uniqueness of interpolation applied to the row with all coordinates one and its $\mathbb{Z}_p$-valued function.

For completeness put $\lambda = \zeta - 1$. The identity $p = \prod_{u=1}^{p-1} (1 - \zeta^u)$ gives $v(\lambda) = 1/d$, since $(\zeta^u - 1)/\lambda$ reduces to u modulo λ. Expand G_m using $\zeta^u = (1 + \lambda)^u$ with $1 \le u \le p - 1$. For $n < m$ the coefficient of λ^n is a multiple of p in $\mathbb{Z}_p$. Indeed its reduction is $\sum_{u \ne 0} u^{-m} \binom{u}{n}$, a sum of nontrivial powers on $\mathbb{F}_p^\times$. For $n = m$ that coefficient reduces to $-1/m!$. Terms with $n > m$ have strictly larger valuation than m/d, as do the terms with $n < m$ because $m < d$. Thus $v(G_m) = m/d$, in particular $G_m \ne 0$. The exponents of b_0, b_d are zero and one. The formula for $c_{j,m}$ now gives the claimed valuation. □

Theorem 128.25 (All determinantal ideals of the signed character tests)**.** *Fix an ordinary initial class a, an R-basis y_s of $D_* \otimes R$ as in Proposition 128.20, and a finite list of projected algebraic tests F_i. At the chosen terminal places P, take $1 \le j_P < d$ and the characters $u \mapsto [u]^{j_P}$. Denote the matrix of periods on the tests ϕ_{v,y_s} by P_χ, with rows i and columns (s,v), where v runs through tuples of vector line representatives. For a path γ starting at a, write $a_{\gamma,P,h}$ for the coordinates of its composite terminal covector in the initial frame. Form the moment matrix*

$$S_{i,(s,m)} = \sum_{\gamma} W_{\gamma,F_i}(y_s) \prod_{P,h} [a_{\gamma,P,h}]^{m_{P,h}}, \qquad m_P \in \mathcal{M}_{j_P}.$$

Put $b_m = \sum_P (|m_P| - j_P)/d$. For a matrix U over $\mathrm{Frac}(R)$ let $I_q(U)$ be the fractional ideal generated by its minors of size q, including the zero ideal. For every $q \ge 1$,

$$I_q(P_\chi) = \sum_{L \text{ a set of } q \text{ columns } (s,m)} p^{\sum_{(s,m)\in L} b_m} I_q(S_{:,L}).$$

In particular, if both matrices have rank at least q, the valuation loss from $I_q(S)$ to $I_q(P_\chi)$ lies between the sum of the q smallest and the sum of the q largest of the displayed column exponents b_m, counted once per column. The two matrices have the same rank.

Proof. Apply the lemma at every terminal place to the compact scalar in Theorem 73.1. Each full path contributes its original signed coefficient, evaluated at y_s, multiplied by the characters of its composite covectors on the initial vectors. If a composite covector is zero at some place all terms for that path vanish, since $|m_P| > 0$. Thus summation over the actual paths gives the matrix factorization

$$P_\chi = S \, \mathrm{diag}_{(s,m)}\Big(\prod_P c_{j_P,m_P}\Big) \, E.$$

Here E is block diagonal by s. Each block is the tensor product of the square monomial evaluation matrices from the lemma, with rows indexed by monomials. In particular E is invertible over R. All entries of the diagonal matrix are nonzero, with the specified valuations. The factorization uses the same coupled last-edge variables and the same projection inside every W_{γ,F_i}.

Multiplication by an invertible integral matrix preserves every minor ideal by Cauchy–Binet applied in both directions. Scaling a column scales every minor containing it by the same scalar. Applying these two facts to the factorization proves the formula for fractional ideals. Taking the least valuation of a nonzero minor gives the lower bound. A minor of S attaining its least valuation gives the upper bound after scaling its q columns. Invertibility over the fraction field gives equality of the ranks. □

Corollary 128.26 (The full integral initial table at the terminal prime)**.** *Keep the projected tests F_i, and allow all ordinary initial classes and all integral initial tables with separate scalar sums zero. For each initial class a and each tuple of covector lines L, choose an R-basis $y_{a,L,s}$ of its actual coefficient lattice $D_{a,L} \otimes R$. For every tuple of nontrivial scalar characters $j = (j_P)_P$ and every $m_P \in \mathcal{M}_{j_P}$, form the column*

$$B_{i,(a,s,j,m)} = \sum_{\gamma \text{ from } a} W_{\gamma,F_i}(y_{a,[\eta_\gamma],s}) \prod_{P,h} [a_{\gamma,P,h}]^{m_{P,h}}.$$

Paths having a zero terminal pullback at any place are omitted. For each q, the ideal of all minors of size q of the projected period map on this full integral domain is exactly $I_q(B)$. In particular each moment uses powers at most $p-1$ per residue coordinate, at every coefficient precision.

Proof. The lattice over an initial class depends only on the tuple of lines, and the values in different refinements can be chosen independently by Theorem 119.11. Fix the bases in the statement. They give integral coordinates for functions valued in these lattices and intertwine local scalar multiplication, since the scalars preserve each tuple of lines. The scalar group at each place has order d, a unit in R, and all its characters take values in $\mathbb{Z}_p^\times$. Its character projectors are integral and split the function space as a direct sum. Summation along a scalar orbit acts, after regarding the sum as constant on that orbit, by d on the trivial character and by zero on the other characters. The separate zero-sum condition therefore retains exactly the tuples with every scalar character nontrivial.

For a fixed such tuple, the lemma and its tensor product show that the monomials in the reduced coordinates of a covector form an integral basis of scalar functions in that character. Multiplying each by the basis section $\eta \mapsto y_{a,[\eta],s}$ and extending by zero at the other classes gives an integral basis of the full domain when all indices vary. Evaluating the original period formula in this basis gives exactly the displayed matrix. Changes of the chosen coefficient bases or of the initial residue frames change the domain basis by an invertible integral matrix. All its minor ideals are therefore intrinsic. The degree bound is the bound on each individual Teichmüller variable in the lemma. □

The last formula includes the line-dependent coefficient lattices even at embeddings of large weight. It locates all possible cancellation for a fixed list of projected tests in explicit sums of the original arithmetic path coefficients. A uniform bound for these signed minors and their comparison to the analytic incoming row on the same intervals are still required in Section 6.6.

128.9 The signed initial table at primes dividing the scalar group order

Fix one of the words, translates and terminal primes T on the common intervals of Section 6.6. Put $d = T - 1$ and let f be the number of terminal halves. Use the actual initial coefficient lattices $D_{a,L}$ of Corollary 128.26. Here a is an ordinary initial class and L a tuple of covector lines. Choose a coefficient field J containing μ_d and the coefficients of the projected tests. Fix a representative of each line. Over a, L a table is a function on G^f with values in $D_{a,L}$, where $G = \mathbb{F}_T^\times$. The lattice is constant on this orbit. A table belongs to Λ if its sum in each scalar coordinate is zero with the others fixed. Take the direct sum over all a, L. For a tuple of characters $\boldsymbol{\chi} = (\chi_1, \dots, \chi_f)$, all nontrivial, let $\Lambda_{\boldsymbol{\chi}}$ be the lattice of integral tables transforming by their product on G^f. Let

$$\Lambda_{\mathrm{ch}} = \sum_{\boldsymbol{\chi}} \Lambda_{\boldsymbol{\chi}} \ \subseteq \Lambda.$$

The summands are independent over J. We compute this inclusion integrally and apply it after contraction with the prescribed signed period.

Lemma 128.27 (Consecutive nodes of prime-power order)**.** *Let $w \mid \ell$ be a finite place of J, let $v_w(\ell) = 1$, and suppose J contains a primitive P-th root ζ, where $P = \ell^a$ with $a \geq 1$. For $1 \leq n \leq P$ take n consecutive nodes $\zeta^b, \dots, \zeta^{b+n-1}$. The elementary divisors of evaluation at these nodes on monomials of degrees $0, \dots, n-1$ have the valuations*

$$s_0 = 0, \qquad s_j = \frac{1}{\varphi(P)} \sum_{u=1}^{j} \ell^{v_\ell(u)} \quad (1 \leq j < n).$$

Proof. If $1 \leq u < P$, the order of ζ^u is $\ell^{a - v_\ell(u)}$. For a primitive ℓ^t-th root η the product of $1 - \eta^c$ over unit classes c modulo ℓ^t is $\Phi_{\ell^t}(1) = \ell$. The reduction of η is one, so the ratio $(1 - \eta^c)/(1 - \eta)$ has reduction $c \neq 0$. All factors have the same valuation. It follows that $v_w(1 - \zeta^u) = \ell^{v_\ell(u)}/\varphi(P)$.

Write $x_i = \zeta^{b+i}$. Replace monomials by the monic Newton polynomials $Q_j(X) = \prod_{i<j}(X - x_i)$, starting with $Q_0 = 1$. This is an integral triangular change of basis with determinant one. The matrix of the new evaluations is lower triangular with diagonal $Q_j(x_j)$ of valuation s_j. For $i \geq j$, the valuation of $Q_j(x_i)$ is $\varphi(P)^{-1}$ times the sum of $\ell^{v_\ell(u)}$ over $u = i - j + 1, \dots, i$, with the empty sum for $j = 0$. Every interval of j consecutive integers contains at least $\lfloor j/\ell^t \rfloor$ multiples of ℓ^t. Expanding $\ell^{v_\ell(u)} = 1 + \sum_{t=1}^{v_\ell(u)} (\ell^t - \ell^{t-1})$ shows that the minimum of the sum occurs on $1, \dots, j$. Dividing column j by $Q_j(x_j)$ therefore produces an integral matrix with diagonal one. It is invertible. The s_j are nondecreasing, which proves the assertion. □

Proposition 128.28 (Elementary divisors of the integral character tables)**.** *For $w \mid \ell$ with $a = v_\ell(d) > 0$, put $P = \ell^a$ and $b = d/P$. At one scalar coordinate the valuation list of the inclusion of the nontrivial character lines into the integral zero-sum lattice is*

$$(b-1) \text{ copies of } (s_0, \dots, s_{P-1}) \quad \text{and one copy of } (s_0, \dots, s_{P-2}).$$

For $\Lambda_{\mathrm{ch}} \subseteq \Lambda$ on one tuple of lines, the valuations are the sums of f entries independently chosen from this list, counted with multiplicities and repeated for each coefficient basis vector. At $w \nmid d$ all the elementary divisors of this inclusion are units.

Proof. Write $G = C_b \times C_P$, with C_j cyclic of order j. The b-character projectors have denominator b, a unit in $\mathcal{O}_{J,w}$. For each nontrivial C_b-character the sum over C_b vanishes, so its part of the zero-sum lattice is a full copy of $\mathcal{O}_{J,w}^P$. Its pure character vectors make the Fourier matrix $(\zeta^{ij})_{0 \leq i,j < P}$. This matrix, or its transpose, is evaluation on P consecutive nodes as in the lemma.

For the trivial C_b-character the remaining P values have sum zero, since b is a unit. Forget their last coordinate. This identifies the zero-sum lattice integrally with $\mathcal{O}_{J,w}^{P-1}$. The nontrivial C_P-characters have columns $(\zeta^{ij})_{0 \leq i < P-1}$, now with $1 \leq j < P$. Its transpose is evaluation at the $P - 1$ nodes starting at ζ on degrees $0, \dots, P - 2$. The lemma gives the shorter list.

For f scalar coordinates the zero-sum lattice on an orbit is the tensor product of the f one-coordinate augmentation lattices with $D_{a,L}$. For example, split each function space as its sum-zero kernel plus the span of the delta function at the identity. In the tensor product the simultaneous kernel consists precisely of the summands using the kernel in every coordinate. The pure character lattices tensor in the same way. Their inclusion is consequently diagonal in the tensor products of the local bases from the lemma. Away from d every character projector is integral and the nontrivial tuples give the full simultaneous kernel. Taking the direct sum over the orbits finishes the computation. □

Theorem 128.29 (The loss in each projected signed minor)**.** *Let $\mathcal{P} : \Lambda \to J^r$ evaluate the original signed period at any fixed list of algebraic projected tests, with all the weights W_{γ,F_i} of Theorem 128.25. Enlarge to a J-linear map on the span. For an integral source lattice H, let $I_q(\mathcal{P}, H)$ be the fractional ideal generated by all coordinate q-minors on H. If the map has rank $\geq q \geq 1$, then for every $w \mid \ell$,*

$$0 \leq v_w I_q(\mathcal{P}, \Lambda_{\mathrm{ch}}) - v_w I_q(\mathcal{P}, \Lambda) \leq q f v_\ell(d).$$

More precisely the two bounds can be replaced by the sums of the smallest and largest q entries of the valuation list of Proposition 128.28 for the whole source. At every complex embedding, in the induced automorphic norms, the Gram operators on the common target satisfy

$$\mathcal{P}\mathcal{P}^* = \sum_{\boldsymbol{\chi}} \mathcal{P}_{\boldsymbol{\chi}} \mathcal{P}_{\boldsymbol{\chi}}^*, \qquad \mathcal{P}_{\boldsymbol{\chi}} = \mathcal{P}|_{\Lambda_{\boldsymbol{\chi}} \otimes \mathbb{C}}.$$

Proof. Choose local bases diagonalizing the inclusion and write u_i for the diagonal multipliers. In the basis for the larger lattice let B be the actual signed period matrix. In the smaller basis its matrix is $B\,\mathrm{diag}(u_i)$. Consequently

$$I_q(\mathcal{P},\Lambda_{\mathrm{ch}})\mathcal{O}_{J,w}=\sum_{|L'|=q}\Big(\prod_{i\in L'}u_i\Big)I_q(B_{:,L'})\mathcal{O}_{J,w}.$$

Changes of either source basis preserve the minor ideals by the two directions of the Cauchy–Binet formula. The minimum among the valuations of the nonzero q-minors for the larger lattice is attained. The displayed equality gives the lower bound for all minors, and using a minor attaining that minimum gives the upper bound.

In the full P-node list one has $s_{P-1}=a$. Indeed $\prod_{j=1}^{P-1}(1-\zeta^j)=P$, by evaluating $(X^P-1)/(X-1)$ at one. The shorter list is bounded by the same endpoint. The tensor product therefore has entries between zero and fa, which proves the stated inequality. This argument uses only a change of initial tests on the very same map. In particular the arithmetic path sums have already been taken when B is formed.

At a complex embedding the scalar action is unitary. On each orbit it permutes values with equal class mass and the same coefficient norm, as in Proposition 128.20. Distinct character spaces are orthogonal, by averaging their inner product over the finite group. Their span in the zero-sum subspace is everything, by character orthogonality at each coordinate. Writing its identity as the sum of the orthogonal projections and composing with $\mathcal{P}$ and $\mathcal{P}^*$ gives the formula. It holds for the chosen tests after their full signed contraction. □

Corollary 128.30 (The total cost on the common period intervals)**.** *For these same tests of rank at least q, the globally normalized logarithmic increase in the finite content of their q-minors, divided by q, is between zero and*

$$f\log(T-1).$$

In the period family of Theorems 9.13 and 93.17 this is $O_{\psi,\varepsilon}(x^{5/2+143\varepsilon/16})$, uniformly for all words and translates on the interval. For the inclusion of the entire domain before applying the period, its exact global logarithmic index per retained dimension, normalized by $[J:\mathbb{Q}]$, is

$$f\,\frac{d-2}{2(d-1)}\log d.$$

Proof. Weight a local valuation over ℓ by $[J_w:\mathbb{Q}_\ell]\log\ell/[J:\mathbb{Q}]$ and sum. At each ℓ the sum of the local degrees equals $[J:\mathbb{Q}]$. The upper bound from the theorem gives $f\log d$ per minor dimension. The bounds $f\ll_\psi x^{2+8\varepsilon}$, $\log T\ll mx^\varepsilon$ and $m\asymp x^{1/2-\varepsilon/16}$ give the specified power. In particular this cost is $o(\exp(x^\varepsilon)/x^A)$ for every fixed $A>0$.

For the exact domain index the full P-node Fourier determinant has valuation $Pa/2$. Indeed the Fourier matrix times the inverse-character matrix is PI, and replacing ζ by ζ^{-1} merely permutes the columns. The shorter list omits $s_{P-1}=a$, so the total of the one-coordinate list is $a(d-2)/2$. Its length is $d-1$. Independent tensor choices add their averages, giving local mean $fa(d-2)/(2(d-1))$. All orbit and coefficient multiplicities repeat this same mean. Summing with the local degrees proves the exact formula. □

At T, Corollary 128.26 evaluates the full table and Theorem 128.21 computes the cost of generating a fixed character lattice by vector tests. At primes dividing $T-1$, the latter tests generate each individual character lattice there, and the calculation above covers their sum inside the full table. The signed Gram formula still involves the sizes of the restricted operators $\mathcal{P}_{\boldsymbol{\chi}}$. Comparing them on the algebraic and analytic incoming rows remains the period estimate in Section 6.6.

128.10 One normalization for the incoming automorphic row

Fix x, the field E_x, a translate and the marked finite level from Section 6.6. For each word u let $\Phi_u(t)$ be its automorphic Poincaré column with class data t. This is the column before pairing with the incoming row. We allow the signed projections in the class data. For a column initially defined by truncation, its pairing means the limit after orthogonal projection as in Theorem 87.26. We compare two ways of evaluating these columns on the *same* automorphic multiplicity space. The calculation also applies to minors with columns from different words.

Write G_∞ for the real group of the row, π for its irreducible unitary discrete series and H_π for its Hilbert space. On the adelic quotient by the rational group take the measure induced by an adelic Haar measure ν and counting measure on the rational group. Let M consist of the bounded intertwiners $S : H_\pi \to L^2_{\mathrm{cusp}}(\nu)$ at the prescribed finite level and in the selected finite types. Finite right translations commute with G_∞, so these conditions can be imposed on the multiplicity space. Throughout, inner products are linear in the second argument.

The comparison uses a second Haar measure on an isomorphic adelic group, a rational-point-preserving isomorphism ι and a fixed finite right translate t_0. Pulling that Haar measure back gives $c\nu$ for some $c > 0$. Use μ for its quotient measure. For $v \in H_\pi$ define $U_v S$ by transporting Sv through ι and evaluating it at $\iota^{-1}(g)t_0$. This defines an element of $L^2(\mu)$ modulo equality almost everywhere. In this notation the level and its character on the second group are transported by $k \mapsto t_0^{-1}\iota^{-1}(k)t_0$.

Lemma 128.31 (Projection between two evaluation lines)**.** *There is a positive definite inner product h_ν on M such that*

$$\langle U_w S, U_v T\rangle_\mu = c\, h_\nu(S,T)\langle w, v\rangle_\pi.$$

For a unit vector w, let $W_w = U_w M$, with M the closed multiplicity space, and let P_w be orthogonal projection onto W_w. Then $P_w U_v T = \langle w, v\rangle_\pi U_w T$, and

$$\|(1 - P_w)U_v T\|^2_\mu = c\, h_\nu(T,T)(\|v\|^2_\pi - |\langle w, v\rangle_\pi|^2).$$

Proof. Since the group acts unitarily, S^*T is a bounded operator commuting with the irreducible representation on H_π. Its real and imaginary self-adjoint parts have invariant spectral projections, each zero or the identity. Both parts are scalars. Define $h_\nu(S,T)$ by $S^*T = h_\nu(S,T)I$. For a unit vector v, $h_\nu(S,S) = \|Sv\|^2_\nu$, and if this scalar vanishes then $S^*S = 0$ and $S = 0$. This proves the inner product assertion before transport. The norm on intertwiners is their operator norm, so the closed conditions defining M give a Hilbert space.

The Haar scalar is common to all pairs of functions. Indeed the quotient integration formula first sums a compactly supported function over the rational subgroup and then integrates the sum on the quotient. The rational subgroup and its counting measure are preserved by ι, so multiplying adelic Haar by c multiplies the quotient measure by c. Right translation preserves that measure by unimodularity of the adelic unitary group. The transported inner product is therefore $c\langle Sw, Tv\rangle_\nu$, which is the asserted formula. For unit w, U_w is a scaled isometry with closed image. The formula shows that $U_v T - \langle w, v\rangle_\pi U_w T$ pairs to zero with every element of this image. Subtraction of the squared norms gives the second identity. □

Let P be the finite subgroup giving the character level in the row model and put $n = |P|$. The multiplicity coordinates of the incoming row are endowed with the Petersson form for ν/n. Write $eF \in M$ for the intertwiner of a row F in these coordinates. The form for ν/n equals h_ν/n: one obtains it by integrating the same product of evaluated functions with the quotient measure divided by n. Thus

$$h_\nu(eF, eF) = n\|F\|^2.$$

The formulas for n and this normalization can be located in `Formalism/LocalToGlobal/IndefiniteReal/LinksDefs.lean`, line 675 and `Development/AbelianVarietyTheory/Proofs/IwasawaDecomposition/LevelRowsReadingEmbeddingGoodRowsModelWithNorm.lean`, line 79.

Take w to be the tensor product of the prescribed unit vectors at the real places. In the common G_∞-type of Theorem 87.26, the row at its line is W_w. The character and type conditions have the same meaning in both evaluations. In fact $(U_w S)(gk)$ evaluates Sw at $\iota^{-1}(g)t_0(t_0^{-1}\iota^{-1}(k)t_0)$. This intertwines every finite right translation with its transported translation. It therefore intertwines the finite character averages, and complex conjugation conjugates all their coefficients. Let C denote pointwise complex conjugation to the conjugate row. It is an antiunitary isomorphism.

Theorem 128.32 (A common phase on the prescribed periodized columns)**.** *In the multiplicity coordinates above, the normalized conjugate row is*

$$F \longmapsto \frac{\delta}{\sqrt{cn}}\, C U_w(eF), \qquad |\delta| = 1.$$

Let a second isometric evaluation of the same rows in W_w be $JF = aU_v(eF)$, with the vector v and scalar a fixed before the word is selected. In particular the choice $J = (cn)^{-1/2}U_w e$ is available in these coordinates. Let τ be the identity or complex conjugation, according to the chosen embedding convention. For a fixed class table t put

$$L_u(F,t) = \langle CJF, \Phi_u(t)\rangle_\mu,$$
$$B_u(F,t) = \tau\!\left(\left\langle \frac{\delta}{\sqrt{cn}}\, C U_w(eF), \Phi_u(t)\right\rangle_\mu\right).$$

There is a single κ, with $|\kappa| = 1$, for which

$$B_u(F,t) = \tau\big(\kappa L_u(F,t)\big)$$

for every word, row and test. If the row space is nonzero, put $b = a\langle w, v\rangle_\pi$. Then $\kappa = \overline{\delta}/(\sqrt{cn}\, b)$.

Proof. On M the evaluation U_w multiplies squared norms by c. On incoming coordinates e multiplies them by n. The unique positive factor that makes the composite an isometry on a nonzero row is $1/\sqrt{cn}$. A choice of phase gives the displayed normalization. For an arbitrary second evaluation in the theorem, $JF \in W_w$, so the projection formula gives $JF = b\,U_w(eF)$. If $F \neq 0$, its norm gives $|b|^2 cn = 1$. Moreover, $(1 - P_w)U_v(eF) = 0$ for a nonzero F. The second identity of the lemma then gives equality in Cauchy–Schwarz for w, v, so v lies in $\mathbb{C}w$.

It remains to check that passing to coordinates on the columns introduces the same test into both inner products. At fixed word, record the real coefficient section on each finite class representative, with the given transformation under its stabilizer. These data determine the section on the whole class by the level character and change of frame. Consistency for two representatives of one argument follows from the stabilizer condition. In a basis of the space of such data, write $t = \sum_i z_i t_i$. Reconstruction gives precisely the section with data t. In τ-coordinates its entries are $\tau(z_i)$, which become z_i on return to the original coordinates. Periodizing the reconstructed section gives $\sum_i z_i \Phi_u(t_i) = \Phi_u(t)$. This argument applies after the original signed projections and on their integral tables. For the projected L^2 limits it follows first on truncations and then by continuity of the row pairing. The coordinate maps used here are specified in `Formalism/LocalToGlobal/IndefiniteReal/SpOfRwRange.lean`, line 87.

Finally $CJF = \overline{b}\, C U_w(eF)$. Conjugate linearity of the first argument gives the factor b in L_u and $\overline{\delta}/\sqrt{cn}$ in the unconjugated value defining B_u. Their ratio proves the formula. Each factor was fixed on the row, so it also works on zero columns. If the row space is zero take $\kappa = 1$. □

Corollary 128.33 (Mixed minors in the two analytic evaluations)**.** *Choose any list of rows and any list of integral projected class tests, allowing different words. Denote the resulting two matrices by L, B. Their column Gram matrices satisfy $B^*B = \tau(L^*L)$. Every j-rowed minor of B is $\tau(\kappa^j)$ times τ of the corresponding minor of L. For integral tables over a coefficient ring $O \subset \mathbb{C}$, the $\tau(O)$-module in $\mathbb{C}$ generated by these minors of B equals $\tau(\kappa^j)$ times the image under τ of the O-module generated by the minors of L.*

Proof. Entrywise the theorem gives $B = \tau(\kappa L)$. Both possible maps τ commute with conjugation. Matrix multiplication cancels $\overline{\kappa}\kappa = 1$ in each term of the Gram matrix, including entries from different words. Multilinearity of the determinant gives the formula for the minors. Taking all finite linear combinations over the indicated rings proves the module identity without assigning a finite-place valuation to a complex phase. □

The two matrices here evaluate the same Poincaré columns. The separate arithmetic column in Theorem 121.7 comes from contracting the chain of CM theta sections. To apply that theorem to these analytic minors one must still identify this contraction on the specified tests with the Poincaré pairing, with its arithmetic scale. In the archive the equality on the independently specified coefficient vectors occurs as the hypothesis on the period values in `Developmen t/AbelianVarietyTheory/Proofs/DecoratedLatticeProofs/JoinSubspaceRealDataExists.lean`, line 101.

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