# The generalized Riemann hypothesis for Dirichlet $L$-functions

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## Volume XXIII

# Integral coefficient lattices and tests for periods

**Abstract**

We construct integral tests for the projected period. Branching between Weyl lattices and elementary divisors of the coefficient tables control the finite projections on the marked rows.

The guide in Volume I lists all twenty-seven volumes.

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# 119 Refined columns and integral class pairings

## 119.1 When two refined columns contribute to the trace

Fix a supported transverse label with $\mathcal{G} > 0$, a coarse rational row cell $tK^*$, and a supporting fine representative $t$, as in Proposition 98.20. Index the two flags by $i = 0, 1$. Their coarse column representatives are $y_i \in H_f$. At the places $v \mid Q$ write $u_{i,v}$ for the middle blocks of the changes of frame in that proposition and $q_v$ for the residue cardinality. Here $U_k$ denotes the upper unitriangular group over the residue field.

Let $K^*_{\mathrm{line}}$ be the coarse column level obtained from $K^*_H$ by allowing the nonzero covector at each $v \mid T$ to be multiplied by a residue unit. It preserves the parent lattice and the kernel of each terminal covector. Put

$$\mathcal{A}_t = \{\alpha \in H(F) : y_1^{-1}\alpha y_0 \in K^*_{\mathrm{line}}\}, \qquad A_{\alpha,v} = u_{1,v}^{-1}\overline{(y_1^{-1}\alpha y_0)_v}\, u_{0,v} \quad (v \mid Q).$$

The bar here means reduction in the first split factor. For $A \in \mathrm{GL}_k(\mathbb{F}_q)$ let $P_k(A; q)$ be the fraction of the cosets $zU_k$ for which $z^{-1}Az \in U_k$.

**Proposition 119.1** (An exact count of scalar-orbit matches)**.** *The fraction of common fine refinements whose column classes belong to the same terminal scalar orbit is*

$$\sum_{\alpha \in \mathcal{A}_t} \prod_{v \mid Q} P_k(A_{\alpha,v}; q_v).$$

*Every matching refinement has a unique rational comparison $\alpha$ in the chosen coordinates. One may restrict the sum by any condition on the conjugacy classes of the $A_{\alpha,v}$. The resulting fraction is also the fraction of the finite covolume over this cell satisfying the condition. Uniformly in the cell,* $\log \max(1, |\mathcal{A}_t|) \ll fk \log^2(2fk)$.

*Proof.* The terminal scalar action multiplies the covectors independently at the places above $T$. Its inverse image in the parent compact is exactly the stabilizer of their lines. Indeed the stabilizer of a nonzero covector has quotient of order $T - 1$ in the stabilizer of its line, with every unit realized by a scalar matrix. Hence two fine column classes have the same scalar orbit precisely when a rational isometry carries the first to the second at the fine level above $Q$, at the line level above $T$, and at the given level elsewhere.

By Proposition 98.20 the common parameters are $z = (z_v U_k)_{v \mid Q}$, and the two column representatives are obtained from $y_i$ by right multiplication by integral lifts of the $u_{i,v} z_v$ at $Q$. Thus the condition at $Q$ for a given $\alpha \in \mathcal{A}_t$ is exactly $z_v^{-1} A_{\alpha,v} z_v \in U_k$ at every marked place. The fraction for this $\alpha$ is the product in the assertion.

If $\mathcal{A}_t$ is nonempty, composition with one of its members identifies it with the rational stabilizer of a coarse projective column. This group embeds in the automorphism group of the positive parent lattice and hence is finite. The underlying integral trace lattice has rank $2fk$. The finite linear group bound used in Theorem 98.41 therefore gives the asserted order bound also at this projective level. Two comparisons for the same $z$ would differ by a rational automorphism of its fine projective column. At each marked prime $g$ this automorphism belongs, up to conjugacy, to the inverse image of $U_k(\mathbb{F}_g)$ in $\mathrm{GL}_k(\mathbb{Z}_g)$. Every finite-order element of this inverse image has order a power of $g$. Indeed all its reductions modulo $g^n$ lie in finite $g$-groups and their kernels intersect in the identity. The embeddings into the local linear groups are faithful. Applying this at the two distinct primes $g_1, g_2$ forces the automorphism to be the identity. The sets of parameters counted for distinct $\alpha$ are disjoint.

For clarity, the class conditions used here are intrinsic. The two middle blocks identify a common residue quotient of the row with the two column coordinates. A change of column

frames transports the rational maps with those frames. A change of the common row frame conjugates their residue endomorphisms. In particular a rational row stabilizer permutes the parameters and transports each comparison with its conjugacy data. By Proposition 98.20 the quotient of the coarse row stabilizer by the common unipotent stabilizer acts freely on the parameters. Every fine orbit has the same size and the same finite covolume. Its size cancels in the fraction, which also proves the assertion with a class condition. □

## 119.2 Integral cross classes in the remaining height range

Fix a supported transverse rational label with $\mathcal{G} > 0$ and $H_* < b^{Mfd}$ as in Theorem 112.6. Write $M_0$ for its monomial cross matrix from Lemma 110.10. Its blocks have sizes $h_i$, with a single pivot scalar $c_i$ times a signed reversal in each block. An integral cross representative $Z$ is the cross matrix of two primitive integral frames for this label. Changing the frames acts on the left by integral lower unitriangular matrices and on the right by integral upper unitriangular matrices.

**Lemma 119.2** (Denominators and the number of integral cross classes)**.** *There exists a nonzero $a \in \mathcal{O}_E$ depending only on the rational label, with $\log |N_{E/\mathbb{Q}} a| \ll f d^3 \log b$, such that every representative has the form $Z = LM_0V$. Here $L$ is lower block unitriangular with identity diagonal blocks, $V$ is upper unitriangular, and $aL, aV$ are integral. The number of integral cross classes for this label is at most $\exp(C f d^6 \log b)$.*

*Proof.* Rational changes of the frames give a lower and an upper unitriangular factor on either side of $M_0$. Factor the lower matrix into a lower block matrix with identity diagonal blocks times its block diagonal part. Conjugating the latter past the reversals turns it into an upper unitriangular matrix, which can be absorbed into $V$. Let $P$ reverse the rows within each block. Then $PZ = (PLP^{-1})(PM_0)V$ is an ordinary triangular decomposition with diagonal middle matrix $PM_0$. For $1 \le p \le d$, the leading $p$-minor $\omega_p$ of $PZ$ is the product of its first $p$ pivots. It is determined by the label. It is a nonzero algebraic integer whenever an integral representative exists. We may assume one exists.

For $i > j$, the entry at $(i, j)$ of the lower factor equals the minor in rows $1, \dots, j-1, i$ and columns $1, \dots, j$ divided by $\omega_j$. The corresponding formula for an entry at $(j, i)$ of the upper factor uses rows $1, \dots, j$ and columns $1, \dots, j-1, i$. These formulas follow by taking determinants of the triangular factors in the indicated rows and columns. Thus $a = \prod_{p=1}^{d} \omega_p$ clears both factors. Within a block every pivot has modulus $|\tau c_i|$ at any one complex embedding. Hence $\log |\tau \omega_p|$ interpolates linearly between its values at the two endpoints, starting at zero. At the proper endpoints the original determinants are $Q^{j_i} a_i$, up to sign. At the last endpoint $D_d = Q^d a_t - T D_s$, with $D_s = 0$ or $D_s = Q^s a_{t-1}$. From $H_i \le H_*^{j_i}$, $\log T = O(d \log b)$ and $\log Q = O(\log b)$, the sum of the positive parts of the logarithms at any endpoint over the $f$ complex embeddings is $O(f d^2 \log b)$. For the last endpoint use $|v + w| \le 2 \max(1, |v|) \max(1, |w|)$. Convexity of the positive part gives the same bound at every $p$. Taking both conjugate embeddings and summing over $p$ proves the norm bound.

Record each free entry of $L, V$ in $a^{-1}\mathcal{O}_E / a^{d-1}\mathcal{O}_E$, a set of size $|N_{E/\mathbb{Q}} a|^d$. If two records agree, the two lower factors differ by left multiplication by an integral lower unitriangular matrix. Indeed the entrywise difference is in $a^{d-1}\mathcal{O}_E$, and inversion of a $d$-square unitriangular matrix with denominator $a$ costs at most $a^{d-1}$. The two upper factors likewise differ by an integral right multiplication. The cross matrices are therefore in one integral class. There are at most $d^2$ recorded entries, which proves the count. □

### 119.3 The conjugates of a prescribed signed class sum

Fix $x, E_x, \xi$ as in Section 6.6. At each $v \in Q(\xi)$ prescribe a nontrivial residue character $\chi_v$, a row $l_v$, and a primitive frequency $\nu_v$ for the signed integral of Theorem 99.12. A rational class may determine positions at several of these places at once. A common cyclotomic action also changes frequencies simultaneously at places above the same rational prime. Both constraints enter the moment at a prescribed tuple.

**Proposition 119.3** (The joint character at the prescribed tuple)**.** *Put $G = \prod_{v \in Q(\xi)} (\mathbb{F}_{p_v}^{\times} \times \mathbb{Z}/p_v^s\mathbb{Z})$. On a product of the prescribed frequency balls consider an integrable weight, or an absolutely summable discrete class weight on the supported positions. Let $H(g)$ be its total weight in the joint fiber $g = ((b_v, y_v))_v$, using the original measure or the original discrete weights. Its integral or sum against the product of local kernels is*

$$\left( \prod_v c_{\chi_v} \chi_v(\bar{\nu}_v)^{-1} \right) \sum_{t \bmod L} C_t \zeta^t, \qquad L = \operatorname*{lcm}_v \{ p_v^s, \operatorname{ord}(\chi_v) \},$$

*where $\zeta$ is a fixed primitive $L$-th root. More precisely,*

$$\lambda(g) = \prod_v \chi_v(b_v) \psi_v(p_v^{-s} \widetilde{\nu}_v \widetilde{y}_v), \qquad C_t = \sum_{\lambda(g) = \zeta^t} H(g).$$

*For the empty tuple take $L = 1$. For every tuple each fiber of $\lambda$ has $|G|/L$ elements.*

*Proof.* On each joint fiber the local scalar has the value of Theorem 99.12. Integration over a finite partition, or regrouping an absolutely convergent series, gives the formula with $H(g)$. Haar fiber volumes, class multiplicities and stabilizers are already included in $H$. The character on each additive factor has order $p_v^s$ because $\nu_v$ is a unit. The restrictions to the other factors have orders $\operatorname{ord}(\chi_v)$. The complete product group contains each factor separately, so its image under $\lambda$ is the finite cyclic group of the asserted order. All of its fibers are cosets of the kernel. This argument imposes no invariance on the weights within those cosets. □

For example, at two places over the same odd $p$, choose the additive characters to give the same primitive $p^s$-th root. Give weight one to $b_1 = b_2 = 1$, $y_1 = y_2$, and zero elsewhere. After removing the displayed row scalar, the value at $(\nu_1, \nu_2) = (1, -1)$ is $p^s$, whereas at $(1, 1)$ it is $\sum_y \psi_p(2p^{-s}y) = 0$. These are two different rows on the same joint array.

For the arithmetic column of Theorem 121.3 take its normalized scalar $Z = \mathcal{A}$, optionally multiplied by the denominator of Theorem 122.10. Under its arithmetic hypotheses the finite class expansion, including the character projector, has algebraic coefficients and CM derivatives. Theorem 122.13 gives the roots of unity in its finite translations. Take a common multiple $L$ of the orders of the chosen finite character and additive phases, or $L = 1$ for an empty list. In each term write their product as $\zeta^t$ and sum the remaining signed coefficients into $C_t$, retaining all original divisors. This gives $Z = \sum_t C_t \zeta^t$ directly on the arithmetic side of the comparison in Section 6.6.

**Theorem 119.4** (Second moment and nonvanishing for a fixed algebraic scalar)**.** *Let $Z = \sum_t C_t \zeta^t$ be either of the preceding sums with algebraic $C_t$, with any common algebraic scalar included in the coefficients if desired. Let $F = \mathbb{Q}(\zeta)$, let $B$ be a number field containing $F$ and all the $C_t$, let $u_1, \dots, u_d$ be an $F$-basis of $B$, and put $n = \varphi(L)$. Write uniquely*

$$C_t = \sum_{i=1}^{d} u_i \sum_{j=0}^{n-1} q_{ijt} \zeta^j, \qquad H_i(k) = \sum_{\substack{j,t \\ t+j \equiv k \bmod L}} q_{ijt}, \qquad q_{ijt} \in \mathbb{Q}.$$

*For a rational prime $p \mid L$ let $E_p$ average the translates of an array on $\mathbb{Z}/L\mathbb{Z}$ by the subgroup generated by $L/p$. Put $P = \prod_{p\mid L}(1 - E_p)$, with $P = 1$ for $L = 1$. Then $Z = 0$ if and only if $PH_i = 0$ for every $i$.*

*For $a \in (\mathbb{Z}/L\mathbb{Z})^\times$ put $b_i(a) = \sum_k H_i(k)\zeta^{ak}$. Define the positive definite Hermitian matrix*

$$G_{ih}(a) = d^{-1} \sum_{\sigma:\, \sigma(\zeta)=\zeta^a} \overline{\sigma(u_i)}\, \sigma(u_h), \qquad \sigma : B \hookrightarrow \mathbb{C}.$$

*The conjugate second moment of this one scalar is*

$$\frac{1}{[B:\mathbb{Q}]} \sum_\sigma |\sigma Z|^2 = \frac{1}{n} \sum_a b(a)^* G(a) b(a). \tag{119.1}$$

*Writing $\lambda_-, \lambda_+$ for the least and greatest eigenvalues over all these matrices, the moment lies between $\lambda_-(L/n)\sum_i \|PH_i\|_2^2$ and $\lambda_+(L/n)\sum_i \|PH_i\|_2^2$, with counting measure on $\mathbb{Z}/L\mathbb{Z}$. For rational $C_t$ take $B = F, u_1 = 1$. In this case the moment is exactly $(L/n)\|PC\|_2^2$.*

*Proof.* The powers of $\zeta$ up to degree $n-1$ are a rational basis of $F$, by irreducibility of the cyclotomic polynomial $\Phi_L$. Substitution gives $Z = \sum_i u_i b_i(1)$. Independence over $F$ shows that $Z$ vanishes exactly when all $b_i(1)$ vanish. As the $H_i(k)$ are rational, their values at the primitive frequencies are precisely the Galois conjugates of $b_i(1)$. Hence vanishing at one primitive frequency is equivalent to vanishing at all of them. On a character of frequency $a$ the multiplier of $E_p$ is $p^{-1}\sum_{j=0}^{p-1} \exp(2\pi i a j/p)$, up to replacing the primitive root by a conjugate. This equals one for $p \mid a$ and zero otherwise. The commuting projections therefore leave in $P$ exactly the frequencies prime to $L$. This proves the vanishing test and, by character orthogonality on $\mathbb{Z}/L\mathbb{Z}$, gives $\sum_a |b_i(a)|^2 = L\|PH_i\|_2^2$, the sum being over units.

Every embedding of $F$ has $d$ extensions to $B$. At such an extension $\sigma Z = \sum_i \sigma(u_i) b_i(a)$, since all coordinates in the $H_i$ are rational. Expanding the squared modulus proves (119.1). For a fixed $a$, the matrix with entries $\sigma(u_i)$ for its $d$ extensions is invertible. For a power basis of a primitive element of $B/F$ this is a Vandermonde determinant on distinct images, since the extension is separable. A change to the given basis preserves invertibility. This proves positive definiteness of $G(a)$. Its two eigenvalue bounds and the orthogonality identity give the inequalities. When $B = F, u_1 = 1$, every $G(a)$ equals one. For $L = 1$ the unit group and frequency set each contain a single element and the same proof applies. □

The test uses rational sums and differences on the original class coefficients after expansion in the stated basis. For rational $C_t$ in particular, if every term occurs with equal total weight in each translate by $L/p$ for one $p \mid L$, the designated sum vanishes. For general algebraic coefficients the rational arrays $H_i$ incorporate the action on the coefficients as well as on their displayed phases. Theorem 83.15 gives the finite valuations of the same $Z$.

**Corollary 119.5** (A lower bound from the orders of the occupied phases)**.** *For each rational array $H_i$ in Theorem 119.4 write $S_i$ for its support. For $t \neq u \bmod L$ write $d_{tu} = L/\gcd(L, t-u)$. Let $\delta$ be the maximum, over all indices $i$ and all $t \in S_i$, of*

$$\sum_{\substack{u \in S_i \\ u \neq t}} \frac{|\mu(d_{tu})|}{\varphi(d_{tu})},$$

*with maximum zero for empty supports. If $\delta < 1$, the moment in (119.1) is bounded below by $\lambda_-(1-\delta)\sum_{i,t}|H_i(t)|^2$. In particular it is positive whenever one array is nonzero. For rational $C_t$ and $L = p^s$, without a sparsity hypothesis, the moment is exactly*

$$\frac{p}{p-1} \sum_{v \bmod p^{s-1}} \sum_{u=0}^{p-1} |C_{v+up^{s-1}} - \overline{C}_v|^2, \qquad \overline{C}_v = p^{-1}\sum_{u=0}^{p-1} C_{v+up^{s-1}}.$$

*Proof.* For fixed $t, u$ the mean of $\zeta^{a(t-u)}$ over units $a$ mod $L$ equals $\mu(d)/\varphi(d)$, where $d = d_{tu}$, also taking $d = 1$ on the diagonal. Indeed reduction of units modulo $d$ is onto by the Chinese remainder theorem, with equal fiber sizes. The sum of the primitive $d$-th roots is $\mu(d)$, by Möbius inversion of the sum of all $d$-th roots. Expand the mean of $|b_i(a)|^2$ into pairs $t, u$. Its diagonal is $\sum_t H_i(t)^2$. Using $2|H_i(t)H_i(u)| \le H_i(t)^2 + H_i(u)^2$ on each unordered off-diagonal pair bounds the absolute value of the remaining sum by $\delta \sum_t H_i(t)^2$. The lower bound now follows from the positive matrices in (119.1). For $L = p^s$ the operator $P = 1 - E_p$ subtracts exactly the indicated progression mean and $L/\varphi(L) = p/(p-1)$. This gives the formula even if entire progressions occur in the support. □

## 119.4 The unit normal at the two marked levels

Fix a period field $E = E_x$, a word, a translate and the prime $T = \pi\bar{\pi}$ of Theorem 22.11. Use the base auxiliary tuple, whose extra set of frames is empty. Write $n = r + mt$ for the smaller terminal rank and $N = n+1$. Let $S_T$ consist of the primes $P$ of $E$ dividing $\pi$. Write $k_v = \mathcal{O}_E/v$ and $A_T = \prod_{P \in S_T} k_P^{\times}$. The two marked rational primes $g_1, g_2$ are different from $T$. In particular $\pi$ is a unit at every marked frame $w$ over either $g_i$.

In the small rank, an object is a unimodular Hermitian lattice $M$ in the prescribed definite rational space together with residue frames $\beta_w : M/wM \longrightarrow k_w^n$, taken modulo the specified groups $J_w$. In the ordinary endpoint, $J_w$ is block upper triangular with blocks $r, t, \dots, t$. Its first diagonal block is upper unitriangular. Its level character is the given additive character of the sum of the first superdiagonal entries in this block times the assigned determinant characters of the $t$-blocks. More generally take a subgroup of $\mathrm{GL}_r(k_w)$ with its specified character on the first block. Taking the prescribed symmetric subgroup in its chosen order gives the symmetric pair of endpoints. We use the following arguments for each pair with matching first block and character.

Write $e$ for the last standard unit vector. The lift $i(M)$ has lattice $M \perp \mathcal{O}_E e$, marked frames $\beta_w^+(u+ye) = (\beta_w u, \pi^{-1}y)$ after reduction, and covectors $f_P(u+ye) = \bar{\pi}y \bmod P$ on its residues. The terminal subgroup at $w$ preserves the extended block flag and fixes its last basis column. It is exactly $\mathrm{diag}(J_w, 1)$. Indeed block upper triangularity annuls the last row to the left of its final entry, and the fixed-column condition gives that last entry and all entries above it. The character restricts to the small character, since its formula uses only the original blocks. These descriptions can also be found in `Formalism/LocalToGlobal/Columns/StdLatUnimodular.lean`, line 168, `Formalism/LocalToGlobal/Columns/ColForms.lean`, line 92 and `Formalism/LocalToGlobal/Columns/LevGSym.lean`, line 81. A morphism between marked lattices is a rational unitary isometry carrying one lattice onto the other, respecting the prescribed orbits of frames and carrying each given covector to the other. Elements of $A_T$ act by multiplying the individual covectors.

**Lemma 119.6** (Recovery of a unit normal from a residue frame)**.** *Let $M' \perp \mathcal{O}_E e$ be such a lattice. A lattice vector $v$ of Hermitian length one whose last coordinate agrees with that of $e$ modulo one $w \mid g_i$ equals $e$, provided $g_i - 1 > [E : \mathbb{Q}]$. This inequality holds on the intervals of Theorem 127.1 for all sufficiently large $x$.*

*Proof.* Write $v = u + \alpha e$ with $u \in M'$ and $\alpha \in \mathcal{O}_E$. The congruence $\alpha \equiv 1 \bmod w$ gives $\alpha \neq 0$. At every complex embedding, orthogonality and positive definiteness give $|\sigma\alpha| \le 1$. The absolute norm of $\alpha$ is a positive integer, so the product of these absolute values is one and each equals one. The orthogonal length identity at one embedding then forces $u = 0$. The same bound holds for every power of $\alpha$. The characteristic polynomials of these powers in $E$ have bounded integer coefficients by the elementary symmetric function formula. Finitely

many roots occur among this finite set of polynomials, so two powers agree and $\alpha$ is a root of unity. Its order $h$ is prime to $g_i$, since otherwise $E$ would contain a primitive $g_i$-th root, of degree $g_i - 1$. If $\alpha \neq 1$, the geometric sum $1 + \alpha + \cdots + \alpha^{h-1}$ is zero. Its residue at $w$ is $h \neq 0$, a contradiction. This proves the assertion. □

**Proposition 119.7** (All scalar translates of the lifted classes)**.** *For small objects $a, b$ and $c, d \in A_T$, a marked isometry from $c\,i(a)$ to $d\,i(b)$ exists only if $c = d$. For $c = d$, extension by the identity on $e$ gives a bijection from the isometries $a \longrightarrow b$ to those $c\,i(a) \longrightarrow c\,i(b)$. In particular the large class set contains $|A_T|$ disjoint copies of the full small class set with its stabilizer orders.*

*Proof.* Apply a putative isometry $\gamma$ to $e$. Its marked residue in the target frame equals $\pi^{-1} e_N$ at every $w$, since the change of frame fixes $e_N$. Its last lattice coordinate is therefore congruent to one. Lemma 119.6 gives $\gamma e = e$. The orthogonal complement of $e$ and its intersection with the lattice must now go to the corresponding complement and intersection. On the small lattice the marked relations are exactly the changes in $J_w$, by the displayed formula for the frames. Evaluating the terminal covector relation at $e$ gives $c_P \bar{\pi} = d_P \bar{\pi} \bmod P$ for every $P$. Here $\bar{\pi}$ is a unit, proving $c = d$. Conversely every small isometry extends by the identity and respects all the indicated data. The residue changes, if recorded as part of a morphism, are determined uniquely because each frame is surjective. Applying this bijection with $a = b$ identifies the stabilizers. □

## 119.5 Simultaneous restriction of terminal class values

Let $\mathcal{H}_N$ be the space of all large coefficient tables at the matched level above, and $\mathcal{H}_n$ the small table on the output side of the restriction. Thus each value transforms by the coefficient representation and level character, and the bilinear testing space uses the dual action and inverse character. At each complex embedding use the normalized Fischer forms of Section 83.13. On each space use the sum of the squared norms of its class values with weight $1/|\Gamma_a|$, where $\Gamma_a$ is the corresponding stabilizer. These give the finite class measures of Theorem 122.17. Write $L = \bigotimes_\tau L_\tau$ for standard lowering, $s = \prod_\tau c_\tau^2 \kappa_\tau > 0$, and $L_z = ZL$ in the rational frames, where $Z$ is given in Corollary 127.2. For $c \in A_T$ define $(R_c P)(a) = L_z P(c\,i(a))$. The smaller block equivariance and the matching characters make this an element of $\mathcal{H}_n$. We write $D = |Z|^2 s$ at the chosen complex embedding.

**Theorem 119.8** (Restriction of all class values at once)**.** *For $c, d \in A_T$ one has $R_c R_d^\dagger = \delta_{cd} D\, I$ on $\mathcal{H}_n$, with $\dagger$ denoting Hermitian adjoint. Consequently the map $P \mapsto (R_c P)_{c \in A_T}$ is onto the direct sum of these small tables. Its minimum-norm right inverse is defined over the coefficient field and preserves the specified integral coefficient lattices at all primes above $g_1, g_2, T$ simultaneously.*

*Proof.* In rational polynomial frames put $B = L^*/s$, where the star denotes the rational Fischer adjoint. Its complexification equals the Hermitian adjoint divided by $s$, since the coefficients of $L$ and the two Gram forms are rational and real. By Theorem 83.28, $LB = I$. To lift a family $(Y_c)_c$, assign $Z^{-1} B Y_c(a)$ to $c\,i(a)$ and zero on all the other classes. The operator $B$ intertwines the small block action by that same theorem. Proposition 119.7 identifies every stabilizer with its small block, including its character. Thus the assignment respects stabilizers and extends by equivariance to the entire large table. The proposition gives disjoint supports for the different $c$. Equality of mass weights shows that $R_c^\dagger$ inserts $\bar{Z} L^*$ on the $c$-copy and zero elsewhere. Hence the asserted operator product follows from $LL^* = sI$. The constructed inverse is the adjoint of the joint map divided by $D$. Its image is orthogonal to the kernel, which proves minimality.

For integrality, put $z_\tau = \tau(\bar\pi)$ and $\delta_\tau = |\lambda_\tau| - |\mu_\tau|$, using the sums of the actual weights with their determinant terms. In the image of $B_\tau = L_\tau^*/(c_\tau^2 \kappa_\tau)$ the normal factor $\operatorname{diag}(I_n, v)$ acts by $v^{\delta_\tau}$. Indeed the smaller block acts by its prescribed type and the large center has weight $|\lambda_\tau|$. Applying the small center with its inverse gives this normal action. Choose an integral frame $G'$ of $M$ at a coefficient place, at each embedding. In the large dual lattice the frame is $G = \operatorname{diag}(G', z_\tau^{-1})$, by Proposition 104.34. Scaling $f_P$ by a unit preserves its kernel and hence preserves this lattice for every $c$. With the transpose action on polynomial covectors, small block equivariance gives

$$\rho_{\lambda_\tau}(G^{\mathsf t})\, z_\tau^{\delta_\tau} B_\tau = B_\tau\, \rho_{\mu_\tau}((G')^{\mathsf t}).$$

In integral frames the right inverse is therefore the tensor product of the standard $B_\tau$. Each preserves the two flag lattices at the stated primes by Theorem 83.28 and the cutoff in Theorem 127.1. This proves simultaneous integrality for the very same inverse, including primes at which $Z^{-1}$ has a denominator in the rational frames. □

**Theorem 119.9** (The last restriction on the separately zero-sum table)**.** *For $P \in S_T$ let $U_h$ denote the action of $h \in k_P^\times$ by translating covectors on large tables. In the complex table let*

$$E^\circ = \prod_{P \in S_T} \left( I - \frac{1}{|k_P^\times|} \sum_{h \in k_P^\times} U_h \right), \qquad d_T = \prod_{P \in S_T} (1 - |k_P^\times|^{-1}).$$

*Then $R_1 E^\circ R_1^\dagger = d_T D\, I$. On its separately zero-sum domain every nonzero singular value of $R_1$ is $\sqrt{d_T D}$, and its image is the full small table. For the integral lattice defined by vanishing of the unnormalized sum in each coordinate, $R_1$ has a single right inverse integral above $g_1, g_2, T$. With $f = |S_T|$ the archimedean loss satisfies $0 \le -\log d_T \le f/(T-2)$.*

*Proof.* Translation permutes the large classes with their stabilizers and preserves their coefficient frames. It is unitary. The factors of $E^\circ$ are commuting orthogonal projections, since each group average projects onto its invariants. On the union of the copies of Proposition 119.7 they act by translation in the separate coordinates of the finite group $A_T$. A vector supported on the identity copy returns there after projection with coefficient $d_T$, the product of the identity coefficients in the factors. The formula follows from Theorem 119.8. The same identity gives the singular values and surjectivity since $T > 2$. All the residue fields at $S_T$ have order $T$, and $-\log(1 - 1/(T-1)) = \log(1 + 1/(T-2)) \le 1/(T-2)$.

At each $P$ choose $h_P \neq 1$. For an integral small table $Y$, first use the section of Theorem 119.8 on the identity copy alone. Apply $\prod_P (I - U_{h_P})$ to this lift. Summing over any one coordinate kills its difference factor. On the identity copy the original value survives with coefficient one by disjointness of the copies. Every translation preserves the integral lattice and every coefficient used is an integer. This gives the claimed section even at primes dividing $T - 1$. □

Theorems 119.8 and 119.9 compute the last map on all admissible class values. In the period of Section 6.6 the input values are supplied jointly by the earlier signed theta transforms. A lower bound on that image after the specified restriction and comparison with the analytic rows is still needed there.

## 119.6 The integral terminal pairing at all primes

Retain the period fields, intervals, matched levels and base tuple of Section 119.5. Fix a coefficient number field $J$ containing all the embeddings and finite character values in use. On the large side take the integral class table with the column Weyl lattice $\Delta$ in its specified

lattice frames. On the small side let $S$ be the integral table of tests with the dual Weyl lattice $\nabla$, the dual action and the inverse finite character. These are the lattices specified in `Formalism/LocalToGlobal/Columns/ColForms.lean`, line 105 and `Formalism/LocalToGlobal/Columns/RowForms.lean`, line 280. Use normalized bilinear Fischer contraction and weight $1/|\Gamma_a|$ on each small class $a$. Thus the functional from a large table $P$ sends $F$ to

$$\sum_a |\Gamma_a|^{-1} B_m\big((R_1P)(a), F(a)\big),$$

where $B_m$ here denotes the tensor of the small forms. This is the finite normalization of Theorem 127.1. For adelic measure multiply by the original small level volume.

Write $e = e_{\mathcal{X}}$ for the prescribed central projection on the two finite marked blocks of Theorem 123.15. The formulas below also allow $e = 1$. Put $S_0 = S \cap e(S \otimes J)$ and $S' = eS$, in the same projected space over $J$. For each of the three choices $T_0 = S, S_0, S'$, let $\mathcal{B}(T_0)$ consist of the restrictions of the displayed functionals to its $J$-span as the integral large table varies. Write $T_0^\vee$ for the lattice of functionals integral on $T_0$.

**Theorem 119.10** (A common integral estimate on the projected test lattice)**.** *At a fixed $x$ let $D_x$ be the integer in Theorem 104.32. Let $d$ bound $n[E_x : \mathbb{Q}]$ for the terminal ranks under consideration, put $M = |\mathrm{GL}_d(\mathbb{F}_3)|$, and write $m_G$ for the order of the two-prime block group of Theorem 123.15. For every word and translate at this $x$ one has the lattice inclusions*

$$\begin{aligned} D_x^2 T_0^\vee &\subseteq \mathcal{B}(T_0) \subseteq M^{-1} T_0^\vee \qquad (T_0 = S,\ S_0), \\ D_x^2 (S')^\vee &\subseteq \mathcal{B}(S') \subseteq (m_G M)^{-1} (S')^\vee. \end{aligned}$$

*Both lines also hold if the large table is required to have separate scalar sums zero at every $P \mid \pi$. For every fixed $A > 0$ the single integer $D_x^2 m_G M$ has logarithm $o(b/x^A)$ on the common intervals, where $b = \exp(x^\varepsilon)$. It bounds the total finite logarithmic loss per exterior degree of this terminal step.*

*Proof.* Fix the word and put $C = \prod_\tau c_\tau$. Work first in the local ring of $J$ at an arbitrary finite prime. Denote by $Y$ the small integral output table with its Weyl lattice and positive finite character. Theorem 120.6 and the change of frame in Proposition 104.34 put $R_1P$ in $Y$. Indeed the matrix on each pair of adapted integral frames is the tensor of the standard $L_\tau$. The formula for the inverse frames in the proof of Theorem 119.8 holds over the coefficient field at every finite prime. In those frames $C^2$ times the section at the identity copy is the tensor of the integral $c_\tau^2 B_\tau$ of Theorem 120.6. Extension by zero commutes with the class stabilizers by Proposition 119.7. The section followed by the product of the integral differences in Theorem 119.9 has the same restriction on this copy. In either large domain the image therefore lies between $C^2 Y$ and $Y$.

At each small class, the two ambient coefficient lattices are exactly dual by Lemma 120.4 and the opposite changes of frame. The invariant test submodule is saturated, being an intersection of kernels of integral endomorphisms of a free module. Its quotient in the ambient test lattice is free over the local ring. Any integral functional on the invariant submodule thus extends to the ambient test lattice and is given by an integral Weyl vector $z$. Take its trace $\sum_{\gamma \in \Gamma_a} \gamma z$, using the coefficient action with the positive finite character. This is a permissible value in $Y$. Against invariant tests its pairing divided by $|\Gamma_a|$ reproduces the given functional, by invariance of the contraction. Doing this separately on all the classes realizes every element of $S^\vee$ using $Y$, even at primes dividing a stabilizer order.

The order of every $\Gamma_a$ divides $M$. Its rational isometry acts faithfully on the underlying integral lattice of rank at most $d$, since its frame changes are determined by that isometry.

Add an identity block to obtain rank $d$. The kernel of reduction of $\mathrm{GL}_d(\mathbb{Z})$ modulo three contains no nontrivial finite-order element. Otherwise it has an element of prime order $q$ of the form $I + 3^j U$, $j \ge 1$, with $U$ nonzero modulo three. For $q \ne 3$, the first binomial term in its $q$-th power is nonzero modulo $3^{j+1}$. For $q = 3$ it is nonzero modulo $3^{j+2}$ while the higher terms vanish there. This proves the asserted divisibility. Integral vectors in $Y$ therefore give functionals integral after multiplication by $M$. Together the two trace assertions give the first line for $S$, since $|C|$ divides $D_x$.

The submodule $S_0$ is saturated in $S$. Extend each integral functional from $S_0$ to $S$ locally, as above. Restriction proves the first line for $S_0$. The finite projection satisfies $m_G e S \subset S_0$ by Theorem 123.15, while $e$ is the identity on $S_0$. Thus $S_0 \subseteq S' \subseteq m_G^{-1} S_0$. Duality on their shared rational span puts $(S')^\vee$ between $m_G S_0^\vee$ and $S_0^\vee$, proving the second line. The class tables, the kernels imposing invariance or the zero sums, and the images of the period pairings commute with localization, by flatness and finiteness of the class tables. The inclusions at all finite primes give the stated inclusions over the ring of integers.

The ranks and field degrees are polynomial in $x$ on the given family. One can take $d$ to be their largest indicated product. Then $\log M \le d^2 \log 3$, and Theorem 123.15 bounds $\log m_G$ by a polynomial in $x$ for the very same marked levels. Theorem 104.32 proves the asserted size. Explicitly, at each prime over $p$, normalize $v(p) = 1$. Every elementary divisor valuation of the image relative to the indicated dual test lattice belongs to $[-v(m_G M), 2v(D_x)]$. Valuations in exterior degree $j$ are sums of $j$ such numbers. After dividing by $j > 0$, the sum over $p$ of the worst absolute bound times $\log p$ is at most $\log(D_x^2 m_G M)$, independently of the word. □

The estimate controls the cost of pairing arbitrary integral large class values with the prescribed projected tests. For the signed theta period those large values belong to the image of the preceding theta transforms. Their restriction to the lifted classes, and agreement with the incoming analytic row, are the remaining period questions in Section 6.6.

## 119.7 Which rational paths interact on the initial column

Use the actual finite path terms of Theorem 73.1 with a fixed testing form $F$. For one unrefined initial class $a$, write $\Xi_a = \prod_{P|\pi} (V_{0,P}^\vee \setminus \{0\})$. For $\xi \in \Xi_a$, let

$$u_{a,\xi} = \sum_{\gamma:\, a_0 = a,\ \eta_\gamma = \xi} W_{\gamma,F},$$

with value zero for an empty sum. Thus $u_{a,\xi}$ is a covector on the initial coefficient module in the fixed class frame. Use the finite class norm and normalized compact coefficient norms. The unrefined ordinary initial classes have trivial stabilizers as in Corollary 9.14. Indeed an automorphism has finite order and acts on an integral lattice of rank $k[E_x : \mathbb{Q}]$. Its order is prime to $g_1$ for $g_1 - 1 > k[E_x : \mathbb{Q}]$, since an element of order $g_1$ would require a primitive $g_1$-th root as a rational matrix eigenvalue. Its reduction at a marked prime over $g_1$ belongs to the upper unitriangular group there, which has $g_1$-power order. It therefore reduces to the identity. For order $h$ prime to $g_1$, the matrix $1 + \gamma + \cdots + \gamma^{h-1}$ is invertible locally since it reduces to $hI$. Multiplication by $\gamma - 1$ gives zero, so $\gamma = 1$. Consequently initial refinements in these coordinates carry unit weights and their coefficient values can be chosen independently in the unrestricted refined table.

**Theorem 119.11** (The squared norm and integral values of the signed path sum)**.** *Let $\mathcal{P}_F^\circ$ be the functional of Theorem 73.1 restricted to the ordinary initial table with separate scalar sums zero, in the arithmetic class frames. For two nonzero covectors $v, w$ over $k_P$ write $[v]$ for the*

*line spanned by $v$ and put*

$$k_P(v,w) = \mathbf{1}_{v=w} - \frac{\mathbf{1}_{[v]=[w]}}{|k_P^\times|}.$$

*In the dual Hilbert norm its square is*

$$\|\mathcal{P}_F^\circ\|^2 = \sum_a \sum_{\gamma,\gamma'\colon a_0=a_0'=a} \langle W_{\gamma,F}, W_{\gamma',F}\rangle \prod_{P\mid\pi} k_P(\eta_{\gamma,P},\eta_{\gamma',P}).$$

*In particular interference can occur only within a common tuple of projective residue lines at the same initial class.*

*There is also an exact formula over the coefficient ring of integers. On each product of scalar orbits choose $v_P, w_P$ in its $P$-th orbit, and in its common integral initial coefficient lattice choose $z$. Put $\xi_{B,P} = w_P$ for $P \in B$ and $\xi_{B,P} = v_P$ otherwise. The fractional ideal of values on the separately zero-sum integral initial table is generated by*

$$\sum_{B\subseteq S_T} (-1)^{|B|}\, u_{a,\xi_B}(z)$$

*over all such choices and all classes $a$, for algebraic $F$ and the arithmetic coefficient maps. The zero ideal is allowed.*

*Proof.* The matrix of the identity minus averaging within a scalar orbit has entry $k_P(v,w)$ on its index set of nonzero covectors. The separate averages commute and are self-adjoint. Their complementary projections have tensor product the orthogonal projection onto the indicated zero-sum space. The squared norm of a functional restricted to an orthogonal summand is the inner product of its full coefficient array with the projection of that array. Apply this to the array $u_{a,\xi}$, using the unit mass of each initial class, and expand both entries as their path sums. All sums are finite, giving the formula.

For integrality, the initial lattice depends on the kernels of the terminal covectors and is identical on each product of scalar orbits. Choose one base point in each factor orbit. Its free coordinate module splits integrally into the line generated by the base point and the augmentation kernel, with basis the differences of all other basis coordinates from the base point. Expanding the tensor product of these decompositions shows that the kernel of summation separately in every coordinate is the tensor of the augmentation kernels: summation in a given coordinate is an isomorphism on its base-point line and zero on its kernel. Tensoring the differences and the common coefficient lattice, then summing over all orbits and classes, therefore generates precisely the integral zero-sum table. Evaluating the period on these generators gives the asserted alternating sums. This calculation uses no inverse of $|k_P^\times|$ in the coefficient ring. □

**Proposition 119.12** (Separation of short composite covectors at the chosen prime)**.** *At an initial representative fix any $E_x$-basis $e_1,\dots,e_k$ contained in its lattice. For a surviving path choose rational representatives for its matrices and set $\ell_\gamma = e_N^\vee x_{m-1}\cdots x_0$. Choose a nonzero $b_\gamma \in \mathcal{O}_{E_x}$, a unit at all $P \mid \pi$, making each $b_\gamma \ell_\gamma(e_j)$ integral. Define*

$$H_\gamma = \max_{\sigma\colon E_x\hookrightarrow\mathbb{C},\ j} |\sigma(b_\gamma \ell_\gamma(e_j))|.$$

*If $\ell_\gamma, \ell_{\gamma'}$ are linearly independent and $2H_\gamma H_{\gamma'} < \sqrt{T}$, their cross term in Theorem 119.11 is zero. Thus for any collection with $H_\gamma < T^{1/4}/\sqrt{2}$, its contribution taken alone is an orthogonal sum over its rational initial covector lines.*

*Proof.* The rational representatives are integral between the local lattices at every $P \mid \pi$. Hence the ideal of simultaneous denominators of the coordinates under consideration localizes to the full ring at each such $P$. It contains an element equal to one modulo their product by the Chinese remainder theorem. This supplies $b_\gamma$.

Suppose the two tuples have the same projective residue lines. The factor $\bar{\pi}$ is a unit on these places and is common to both. Thus for any $i, j$ the algebraic integer

$$a_{ij} = b_\gamma b_{\gamma'}(\ell_\gamma(e_i)\ell_{\gamma'}(e_j) - \ell_\gamma(e_j)\ell_{\gamma'}(e_i))$$

belongs to every $P \mid \pi$. Independence gives at least one nonzero $a_{ij}$. For this pair its principal ideal is divisible by $(\pi)$, the product of those distinct primes. Writing $d = [E_x : \mathbb{Q}]$, the absolute ideal norm of $(\pi)$ is $T^{d/2}$, since $\pi\bar{\pi} = T$ and conjugation preserves ideal norm. At each complex embedding the triangle inequality bounds the absolute value of $a_{ij}$ by $2H_\gamma H_{\gamma'}$. Taking norms gives

$$T^{d/2} \le |N_{E_x/\mathbb{Q}}(a_{ij})| \le (2H_\gamma H_{\gamma'})^d,$$

a contradiction. Therefore some local projection entry vanishes. This proves both claims upon expanding the squared norm. □

**Corollary 119.13** (Separation when choosing the common terminal prime)**.** *At a given $x$ fix a finite list of rational matrix paths on the period field, before choosing $T$ from the candidate primes of Theorem 22.11. The list may include paths for several words. Retain the nonzero composite normal rows. Take one positive integer $D_0$ clearing the denominators of all the matrix maps between their lattices and of these rows on their initial lattices. In the preceding height definition use $b_\gamma = D_0$ for the entire list. Let $\mathcal{J}$ consist of the unordered pairs of independent rows in matching initial coordinates. Write $d = [E_x : \mathbb{Q}]$ and set*

$$\mathcal{L} = \log D_0 + d\sum_\gamma \log H_\gamma + 2\sum_{\{\gamma,\gamma'\}\in\mathcal{J}} \log(2H_\gamma H_{\gamma'}).$$

*Among candidate primes $T > Y > 1$, at most $\mathcal{L}/\log Y$ have to be excluded in order that all the matrices reduce, that every listed row survives at every terminal place, and that each pair in $\mathcal{J}$ gives zero cross term. If $\mathcal{L} \le Y^{1-\rho}$ for fixed $\rho > 0$, there remain $Y^{1-o(1)}$ choices on the cohomological intervals with the original word and marked-prime conditions.*

*Proof.* Exclude the primes dividing $D_0$. For each row choose a nonzero integral coordinate $D_0\ell_\gamma(e_j)$. If the row vanishes at any one terminal place away from $D_0$, the underlying rational prime divides the absolute norm of this coordinate. The sum of logarithms of these primes for this row is at most $d\log H_\gamma$.

For each pair in $\mathcal{J}$ fix a nonzero integral minor $a_{ij}$ as in Proposition 119.12. At every colliding prime its ideal is divisible by the full chosen $(\pi)$. The ideals obtained for distinct rational $T$ are pairwise coprime. Taking the product of their norms bounds the sum of their $\log T$ by $(2/d)\log|N_{E_x/\mathbb{Q}}(a_{ij})| \le 2\log(2H_\gamma H_{\gamma'})$. Adding these bounds and $\log D_0$, then dividing by $\log Y$, proves the stated count. The original common selection leaves $Y^{1-o(1)}$ primes by Theorem 22.11. For fixed $\rho > 0$ the exclusions are a vanishing fraction of this number. All the stated cohomological conditions persist on the remaining primes. □

The height and prime selection arguments control the indicated cross terms in the actual finite array. A lower bound for the complete signed norm must also control the other paths and the signed sums inside a projective class, including rationally proportional paths.

# 120 Branching and elementary divisors of coefficient lattices

## 120.1 Relative Fischer norms of products of flag minors

Use polynomials in an $N$-square variable matrix with $N = n + 1$ and $n \geq 1$. For $1 \leq i \leq n$, let $D_i, D_i'$ have first $i$ columns and rows $(1, \dots, i)$, respectively $(1, \dots, i-1, N)$. For nonnegative exponent arrays write

$$X(a,b) = \prod_i D_i^{a_i} (D_i')^{b_i}, \qquad q(X) = \sum_\alpha \alpha! X_\alpha^2.$$

All these polynomials have integral coefficients. Fix a box of exponent arrays bounded coordinatewise by a lower array and an upper array. Write $X_-$ and $X_+$ for the two extremal products, $D = \deg X_+$, and $L = \deg X_+ - \deg X_-$. The degree $D$ can be much larger than the number of boxes in the Young diagrams that change, measured here by $L$.

**Lemma 120.1** (An integral multiple of the ratio of two squared norms)**.** *There is one positive integer $A$ depending on the box with*

$$\log A \ll N^2 (1+L)^2 \log(2 + N + D).$$

*If the arrays of $X, Y$ in the box are ordered coordinatewise in that order, then $Aq(Y)/q(X)$ is a positive integer and*

$$q(Y)/q(X) \leq (N^2 \max(1, D))^{\deg Y - \deg X}.$$

*Proof.* The variables carry the defining column action of $\mathrm{GL}_N$ and, on the rows, the defining action of $\mathrm{GL}_n$ on the first $n$ rows and the identity on the last. Choose the row and column triangular subgroups for which initial minors are highest vectors. Every indicated product is a joint highest vector. Adding an earlier column to a later one leaves each used minor unchanged. The same applies to the first $n$ rows because each minor uses an initial segment of them. Its joint diagonal weight $\eta = (\eta^{\mathrm{col}}, \eta^{\mathrm{row}})$ counts the occurrences of these columns and rows. Each such highest line in its joint type is unique. In degree $d$, the Cauchy decomposition pairs one column module of partition $\lambda$ with one large row module of that partition. Restricting the latter to the first $n$ rows has multiplicity one, by the interlacing rule. Both statements are also proved by the character arguments used in Lemma 104.6 and Proposition 94.29. The polynomial spanning the line is primitive. Extract the terms in the last row in decreasing column order, as in Lemma 83.27. This gives $\prod_i D_i^{a_i} D_{i-1}^{b_i}$, whose diagonal monomial has coefficient one. This argument works for all the arrays in the box.

Write $Y = SX$, where $S$ is the product of the extra minors and $e = \deg S \leq L$. The Fischer adjoint of multiplication by $S$ is the constant-coefficient differential operator $S(\partial)$. Indeed multiplication by a variable is adjoint to its derivative on the factorial monomial basis, and the coefficients here are real. The polynomial $v = S(\partial) Y$ is integral. Its weight for the two diagonal tori is the weight of $X$. It belongs to the image of the contraction map from the module generated by $Y$ tensored with the dual of all degree-$e$ polynomials. This map is equivariant because a linear change of variables acts dually on constant derivatives.

For either factor of rank $j = N, n$, write $\eta(Y)$ for its part of the highest weight. Every irreducible type $\gamma$ in that contraction satisfies

$$\eta(Y)_i - e \leq \gamma_i \leq \eta(Y)_i.$$

In fact the dual of degree-$e$ polynomials embeds in the $e$-fold dual tensor power by symmetrization in characteristic zero. For columns each factor is a sum of dual defining modules. For

the smaller row group trivial summands also occur. Tensoring an irreducible module by a dual defining module subtracts one from one coordinate in all ways that keep the entries dominant. To verify this, multiply its Weyl character numerator $\det(t_j^{\eta_i+r-i})$ in rank $r$ by $\sum_j t_j^{-1}$. Expansion of the determinant groups the terms by the exponent lowered. Equal resulting exponents give zero, and every other determinant remains in decreasing order. Division by the same Weyl denominator proves the rule, including determinant twists. Iteration proves the two bounds.

Let $\eta_-$ be the weight of $X_-$ in the factor of rank $j$. Every coordinate of $\eta(Y)-\eta_-$ lies between zero and $L$, by counting the added variables. All shifted entries $\gamma_i+j-i$ therefore belong to

$$I_j=\bigcup_{i=1}^{j}[\eta_{-,i}+j-i-L,\eta_{-,i}+j-i+L]\cap\mathbb{Z}.$$

For the desired type $\eta(X)$ let $S_j=\{\eta(X)_i+j-i:1\le i\le j\}$. Apply the single Capelli projection of (70.2), taking its product over $b\in I_j\setminus S_j$. On a type $\gamma$ its numerator has eigenvalue $\prod_{b\notin S_j}\prod_i(\gamma_i+j-i-b)$ where $b$ runs through $I_j$. Dividing by the value on $\eta(X)$ makes the operator the orthogonal isotypic projection on the entire contraction space. Every other type has a shifted entry outside $S_j$, as the strictly decreasing shifted entries determine the type. The numerator is integral on the full polynomial ring in view of the integral matrix differential operators in the Capelli formula of Theorem 70.1. A common multiple of the absolute denominators for both projections is

$$A=\prod_{j\in\{n,N\}}\ \prod_{a,b\in I_j,\ a\neq b}|a-b|.$$

There are at most $j(2L+1)$ entries in $I_j$, and every absolute difference is at most $2L+D+N$. This gives the claimed logarithmic bound.

The projections for columns and rows commute. Applied to $v$ their product is $tX$, since the torus weight already is the joint highest weight and that line occurs once. Pair with $X$ in the Fischer form. Orthogonality gives

$$tq(X)=(S(\partial)Y,X)_{\mathrm{F}}=(Y,SX)_{\mathrm{F}}=q(Y).$$

Since $AtX$ is integral and $X$ has a coefficient one, $At$ is a positive integer. Finally, on degree $j$ the norm of multiplication by a coordinate is at most $\sqrt{j+1}$ by the monomial basis. The sum of the absolute coefficients of an order-$i$ determinant is $i!\le N^i$. Applying the triangle inequality to all factors of $S$ gives $\|SX\|_{\mathrm{F}}\le N^e\max(1,D)^{e/2}\|X\|_{\mathrm{F}}$, as asserted. □

**Theorem 120.2** (One integer for all the products in a box)**.** *For all products in the box there are positive divisors $a_X$ of a single positive integer $J$ such that*

$$\frac{q(X)}{q(X')}=\frac{a_X}{a_{X'}},\qquad \log J\ll N^2(1+L)^2\log(2+N+D).$$

*Proof.* Apply Lemma 120.1 in both directions along the ordered triple $X_-,X,X_+$. The positive integers $a_X=Aq(X)/q(X_-)$ and $Aq(X_+)/q(X)$ have product $J=A^2q(X_+)/q(X_-)$, independent of $X$. Applying the upper bound of the lemma to the two extremal products gives the estimate for $\log J$. Dividing the two formulas for $a_X,a_{X'}$ proves the identity. □

## 120.2 The three large powers at the last hyperplane

Keep the initial minors $D_i,D_i'$ of Section 120.1 in rank $N=n+1$. For the actual weights, only three of their exponents can grow faster than every fixed power of $x$. The following calculation keeps their full multiplicities. Write $q$ for the squared Fischer norm of a real polynomial.

**Lemma 120.3** (The normalizing ratio for three minor powers)**.** *Let $3 \le k \le n$ and $A, B, C$ be nonnegative integers. Define two polynomials*

$$X_0 = (D'_1)^B D_{k-1}^A (D'_k)^C, \qquad P_0 = D_1^B D_{k-1}^A D_k^C .$$

*Then*

$$\frac{q(X_0)}{q(P_0)} = \frac{(A+k-2)!(B+k-2)!}{(k-2)!(A+B+k-2)!} = \prod_{j=1}^{A} \frac{k-2+j}{B+k-2+j}.$$

*The numerator and denominator of this reduced fraction both divide an integer of logarithm at most $2A\log(2+A+B+k)$. Its value is one whenever $A = 0$.*

*Proof.* The variables used by $X_0$ are in the $k$-square matrix $Y$ on rows $1, \dots, k-1, N$ and the first $k$ columns. Rename the last of these rows as row $k$, and write $D$ for the leading minor of order $k-1$ in $Y$. Then $X_0 = (\det Y)^C Q$ with $Q = Y_{k1}^B D^A$. The variable $Y_{k1}$ is disjoint from those in $D$, so monomial factorials factor and $q(Q) = B! H_{(A^{k-1})}$, using Lemma 92.4. Here $H_v$ denotes the hook product of the partition $v$.

For column operations $Q$ is highest of weight $v = (A+B, A^{k-2}, 0)$. Its column degrees give these entries, and adding earlier columns to later ones fixes both factors. By the Cauchy decomposition the full space of highest column polynomials of this weight is one irreducible row module. Multiplication by $(\det Y)^C$ changes its row action by a determinant character, which has absolute value one on the unitary group. Consequently the squared norm after multiplication is a scalar times the old squared norm on that module. Evaluate on its leading-minor polynomial $p_v$. This gives the scalar $H_{v+C\mathbf{1}}/H_v$, because $(\det Y)^C p_v = p_{v+C\mathbf{1}}$. The polynomial $P_0$, on the first $k$ original rows, is also this leading-minor product. Thus the ratio in the statement is $B! H_{(A^{k-1})}/H_v$.

In rank $k-1$ the hook formula is

$$H_\nu = \frac{\prod_{i=1}^{k-1} (\nu_i + k - 1 - i)!}{\prod_{i<j} (\nu_i - \nu_j + j - i)}.$$

Increasing the first entry of the rectangle by $B$ changes only its first factorial and the differences involving the first row. Their ratios give the two factorials in the numerator and the two in the denominator of the asserted expression. In its product over $j$, each of the $2A$ integer factors is at most $2 + A + B + k$. Their product is a common multiple of both parts of the reduced fraction, which proves the bound. □

## 120.3 Integral branching on the two polynomial lattices

Let $O$ be the valuation ring at any finite place of a number field and $K$ its fraction field. In a rank $q$ polynomial model of partition $v$, write $V_v$ for the flag space over $K$, $p_v$ for the product of leading minors, and $B_v$ for the bilinear Fischer form normalized by $B_v(p_v, p_v) = 1$. Put $\nabla_v = V_v \cap O[X]$. Write $\Delta_v$ for the $O$-span of all iterated coefficients of elementary row substitutions on $p_v$. In a substitution one replaces a row $X_j$ by $X_j + tX_i$ and takes any coefficient in $t$. This includes $i = j$. These are the dual Weyl and Weyl lattices used in the terminal test and column, respectively. Their definitions in the supplied polynomial model are in `Formalism/GradedWeylDivisibility/StatementA.lean`, line 24 and lines 63–74 there. For general integral highest weights we use an integral frame for the separate determinant line.

**Lemma 120.4** (Exact duality in the highest-vector normalization)**.** *The $B_v$-dual of $\Delta_v$ is exactly $\nabla_v$, and the $B_v$-dual of $\nabla_v$ is exactly $\Delta_v$, in every residue characteristic.*

*Proof.* For $P \in V_v$ one has $B_v(P, p_v) = P(I_q)$. Both functionals vanish on row torus weights different from $v$. For evaluation, a surviving monomial at the identity uses only diagonal entries, whose exponents are its prescribed column degrees. For the pairing, distinct monomials are orthogonal. The row weight $v$ is the one-dimensional highest line and both functionals equal one on $p_v$. If the matrix $G$ substitutes $X \mapsto GX$, its Fischer adjoint substitutes $X \mapsto G^{\mathsf{t}}X$. This follows on a product of linear forms by contracting each factor with one in the second product and summing over permutations. Thus, as a polynomial identity in the entries of $G$,

$$B_v(P, p_v(GX)) = P(G^{\mathsf{t}}).$$

In particular every integral-coefficient $P$ pairs integrally with every iterated row coefficient of $p_v$.

Conversely assume that all these pairings belong to $O$. Use $G$ equal to a product of elementary row substitutions, with independent parameters. Every coefficient of $P(G^{\mathsf{t}})$ then belongs to $O$ by the identity. Choose a product parametrizing a lower unitriangular matrix, a diagonal matrix with entries $1 + t_i$, and an upper unitriangular matrix. Such products of elementary matrices give all the indicated triangular factors by row elimination. Over the algebraic closure of the residue field their image contains every matrix with nonzero leading principal minors, by triangular factorization. This is a nonempty Zariski open subset of the matrix space, and its transpose is dense as well. If $P$ has a nonintegral coefficient, multiply $P$ by a positive power of a uniformizer making its coefficients integral and at least one a unit. Its reduction is a nonzero polynomial. In the displayed substitution the reduction is identically zero, by integrality of the original substituted coefficients. Density gives a contradiction. This proves the first duality. Both modules are full lattices. For $\Delta_v$ the same argument over $K$ shows that a vector orthogonal to all its generators vanishes on a dense set, so is zero. Finite generation follows from containment in the finite free monomial module over $O$. Lattice biduality over a discrete valuation ring proves the second equality. □

Use the interlacing weights $\lambda, \mu$ of Section 83.13 in ranks $N = n + 1, n$. Their last entries agree. Let $l, m$ denote their partitions after subtracting this common entry. Write $C_b$ for extraction of the last-row monomial of exponents $b_i = \lambda_i - \mu_i$, with the final column set to zero. Write $A = P_m C_b$, where $P_m$ is the projection onto the small flag space in its column multidegree. Let $I : V_m \longrightarrow V_l$ be the small-block intertwiner carrying $p_m$ to the interlacing highest vector $X$ of Lemma 83.27. The branching rule gives uniqueness of its copy, and $AI = 1$ by $C_b X = p_m$. Set $c = c_\tau$ and $\kappa = B_l(X, X)$ in this single-embedding calculation.

**Proposition 120.5** (Integral maps on the horizontal strip)**.** *There is an $O$-linear map $H : \nabla_m \longrightarrow \nabla_l$ with $C_b H = 1$, taking values in last-row degree $\sum_i b_i$. The two equivariant maps $cA$ and $cI$ preserve the dual Weyl lattices in their respective directions.*

*Proof.* For a semistandard tableau of shape $m$ on $1, \dots, n$, form the product of its column minors, using the entries of each diagram column as the row list and the initial matrix columns of the same length. These products form an $O$-basis of $\nabla_m$. Indeed, in lexicographic order read the matrix columns successively, putting smaller row indices first in each. A minor with increasing row list has its diagonal product as leading monomial with coefficient one. In a tableau product this monomial records the multiset of entries in each diagram row. Since the row is sorted, it recovers the tableau. The products are independent and their number equals the dimension by the Schur character formula for semistandard tableaux. [1, Chapters 1 and 8] They therefore span over $K$. For a polynomial with integral coefficients, successively subtract its largest remaining basis monomial coefficient times its tableau product. At each stage that coefficient lies in $O$, giving the asserted $O$-basis.

Fill the boxes of $l/m$ with the new letter $N$. They are at row ends, and at most one lies in any diagram column by interlacing. The result is semistandard. Define $H$ on the basis by the new product of minors. There are exactly $b_i$ modified minors of height $i$, each containing the last row once. To obtain all the required $X_{Nn}$ factors one must choose the final-column term in every modified minor of height $n$. Proceed in decreasing height. At height $i$ the same argument forces the choice of $X_{Ni}$ in every remaining modified minor of that height. Each sign is positive because the last row occurs last in the row list. The cofactors give the original tableau product. This proves $C_b H = 1$.

On the large module keep just the indicated last-row degree, by the integral monomial projection $Q$. Among the small types $\nu$ on this slice, interlacing and equality of the small central weight give $\nu_i \in I_i$, where $I_i$ is the interval in Section 83.13. In detail the nonnegative differences $\nu_i - \lambda_{i+1}$ have sum $\sum_i(\mu_i - \lambda_{i+1})$. For integral $y$, use the central block operator $K_n(y) = (-1)^n C_n(-y-n)$, in the Capelli notation of Theorem 70.1. Its eigenvalue on type $\nu$ is $\prod_j(y - \nu_j + j)$. Its entries are integral polynomial differential operators, including the integral diagonal shift from the determinant line. Thus the product

$$\prod_{i=1}^{n} \prod_{v \in I_i,\ v \neq \mu_i} K_n(v - i)$$

preserves $\nabla_l$. On the slice it acts as $c$ on type $\mu$ and zero on every other type, using any differing coordinate of $\nu$. If $P$ denotes the large projection onto this copy on the slice, it follows that $cPQ$ is integral.

The map $A$ kills every other type by equivariance, and vanishes outside the slice by extraction. On that copy $C_b$ carries $X$ to $p_m$ and intertwines the smaller group, so it already takes its values in $V_m$. Consequently $A = C_b PQ$ and $PQ = IA$. Extraction is integral, proving that $cA$ preserves integral coefficients. For $Y \in \nabla_m$, $cIY = cPQH(Y)$ since $AH(Y) = Y$. This proves integrality of $cI$. □

**Theorem 120.6** (The column Weyl lattice at every finite prime)**.** *For the prescribed standard last restriction $L = c\kappa A$, put $B = L^*/(c^2\kappa)$. In integral frames at every finite place one has*

$$L(\Delta_l) \subseteq \Delta_m, \qquad c^2 B(\Delta_m) \subseteq \Delta_l, \qquad LB = 1.$$

*In particular each elementary divisor of the cokernel on these lattices divides $c^2$. Under tensor product over the CM embeddings this holds with $c^2$ replaced by $(\prod_\tau c_\tau)^2$.*

*Proof.* The calculation on the highest line in Theorem 83.28 gives $A^* p_m = X/\kappa$. Since the adjoint intertwines the small block, this proves $I = \kappa A^*$. Hence $L = (cI)^*$ and $c^2 B = (cA)^*$. Dualizing the two integral maps of Proposition 120.5 with Lemma 120.4 gives both inclusions. The identity $LB = 1$ is the adjoint identity of Theorem 83.28. Multiplication by $c^2$ therefore annihilates the cokernel. Tensoring these explicit maps proves the final assertion, including after extension of coefficient rings. □

## 120.4 Coefficient lattices over residue hyperplanes

Let $L$ be a free $\mathbb{Z}_T$-lattice of rank $k$, where $T > 2$ is prime. Write $W_\lambda(L)$ for the Weyl lattice $\Delta$ of Section 120.3 with its determinant power, for an integral decreasing highest weight $\lambda$. For a nonzero residue covector $a$ put $L_a = \{x \in L : a(\bar{x}) = 0\}$. This is independent of scaling $a$. For a finite list $\mathcal{U}$ of nonzero vectors in $L/TL$, let $A$ consist of the $a$ with $a(u) \neq 0$ for every $u \in \mathcal{U}$.

**Lemma 120.7** (Sum and intersection over the allowed hyperplanes)**.** *Put $s = \lambda_k$ and $h = \lambda_1 - \lambda_k$. Assume $T > h + k$ and $T - |\mathcal{U}| > h$, with $A$ restricted to nonzero covectors also for*

*an empty list. Then*

$$\sum_{a\in A} W_\lambda(L_a) = T^{\lambda_k} W_\lambda(L), \qquad \bigcap_{a\in A} W_\lambda(L_a) = T^{\lambda_1} W_\lambda(L).$$

*The identities persist over the valuation ring of any finite extension of* $\mathbb{Q}_T$.

*Proof.* Since $\det L_a = T \det L$, remove the factor $T^s$ and write $\nu_i = \lambda_i - s$. Its Young diagram has $h$ columns and at most $k-1$ rows. In a basis of $L$ adapted to $L_a$, one basis vector is multiplied by $T$. Every normal-coordinate weight of $W_\nu$ is between zero and $h$: in a product of its $h$ column minors the vector occurs at most once in each minor. The integral Weyl lattice is the direct sum of its torus weight lattices, since each iterated row coefficient of the highest vector is homogeneous for the diagonal torus. These facts give the two immediate inclusions of each lattice between $T^h W_\nu(L)$ and $W_\nu(L)$.

The translates of the highest vector by residue bases span the reduction of $W_\nu(L)$. Indeed each elementary row substitution applied to any Weyl vector has degree at most $h$ in its parameter. This follows from the minor products over the fraction field, since a rank-one change of a minor is at most linear. Its coefficients lie in the Weyl lattice by definition of the divided row operators. Modulo $T$ each coefficient is a linear combination of values at invertible elementary matrices, by interpolation at $h+1$ distinct parameters. In a diagonal substitution omit the parameter $-1$. The hypothesis on $T$ leaves enough choices. Iteration proves the spanning assertion.

In such a translate the last basis vector is unused. Let $a_0$ vanish on the span of the first $k-1$. If $a_0 \in A$ the translate is already in the image of $W_\nu(L_{a_0})$ modulo $T$, by lifting these basis vectors into $L_{a_0}$. Otherwise choose a covector $b$ nonzero on every $u \in \mathcal{U}$ with $a_0(u) = 0$. Fewer than $T$ hyperplanes are excluded, so their union has fewer than $T^k$ elements. The covectors $a_0, b$ are independent since $a_0$ was excluded. Choose $y$ with $a_0(y) = 1,\ b(y) = 0$. The transvection $g_t = 1 - tyb$ sends $\ker a_0$ onto $\ker(a_0 + tb)$. At most $|\mathcal{U}|$ parameters $t$ are forbidden. The translate of our highest vector under $g_t$ is polynomial of degree at most $h$ in $t$. Interpolation from the allowed values recovers its value at zero. Thus the sum of the allowed lattices surjects modulo $T$ onto $W_\nu(L)/T$. Nakayama's lemma gives equality.

For the intersection use the normalized Fischer duality. All hooks of $\nu$ are $< T$, so the polynomial Weyl and dual Weyl lattices agree in a standard frame by Lemma 120.4 and Theorem 122.6. Their pairing is perfect there. The transpose adjoint for a change of frame identifies the dual lattice of $W_\nu(L_a)$ with $W_\nu(L_a^\vee)$ in the dual frame. Here $L_a^\vee = L^\vee + T^{-1}\mathbb{Z}_T \tilde{a}$, with $\tilde{a}$ any lift of $a$. Thus $T^h W_\nu(L_a^\vee) \subset W_\nu(L^\vee)$. Its reduction contains every highest-vector translate whose first basis vector is $a$: lift that residue basis and use $T^{-1}\tilde{a}$ as the first vector in the enlarged lattice. Each of the $h$ minors uses this vector once.

These translates again span even with $a$ restricted to $A$. Starting with an excluded first vector $a_0$, use $b, y$ as above and the transvection on $L^\vee/T$ given by $z \mapsto z + tz(y)b$. The first vector becomes $a_0 + tb$. Interpolation at the same allowed parameters proves the assertion. Nakayama gives $\sum_{a\in A} T^h W_\nu(L_a^\vee) = W_\nu(L^\vee)$. Taking lattice duals gives the claimed intersection. Reinstating the determinant gives $s + h = \lambda_1$. Finite flat scalar extension preserves the sums and intersections, proving the last assertion. □

## 120.5 Elementary divisors of the transformed coefficient table

At a fixed ordinary initial class $a$, let $S_{\mathrm{cov}}$ and $S_{\mathrm{vec}}$ be the separately zero-sum tables on covectors and vectors, respectively, taking all values in the single unrefined lattice $D_a$ of Theorem 99.20. Use $S_a^\circ$ for the actual lattice on the covector side. These three lattices have the same

rank $d$. We compute the elementary divisors of $S_a^\circ$ relative to $S_{\mathrm{cov}}$ and bound the change in each of them when the target becomes $S_{\mathrm{vec}}$ under Fourier transformation. Elementary-divisor exponents are allowed to be negative for fractional lattices.

**Theorem 120.8** (The full list of weight exponents and the Fourier contribution)**.** *Fix a coefficient prime above $T$, with valuation $v(T) = 1$. For each $\tau$ with $v(\tau(\bar\pi)) = 1$, list the weights of a single diagonal coordinate of $W_{\lambda_\tau}$ with multiplicity. For every other $\tau$, use zero with multiplicity $\dim W_{\lambda_\tau}$. Form all sums of one entry per list and repeat the multiset once per basis element of the scalar separately zero-sum table. Order the result as $\alpha_1 \le \cdots \le \alpha_d$. These are the exponents of the identity $S_a^\circ \longrightarrow S_{\mathrm{cov}}$. If $\beta_1 \le \cdots \le \beta_d$ are the exponents of $\mathcal{F}_0 : S_a^\circ \longrightarrow S_{\mathrm{vec}}$, then*

$$\alpha_i \le \beta_i \le \alpha_i + fk \qquad (1 \le i \le d).$$

*In every exterior degree $j$, the valuations of the two ideals generated by the $j$-rowed minors differ by a number in $[0, jfk]$, with the larger valuation for the transform. At each finite prime away from $T$, both lists of exponents consist of zeros.*

*Proof.* Fix a tuple of covector lines and choose bases adapted to their kernels as in Theorem 99.20. In each affected coefficient factor a single basis vector has been multiplied by $T$. Its matrix on the direct sum of torus weight lattices is diagonal with entries $T^w$, one for each weight $w$ of this coordinate including the determinant twist. The existence of the integral weight decomposition was proved in Lemma 120.7. A change of the chosen line or of its integral adapted frame leaves the elementary divisors unchanged. Tensoring adds the exponents. On a product of scalar orbits the table of coefficients is a tensor of augmentation modules with this single coefficient lattice. The augmentation of an orbit of size $T-1$ has a basis of the $T-2$ differences against a fixed orbit member. Consequently this tuple contributes $(T-2)^f$ copies of the listed coefficient weights. The different line tuples are independent direct summands. This proves the assertion for the identity and justifies every multiplicity.

In the two unrefined lattices the Fourier matrix and its opposite are integral, including on the augmentation modules. The two compositions are $QI$ for $Q = T^{fk}$ by Theorem 73.2. For a lattice $L$ over the local valuation ring define $\mathrm{ord}_L(z)$, for $z \neq 0$, as the minimum valuation of its coordinates in an integral basis. The preceding two inclusions give

$$\mathrm{ord}_{S_{\mathrm{cov}}}(z) \le \mathrm{ord}_{S_{\mathrm{vec}}}(\mathcal{F}_0 z) \le \mathrm{ord}_{S_{\mathrm{cov}}}(z) + fk.$$

For an invertible linear map $A$ between two lattices of rank $d$, its $i$-th increasing exponent equals

$$\max_{\dim U = d-i+1} \; \min_{\substack{z \in U \\ \mathrm{ord}_{\mathrm{source}}(z)=0}} \mathrm{ord}_{\mathrm{target}}(Az),$$

where $U$ runs over subspaces of the fraction-field source. Indeed, in Smith bases the span of the last $d-i+1$ vectors attains the $i$-th exponent. Every competing subspace meets the first $i$ coordinates nontrivially. Scaling a vector of this intersection to be primitive gives an image of order at most the $i$-th exponent. Applying this formula with the same source $S_a^\circ$ to the two orders gives the claimed pointwise inequalities. The valuation of the $j$-minor ideal is the sum of the $j$ smallest exponents, by diagonalization. Away from $T$ the three coefficient frames agree and the opposite transform divided by $Q$ is integral. □

**Corollary 120.9** (The mean exponent and its global normalization)**.** *At each prime over $T$,*

$$\frac{1}{d}\sum_{i=1}^{d} \beta_i = \frac{fk}{2} + \frac{1}{k} \sum_{\tau\colon v(\tau(\bar\pi))=1} |\lambda_\tau|, \qquad |\lambda_\tau| = \sum_{j=1}^{k} \lambda_{\tau,j}.$$

*The vertical bars here denote the signed sum of the entries of a weight. In the class Hilbert metrics at every complex embedding, every singular value of $\mathcal{F}_0$ is $T^{fk/2}$. Consequently the normalized sum of the mean finite exponents times $\log T$, less the normalized archimedean log determinant per rank, is*

$$\frac{\log T}{2k}\sum_{\tau}|\lambda_\tau|.$$

*For each fixed translate and terminal prime, this quantity and each local mean in the first formula agree for all the equal-weight words of Theorem 9.13.*

*Proof.* In each representation the coordinate permutations interchange the diagonal coordinates, preserving multiplicities. Scalar matrices act through the power $|\lambda_\tau|$, so the sum of all $k$ coordinates of every torus weight is $|\lambda_\tau|$. Thus the average of the chosen coordinate is $|\lambda_\tau|/k$. Taking the mean of the list in the theorem gives the second summand in the local formula.

For the determinant between the two unrefined lattices, identify each residue space with its dual using a chosen basis and the associated symmetric dot product. This identification permutes the coordinates of the augmented tables integrally. The resulting Fourier endomorphism squares to $Q$ times reflection in zero by character orthogonality. Reflection is an integral automorphism of the augmented table. The valuation of the determinant of the Fourier endomorphism is therefore $dfk/2$. Composition with the identity of lattices accounts for both terms of the displayed formula. The archimedean assertion follows from Theorem 73.2, since Galois conjugation preserves nontriviality of every residue additive character.

In the finite mean give a prime $w \mid T$ weight $[J_w : \mathbb{Q}_T]/[J : \mathbb{Q}]$, and average over all embeddings of $J$ for the archimedean term. The constant Fourier contributions cancel. For each $\tau$, $\tau(\bar{\pi})$ is supported above $T$ and all its archimedean absolute values are $\sqrt{T}$, since $\pi\bar{\pi} = T$ in the imaginary quadratic subfield. The norm formula shows that the weighted finite mean of $v_w(\tau(\bar{\pi}))$ is $1/2$. This gives the global formula, with signed exponents also for negative weights. Finally (9.1) shows that the variation of the weight sum at each embedding depends on the word only through its Hamming weight. This proves the last assertion even when the ranks of the different word spaces differ. □

## 120.6 The distribution of one coordinate of a Weyl weight

The elementary divisors in Theorem 120.8 are controlled by torus weights counted with their actual multiplicities. For a decreasing integral weight $\lambda$ in rank $k \geq 2$, choose one weight of $W_\lambda$ at random, giving it probability equal to its multiplicity divided by the dimension. Let $w$ be its first coordinate. A permutation of the coordinates gives the same distribution for any other one.

**Lemma 120.10** (An exponential moment for a coordinate of a Weyl weight)**.** *Put $l_j = \lambda_j + k - j$. If $p = (p_1, \dots, p_k)$ is uniform on the simplex $p_i \geq 0$, $\sum_i p_i = 1$, then for $t > 0$*

$$\mathbb{E}(e^{tw})\left(\frac{e^t-1}{t}\right)^{k-1} = \mathbb{E}\exp\left(t\sum_j p_j l_j\right).$$

*Suppose $l_1 \leq A$, $l_j \leq C$ for $j \geq 2$, with $A > 0$, $C \geq 0$. Then for $t = (k-1)/(2A)$ one has $\mathbb{E}(e^{tw}) \leq 2e^{tC}$. When also $l_1 \leq C$ the bound improves to $e^{tC}$.*

*Proof.* The left character evaluated at $(e^t, 1, \dots, 1)$, divided by its dimension, is $\mathbb{E}(e^{tw})$. In the Weyl character formula let the last $k-1$ variables tend to one, writing the variables as exponentials. The rows for these variables tend, after divided differences, to the successive

derivatives of $e^{yl_j}$ at $y=0$ of orders $0,\dots,k-2$, divided by their factorials. Expanding the determinant along its first row gives the Vandermonde minors of those $k-1$ rows. Relative to the full Vandermonde in the $l_i$, the signed factor at column $j$ is $1/\prod_{i\neq j}(l_j-l_i)$ up to a sign common to the columns. The denominator alternant in the original variables leaves the factor $(e^t-1)^{k-1}$; all other factors are independent of $t$. Thus the normalized character is

$$\frac{(k-1)!}{(e^t-1)^{k-1}}\sum_{j=1}^{k}\frac{e^{tl_j}}{\prod_{i\neq j}(l_j-l_i)}.$$

To verify its constant, the partial fraction expansion of $\prod_i (z-l_i)^{-1}$ has simple-pole coefficients $1/\prod_{i\neq j}(l_j-l_i)$. Expansion at infinity shows that the sums with numerator $l_j^m$ vanish for $m<k-1$ and equal one for $m=k-1$. The displayed character therefore tends to one as $t$ tends to zero. The $l_j$ are distinct even when some highest weight entries coincide. A determinant power also allows arbitrary negative entries in $\lambda$.

For nonnegative integers $d_j$ of sum $n$, integration of the monomial $\prod p_j^{d_j}$ against uniform probability on the simplex gives $(k-1)!\prod d_j!/(n+k-1)!$. This follows by integrating one coordinate at a time, rescaling the remaining simplex after each integration, and using the beta integral for integer exponents. Thus the coefficient of $t^n$ in the simplex expectation is $(k-1)!h_n(l)/(n+k-1)!$, where $h_n(l)=\sum_{\sum d_j=n}\prod l_j^{d_j}$. In the same partial fraction expansion at infinity the coefficient of $z^{-k-n}$ is precisely $h_n(l)$. Expanding the exponentials in the preceding display proves the first identity coefficient by coefficient after multiplying by its stated factor.

Now $\sum p_j l_j \le C + Ap_1$, and $\Pr\{p_1>v\}=(1-v)^{k-1}$ for $0\le v\le 1$, by rescaling the simplex. For $s=(k-1)/2$, integrating the tail gives

$$\mathbb{E}(e^{sp_1})=1+s\int_0^1 e^{sv}(1-v)^{k-1}\,dv \le 1+s\int_0^\infty e^{-(k-1-s)v}\,dv = 2.$$

Use $(e^t-1)/t\ge 1$ in the identity. When every $l_j\le C$ the simplex average is at most $e^{tC}$ directly. □

## 120.7 Divisibility of a theta operator in the Weyl lattice

Fix $p$ and a class section $F=F_{ch}$ as in Theorem 66.24. Assume that $\pi=\sigma_p c_E$ interchanges two sets $H,L$ partitioning the embedding labels. This occurs for $H=\Phi_0,\ L=\Phi_1$ at the good primes inert in $K_d$ and split in $K_q$. Work at any coefficient place above $p$, in its valuation ring $O$, with $v(p)=1$. There are $s$ rows and $r$ columns of matrix variables per embedding. Our aim is to retain the individual row lengths of a highest weight when a derivative is paired with an integral test.

Put $f(m)=\sum_{a\ge 1}\lfloor m/p^{2a}\rfloor$. For a list $Z$ of high variables, let $\mathcal{A}(Z)$ be the ring of series whose coefficient at $Z^\alpha$ has valuation at least $-\sum_i f(\alpha_i)$. We also use polynomial parameters. The same bound is then required for every coefficient in those parameters. Superadditivity of $f$ proves closure under multiplication. Integral linear substitutions preserve this ring. Indeed for $\sum_i b_i=m$ the multinomial valuation is $v(m!)-\sum_i v(b_i!)$, and $v(m!)-f(m)$ is the sum of the floors at odd powers of $p$, hence superadditive. Applying this to each factor proves the substitution assertion.

**Lemma 120.11** (The exceptional part after a change of high frame)**.** *Let $A_h=(a_{h,li})$ be $r$-by-$r$ matrices over an integral polynomial ring over $O$. Replace the high variables by*

$$Z_{h,bi}=\sum_{l=1}^{r} a_{h,li}X_{h,bl}.$$

*Write $y_{\mu,bi}$ for the low variables, and $g_M = [z^M](\vartheta'/\vartheta)(z)$ for the one-variable marked theta. To any fixed total degree in $X$ define matrices*

$$(U_{hlj})_{bi} = X^M_{h,bl} a^M_{h,li}, \qquad M = p^j, \quad j \geq 1 \text{ odd}, \qquad S_\mu = \sum_{\pi^j h = \mu} \sum_{l=1}^{r} g_{p^j} U_{hlj}.$$

*The sum uses the odd $j$ up to that degree. Every undivided low derivative at $y = 0$ of*

$$F(Z(X), y) \exp\left( - \sum_{\mu,b,i} y_{\mu,bi} (S_\mu)_{bi} \right)$$

*belongs through that degree to $\mathcal{A}(X)$. Each $U_{hlj}$ is an outer product.*

*Proof.* Factor the section as in Theorem 66.24. The initial integral elliptic germ and all its low derivatives have bounded coefficients on the high variables. Each vacuum restricted to its chosen high half has the same bound by isotropy and Theorem 65.27. For a low derivative in direction $(\mu, b, i)$, the first logarithmic derivative of the product of vacua on this half is $E_{\mu,bi}$ of Lemma 66.23. All logarithmic derivatives of order at least two satisfy the bound. In fact the first ordinary derivative of $\vartheta'/\vartheta$ is an integral elliptic germ, and the invariant elliptic derivation preserves integrality. Passing from elliptic to additive coordinates uses the substitution estimate above.

In $E_{\mu,bi}$ the coefficient at $Z^M_{h,bi}$ is $g_M \sum_d \overline{V}_{\mu bi,d} \, \overline{V}^{\,M}_{hbi,d}$. For odd $j$ this sum is congruent modulo $pO$ to one when $\mu = \pi^j h$ and to zero otherwise. Indeed Frobenius sends the second factor modulo $p$ to $V_{\pi^j hbi,d}$, and the rows of $V$ are orthonormal. All entries before the congruence lie in an unramified ring. This identifies the same scalar $g_M$ for each matching matrix entry. Since $v(g_M) \geq -f(M) - 1$, the error from this congruence belongs to $\mathcal{A}(Z)$. All the other coefficients satisfy its bound by Lemma 66.23.

For a mixed term of $(\sum_l a_l X_l)^M$ with exponent vector $\beta$, each $\beta_l < M$. In

$$v\binom{M}{\beta_1, \dots, \beta_r} - f(M) + \sum_l f(\beta_l) = \sum_{k \text{ odd}} \left( \lfloor M/p^k \rfloor - \sum_l \lfloor \beta_l / p^k \rfloor \right)$$

the term $k = j$ equals one and all terms are nonnegative. Consequently multiplication by $g_M$ leaves every mixed term in $\mathcal{A}(X)$. Removing the pure terms from each first logarithmic derivative therefore leaves a member of this ring. Those removed terms form precisely $S_\mu$. The exponential in the statement subtracts them from the first logarithmic derivatives and has zero higher logarithmic derivatives. For any undivided derivative of a vacuum, differentiating a product successively gives the vacuum times a sum of products of its logarithmic derivatives with integer multiplicities. Apply this to the corrected vacua, and use the ordinary product rule on the initial integral germ. Closure under products proves the claim, including sections which vanish at the origin. All calculations in truncated series take place in characteristic zero. □

For $\nu_\tau = (\lambda_{\tau,i} + n_\tau)_{1 \leq i \leq r}$ in (66.22), let $\nu_\tau^+$ denote its zero padding to length $s$. Write $\Delta_\lambda = \bigotimes_\tau \Delta_{\lambda_\tau}$ for the Weyl lattice in the chosen integral frames. Put $[u]_+ = \max(u, 0)$ and $d_M(t, s) = \max(0, \lceil (t - s(M-1))/M \rceil)$.

**Theorem 120.12** (A valuation bound in terms of the rows of the partition)**.** *For the class section and the two halves fixed above set*

$$G_p(\nu) = \sum_{h \in H} \sum_{l=1}^{r} \sum_{\substack{j \geq 1 \\ j \text{ odd}}} \left[ \, d_{p^j}(\nu_{h,l}, s) - \nu_{\pi^j h, 1} \, \right]_+ .$$

*The class operator of* (66.22) *satisfies*

$$\mathcal{T}_{F,n}(\Delta_\lambda) \;\subseteq\; p^{G_p(\nu)}\Delta_{\nu^+}.$$

*Proof.* Write $p_\lambda$ for the product of leading minors. Generators of the source in one factor are coefficients of $p_\lambda(AX)$ for products $A$ of elementary off-diagonal row additions with independent parameters. To check this from the definition of $\Delta$, start with the row-homogeneous highest vector. An off-diagonal coefficient of order $d$ moves $d$ units of row degree from one row to another. A diagonal substitution on a row of degree $t$ scales by $(1+z)^t$, so its coefficients give integral multiples of the previous vector. This proves the generator assertion by induction. Since $\det A = 1$, multiplying by $\det(X)^n$ gives $p_\nu(AX)$.

Pair at each output embedding with an arbitrary $P_\tau \in \nabla_{\nu_\tau^+}$. Such tensor tests span the dual lattice by Lemma 120.4. For the normalized bilinear Fischer form the resulting differential symbol is

$$K_\tau(D) = P_\tau([DA_\tau^{\mathsf{t}}, 0]).$$

Indeed, place the $r$ rows of $A_\tau D^{\mathsf{t}}$ in the top of an $s$-square matrix $B$, filling the rest with zeros. Then the polynomial in the output frame is $p_{\nu_\tau^+}(BY)$, and its pairing with $P_\tau$ is $P_\tau(B^{\mathsf{t}})$ by the polynomial identity in the proof of Lemma 120.4. Thus $K_\tau$ has integral coefficients even when $p$ is small relative to the weight. The scalar to be bounded is $(\prod_\tau K_\tau(\partial))F(0)$.

At $h \in H$ perform the substitution of Lemma 120.11 with this $A_h$. By the chain rule the high symbol becomes $P_h([D,0])$ in the new variables. Every monomial $\partial_X^\alpha$ in these high symbols satisfies $\sum_b \alpha_{h,bl} = \nu_{h,l}$ by column homogeneity. At $\mu \in L$, the degree of $K_\mu(D + tuv^{\mathsf{t}})$ in $t$ is at most $\nu_{\mu,1}$ for arbitrary column vectors $u, v$ of the indicated matrix sizes. Indeed $P_\mu$ is a linear combination of tableau products of flag minors, as proved in Proposition 120.5. A minor is affine in $t$, since any term using two perturbed columns has proportional columns. The right multiplication by $A_\mu^{\mathsf{t}}$ and zero padding preserve the outer-product form of the perturbation. The number of minor factors is the number $\nu_{\mu,1}$ of columns of the Young diagram.

Let $F_0$ be the corrected germ in Lemma 120.11. Conjugation by its removed exponential replaces the low operator by $\prod_{\mu\in L} K_\mu(\partial_y + S_\mu)$ on $F_0$ at $y = 0$. This identity holds monomial by monomial because $S_\mu$ is independent of $y$. Expand first with a separate indeterminate in place of every $g_{p^j}$ multiplying $U_{hlj}$. The degree in each such indeterminate is at most $\nu_{\pi^j h,1}$ by the outer-product calculation, even with all the other perturbations present. Coefficients in this expansion are integral polynomials in the matrix entries, the frame parameters and the commuting low derivative symbols. If one indeterminate occurs $k_{hlj}$ times, each monomial in its coefficient uses $k_{hlj}$ factors $X_{h,bl}^{p^j}$, distributed over $b$. Thus a term contributing to the high index $\alpha$ satisfies both bounds

$$k_{hlj} \le \nu_{\pi^j h,1}, \qquad k_{hlj} \le \sum_{b=1}^{s} \lfloor \alpha_{h,bl}/p^j \rfloor.$$

Restore the scalars $g_{p^j}$ and differentiate $F_0$ in the low variables. Its resulting high coefficients satisfy the elliptic factorial bound. Superadditivity of $f$ absorbs the cost $f(p^j)$ of each exceptional factor together with the factorial cost of the remaining coefficient into $\sum_{h,b,l} f(\alpha_{h,bl})$. The additional valuation loss is at most $\sum k_{hlj}$. Multiplication by the undivided high factorials leaves at least

$$\sum_{h,l} \sum_{j \text{ odd}} \left[ \sum_b \lfloor \alpha_{h,bl}/p^j \rfloor - \nu_{\pi^j h,1} \right]_+ .$$

For a column of total degree $\nu_{h,l}$, division with remainder bounds its floor sum below by $d_{p^j}(\nu_{h,l}, s)$. This gives $G_p(\nu)$. A truncation at the total high degree suffices throughout.

Every coefficient in the independent parameters obeys this valuation bound. Extract the source generators and use the exact integral duality with the output tests to obtain the lattice inclusion. □

The estimate applies separately to each original section in the local frames of Theorem 66.24. A sum of sections with integral scalar coefficients inherits it by linearity. The first row in a low factor occurs here because every column minor is affine along a matrix of rank one.

## 120.8 A rank-one terminal frequency on the integral Weyl lattice

At a split terminal prime $T$, the Fourier sum of Lemma 141.20 involves characters $X \mapsto \psi_T(\xi' \bar{X} u)$ at the position place $P$. Their frequency matrices have rank at most one. To use this in estimating an individual summand, first compare two choices of elliptic product basis at an individual coefficient place.

**Lemma 120.13** (The phase between two locally integral product bases)**.** *Let $p$ be a rational prime. Suppose $B, B'$ are rational orthonormal product bases both generating the fixed trace lattice $M \otimes \mathbb{Z}_p$ over $\mathcal{O}_K \otimes \mathbb{Z}_p$. In the fixed marked differential the phase $u(B', B)$ of Theorem 122.3 is a unit at every place above $p$.*

*Proof.* In the product basis $B$ apply the reverse normalized lattice sum to the standard vacuum to obtain the finite test $\phi_B$ used at the end of the proof of Theorem 67.8. Its section $G^B_{\phi_B}$ is the marked product section, which has value one at the origin. Indeed the forward and reverse intertwiners compose to the identity on the vacuum line by finite character orthogonality and equality of the two indices for self-dual lattices. The finite vector $\phi_B$ is a sum of translates of the standard vacuum with root-of-unity coefficients times the inverse square root of a lattice index. That index is prime to $p$, since the lattices coincide there, and the shifts have order prime to $p$. Expanding that same standard vacuum in the $B'$ product model has the same two properties. Composing the sums thus expresses $G^{B'}_{\phi_B}(0)$ as an integral linear combination at $p$ of marked elliptic products translated by prime-to-$p$ torsion. Each has integral value there by Theorem 66.35 in order zero. Since $G^{B'}_{\phi_B} = u(B', B) G^B_{\phi_B}$, it follows that $u(B', B)$ is integral there. Using $\phi_{B'}$ in the opposite order proves integrality of its inverse. □

For one Fourier summand of one edge write $\eta_v$ for the residue frequency functional at $P \in S_T$ over $v \mid T$; it is the map $\bar{X} \mapsto \xi' \bar{X} u$. At a coefficient place $w \mid T$ set

$$H_w(\eta) = \{\tau : P_{\tau,w} \in S_T,\ \eta_{P_{\tau,w} \cap F} \neq 0\}.$$

Here $P_{\tau,w}$ is the place pulled back by $\tau$ as in Theorem 141.21. On a conjugated edge use the conjugate coefficient place.

**Theorem 120.14** (The cost of a terminal Fourier character)**.** *Use unrefined integral coefficient frames at $T$. Remove the explicit coefficient of the Fourier summand. In a product basis generating $M \otimes \mathbb{Z}_T$, each matrix coordinate of the class operator between the Weyl lattices has valuation at least*

$$- \sum_{\tau \in H_w(\eta)} \nu_{\tau,1}.$$

*Prime-to-$T$ shifts with coefficients integral at $w$ may be included in the class section.*

*Proof.* At a place $v$ with $\eta_v \neq 0$, lift $\xi'$ and $u$ to primitive integral vectors at the position place. Their product gives the primitive functional $\widetilde{\eta}_v(X) = \widetilde{\xi}' X \widetilde{u}$ of rank one on the position lattice. Extend this element to a basis of the dual position lattice, choosing it as the first basis vector.

This is possible since one of its coefficients is a unit. In the position lattice use the exact dual basis. The sums of corresponding dual and position vectors are an orthonormal basis in the split Hermitian trace model. A transpose in identifying a dual matrix with the opposite matrix component preserves rank. At $v$ with $\eta_v = 0$ use any such pair of dual bases. Approximate these bases at $T$ by a single rational orthonormal basis to precision $T$, as in Theorem 141.21. This choice can depend on the Fourier summand. Lemma 120.13 shows that changing to it multiplies the section by a unit at the given place, for every differential order.

In the product coordinates of this basis, the $T$-primary part of the shift over each $v$ is concentrated in its first coordinate if $\eta_v \neq 0$. In fact the class is represented in the dual position lattice by $T^{-1}\widetilde{\eta}_v$ modulo that lattice, and reducing the change of basis modulo $T$ suffices to transport a class killed by $T$. By the conjugate annihilator formula of Theorem 66.35, the only coordinates at which one may lose one unit of valuation per differentiation are the first coordinates belonging to $v = P_{\tau,w} \cap F$ with $\tau \in H_w(\eta)$. The map $\tau \mapsto P_{\tau,w} \cap F$ is bijective since the CM type induces all the embeddings of $F$ and $T$ splits completely. Call this list of first coordinates $I_w$. All other product coordinates have integral undivided derivatives in every order. For a multi-index in product coordinates this bounds the loss by the total order on $I_w$, because the translated section is a product of one-variable sections. Finite prime-to-$T$ intertwiners produce sums of such products with the same bounds.

Let $D_\tau$ denote the matrix of physical differential symbols at $\tau$. In the chain rule a product differential $\partial_a$ has a coefficient matrix $M_{\tau a}$ in $D_\tau$, consisting of conjugate embedded basis entries. For $a \in I_w$ in a different $v$-block this matrix is zero modulo $T$. For the first coordinate $a(\tau)$ of the matching block, when $\tau \in H_w(\eta)$, the matrix is congruent modulo $T$ to an integral rank-one matrix $U_\tau$. Indeed conjugation selects the dual position component for this embedding, and in the exact local basis its entries form $\widetilde{\eta}_v$, up to the transpose identification. The rational basis agrees with it modulo $T$. Consequently $D_\tau$ is the sum of a matrix $D_\tau^0$ and, just when $\tau \in H_w(\eta)$, the term $U_\tau \partial_{a(\tau)}$. All entries of $D_\tau^0$ are integral linear combinations of the commuting operators

$$\partial_a \quad (a \notin I_w), \qquad T\partial_a \quad (a \in I_w).$$

For an elementary-parameter generator of the source Weyl lattice and a dual target test, the differential polynomial on this block is $K_\tau(D) = P_\tau([DA_\tau^{\mathsf{t}}, 0])$ from Lemma 141.17. Its degree along $D + yU_\tau$ in the indeterminate $y$ is at most $\nu_{\tau,1}$. Indeed $P_\tau$ is an integral linear combination of products of flag minors, with one minor for each column of the Young diagram. Right multiplication and zero padding preserve rank at most one. Each minor of a rank-one perturbation is affine in $y$ by multilinearity. This proof applies with commuting differential operators for the entries of $D$.

Introduce a separate $y_\tau$ in every affected block and expand the product of all $K_\tau$ around $D_\tau^0$. Every coefficient is an integral polynomial in the regular operators displayed above and the source parameters. Substitute $\partial_{a(\tau)}$ for $y_\tau$. In each term every occurrence of a derivative on $I_w$ coming from a regular operator has its compensating factor $T$. The remaining number of such derivatives is at most $\sum_{\tau \in H_w(\eta)} \nu_{\tau,1}$. Applying the product valuation bound and extracting the source parameters gives the claimed result for every integral dual test and hence the full Weyl lattice. Finally one can return to the same locally integral product basis for all Fourier summands by the lemma. □

**Corollary 120.15** (First-row lengths in the centered terminal cost)**.** *In Theorem 141.21 replace $d_{ij}/2$ by $\frac{1}{2}\sum_\tau \nu_{ij,\tau,1}$. In the centered inequality of Corollary 141.22 its terminal term*

*can consequently be replaced by*

$$\frac{\log T}{q}\sum_{i,j}\left(2fr_j+\sum_{\tau}\nu_{ij,\tau,1}\right).$$

*The same rescaled Gram matrices, or the same explicit correction to the original Gram matrices, occur in that inequality.*

*Proof.* For every summand enlarge $H_w(\eta)$ to all $\tau$ with $P_{\tau,w}\in S_T$. This makes the loss independent of the summand. Restore the $f$ factors $T^{-r_j}$ per edge and use the integral coefficients at the other marked places. Summing the Fourier terms, composing the edges with their fixed covectors, and summing the original paths thus preserve the lower bound. The initial and terminal scalars, class masses and test projections are exactly those of Theorem 141.21. Each $\tau$ occurs with total local degree weight $1/2$ on averaging over $w\mid T$. This gives the first assertion for both test lattices. Adding their two bounds in the product formula gives the displayed contribution. □

## 120.9 Elementary divisors of split matrix shifts

Fix a prime $p$ splitting completely in the period field $E$. For a class of $s$-by-$r$ matrices $b$ over $\mathbb{Q}_p$ modulo the integral matrices, let

$$a_1(b)\ge\cdots\ge a_r(b)\ge 0$$

be the exponents of the elementary divisors of the finite image of $\mathbb{Z}_p^r$ under $b$ in $(\mathbb{Q}_p/\mathbb{Z}_p)^s$, padded by zeros. We use $r\le s$; transposition gives the same nonzero exponents. There are integral row and column bases in which $b$ has a representative diagonal with entries $p^{-a_t(b)}$ for $a_t(b)>0$ and zero elsewhere. In fact one can clear the denominator $p^A$, put the resulting integral matrix in Smith normal form, divide by $p^A$, and discard the integral entries. This also proves that the exponents are independent of the lift.

**Lemma 120.16** (Degrees of a flag polynomial in several rank-one directions)**.** *Let $K(D)=P([DA^{\mathrm t},0])$ be the integral differential symbol for a partition $\nu$ as in Lemma 141.17. If $U_1,\dots,U_r$ are rank-at-most-one matrices and the coefficient of $y_1^{m_1}\cdots y_r^{m_r}$ in $K(D+\sum_t y_tU_t)$ is nonzero, then*

$$\sum_{t=1}^{k}m_t\le\sum_{t=1}^{k}\nu_t\qquad(1\le k\le r).$$

*Consequently $\sum_t a_tm_t\le\sum_t a_t\nu_t$ for all $a_1\ge\cdots\ge a_r\ge 0$.*

*Proof.* Hold $y_{k+1},\dots,y_r$ fixed. Over the field of rational functions, the column space of $\sum_{t\le k}y_tU_t$ has dimension at most $k$. The same holds after the fixed right substitution and padding. In a minor of height $h$, any term using more than $k$ perturbed columns vanishes by linear dependence. The degree in $y_1,\dots,y_k$ is therefore at most $\min(k,h)$. A tableau monomial of shape $\nu$ is a product of one minor per diagram column. The sum of $\min(k,h)$ over their heights is the number of boxes in the first $k$ rows. Passing to a linear combination proves the degree bounds, as polynomial identities also after integral specialization. With $a_{r+1}=0$, multiply the $k$-th inequality by $a_k-a_{k+1}$ and sum. The weighted degree is attained by the highest polynomial on a diagonal matrix with entries $y_t$, where the product of its leading minors equals $\prod_t y_t^{\nu_t}$. □

Write a split Weyl shift on a given edge as a tuple of matrix classes $b_Q$, one for each $Q \mid p$ in $E$. At the chosen position place of a split pair, $b_Q$ is a position class. At its conjugate place it is the frequency class in the dual matrix lattice, using the trace pairing. Since $E$ splits completely at $p$, these lattices are the ordinary integral matrix lattices in dual local frames. We work with the marked CM model and its unrefined physical frames.

**Theorem 120.17** (The valuation of a translated Weyl operator at a split prime)**.** *For a finite shift $b$, at every coefficient place $w \mid p$ every coordinate of its class operator on the Weyl lattices has valuation at least*

$$-\sum_{\tau}\sum_{t=1}^{r} \nu_{\tau,t}\, a_t(b_{\overline{P_{\tau,w}}}).$$

*This holds in a product basis generating the local trace lattice. A conjugated class operator uses the conjugate place. A scalar coefficient $c$ adds $v_w(c)$ to the lower bound.*

*Proof.* Fix $w$. For each $v \mid p$ in $F$, precisely one embedding $\tau$ in the CM type induces $v$ under $w$. At the opposite component $\overline{P_{\tau,w}}$ take Smith row and column bases for the shift as above. The matrix units in these bases form a basis of this component of the trace lattice, each member of rank one in the physical coordinates. The corresponding diagonal matrix units carry the components of positive order $p^{a_t}$. Use the exact trace-dual basis at $P_{\tau,w}$ and take the sums of dual partners. This is a local orthonormal basis. If $A \geq 1$ bounds all exponents of both components at every $v$, choose a rational orthonormal basis agreeing with it modulo $p^A$ at both places of $K$, by the Cayley approximation in Theorem 122.14. The inverses agree to the same precision. Multiplying the difference by a shift of order dividing $p^A$ therefore changes its product coordinates by integral elements. The choices may be made for this one shift and this one place, by Lemma 120.13.

In the resulting elliptic coordinates, label a diagonal coordinate of positive order in the opposite component by $(v, t)$, and give it cost $a_t$; give all other coordinates cost zero. The conjugate annihilator bound of Theorem 66.35 says exactly that an undivided derivative of order $n$ on a coordinate of cost $a$ has valuation at least $-na$. Indeed the annihilator over the opposite split component contains $p^a$, and $\kappa_p = 0$. The component on the other side of the same elliptic coordinate imposes no condition on this local exponent. The normalized intertwiners for the local trace lattice add shifts of order prime to $p$, with integral coefficients. Each resulting elliptic product obeys the additive bound on all its separate derivative orders.

Call $p^a \partial_{v,t}$ the scaled variable on a coordinate of cost $a$, and use the ordinary derivative on a zero-cost coordinate. Each matrix $D_\tau$ of physical differential symbols is a matrix $D_\tau^0$ of integral combinations of these variables, plus

$$\sum_{t:\, a_t > 0} U_{\tau t}\, \partial_{v,t}, \qquad v = P_{\tau,w} \cap F,$$

where $U_{\tau t}$ has rank one. To check this, in the exact local basis the coefficient matrices of the indicated derivatives are the embedded conjugate entries of the Smith matrix units, up to transposition; entries on other $v$-blocks vanish. In the rational basis used for differentiation the coefficients differ by multiples of $p^A$. Since $A \geq a$ for every coordinate, all these differences multiply scaled variables with integral coefficients. Thus the asserted decomposition is exact, with the difference absorbed into $D_\tau^0$.

Now start from a Weyl generator and pair with any integral dual test as in Lemma 141.17. Expand its $K_\tau(D_\tau)$ in independent symbols $y_t$ for the unscaled $\partial_{v,t}$ in the displayed sum. The coefficients are polynomials in the scaled variables with integral coefficients. Lemma 120.16 bounds the valuation loss on substituting $\partial_{v,t}$ for $y_t$ by $\sum_t a_t \nu_{\tau,t}$ in that block. The compensating powers in all scaled variables cover their own differentiations, even when they land on the

same elliptic factor as an unscaled variable. Multiplication over the embeddings, evaluation on the product sections, and extraction of the source-generator coefficients prove the bound for all pairings. Duality gives it on the integral target lattice. Returning to any common locally integral basis only multiplies the section by a unit at $w$. □

For a prescribed column $i$, edge $j$, and nonterminal row prime $p$, use the normalized split functions of Section 141.6. Let $L$ be the product of their position lattices over $v \mid p$ and choose a common period sublattice $L' \subset L$. For each function $\phi$ in the list, each supported coset $z+L$ and each $\eta \in (L/L')^\vee$, use the coefficient

$$c_{\phi,z,\eta} = \frac{1}{[L:L']} \sum_{x \in L/L'} \phi(z+x)\overline{\eta(x)}.$$

The bar on a finite character denotes its inverse. In the symmetric Weyl convention there can also be a root-of-unity phase, which has valuation zero. Write $b(z,\eta)$ for the position and frequency pair. Its two local matrix classes are evaluated in the indicated split frames. In each fixed $w$ maximize over the nonzero coefficients and include 0 among the candidates; set

$$J^{\mathrm{mat}}_{ij,p} = \sum_{w \mid p} \omega_w \max\left( \{0\} \cup \left\{ \sum_{\tau,t} \nu_{ij,\tau,t}\, a_t\Big(b(z,\eta)_{\overline{P_{\tau,w}}}\Big) - v_w(c_{\phi,z,\eta}) \right\} \right).$$

Conjugate the local calculation on a conjugated edge.

**Corollary 120.18** (The original row-prime costs from finite Fourier coefficients)**.** *In Corollary 141.22, with the terminal improvement of Corollary 120.15, one can replace every $J^{\mathrm{row}}_{ij,p}$ by $J^{\mathrm{mat}}_{ij,p}$. These quantities satisfy $J^{\mathrm{mat}}_{ij,p} \le J^{\mathrm{row}}_{ij,p}$.*

*Proof.* Finite Fourier inversion expresses $\phi$ exactly as the specified linear combination of shifted unit vacua. Taking a smaller period lattice adds characters with zero coefficient by orthogonality. Changing a lift of $z$ changes coefficients only by phases and changes its matrix by an integral matrix. Use Theorem 120.17 on each nonzero summand. Maximizing its loss and then averaging over $w$ is valid in the proof of Theorem 141.18; all path operations there stay linear and keep the specified coupling coset. This proves the replacement. For the estimate against the previous bound take $L' = \prod_v p^{b_v} L_v$. Every $c_{\phi,z,\eta}$ has valuation at least $-c_{ij,p} - \sum_v b_v D_{jv}$ by the defining finite sum. Over either place $\epsilon$ of $K$, all the relevant matrix exponents at the opposite place are at most $h_{-\epsilon}$. Thus their weighted partition sum is at most $d_{ij} h_{-\epsilon}$. Each $\epsilon$ has total local degree weight $1/2$. This gives the claimed inequality. □

# 121 Arithmetic functionals on the prescribed rows

## 121.1 The effect of theta lifting on the Gram estimates

The diagonal and mixed estimates preceding Section 6.6 use different dimension factors. We compare them with the same factors on the period and theta sides, on the entire interval.

**Corollary 121.1** (Transport of the two normalized Gram estimates)**.** *Use the original data and the maps $S, \widehat{\Theta}$ of Theorem 83.9. For each block choose a finite Hilbert space $V_i$ of dimension $1 \le R_i \le \dim D_{\mathrm{all}}$ and a map $X_i : V_i \to D_{\mathrm{all}}$. These maps can include the column restrictions and chosen scalar normalizations. Put $B_i = SX_i$, $C_i = \widehat{\Theta} B_i$ and $G_{ij} = B_i^* B_j$, $H_{ij} = C_i^* C_j$. Use $D_*$ from Lemma 94.22. Suppose for one of the two families $K = G, H$ the diagonal*

*quantities* $\|K_{ii}-1\|_{\mathrm{HS}}/\sqrt{R_i}$ *are at most one. Then on every sufficiently distant common interval,*

$$\frac{\|H_{ii}-G_{ii}\|_{\mathrm{HS}}}{\sqrt{R_i}}\le\eta_x,\qquad\sqrt{\frac{D_*}{R_iR_j}}\,\|H_{ij}-G_{ij}\|_{\mathrm{HS}}\le\eta_x,\qquad\eta_x=\exp(-(\log 2)x^{377}/3).$$

*Hence the normalized diagonal errors, and the normalized mixed Hilbert–Schmidt norms, differ by at most* $\eta_x$ *between the two families. The estimate applies to all pairs with the same choice of theta tests.*

*Proof.* On the span of the images write $L$ for the compressed normalized theta Gram operator from Theorem 83.9. Let $g_i=R_i^{-1}\operatorname{tr}G_{ii}$. Cauchy–Schwarz for the matrix trace shows $|\operatorname{tr}(K_{ii}-1)|\le\sqrt{R_i}\|K_{ii}-1\|_{\mathrm{HS}}$. The hypothesis gives $\operatorname{tr}K_{ii}\le 2R_i$. The quadratic comparison in that theorem consequently gives $g_i\le 2e^{\Delta_x}$ in either case.

Each entry of $H_{ij}-G_{ij}$ in orthonormal bases is a pairing of one column of $B_i$ with $L-1$ applied to a column of $B_j$. Summing the squared Cauchy–Schwarz estimates for all entries gives

$$\|H_{ij}-G_{ij}\|_{\mathrm{HS}}\le(e^{\Delta_x}-1)\sqrt{R_iR_jg_ig_j}.$$

Divide by $\sqrt{R_i}$ for the diagonal bound and multiply by $\sqrt{D_*/(R_iR_j)}$ for the mixed bound. The remaining dimension factors are at most $\exp(x^{40}/2)$ by Lemma 94.22. The estimate for $\Delta_x$ in the proof of Theorem 83.9 absorbs these factors and $2e^{\Delta_x}$ in the stated $\eta_x$. The reverse triangle inequality gives the final assertion. □

In particular an estimate for the signed orbital Gram matrices at $K^0,\varpi$ would control these complex theta Gram matrices at the precision used in Section 6.6. The signed orbital estimate itself, and the lower bound for the pairing with the designated integral row in Corollary 83.7, are still required there.

## 121.2 The arithmetic period of the definite column tower

Use the fields and the two towers of Proposition 92.1 throughout the interval of Section 6.6. All the column groups $H_i=U(X_i)$ are definite. The last row pair $U(Y_{m-1}),U(Y_m)$ is definite as well. The column ranks are even, and the last two row ranks differ by the odd integer $t_{\mathrm{r}}$. Thus the rank product in each of these pairs is even, as required in Theorem 67.8. For the signed column operator use (70.6) with its fixed splittings and finite-order intermediate determinant characters.

We use a single marked elliptic differential on each $E_x$. Theorem 67.9 applies in the oriented matrix coordinates. In the tensor convention the action on a coefficient matrix is $Z\mapsto hZg^t$, which is the oriented action evaluated at $((g^{-1})^t,h)$ in the standard definite bases. If the alternating form has the opposite sign, $Z\mapsto\bar{Z}$ on rational matrices is an isometry to the chosen form, since it negates $B$. Carrying both the group action and the finite and real tests by this isometry gives the same theta sum by the bijection on rational summation variables. Both descriptions use the original integration variables and determinant characters.

**Lemma 121.2** (The real central characters fix the surviving degrees)**.** *Suppose the initial function in the definite column integral has one character on the real center and each real Schwartz vector is a Fock polynomial homogeneous at each place. With the splittings fixed, every column giving a nonzero integral has exactly the same degree* $d_{i,\tau}$ *at edge* $i$ *and real place* $\tau$*. This applies throughout the determinant-twist family of Theorem 92.2.*

*Proof.* Fix $\tau$. On the matrix coordinates the central circle at either end of an edge acts by $z$ or $z^{-1}$. Rotation preserves the vacuum line. Hence on homogeneous degree $d_i$ that circle acts by $z^{a_{i,e}+\epsilon_{i,e}d_i}$, where $e = L, R$ denotes the end, $a_{i,e}$ is the vacuum weight, and $\epsilon_{i,e} \in \{-1, 1\}$ is determined by the action on the degree-one variables. The vacuum weight is an integer because the splitting gives a circle character. The argument also works after transporting the action as above. Write $b_0$ for the input weight at $\tau$. Translating the first compact integration variable by this central circle shows that nonvanishing requires $b_0 + a_{1,L} + \epsilon_{1,L}d_1 = 0$. At each interior variable the determinant twist is trivial on the connected circle, because it has finite order. Translation there requires

$$a_{i,R} + \epsilon_{i,R}d_i + a_{i+1,L} + \epsilon_{i+1,L}d_{i+1} = 0.$$

Indeed any nontrivial power of $z$ has circle average zero. The initial equation fixes $d_1$ uniquely, and each succeeding one fixes $d_{i+1}$ uniquely. A negative or nonintegral required degree gives zero for every column. Applying the argument separately at each real place proves the lemma. □

**Theorem 121.3** (CM derivatives for the projected definite column period)**.** *Pair the restriction of (70.6) to $H_* = U(Y_m) \subset U(X_m)$ with a fixed vector $b$ of the chosen terminal test space. An orthogonal projection onto that space may be taken before pairing. Allow any finite signed sum of the determinant twists and retain all their prescribed finite tests. Write $a$ for the initial vector. Use the polynomial coordinates of Theorem 67.8. For an arithmetic statement suppose the finite tests have algebraic coefficients in the finite lattice model, the determinant factors and split Weil actions are as in Theorem 65.13, and the finite class values of $a, b$ are algebraic matrix coefficients at infinity. Take algebraic coefficients in the real test polynomials and in the signed sum. Under these hypotheses the actual period has the form*

$$\mathcal{P} = V_J \left( \prod_{i=1}^{m} C_i \right) c^{-t/2} \mathcal{A}, \qquad t = \sum_{i,\tau} d_{i,\tau}, \qquad |C_i| = \nu^{-[E:K]\, N_{i-1} N_i / 2}.$$

*Here $N_i = \dim_E X_i$, $c = \pi/A$ and $\nu$ are those of Theorem 67.8. The factor $V_J$ is the product of the volumes of the real group times the finite integration level at the vertices $0, \dots, m-1, *$. The scalar $\mathcal{A}$ is an explicit sum of products of ordinary CM derivatives, with the original signs. If no degree profile survives then $\mathcal{P} = 0$.*

*More precisely, at common levels fixing the integrands let $A_j$ be the least common multiple of the rational stabilizer orders at vertex $j$. For each $j, \tau$ let $L_{j,\tau}$ bound the degrees in the ordinary and the conjugate entries of that compact matrix variable in the expansion below. Write $n_j$ for the dimension of the group being integrated, so $n_* = N_m - 1$. Let $I$ be the fractional ideal generated by the nonzero coefficients of this expansion, including its CM derivatives, in any common number field. Use the zero ideal if all the coefficients vanish. Then*

$$\left( \prod_{j} A_j \right) \left( \prod_{j,\tau} \mathcal{D}(n_j, L_{j,\tau}) \right) \mathcal{A} \ \in I.$$

*In particular the compact integrations introduce no denominator at a prime exceeding every $n_j + L_{j,\tau} - 1$ and prime to $\prod_j A_j$.*

*Proof.* Fix the endpoint argument $g_* \oplus 1$ in the last kernel, multiply by $\overline{b(g_*)}$, and integrate over $[H_*]$. For an orthogonal projection $p$ and $b$ in its range, $\langle pu, b\rangle = \langle u, b\rangle$. If a fixed test outside its range is used, replace $b$ by $pb$ and use its actual coefficients. Intersect the finite stabilizers of the data on all incident edges, the character kernels, and the levels of $a, b$. At the endpoint take the inverse image of the last test's stabilizer. This gives compact open subgroups inside

the prescribed ones because the number of tests is finite. Splitting the original quotients into classes at these levels evaluates the same integrals. In particular a finite character projector prescribed in the data keeps every term of its defining finite average and its divisor.

For each tuple $\mathbf{u}$ of finite representatives, choose the representatives in Theorem 65.11 away from the finitely many bad places. Denote by $\phi_{i,\mathbf{u}}$ the actual finite translate of the test on edge $i$. Write $k_j$ for the compact real matrix variables, using $k_* \oplus 1$ at the endpoint. On their product the integrand after removal of the common scalar is

$$a(k_0,u_0)\,\overline{b(k_*,u_*)}\prod_{j=1}^{m-1}\chi_j(\det u_j)\prod_{i=1}^{m}[\omega_{i,\infty}(k_{i-1},k_i)P_i](\partial)G_{\phi_{i,\mathbf{u}}}(0).$$

The symbol $k_m$ here means $k_*\oplus 1$. Form this expression for every summand with its coefficient, omitting the degree profiles that integrate to zero by the lemma. Expand it as a polynomial in all the entries $k_j,\bar{k}_j$. Define $\mathcal{A}$ by taking the probability Haar average of each such polynomial, dividing by the product of its vertex stabilizer orders, and summing over tuples and signed terms. This is the expansion defining $I$, before averaging and division by stabilizers.

To verify the period formula apply Theorem 67.8 pointwise to every kernel, then use the class integration argument of Theorem 67.9 at every vertex. Each kernel contributes $C_i c^{-\sum_\tau d_{i,\tau}/2}$. The lemma allows their product to be taken outside the signed sum. Finite sums, compact integrals, and the uniformly convergent theta kernels admit these interchanges. The same calculation for tests written as $P_i(\overline{\Phi(a')})$ in raw matrix coordinates uses $(c\bar{\delta})^{-t}$ in place of $c^{-t/2}$, since $\bar{w}=\bar{\delta}\,\overline{\Phi(a')}$. If the original theta character is used, first transport its vectors and splittings by Theorem 94.16. That identity holds pointwise and retains the given terminal character.

It remains to justify the coefficient assertion. At good split places a class translate acts on a unit lattice vector by normalized change of variables and its algebraic character factor. The change of lattice is a finite sum with coefficients that are roots of unity times inverse square roots of indices, by Lemma 65.12 and Theorem 65.13. The same argument applies to a translated vector by conjugating its translation. Any finite adelic translate of the new vacuum has a rational lift modulo its periods, and changing the lift only introduces a root of unity. At the other places the chosen representatives act trivially. Thus the resulting sections have algebraic Taylor coefficients by Theorem 65.9. At finer levels a rational change of representative acts on endpoint matrix coefficients by an algebraic matrix. The old level factors on algebraic endpoint data are retained in their class coordinates.

On the compact real groups the polynomial action is substitution of unitary matrices, times the vacuum character. This character is a product of integral powers of determinants, since every character of $U(n)$ factors through its determinant. Inverses of determinants on a unitary group are conjugate determinants. The same description holds for the stipulated algebraic matrix coefficients, since every algebraic representation of $\mathrm{GL}_n$ has regular matrix coefficients in the entries and the inverse determinant. Complex conjugation preserves algebraicity. All coefficients just described, and all ordinary derivatives occurring here, therefore belong to a single number field for the fixed finite family. Proposition 94.23 puts the compact average in $I/\prod_{j,\tau}\mathcal{D}(n_j,L_{j,\tau})$. Clearing the finitely many stabilizer denominators proves the assertion for $\mathcal{A}$. □

At the last pair of the row tower Theorem 67.9 also evaluates the integral against the incoming form, with its actual coefficients. The earlier row passes through indefinite groups. The size and nonvanishing of its pairing with the designated column in Corollary 83.7 still require a quantitative argument on the common interval.

### 121.3 The two integral positions along the row

Fix a field and interval of Theorem 74.6. At a horizontal edge put $S = V_{i,0}$, $T = V_{i+1,0}$, with $2 \le i < m$. The staircase of $S$ has $c = i - 1$ columns $b_j$, and that of $T$ has $c + 1$ columns $a_j$, numbered from one. Write $r = \lfloor c/2 \rfloor$ and $\ell = \lfloor (c+1)/2 \rfloor$. Let $S_0, T_0$ be the orthogonal complements of the $r, \ell$ hyperbolic planes, respectively. Their lattices are the intersections with the given full lattices. In either staircase opposite columns pair to one, and the middle column in odd length has norm $-1$. Put

$$d = -\xi_0, \qquad A = \mathfrak{A}/\bar{\mathfrak{m}}, \quad B = \bar{\mathfrak{A}}/\mathfrak{m}, \quad I_j^{(c)} = A^{c+1-j} B^{-j}, \quad \mathfrak{j} = \mathfrak{m}\bar{\mathfrak{m}}.$$

Thus $\Lambda_S$ contains the direct summands $I_j^{(c)} b_j$; the expression with $c + 1$ gives the analogous summands in $\Lambda_T$. These formulas describe fractional ideals and require no global choice of generators.

Use two successive polarizations of $\mathrm{Hom}(-T, S)$. First retain the positions $x_j = X(a_j)$, $1 \le j \le \ell$. In the remaining map from $T_0$ to $S$ retain the coefficient $p_k \in T_0^\vee$ of $b_{c+1-k}$, $1 \le k \le r$. Here $T_0^\vee$ is the space of $E$-linear functionals. The remaining oscillator is that on $\mathrm{Hom}(-T_0, S_0)$. In the scaled frame used for the Fourier character the first coordinates are evaluated on $d^j a_j$; their base values are $d^j b_j$. Denote the corresponding unscaled row values by $p_{*,k}$.

**Proposition 121.4** (Integral tests in the two polarizations)**.** *Put $L_j = (I_j^{(c)})^{-1} \Lambda_S$ and $M_k = \mathfrak{A} I_{c+1-k}^{(c)} \Lambda_{T_0}^\vee$, where $\Lambda_{T_0}^\vee$ is the integral linear dual. At all finite places take the product of the indicators*

$$x_j \in b_j + \mathfrak{j}\widehat{L}_j, \qquad p_k \in p_{*,k} + \mathfrak{m}\widehat{M}_k.$$

*For $c = 2r + 1$ all $p_{*,k}$ are zero. For $c = 2r$ the only nonzero row, $p_{*,r}$, vanishes on the core in $T_0$ and takes a value in $\{1, -1\}$ on the middle column $a_{r+1}$. These tests tensored with an original residual lattice test have the prescribed depth. If the residual test has one marked coset per place, the change to the original split polarizations has exactly*

$$(N\mathfrak{m})^{\ell \dim_E S + r \dim_E T_0}$$

*distinct products of marked integral coset indicators. In the measures of Proposition 75.5 their coefficients are roots of unity. At every unmarked place the resulting test is the original lattice vacuum, with the lattice normalization.*

*Proof.* The successive frame scalars are $d^j$ on both sides, including the last vector $d^\ell b_\ell$ when it has norm of nonzero absolute value. They cancel in $X(a_j)$. For $c = 2r$ the last row of the second position uses the vector $-d^{r+1} a_{r+1}$ and the adjoint multiplier $-\xi_0^{-1}$. The physical opposite column paired to $d^r b_r$ is $\bar{d}^{-r} b_{r+1}$. Thus the scalar multiplying the row $h_{T_0}(a_{r+1}, \cdot)$ is $\bar{d}^{-r}(-\xi_0^{-1})(-\bar{d}^{r+1}) = 1$. Reversing the input Hermitian form reverses the row's sign. This gives a unit at every finite prime, including primes dividing the absolute different. For $c = 2r + 1$ the grid prescribes the zero second position. `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/IsAuxiliaryVectorWittDec.lean`, line 160–194. `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/WittDecompSymplInclIsotropicInclAnisotropic.lean`, line 183–201.

The undeepened map lattice on the $j$-th input column is $\mathfrak{A}(I_j^{(c+1)})^{-1} \Lambda_S = \bar{\mathfrak{m}} L_j$, because $I_j^{(c+1)} = A I_j^{(c)}$. On the retained output row it is $M_k$. Each is paired with its opposite column or row by exact trace duality, by Proposition 122.7. Different hyperbolic planes are mutually orthogonal, and the second change uses only the residual map space after the first. For a

retained lattice $D_1$ of either kind the allowed translations are $\mathfrak{m}D_1$. The allowed support is $\bar{\mathfrak{m}}^{-1}D_1$, the annihilator of the deepened opposite lattice. We have $b_j \in L_j$. In the case of the nonzero row its values on $a_{r+1}$, as elements of $M_r$, range over the ideal

$$\mathfrak{A} I_{r+1}^{(2r)} (I_{r+1}^{(2r+1)})^{-1} = \bar{\mathfrak{m}}.$$

The middle line is a direct summand and the row vanishes on its complement, so $p_{*,r} \in \bar{\mathfrak{m}}^{-1} M_r$. All the indicated cosets therefore satisfy both depth conditions.

At a marked pair $w, \bar{w}$ of depth $a$, retain the given coset on the $w$-half of each position. On the other half the test is the translate of the full unshifted position lattice, since $\mathfrak{m}_{\bar{w}}$ is the unit ideal. Its inverse Fourier transform is the full opposite lattice at $w$ times the character of the center of that translate. This follows by integrating the character over the lattice in dual measures with that lattice of volume one. The character is trivial on the sublattice of depth $a$ by the support calculation above. Splitting the opposite lattice into these cosets gives $(Nv)^{ae}$ terms for a position of $E$-dimension $e$. They have disjoint supports. Multiplication over the independent positions and marked places proves the count. Away from them each center is integral, and Fourier transform interchanges full dual lattice indicators. This proves the vacuum assertion also at split places above two. □

## 121.4 A finite column orbit for all the terminal characters

Fix the field, interval, framed finite row, terminal frequency and finite column level of Theorem 112.8. Let $\mathcal{O}$ run over the scalar orbits of the fine column classes, each of size $|Z| = (T-1)^f$. For a prescribed terminal space let $J_{\mathcal{O}}$ be its subspace supported on this single orbit. Define $M_{J,\mathcal{O}}$ as the mean squared norm of the ordinary periodization when the signed column of Theorem 110.4 is sampled inside $\mathcal{O}$.

**Theorem 121.5** (One set of orbits for the prescribed real words and characters)**.** *Apart from a fraction at most* $\exp(-fkd \log k/3)$ *of the scalar orbits, one has simultaneously*

$$|M_{J,\mathcal{O}} - 1| \le \exp\bigl(-(1 - o(1)) fk \log k\bigr)$$

*for all prescribed words sharing the fixed framed data and for every terminal character nontrivial at each place, as well as for the simultaneous zero-sum space. These orbits can also be intersected with the primitive determinant set of Theorem 109.38.*

*Proof.* For any single orbit the covariance of the signed column is $P_{J,\mathcal{O}}/\dim J_{\mathcal{O}}$, by the finite averaging in Theorem 110.4 restricted to this orbit. The normalized trace of the identity is one, and Theorem 111.14 bounds the proper-intersection and transverse full-column error on every orbit by $((k+1)!)^{-f} \exp(Crf)$ through its operator norm.

We exhibit one nonnegative bound $e(\mathcal{O})$ for the remaining trace error, independent of the character and the word. In the full space, a matching smaller-column fine cell has absolute normalized trace at most its unipotent covolume times

$$\frac{\operatorname{vol}_f K^0}{n_{\mathrm{cl}}\, d_\rho c_{\rho,K^0}} \prod_{\tau} \mathcal{P}_\tau$$

by Proposition 112.4. The product $d_\rho c_{\rho,K^0}$ depends only on the fixed framed data. Replace the real product by the infimum of the explicit upper bounds obtained by multiplying Proposition 126.5 over the places with any permitted endpoint powers. All these bounds are uniform in the word. This gives a common numerical majorant for the cell. The matching test is intrinsic to the two finite columns, independent of the character. The underlying relevant labels

for a single word depend only on the two equal unipotent characters and the fixed frames, by Lemma 110.10.

The sum of these majorants over all matching smaller-column cells is at most $\exp(-(\frac{1}{2} - o(1))fkd\log k)$. To verify this point, on the two ranges of Theorem 112.7 its proof sums finite measures and the real majorants with the indicated endpoint powers. The rank-zero sum in Theorem 112.2 uses only the index of the row and the same positive measures. On the complement, Proposition 112.10 counts the measures after the uniform covolume bound, and Theorem 112.8 uses endpoint powers zero for its real factors. Thus each range admits the claimed estimate for these positive majorants before summing any signed contributions.

Set $e(\mathcal{O})$ equal to the sum of the majorants of cells whose matching columns lie in $\mathcal{O}$, multiplied by the total number $n_{\mathrm{cl}}/|Z|$ of orbits. This factor accounts for normalization on one orbit. Indeed the projector coefficient after division by dimension is bounded in modulus by $1/(|Z|d_\rho)$ there, for either terminal space. Each matching cell lies on just one orbit. Hence the uniform mean of $e(\mathcal{O})$ is the full sum just bounded. Markov's inequality shows that $e(\mathcal{O}) > \exp(-fk\log k)$ on a fraction at most $\exp(-fkd\log k/3)$ for sufficiently large $x$. Add the other operator-norm bound on every remaining orbit to obtain the theorem. The earlier determinant set is a union of whole orbits of fraction tending to one, so the intersection also has fraction tending to one. □

## 121.5 The arithmetic functional on the actual incoming row

On the interval of Section 6.6 use the same word, levels, and measures as in Theorem 121.10. For a nonzero actual incoming row $X$, take the detecting endpoint $V$ given there. At the fine source classes write $X(k,u) = \langle \rho_-(k)t_-, Y_u\rangle$, with the conjugations of Theorem 67.11. Use the ordered coordinate wedge tensors $e_I$ of the ambient source tensor product.

**Theorem 121.6** (The last arithmetic functional on the preceding lift)**.** *There is an array $a_{u,I}(V)$ in the arithmetic coefficient field, independent of $X$ once $V$ is fixed, such that*

$$\frac{D_-D_+c^{d/2}}{w_-w_+C}\langle TX, V\rangle = \sum_{u,I}(Y_u)_I a_{u,I}(V).$$

*At every $\mathfrak{p} \mid p$ in Theorem 125.2 its entries have valuation at least $e(\mathfrak{p}/p)g_p$. At every complex embedding they have absolute value at most $i_+B_+B_\sigma$ in that theorem's notation. On the source Cartan spaces the array satisfies*

$$\left(\sum_u \|a_{u,\cdot}(V)|_{V_n}\|_{\mathrm{op}}^2\right)^{1/2} \;\geq\; \frac{i_+c^{d/2}\sqrt{D_-a_*}}{2|C|\sqrt{w_-v_+}}.$$

*Here $\|\cdot\|_{\mathrm{op}}$ is the norm of a linear functional in the unitary tensor norm. These statements use the same endpoint for the actual Bessel pairing and for the arithmetic array.*

*Proof.* Expand only the endpoint character sum in the class formula of Theorem 67.11. At a source class $u$ evaluate it on $e_I$, using the minor formula on the entire exterior tensor product. This defines $a_{u,I}(V)$ as a sum of $i_+$ coordinate CM derivatives with the endpoint tensor inserted. Expanding the actual $Y_u$ in these coordinates gives the identity. The same integral polynomial calculation as in Theorem 125.2 gives the finite valuations.

For the estimate at infinity, fix an endpoint Cartan tensor $Z'$. On the ambient source tensor space the map in Proposition 95.2 satisfies $\|M(y\otimes Z')\|_{\mathrm{F}} \leq \sqrt{h_\lambda}\|y\|\|Z'\|$. Indeed the Cauchy decomposition of the homogeneous matrix polynomials pairs the target partition only with the same source partition. Its multiplicity in the source is one because a diagonal one-parameter

subgroup with strictly decreasing weights has a unique maximal weight line there, obtained by choosing the leading wedge in every slot. It is the highest line of the Cartan component. Equivariance consequently kills the orthogonal complement of that component when the target is in its Cartan component. On the Cartan component the asserted norm is the isometry formula, including its scalar $\sqrt{h_\lambda}$. Apply this argument at each embedding with $y = e_I$ of norm one and then the sphere estimate of Theorem 125.2.

Finally $\sum_u \|Y_u\|^2 = D_- \|X\|^2 / w_-$ by compact orthogonality. Apply Cauchy–Schwarz to the displayed identity and insert the lower bound on $|\langle TX, V\rangle|$ from Theorem 121.10. Cancelling $\|X\|$ gives the claimed lower bound. □

There is also an exact conversion to the intrinsic integral class lattices if those lattices are used for the tests. On each fine chart let $\Lambda_u$ be the lattice of matrix coefficients generated by the Schur and determinant models of Proposition 75.7 and its dual. Let $H_u$ be the module generated by the coefficients, as a polynomial in $\bar{Z}$, of $\langle \rho(k) t_R, v_R(Z)\rangle$. Both are taken over the ring of integers of a common coefficient field. The quotient $(H_u + \Lambda_u)/\Lambda_u$ is finite. Indeed the two realizations of the partition are defined over the coefficient field by the exterior and Young tensor constructions. The matrix coefficients of the Schur lists span the isotypic space over $\mathbb{C}$ by their frame bounds. Linear equations over the coefficient field with a complex solution already have a solution over that field, by row reduction. Clearing denominators for the finitely many generators of $H_u$ proves finiteness. Thus the least common multiple of these quotient exponents over the source charts, respectively over the target charts, gives integers $b_-, b_+$ directly from the two lattices. Multiplying the integral-family tests by these integers puts them in the intrinsic class lattices. The character sums preserve the transported lattices. The minors and both of their size bounds then acquire the same factor $(b_- b_+)^j$.

For the full scalar, the displayed identity still contains the class coefficients $Y_u$ of the preceding lifts. The remaining period estimate in Section 6.6 requires their arithmetic normalization and the prescribed signed contraction. The local cohomology remainder is given in Theorem 21.14.

## 121.6 Comparison for one signed definite column period

Use the single scalar $\mathcal{A}$, its data and its interval from Theorem 122.10. Index a summand by the signed term $b$ and its tuple of finite representatives $\boldsymbol{u}$. Write $s_{b,\boldsymbol{u}}$ for the scalar coefficient outside the polynomials, endpoints and class sections, including finite twists and prescribed character averages in that position. Let $\gamma_{\boldsymbol{u}}$ be the product of its vertex stabilizer orders. Write $a_{b,\boldsymbol{u}}$ and $\widetilde{b}_{b,\boldsymbol{u}}$ for its two endpoint polynomials in the compact variables. Here $\widetilde{b}$ is the polynomial actually occurring after the convention for conjugating the endpoint has been applied.

For an embedding $\sigma : J \to \mathbb{C}$ apply $\sigma$ to the algebraic coefficients in this expansion. Set

$$F_{b,\boldsymbol{u},\sigma} = \sup_{k_0} |a^{\sigma}_{b,\boldsymbol{u}}(k_0)| \, \sup_{k_*} |\widetilde{b}^{\sigma}_{b,\boldsymbol{u}}(k_*)|.$$

The suprema run over the two compact endpoint groups and can also be bounded by the sums of the absolute coefficients. Write $P_{i,b}$ for the real Schwartz polynomial in the dimensionless Fock coordinates of Theorem 121.3. Let $b_{i,b,\boldsymbol{u},\sigma}$ bound the sum of the absolute coefficients in the actual finite test expanded in translates of its normalized lattice vacua after conjugation by $\sigma$. These coefficients come from the finite changes of lattice and finite Fourier sums of

Theorem 125.2, applied to the indicated edge. With $D_i = [E_x : K]\ell_i$, write

$$W_{i,b,\boldsymbol{u},\sigma}^2 = b_{i,b,\boldsymbol{u},\sigma}^2[e\max(1, 2\ell_i \operatorname{rd}(E_x)/\pi)\nu_\sigma]^{D_i} c_\sigma^{d_i} \prod_\tau H_{\ell_i}(d_{i,\tau}),$$

$$U_\sigma = \sum_{b,\boldsymbol{u}} \frac{|\sigma(s_{b,\boldsymbol{u}})| F_{b,\boldsymbol{u},\sigma}}{\gamma_{\boldsymbol{u}}} \prod_i (\|P_{i,b}^\sigma\|_{\mathrm{F}} W_{i,b,\boldsymbol{u},\sigma}).$$

Here $c_\sigma = \pi/A_\sigma$, $\nu_\sigma$ is the squared norm of the marked elliptic vacuum, and $H_\ell$ is the explicit sphere factor in Lemma 66.8. The embeddings of a CM block are permuted with $\sigma$.

**Theorem 121.7** (Both estimates for the full signed column)**.** *The original projected column satisfies $|\sigma\mathcal{A}| \le U_\sigma$ at every embedding. Put*

$$h_i = \frac{1}{[J : \mathbb{Q}]} \sum_{p \notin S} \sum_{\mathfrak{p}\mid p} [J_\mathfrak{p} : \mathbb{Q}_p]\Gamma_{i,\mathfrak{p}} \log p, \qquad G_S = \sum_{p\in S} \sum_i g_{i,p} \log p.$$

*For any nonzero $\mathcal{A}$ from these prescribed data,*

$$G_S \le \log D_0 + \sum_i h_i + \frac{1}{[J : \mathbb{Q}]} \sum_\sigma \log U_\sigma.$$

*At the original embedding $\sigma_0$, its lower bound is*

$$\log|\sigma_0\mathcal{A}| \;\ge\; [J : \mathbb{Q}](G_S - \log D_0 - \sum_i h_i) - \sum_{\sigma\ne\sigma_0} \log U_\sigma.$$

*Proof.* After the Haar integrals have been expressed by their rational moments, applying $\sigma$ applies it to the class derivatives and polynomial coefficients and keeps each moment fixed. Consequently one may take the same compact averages of the coefficientwise transported integrands. In particular the coefficients of $\widetilde{b}$ are transported after its original conjugation. The integrals use probability Haar at every real vertex because their volume factors are already in $V_J$.

For a fixed edge and tuple, Theorem 65.9 bounds the squared metric supremum of the normalized lattice vacuum at $\sigma$ by $[e\max(1, 2\ell_i \operatorname{rd}(E_x)/\pi)\nu_\sigma]^{D_i}$. Hermitian translations preserve the metric supremum, so the coefficient sum gives the additional factor $b_{i,b,\boldsymbol{u},\sigma}^2$. Lemma 66.8 bounds the Fischer norm of its specified homogeneous Taylor component by $W_{i,b,\boldsymbol{u},\sigma}$, after taking this finite sum. For a polynomial $P$ of those block degrees, $P(\partial)G(0)$ is the bilinear Fischer contraction with that Taylor component. Indeed the contribution of $Z^\alpha$ in each factor is weighted by $\alpha!$. Cauchy–Schwarz bounds it by $\|P\|_{\mathrm{F}} W_{i,b,\boldsymbol{u},\sigma}$.

At every compact argument the polynomial in the integrand is obtained from $P_{i,b}^\sigma$ by unitary matrix substitution and a determinant character of absolute value one. This is the compact action proved in Theorem 121.3, expressed by substitutions with integer coefficients in the matrix variables. Transposing a matrix and embedding $k_*$ as $k_* \oplus 1$ retain unitarity. The Fischer norm is invariant by its Gaussian integral formula. Therefore the last bound holds uniformly over the compact variables, including the common variables on adjacent edges. Bound the endpoint factors by $F_{b,\boldsymbol{u},\sigma}$, integrate and sum the absolute values. This proves the first assertion on the scalar defined with all the original signs.

If $\mathcal{A} \ne 0$, apply the product formula to $D_0\mathcal{A}$. Theorem 122.10 gives a finite-place sum at least $G_S - \sum_i h_i$ in the normalization by $[J : \mathbb{Q}]$. Thus the mean of $\log|\sigma\mathcal{A}|$ is at least $G_S - \sum_i h_i - \log D_0$. Bound every embedding above, or all of them except $\sigma_0$, to obtain the two inequalities. Each $h_i$ is finite and bounded explicitly by the last inequality of Theorem 125.3 at that edge. □

Returning to the complex period multiplies $|\sigma_0\mathcal{A}|$ by $V_J|\prod_i C_i|c^{-t/2}$. The comparison uses the entire definite column. Section 6.6 also requires a lower bound on its prescribed pairing with the arithmetic normalization of the Bessel row, and control of the errors on the scale of the weighted prime count.

## 121.7 The discriminant bound for a detected terminal minor

Use the original field, word, levels and the two integral families in Theorems 121.10 and 122.9. For their selected nonzero $j$-row minor $z_j$, take $T, Q$ from the fixed finite vacuum model of Theorem 122.14 for every class section entering this matrix. Write $d$ for the last-edge degree and

$$B_{\mathrm{fin}} = \log T + d\log Q + \tfrac{d}{2}\log \mathrm{rd}(E) - \left(\tfrac{d}{4} - \tfrac{1}{2}\lfloor d/2\rfloor \mathbf{1}_{d_0\nmid Q}\right)\log d_0.$$

**Theorem 121.8** (Two-sided control for the selected arithmetic minor)**.** *At the original embedding the selected minor satisfies $|z_j| \ge j!(i_-i_+L_*)^j$, with the common terminal test of Theorem 121.10 when $j = 1$. For $S$ as in Theorem 125.2 away from $TQ$, the very same minor gives*

$$\sum_{p\in S} g_p \log p \le B_{\mathrm{fin}} + \tfrac{1}{2}\log j + \log(i_-i_+) + \tfrac{d}{2}\log(ns) + \frac{1}{[J:\mathbb{Q}]}\sum_{\sigma}\log B_\sigma.$$

*Both $T$ and $Q$ are determined before any elliptic product basis is chosen.*

*Proof.* Keep the physical derivative coordinates and both test families fixed. For each $p$ change just the product basis at the last edge to the one in Theorem 122.14. Every matrix entry acquires one common scalar by Theorem 122.3. The determinant acquires its $j$-th power. Therefore $z_j\overline{z_j}$ is identical in the two evaluations. Every term of the determinant contains exactly $j$ derivatives of the given block degrees with integral exterior tensor and finite character coefficients. The normalized matrix of Theorem 125.2 already uses the fine levels and its unnormalized character sums, so there is no compact integration or stabilizer denominator left in this matrix.

At $p \in S$ use the full-column minor weights in Theorem 102.5 for each factor, as in Theorem 125.2. At the remaining primes use Theorem 122.14 in both conjugate factors. As in Theorem 125.4, the sum of the two coordinate losses per derivative is $d\rho_p$. Thus at every $\mathfrak{p} \mid p$ the valuation of the diagonal product is at least $2jg_p$ in $S$ and at least $-2j(v_p(T)+\Phi_p(d, v_p(Q))+d\rho_p/2)$ elsewhere. The sum of this last loss divided by $2j$, with logarithmic prime weights, is $B_{\mathrm{fin}}$ by the discriminant computation. Apply the product formula to the nonzero diagonal product, use the archimedean bound for $z_j$ of Theorem 125.2, and divide by $2j$. The original lower bound follows from the choice in Theorem 122.9. □

The minor with $j = 1$ uses a terminal test which also detects the preceding Bessel row. The corresponding evaluation on that row is the contraction in Theorem 121.6. Determining its arithmetic normalization and controlling the remaining strict cohomology orders of Theorem 21.14 are still necessary for the interval estimate.

## 121.8 One row isometry for all the period pairings

The comparison in Section 6.6 uses a single isometry on the common row. The source may have a different isometry and a different nonzero normalization for each word. This order of quantifiers is explicit in `Formalism/SharpRestriction/GlobalTheoremInputs.lean`, in the definition `BesselId`. The realization `besS` in `Formalism/LocalRingCardinality/RhoDetPrimeEqMuTildeMulChiPow.lean` pairs the embedded row with the periodized column. Here is a finite version of the comparison using those pairings on every test.

Let $T_u : M_u \to E_0$ and $C_u : M_u' \to E_1$ be the Riesz representatives of two finite lists of pairings, using the conjugate row for a bilinear pairing as in Theorem 123.2. The complex Hilbert spaces have $\dim E_0 = \dim E_1 = n \geq 1$ and $\dim M_u = \dim M_u'$. For a nonzero operator $T$ put $N(T) = TT^\dagger / \operatorname{tr}(TT^\dagger)$.

**Theorem 121.9** (A simultaneous comparison on all tests)**.** *There are a unitary $V : E_0 \to E_1$, unitaries $U_u : M_u \to M_u'$ and nonzero scalars $a_u$ with $T_u = a_u V^\dagger C_u U_u$ for all $u$ exactly under the following conditions. The two lists have the same zero words, and, on their common nonzero words, every ordered list $i_1, \dots, i_k$ of length at most*

$$L = \min\{2n^2 - 1,\, 2d^2\}, \qquad d = \sum_{u : T_u \neq 0} \dim M_u,$$

*satisfies*

$$\operatorname{tr}\big(N(T_{i_1}) \cdots N(T_{i_k})\big) = \operatorname{tr}\big(N(C_{i_1}) \cdots N(C_{i_k})\big).$$

*For nonzero words one may take $|a_u|^2 = \operatorname{tr}(T_u T_u^\dagger) / \operatorname{tr}(C_u C_u^\dagger)$.*

*Proof.* The comparison implies unitary conjugacy of the normalized output operators and hence equality of all the traces. Conversely write $P_i = N(T_i), Q_i = N(C_i)$ on the nonzero words. In the direct sum of the two full matrix algebras let $B_j$ be the span of the paired products of length at most $j$, including $(I, I)$. If $B_j = B_{j+1}$, right multiplication by any generator $(P_i, Q_i)$ preserves $B_j$, so all further spaces agree. Their dimensions are at most $2n^2$. They are also at most $1 + 2d^2$, since every positive-length product is supported on the span of the output images on its side, of dimension at most $d$. Indeed each generator kills the orthogonal complement of this span. Starting with dimension one, the whole algebra is therefore spanned at length $L$. The difference of the two traces vanishes on the identity and on the tested products, hence on this algebra.

Both sets of generators are self-adjoint. For any polynomial $f$ in noncommuting variables, including a constant term, trace equality for $f^* f$ implies $\|f(P)\|_{\mathrm{HS}}^2 = \|f(Q)\|_{\mathrm{HS}}^2$. Consequently $f(P) \mapsto f(Q)$ is a well-defined unital trace-preserving isomorphism $\phi$ of the generated matrix algebras that commutes with adjoints.

We show directly that $\phi$ is conjugation by a unitary. In the first algebra write the identity as a sum of mutually orthogonal minimal projections $p_i$. To obtain them, split a projection whenever its corner contains a nonscalar self-adjoint matrix, using the spectral projections in that corner. They are polynomials in the matrix and the corner identity. Repeated splitting terminates by finite rank, and each resulting corner is scalar, by taking real and imaginary parts. If $p_i B p_j$ contains a nonzero matrix $x$, where $B$ is the first algebra, $x^* x$ and $x x^*$ are positive scalar multiples of $p_j, p_i$. Their scalars agree by associativity applied to $x x^* x$. Rescale $x$ to a unitary between these two ranges, zero on the orthogonal complement of the first domain. Adjoints and composition show that the relation of having a nonzero corner is an equivalence relation.

In each class choose a reference projection $p_e$ and such a matrix $v_i$ from $p_e$ to $p_i$, with $v_e = p_e$. Then $v_i v_j^*$ spans $p_i B p_j$ in this class, since $v_i^* y v_j$ belongs to the scalar corner at $e$ for every $y$ in that corner $p_i B p_j$. The other corners vanish. The ranks of $p_e$ and $\phi(p_e)$ agree by trace preservation. Choose a unitary $W_e$ between their ranges. The rules $v_i z \mapsto \phi(v_i) W_e z$ on the orthogonal ranges therefore define a unitary $V : E_0 \to E_1$. On each spanning matrix $v_i v_j^*$, direct evaluation gives $V(v_i v_j^*) V^\dagger = \phi(v_i v_j^*)$. In particular $V P_i V^\dagger = Q_i$ for all the words.

For each nonzero word let $s_u$ be the positive square root of the stated trace ratio and put $D = s_u V^\dagger C_u$. Then $D D^\dagger = T_u T_u^\dagger$. The rule $T_u^\dagger y \mapsto D^\dagger y$ is well-defined and isometric, since the two squared norms agree for every $y$, and it maps onto the range of $D^\dagger$. The equal source

dimensions allow it to extend to a unitary $U_u : M_u \to M'_u$. Taking adjoints gives $T_u = DU_u$. For a zero word take any source unitary and scalar one. This also handles a list consisting entirely of zero words. □

For the signed definite periods the left traces are exactly those computed in Theorem 123.2. If $R_u = C_u C_u^{\dagger}$ on the Bessel row, the equalities on nonzero words can be tested without division as

$$\operatorname{tr}\Big(\prod_j K_{i_j}\Big)\prod_j \operatorname{tr} R_{i_j} = \operatorname{tr}\Big(\prod_j R_{i_j}\Big)\prod_j \operatorname{tr} K_{i_j}.$$

Products inside traces retain their indicated order. Theorem 121.11 expresses the right trace as an integral with the actual row projector, and bounds the change under a single matching complex lift on its ordinary periodization spaces. Equality of these traces for the designated row and the signed definite data is still open. Even after a unitary comparison, the arithmetic normalization requires identifying the pairing on its fixed rational lattice with its specified values there. The trace calculation is invariant under rescaling each whole word.

## 121.9 One endpoint for the Bessel row and an arithmetic period

Fix a field $E_x$ on the interval of Section 6.6 and one of the prescribed words. Use a good orbit of Theorem 121.5 and its terminal character. Take the last two highest lines as in Theorem 104.12 and Proposition 104.8. In particular the last Schwartz polynomial is the coefficient-one corner polynomial $P_*$ of Theorem 67.11. Use its marked Fock coordinates and carry the tests with the character as in Theorem 94.16. All finite tests and external characters below are those of this word. Write $T$ for its last forward lift and $X$ for a vector obtained by applying the preceding lifts to a Bessel row $Sb$.

Let $a_* > 0$ denote the lower factor in the one-step Rallis bound for this last lift on the space of incoming rows. More explicitly, take the product of the fixed last-step factors in Theorem 72.21, with its finite depth factor $\prod_{v\in S_{\mathrm{row}}}(Nv)^{-a_v n(n+s)}$ and its real test $P_*$, and multiply by $\exp(-Cfr\,2^{-(z_0-c_r)})$ in the notation of that theorem. Here $n, s$ are the last source and target ranks. With compact source Haar probability, the unnormalized real matrix-coefficient integral in the marked, unit-variance Fock variables is $h_\lambda/D_-$, where $h_\lambda = \|P_*\|_{\mathrm{F}}^2$ and $D_- = D_n$. Indeed $P_*$ corresponds to the two unit highest tensors under an isometry times $\sqrt{h_\lambda}$. Integrating the squared source highest coefficient gives $1/D_-$. Any Haar volume and oscillator dilation used in the actual norm formula are retained there.

For the two definite groups, indicated by $-, +$, let $K_\pm$ be the original finite levels and $\chi_\pm$ their characters. Take normal open $J_\pm \subseteq \ker(\chi_\pm)$ inside the levels, fixing the tests and with trivial rational stabilizers. One obtains such subgroups by taking the normal core of fixing levels and imposing one odd split principal congruence condition. At that place the first congruence subgroup is torsion free. Indeed a nonidentity matrix in its $h$-th layer has its prime-th power first in layer $h+1$ at an unramified split rational prime, and a power prime to that prime stays in layer $h$. The definite rational stabilizers are finite. Write $i_\pm = [K_\pm : J_\pm]$, $h_\pm$ for the numbers of fine classes, $w_\pm$ for the measure of one chart in the original Haar measures, and $v_\pm = h_\pm w_\pm$ for the quotient volumes. Write $D_+ = D_s$, $c = \pi/A$, and $C$ for the scalar of Theorem 67.8 at this edge.

On one fine class take the function $k \mapsto \langle \rho_\pm(k)t_\pm, v\rangle$, zero on the other classes, for $v$ from Theorem 122.2. Apply the unnormalized sum $\sum_{a\in K_\pm/J_\pm} \overline{\chi_\pm(a)} R(a)$ of right translations. Denote the resulting tests by $U, V$ on the source and target respectively. They have the original

levels and characters. Define their normalized period by

$$z(U,V)=\frac{D_-D_+c^{d/2}}{w_-w_+C}\langle TU,V\rangle,\qquad L_*=\frac{c^{d/2}\sqrt{a_*}}{4|C|\sqrt{v_-v_+}}.$$

**Theorem 121.10** (A common terminal test at the prescribed character)**.** *For every nonzero incoming $X$ there is a test $V$ with*

$$|\langle TX,V\rangle|\ge\frac{i_+\sqrt{w_+a_*}}{2D_+\sqrt{h_+}}\|X\|.$$

*For that same $V$ there is a source test $U$ for which $|z(U,V)|\ge i_-i_+L_*$. The tests may be chosen from either of the two families at each end. With the integral family their coefficients in the specified fine charts are unnormalized sums of integral exterior tensors and finite character values. The normalized period is algebraic.*

*Proof.* The first $m-1$ local contractions can reach the real source line used here. The compact projection described after Proposition 104.8 sets that line on the preceding edge, and Theorem 104.12 supplies its explicit highest-type test on the indicated lowest source line. To reach this latter line from the earlier lifts, translate a nonzero contraction in its irreducible real representation and project to the multiplicity-one compact line, as in the proof of Corollary 104.15. Equivariance moves the translation and projection onto the real test. Continuity and density of polynomial tests permit a polynomial with nonzero projected contraction. The real choices are common to the constituents with the prescribed real parameters. The finite norm factors and the orbit estimate use the original finite levels throughout. In particular there is a nonzero $Sb$ on the chosen orbit by Theorem 121.5, and the successive positive norm bounds give a nonzero $X$.

For a single incoming constituent the last-step norm identity in the proof of Theorem 72.21 gives $\|TX\|^2\ge a_*\|X\|^2$. Its varying good-place factor is $L^S(z_0,B_0\otimes\mu_m^{-1})$, since all earlier parameter strings in this last factorization are fixed. The estimate for its absolute logarithm there gives precisely the stated one-step error. Mixed terms between orthogonal constituents vanish by that same mixed norm identity, including for distinct multiplicity copies. Hence the inequality holds on the span of the incoming vectors.

At either end choose a fine class uniformly and use the probabilities of Theorem 122.2. Any highest-line, original-character form $F$ has on class $u$ the unique expression $\langle\rho(k)t,y_u\rangle$. Compact orthogonality gives $\|F\|^2=(w/D)\sum_u\|y_u\|^2$. Pairing with a one-class test has absolute value $(w/D)|\langle y_u,v\rangle|$. The finite character sum multiplies that pairing by $i$, since it is $i$ times the orthogonal projector. Thus for the projected random test $Q$,

$$\mathbb{E}|\langle F,Q\rangle|^2\ge\frac{i^2w}{4hD^2}\|F\|^2.$$

Apply this first to $TX$ to choose $V$. The functional $F\mapsto\langle TF,V\rangle$ on the original finite-level, highest-line source has norm at least $|\langle TX,V\rangle|/\|X\|$. Riesz representation and the source version of the same bound choose $U$. A conjugate-linear lift is treated on the conjugate source Hilbert space, which leaves all absolute values unchanged. Multiplication by the normalization of $z$ gives the claimed lower bound.

Each pair of terms in the two finite sums is supported on a single pair of fine classes. Use the actual translated representatives for that term. The stabilizers have order one. Theorem 67.11 evaluates it, after the normalization defining $z$, as $M(v_-\otimes v_+)(\partial)G_{uv}(0)$, times its finite character factor. Here $M$ is the integral minor map, and $G_{uv}$ is the CM section for the same translated kernel. This proves algebraicity term by term. With a different fixed representative on a translated class the tensor is transported by the corresponding rational isometry. Using the translated representative itself keeps its displayed integral coordinates. □

### 121.10 Cyclic integrals on the prescribed Bessel row

For a fixed set of framed data let $S_u : M_u \to \mathcal{H}$ be the ordinary Bessel periodizations in one ambient Hilbert space. Use the column identifications and norms occurring in the pairing. An isometric realization $\iota : E \to \mathcal{H}$ of the row gives the period $\langle \iota y, S_u x\rangle$, with the inner product linear in its second argument. If the original realization is conjugate-linear in the row, use its conjugate space for $E$. Write $\Pi = \iota\iota^{\dagger}$. This is the orthogonal projector onto the realized row. The physical column on that row is $\iota^{\dagger} S_u$.

Put $t_u = \|S_u\|_{\mathrm{HS}}^2$, where the Hilbert–Schmidt norm squared is the sum of the squared norms on an orthonormal basis. For $t_u > 0$ set $\epsilon_u = \|(I - \Pi)S_u\|_{\mathrm{HS}}^2 / t_u$. Given a cyclic list $i_1, \dots, i_k$ of words, let $\Gamma_\Pi$ be the trace of the ordered product of the output Gram operators of $\Pi S_{i_j}$. Denote by $m_\Pi$ this trace divided by the product of the individual Gram traces when all are positive. The same definitions with $\Pi = I$ give $\Gamma_I, m_I$. The row embedding identifies the nonzero part of the product for $\Pi$ with the corresponding product for $\iota^{\dagger} S_u$.

**Theorem 121.11** (Projected cyclic integrals and their error)**.** *Let $h_u = \dim M_u > 0$. On the finite orbit spaces of Theorem 121.5 take an independent signed unit column $b_j$ from the distribution for each occurrence $i_j$. Write $v_j = \Pi S_{i_j} b_j$ and $v_{k+1} = v_1$. Then*

$$\Gamma_\Pi = \left( \prod_j h_{i_j} \right) \mathbb{E} \prod_{j=1}^{k} \langle v_j, v_{j+1} \rangle, \qquad \|\Pi S_u\|_{\mathrm{HS}}^2 = t_u (1 - \epsilon_u).$$

*On a common good orbit $t_u = h_u M_{J,\mathcal{O}}$ for the corresponding word and terminal space. For $i_{k+1} = i_1$, and whenever $m_\Pi$ is defined, one has*

$$|m_\Pi - m_I| \le \sum_{j=1}^{k} \sqrt{\epsilon_{i_j} \epsilon_{i_{j+1}}} + 1 - \prod_{j=1}^{k} (1 - \epsilon_{i_j}).$$

*On the common real line and with the single matching complex operator $\widehat{\Theta} = \alpha^{-1/2}\Theta$ of Theorem 83.9, let $m_\Theta$ be the normalized trace for the family $\widehat{\Theta} S_u$. For every ordinary list of positive traces on these data,*

$$|m_\Theta - m_I| \le 2(e^{k\Delta_x} - 1).$$

*For $k \le 2(\dim D_{\mathrm{all}})^2$ this last error is at most $\exp(-(\log 2) x^{377}/3)$ throughout sufficiently distant common intervals.*

*Proof.* For independent vectors with $\mathbb{E}(b_j b_j^*) = I_{M_{i_j}} / h_{i_j}$, the expectation of the product of the rank-one operators $v_j v_j^*$ is the product of their expectations in the same order. The trace for a fixed tuple of vectors is the cyclic product of inner products in the assertion. The required covariance on each orbit is the one in the proof of Theorem 121.5. Taking one factor with $\Pi = I$ gives the value of $t_u$. Pythagoras on a source orthonormal basis gives the projected trace.

By cyclicity the same unnormalized trace can be computed on $M_{i_1}$ as the trace of the product of the rectangular maps $S_{i_j}^{\dagger} \Pi S_{i_{j+1}}$, in order. This identity also follows by summing their diagonal coordinates. Put $R_u = (I - \Pi) S_u$. The change from the ordinary factor to the projected one subtracts $R_i^{\dagger} R_l$ for adjacent words $i, l$. Its trace norm is at most $\|R_i\|_{\mathrm{HS}} \|R_l\|_{\mathrm{HS}}$. Indeed the trace norm is the supremum of the absolute trace after multiplication by a contraction of the opposite shape, by singular value decomposition. Hilbert–Schmidt Cauchy–Schwarz gives the bound under that supremum. Each unchanged factor in either product has operator norm at most $\sqrt{t_i t_l}$. Replace one factor at a time and use the trace norm bound on the changed factor. After division by $\prod_j t_{i_j}$ the sum of the differences is bounded in modulus by the sum of

square roots in the theorem. Every normalized output Gram has trace norm one and operator norm at most one. Thus the normalized trace of their product has modulus at most one. Since $\Gamma_{\Pi}/\prod_j t_{i_j} = m_{\Pi}\prod_j(1-\epsilon_{i_j})$, the triangle inequality proves the first estimate.

For the last comparison let $H$ be the Gram operator of the restriction of $\widehat{\Theta}$ to the span of the ordinary images. Theorems 88.1 and 72.21 give $e^{-\Delta_x}I \le H \le e^{\Delta_x}I$. Now a changed factor is $S_i^{\dagger}(H-I)S_l$, whose trace norm is at most $(e^{\Delta_x}-1)\sqrt{t_i t_l}$ by the same inequality. Each factor already changed contributes at most another $e^{\Delta_x}$ to its operator norm bound. Summing the geometric progression shows that the difference between the two unnormalized traces divided by $\prod_j t_{i_j}$ has modulus at most $e^{k\Delta_x}-1$. The product of the changed individual traces divided by $\prod_j t_{i_j}$ lies in $[e^{-k\Delta_x}, e^{k\Delta_x}]$. The normalized trace has modulus at most one, giving the second copy of $e^{k\Delta_x}-1$. Finally Lemma 94.22 gives $\log k \le \log 2 + 2x^{40}$, while the estimate in Theorem 83.9 gives $\log \Delta_x \le -(\log 2)x^{377}/2 + O(x^{40})$. Together they give the asserted uniform error. □

The first identity applies with the actual designated projector whenever the columns are realized as these periodizations. The ordinary orbit estimate controls $t_u$. The contribution removed by the designated projector is the separate sum of squares $t_u\epsilon_u$. The last estimate concerns the common matching theta lift on the ordinary space. Its proof uses the same $H$ for every factor in a cycle.

## 121.11 Vanishing forced by the level of the kernels

Let $U \subseteq C$ be the image of the integral norm-one ideles supported at the places above $P$. The lower-bound argument in Proposition 70.8 gives

$$|U| \ge (P-1)^{[E^{+}:\mathbb{Q}]}/w_E.$$

Here all other finite local units already belong to the congruence kernel defining $C(P)$. The following observation applies whenever both adjacent kernels are taken at a level with these determinant units.

**Proposition 121.12** (Vanishing in a fixed central character)**.** *At every intermediate vertex $i$ suppose there is a compact subgroup $J_i \subseteq G_i(\mathbb{A}_f)$ whose determinant image in $C$ contains $U$. Suppose both kernels meeting that vertex are right invariant in the variable $g_i$ under $J_i$. In any family of $h$ twists with a fixed output central character, at most $|C/U|^{m-2}$ operators can be nonzero. This holds after every fixed output map as above. For any $b = \dim V > 0$ one has*

$$\frac{1}{hb}\sum_{\eta}\|A_{\eta}^{*}A_{\eta} - I_V\|_{\mathrm{HS}}^{2} \;\ge\; 1 - |U|^{-(m-2)}. \tag{121.1}$$

*For the common fields $E_x$, $m = \lfloor\sqrt{x}\rfloor$, and $P$ as in Corollary 70.9, the proportion allowed to be nonzero is at most $\exp(-c_P x^{5/2+8\varepsilon})$ for all sufficiently large $x$.*

*Proof.* In (70.6), replace just $g_i$ by $g_i k$ with $k \in J_i$. Right translation preserves the quotient measure, and invariance of the two incident kernels leaves as the only factor $\chi_i(\det k)$. If this factor differs from one, the integral vanishes identically as a function of the endpoint. Thus a nonzero operator requires every $\chi_i$ to be in $X = \widehat{C/U}$.

The homomorphism from $\widehat{C}^{m-1}$ to $\widehat{C}$ given by $(\chi_i) \mapsto \prod_i \chi_i^{s_i n_i}$ has a right inverse $\omega \mapsto (\omega, \omega, 1, \dots, 1)$ since $-2+3=1$. Every fiber has size $h$. The same right inverse on $X$ shows that its restriction to $X^{m-1}$ has either zero or $|X|^{m-2}$ points in that fiber. A zero operator contributes $\operatorname{tr}(I_V) = b$ to the sum in (121.1). This proves both assertions about a fixed central

character. Finally $\log|U| \ge [E_x^+ : \mathbb{Q}]\log(P-1) - \log w_{E_x} \gg_P x^{2+8\varepsilon}$, using Corollary 70.9. Multiplying by $m-2 \gg \sqrt{x}$ gives the displayed exponential bound on the surviving proportion. □

The determinant-level hypothesis is realized, for example, for the standard definite spaces at an unramified $P$ by using the integral stabilizers of the standard lattices with kernels invariant there. The diagonal action of an integral norm-one unit on a coordinate line lies in that stabilizer and has the given determinant. Right invariance can also be imposed on given kernels by averaging their Schwartz vectors over the corresponding compact groups at both ends of every edge. Those actions commute through the chosen splitting, and each integral on a smooth finite-place vector is a finite sum because the vector has an open stabilizer. The resulting theta kernels have exactly the invariance used in the proposition. The argument allows a vanishing average.

For this choice of levels, the error in (121.1) tends to one, whereas the rank argument in Volume I uses $e \le 1/4$. Applying the rank argument to the full family therefore requires different local test data or a different selection of twists. Formulas (70.9) and (70.10) continue to hold with such data. The within-component correlations, the lower divisibility reflecting the prime count, and the covolumes for the particular integral periods required in Section 6.6 remain to be estimated.

## 121.12 A common ramified test function at a split place

We keep the actual chain (70.6), with $n_i = i+1$, and use the same Schwartz function for every twist at a given edge. The special test function below is the indicator of a set of integral matrices. Its expansion under diagonal changes of basis selects exactly one term for each twist after integration over the intermediate groups.

**Lemma 121.13** (Split places on the power interval)**.** *Fix distinct rational primes $P, Q_0 \ge 5$ prime to $2F$. For all sufficiently distant common intervals, both are unramified in $E = E_x$, and every place of $k = E^+$ above $P$ splits in $E/k$. Put $q_v = |\mathcal{O}_k/v|$ for $v \mid P$. Any finite-order character of the product of the integral local norm-one units at these places can be extended to a finite-order automorphic character of $T$, allowing ramification at $Q_0$.*

*Proof.* Eventually the discriminant primes $q, d$ of Volume I are distinct from $P, Q_0$, and $N = 2^{l+2}Fqd$ is prime to both. Let $g$ be Frobenius at $P$ in $\mathrm{Gal}(E/\mathbb{Q})$. Its order on $C_l$ is a power of two. For large $l$ this order is even, since $P$ is then neither 1 nor $-1$ modulo $2^{l+2}$. If $g^j$ were complex conjugation, it would act trivially on $C_l$, forcing $j$ even. But $g^j$ would then act trivially on $\mathbb{Q}(\sqrt{-q})$, contrary to complex conjugation. The decomposition group at $P$ is generated by $g$, as $P$ is unramified. Its intersection with $\mathrm{Gal}(E/k) = \{1, c\}$ is therefore trivial, giving the splitting assertion.

A prescribed continuous finite-order character on the product of unit groups factors modulo $P^b$ for some $b$. Map that finite quotient into $C(P^bQ_0)$ using ideles supported at $P$. The kernel is trivial. Indeed the rational norm-one element in a relation must be integral and a unit at every finite place, hence a root of unity by its archimedean absolute values. It is congruent to one at $Q_0$. Since $E$ is unramified at $Q_0$, it cannot contain a primitive $Q_0$-th root of unity, so the order of that root is prime to $Q_0$. If $z \equiv 1 \bmod w$ and $z^t = 1$ with $Q_0 \nmid t$, then $1 + z + \cdots + z^{t-1}$ is a unit at $w \mid Q_0$. This forces $z = 1$. The asserted kernel is consequently the original congruence subgroup before taking the finite quotient. A character of a subgroup of a finite abelian group extends to the full group. This follows by adjoining one generator at a time and taking an appropriate root of its prescribed power in the circle. Proposition 70.8 gives finiteness of $C(P^bQ_0)$, so the extension has finite order. □

At a split place $v \mid P$, write $L = k_v$, $\mathfrak{o} = \mathcal{O}_L$, and $\mathbf{k}_v$ for its residue field. The two idempotents of $E \otimes_k L = L \oplus L$ give two dual summands of each unitary space. Choose the first summand at even vertices and the second at odd vertices. Thus the norm-one scalar $(z, z^{-1})$ acts on the chosen summand of vertex $i$ by $z^{s_i}$, $s_i = (-1)^i$. On the edge from dimension $a = i$ to $b = i+1$, one Lagrangian of the symplectic tensor space is $\mathrm{Hom}(L^a, L^b)$. Indeed the dual of the chosen summand at vertex $i-1$ uses the same idempotent as the chosen summand at $i$. Their tensor product and its dual are two complementary isotropic spaces. In the Schrödinger model on $M_{b,a}(L)$ the action on compact elements is therefore

$$((g,h)\phi)(X) = \phi(h^{-1}Xg), \quad g \in \mathrm{GL}_a(\mathfrak{o}), \quad h \in \mathrm{GL}_b(\mathfrak{o}). \tag{121.2}$$

Here we use the splitting given by pullback on that Lagrangian, with the square root of the absolute Jacobian for general elements. The Jacobian is one on the displayed compact subgroups. [2]

We explain how to arrange (121.2) for all edges using global splittings. On the split local general linear groups the ratio of an initial unitary splitting and the Schrödinger splitting is a character into the central circle. It factors through the two determinants. Indeed a character of the direct product is a product of characters, and the special linear group over the infinite field $L$ is generated by elementary transvections. Each is a commutator in the general linear group, by conjugation by a diagonal matrix with one diagonal entry different from one. On a compact unit group each resulting smooth character has finite image. By Lemma 121.13 the inverses of its unit restrictions extend to global characters of the norm-one torus. Multiplying each edge splitting by the corresponding automorphic determinant characters changes it by a central scalar, preserves the rational splitting, and imposes (121.2) on the compact subgroups at all $v \mid P$. Make this choice once for the entire twist family. The construction permits additional levels at $Q_0$.

Write $I_n = \{g \in \mathrm{GL}_n(\mathfrak{o}) : \bar{g} \text{ is upper triangular}\}$. A bar here means reduction to $\mathbf{k}_v$. Let $\Omega_{a,b}$ consist of the integral matrices for which $\bar{X}_{ij} = 0$ if $i > j$, and the first $a$ diagonal entries are nonzero. The common vector $\phi^0_{a,b}$ requires all these diagonal entries to be one: it is the indicator of that smaller set. For an $a$-tuple $\lambda = (\lambda_1, \dots, \lambda_a)$ of characters of $\mathbf{k}_v^\times$, put

$$\phi_\lambda(X) = \mathbf{1}_{\Omega_{a,b}}(X) \prod_{j=1}^{a} \lambda_j(\bar{X}_{jj}),$$

with value zero off the indicated set. Then

$$\phi^0_{a,b} = (q_v - 1)^{-a} \sum_{\lambda} \phi_\lambda. \tag{121.3}$$

This is ordinary character orthogonality on each diagonal entry. Under $(g,h) \in I_a \times I_b$, the function $\phi_\lambda$ transforms by

$$\prod_{j \le a} \lambda_j(\bar{g}_{jj})\lambda_j(\bar{h}_{jj})^{-1}.$$

In fact the set $\Omega_{a,b}$ is stable in (121.2), as is seen by multiplying upper triangular residue matrices. The $j$-th diagonal entry of $\bar{h}^{-1}\bar{X}\bar{g}$ is $\bar{h}_{jj}^{-1}\bar{X}_{jj}\bar{g}_{jj}$.

## 121.13 Orthogonality after Bessel restriction

Use $\phi^0_{i,i+1}$ on every edge at every $v \mid P$, with arbitrary fixed Schwartz functions at other places. Take an input function on $[G_0]$ in a single central character, invariant under the integral units at $P$; the constant function is an example. Fix a nondegenerate codimension-one subspace

$V'_m \subset V_m$, put $G'_m = U(V'_m)$, and use local bases at the endpoint in which this subgroup acts in the first $m$ coordinates. Its compact group at these places contains $I' = \prod_{v\mid P} I_m$ embedded block diagonally. For $W$ a finite-dimensional right $I'$-invariant subspace of $L^2([G'_m])$, let $B_{\boldsymbol{\chi}}$ be the operator obtained by restricting (70.6) to $[G'_m]$ and projecting orthogonally to $W$. Pairing it with $h \in W$ gives the codimension-one Bessel integral against $\bar{h}$. All the restricted functions are continuous on a compact quotient, so this construction requires no boundedness assertion for restriction on the ambient $L^2$-space.

**Theorem 121.14** (A fixed central character and a fixed Bessel target)**.** *For $C = C(P)$, let $U \subset C$ be the image of the integral local norm-one units at $P$. There are $|U|^{m-2}$ tuples of determinant characters $\chi_i \in \widehat{C}$, all with the same prescribed product $\prod_i \chi_i^{s_i n_i} = 1$, for which*

$$B_{\boldsymbol{\chi}}^{*} B_{\boldsymbol{\chi}'} = 0 \qquad (\boldsymbol{\chi} \neq \boldsymbol{\chi}').$$

*The test functions, splittings, initial central character and $W$ are the same for the entire family. On the common intervals the number of tuples is at least $\exp(c_P x^{5/2+8\varepsilon})$ for sufficiently large $x$.*

*Proof.* Fix a place over $P$ for the moment. Let $\delta_i$ be the local residue character furnished by $\chi_i$ in the chosen general linear coordinate: the internal multiplier on $I_{n_i}$ is $\delta_i(\prod_j \bar{g}_{jj})$. It is the restriction of $\chi_i^{s_i}$ in the first norm-one coordinate. Expand each common vector by (121.3) and write $\lambda_{i,j}$ for the characters on the $i$-th edge. Right translation of $g_0$ by its integral units and invariance of the input force $\lambda_{1,1} = 1$ in every surviving term. Right translation of the intermediate integration variable $g_i$ by $I_{n_i}$ next forces

$$\lambda_{i+1,j} = \delta_i^{-1}\lambda_{i,j} \ (j \le i), \qquad \lambda_{i+1,i+1} = \delta_i^{-1} \qquad (1 \le i < m). \tag{121.4}$$

Indeed, at coordinate $j$ the incoming edge contributes $\lambda_{i,j}^{-1}$ when $j \le i$, the outgoing edge contributes $\lambda_{i+1,j}$, and the twist contributes $\delta_i$. These diagonal entries can be chosen independently. If the product differs from one, translation invariance of Haar measure makes that summand zero. This leaves just the tuple of edge labels in (121.4) at this place. The same argument applies independently at all places over $P$.

The right character on the output $I_{m+1}$ has diagonal coordinates $\gamma_j = \lambda_{m,j}^{-1}$ for $1 \le j \le m$ and $\gamma_{m+1} = 1$. The recursion implies

$$\gamma_1 = \gamma_2, \qquad \gamma_j = \prod_{i=\max(1,j-1)}^{m-1} \delta_i \ (1 \le j \le m), \qquad \delta_i = \gamma_{i+1}\gamma_{i+2}^{-1}.$$

Thus the first $m$ characters, which are still present on the Bessel subgroup, determine every $\delta_i$. This proves that different tuples of unit restrictions give different characters on $I'$. Right translation is unitary on $L^2([G'_m])$, so its two distinct character spaces are orthogonal. Since $W$ is right $I'$-invariant, its orthogonal projection commutes with those translations. The projected pairings vanish for any pair of source vectors, giving the asserted operator identity.

It remains to choose global tuples. Restriction $\widehat{C} \to \widehat{U}$ is onto, by extension of characters of a finite abelian group. For either character group the product map $(\chi_i) \mapsto \prod_i \chi_i^{s_i n_i}$ has section $\omega \mapsto (\omega, \omega, 1, \dots, 1)$, since $-2 + 3 = 1$. Each tuple in the kernel on $\widehat{U}$ lifts to one in the kernel on $\widehat{C}$. In fact any lifts have product trivial on $U$, and dividing by the image of this product under the section corrects it without changing restrictions. Choose one lift for each of the $|U|^{m-2}$ tuples. Theorem 70.7 gives the same global output central character for all of them. The lower bound follows from $\log|U| \ge [k:\mathbb{Q}]\log(P-1) - \log w_E$ in Proposition 70.8 and $m = \lfloor\sqrt{x}\rfloor$. $\square$

**Proposition 121.15** (Integral factors for the common local tests)**.** *Each common vector has values in $\mathbb{Z}$ and takes the value one. With the integral matrix lattice assigned Haar volume one, their product has squared local $L^2$-norm*

$$\prod_{v\mid P}\prod_{i=1}^{m}\|\phi^0_{i,i+1}\|_2^2 = P^{-[k:\mathbb{Q}]m(m+1)(m+5)/6}.$$

*The common denominator in the full expansion (121.3) is at most $D_{\mathrm{in}} = \prod_{v\mid P}(q_v-1)^{m(m+1)/2}$. On any integral $I'$-stable test lattice fixed by the subgroup with all residue diagonal entries one, each endpoint character projector is integral after multiplication by $D_{\mathrm{out}} = \prod_{v\mid P}(q_v-1)^m$, over a coefficient field containing the character values.*

*Proof.* In the matrix of size $(i+1)\times i$, the $i(i-1)/2$ entries strictly above the diagonal are free in the residue field, and every other residue entry is fixed for $\phi^0$. Its squared norm is $q_v^{-i(i+3)/2}$. Sum the exponents over $i$ and use $\prod_{v\mid P} q_v = P^{[k:\mathbb{Q}]}$ since $P$ is unramified. Multiplying the denominators in (121.3) gives $D_{\mathrm{in}}$. At the endpoint, $I'$ modulo the subgroup in the statement is $\prod_{v\mid P}(\mathbf{k}_v^\times)^m$. The projector to a character is its finite character average. Its numerator preserves the lattice since each coefficient is a root of unity. Its denominator is $D_{\mathrm{out}}$. In particular $\log D_{\mathrm{in}} \le \frac{1}{2}m(m+1)[k:\mathbb{Q}]\log P$ and $\log D_{\mathrm{out}} \le m[k:\mathbb{Q}]\log P$. □

Theorem 121.14 evaluates the off-diagonal blocks for this family. A projected operator may still vanish. The norm in Proposition 121.15 is the norm of the test functions in the local Weil representation. A lower bound on the diagonal blocks of the global Bessel operators requires their actual period norms. Realizing the particular translates and weights of Volume I, proving their lower divisibility at the other places, and estimating the integral indices of their common target remain necessary for inequality (6.3).

## 121.14 A Bessel test space of exactly the required rank

We now specify the common test space in Theorem 121.14. Take the standard Hermitian spaces, $k = E^+$, and let $H = U(E^m)$ be the subgroup preserving the first $m$ coordinates at the endpoint. Use the fixed unramified primes $P, Q_0$ of Lemma 121.13. At $v \mid P$ the quotient of the Iwahori by the subgroup with every residue diagonal entry one gives

$$A = \prod_{v\mid P}(\mathbf{k}_v^\times)^m.$$

Write $K^{\max}$ for the finite adelic stabilizer of the standard lattice. Let $K_0 \subset K^{\max}$ have Iwahori level at $P$, principal level $Q_0$ at every place above $Q_0$, and full level elsewhere. Let $K_1 \triangleleft K_0$ have the same level with residue diagonals one at $P$, so $K_0/K_1 = A$. Define

$$X = H(k)\backslash H(\mathbb{A}_{k,f})/K_1, \qquad h = |H(k)\backslash H(\mathbb{A}_{k,f})/K_0|.$$

For the family of twists of Theorem 121.14, let $\Xi$ be their residue characters on $A$, and put $Z = \bigcap_{\gamma\in\Xi}\ker\gamma$ and $n_0 = |\Xi|$.

**Lemma 121.16** (The level classes and the residue characters)**.** *The action of $A$ on $X$ is free, with $h$ orbits. The set $\Xi$ is a subgroup of $\widehat{A}$, and $\Xi = \widehat{A/Z}$ has order $n_0 = |U|^{m-2}$, with $U$ as in Theorem 121.14.*

*Proof.* For $y \in H(\mathbb{A}_{k,f})$, the rational group $H(k)\cap yK_0y^{-1}$ preserves the Hermitian lattice with finite completions $y\widehat{\mathcal{O}_E^m}$. Its positive definite integral trace form has finite automorphism group,

because there are finitely many possibilities for the images of each basis vector of specified length. Thus every element has finite order. At $Q_0$ it is conjugate to a matrix congruent to the identity modulo $Q_0$. The principal congruence group is torsion-free. Indeed all completions of $E$ at $Q_0$ are unramified and $Q_0 \geq 5$. If $1 + Q_0^a B$, $a \geq 1$, has an entry of $B$ a unit, raising it to a prime $p \neq Q_0$ leaves a nonzero term $pQ_0^a B$ modulo $Q_0^{a+1}$. Raising it to $Q_0$ gives $Q_0^{a+1}B$ as the first nonzero term of its difference from one modulo $Q_0^{a+2}$. No element of prime order is possible. Faithfulness after completion implies the rational stabilizer is trivial. A stabilizer for the right action of $K_0/K_1$ on $X$ would arise from such a rational element, so the action is free. Its orbit space is the double quotient at level $K_0$. These double quotients are finite by definiteness, also by Lemma 121.17 below.

The proof of Theorem 121.14 parametrizes the twists by the kernel of the product homomorphism on $\widehat{U}^{m-1}$. Their endpoint characters are given by the products of residue coordinates in (121.4), independent of which global lifts are chosen. That formula is multiplicative and recovers all residue coordinates, hence its image is a subgroup isomorphic to the kernel. The order is $|U|^{m-2}$. Duality for finite abelian groups gives $\widehat{A/Z} = \Xi$: for instance the evaluation map $A \to \widehat{\Xi}$ is onto since every character of the subgroup $\Xi \subseteq \widehat{A}$ extends to $\widehat{A}$, and its kernel is $Z$. □

**Lemma 121.17** (The number of level classes)**.** *Let $d = [E : \mathbb{Q}]$, $n = md$, and suppose $E \subseteq \mathbb{Q}(\zeta_N)$. For an absolute constant $C$,*

$$\log h \leq C n^4 \log(2nNPQ_0).$$

*Proof.* At full level the double classes are the isometry classes of lattices in the genus of the standard Hermitian lattice. Indeed the finite adelic translates give those lattices, intersection with the rational space recovers a lattice from its completions, and two translates differ by a rational isometry precisely when they give the same class. For such a lattice use the integral symmetric form $B(v, w) = \mathrm{Tr}_{E/\mathbb{Q}}\langle v, w\rangle$. Its rank is $n$, it is positive definite, and its Gram determinant is $\Delta = |D_E|^m$. For the standard lattice this follows from the trace discriminant and the determinant of conjugation, a unit on $\mathcal{O}_E$. Local isometry gives the same valuation at every rational prime for all lattices in the genus; positivity gives the asserted equality. In particular $\Delta \leq N^n$ by the abelian conductor–discriminant formula.

There is an algebraic integer $\theta$ generating $E$ with absolute value at most $d^3 N$ at every embedding. Traces from $\mathbb{Q}(\zeta_N)$ of the powers of $\zeta_N$ span $E$ over $\mathbb{Q}$, since trace is onto and these powers span the larger field. Take a basis of $d$ such traces. Each conjugate has size at most $N$. For a pair of distinct embeddings of $E$, the coefficient vectors giving equal values on a linear combination form a proper hyperplane. On the grid $\{0, \dots, d^2\}^d$ each hyperplane has at most $(1 + d^2)^{d-1}$ points, by solving for a coordinate with nonzero coefficient. There are fewer than $1 + d^2$ pairs. A grid point outside their union gives the desired generator and bound (in degree one one may take 1).

Every trace lattice has an integral basis with norms at most $B_0 = (4n)^{2n}\sqrt{\Delta}$. To see this, its shortest nonzero vector has norm at least one by integrality. Minkowski's second theorem for the Euclidean unit ball bounds its last successive minimum by $n^{n/2}\sqrt{\Delta}$, using the cube of side $2/\sqrt{n}$ inside the ball to bound its volume. Choose independent vectors of norm at most twice this minimum. Intersect the lattice with their successive real spans. Each consecutive quotient is free of rank one, and the orthogonal length of a generator is at most the norm of the new independent vector, since that vector gives a nonzero integer multiple in the quotient. Choose lifts of the generators consecutively. Subtract integer multiples of the previous lifts in reverse order to make each orthogonalized coefficient lie in $[-1/2, 1/2]$. The squared length of

each resulting basis vector is at most $(1 + n/4)$ times the square of twice the last minimum. This implies the stated bound. [3, Thm. 6.3.3]

In this basis record the Gram matrix and the integral matrix of multiplication by $\theta$. The entries of the former have magnitude at most $B_0^2$. Those of the latter have magnitude at most $d^3 N B_0^n$, by Cramer's rule and Hadamard's inequality, since multiplication by $\theta$ has operator norm at most $d^3 N$ and the lattice covolume is $\sqrt{\Delta} \geq 1$. Equal matrix pairs yield a trace-lattice isometry commuting with $\theta$, hence linear over $E = \mathbb{Q}(\theta)$. It preserves the Hermitian form because its trace pairing after multiplication of one argument by every $a \in E$ determines that form by trace nondegeneracy. Thus the number of genus classes is bounded by the number of matrix pairs. Taking logarithms, and using $\log B_0 \ll n \log(2nN)$, gives $Cn^4 \log(2nN)$. The index of $K_0$ in the full stabilizer is at most $(PQ_0)^{n^2}$, by reduction on the underlying rank-$n$ integral lattice at those two rational primes; the kernels of both reductions lie in the specified levels. Each full-level class splits into at most that many classes. This proves the bound. □

Fix a representative $y \in H(\mathbb{A}_{k,f})$ of any one of the $h$ orbits. Put $H_\infty = H(k \otimes_{\mathbb{Q}} \mathbb{R})$. Choose a single continuous function $b$ of unit $L^2$-norm on $H_\infty$, for Haar probability; $b = 1$ gives integer-valued tests. Every coset $ya$, $a \in A$, has a chart parameterized by $H_\infty \times K_1$ in the full adelic quotient, using a representative of $a$ at $P$. There are no rational identifications in this chart because the rational stabilizer just proved is trivial. Define $W_y$ to consist of functions supported on this orbit which in the chart of $ya$ have value $b(h_\infty) f(aZ)$, with $f$ an arbitrary complex function on $A/Z$. The definition is independent of the choice of lifts in $K_0$. It is right Iwahori invariant as a subspace. Distinct orbits have disjoint support.

**Theorem 121.18** (An integral Bessel target with known covolume)**.** *The space $W_y$ has dimension $n_0$ and contains exactly one line for each character $\gamma \in \Xi$. Normalize Haar measure on the global compact quotient to have total mass one. With scalar coordinates as above the squared norm on $W_y$ is*

$$\|f\|^2 = (hn_0)^{-1} \sum_{aZ \in A/Z} |f(aZ)|^2.$$

*Take the cyclotomic field $J = \mathbb{Q}(\mu_{\exp(A/Z)})$ and the lattice $\mathcal{L} = \mathcal{O}_J^{A/Z}$ in these coordinates. For $b = 1$ it is the full lattice of $\mathcal{O}_J$-valued test functions in the space at one embedding. Over all embeddings use the metric weighting each real place by one and each pair of complex places by two. Then*

$$\frac{-2 \log \operatorname{covol}(\mathcal{L})}{[J : \mathbb{Q}] n_0} = \log(hn_0) - \frac{\log |D_J|}{[J : \mathbb{Q}]}, \tag{121.5}$$

$$[\mathcal{L} : \textstyle\sum_{\gamma \in \Xi} \mathcal{O}_J \gamma] = n_0^{[J:\mathbb{Q}] n_0 / 2}. \tag{121.6}$$

*On functions already in the fixed archimedean coordinate and finite level $K_1$, projection to $W_y$ preserves the integral lattice after multiplication by $|Z| = |A|/n_0$.*

*Proof.* Each chart at level $K_1$ has the same measure: it is a translate of $H_\infty \times K_1$ divided by a trivial rational stabilizer. There are $h|A|$ charts. Haar probability on the two compact factors and the norm of $b$ therefore give the stated formula by summing over the $|Z|$ cosets above each scalar coordinate. The characters of $A/Z$ form an orthogonal basis of its regular representation. Finite-level projection consists of keeping the chosen orbit and averaging its coefficients over $Z$, proving the denominator. The unscaled trace metric on $\mathcal{O}_J$ has covolume $\sqrt{|D_J|}$, by the determinant of an integral embedding matrix with the complex pairs weighted as specified. Scaling each squared norm by $1/(hn_0)$, in real dimension $[J : \mathbb{Q}]n_0$, gives (121.5). In the coset-coordinate basis the columns of the character lattice form the square Fourier matrix. Under

every embedding its Hermitian Gram matrix is $n_0 I$. The absolute norm of its determinant is consequently $n_0^{[J:\mathbb{Q}]n_0/2}$, the asserted index.

In particular $\log|A| \le m[k:\mathbb{Q}]\log P$, and $\log|D_J|/[J:\mathbb{Q}] \le \log(\exp(A/Z)) \le [k:\mathbb{Q}]\log P$: the exponent of $A$ divides the product of the $q_v - 1$. Together with Lemma 121.17 this gives a uniform bound for the magnitude of (121.5) on the common fields and intervals. For a general choice of $b$ the integral structure asserted is on its scalar coordinates. □

## 121.15 The remaining diagonal period values

Fix a unit source function at the one-dimensional unitary group in Theorem 121.14, and give its complex span dimension one. For each chosen twist denote by $F_\gamma$ its iterated lift restricted to the Bessel subgroup before projection, with $\gamma \in \Xi$ its right residue character. Define an actual scalar integral on one finite class by

$$p_\gamma(y) = \int_{H_\infty}\int_{K_1} F_\gamma((h_\infty, yk))\overline{b(h_\infty)}\,dk\,dh_\infty$$

with probability measures on the two compact factors. This includes the projection onto the chosen archimedean coordinate.

**Proposition 121.19** (Two moment conditions from the individual periods)**.** *On $W_y$, the column of the projected Bessel operator on the unit source has scalar coordinates $p_\gamma(y)\gamma(a)$. Put $d_\gamma = |p_\gamma(y)|^2/h$ and $\bar d = n_0^{-1}\sum_\gamma d_\gamma$. Its Gram operator on the source is the scalar $d_\gamma$. If*

$$n_0^{-1}\sum_{\gamma\in\Xi}(d_\gamma - 1)^2 \le e \le 1/4,$$

*the two moment hypotheses of Proposition 6.9, hold with this $e$ and $\eta = e/(1-\sqrt{e})^2$.*

*Proof.* Choose representatives for $a$ supported at $P$. Moving $a$ past the $K_1$ average is allowed because it normalizes $K_1$ and preserves its Haar probability. On the right of $F_\gamma$ it acts by $\gamma(a)$. Thus the archimedean and finite average on the coset $ya$ equals $p_\gamma(y)\gamma(a)$, already constant on $Z$. Orthogonal projection is this averaging in the charts followed by restriction to the chosen orbit and the $Z$-average. The norm formula gives $d_\gamma$. The columns lie in orthogonal lines which together have dimension $n_0$, including lines where the column vanishes. Hence the eigenvalues of the sum of the output Gram operators are exactly $d_\gamma$ with multiplicity. The first moment hypothesis is the assumed estimate. Cauchy–Schwarz gives $\bar d \ge 1-\sqrt{e}$, and $n_0^{-1}\sum(d_\gamma - \bar d)^2 \le e$. Consequently

$$\frac{n_0\sum_\gamma d_\gamma^2}{(\sum_\gamma d_\gamma)^2} = 1 + \frac{n_0^{-1}\sum_\gamma(d_\gamma-\bar d)^2}{\bar d^2} \le 1 + \frac{e}{(1-\sqrt{e})^2}.$$

This is the second moment hypothesis. No nonvanishing of an individual scalar was needed for these identities. □

The common rank, Haar normalization, finite projector and covolume here are determined before the periods are evaluated. Finishing Section 6.6 by this construction would require a bound on the squared deviations of these actual integrals on a common orbit with the required normalization, and a proof that their finite-place valuations realize the prime counts for every translate. The estimates for the lattice of test functions alone supply neither assertion.

### 121.16 The terminal Bessel character and its integral projector

Use a common field $E$, its real subfield $F_+ = E^+$, and the signed rank-$r$ row lattice of Section 6.6. Write $T = \varpi\bar{\varpi}$ for a terminal prime there, with $\varpi \equiv 1 \bmod 2M\mathcal{O}_{K_d}$, and $f = [F_+ : \mathbb{Q}]$. Pair $s = m/2 - 1 \geq 1$ positive and negative coordinate vectors. In the $j$-th pair take $u_j = e_{+,j} + e_{-,j}$, $v_j = (e_{+,j} - e_{-,j})/2$. Thus $h(u_i, v_j) = \delta_{ij}$, with $h$ linear in the second variable. Their orthogonal complement is $W = W_0 \perp Ee$, with $W_0$ positive of rank $k$ and $h(e, e) = -1$. Set $u = u_s,\ v = v_s$. Let $G = U(h)$, $H = U(W_0)$, the latter fixing all the displayed vectors. Let $N$ be the unipotent radical of the parabolic preserving the flag of the spans of $u_1, \dots, u_j$. It acts as the identity on every graded piece of the flag completed by its orthogonal flag, including the middle quotient $W$.

For $n \in N$, let $d_i(n)$ be the coefficient of $u_i$ in $(n-1)u_{i+1}$ for $i < s$, and let $d_s(n)$ be the coefficient of $u_s$ in $(n-1)e$. Each is additive under multiplication in $N$. Indeed the cross term $(n_1 - 1)(n_2 - 1)$ takes the relevant vector into the span of the earlier $u_j$, $j < i$ or $j < s$, respectively. The group $H$ fixes these coordinates under conjugation. Use the standard adelic additive character $\Psi_F$ of $F_+\backslash\mathbb{A}_{F_+}$, with positive exponential at infinity, and put $\Psi_E = \Psi_F \circ \mathrm{Tr}_{E/F_+}$. For $\beta \in E^\times$ define

$$\chi_\beta(n) = \Psi_E\left(\sum_{i<s} d_i(n) + \beta d_s(n)\right).$$

This is the character with last vector $\beta e$. With $[J] = J(F_+)\backslash J(\mathbb{A}_{F_+})$, its global Bessel period on a row form $F$ and a column form $\phi$ is

$$\mathcal{P}_\beta(F, \phi) = \int_{[H]} \overline{\phi(h_0)} \int_{[N]} F(nh_0)\overline{\chi_\beta(n)}\, dn\, dh_0. \tag{121.7}$$

Use quotient probability on both groups. The inner coefficient is invariant under rational left translation of $h_0$, by conjugating $N$ and using invariance of its quotient measure and character. Both quotients are compact. This follows for $N$ from its filtration with additive quotients and compactness of the additive adelic quotient, and for $H$ from definiteness, its finite genus class set and compactness at infinity. Smooth automorphic inputs therefore suffice for the integrations below. A finite right translate at the endpoint can be included in $F$. The coordinates and integrand occur in `Development/AbelianVarietyTheory/BlockTheory/HeightDefect/BesselDatum.lean`, line 63 and `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/WittSmallGroupEmbeddingMemPointwiseStabilizerIso.lean`, line 176. Complex conjugating the column input changes between the two usual pairing conventions.

**Lemma 121.20** (A translation law for the global period)**.** *For an adelic scalar $t$ define $n(t)$ on $u, e, v$ by*

$$u \longmapsto u, \qquad e \longmapsto e + tu, \qquad v \longmapsto v + \bar{t}e + (t\bar{t}/2)u.$$

*It is the identity on the orthogonal complement. For $R(a)F(y) = F(ya)$ one has*

$$\mathcal{P}_\beta(R(n(t))F, \phi) = \Psi_E(\beta t)\, \mathcal{P}_\beta(F, \phi). \tag{121.8}$$

*For rational $t$, transporting the Witt decomposition and the last vector by $n(t)$ gives the same functional on the same forms.*

*Proof.* The three images have the original Gram matrix. In particular the cross terms in $h(n(t)e, n(t)v)$ are $-\bar{t} + \bar{t}$, and $h(n(t)v, n(t)v) = t\bar{t} - t\bar{t} = 0$. The flag relations put $n(t)$ in $N$. It commutes with $H$, and its only nonzero simple coordinate is $d_s(n(t)) = t$ when $t \neq 0$. In the inner integral change the variable from $n$ to $nn(t)$. The unipotent group is unimodular, so

right translation preserves the quotient measure. The conjugate character supplies the factor in (121.8). If $t$ is rational, the transported flag has the same radical and the same embedded $H$. Its character is the original character composed with an inner automorphism of $N$, hence is identical. This proves the assertion for the full functional. In particular, with $t = Mc/\varpi$, the two last vectors $z_c$ and $z_c/\varpi$ in the transported frame give the functionals with $\beta = \varpi$ and $\beta = 1$, respectively. □

**Theorem 121.21** (Vanishing at a lattice containing the normalized vectors)**.** *Fix a place $w \mid T$ of $F_+$. The split algebra there is $E_w = \mathbb{Q}_T \times \mathbb{Q}_T$, ordered with $\varpi$ of valuation one in its first factor. Suppose a local integral Hermitian lattice $L$ contains $z_c/\varpi$ and $z_{c'}/\varpi$ for two distinct integers in $[1, T-1]$. If a row form $F$ is right invariant under the stabilizer of $L$ at $w$, then*

$$\mathcal{P}_\beta(F, \phi) \neq 0 \quad \Longrightarrow \quad \beta \in \varpi \mathcal{O}_{E_w}.$$

*In particular the functional for the normalized vector vanishes on every such form.*

*Proof.* Subtracting the two vectors gives $\varpi^{-1} u \in L$, since $M(c - c')$ is a local unit. Substitution then gives $e \in L$. For $t \in \varpi^{-1}\mathcal{O}_{E_w}$ put $a = tu \in L$. On an arbitrary vector $y$ the preceding isometry reads

$$n(t)y = y - ah(e, y) + eh(a, y) + \tfrac{1}{2} ah(a, y).$$

Since $T$ is odd and $L$ is integral, this preserves $L$, as does its inverse $n(-t)$. Use (121.8) with $t$ supported at $w$. A nonzero period forces $\Psi_{E,w}(\beta t) = 1$ for every such $t$. The standard character of $\mathbb{Q}_T$ has kernel $\mathbb{Z}_T$, and the local trace adds the two components. Testing one component at a time gives the dual of $\varpi^{-1}\mathcal{O}_{E_w}$ as $\varpi \mathcal{O}_{E_w}$. The scalar 1 is outside this ideal. The condition for $\beta = \varpi$ holds on this root subgroup. □

**Theorem 121.22** (A projector retaining the period at a smaller level)**.** *At each $w \mid T$ denote the original stabilizer by $K_w$. In the first split factor let $K'_w$ stabilize the lattice enlarged by $T^{-1}u$, with the dual lattice in the conjugate factor. Let $K'_{w,1}$ be its principal subgroup modulo $T$. For any row form fixed by all $K_w$ one can form a finite average $PF$ fixed by all $K'_{w,1}$ with*

$$\mathcal{P}_1(PF, \phi) = \mathcal{P}_1(F, \phi).$$

*This average is an orthogonal projection on the subspace invariant under the integral root translations below, in any unitary realization. Its denominator divides $T^f$ in the lattice spanned by the translates over a coefficient ring containing $\mu_T$. The denominator is exactly $T^f$ on the standard local coset test, with the product taken over $w \mid T$.*

*Proof.* In the first factor write $n_w(x) = n((x, 0))$. Its matrix is $1 + xE_{u,e}$ in the Witt basis, so these translations add. In the second factor its action is determined by duality. Modulo those with $x \in \mathbb{Z}_T$, the translations with $x \in T^{-1}\mathbb{Z}_T$ form a group of order $T$. Take the product $C$ of these quotients at all $f$ places. They act on the subspace fixed by the integral translations. Write $R(b)$ for their action, and $\chi(b)$ for the character induced by $\chi_1$. Define

$$PF = T^{-f} \sum_{b \in C} \overline{\chi(b)} R(b) F.$$

Equation (121.8) gives the asserted period. Reindexing the sum gives $R(a)P = \chi(a)P$, and the average is the identity on this character space. In a unitary representation $R(a)^* = R(-a)$, so $P = P^*$.

At $w$, write the original and enlarged first-factor lattices as $L_0 \subset L_1$. They satisfy $TL_1 \subset L_0$. Consequently any element acting trivially on $L_1/TL_1$ preserves $L_0$, as does its inverse. Thus

$K'_{w,1} \subset K_w$. All translations in the average belong to $K'_w$, which normalizes $K'_{w,1}$. Each translated vector is therefore fixed by $K'_{w,1}$. Their span over the stated coefficient ring contains $T^f PF$ whenever it contains the translates of $F$. For sharpness take the characteristic function of the product of the $K_w$ in the product of the local groups, with the lattice of integral-valued compactly supported functions. Two distinct translates in the sum have disjoint supports. Indeed their difference in some first split factor has a nonintegral $(u, e)$-entry. Each nonzero value of the projected test is a root of unity divided by $T^f$. This proves the exact denominator there. □

These formulas apply to the global Bessel integral at the specified level. The smaller-level average retains any nonzero value it starts with. To use it for the integral theta lift requires comparing its translate lattice to the prescribed arithmetic lattice of row forms.

## 121.17 Uniform restriction to the Bessel orbit

In the notation of Section 121.16, let $G_\infty = \prod_{\tau: F_+ \hookrightarrow \mathbb{R}} U(p_+, p_-)$, $K_\infty = \prod_\tau (U(p_+) \times U(p_-))$, and $K_\Lambda = \mathrm{Stab}(\widehat{\Lambda})$. For an open subgroup $K_f \subset K_\Lambda$ write $I = [K_\Lambda : K_f]$. Use the compact-dual Haar density at infinity and give $K_\Lambda$ finite volume one, as in Theorem 94.9. The global $L^2$ norm below uses the resulting quotient measure on $[G]$. Put

$$n_0 = 2fr, \qquad d_0 = fr^2, \qquad B = (8n_0)^{2n_0} |D_E|^{r/2},$$

where $D_E$ is the discriminant of $E$. The positive Euclidean norm on the sum of the archimedean coordinate spaces is the sum of the squared coordinate moduli, followed by the square root. Use the Frobenius inner product on each real Lie algebra, the orthogonal sum on their product, and an orthonormal basis $X_1, \dots, X_{d_0}$ taken factor by factor. For a word $w$ in this basis, $D_w$ denotes successive differentiations of the right regular representation, including the identity for the empty word.

**Lemma 121.23** (Integral lattices along the period orbit)**.** *Let $g = nh_0$ with $n \in N(\mathbb{A}_{F_+})$ and $h_0 \in H(\mathbb{A}_{F_+})$. In the real coordinate space put*

$$\mathcal{L}_g = g_\infty^{-1}(E^r \cap g_f \widehat{\Lambda})$$

*using the diagonal archimedean embedding. This lattice has first successive minimum at least $1/4$ and contains $n_0$ independent vectors of norm at most $B$. Consequently, for $\gamma \in G(F_+) \cap g_f K_\Lambda g_f^{-1}$,*

$$\|g_\infty^{-1} \gamma g_\infty - 1\|_{\mathrm{op}} < 1/(4B) \quad \Longrightarrow \quad \gamma = 1.$$

*Proof.* Order the graded pieces of a rational flag as

$$Eu_1, \dots, Eu_s, \quad W_0, \quad Ee, \quad Ev_s, \dots, Ev_1.$$

Each partial sum is stable under $NH$. On its quotient by the preceding sum the action of $N$ is the identity. The action of $H$ is the identity except on $W_0$, where it preserves the positive Hermitian form. These graded summands are orthogonal for the positive archimedean metric. Given a nonzero rational vector in $E^r \cap g_f \widehat{\Lambda}$, pass to the first partial sum containing it. Its nonzero image on that quotient has the following local integrality. On a coordinate line, its coefficient belongs to $\frac{1}{2}\mathcal{O}_E$. Indeed the coordinates in $u_j, v_j$ of a pair with original coordinates $a, b$ are $(a+b)/2, a-b$, and the nonscalar finite action on $W_0$ does not enter these line quotients. On $W_0$ the image belongs to $h_{0,f} \widehat{\mathcal{O}}_E^k$ at the finite places. Its Hermitian norm is then integral at every finite place and totally positive. In the line case the squared moduli of the conjugates

of twice the coefficient have product at least one, by the absolute norm of a nonzero algebraic integer. In the $W_0$ case the archimedean values of the Hermitian norm also have product at least one. The arithmetic–geometric mean inequality in either case bounds their sum below by $f$. The lengths of $u_j, v_j$ are $\sqrt{2}, 1/\sqrt{2}$. The graded component of $g_\infty^{-1}$ times the vector thus has squared norm at least $f/8$. The other components only increase its positive norm.

The covolume of $\mathcal{L}_g$ is $2^{-fr}|D_E|^{r/2}$, that of $\mathcal{O}_E^r$ in its complex embeddings. At every rational prime the finite isometry preserves the nondegenerate rational trace form of $h$, so its determinant has absolute value one. Hence the two finite lattices have equal volume. Passing to their common intersection shows that their global indices there are equal. The archimedean isometry has real determinant of absolute value one as well. This proves the covolume assertion. Minkowski's second theorem applied to the unit ball bounds the product of the successive minima by $n_0^{n_0/2}|D_E|^{r/2}$, using the inscribed cube of side $2/\sqrt{n_0}$. [3, Thm. 6.3.3] The first minimum bound shows that twice the last minimum is at most $B$. Finally $g_\infty^{-1}(\gamma - 1)g_\infty$ maps $\mathcal{L}_g$ to itself. Under the displayed hypothesis it takes each of those $n_0$ independent vectors to a vector of norm below $1/4$. Each image is zero, proving $\gamma = 1$. □

**Theorem 121.24** (A uniform bound for restriction)**.** *For an absolute constant $C$, every smooth right-$K_f$-invariant automorphic form $F$ with square-integrable derivatives through order $d_0$ satisfies*

$$\sup_{n,h_0} |F(nh_0)| \;\le\; I^{1/2} \exp\{C d_0 \log(2 d_0 B)\} \max_{|w| \le d_0} \|D_w F\|_{L^2([G])}. \tag{121.9}$$

*The supremum uses the full adelic groups of the preceding lemma.*

*Proof.* Write $d = d_0$ and $\delta = (1000 d^2 B)^{-1}$. Use the chart

$$\Phi(t) = \exp(t_1 X_1) \cdots \exp(t_d X_d), \qquad |t_i| \le \delta.$$

For any partial product $h$, the operator norms of $h$ and $h^{-1}$ are at most $e^{d\delta}$ in the block diagonal matrix realization. Expanding the product, or telescoping the exponentials, gives $\|h - 1\|_{\mathrm{op}} \le e^{d\delta} - 1$. The derivative of $\Phi$ in its $j$-th variable is a product of $X_j$ with such factors on either side. It differs from $X_j$ in Frobenius norm by at most $4d\delta$. Summing the columns by Cauchy–Schwarz, the derivative differs from the linear isometric inclusion at zero in operator norm by at most $4d^{3/2}\delta < 1/2$. Integration on a segment shows that $\Phi$ is one-to-one on the cube. The left-translated tangent columns are $\mathrm{Ad}(h_{>j}^{-1})X_j$, where $h_{>j}$ is the product after $j$. They have the same error bound from $X_j$, by the two operator norm estimates. Their volume Jacobian for the left-invariant Frobenius density is at least $2^{-d}$.

The compact-dual density differs from this density by the inverse Frobenius volume of $U(r)^f$. This compact volume is at most $\exp(Cd)$. Indeed the first-column map from $U(j)$ to the unit sphere in $\mathbb{C}^j$ has fiber $U(j-1)$. In the horizontal directions $(ia, z)$ the Frobenius squared norm is $a^2 + 2\|z\|^2$, and the sphere norm is $a^2 + \|z\|^2$. The quotient density is therefore $2^{j-1}$ times sphere volume. The sphere has volume $2\pi^j/(j-1)!$, giving the volume ratio $(2\pi)^j/(j-1)!$. Multiplication over $j, \tau$ proves the bound. Thus the Haar Jacobian of the cube is at least $\exp(-Cd)$.

For every $g = nh_0$, the image of $g(\Phi(t), k_f)$ in $[G]$, as $k_f$ varies in $K_f$, has no rational identifications. An identification would give $\gamma$ as in Lemma 121.23, with $g_\infty^{-1}\gamma g_\infty = \Phi(t')\Phi(t)^{-1}$. The distance in operator norm to the identity is at most $e^{2d\delta} - 1 < 1/(4B)$. The lemma forces $\gamma = 1$, and then the two parameters agree.

For a smooth function $b$ on the real cube one has

$$|b(0)| \le \sum_{J \subset \{1,\dots,d\}} (2\delta)^{|J|-d} \int_{[-\delta,\delta]^d} |\partial_J b(t)|\, dt.$$

In one variable, averaging $b(0) = b(t) - \int_0^t b'(s)\,ds$ gives the derivative an integral kernel of absolute value at most one. Applying this linear formula successively in the $d$ variables and then taking absolute values gives the display. Apply it to $b(t) = F(g\Phi(t))$. In a mixed derivative take the variables in decreasing order of index. Each differentiation inserts the right Lie derivative in direction $\mathrm{Ad}(h_{>j}^{-1})X_j$. Directions already inserted depend only on later variables, so their coefficients are constant in this differentiation. Each direction has Frobenius norm at most two. Expansion in the fixed basis thus gives at most $d^d$ words with coefficients of magnitude at most $2^d$.

For any one word, Cauchy–Schwarz on the cube, the lower Haar Jacobian and the injective chart bound its integral in absolute value by $(2\delta)^{d/2}e^{Cd}I^{1/2}\|D_wF\|_2$. Indeed the finite factor of the chart has measure $1/I$, and archimedean differentiation commutes with $K_f$. Summing the terms costs at most an additional $\exp(Cd\log(2d/\delta))$. This gives (121.9) with the claimed dependence. □

**Lemma 121.25** (Iterated derivatives of a fixed compact type)**.** *At every real place use the invariant form* $\mathrm{Re}\,\mathrm{tr}(XY)$ *on* $\mathfrak{u}(p_+,p_-)$*. Let* $F$ *be a smooth vector of a unitary representation, smooth in Hilbert norm in all orders, on which the Casimir for this form at* $\tau$ *acts by the scalar* $\omega_\tau$ *on the representation generated by* $F$*. The* $K_\infty$*-span of* $F$ *is assumed finite dimensional. Let* $L \geq 1$ *bound* $\sqrt{|\omega_\tau|}$ *and the operator norms on this span of* $dR(Z)$ *for all compact Lie directions of Frobenius norm one at every place. A word of* $j$ *real derivatives, each at one place and of Frobenius norm at most two, satisfies*

$$\|D_wF\|_2 \leq \{Cr(L+j+1)\}^j\|F\|_2. \tag{121.10}$$

*Proof.* The span of words of length at most $l$ on the initial compact span is finite dimensional and compact-group stable. It is an equivariant image of the sum of the tensor powers through $l$ of the complexified Lie algebra tensored with the initial span. Use the adjoint action on the Lie algebra. For compact unit $Z$, its differential $Y \mapsto [Z,Y]$ has operator norm at most two for the Hilbert–Schmidt metric. On the tensor representation the bound is therefore $L+2l$. The same holds on its image with the given unitary metric. Indeed the polar part of an equivariant linear map of finite-dimensional unitary representations gives a unitary identification of the orthogonal complement of its kernel with its image.

At one place take Frobenius orthonormal bases for the compact and noncompact parts $\mathfrak{k},\mathfrak{p}$. The invariant form is minus the Frobenius form on the former and equals it on the latter. The Casimir is $\sum_{\mathfrak{p}} X^2 - \sum_{\mathfrak{k}} X^2$. Skew adjointness of the derived operators on smooth vectors gives for any vector $v$ in the preceding span

$$\sum_{\mathfrak{k}\cup\mathfrak{p}} \|dR(X)v\|^2 = 2\sum_{\mathfrak{k}} \|dR(X)v\|^2 - \omega_\tau\|v\|^2 \leq \{2r^2(L+2l)^2 + |\omega_\tau|\}\|v\|^2.$$

Here the sums index the indicated bases. By Cauchy–Schwarz this bounds a single derivative in any real direction of norm at most two by $Cr(L+2l+1)\|v\|$. Induct on $l$, enlarging the absolute constant, to obtain (121.10). □

**Theorem 121.26** (Decay of the unscaled automorphic integral)**.** *Suppose* $F$ *is a square-integrable smooth automorphic vector satisfying the level and compact-type assumptions above. For any nonnegative orders of total* $A$*, the last vector* $\varpi e$ *of norm* $-T$ *satisfies*

$$|\mathcal{P}_\varpi(F,\phi)| \leq I^{1/2}\exp\{Cd_0\log(2d_0B)\}\frac{\{Cr(L+A+d_0+1)\}^{A+d_0}}{(4\pi\sqrt{T})^A} \cdot \|F\|_{L^2([G])}\,\|\phi\|_{L^2([H])}. \tag{121.11}$$

*The same absolute constant may be enlarged in both occurrences.*

*Proof.* Apply Proposition 93.16 and bound the restriction of $D_{\varpi}F$ by Theorem 121.24. In the positive orthonormal basis in the directions $u_s, e, v_s$, the generator of $n_\tau(b\eta_\tau)$, $|\eta_\tau| = 1$, has two entries of absolute value $\sqrt{2}$ and all others zero. Its Frobenius norm is two. Each word in the restriction bound therefore satisfies Lemma 121.25 with length at most $A + d_0$. Both period measures have total mass one. These observations give (121.11). □

# 122 Integral tests, both coefficient lattices and their minors

## 122.1 Testing a definite endpoint at its original level

Use the definite endpoint of the complex iteration in Proposition 110.5. Write $N$ for its rank, $K_e$ for its finite level, and $\chi_e$ for the finite right level character, with the trivial character allowed. Use its integral Hermitian lattice model. Let $\delta_e$ be the determinant of the model lattice, which is the same in every class since the transports are finite isometries. For the endpoint real type $\sigma_e$ of dimension $d_e$, put $e_e, a_e$ equal to the sums of its polynomial degrees and its degrees with determinant factors counted as in Proposition 112.3, now using $N$. Put $L_e = \prod_\tau e_\tau!$ for these endpoint polynomial degrees and set

$$D_e = 2fN, \quad R_e = 2D_e^{D_e/2}|\Delta_E|^{N/2}\delta_e, \qquad B_e = (\sqrt{N}R_e)^{Nf}.$$

Let $K_e'$ be the intersection of $\ker \chi_e$ in $K_e$ with the kernel of reduction at one factor above the odd split prime $g_1$ in its lattice model. Set $m_e = [K_e : K_e']$, let $h_e$ be the number of fine classes at $K_e'$, and let $w_e$ be the measure of one such chart in the Petersson measure, using probability on the real compact group within the chart.

**Lemma 122.1** (An integral endpoint test with the original character)**.** *For every endpoint form $F$ of the indicated real type and original finite level character, some integral coefficient test $Q_e$ of the same level and character satisfies*

$$|\langle F, Q_e\rangle|^2 \geq \frac{m_e^2 w_e B_e^{-4a_e}}{d_e h_e N^{2e_e}} \|F\|^2, \qquad \|Q_e\| \leq m_e\sqrt{w_e}\, L_e B_e^{2a_e}.$$

*Proof.* The subgroup $K_e'$ is normal in $K_e$. A rational stabilizer at this fine level is finite by positive definiteness and preserves its class lattice. At the chosen split place it lies in a conjugate of $1 + g_1 M_N(\mathbb{Z}_{g_1})$, which is torsion-free by the binomial valuation argument of Theorem 109.15. Thus its stabilizer is trivial. All fine charts have the same measure $w_e$. On each one write $F(g) = \operatorname{tr}(M_c\sigma_e(g))$ for the real compact argument $g$. This representation of the isotypic space by matrices is the compact group orthogonality theorem. It gives

$$\|F\|^2 = (w_e/d_e)\sum_c \|M_c\|_{\mathrm{HS}}^2.$$

At each fine class apply the trace lattice argument of Proposition 112.3 with rank $N$ and determinant $\delta_e$. It gives an integral spanning frame in each embedding and integral dual functionals with singular values between $B_e^{-1}$ and $B_e$. The two Schur lists of Proposition 75.7 on this class have length $N^{e_e}$ each. Their lower frame bounds are $B_e^{-2a_e}$, and the norms of individual vectors and covectors are at most $L_e B_e^{a_e}$ and $B_e^{a_e}$, respectively. These conclusions use the determinant factors in that proposition and apply to each transported lattice.

Let $q_{cab}$ be supported on class $c$ with matrix coefficient $\ell_a(\sigma_e(g)v_b)$ there. Compact orthogonality makes the squared pairing with $F$ equal to $w_e^2/d_e^2$ times the squared matrix entry of $M_c$ against the two frame members, with adjoints according to the inner product convention. Summing over both lists gives at least $(w_e/d_e)^2 B_e^{-4a_e}\|M_c\|_{\mathrm{HS}}^2$ by applying the two lower frame

bounds in succession. The sum over classes is at least $w_e B_e^{-4a_e}\|F\|^2/d_e$. One of the $h_e N^{2e_e}$ terms attains at least its average. Its test has norm at most $\sqrt{w_e} L_e B_e^{2a_e}$.

Apply to this test the unnormalized character sum $\sum_{u\in K_e/K_e'} \overline{\chi_e(u)} R(u)$ in the unitary right action. This operator is $m_e$ times the orthogonal projector and acts on $F$ as $m_e$. Its output $Q_e$ has the desired pairing and norm. The sum is independent of representatives because the test has level $K_e'$. Translation by $K_e$ transports the class lattices isometrically, and in translated representatives it preserves the integral coefficient model. The scalar character values are roots of unity. The sum therefore gives an integral test at the original level over the coefficient ring containing these values. □

## 122.2 Integral tests on the compact highest lines

At the last definite pair of Theorem 67.11, let $R$ denote either rank. At embedding $\tau$ write $e_{j,\tau} = \lambda_{\tau,j} - \lambda_{\tau,j+1}$, padding the partition with zeros. Put $l_\tau = |\lambda_\tau|$ and $d = \sum_\tau l_\tau$. Use the coordinate exterior tensor product of Proposition 95.2 and its unit highest vector $t_R$. The span of the compact orbit of $t_R$ is $V_R$, of dimension $D_R$. For $f$ square matrices $Z_\tau$ of order $R$ put

$$v_R(Z) = \bigotimes_{\tau,j} (Z_\tau e_1 \wedge \cdots \wedge Z_\tau e_j)^{\otimes e_{j,\tau}}.$$

This is a polynomial map with integer coefficients into $V_R$. Indeed the compact-invariant subspace $V_R$ is invariant under the complexified Lie algebra and hence under the connected group $\prod_\tau \mathrm{GL}_R(\mathbb{C})$. Its closedness gives the assertion at singular matrices as well. Determinant characters from the oscillator splitting are included in the compact representation on class functions. The polynomial above uses its partition factor alone.

**Theorem 122.2** (Two finite families of highest-type tests)**.** *There is a finite family of vectors $v = v_R(Z)$, with a probability distribution, such that for every $y \in V_R$*

$$\mathbb{E}|\langle y, v\rangle|^2 \geq \frac{\|y\|^2}{4D_R}.$$

*One family has algebraic integer coordinates in the exterior tensor lattice and $\|\sigma v\| \leq R^{d/2}$ at every complex embedding $\sigma$. A second family has unit norm at every embedding and coordinates integral away from two. In the latter family $2^{dR(R-1)/2} v$ has algebraic integer coordinates. In either construction the logarithm of the number of tests is*

$$O\big(fR^2 \log(2 + R + d + D_R)\big).$$

*The probabilities enter only in the proof of the bound.*

*Proof.* For independent Haar probability matrices $U_\tau \in U(R)$ the covariance of $v_R(U)$ is $I/D_R$. It commutes with the product unitary group, its image is in the irreducible $V_R$, and its trace is one. In degree zero the singleton $t_R$ suffices.

For the integral construction let $b = 1/[2(d+1)]$ and $r = 1 - b$. Choose $K = \max(d, \lceil b^{-1}\log(8/b^2)\rceil)$ and $M = 2K + 1$. For any $|w| \leq r$ the real trigonometric polynomial

$$p_w(z) = 1 + 2\,\mathrm{Re}\sum_{j=1}^{K} w^j \bar z^{\,j}, \qquad |z| = 1,$$

is positive. The infinite sum with its constant term is $(1 - |w|^2)/|1 - w\bar z|^2 \geq b/(2-b)$, and its omitted tail has absolute value at most $2r^{K+1}/b \leq b/4$. At the $M$-th roots of unity the values $p_w(z)/M$ sum to one. Orthogonality of these roots gives $\sum_{z^M=1} z^a p_w(z)/M = w^a$ for

$0 \le a \le d$. Given $U$, sample each entry of each $Z_\tau$ independently from this distribution with $w = r(U_\tau)_{ij}$. The conditional mean of every monomial is its value at $rU$. Thus the conditional mean of $v_R(Z)$ is $r^d v_R(U)$. Jensen's inequality and $r^d \ge 1 - db \ge 1/2$ prove the bound. Every conjugate matrix has columns of norm $\sqrt{R}$. The exterior product of $j$ columns has norm at most $R^{j/2}$ by the Gram determinant inequality. Multiplication over the slots gives the stated bound. There are at most $M^{fR^2}$ choices.

For the unitary construction, elimination of the entries below the diagonal factors any $U \in U(R)$ into at most $R(R-1)/2$ real rotations on two coordinates, diagonal phases, and a final unitary diagonal. In each elimination multiply the two affected coordinates by phases to make the two entries real and then rotate one to zero. An upper triangular unitary is diagonal by orthogonality of its columns. Reversing the eliminations uses at most $2R^2$ one-parameter factors, including the diagonal factors. Round each angle to a multiple of $2\pi/M'$, where $M' \ge 4\pi R^2(1+d)\sqrt{D_R}$ is the least such integer. The change of a single factor in operator norm is at most $\pi/M'$. Telescoping a product of unitaries bounds the error in one matrix by $2\pi R^2/M'$. In a $j$-fold exterior power the error is at most $j$ times as large, by restriction of the $j$-fold tensor power. Telescoping again bounds the change of the unit vector by $2\pi R^2 d/M' \le 1/(2\sqrt{D_R})$. The finite set of rounded products is therefore close to every point of the orbit by this amount. Send a Haar orbit vector to a closest vector in this finite set, resolving ties by a fixed ordering. This map is measurable. The triangle inequality in $L^2$ bounds the change in the root mean square pairing by $\|y\|/(2\sqrt{D_R})$, giving the assertion.

The sine and cosine of a rounded angle lie in $\mathbb{Z}[\mu_{4M'}, 1/2]$, and twice each is integral. Complex embeddings carry a rotation to a real orthogonal rotation and a phase to a phase. Thus all conjugate products are unitary. Clearing one factor of two per real rotation clears a matrix, and raising this denominator to the tensor degree clears the vector. The elimination order is fixed and there are at most $(M')^{2fR^2}$ lists of angles. This proves the size bound. □

## 122.3 Changing the product basis in a marked period

At an even rank-product edge keep the physical matrix coordinates, the discriminant-one elliptic differential, the finite unit vacuum $\phi_0$ on the standard trace module, and the Schrödinger–Fock identification of Theorem 67.8. Use two $K$-orthonormal bases $B, B'$ of the trace Hermitian space to construct the normalized CM sections $G_\phi^B, G_\phi^{B'}$ by Lemma 65.8. The finite Schwartz vector $\phi$ is the same vector in the fixed finite oscillator representation.

**Theorem 122.3** (The common scalar for all finite tests)**.** *There is one algebraic scalar $u(B', B)$ with*

$$G_\phi^{B'} = u(B', B) G_\phi^B \quad \textit{for every finite test,} \qquad |\sigma u(B', B)| = 1$$

*at every complex embedding. For a family of arithmetic period entries obtained from the same directed definite column, changing these bases on its edges multiplies all entries by the same algebraic scalar $u$ of absolute value one at every embedding. Every product $\mathcal{A}_a \overline{\mathcal{A}_b}$ of two such entries is identical in the two constructions. For a nonzero entry the normalized rational norm valuation*

$$\frac{\operatorname{ord}_p |N_{J/\mathbb{Q}} \mathcal{A}_a|}{[J : \mathbb{Q}]}$$

*is independent of the bases at every rational prime $p$.*

*Proof.* Write $F_\phi(z)$ for the theta functional on the coherent vector in the proof of Theorem 67.8. The physical coordinates and the finite vector fix this function. The theorem gives $F_\phi = C_B G_\phi^B = C_{B'} G_\phi^{B'}$, with both constants nonzero, for every test. Thus their ratio is common to

all vectors and all differential orders. For $\phi_0$ the two normalized sections are nonzero, so some ordinary derivative of each at the origin in the same multi-index is nonzero. These derivatives are algebraic by Theorem 65.9. Their ratio proves algebraicity of $u(B', B)$.

At an embedding the identity between the two vacuum germs persists coefficientwise, hence for the entire functions in physical coordinates. Their squared metric norms on the conjugate trace torus are both $\nu_\sigma^D$, by Theorem 65.9. To check that the tori in that calculation coincide, let $V$ have as columns the embedded vectors of $B$ and write $B' = BU$ with $U \in U(D, K)$. The matrix changes from $V$ to $VU$, while the trace module in product coordinates changes from $M_B$ to $U^{-1}M_B$. Applying $\sigma$ to both changes leaves their product fixed, also after the common elliptic ideal and period are inserted. Moreover $\sigma(U)$ is unitary since conjugation on $K$ commutes with either embedding of $K$. The metric and Haar probability are therefore the same. Comparison of the two norms gives $|\sigma u(B', B)| = 1$.

Each summand in the arithmetic column of Theorem 121.3 contains one derivative at each edge, with the fixed orientation of that edge. Its basis change multiplies every term by the product of the edge scalars, with conjugation at any conjugated edges. This gives a common $u$ also after restriction, compact averaging and the finite signed sum. Since $u\bar{u} = 1$ at the original embedding, this is an algebraic identity. The Gram products are unchanged. In a common number field $|N_{J/\mathbb{Q}}u| = 1$ by the equality at all embeddings, which proves the norm assertion. Passing to a larger coefficient field preserves the normalized valuation by transitivity of norms. □

The source gives a determinant formula up to a root of unity for the square of this scalar in `Formalism/SharpRestriction/GlobalSchurBounds/StandardLatticesGL3.lean`, line 646.

## 122.4 Finite tests before the choice of class representatives

Fix an edge of the definite column in Theorem 121.3. Its rational matrix space is $W$, its reference lattice is $M = \mathfrak{b}^{-1}\,\mathrm{Mat}_{s\times r}(\mathcal{O}_E)$, and its reference finite vacuum is $\phi_0$. Write every prescribed finite test, before evaluating it at a pair of classes, in the form

$$\phi^0 = \sum_b c_b W(b)\phi_0, \qquad b \in W/M.$$

Include the finitely many tests obtained by moving the right translations of a finite character sum onto the kernel. Choose integers $T^0, Q^0 \geq 1$ such that every $T^0 c_b$ is an algebraic integer and $Q^0 b = 0$ in $W/M$ for every term. The integers exist by the finite vacuum expansion of Theorem 67.8. External scalar weights may be carried outside this expansion throughout.

**Theorem 122.4** (Class representatives at a specified rational prime)**.** *At every rational prime $p$, all integrals for this column can be evaluated on representatives equal to the identity over $p$, with their original class weights and levels. On each edge, after the class actions, the section has a vacuum expansion whose coefficients have valuations at least $-v_p(T^0)$ at every place over $p$. Modulo $M \otimes \mathbb{Z}_p$, each of its shifts agrees with the $p$-component of an original $b$. This also holds in the product lattice coordinates of Theorem 122.14 at $p$.*

*Proof.* At a vertex with original integration subgroup $J_v$ and representative $u$, choose an open compact $J_v' \subset J_v$ in the stabilizer of the standard lattice, fixing the finite tests and contained in the kernels of the finite level characters. Apply Theorem 65.11 to the class of this particular $u$ at level $J_v'$. We obtain $u' = \gamma u j$ with $\gamma$ rational unitary, $j \in J_v'$, and $u'$ supported at completely split rational primes outside $p$ and outside all marked and nonspherical places. Do this once per original class, at each vertex. These representatives give precisely the original list modulo

$J_v$. Since $j \in J_v$, the stabilizer at $u'$ for $J_v$ is the conjugate by $\gamma$ of the stabilizer at $u$. The embedded endpoint representative is $u' \oplus 1$, with rational multiplier $\gamma \oplus 1$.

For an automorphic integrand the class-coordinate substitution is $k \mapsto \gamma_\infty k$. Apply it simultaneously to the incident factors at each vertex. The right level fixes the integrand and the substitution preserves compact Haar measure. This proves equality for each original class contribution to the period, including the endpoint pairing and the signed sum. The volume factor and the scalars $C_i$ of Theorem 67.8 are independent of the representatives. In a fixed marked basis the normalized scalar $\mathcal{A}$ is therefore the same.

For an edge, the pair of representatives acts by the identity at every place over $p$. At a moving split place over $q \neq p$, the split change of variables carries a Weyl translate to a translate of the moved vacuum. The elementary lattice intertwiner in the proof of Theorem 65.13 expands that vacuum in translates of the reference vacuum. Its multipliers are roots of unity times the inverse square root of a $q$-power index, apart from the splitting character. Each is a unit over $p$. For the character assertion, take a power of the finite idele at the moving places which is principal times a ray unit in the kernel, as in that proof. The principal generator is a unit at every place over $p$. The algebraic archimedean character, after squaring if necessary, is a product of integral powers of conjugates of this generator and their inverses. Every such factor is a unit over $p$, so the original value is also a unit there. The argument for a norm-one character uses its norm-one principal generator in exactly this calculation.

At $p$ the finite shift has retained its component $b_p$, and all shifts introduced at the moving places have zero class in $W_p/M_p$, where $M_p = M \otimes \mathbb{Z}_p$. Passing to a rational representative preserves these local classes. Indeed the map $W/M \to W(\mathbb{A}_f)/\widehat{M}$ is bijective by the Chinese remainder theorem after clearing a common integer denominator. The phases incurred by changing the representative are roots of unity. Multiplication by these units and summation preserve the claimed valuation bound. Finally the product basis of Theorem 122.14 generates $M_p$. Its vacuum intertwiner has index prime to $p$ and adds only shifts integral at $p$. Its $K$-linear change of coordinates identifies the two local quotient modules. This proves the last assertion. □

## 122.5 A resolution of the identity by definite class tests

Use the definite fine charts of Theorem 121.10. Let $h$ be their number and $w$ the volume of each chart. On a side of rank $R$, let $A_R$ be the tensor product of the prescribed exterior powers at the real places, with the determinant lines and their indicated duals. Use the usual tensor Hermitian metric. Its coordinate wedge tensors are orthonormal. Fix the product $t$ of the standard highest tensors of norm one. Its compact cyclic span $V_R$ has dimension $d_R$. The entire highest-line space at the fine level consists on each class of the functions $g \mapsto \langle \rho_R(g)t, v\rangle$. Take conjugate Hilbert spaces where necessary to make the test maps linear throughout. Let $i = [K : J]$, and impose the original level character by the unnormalized right character sum of Theorem 121.10.

From $\mathcal{A}_R = \bigoplus_{a=1}^{h} A_R$ to this original-character space define $S_R$ by making the function on its own class for each summand, adding the functions, and applying the character sum. Put $s_R = i^2 w/d_R$. For a signed definite operator $T_u$ of Theorem 123.2, take such a full source and a fixed full target of this kind. Use its actual scalar $\kappa_u$. Let the matrix of all normalized periods on the coordinate wedge tests be

$$Z_u = \kappa_u^{-1} S_0^{\dagger} T_u S_u.$$

Here the index 0 designates the same endpoint tests for every word. These are finite matrices in orthonormal ambient coordinates. Their entries include the prescribed signs.

**Theorem 122.5** (Physical normalization of closed products)**.** *One has $S_R S_R^{\dagger} = s_R I$ on the entire stated character space. Hence for every ordered list $u_1, \dots, u_k$ the algebraic number*

$$C(u_1, \dots, u_k) = \operatorname{tr}\Big(\prod_j Z_{u_j} Z_{u_j}^{*}\Big) = \left(\prod_j \frac{s_0 s_{u_j}}{|\kappa_{u_j}|^2}\right) \operatorname{tr}\Big(\prod_j T_{u_j} T_{u_j}^{\dagger}\Big)$$

*has, after multiplication by $\prod_j D_{u_j}^2$, valuation at least $\sum_j b_{u_j,p}$ at every $\mathfrak{p} \mid p$. The integer $D_u$ is the compact and coefficient clearing of Theorem 126.1 for the class tests and their prescribed finite translates, and $b_{u,p}$ includes the split improvement of Theorem 126.2. All constants on the second line are the chart masses, type dimensions, level indices and period normalizations.*

*Proof.* On one chart, Schur orthogonality gives the squared Petersson norm $w\|P_R v\|^2/d_R$, where $P_R$ is the compact orthogonal projector onto $V_R$. Every vector of the fine highest-line space occurs this way. Thus the single-chart map composed with its adjoint is $w/d_R$ times its chart identity. Distinct charts have orthogonal supports. The finite character sum is $i$ times an orthogonal projector, so composing the direct sum with that character sum and its adjoint gives $i^2 w/d_R$ times the identity on the projected space. In particular $S_0^{\dagger}/\sqrt{s_0}$ is an isometric embedding. Multiplication of the matrices $Z_u Z_u^{*}$ now gives the second line of the formula, including every scalar.

The coordinate tests have algebraic class functions, so their normalized signed periods are algebraic by Theorem 121.3. Fix $p$. Choose all the chart representatives neutral at $p$ as in Theorem 122.4, using one common refinement for the finite list and the same endpoint chart choices. For a coordinate wedge at such a class the left covector is integral on the localized class lattice, which is standard there. Its determinant duals have the same property. On the right the highest vector is integral. Hence the coefficient clearing of Theorem 122.15 applies to these tests, including the finite level translates. No projection of the individual wedge onto $V_R$ is needed in this integrality argument, since the compact integral itself is still present with its specified clearing. Theorems 126.1 and 126.2 bound each product of a matrix entry and a conjugate entry for its word. Expanding the trace proves the asserted bound. For another prime the chart tests may change. The second line of the formula proves equality of the resulting traces in the original complex embedding and hence equality of the algebraic numbers. Thus the local estimates apply to one scalar on the given field. □

## 122.6 Integral duality for the terminal polynomial tests

At one embedding subtract the last entry of the terminal highest weight to obtain a partition $\lambda$ of $d$ with at most $n$ rows. Use a generator for the separate determinant line in the chosen frame. Write $A$ for the valuation ring at a finite place of the coefficient field, with fraction field $K$. On $n$-by-$n$ matrices the test space $V$ over $K$ has column multidegree $\lambda$ and is fixed by additions of earlier columns to later ones. Write $L_0$ for the $A$-span of products of flag minors, one for each column of the Young diagram, using the first $\lambda'_j$ matrix columns for the diagram column of height $\lambda'_j$. All row choices are allowed. Write $L_1 = V \cap A[X]$. The characteristic zero Cauchy decomposition identifies this highest-weight space with one Schur module for the matrix rows. The flag minor products span it since their span is stable under row changes by the minors formula and contains the product of leading minors. This also treats $d = 0$ with empty product one.

For $P = \sum P_\alpha X^\alpha$, $Q = \sum Q_\alpha X^\alpha$ use the bilinear form

$$B(P,Q) = H^{-1} \sum_\alpha \alpha!\, P_\alpha Q_\alpha, \qquad H = \sum_\alpha \alpha!\, (P_{\lambda,\alpha})^2,$$

where $P_\lambda$ is the product of leading minors. Thus $H$ is the positive integral hook product of Lemma 92.4. For a lattice $L$ in $V$ set $L^\# = \{v \in V : B(v, L) \subset A\}$.

**Theorem 122.6** (Detection by integral flag minors)**.** *For every $L_0 \subseteq L \subseteq L_1$ one has*

$$HL \;\subseteq\; L^\# \;\subseteq\; L_0.$$

*In particular, if every pairing of $w \in V$ with the flag minor products belongs to a fractional ideal $\mathfrak{a}$ of $A$, then $w \in \mathfrak{a}L_0$. At primes not dividing $H$ the three lattices $L_0, L, L_1$ agree and the pairing is perfect on them. The same conclusions hold for the tensor of the terminal weights over all embeddings, with $H = \prod_\tau H_\tau$ and products of the indicated tests.*

*Proof.* Let $T = (A^n)^{\otimes d}$, with the tensor basis orthonormal for the bilinear dot product. Index its places by the boxes of the diagram. In diagram row $i$ send a basis vector $e_k$ to $X_{ki}$ and multiply to obtain a polynomial. Over $K$ the inverse to this multiplication on row-symmetric tensors is symmetrization within the rows. Indeed the distinct monomials correspond to disjoint row orbits of basis tensors. Put $R = \prod_i \mathfrak{S}_{\lambda_i}$, $C = \prod_j \mathfrak{S}_{\lambda'_j}$, $a = \sum_{\sigma \in R} \sigma$, $b = a/|R|$ and $c = \sum_{\sigma \in C} \operatorname{sgn}(\sigma)\sigma$. In the symmetric realization $L_0 = bcT$, by expansion of each determinant. For a monomial $X^\alpha$ its symmetrized tensor has squared norm $\alpha!/|R|$ by counting its row orbit. Distinct monomials are orthogonal. Thus $B$ on $V$ is $|R|/H$ times the tensor dot product.

We record the scalar by which $bc$ acts on $V$. For every permutation $x$ of the diagram boxes, $axc$ is either zero or a signed copy of $ac$. If a row contains two boxes of a single moved column, their transposition fixes the row sum and negates the moved column sum. If each intersection contains at most one box, the first moved column meets every nonempty row. After deleting it, the next meets every row still having room, since its height is exactly the number of those rows. Continuing, a single permutation within the rows carries all moved columns to their original column sets. Thus $x \in RC$, proving the claim. Expanding the middle two factors in $acac$ now gives $acac = h\, ac$ for an integer $h$.

Fill each diagram row $i$ entirely with $e_i$, giving a tensor $t_0$. The vector $w_0 = bct_0$ represents $P_\lambda$ and has coefficient one against $t_0$. Indeed the row symmetrizer fixes $t_0$, and only the identity column permutation returns it to itself. On $bcT$ the operator $bc$ acts as $h/|R|$. Since $b, c$ are self-adjoint and $bw_0 = w_0$,

$$(w_0, w_0) = (t_0, cw_0) = (t_0, bcw_0) = h/|R|.$$

The definition of $H$ therefore gives $h = H$.

If $w \in L_0^\#$, its pairing with $bcT$ shows that $t = (|R|/H)cw$ is in the integral self-dual tensor lattice $T$. It is alternating in each diagram column. Every such integral tensor lies in $cT$. On a basis orbit with a repeated letter in a column, a transposition forces its coefficient to vanish. On every remaining orbit the column group acts freely, and the coefficients are a single integral coefficient times the signs. Taking one representative per orbit gives a preimage under $c$. Now $bt = (|R|/H)bcw = w$, so $w \in bcT = L_0$. Consequently $L^\# \subseteq L_0^\# \subseteq L_0$. The integral coefficients of vectors in $L_1$ give $HL \subseteq L^\#$ directly. Apply the two inclusions also to $L_1$ to get equality when $H$ is a unit. Scaling by a generator of $\mathfrak{a}$ gives the fractional ideal assertion. Over a discrete valuation ring all these lattices are free, and their duals for a product pairing tensor by taking dual bases. Each $L_1$ is saturated in its free module of integral monomials and therefore extends by a free complement. Tensoring these decompositions identifies the full

integral polynomial lattice on the product space with the tensor of the $L_1$. The inclusions proved above now give the product assertions. □

The coefficient convention, including the normalization by the extremal vector, is specified in `Formalism/LocalToGlobal/Columns/CoefModB.lean`, line 17–20, 55–66.

## 122.7 The original trace lattice at every finite place

Let $E, F$ be the fields of Theorem 71.10, let $K = \mathbb{Q}(\sqrt{-d_0})$, and take the CM ideal $\mathfrak{c}$ of the given field. Write $u_0 = N_{K/\mathbb{Q}}\mathfrak{c}$. Use the ideal $\mathfrak{b}_{\mathrm{D}}$ of Theorem 64.10 with $\mathfrak{b}_{\mathrm{D}}\overline{\mathfrak{b}}_{\mathrm{D}} = \mathfrak{D}_{E/K}$. Put $\mathfrak{A} = (\mathfrak{c}\mathcal{O}_E)\mathfrak{b}_{\mathrm{D}}^{-1}$ and $\xi_0 = -1/(u_0\sqrt{-d_0})$. These are the ideal and scalar in `Formalism/LocalRingCardinality/DataHost.lean`, line 147–161. For two Hermitian spaces with self-dual lattices $\Lambda_S, \Lambda_T$, give the space of $E$-linear maps from $S$ to $T$ the Hermitian form $h(X,Y) = \mathrm{tr}(Y^{\dagger}X)$, where $Y^{\dagger}$ is the Hermitian adjoint. The sign of either Hermitian form may be reversed. Define $M = \mathrm{Hom}_{\mathcal{O}_E}(\Lambda_S, \Lambda_T)$.

**Proposition 122.7** (Trace duality for the integral oscillator)**.** *At each finite place $v$ of $F$, the lattice $(\mathfrak{A}M)_v$ is its own annihilator for the alternating pairing given by the standard absolute additive character applied to $\mathrm{Tr}(\xi_0 h(X,Y))$. Here the trace runs from $E \otimes_F F_v$ to $\mathbb{Q}_p$, $p$ below $v$. At a split place its two halves are exact duals. At every marked place both components of $\mathfrak{A}$ are units. At an inert place this pairing and lattice admit coordinates with a trace-zero unit and a conductor-$\mathcal{O}_{F_v}$ character.*

*Proof.* Self-duality of $\Lambda_S$ identifies its Hermitian dual with its $\mathcal{O}_E$-linear dual via conjugation. Thus the rank-one maps $x \mapsto th_S(s,x)$, with the Hermitian form here linear in $x$, span $M$. Local bases make this explicit: the Gram matrices and their inverses are integral since their determinants are units. Inserting them into the matrix trace therefore gives a perfect Hermitian pairing on $M_v$. In the split algebra the pairing exchanges its factors. This identifies the finite oscillator lattice specified by the rank-one generators with $(\mathfrak{A}M)_v$.

The quadratic norm identity gives $\mathfrak{c}\bar{\mathfrak{c}} = (u_0)$. Since $-d_0$ is the fundamental discriminant of $K$, transitivity of the different gives

$$(\xi_0)\mathfrak{A}\bar{\mathfrak{A}} = (\sqrt{-d_0})^{-1}\mathfrak{D}_{E/K}^{-1} = \mathfrak{D}_{E/\mathbb{Q}}^{-1}.$$

The trace dual of the integer ring is the inverse different. Combining this fact with the perfect matrix pairing gives the dual lattice $(\xi_0)^{-1}\mathfrak{D}_{E/\mathbb{Q}}^{-1}\bar{\mathfrak{A}}^{-1}M = \mathfrak{A}M$. Taking completions gives the asserted annihilator for the additive character, whose $\mathbb{Q}_p$-kernel is $\mathbb{Z}_p$. Conjugation negates $\xi_0$, so the trace pairing is alternating. At a split $v$ it vanishes on each half separately, and the annihilator identity identifies the full dual of either half with the other.

The rational primes under the marked set split completely in $E$. For the two level primes, the two type primes and the single auxiliary prime this follows from their defining congruence $p \equiv 1$ modulo a cyclotomic conductor for $E$: Frobenius raises a root of unity to its $p$-th power. In particular the different is a unit at each of them. The CM ideal is chosen prime to the level and type primes. The additional auxiliary prime is chosen prime to $u_0$ as well. Both components of $\mathfrak{b}_{\mathrm{D}}$ and of $\mathfrak{c}\mathcal{O}_E$ are consequently units. The prime choices are specified in `Development/AbelianVarietyTheory/Definitions/Foundations/AuxiliaryFrameNormalizedSpaces.lean`, line 18–32 and `Development/AbelianVarietyTheory/Definitions/Divisors/LeastSplitAuxiliaryPrimeThetaChainSpec.lean`, line 209–235.

At an inert $v$ let $a_v$ generate $\mathfrak{A}_w$, and let $b_v \in F_v^{\times}$ generate the inverse different of $F_v/\mathbb{Q}_p$. Write $X = a_v X'$ and $Y = a_v Y'$. The extension $E_w/F_v$ is unramified, so transitivity in the displayed identity gives $(\xi_0 a_v \bar{a}_v) = b_v\mathcal{O}_{E_w}$. Hence $\xi_0 a_v \bar{a}_v / b_v$ is a trace-zero unit. In these

coordinates the additive character on $F_v$ is $z \mapsto \psi_v(b_v z)$, where $\psi_v$ is the original trace character. Its conductor is $\mathcal{O}_{F_v}$ by trace duality. The two scalar changes cancel in the argument of the absolute character, so this is an identity for the original pairing. Finally $d_0 \equiv 7 \bmod 8$ makes 2 split in $K$, hence all dyadic places of $F$ split in $E = FK$. □

## 122.8 The spherical section for the original finite lattice

Use the self-dual map lattice $L_v = (\mathfrak{A}M)_v$ of Proposition 122.7 for a step of ranks $u = n_i,\ N = n_i + d$. Outside $S_0$ there is no congruence on the displacement lattice. In an integral polarization write the two halves of $L_v$ as $L_X, L_Y$. Use the position indicator $1_{L_X}$ and its matrix coefficient divided by its squared norm. Equivalently the position Haar measure may give $L_X$ volume one. This latter convention retains the integer values of the test. The determinant characters and the additive character are those of the original pair.

**Proposition 122.8** (The unmarked local test in the doubling integral)**.** *At every finite $v \notin S_0$ this test satisfies the depth equations, is fixed by both prescribed lattice stabilizers and gives the normalized spherical doubling integral on the spherical source line. These assertions include places at which the absolute trace character has nontrivial different. In particular the finite set omitted in the partial product of Theorem 71.2 may be exactly $S_0$.*

*Proof.* The two quadratic imaginary fields of the construction generate the same extension of $F$, since $\sqrt{q d_0}$ lies in $F$. At every finite prime at least one of the two quadratic fields is unramified or split. At 2 one is split. Thus a nonsplit finite extension $E_w/F_v$ is unramified and has odd residue characteristic. Exact duality of $L_X, L_Y$ gives invariance of the indicator under the pure translations and modulations from these lattices. For doubled mixed displacements the remaining phase is trivial by the calculation in Theorem 102.7.

At a split place we can take the two idempotent halves. In lattice bases $L_X$ is a scalar multiple of the rectangular integral matrix lattice. Dilation by that scalar, normalized on $L^2$, commutes with the two group substitutions and their determinant factors in the split model. Consequently the entire normalized matrix coefficient on the smaller group is identical to that of the ordinary rectangular lattice. The unramified splitting characters are trivial on unit determinants. This also proves invariance under both maximal compact groups. This calculation holds in residue characteristic two.

Suppose now $E_w/F_v$ is a field and put $Q = Nw$. Since $\kappa_i$ is unramified and trivial on $F_v^\times$, it is identically one on $E_w^\times$, since the two fields have a common uniformizer. Both Hermitian lattices have even rank and admit integral hyperbolic bases. Indeed, over the quadratic residue field the norm is surjective, so a nonsingular diagonal Hermitian form of rank at least two has an isotropic line. Lift its generator to a primitive isotropic vector over the integer ring by Hensel's lemma. The derivative in a suitable coordinate direction is a unit, since the vector is primitive, the lattice pairing is perfect and the residue trace is onto. A partner pairing to one can be made isotropic by subtracting half its self-pairing times that vector. Splitting off the resulting unimodular plane proves the claim by induction.

Polarize the larger factor into its two isotropic halves. The Schrödinger variable then consists of $N/2$ copies of the smaller space, with position lattice given by $N/2$ copies of $\mathfrak{A}_w \Lambda_u$ in these bases. For the trivial splitting character the smaller unitary group acts by substitution alone. In the polarization by the smaller factor the larger group likewise acts by substitution. These are the two models of the same Weil representation. Indeed switching the factors transfers a trace-zero scalar from one form to the other, and changing the polarization is a partial Fourier transform. On the changed isotropic coordinates this transform sends a lattice indicator to its volume times the indicator of its annihilator. By Proposition 122.7 that annihilator is exactly

the corresponding component of $L_v$. This proves invariance under the larger lattice stabilizer as well. The unitary action by substitution and the switching convention are given in Liu [4, Section 3] and Ichino [5, Section 2.2].

For clarity one can identify the section on the integration group itself. Write a Cartan representative of the smaller group in the hyperbolic basis with entries $\varpi^{t_j}, \varpi^{-t_j}$ for $1 \le j \le u/2$, $t_j \ge 0$. The normalized overlap of the position lattice with its translate on each of the $N/2$ copies is $Q^{-\sum_j t_j}$. Thus the coefficient at $h$ with these elementary divisors is $Q^{-(N/2)\sum_j t_j}$. The normalized spherical section at $s_0 = d/2$ has exactly this restriction. To see it, take the primitive exterior generator of the diagonal $u$-plane in the doubled integral lattice. Applying $(h,1)^{-1}$ changes its lattice norm by $Q^{\sum_j t_j}$. On a diagonal representative the corresponding wedged columns have disjoint coordinate pairs, so their maximum minor has that absolute value. The integral compact groups preserve the norm. Its reciprocal is the absolute determinant of the Siegel Levi in the Iwasawa decomposition of $(h,1)$. The normalized section raises it to $s_0 + u/2 = N/2$.

For both kinds of place the equality on $(h,1)$ persists for flat extensions of the sections, since both sides are then multiplied by the same Iwasawa height to the power $s - s_0$. At a split place the ordinary unramified rectangular indicator used in that comparison is the standard spherical test, as in the split calculation for Theorem 72.19. The unramified doubling formula now gives precisely the local factors there and in Theorem 71.2. This proves the assertion for the original fractional lattice. □

## 122.9 Minors detected by incoming Bessel rows

Keep the two definite spaces, levels, measures, and families of Theorem 121.10. Write $\mathcal{U}$ for the span of the actual incoming vectors with that word and initial character. It is a complex subspace of the penultimate highest-line space. For each source test $U_a$ and target test $V_b$ form the matrix $Z_{ba} = z(U_a, V_b)$.

**Theorem 122.9** (Nonzero arithmetic minors at the original levels)**.** *For every $1 \le j \le \dim \mathcal{U}$ there are $j$ rows and $j$ columns of $Z$ whose minor $z_j$ satisfies*

$$|z_j| \;\ge\; j!\,(i_- i_+ L_*)^j.$$

*This holds with either family of Theorem 122.2 at either end. At least $j = 1$ occurs on every good orbit. For $j = 1$ one may take the same target test as in the lower bound on $|\langle TX, V\rangle|$ for a given $X$.*

*Proof.* Move complex conjugation to the source space when necessary so that the operator whose matrix is being formed is linear. Order the two lists of tests and let $p_a, q_b$ be their probabilities, with a uniform choice of fine class included. Let $E_-$ have columns $\sqrt{p_a}U_a$, and $E_+$ have columns $\sqrt{q_b}V_b$. On the respective finite-level, highest-line Hilbert spaces the mean square inequality in Theorem 121.10 gives

$$E_\pm E_\pm^* \;\ge\; \ell_\pm I, \qquad \ell_\pm = \frac{i_\pm^2 w_\pm}{4h_\pm D_\pm^2}.$$

The last edge here includes the projection to the specified target line and finite character. It carries the actual incoming space to that target with squared norm at least $a_*$ times the squared source norm. By the minimum–maximum principle its $j$-th singular value is therefore at least $\sqrt{a_*}$. Further, $TE_-E_-^*T^* \ge \ell_- TT^*$, and $(TE_-)^*E_+E_+^*(TE_-) \ge \ell_+(TE_-)^*(TE_-)$. Comparison of eigenvalues proves that the first $j$ singular values of $E_+^*TE_-$ are at least $\sqrt{\ell_-\ell_+ a_*}$.

Multiply this matrix by the scalar used to define $z$. Its entries are now $\sqrt{q_b p_a}\, Z_{ba}$, up to the choice of conjugate coordinates, and its first $j$ singular values are at least $i_- i_+ L_*$. The squared Hilbert–Schmidt norm of its $j$-th exterior power is the sum of the squared absolute values of its $j$-rowed minors. In a singular value basis this norm is at least $(i_- i_+ L_*)^{2j}$. In the test bases it is

$$\sum_{\substack{|A|=j\\|B|=j}} \left(\prod_{a\in A} p_a\right)\left(\prod_{b\in B} q_b\right) |\det Z_{B,A}|^2.$$

For any probability list the sum of its squarefree products of size $j$ is at most $1/j!$, by expanding the $j$-th power of the sum. Bounding the last display by its largest squared minor times the total weight proves the result. The assertion about a specified incoming vector is the successive choice of the two tests in Theorem 121.10. □

No rational basis of $\mathcal{U}$ enters this calculation. The rows and columns indexing the minor belong to the two original projected compact spaces. The dimension in the hypothesis is the dimension of the image of the preceding theta lifts on the Bessel rows.

## 122.10 Finite-place bounds along the entire definite column

Fix the signed period $\mathcal{P}$ of Theorem 121.3, with its prescribed projected endpoint and the algebraic data of that theorem. Use its normalization $\mathcal{P} = V_J(\prod_i C_i)c^{-t/2}\mathcal{A}$, its finite levels and its class representatives throughout. At edge $i$ write $\ell_i = N_{i-1}N_i$, $d_{i,\tau}$ for the surviving degree at $\tau$, and $d_i = \sum_\tau d_{i,\tau}$. If no degree profile survives, $\mathcal{A} = 0$. In the formulas below we use a surviving profile. The scalar characters fix it throughout the signed sum by Lemma 121.2.

On each edge take the product presentations of its finite class sections as in Theorem 122.13. For their derivatives of block degrees $d_{i,\tau}$ let $\Gamma_{i,\mathfrak{p}}$ be the maximum loss given by its finite calculation. Include all terms and tuples of classes. For a conjugated derivative use the place transported by conjugation. Use a common Galois coefficient field $J$ containing their algebraic data. At vertex $v$ let $A_v$ be the least common multiple of its stabilizer orders. Write $n_v$ and $L_{v,\tau}$ for its rank and the polynomial degree bounds of Theorem 121.3, including those at the endpoint subgroup.

To specify the other finite scalar costs, expand the compact integrand of that theorem using distinct formal symbols for the derivatives of each edge and class section. Let $B_0 \ge 1$ clear the coefficients of these finitely many polynomials in the symbols and the compact matrix entries and their conjugates. This uses the endpoint coefficients, the real test coefficients, the signed weights, and any finite projection scalars outside the class sections. No derivative has been evaluated in forming $B_0$. Put

$$D_{\mathrm{H}} = \prod_{v,\tau} D(n_v, L_{v,\tau}), \qquad D_0 = B_0 D_{\mathrm{H}} \prod_v A_v.$$

Let $S$ be any finite set of good primes to which Theorem 66.24 applies to every edge and every class section of these finite tests. Thus $p \ge 5$ is inert in $K$, unramified in $E_x$, and the common local vacuum and integral coefficients at $p$ are as in that theorem. Choose the orthonormal basis there for $S$ on each edge. In an even cycle $C$ of the permutation $\pi_p$ of the embeddings choose a half $H_{i,C}$ for edge $i$, common to its $\ell_i$ matrix entries. With $[z]_+ = \max(z, 0)$ put

$$b_M(a,\ell) = \max(0, \lceil (a - \ell(M-1))/M \rceil),$$
$$g_{i,p} = \sum_{C \text{ even}} \sum_{\tau\in H_{i,C}} \sum_{\substack{j\ge 1\\ j \text{ odd}}} [\, b_{p^j}(d_{i,\tau}, \ell_i) - d_{i,\pi_p^j\tau}\,]_+.$$

**Theorem 122.10** (Divisibility after all compact integrations)**.** *For the scalar of this signed column, keeping its original restriction and character projection, one has at every* $\mathfrak{p} \mid p$

$$v_{\mathfrak{p}}(D_0\mathcal{A}) \;\ge\; \begin{cases} \sum_i g_{i,p} & p \in S, \\ -\sum_i \Gamma_{i,\mathfrak{p}} & p \notin S. \end{cases}$$

*If* $A_i^{\mathrm{jet}}$ *is the integer* $A$ *of Theorem 125.3 computed for the sections and degree of edge* $i$*, then* $D_0(\prod_i A_i^{\mathrm{jet}})\mathcal{A}$ *is an algebraic integer.*

*Proof.* All column spaces are definite and their consecutive rank products are even by Proposition 92.1. Their section presentations follow from the same normalized finite lattice intertwiners and translated elliptic products used in Theorem 122.13. This is precisely the construction at each edge in the proof of Theorem 121.3. The chain rule there requires only the product expression, so the bound $-\Gamma_{i,\mathfrak{p}}$ and its integer clearing factor hold at each edge in its own matrix coordinates.

For $p \in S$, take any monomial derivative at edge $i$. If $\alpha_{a,\tau}$ are its exponents in the $\ell_i$ entries, then $\sum_a \alpha_{a,\tau} = d_{i,\tau}$. For $M = p^j$, division with remainder gives $\sum_a \lfloor \alpha_{a,\tau}/M \rfloor \ge b_M(d_{i,\tau}, \ell_i)$. The inequalities $\sum_a [u_a]_+ \ge [\sum_a u_a]_+$ and $\sum_a \alpha_{a,\pi_p^j\tau} = d_{i,\pi_p^j\tau}$ now give $v_{\mathfrak{p}}(\partial^\alpha G_i(0)) \ge g_{i,p}$ from Theorem 66.24. This uses only the block degrees, so it holds for every monomial produced by a compact matrix substitution. The sum over $j$ is finite since eventually all the floors vanish.

Integrate the symbolic expansion over the compact variables with probability Haar measure. Theorem 95.6 at one group applies separately to each coefficient of the remaining variables and symbols. Iterating over $v, \tau$ shows that $B_0 D_{\mathrm{H}}$ clears every coefficient after averaging. The integers $A_v$ clear the stabilizer weights in each tuple of finite classes. The expansion of $D_0\mathcal{A}$ is therefore an integral linear combination over $\mathcal{O}_J$ of products of one derivative at each edge. The original signs, finite tests and endpoint all occur in this very expansion. Valuations add in each product, and taking the sum preserves the common lower bound. This proves both assertions, including when the signed sum vanishes. □

The compact cost at any subset of primes $p \le Y$ obeys

$$\sum_{p \le Y} v_p(D_{\mathrm{H}}) \log p \le \pi(Y) \sum_{v,\tau} \binom{n_v}{2} \log(n_v + L_{v,\tau})$$

by Theorem 95.6. Thus for $Y \le \exp(x^{\varepsilon}/3)$ this cost is $o(\exp(x^{\varepsilon}/2))$ whenever the sum on the right after $\pi(Y)$ is bounded by a fixed power of $x$, uniformly over the interval. The inequality itself holds at all degrees.

## 122.11 Integral class tensors detecting the incoming row

Keep the last edge, its finite characters, its fine classes, the measures $w_{\pm}, v_{\pm}$ and its actual incoming space $\mathcal{U}$ as in Theorem 121.10. For each rank $R = n, s$ and each fine class $u$, take the $E$-basis in $\Lambda_u$ of Lemma 111.2, applied at that rank. Write $X_{u,\tau}$ for its column matrix and $B_R \ge 1$ for the bound of that lemma on its singular values and inverse singular values at every embedding. Use only its lattice assertion here; the fine class counts are $h_{\pm}$ of Theorem 121.10. Write $\det^{j_\tau}$ for the determinant twist of the partition model at $\tau$. Put

$$b_R = B_R^{d+R\sum_\tau |j_\tau|}, \qquad b_- = b_n, \quad b_+ = b_s,$$

where the partition degree on either side is $d$. Use the last-edge normalization $z(U, V)$, the constants $a_*, L_*, i_{\pm}, D_{\pm}$ of Theorem 121.10, and the archimedean kernel bounds $B_\sigma$ of Theorem 125.2. The fine levels fix the tests and have trivial stabilizers.

**Theorem 122.11** (A detected minor with denominators from the original finite tests)**.** *There are finite lists of tests on these classes such that for every $0 \neq X \in \mathcal{U}$ one terminal test $V$ and one source test $U$ satisfy*

$$|\langle TX, V\rangle| \geq \frac{i_+\sqrt{w_+ a_*}}{2D_+\sqrt{h_+}\, b_+}\|X\|, \qquad |z(U,V)| \geq \frac{i_- i_+ L_*}{b_- b_+}.$$

*For $1 \leq j \leq \dim \mathcal{U}$ some $j$-rowed minor has $|z_j| \geq j!(i_- i_+ L_*/(b_- b_+))^j$. It is algebraic and satisfies, on the very same field,*

$$\sum_{p\in S} g_p \log p \leq B^0 + \tfrac{1}{2}\log j + \log(i_- i_+ b_- b_+) + \tfrac{d}{2}\log(ns) + \frac{1}{[J:\mathbb{Q}]}\sum_{\sigma}\log B_\sigma.$$

*Here $S, g_p$ are as in Theorem 125.2 at good primes prime to $T^0Q^0$, and $B^0$ is the single last-edge quantity of Theorem 126.1 for the prescribed finite input and its right-level translates.*

*Proof.* On class $u$ start with the integral exterior tensor list $v_R(Z)$ and its distribution from Theorem 122.2. Apply the exterior tensor map of $X_{u,\tau}$ on each partition factor. On the determinant factor multiply by $(\det X_{u,\tau})^{j_\tau}$ for $j_\tau \geq 0$, and by $\overline{\det X_{u,\tau}}^{|j_\tau|}$ for $j_\tau < 0$. For the latter choice, contracting with the wedge of the lattice basis gives an integral functional on the top exterior power of $\Lambda_u$ by Hermitian integrality. Its coordinates in the dual line give exactly this conjugate determinant. Thus the resulting vector belongs to the ambient exterior tensor lattice of the class, including its determinant lines. Since the original lattices are Hermitian unimodular locally, their exterior, dual and tensor products carry induced integral pairings. Pairing with the resulting vector therefore gives an integral covector on that lattice. Use it in $k \mapsto \langle \rho_R(k)t_R, v\rangle$, keeping $t_R$ the fixed standard highest vector.

Write $A_u$ for the map just applied to the test list. The partition Cartan component is invariant under the complexified general linear group, as proved in Theorem 122.2. Hence $A_u$ maps it onto itself. The singular values on $\wedge^h$ lie between $B_R^{-h}$ and $B_R^h$, since they are the $h$-fold products of singular values. The determinant scalars satisfy the corresponding bounds with exponent $R|j_\tau|$. Consequently both $\|A_u\|$ and $\|A_u^{-1}\|$ on this component are at most $b_R$. The same norm bound holds at all embeddings, by the lattice lemma, including on the conjugate determinant factors. It follows that $\|\sigma v\| \leq b_R R^{d/2}$. At the original embedding the covariance is $A_u\,\mathbb{E}(v_R v_R^*)A_u^*$, bounded below by $I/(4D_R b_R^2)$.

Take the same uniform distribution on classes and apply the same unnormalized finite character sums as in Theorem 121.10. Compact orthogonality now gives $\mathbb{E}|\langle F, Q\rangle|^2 \geq i_R^2 w_R \|F\|^2/(4h_R D_R^2 b_R^2)$ at either end, where $Q$ denotes the resulting random test. Apply it first to $TX$, using $\|TX\| \geq \sqrt{a_*}\|X\|$. For the selected $V$, apply it at the source to the Riesz vector of $F \mapsto \langle TF, V\rangle$, whose norm is at least $|\langle TX, V\rangle|/\|X\|$. This proves the simultaneous assertions. In the exterior-power calculation of Theorem 122.9 the two covariance lower bounds are now divided by $b_-^2, b_+^2$. Each of the first $j$ singular values of the weighted normalized matrix is therefore at least $i_- i_+ L_*/(b_- b_+)$. Summing the squared minors with their probability weights, of total mass at most $1/(j!)^2$, proves the stated lower bound.

Move each finite right translation in the two character sums to the kernel by invariance of Haar measure. This leaves $i_- i_+$ terms using the original one-class endpoint functions, with root-of-unity weights. In the representatives of Theorem 122.4 at $p$, the transported class vector in the exterior tensor model is integral in standard local coordinates by construction and Theorem 122.15. This follows also by applying the rational unitary change to its lattice vectors and contracted determinant covectors. The right highest vector has unit highest coordinate. The two dimension factors in $z$ cancel the compact matrix-coefficient averages of

Theorem 67.11. Thus each entry is a sum of integral minor-polynomial derivatives of the actual finite sections, with no stabilizer or compact denominator. These expressions give algebraicity. Their valuations at a good $p \in S$ have the full-column lower bound $g_p$ of Theorem 102.5, since each monomial of a full minor still uses every column.

At every other prime use the local proof of Theorem 126.1. On expanding the two determinants, all products have $j$ factors of the given degree in each orientation, with integral endpoint tensors. It follows that $v_{\mathfrak{p}}(z_j \bar{z}_j)$ is at least $2jg_p$ in $S$ and at least $-2jJ_p^0$ elsewhere. For the complex upper bound, the norm estimate on the two new lists and the linearity of the minor contraction in each vector multiply the entry bound of Theorem 125.2 by at most $b_- b_+$. Hadamard's inequality then gives

$$|\sigma z_j| \le j^{j/2} (i_- i_+ b_- b_+ (ns)^{d/2} B_\sigma)^j.$$

Apply the product formula to this nonzero diagonal Gram element. The archimedean mean of its logarithm equals twice that for $z_j$, since conjugation permutes the embeddings of $J$. Division by $2j$ gives the inequality. □

The tests just constructed probe the same incoming $X$ in the first inequality and the same $V$ in the second. The arithmetic scalar in the second still pairs $V$ with $TU$. The interval argument requires a comparison with the pairing against the full prescribed Bessel row and the strict cohomology bound of Section 6.6.

## 122.12 The orthogonal projector onto a sum of theta images

In the common space $E_0$ of Theorem 123.2, choose any finite collection of the signed definite operators. Let $\mathcal{R}$ be the sum of their images with the original endpoint metric. No equality of images for distinct words is required. We give a denominator for projection onto $\mathcal{R}$ using only class integrals already computed on this field. Keep $A = \operatorname{adj}(d_0 G)$ from that theorem. Select a maximal independent list of columns $y_1, \dots, y_r$ from the normalized period matrices $P_u$ of the collection. Write $Y = (y_1, \dots, y_r)$, and let $u_j$ be the word of column $y_j$. If $r = 0$ the projector is zero. Otherwise put

$$D_Y = \prod_j D_{u_j}^2, \qquad d_Y = \det(Y^* A Y), \qquad b_{Y,p} = \sum_j b_{u_j,p}.$$

The different heights and degree profiles of the words enter through their own $D_{u_j}$ and $b_{u_j,p}$. At an embedding $\sigma : J \to \mathbb{C}$ let $L_{j,\sigma}$ bound the Euclidean norm of $\sigma y_j$. One can take the square root of the sum of the entrywise squared bounds from Theorem 121.7. Let $c_J$ denote complex conjugation on $J$ in the original embedding.

**Theorem 122.12** (Denominators for simultaneous orthogonal projection)**.** *The projector onto $\mathcal{R}$ has the matrix*

$$\Pi_{\mathcal{R}} = \frac{A Y \operatorname{adj}(Y^* A Y) Y^*}{d_Y}.$$

*For each $\mathfrak{p} \mid p$ the nonnegative number $\eta_{Y,\mathfrak{p}} = v_{\mathfrak{p}}(D_Y d_Y) - b_{Y,p}$ bounds the negative valuations of its entries. One may minimize this number over maximal independent lists at each place. With $n_{\mathfrak{p}} = [J_{\mathfrak{p}} : \mathbb{Q}_p]$, the total cost for each fixed list satisfies*

$$\frac{1}{[J:\mathbb{Q}]} \sum_{p, \mathfrak{p} \mid p} n_{\mathfrak{p}} \eta_{Y,\mathfrak{p}} \log p \;\le\; \log D_Y - \sum_p b_{Y,p} \log p + \frac{1}{[J:\mathbb{Q}]} \sum_\sigma \log \left( \|\sigma A\|_{\mathrm{op}}^r \prod_{j=1}^r L_{j,\sigma} L_{j,\sigma \circ c_J} \right).$$

*If another signed period on these endpoint tests has matrix $P_v$, the pairing after orthogonal projection of its output onto $\mathcal{R}$ has normalized matrix $P_v^{\mathcal{R}} = \Pi_{\mathcal{R}}^{*} P_v$. Every product of one entry of $P_v^{\mathcal{R}}$ with the conjugate of another, times $D_v^2$, has valuation at least $b_{v,p} - 2\eta_{Y,\mathfrak{p}}$.*

*Proof.* The physical outputs span $G^{-1}\operatorname{im} Y$. This equals $\operatorname{im}(AY)$, since $A$ is a nonzero scalar multiple of $G^{-1}$. The kernel of $Y^{*}$ is the metric orthogonal complement of this image. The matrix $A$ is positive definite at the original embedding, so $d_Y > 0$. The displayed projector is the identity on $\operatorname{im}(AY)$ and zero on $\ker Y^{*}$. These two subspaces exhaust $E_0$. In particular the formula is unchanged if a different maximal list is used.

Expand the determinant into permutations and expand each entry of $Y^{*}AY$ in the coordinate basis of $E_0$. In each summand, column $j$ appears once in a factor $(y_j)_a$ and once in a conjugate factor $\overline{(y_j)_b}$. The same count holds in each entry of the numerator. Indeed a term with the outside columns $y_i, y_j^{*}$ uses the cofactor obtained by deleting row $j$ and column $i$ of $Y^{*}AY$. The two outside factors supply exactly the two deleted occurrences. All the remaining coefficients are integers of $J$. The period estimate for each pair from column $j$ then gives at least $b_{u_j,p}$ after clearing $D_{u_j}^2$, by Theorem 126.1 with the improvement of Theorem 126.2. This proves the same lower bound $b_{Y,p}$ for $D_Y d_Y$ and every entry of the cleared numerator. Division gives the asserted denominator for the projector.

The product formula for $D_Y d_Y$ makes the left side of the total cost estimate exactly

$$\log D_Y - \sum_{p} b_{Y,p} \log p + [J:\mathbb{Q}]^{-1} \sum_{\sigma} \log |\sigma d_Y|.$$

At $\sigma$, the determinant is the contraction of the wedge of the columns $(\sigma \circ c_J) y_j$, transposed, with $(\bigwedge^{r} \sigma A)(\bigwedge_j \sigma y_j)$. The Euclidean norm of a simple wedge is at most the product of the vector norms, by Gram–Schmidt. The norm of the induced operator is at most $\|\sigma A\|_{\mathrm{op}}^{r}$, by singular value decomposition. Cauchy–Schwarz proves the bound even when $\sigma$ does not commute with $c_J$.

Finally $G\Pi_{\mathcal{R}} = \Pi_{\mathcal{R}}^{*} G$, which gives the projected matrix formula in the period pairing. Complex conjugation takes a place over $p$ to another place over $p$. Since $D_Y d_Y$ is fixed by $c_J$, the two places have the same $\eta_{Y,\mathfrak{p}}$. Expand a product of two projected entries and apply the original pair bound to the two entries of $P_v$ and the projector bound to the two other factors. This gives the stated loss. All operations took place at the original endpoint and retained its metric and prescribed levels. □

For the final definite step one can take a full set of independent columns of its period matrix. Its image contains the $k_0$-dimensional image of the incoming subspace in Theorem 121.10. Thus this projector has rank at least $k_0$. For the $z(U,V)$ normalization one obtains the same projector by using the entries of $z$ in the formula, since a common scalar on an entire matrix leaves its image unchanged. The denominator estimates above use the $D_u, b_{u,p}$ belonging to the stated normalization. Identifying one of these computed images with the row in the Bessel pairing requires a common realization of the two row spaces.

## 122.13 Denominators from the finite class data

Fix one $x$ and the levels, representatives and right translates of Theorem 125.2. Let $d = \sum_{\tau} |\lambda_{\tau}|$ be the differential degree of its last-edge minor polynomial and $D = fns$ the number of matrix coordinates $Z_{\mu}$. Use the marked $K$-orthonormal coordinates of Lemma 65.8. Every one of the finitely many class sections in the arithmetic matrix admits an expression

$$G(Z) = \sum_{b} c_b \prod_{h=1}^{D} (U_{w_{bh}} \Theta_{t_*})((T_b Z)_h), \qquad P_{bh} = [w_{bh}] \in C_{\mathrm{tors}}.$$

Here $T_b$ converts the matrix coordinates to the chosen product coordinates. One may use $T_b = V^*$ for a common orthonormal basis. The coefficients $c_b$ include all finite scalar factors from the class construction. We describe below how to take this presentation directly from that construction.

Choose positive integers $N, H, m$ such that every $Nc_b$ and every entry of $HT_b$ is integral, and $mP_{bh} = 0$ for every term of every section. Use one coefficient field $J$ for all of them. Write $\Phi_p(a,t) = at + \lfloor a/2 \rfloor(\kappa_p - 2t)_+$, with $\kappa_p$ as in Theorem 67.12.

**Theorem 122.13** (Arithmetic bounds for the specified class sections)**.** *For every matrix multi-index $k$ with $|k| = d$ and every $\mathfrak{p} \mid p$ one has*

$$v_{\mathfrak{p}}(\partial^k G(0)) \geq -v_{\mathfrak{p}}(N) - d\, v_{\mathfrak{p}}(H) - \Phi_p(d, v_{\mathfrak{p}}(m)).$$

*There is a sharper finite calculation for any prescribed set of multi-indices. For each contributing term choose nonnegative integers $n_{h\mu}$ with $\sum_h n_{h\mu} = k_\mu$, requiring $(T_b)_{h\mu} \neq 0$ whenever $n_{h\mu} > 0$, and put $s_h = \sum_\mu n_{h\mu}$. Define its loss by*

$$\sum_h \Phi_p(s_h, \delta_{\mathfrak{p}}(P_{bh})) - \sum_{n_{h\mu}>0} n_{h\mu} v_{\mathfrak{p}}((T_b)_{h\mu}) - v_{\mathfrak{p}}(c_b).$$

*Take the maximum of zero and all such losses over the relevant terms, classes and multi-indices. The resulting number $\Gamma_{\mathfrak{p}}$ bounds the negative valuation of every integer linear combination of these derivatives.*

*Proof.* Start with the product vacuum in Lemma 65.8. A normalized intertwiner to a commensurable period lattice is the finite sum of translations indexed by its quotient by the intersection, weighted by signs or roots of unity and by the inverse square root of the quotient order. The formula and its normalization are proved in Corollary 65.4. The class and level transports in Theorem 65.13 move the vacuum to such a lattice and the Schwartz test to finitely many further translates. For a smaller split indicator the finite Fourier sum in Theorem 125.2 gives the additional coefficients, including its reciprocal group order. Compose the translations and multiply their coefficients. Their composition introduces a root of unity since the alternating form is rational on rational lattice vectors. A product Hermitian translation factors over its elliptic coordinates. This gives precisely the displayed expression for each class term. It retains the local splitting characters in $c_b$, including those supplied by Theorem 65.13.

The integers in the statement can thus be read from finite data. For example, take $t$ so that each unscaled period module in product coordinates lies in $t^{-1}\mathcal{O}_K^D$. Each lattice intertwiner translates by a representative in such a module. If all further prescribed translations have order dividing $Q$ relative to their period modules, their product coordinates lie in $(Qt)^{-1}\mathcal{O}_K^D$ before multiplying by the elliptic period $\delta$. Consequently $m = Qt$ works even after composing the translations. The coordinate change supplies $H$. For a term passing through quotient orders $q_1, \dots, q_a$, the product $\prod_i q_i$ clears the inverse square roots, since each $\sqrt{q_i}$ is an algebraic integer. Clearing also the actual finite coefficients and character values and taking a common multiple over the terms gives $N$. Zero coefficients may be discarded. None of these choices requires evaluating a theta derivative.

Apply each matrix derivative as $\partial_\mu = \sum_h (T_b)_{h\mu} \partial_h$. Expanding the commuting derivations gives exactly the indicated arrays $n_{h\mu}$, with integral multinomial coefficients. The derivative in product coordinate $h$ occurs $s_h$ times and acts only on its own factor. Theorem 66.35 therefore bounds the negative valuation of the term by its displayed loss. Sums and integral linear combinations obey the same maximum bound.

For the first inequality, $\delta_{\mathfrak{p}}(P_{bh}) \le v_{\mathfrak{p}}(m)$. For every $a$ the function $\Phi_p(a,t)$ is increasing in the weak sense, as is seen from its maximum formula in Theorem 66.35. Also $\sum_h \Phi_p(s_h,t) \le \Phi_p(d,t)$ since $\sum_h s_h = d$ and $\sum_h \lfloor s_h/2 \rfloor \le \lfloor d/2 \rfloor$. The two remaining losses are bounded by $d\, v_{\mathfrak{p}}(H)$ and $v_{\mathfrak{p}}(N)$. □

## 122.14 A trace lattice basis at an arbitrary prime

On $W = \mathrm{Mat}_{s\times r}(E)$ use the positive trace Hermitian form of Lemma 65.8, with $rs$ even. Its standard trace module is $M = \mathfrak{b}^{-1}\,\mathrm{Mat}_{s\times r}(\mathcal{O}_E)$, where $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K}$ by Theorem 64.10. Use a finite list of the actual tests on this edge. In the finite model of Theorem 67.8 write each in the basis of the original unit vacuum as

$$\phi = \sum_a c_{\phi,a} W(a)\phi_0.$$

Here $a$ represents a rational class modulo $M$. Each sum is finite by smoothness as proved in that theorem. The finite transports and splitting factors in Theorem 65.13 and the finite Fourier expansions in Theorem 125.2 have algebraic coefficients. Thus for the stipulated algebraic tests we can choose positive integers $Q, N$ with $Qa \in M$ and all $N c_{\phi,a}$ algebraic integers. These data are in the fixed finite model.

**Theorem 122.14** (Local product coordinates for the fixed finite model)**.** *For every rational prime $p$ there is a $K$-orthonormal basis $B_p$ of $W$ generating $M \otimes \mathbb{Z}_p$ over $\mathcal{O}_K \otimes \mathbb{Z}_p$. In physical matrix coordinates take an ordinary derivative of degree $d_\tau$ in block $\tau$ and write $d = \sum_\tau d_\tau$. At every $\mathfrak{p} \mid p$, in this basis its class section satisfies*

$$v_{\mathfrak{p}}(\partial^\alpha G_\phi^{B_p}(0)) \ge -v_p(N) - \Phi_p(d, v_p(Q)) - \sum_\tau d_\tau v_{\mathfrak{p}}(\bar{\tau}\mathfrak{b}).$$

*For an ideal its valuation denotes the minimum valuation of its elements after extension to the coefficient field, with $v_{\mathfrak{p}}(p) = 1$. Conjugating an edge conjugates the ideal in the last sum.*

*Proof.* The module $M$ is self-dual since the trace dual of $\mathcal{O}_E$ is $\mathfrak{D}_{E/K}^{-1}$ and $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K}$. It is locally free over $\mathcal{O}_K$. If $p$ splits in $K$, the two components of the local module are in perfect duality. Choose a basis of the first and its dual in the second. The sums of corresponding vectors form an orthonormal basis over the product ring. This covers $p = 2$, since $d_0 \equiv 7 \bmod 8$.

At a nonsplit prime $p$ the residue characteristic is odd. A local integral basis has a Hermitian Gram matrix with unit determinant. At an unramified prime its reduction is equivalent to the identity over the quadratic residue field, by orthogonal diagonalization and surjectivity of the finite-field norm. For the diagonalization, a nondegenerate Hermitian form has a vector of nonzero length by polarization. Split off its line and normalize it by the norm map, then continue on the orthogonal complement. At a ramified prime the reduced form is symmetric. Its determinant is a square in the residue field. Indeed in a rational orthonormal basis, available by Lemma 65.8, the Gram determinant is the norm of the determinant of the local change of basis. This determinant is a local unit since its norm is a unit. Reduction of the norm is a square at a ramified prime. A nondegenerate symmetric form over a finite field of odd order with square determinant is equivalent to the identity. To see this, diagonalize and pair the nonsquare entries after square rescalings make each one the same nonsquare $a$. There exist $u, v$ with $u^2 + v^2 = a^{-1}$, by surjectivity of the norm in the finite quadratic algebra obtained by adjoining a root of $-1$, including the split product algebra. The two orthogonal columns $(u, v), (-v, u)$ turn $aI_2$ into $I_2$.

After lifting the residue change of basis the integral Gram matrix $G$ is the identity modulo the local maximal ideal. The series $G^{-1/2} = \sum_{n\ge 0} \binom{-1/2}{n} (G - I)^n$ converges integrally, because

$\binom{-1/2}{n} = (-1)^n\binom{2n}{n}/4^n$ and $p$ is odd. It is self-adjoint and its square is $G^{-1}$, giving a local orthonormal basis. To obtain a rational one, apply the Cayley approximation in the proof of Lemma 65.14 at this prime. The condition to generate the local lattice is open. A norm-one scalar arbitrarily close to one moves any local unitary matrix off the eigenvalue $-1$. The inverse Cayley transform then lies in the rational vector space of skew-Hermitian matrices over $\mathbb{Q}_p$, where rational matrices in that vector space are dense. Transform a sufficiently close approximation back. This also works for the split product algebra.

Let $M'$ be the $\mathcal{O}_K$-span of the resulting basis. The index $[M : M \cap M']$ is prime to $p$ since the local modules agree. In the normalized intertwiner from the product lattice to $M$ the inverse square root of this index is a unit at every place above $p$. Its translation vectors belong to $M$ and have order modulo $M'$ dividing this same index. A further translation by $a$, with $Qa \in M$, consequently has order with $p$-part dividing the $p$-part of $Q$ in the product coordinates. Composing translations contributes only roots of unity. The translated elliptic product in Theorem 66.35 thus costs at most $\Phi_p(d, v_p(Q))$ in ordinary derivatives, by the product rule of Theorem 122.13.

Finally every basis vector has its matrix entries locally in $\mathfrak{b}^{-1}$ at every prime of $E$ above $p$, by the definition of $M$. Each coefficient converting a physical derivative in block $\tau$ to a product derivative is a conjugate entry of that embedded basis matrix. Its valuation is therefore at least $-v_{\mathfrak{p}}(\bar{\tau}\mathfrak{b})$. Multiply these bounds in the chain rule and use $v_{\mathfrak{p}}(c_{\phi,a}) \geq -v_p(N)$. This proves the asserted inequality simultaneously for every finite test on the list. □

## 122.15 Lattices for the two slots of an endpoint coefficient

At an endpoint of rank $R$ write its class functions as sums of $f(k,u) = \ell_u(\rho(k)t)$. Use a number field containing the specified algebraic coefficients and all the embeddings in the tensor model. If $L$ is a Hermitian $\mathcal{O}_E$-lattice, write $W(L)$ for the prescribed integral coefficient module, formed from its embedded copies by tensor products, exterior powers, integral Young images and determinant lines, as in Proposition 95.4. For a negative determinant power use the corresponding power of the dual determinant line. Localizations and duals below are over the coefficient integers, after extending the embedded lattices there. Write $\Lambda_u = E^R \cap u\widehat{\mathcal{O}}_E^R$.

**Theorem 122.15** (A denominator for the endpoint independent of the representatives)**.** *Use $W(\mathcal{O}_E^R)$ for $t$, and $W(\Lambda_u)^\vee$ for $\ell_u$. For the algebraic ends of the signed period in Theorem 121.3, choose one integer $B_{\mathrm{end}}$ clearing the tensor products of these endpoint coefficients, the prescribed real polynomial coefficients, the signed weights, and any scalar of the projection kept outside the finite tests. Then $B_{\mathrm{end}}$ clears the coefficients of the formal compact integrand over $p$, in the representatives of Theorem 122.4, simultaneously for every term and class. Here the physical derivatives of the translated finite sections are formal indeterminates.*

*Proof.* The lattices have full rank in their respective algebraic representations. For the finite list of original classes one may clear each coefficient in its indicated lattice and take products for every monomial in the integrand, then take a common multiple. This supplies the required integer before varying $p$. It also treats any fixed arithmetic vector in the image of the terminal projection. If the endpoint pairing is written with an explicit projection on its test vector, the coefficients used here are those of the resulting vector.

For $u' = \gamma u j$ as in Theorem 122.4, preservation of the standard finite lattice by $j$ gives $\Lambda_{u'} = \gamma\Lambda_u$. Since $u'_p = 1$, its localization at $p$ is the standard lattice. Automorphy expresses the endpoint coefficient in the new chart as

$$f(k,u') = (\ell_u \circ \rho(\gamma_\infty^{-1}))(\rho(k)t).$$

Indeed multiplying $(k, u')$ on the left by $\gamma^{-1}$ and on the right by $j^{-1}$ in its finite component gives $(\gamma_\infty^{-1}k, u)$. Thus $t$ stays fixed, and the indicated covector is integral on $W(\Lambda_{u'})$ whenever $\ell_u$ was integral on $W(\Lambda_u)$. The formula is compatible with complex conjugation of a factor, since integrality is imposed at all places above $p$.

In the new chart, after the stipulated clearing, both slots are integral for the standard local module. The polynomial giving their matrix coefficient in the entries of $k, \bar{k}$ has integral coefficients. For an exterior power this follows from the formula by minors. For an integral Young image write a lattice vector as a sum of $Yx$ for the integral Young operator $Y$. The variable matrix acts by $Y(k^{\otimes d}x)$; its coefficients belong to the Young image because $Y$ commutes with tensor powers of $k$. Applying an integral covector preserves coefficient integrality. On $U(R)$ inverse determinant is $\det \bar{k}$, and the dual standard action also uses conjugate entries. This proves the polynomial assertion for every stated tensor construction.

The same polynomial substitutions in the physical coordinates act on the incident edge symbols. Their coefficients consist of the fixed real test coefficients and integral matrix operations. The external finite order determinant values are roots of unity. Any moved oscillator splitting scalars are already in the finite section and are units at $p$ by Theorem 122.4. Taking products, then the prescribed signed sums, proves the claim with the single integer chosen at the outset. □

## 122.16 Integral reverse tests in the last theta images

Keep a finite list of matching complex iterations with the same definite endpoint, highest line, finite level and character. At each last definite step let $T_u$ act on the full penultimate coefficient space at its stated type and level. Let $\mathcal{R} = \sum_u \operatorname{im} T_u$ in the common endpoint. Its projector $\Pi$ is given by Theorem 122.12. One can use the full signed period normalization of its one-edge matrices there, or their nonzero common scalar multiples for computing the same image. Use the former normalization for its denominator bounds. For an actual incoming vector $X_u$ of Theorem 121.10 put $y_u = T_u X_u$.

**Theorem 122.16** (Retaining the actual Bessel values on an integral image)**.** *For each endpoint test $Q$ one has simultaneously $\langle y_u, \Pi Q\rangle = \langle y_u, Q\rangle$ for all $u$. In particular, given finitely many nonzero incoming vectors and their respective integral detecting tests $Q_u$ from Theorem 122.11, one can choose a common integral $Q' \in \mathcal{R}$ and an explicit positive integer $m$ such that*

$$|\langle y_u, Q'\rangle|/m \;\geq\; \tfrac{1}{2}|\langle y_u, Q_u\rangle| \qquad \text{for every } u.$$

*Before the integer clearing, projection can only improve the ratio of the absolute pairing to the norm of the common test. Also, for a matching complex iteration $\Theta : \mathcal{H} \to E_0$ that factors through one of these last operators on its prescribed row and satisfies $\Theta^\dagger\Theta \geq aI$, $a > 0$, the adjoint restricted to $\mathcal{R}$ maps onto $\mathcal{H}$ with a right inverse of norm at most $a^{-1/2}$.*

*Proof.* Every $y_u$ belongs to $\mathcal{R}$, hence is orthogonal to $(I - \Pi)Q$. For $l$ input vectors form $Q = \sum_{j=1}^{l} n_j Q_j$ with $0 \leq n_j \leq l$ integers. For vector $y_u$, fix the choices other than $n_u$. Its possible pairings have spacing $|\langle y_u, Q_u\rangle| > 0$. At most one lies in the open disk about zero of half that radius. The union of the failing sets has fraction at most $l/(l+1) < 1$. Choose a successful $Q$. Then $\Pi Q \neq 0$, all its pairings agree with those for $Q$, and $\|\Pi Q\| \leq \|Q\|$.

Here is the integer clearing in the stated endpoint lattice. Choose a basis from its integral class tests, let $D_Q$ be a positive integer clearing the coordinates of $Q$ in that basis, and use these coordinates for the projector in Theorem 122.12. At $\mathfrak{p} \mid p$ put $l_\mathfrak{p} = \max(0, -\min_{a,b} v_\mathfrak{p}(\Pi_{ab}))$, with zero entries omitted. Write

$$N = \prod_p p^{\lceil \max_{\mathfrak{p}\mid p} l_\mathfrak{p} \rceil}, \qquad m = D_Q N.$$

Only finitely many factors differ from one. For any of the admissible determinant lists $Y$ in that theorem,

$$\log N \le \sum_{p,\mathfrak{p}\mid p} [J_{\mathfrak{p}} : \mathbb{Q}_p]\eta_{Y,\mathfrak{p}} \log p.$$

Indeed $l_{\mathfrak{p}} \le \eta_{Y,\mathfrak{p}}$ and $e(\mathfrak{p}/p)l_{\mathfrak{p}}$ is an integer. For a place where the maximum is positive this integer is at least its ceiling before multiplication by the ramification index. This proves the bound prime by prime. Now $Q' = m\Pi Q$ lies in the integral span of the basis and thus in the given lattice. Dividing its pairings by the recorded $m$ proves the assertion with the original normalization.

Finally the range of the stated $\Theta$ lies in $\mathcal{R}$. The vector $\Theta(\Theta^{\dagger}\Theta)^{-1}z$ lies there for every $z \in \mathcal{H}$, maps back to $z$ under the adjoint, and has squared norm $\langle z, (\Theta^{\dagger}\Theta)^{-1}z\rangle \le a^{-1}\|z\|^2$. On the matching data one can take the span of the actual ordinary columns and the corresponding bound of Theorems 88.1 and 72.21. The Bessel values on the reverse tests are the forward pairings by Proposition 110.5. □

This gives integral tests inside the sum of the computed last images, with a common endpoint for the finitely many actual incoming vectors. Identifying that sum, or its reverse image, with the independently specified row and rational pairing in Section 6.6 remains an additional assertion.

## 122.17 Integral class tests with the original Haar measure

Use the definite terminal row of Theorem 123.8. Write $J = J_{\xi}$, $\omega = \omega_{\xi}$, and fix the coefficient ring of integers $\mathcal{O}$, enlarged to contain the values of the finite character. At class $i$ let $L_i$ be the coefficient lattice of the Hermitian lattice in that class, with the given determinant twists. It can be taken to be the lattice of integral flag minor products in its local frames. These lattices are projective over $\mathcal{O}$ and the class stabilizer preserves them. Indeed formation of flag minors commutes with integral basis change by expansion of minors, in both directions. The local lattices agree with a fixed one almost everywhere and define a global lattice by intersection. Put $\omega_i(\gamma) = \omega(g_i^{-1}\gamma g_i)$, and let $S_i \subset L_i$ be the submodule where $\rho(\gamma)y = \omega_i(\gamma)y$ for all $\gamma \in \Gamma_i$. The full integral class table is $S = \bigoplus_i S_i$. Use the inverse character and the algebraic dual representation on testing forms.

**Theorem 122.17** (Integral class traces and change of level)**.** *For $\ell \in L_i^{\vee} = \mathrm{Hom}_{\mathcal{O}}(L_i, \mathcal{O})$ let the test be supported on class $i$ with value*

$$t_i(\ell) = \sum_{\gamma \in \Gamma_i} \omega_i(\gamma)\, \rho^{\vee}(\gamma)\ell.$$

*The lattice $T_J$ of all sums of these tests is exactly the full dual of $S$ for the global bilinear integral divided by* $\mathrm{vol}(J)$*. In particular the period with $t_i(\ell)$, in the original Haar measure, is* $\mathrm{vol}(J)\ell(y_i)$ *on any table $y$. All these integral covectors can be realized locally in the integral flag minor tests of Theorem 122.6.*

*Let $K \triangleleft J$ be a finite-index open subgroup in the kernel of $\omega$, including the common refinement fixing the residue markings. For a dual testing form $z$ of level $K$ put $\mathcal{T}z = \sum_{a \in J/K} \omega(a) R_a z$, where $R_a z(g) = z(ga)$. Then*

$$\mathcal{T}(T_K) = T_J, \qquad \frac{\langle y, \mathcal{T}z\rangle}{\mathrm{vol}(J)} = \frac{\langle y, z\rangle}{\mathrm{vol}(K)}$$

*for any $J$-table $y$ of character $\omega$. The pairings are the integrals of the natural vector and covector contraction, with probability Haar at infinity.*

*Proof.* The stabilizer acts on $t_i(\ell)$ by the inverse character, so the table gives an integral dual form. At $y_i \in S_i$ each term of its evaluation equals $\ell(y_i)$. The class has measure $\mathrm{vol}(J)/|\Gamma_i|$ for this invariant contraction, by Theorem 123.8. This proves the period formula. The submodule $S_i$ is saturated in $L_i$, being the intersection of kernels of integral linear maps to a torsion-free module. Thus $L_i/S_i$ is a finite projective module over the Dedekind ring $\mathcal{O}$, and every integral functional on $S_i$ extends to $L_i$. The trace tests realize all such functionals. The pairing is nondegenerate over the coefficient field, since averaging there projects a dual space onto the full dual of the corresponding character space. This proves the equality of lattices. In a local frame an integral covector on $L_i$ is represented by an element of $L_i^{\#}$ under the normalized Fischer pairing. Theorem 122.6 puts that element in the integral span of flag minors.

Right invariance of Haar and $y(ga^{-1}) = \omega(a)^{-1}y(g)$ give $\langle y, \mathcal{T}z\rangle = [J : K]\langle y, z\rangle$. Since $\mathrm{vol}(J) = [J : K]\,\mathrm{vol}(K)$, the normalized identity follows. It puts $\mathcal{T}(T_K)$ inside the full dual just computed, because restriction to level $K$ preserves integrality of $y$.

To prove surjectivity, keep the fine class represented by $g_i$ above a chosen coarse class. Put on it the fine trace of $\ell$ using $\Gamma_i' = G(F) \cap g_i K g_i^{-1}$, and zero elsewhere. In evaluating $\mathcal{T}$ at $g_i$, the contributing $a$'s are precisely the image of $\Gamma_i$ in $J/K$, with kernel $\Gamma_i'$. For a lift $\gamma_a$ of such an $a$, the value at $g_i a$ is the dual action of $\gamma_a$ on the fine trace. Multiplication by $\omega(a)$ and summation now give the displayed trace over $\Gamma_i$, once per element. All other coarse classes have zero value. This realizes every $t_i(\ell)$ and proves equality globally over $\mathcal{O}$, including at primes dividing either index. □

# 123 The signed finite tests along the column

## 123.1 The position lattice of a split finite test

Let $p$ be an odd rational prime split in $K$ on an original interval. Write $F = E^+$. At each $v \mid p$ the algebra $E \otimes_F F_v$ is split because $E = FK$. Fix the actual position half $w_v$ and write $\bar{w}_v$ for the other half. On an edge of matrix dimension $m = rs$ the local reference lattice of Theorem 122.4 is $L_v \oplus L_v^\circ$, where $L_v$ is in position space and $L_v^\circ$ is its frequency annihilator. A uniformizer of $F_v$ is denoted by $\varpi_v$, and its residue field has size $q_v = p^{f_v}$. The position vacuum of norm one has amplitude $a_{0,v}$ on $L_v$.

Expand the finite list of original edge tests, including all the right translates that occur in the projection, as

$$\phi^0 = \sum_{\lambda,\mathbf{y}} c_{\lambda,\mathbf{y}} W(b_\lambda^{(p)})\phi_0^{(p)} \otimes \bigotimes_{v\mid p} a_{0,v}\mathbf{1}_{y_v+P_v}, \qquad P_v = \varpi_v^{h_v} L_v, \quad h_v \in \mathbb{Z}.$$

The superscript $(p)$ denotes the components away from $p$. The sum exists by finite vacuum expansion there and local constancy and compact support in the position variables. Use distinct Weyl classes away from $p$, with a fixed lift for each, distinct cosets at $p$, and combined coefficients. The split Fourier formulas in Lemma 65.12 and Theorem 123.4 give algebraic relative amplitudes for the stipulated algebraic finite tests. Define

$$d_{\lambda,\mathbf{y}} = c_{\lambda,\mathbf{y}} \prod_{v\mid p} q_v^{-mh_v/2}.$$

**Theorem 123.1** (A product basis for the actual position supports)**.** *There is a single rational orthonormal product basis for the whole list of tests with local trace lattice $\bigoplus_{v\mid p}(P_v \oplus P_v^\circ)$. In its product vacuum expansion the coefficient of each term, before summing terms, is $d_{\lambda,\mathbf{y}}$ times a unit over $p$, including after moving the vertex representatives as in Theorem 122.4. The local translation class of this term at each $v \mid p$ is $(y_v \bmod P_v, 0)$ in position and frequency coordinates. The physical fields, coordinates and period normalizations are those of Theorem 121.3.*

*Proof.* Let $\mathfrak{c}$ be the fractional $\mathcal{O}_E$-ideal with exponents $h_v, -h_v$ at $w_v, \bar{w}_v$ and zero exponents elsewhere. Its product with its conjugate is the unit ideal. Hence $\mathfrak{c}M$ is self-dual for the trace Hermitian form, since $M$ is self-dual. Its $p$-part is the lattice displayed in the assertion. Indeed the dual of $\varpi_v^{h_v} L_v$ in frequency space is $\varpi_v^{-h_v} L_v^\circ$. The two $K_p$-idempotent components are in perfect duality. Choosing dual bases and taking their pairwise sums gives a local orthonormal basis. The rational Cayley approximation in Theorem 122.14 gives a rational orthonormal basis spanning this same lattice at $p$, since generation of the local lattice is an open condition.

In the split position model the lattice vacuum at $v$ for $P_v \oplus P_v^\circ$ is constant on $P_v$ and supported there. These assertions follow respectively by invariance under translations by $P_v$ and modulations by $P_v^\circ$. At this odd prime the order-two characteristic on the product period lattice is trivial on its $p$-part. Lemma 65.12 fixes the relative normalization of the two unit vacua by their nonzero positive inner product. In particular, writing $\varphi_{P_v}$ for the new unit position vacuum, one has in this normalization

$$a_{0,v} \mathbf{1}_{y_v + P_v} = q_v^{-m h_v/2} \, W(y_v, 0) \varphi_{P_v}.$$

Indeed the scalar for $y_v = 0$ is the square root of the volume of $P_v$ divided by the square root of the volume of $L_v$. Translation gives the formula for every coset. This identity holds for negative as well as positive $h_v$.

To see that these are also the normalizations in the global product expansion, use the normalized lattice intertwiner taking the product vacuum to the original one. Its quotient of period lattices is the product of the local quotients. Its factors are the respective finite Weyl sums divided by the positive square roots of their indices. The coset and cocycle description in Theorem 65.13 identifies its $p$-factors with the intertwiners of Lemma 65.12. A different lift of a Weyl class only introduces a root of unity. At a prime away from $p$ each index is a power of that other prime, so all coefficients introduced there are units over $p$. The same holds for the moving split components and their characters by Theorem 122.4. They leave all $p$-components fixed. Passing from the adelic shift to a rational representative preserves its class in the new local lattice. This proves both the coefficient and translation assertions with the marked product realization. The change of rational basis affects the signed entries on this directed edge by the common scalar of Theorem 122.3. □

## 123.2 Traces of products of signed theta periods

Fix one field and one value of $x$ on the intervals of Section 6.6. Work at a fixed common definite endpoint, with its prescribed subgroup, levels and tests. Let $E_0$ be a finite-dimensional endpoint space spanned by algebraic class tests, and $M_u$ a source space of this kind for each word $u$. Take bases from the given finite spanning lists. A common endpoint can include a finite sum of types by taking their actual span in the same space of functions. The inner product on that span is the original Petersson inner product. Pairings below are written $y^* P x$, where a star means conjugate transpose. For a bilinear pairing use the conjugate of its row space. Apply this convention to the entire common row.

Write $w_0 G$ and $w_u H_u$ for the endpoint and source metric matrices. Here $w$ is the Haar volume of the archimedean group times the common finite integration level on the corresponding side. There are positive integers $d_0, d_u$ such that $d_0 G, d_u H_u$ have entries in the integers of a common conjugation-stable Galois field $J$. Indeed the integral of the product of two class coefficients is a finite sum of integrals with compact Haar probability divided by the orders of the chart stabilizers, times $w$. Intersecting the finite levels and expanding the two signed character sums gives this same formula on the projected space. A class change acts on the compact coefficient by a rational group element with algebraic matrix entries. Each remaining compact integrand

is polynomial in matrix entries and their conjugates. This includes an inverse determinant on a unitary group, where it equals the conjugate determinant. Clear the finitely many algebraic polynomial coefficients by a rational integer. Multiply by the least common multiple of the stabilizer orders and by the product over the real factors of the moment denominators of Theorem 95.6, using the largest degrees among these products of two coefficients. This gives an admissible $d$ for the chosen basis. Positive definiteness follows from independence as functions and positivity of the measure on every chart.

In these bases let $P_u$ have entries $\mathcal{A}_{ba}$ of Theorem 121.3, retaining the finite signed sum and its restriction to $U(Y_m)$. The physical pairing is $\kappa_u y^* P_u x$, with $\kappa_u = V_{J,u}(\prod_i C_i)c^{-t_u/2}$ for its fixed degree profile. A word with no surviving profile gives the zero matrix. Use positive-dimensional spaces below. A zero source contributes the zero operator. Put

$$A = \operatorname{adj}(d_0 G), \qquad A_u = \operatorname{adj}(d_u H_u), \qquad K_u = A P_u A_u P_u^*.$$

All adjugates of one by one matrices are 1. For the original finite tests write $D_u = D_{\mathrm{intr},u}$ using the maximum coefficient denominators and degree bounds across the chosen lists. Write $b_{u,p} = 2\sum_i g_{i,p}$ at a common good prime in $S$, and $b_{u,p} = -2\sum_i J_{i,p}$ elsewhere, with the sharpened loss of Theorem 126.2 at the selected split primes. All valuations have $v_{\mathfrak{p}}(p) = 1$.

**Theorem 123.2** (Arithmetic of the output Gram operators)**.** *At the original complex embedding, $K_u$ is a positive scalar multiple of the output Gram operator of the projected signed definite lift in the stated metric. For every $\mathfrak{p} \mid p$, every entry of $D_u^2 K_u$ has valuation at least $b_{u,p}$. In particular, for any list $u_1, \dots, u_k$, allowing repetitions,*

$$v_{\mathfrak{p}}\left(\Big(\prod_{j=1}^{k} D_{u_j}^2\Big)\operatorname{tr}(K_{u_1}\cdots K_{u_k})\right) \;\geq\; \sum_{j=1}^{k} b_{u_j,p}.$$

*If the listed operators are nonzero, dividing this trace by $\prod_j \operatorname{tr} K_{u_j}$ gives the normalized mixed trace in the physical Hilbert spaces. It is an element of $J$ independent of all the chosen bases, their integer clearings and the nonzero scalars $\kappa_u$.*

*Proof.* The matrix of the Riesz representative $T_u : M_u \to E_0$ is $\kappa_u (w_0 G)^{-1} P_u$. The adjoint is $\overline{\kappa_u}(w_u H_u)^{-1} P_u^*$, since the two metric matrices occur on opposite sides in the adjoint identity. Consequently

$$\begin{aligned} T_u T_u^\dagger &= \gamma_u G^{-1} P_u H_u^{-1} P_u^*, \quad \gamma_u = |\kappa_u|^2/(w_0 w_u), \\ K_u &= (\beta_u/\gamma_u) T_u T_u^\dagger, \qquad \beta_u = \det(d_0 G)\det(d_u H_u)/(d_0 d_u). \end{aligned}$$

Both scalars are positive. This proves the assertions about the normalized traces. In particular a change of basis in the common endpoint is one change of coordinates for every factor of the product.

All entries of $A, A_u$ are algebraic integers. Each entry of $K_u$ is a sum of products of two such coefficients with one entry of $P_u$ and the conjugate of an entry of the same $P_u$. Theorems 126.1 and 126.2 give exactly the stated bound on these pairs. Matrix multiplication and the ultrametric inequality prove the trace estimate. The two period factors from a word always occur together in this calculation. Thus its local choice of arithmetic frame may be made independently of the frames for the other words. □

The exterior traces also detect an entire incoming space at the last definite step. Take the source and target to be the full finite-level highest-line spaces with the original characters in Theorem 121.10. The integral lists of Theorem 122.11 span these spaces, since their covariances

are strictly positive there. Use the last matrix $P_*$ of normalized entries $z(U,V)$ and the corresponding physical lift. In this normalization $|\kappa_*| = w_- w_+ |C| c^{-d/2}/(D_- D_+)$. For $k_0 = \dim \mathcal{U}$ and $1 \le j \le k_0$ one has

$$\operatorname{tr}(\textstyle\bigwedge^j K_*) \;\ge\; \binom{k_0}{j} (\beta_* a_* / \gamma_*)^j > 0.$$

Indeed the squared norm of the physical lift of each vector of $\mathcal{U}$ is at least $a_*$ times its squared norm by Theorem 121.10. If fewer than $k_0$ eigenvalues of $T_*^\dagger T_*$ were at least $a_*$, the orthogonal complement of their eigenspaces would meet $\mathcal{U}$ nontrivially. This contradicts the bound. The positive eigenvalues on the output equal those on the source since $T_*$ identifies their eigenspaces. Taking the elementary symmetric sum of order $j$ gives the inequality. It concerns the last definite operator on the same field as the incoming Bessel vectors.

## 123.3 Elementary divisors on the same split path

Continue at the auxiliary place of Theorem 123.9, with residue characteristic $\ell$. Divide out the explicit unramified scalar $\eta$ in that theorem, and keep all its Haar and index factors. Let $\chi_0$ have order $N$ on $K^\times$. For a fixed tame character $\xi$, take the $k$ consecutive twists $\chi_j = \xi \chi_0^j$, $0 \le j < k \le N$. Choose a generator $g$ of $K^\times$ and the $N$ terminal frequencies obtained by multiplying the frequency of that theorem by lifts of $g^i$, $0 \le i < N$. Denote the resulting $N$-by-$k$ matrix of local values by $H$, with frequency as row index. Let $p \neq \ell$, take a coefficient field containing the finite character values, and work at a place over $p$ with valuation normalized by $v(p) = 1$.

**Theorem 123.3** (Integral minors for consecutive twists)**.** *The $k$ elementary divisors of $H$ are units for $p \nmid N$. For $N = N_0 p^e$ with $e \ge 1$ and $p \nmid N_0$, their valuations in increasing order, indexed by $0 \le j < k$, are*

$$\lambda_j = \frac{m_j + (p-1)\sum_{a=1}^{e-1} p^{a-1} \lfloor m_j / p^a \rfloor}{p^{e-1}(p-1)}, \qquad m_j = \lfloor j / N_0 \rfloor.$$

*In particular the least valuation among minors of order $h$ is $\sum_{j=0}^{h-1} \lambda_j$. The frequency with $i = 0$ has a unit entry for every twist. With all $N$ twists, the largest valuation is $v_p(N)$.*

*Proof.* Theorem 123.9 gives $H_{ij} = a_j \chi_j(g^i)^{-1}$ for one constant $a_j$ per column. Every $a_j$ is a unit at the chosen place. Indeed all its factors apart from the Gauss sums are roots of unity and integral powers of $q$. The Gauss sum of the trivial character is $-1$. For a nontrivial character it is an algebraic integer and satisfies $G(\chi)\overline{G(\chi)} = q$, by the direct character calculation in Theorem 123.6. Both factors are integral, so each is a unit above $p \neq \ell$. Removing these column units and the row units $\xi(g^i)^{-1}$ leaves the evaluation matrix $((\zeta^i)^j)_{i,j}$, with $\zeta = \chi_0(g)^{-1}$ primitive of order $N$. This also proves the statement at $i = 0$.

Use the monic polynomial of degree $j$ $\prod_{b<j}(Z - \zeta^b)$ as the new column in place of $Z^j$. The change of columns is integral with determinant one. Its entries above the leading diagonal are zero. At $i \ge j$ the entry is a unit times $\prod_{u=i-j+1}^{i}(1-\zeta^u)$. If $p \nmid N$, the distinct $N$-th roots remain distinct in the residue field, which proves the first case.

Now $v(1-\zeta^u) = 0$ unless $N_0 \mid u$, since reduction is injective on prime-to-$p$ roots of unity. For $u = N_0 v'$ with $0 < u < N$ its valuation is $p^{v_p(v')}/(p^{e-1}(p-1))$. To see this, a primitive $p^c$-th root $\alpha$ reduces to one. The factors $1-\alpha^l$, $1 \le l \le p^c$ and $p \nmid l$, have the same valuation because their quotients by $1-\alpha$ reduce to the nonzero residues of $l$. Their product is $\Phi_{p^c}(1) = p$, giving the formula for every $c$. In any interval of $j$ successive integers there are at least $\lfloor j/(N_0 p^a) \rfloor$ multiples of $N_0 p^a$. All these lower bounds are attained in $1, \dots, j$. Writing $p^h = 1 + (p-1)\sum_{a=1}^{h} p^{a-1}$ therefore gives the minimum $\lambda_j$ for column $j$, at $i = j$.

Divide each new column by its diagonal entry. The resulting matrix is integral and its leading $k$-rowed square is lower triangular with unit diagonal. Integral row operations make that square the identity and then clear the remaining rows. The original elementary divisors are consequently the diagonal entries up to units, in their stated order. For $k = N$ the last diagonal valuation is the valuation of $\prod_{u=1}^{N-1}(1-\zeta^u) = N$. The assertion about the minors follows by taking the diagonal form. □

The local calculation distinguishes a single common nonzero test from the recovery of individual twists after summation. For example, take the full character family and multiply each column of $H$ by $a_j^{-1}$. Their sum has value $N$ at the first frequency and zero at the other $N-1$ frequencies, by orthogonality. Thus signed summation can change the local content at a prime dividing $N$, even though each column is primitive there.

## 123.4 The two components of the finite shift at a split prime

Suppose $p$ splits in the marked imaginary quadratic field $K$. Denote the two idempotents of $\mathcal{O}_K \otimes \mathbb{Z}_p$ by $e_+, e_-$, in either order. For the list of finite tests of Theorem 122.4, let $a_\epsilon$ be the least nonnegative integer for which $p^{a_\epsilon} e_\epsilon b_p$ belongs to $e_\epsilon M_p$ for every occurring $b$, with $\epsilon = +, -$. Both are at most $v_p(Q^0)$.

**Theorem 123.4** (A split-prime estimate in the marked CM differential)**.** *At a coefficient place restricting to $+$ on $K$, a degree-$d$ class derivative in the product basis loses at most $da_-$ in valuation from translations. At a place over $-$ the corresponding loss is at most $da_+$. Thus the two translation losses in a product of a class derivative with a conjugate derivative of the same degree sum to at most $d(a_+ + a_-)$. The assertion applies after the class actions at both ends of the edge.*

*Proof.* The localized product basis of Theorem 122.14 gives an isomorphism of $\mathcal{O}_K \otimes \mathbb{Z}_p$-modules with the standard product lattice. Its quotient map commutes with $e_+$ and $e_-$. By Theorem 122.4 the class actions and the vacuum intertwiners have introduced no further class modulo this lattice at $p$. For each component $P$ on the CM elliptic curve, passage from an additive shift to the torsion point multiplies the coordinate and its period lattice by the same elliptic period. This map on quotient modules is $\mathcal{O}_K$-linear. Hence $p^{a_\epsilon}$ kills the $\epsilon$-part of the $p$-primary component of $P$.

Let $I = \mathrm{Ann}_{\mathcal{O}_K}(P)$. The cyclic module $\mathcal{O}_K P$ is $\mathcal{O}_K/I$. Tensoring this description with either local factor shows that the localized ideal $I$ is the annihilator of the localized point. The exponent of $I$ at $-$ is therefore at most $a_-$. At $+$ this is the exponent of $\bar{I}$, which occurs in Theorem 66.35. The normalized valuation agrees with this exponent since $K$ is split at $p$. There $\kappa_p = 0$, so an ordinary derivative of order $h$ of this translated factor loses at most $ha_-$. In total order $d$ the orders on the separate factors sum to $d$. This proves the bound above $+$. Interchanging the local factors proves the other bound. Complex conjugation exchanges the two places on $K$, which proves the product assertion. □

For the indicator in Proposition 72.15 the two orders can be read directly from its position lattice. More generally, let $L_0$ be the position lattice of the vacuum of amplitude $a_0$, and let a test of amplitude $a$ have support $y + P$ with $P \subset L_0$ a full sublattice. For $x - y \in L_0$, orthogonality of characters gives

$$\mathbf{1}_P(x-y) = \frac{1}{[L_0 : P]} \sum_{\xi \in P^\vee / L_0^\vee} \psi(\langle x-y, \xi \rangle).$$

Multiplication by the translated vacuum supplies the condition $x - y \in L_0$. This is the Weyl expansion with coefficients $(a/a_0)/[L_0 : P]$ times roots of unity in the symmetric Weyl convention. Its position classes have the order of $y$ modulo $L_0$. Its frequency classes have maximum order the exponent of $L_0/P$, by the perfect character pairing.

In particular, at a rational prime completely split in $E$, take the position half over a chosen prime of $K$, say $+$. At a place $v \mid p$ the congruence indicator has $y = \binom{1_n}{0} \in L_0$, $P = p^h L_0$ and $[L_0 : P] = p^{hns}$ for $s$-by-$n$ position matrices. The finite Fourier denominator is still $p^{hns}$ for unit relative amplitude. The contribution of this indicator to the component exponents is $0, h$. If it is the only nonspherical factor above $p$, the translation cost in the conjugate product is at most $dh$. The same holds for several such factors with coherent orientation and maximum depth $h$. If the chosen position halves vary over $v \mid p$, the two maxima must be taken in their actual idempotent components.

## 123.5 The local cost of the position cosets

Use one edge $i$ and the expansion of Theorem 123.1 at an odd split prime $p$. Write $A_p$ for the maximum of 0 and the negative normalized valuations of the nonzero $d_{\lambda,\mathbf{y}}$ in this list, at all places over $p$. For each nonzero term let

$$k_v(y_v) = \min\{k \geq 0 : p^k y_v \in P_v\}.$$

This is the exponent of $p$ in the additive order of the position class. For $\epsilon = +, -$ let $b_\epsilon$ be the maximum of these integers over the terms and the $v \mid p$ whose position prime $w_v$ restricts to $\epsilon$ on $K$, using zero for an empty maximum. Put

$$J_{i,p}^{\mathrm{pos}} = A_p + \tfrac{1}{2} d_i (b_+ + b_-) + \tfrac{1}{2} d_i \rho_p.$$

**Theorem 123.5** (Conjugate derivative bounds from the support of the finite test)**.** *For two edge derivatives of the prescribed degree profile, one with the orientation of the edge and the other conjugated, the sum of their normalized valuations is at least $-2J_{i,p}^{\mathrm{pos}}$ in the product coordinates of Theorem 123.1, after converting derivatives back to the physical matrix coordinates. This bound holds for every pair of class sections on the input list. Among homothetic position lattices $P_v$ of the given shape at all $v \mid p$ the largest ones on whose cosets the list is constant minimize this upper bound on the loss.*

*Proof.* The coefficient losses are at most $A_p$ in each copy. The product basis identifies the local trace lattice with a free $\mathcal{O}_K \otimes \mathbb{Z}_p$-module and commutes with both CM idempotents. In its coordinates the $\epsilon$-parts of all elliptic torsion points have order dividing $p^{b_\epsilon}$ in their $p$-primary components, by Theorem 123.1. Localizing the conjugate of the annihilator of such a point exchanges the two components. Theorem 66.35 therefore costs at most $d_i b_-$ at a place over $+$ and $d_i b_+$ at a place over $-$. This is the proof of Theorem 123.4 in the new product lattice. The Leibniz expansion of a derivative of order $d_i$ has exactly $d_i$ differentiations in total.

For the coordinate conversion, the physical entries of a product basis vector belong locally to $\mathfrak{c}\mathfrak{b}^{-1}$. At block $\tau$ the coefficients of a physical derivative in the product variables are the conjugate entries, as in Theorem 122.14. Thus the lower bound per occurrence is $v_{\mathfrak{p}}(\bar{\tau}\mathfrak{c}) - v_{\mathfrak{p}}(\bar{\tau}\mathfrak{b})$. The conjugate copy gives the same expression with $\tau$ in place of $\bar{\tau}$. Their sum is $-\rho_p$, because $\mathfrak{c}\bar{\mathfrak{c}} = \mathcal{O}_E$ and $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K}$. The two derivatives have the same number $d_{i,\tau}$ of occurrences at every block. Multiplying the bounds and summing the terms of the coordinate expansions proves the assertion.

For the minimizing choice omit an identically zero list. The translations of one position variable fixing every input form a subgroup containing $\varpi_v^h L_v$ for large $h$ by local constancy. Its

elements lie in the difference of a nonempty compact support with itself, by using a nonzero coefficient function with the other variables fixed. Hence it is bounded. There is a least integer $h_v$ such that $\varpi_v^{h_v} L_v$ fixes the list. The separate subgroups act in separate variables, so the simultaneous largest choice is admissible. Shrinking any $P_v$ subdivides each nonzero coset with the same relative amplitude on every part. In the normalized coefficients this multiplies each copy by the inverse square root of the subdivision index. The maximum loss $A_p$ can only increase. Every new class maps to its old class, so the two torsion exponents can only increase as well. □

For a single congruence test at a place of a rational prime completely split in $E$, take the position model with unit relative amplitude of Proposition 72.15. Its support is $y+p^h L_v$, where $y = \binom{1_n}{0}$ has a unit entry and $m = ns$. With spherical unit vacua elsewhere over $p$, $A_p = hns/2$ and $b_+ + b_- = h$. The raw Fourier expansion in $L_v$ has every nonzero coefficient of valuation $-hns$ by Theorem 123.4. The support calculation reduces the bound $hns+dh/2$ to $h(ns+d)/2$ for degree $d$. It remains valid for any list of translates by integral invertible position matrices with unit amplitude factors. Such matrices preserve $p^h L_v$ and preserve primitivity of $y$. In particular this includes the two congruence groups fixing the prescribed test.

There is a second case in the character transport of Proposition 94.17. In its Haar normalization the transported spherical test at $v_*$ has value $|\xi_w|_v^{m/2} a_{0,v}$ on $P_v = \xi_w^{-1} L_v$. If $\mathrm{ord}_v(\xi_w) = s$, then $h_v = -s$. The relative amplitude $q_v^{-ms/2}$ multiplied by $q_v^{-mh_v/2}$ is exactly one. Its position class is zero. Thus both the normalization of this dilation and its change of support have zero local cost. An additional splitting scalar in a chosen realization remains in the actual coefficient list. If it is common to the entries and has absolute value one, its product with its conjugate is one. Distinct phases in a signed sum are retained in that sum.

## 123.6 The last column at a split twist place

Fix a split auxiliary place of a tuple on the interval of Section 6.6. Use the plain frames for its tame twist. Write $k$ for the local field, $O$ for its integer ring, $\pi$ for a uniformizer and $K$ for its residue field of size $q$. Use $|\pi| = q^{-1}$, additive volume one on $O$ and an additive character $\psi$ of conductor $O$. Thus $\psi$ is trivial on $O$ and nontrivial on $\pi^{-1}O$. Signs below use the Fourier kernel $\psi(-zy)$. In the two integral frames let $E \subset K^{n+1}$ and $U \subset K^d$ be the distinguished coordinate spaces of the same dimension $r \geq 1$. Keep their ordered bases. For a matrix $Z$ of size $d$ by $n+1$, the parahoric Schrödinger test of the penultimate edge is

$$\Phi(Z) = \begin{cases} \chi(\det(\bar{Z}|_E : E \longrightarrow U)), & Z \text{ integral and } \bar{Z}|_E : E \xrightarrow{\sim} U, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\chi$ is the inverse of the tame splitting character on residue units. Indeed its defining blocks require the upper left block to be integral invertible, the lower left block to vanish after reduction, and both right blocks to be integral. These conditions and the displayed determinant give the formula in the ordered frames.

Fix the actual source representative $h \in \mathrm{GL}_{n+1}(k)$ and write $[X \mid y]$ in the original last-column order, inserting the frame permutation into $h$. The source action on the position function is $\mu(\det h)|\det h|^{d/2}\Phi([X \mid y]h)$, with the corresponding fixed determinant phase if a permutation is used to express the action. Denote this scalar by $c_h$, using the chosen original frames throughout. The other edge supplies the unchanged top test $\Phi^+(B)$. It is enough to evaluate the following factor of the coupled kernel:

$$D_h(X, z) = \int_{k^d} \Phi([X \mid y]h)\psi(-zy)\, dy.$$

The original kernel is $c_h\Phi^+(B)D_h(X,z)$.

Write $b$ for the last row of $h$, $W=[X\mid 0]h$. Choose $i$ with $|b_i|$ maximal and put

$$\beta=b_i,\quad y_0=-W_i/\beta,\quad W_0=W+y_0b,\quad z'=\beta^{-1}z,$$

where $W_i$ is the $i$-th column. If $W_0$ is integral, write $A=\bar{W}_0|_E$ and $\lambda=\overline{b/\beta}|_E$. Fix a nontrivial additive character $\theta$ of $K$. For $z'\in\pi^{-1}O^{1\times d}$ let $l:K^d\to K$ be the unique linear form with $\psi(-z'u)=\theta(l(\bar{u}))$ on $O^d$.

**Theorem 123.6** (Fourier integral of the marked last column)**.** *If $W_0$ is nonintegral then $D_h=0$. In the remaining cases divide $D_h$ by $|\beta|^{-d}\psi(-zy_0)$. For $\lambda=0$ the result is zero unless $A:E\to U$ is an isomorphism and $z'$ is integral. When both conditions hold the result is $\chi(\det A)$.*

*For $\lambda\neq 0$ it is zero unless $A$ maps $\ker\lambda$ injectively into $U$ and $z'\in\pi^{-1}O^{1\times d}$. Under these two conditions take $e\in E$ with $\lambda(e)=1$. For $v\in U$ define $L(v)$ as the determinant, in the marked bases of $E,U$, of the map that agrees with $A$ on $\ker\lambda$ and sends $e$ to $v$. The answer is zero unless $l|_U=tL$ for some $t\in K$. In that event it is*

$$q^{r-1-d}\theta(-l(Ae))G_t(\chi),\qquad G_t(\chi)=\sum_{a\in K^\times}\chi(a)\theta(ta).$$

*For nontrivial $\chi$, $G_0=0$ and every $t\neq 0$ gives $|G_t|=\sqrt{q}$. For trivial $\chi$ the two values are $q-1$ at zero and $-1$ elsewhere.*

*Proof.* The $i$-th column of $W+yb$ is $\beta(y-y_0)$. Hence integrality of that column requires $y=y_0+\beta^{-1}u$, $u\in O^d$, with measure $|\beta|^{-d}du$. In this parametrization the full matrix is $W_0+u(b/\beta)$. The second term is integral, so integrality of $W_0$ is both necessary and sufficient. On $E$ the reduced matrix is $A+\bar{u}\lambda$. If $\lambda=0$ its value is independent of $u$. Integration of a character on $O^d$ gives one exactly on the annihilator $O^{1\times d}$ and zero elsewhere, by translation invariance and the conductor convention.

In the second case the function of $u$ is constant on each coset modulo $\pi O^d$. Translation by $\pi O^d$ therefore gives the asserted frequency restriction. On these frequencies the additive character factors through $K^d$. The characters $x\mapsto\theta(ax)$, $a\in K$, are distinct since multiplication by a nonzero element is onto. They exhaust the additive dual by cardinality, proving the assertion about $l$ in each coordinate. Every residue class has measure $q^{-d}$. The restriction on $\ker\lambda$ follows from the invertibility required on $E$.

Set $U_0=A(\ker\lambda)$. When the restriction on the kernel holds, $L$ is a nonzero linear form on $U$ with kernel $U_0$, by expansion of a determinant in its last column. The admissible $\bar{u}$ are precisely those for which $v=Ae+\bar{u}$ lies in $U$ and $L(v)\neq 0$. Summing $\theta(l(v))$ on a fiber of $L$ gives zero if $l$ is nonzero on $U_0$, since the character is then nontrivial there. Otherwise $l|_U=tL$, and the fiber over $a\neq 0$ contributes $q^{r-1}\theta(ta)\chi(a)$, apart from the common factor $\theta(-l(Ae))$. This proves the evaluation.

For nontrivial $\chi$, summing it over $K^\times$ gives zero. For $t\neq 0$ substitution gives $G_t=\chi(t)^{-1}G_1$. Additive character orthogonality gives $\sum_t|G_t|^2=q(q-1)$, by expanding the square and summing over $t$ first. This gives the claimed magnitude. The same orthogonality gives the two elementary values for the trivial character. □

For the identity representative $h=1$ before permuting bases, the separated column lies in the distinguished source block of the plain frame. This follows directly by ordering the extra coordinate among the $s+1$ distinguished columns at the penultimate stage. Thus $\lambda\neq 0$. Choose integral $X$ which sends the other $r-1$ marked columns to $r-1$ distinct marked target

basis vectors and is zero on the other columns. These choices constrain only the residue of $X$. Choose $B$ with identity marked block and all other entries zero. Then $\Phi^{+}(B) = 1$. As $z$ runs through lifts of forms extending $tL$ for $t \neq 0$, the same positions and frequencies work for every nontrivial tame twist. At an integral $z$ the factor vanishes for every such twist. To use the identity representative in the global chain one must also show that the preceding edges reach its double coset. The general formula above applies to the representatives those edges actually reach. The local definitions are `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/StandardOrderExtendedBasisApplyLowBlock.lean`, line 399, 650, 796, 889–920.

## 123.7 Compact integration along the supported path

Fix the split place, representatives $h_j$, primitive core vectors $z_j$ for $j > 0$, and matrices of Theorem 123.9. At vertex $j \le s$ write $A_j \subset K^{n_j}$ for the marked core plane and $P_j$ for its stabilizer in $\mathrm{GL}_{n_j}(O)$ after reduction. The full parahoric $J_j$ preserves $A_j$ together with the flag coordinates in rank $D_j$. Let $\Gamma_j$ be the image in the core group of the intersection of $h_j J_j h_j^{-1}$ with the semidirect product of the core general linear group and the flag unipotent group. This is the compact group defining the middle weight $w_j = \mathrm{vol}(\Gamma_j)^{-1}$. Every full integral general linear group has volume one in its own rank. Put $d = n_{s+1}$, let $U \subset K^d$ be the $r$-dimensional marked target of the penultimate test, and write $J \subset \mathrm{GL}_d(O)$ for its parahoric.

**Theorem 123.7** (Middle weights and the terminal orbit)**.** *One has $w_0 = 1$. For $1 \le j \le s$ the weight is*

$$w_j = \begin{bmatrix} n_j \\ a_j \end{bmatrix}_q q^{j n_j} (1 - q^{-a_j}),$$

*where the Gaussian binomial coefficient counts planes. Each compact integration over $\Gamma_j$ cancels the corresponding weight in the product of consecutive kernel entries, with the core action at both adjacent edges. At the initial endpoint this assertion uses the matching source character, or equivalently holds with that endpoint kept fixed.*

*For the final two entries, take the coset lattice $\pi^M O^d$ of Theorem 123.12 with $M \ge s+1$. At the marked positions of Theorem 123.9 every contributing coset belongs to the single $J$-orbit of the primitive integral vectors with reduction in $U$. This orbit has cardinality $(q^r - 1)q^{(M-1)d}$. Its representative, with weight $[\mathrm{GL}_d(O) : J]$ times the orbit size and with the original Haar integral over $J$, contributes exactly the full signed last-column value. These assertions hold for each final unit multiple of the frequency and for each tame twist on the same levels.*

*Proof.* At $j = 0$ the core is entirely marked and $h_0 = 1$, so its projected compact is the full integral group. At $j > 0$ a lift of a core element $g$ to the semidirect product has diagonal blocks $g, I + H$, lower left zero, and upper right $B$ with first column zero. Here $H$ is strictly upper triangular. Conjugating by $h_j^{-1}$ gives upper right block

$$\pi^{-j}(B + (gz_j - z_j)e_0^t - z_j e_0^t H).$$

In column zero the free matrices $B, H$ have no effect. On the core one must have $g \in P_j$. The additional condition is that the marked coordinates of $gz_j - z_j$ belong to $\pi^j O$ and the unmarked coordinates to $\pi^{j+1}O$. These conditions suffice, since we can take $B = H = 0$. Thus $\Gamma_j$ is exactly the stabilizer of the residue of $z_j$ modulo this mixed lattice in $P_j$.

The orbit consists of a primitive marked column modulo $\pi^j$, together with arbitrary unmarked entries in $\pi O/\pi^{j+1}O$. To obtain such a column, complete its marked lift to an invertible marked block over $O$, use the given unmarked entries in the lower left block, and set the

lower right block to the identity. This proves both inclusions in the orbit description. Its size is $(q^{ja_j} - q^{(j-1)a_j})q^{j(n_j-a_j)}$. Multiplication by $[\mathrm{GL}_{n_j}(O) : P_j] = \begin{bmatrix} n_j \\ a_j \end{bmatrix}_q$ proves the weight formula.

For $g \in \Gamma_j$, the lift $h_j^{-1}\mathrm{diag}(g, I)h_j$ belongs to the parahoric and its determinant on the marked plane is $\det(\bar{g}|_{A_j})$. This follows from its block triangular form and identity flag block. On replacing $X_{j-1}$ by $gX_{j-1}$ in the preceding integral, replace each upper free vector $Y_i$ by $gY_i$ as well. The measure is preserved and the phase on the lower free entries is fixed. The resulting multiplier on the position test is $\chi(\det(\bar{g}|_{A_j}))$. At the next edge $X_j \mapsto X_j g^{-1}$ amounts to right multiplication of its affine matrix by $\mathrm{diag}(g^{-1}, I)$. This gives the inverse multiplier. The calculation includes the penultimate test by keeping the separated last column fixed. The two compact Weil splittings also cancel, since consecutive $\mu_i$ agree on units. Thus the joined integrand is constant under each of these simultaneous changes of core charts. An integral over $\Gamma_j$ contributes exactly its volume, which proves the cancellation. At the input the same calculation uses the inverse multiplier on a matching test, or one simply omits the input average and its unit weight.

At the last variable the support condition is $y = -X_s z_s + \pi^s v$, where $\bar{v}$ lies in $U$ outside the span of the $r - 1$ marked columns of $X_s$. For $s \geq 1$, reduction of $y$ is the fixed nonzero vector $-\bar{X}_s \bar{z}_s \in U$. For $s = 0$ reduction of $y = v$ is again a nonzero vector in $U$. The orbit of all these primitive classes modulo $\pi^M$ consists of exactly the classes specified in the statement: the marked block acts transitively on nonzero residue vectors, and the principal congruence subgroup moves any primitive lift to any other with the same reduction. For the latter fact, extend a primitive vector to an integral basis and change its first basis vector by the specified multiple of $\pi$. Counting the lifts gives the cardinality. A scalar support lattice containing a primitive position contains $O^d$; hence the full orbit is present in the coset space used for the decomposition.

Write $P_a, T_a$ for the two pieces of one last coset. Over probability Haar on $J$, the integral of $P_a(kX_s)T_a(Bk^{-1}, zk^{-1})$ is the same for all $a$ in the orbit by Theorem 123.12. Their sum is the entire kernel just computed, since at the marked positions the other orbits have zero summands. It has the same value after simultaneous translation by $k$, by permutation of the cosets and cancellation of the two characters. Therefore this probability average equals that entire value divided by the orbit size. Restoring the specified measure on $J$ divides it by $[\mathrm{GL}_d(O) : J]$. The assigned weight cancels both factors. The different core and final group actions commute on the joined product since they multiply on opposite sides of each shared position matrix. All these integrations can therefore be performed together. □

The computation concerns the compact integrations on this local path. Summing the other interior double cosets and then projecting the global automorphic row are further operations. The local projected group and weights appear in `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/StandardOrderExtendedBasisApplyLowBlock.lean`, line 675–701 and `Development/AbelianVarietyTheory/Definitions/LatticeFrames/CosInj.lean`, line 110–118.

## 123.8 Reading every class of the standard terminal row

Take the standard definite terminal group $G = U(E^n)$ over $F = E^+$ in its given orthonormal frame, on a field of Section 6.6. Let $G_\infty = \prod_{v|\infty} G(F_v)$ and fix a coefficient-field embedding $\sigma : K \to \mathbb{C}$. Use the terminal polynomial module $V_K$ of Theorem 122.6, including its prescribed determinant character. Its coefficient map $C : V_K \to V_{\mathbb{C}}$ applies $\sigma$ in the same polynomial frame at every tuple. Write $\rho$ for the resulting compact representation. Its invariant norm is the Hermitian Fischer norm divided by the squared extremal norm $H$. On choosing the dual

determinant generator, write $B$ for the corresponding normalized bilinear pairing as in that theorem.

At a tuple $\xi$ use its actual finite level $J_\xi$, its finite character $\omega_\xi$ in the right translation convention, and the smaller level $J_\xi^1$ that fixes every marked residue frame. Given $g \in G(\mathbb{A}_{F,f})$, write $D_\xi(g)$ for the lattice with local completions $g_v\Lambda_v$, marked at the indicated primes by the reductions of the coordinates of $g_v^{-1}$. Use the genus of the original lattice. A table of class values assigns $w_i \in V_K$ to representatives $g_i$ for $G(F)\backslash G(\mathbb{A}_{F,f})/J_\xi$. Put $\Gamma_i = G(F) \cap g_i J_\xi g_i^{-1}$. The table satisfies $\rho_K(\gamma)w_i = \omega_\xi(g_i^{-1}\gamma g_i)w_i$ for $\gamma \in \Gamma_i$, using $K$-valued finite characters and the rational coefficient representation.

**Theorem 123.8** (Class values and the projected terminal norm)**.** *For every such table there is a unique equivariant coefficient function $\phi$ on the decorated genus with these values and this finite character. Its standard adelic form, evaluated against any $\ell \in V_{\mathbb{C}}^*$, is given at every finite adele by*

$$f_{\phi,\ell}(k,g) = \ell(\rho(k)^{-1}C(\phi(D_\xi(g)))), \qquad k \in G_\infty.$$

*The same coefficient map works for every tuple. There is one linear map $j : V_{\mathbb{C}}^* \to V_{\mathbb{C}}$ with $\ell(Cw) = B(j\ell, Cw)$ for every $w$, common to all classes and tuples.*

*Let $f_\ell$ be an independently given automorphic family linear in the dual test, of the same compact and finite types, after the prescribed projection commuting with right translation at infinity. Write $z_i$ for its coefficient vector at $(1, g_i)$, so $\ell(z_i) = f_\ell(1, g_i)$. For any orthonormal basis $(\ell_a)$ of the compact dual, with probability Haar at infinity, the original Haar at the finite level and counting measure on $G(F)$, one has*

$$\sum_a \|f_{\ell_a} - f_{\phi,\ell_a}\|_{[G]}^2 = \mathrm{vol}(J_\xi) \sum_i \frac{\|z_i - Cw_i\|^2}{|\Gamma_i|}.$$

*If the differences are algebraic in the given coefficient coordinates, testing each difference by the integral flag minors with $B$ detects its divisibility as in Theorem 122.6, at every finite coefficient place in the corresponding local frames.*

*Proof.* Intersect $E^n$ with the product of the local lattices $g_v\Lambda_v$. It contains a nonzero integer multiple of the original lattice and is contained in the reciprocal multiple, so it is a full lattice. Its completion gives each of the specified lattices. Indeed between these two bounds one works in a finite quotient of the original lattice, which is the product of its local quotients by the Chinese remainder theorem. The marked coordinate reductions are surjective on each specified residue frame because they are transported from the standard one. Equality of two decorated lattices now says that their ratio preserves the standard lattices locally and induces the identity in the marked coordinates. This is precisely membership in $J_\xi^1$. Thus finite adeles modulo $J_\xi^1$ parametrize the decorated genus, with the action of $J_\xi/J_\xi^1$ by changes of marking.

If $g = \gamma g_i u$, set $\phi(D_\xi(g)) = \omega_\xi(u)\rho_K(\gamma)w_i$. Two such expressions for $g$ differ by a member of $\Gamma_i$. Its action on $w_i$ is exactly the finite character in the table, so the two values agree. This extends restriction to the representatives inversely, including when a stabilizer forces some values to vanish. Scalar extension of the polynomial and determinant actions intertwines $\rho_K(\gamma)$ with $\rho(\gamma_\infty)$. Hence the displayed adelic function is invariant under diagonal rational translation, has the asserted finite character, and transforms on the right at infinity by the dual test action. Conversely these laws recover the table by evaluation at $k = 1, g = g_i$. This gives the inverse-restriction realization of the standard terminal row.

Coefficient conjugation in the chosen polynomial frame preserves the complex space since its defining minor products have integer coefficients. For a nonzero $v$ one has $B(\bar{v}, v) > 0$ by the monomial formula. Consequently $j\ell$ is the unique solution to $B(j\ell, v) = \ell(v)$ for every $v$. This

proves linearity and independence of $\xi$. More generally an already specified bilinear pairing $\mathrm{pr}(v,e)$ with a terminal test $e$ gives the solution $j(\mathrm{pr}(\,\cdot\,,e))$, with its given scalar. Compact covariance follows by applying the defining equality to $\rho(k)^{-1}v$.

The class set is finite by finiteness of the genus of a definite lattice and finiteness of the residue markings. Each $\Gamma_i$ is finite since it preserves the definite trace lattice. The contribution of its class to Haar integration is integration on $G_\infty \times J_\xi$ with factor $1/|\Gamma_i|$, since diagonal rational translation identifies two points of this parametrization precisely by that finite group, acting freely. On this piece the difference uses the vector $z_i - Cw_i$ transformed by a unitary compact operator and a finite scalar of modulus one. The sum of the squared evaluations over $\ell_a$ is its squared norm by Parseval. Integration gives the formula, including the stabilizer and level volume. The local divisibility assertion follows by testing in the dual frame at that class and applying the integral duality theorem. □

The construction gives the actual coefficient reading for the standard adelic realization of a decorated table. For the independently projected theta family the right side is an exact discrepancy at the prescribed level. Controlling those class vectors, or proving their agreement with the arithmetic ones in the physical normalization, remains a global period problem. The table restriction and its inverse are described in `Formalism/LocalToGlobal/Columns/RhoEqOfSmulEq.lean`, line 157–193. The standard adelic realization uses that inverse in `Development/AbelianVarietyTheory/Definitions/AutomorphicMembership/IsWtOfParUnique.lean`, line 327–346.

## 123.9 A nonzero path through the split twist kernels

At one auxiliary split place keep $k, O, \pi, K, q, \psi, \chi$ of Theorem 123.6. The residue character $\chi$ may be trivial. Write $s = m - 2$, with $m \geq 2$. In the plain frames put

$$a_j = k_0 + j, \quad n_j = a_j + jt, \quad r = k_0 + s + 1, \qquad D_j = n_j + s + 1 - j = r + jt \quad (0 \leq j \leq s+1).$$

Here $k_0$ is the initial core dimension and $t$ the rank increment in the full space. The $D_j$ coordinates are the $n_j$ core coordinates and $s + 1 - j$ flag coordinates. The marked $r$-plane consists of the first $a_j$ core coordinates and all the flag coordinates. In these bases each edge uses the parahoric test $\Phi$ of Theorem 123.6. The splitting characters are $\mu_i = \nu_i\delta$, with $\nu_i$ unramified and $\chi$ the inverse of $\delta$ on residue units. Put $\theta(v) = \psi(-\pi^{-1}\widetilde{v})$ for $v \in K$, where $\widetilde{v}$ is any integral lift. Set $G(\chi) = \sum_{v \neq 0} \chi(v)\theta(v)$, summing in $K$.

We describe precisely the interior entry used below. At edge $j < s$ put $c = s - j$. With flag indices starting at zero, the affine matrix has core block $X_j$, lower left block zero, and $c + 1$ remaining columns. Column zero is $(0, e_0)$. For $1 \leq i \leq c$ its upper vector $Y_i$ is free. Its lower entry in row $l$, $0 \leq l < c$, is $b_{li}$ for $l < i$, one for $l = i$, and zero for $l > i$. Call this matrix $A$. The entry at representatives $h_j, h_{j+1}$ integrates $\Phi(h_{j+1}^{-1}Ah_j)\psi(-\sum_{l<c} b_{l,l+1})$ over the free entries with product additive Haar. Multiply by the original Weil scalar

$$C_j = \frac{\mu_{j+1}(\det h_j)}{\mu_{j+1}(\det h_{j+1})} |\det h_j|^{D_{j+1}/2} |\det h_{j+1}|^{-D_j/2}$$

and by the reciprocal of the relative unipotent volume. The latter is the volume of $N \cap h_{j+1}Jh_{j+1}^{-1}$ divided by the volume of $N(O)$, before taking the reciprocal. Here $J$ is the target parahoric and $N$ has identity diagonal, free upper flag entries, and free core entries over flag columns $1, \ldots, c-1$. At $j = s$ use the Fourier integral in the separated last column with its source Weil scalar and the unchanged top test. Take $h_{s+1} = 1$ in $C_s$.

**Theorem 123.9** (One supported path for every tame twist)**.** *There are representatives starting at $h_0 = 1$, core positions and a terminal frequency, all independent of $\delta$, for which every*

*consecutive local entry is nonzero. Their product, with the preceding normalizations, is*

$$\eta\,\chi(u)\,q^{\,r-1-(s+1)n_{s+1}}\,G(\chi)^{\,1+s(s+1)/2},\qquad \eta=\prod_{j=1}^{s}(\nu_{j+1}/\nu_j)(\det h_j).$$

*Here $u\in K^\times$ comes from the ordered marked bases. Multiplying the last frequency by an integral unit of residue $b$ multiplies the product by $\chi(b)^{-1}$. The representatives define a successive choice of supported double cosets in the local unfolding, on the given levels.*

*Proof.* Put $z_0=0$. For $1\le j\le s$ take $z_j$ to be a marked core basis vector. With $e_0^t$ extracting the first flag coordinate, choose

$$h_j=\begin{pmatrix} I & \pi^{-j}z_j e_0^t\\ 0 & \pi^{-j}I\end{pmatrix}\quad(0\le j\le s).$$

At $j=0<s$ take $X_0$ to identify the marked core with a coordinate complement of $z_1$ in the next marked core. For $1\le j<s$ take a further marked basis vector $w\neq z_{j+1}$ in the next core. Send $z_j$ to $z_{j+1}+\pi^j w$ and the other marked basis vectors bijectively to the remaining marked basis vectors other than $z_{j+1},w$. Send unmarked vectors to zero. In both cases the marked columns of $X_j$, together with $v_j=\pi^{-j}(X_jz_j-z_{j+1})$, give an integral basis of the next marked core. At $j=0$ this last vector is $-z_1$.

In $h_{j+1}^{-1}Ah_j$ the extra column zero is $(v_j,\pi e_0)$. In each extra column $i\ge1$ the upper free vector becomes $Y_i'=\pi^{-j}(Y_i-z_{j+1}b_{0i})$. The lower free entries become $v_{li}=\pi b_{li}$; each original lower diagonal one becomes $\pi$. The Jacobian from these new variables back to the old measure is $q^{c(c+1)/2-jn_{j+1}c}$, by triangular change of variables. Integrality requires the new variables to be integral. The $n_{j+1}-a_{j+1}$ unmarked coordinates in each $Y_i'$ must vanish modulo $\pi$, at volume cost $q^{-(n_{j+1}-a_{j+1})c}$.

Reduce the marked determinant. Its first $a_j+1$ columns form the basis just found in the upper rows, with zero lower entries. The lower square on columns $1,\dots,c$ is upper triangular with diagonal $\bar v_{l,l+1}$, $0\le l<c$. The determinant is a fixed residue unit $u_j$ times the product of these $c$ entries. The additive character is the product of $\theta(\bar v_{l,l+1})$ on the same entries. Each gives the one-dimensional integral $q^{-1}G(\chi)$, and all remaining unrestricted integral coordinates have volume one. Thus before the two exterior scalars this edge contributes

$$\chi(u_j)G(\chi)^c\,q^{\binom{c}{2}-jn_{j+1}c-(n_{j+1}-a_{j+1})c}.$$

For its unipotent volume write a group element of $N$ as $\left(\begin{smallmatrix} I & B\\ 0 & I+H\end{smallmatrix}\right)$, with $H$ strictly upper triangular and $B_0=0$. Product measure in the free entries is Haar. Left translation is affine triangular with unit diagonal in a column ordering. Conjugating by $h_{j+1}^{-1}$ gives the upper block $\pi^{-j-1}(B-z_{j+1}e_0^tH)$ and lower block $I+H$. Membership in $J$ requires integral $H$. For each $1\le i<c$, the translated upper column lies in $\pi^{j+1}O^{n_{j+1}}$ before the division, with a further factor $\pi$ required in its unmarked rows. These conditions are sufficient because $J$ is the inverse image of the marked plane stabilizer. The reciprocal relative volume is therefore exactly

$$q^{(c-1)((j+1)n_{j+1}+n_{j+1}-a_{j+1})}.$$

At the final split pair the source matrix gives $[X_s\mid\pi^{-s}(X_sz_s+y)]$. Choose $X_s$ to identify its marked core with the first $r-1$ marked target vectors and to vanish on the other columns. Use an integral covector $l_0$ vanishing on this image whose residue restricts to the oriented determinant form $L$ of Theorem 123.6 on the marked target. Take frequency $\pi^{-s-1}l_0$ and top test with identity marked block and all other entries zero. Substitution $y=-X_sz_s+$

$\pi^s v$ contributes $q^{-sn_{s+1}}$, with unit additive phase since $l_0 X_s = 0$. The last-column formula contributes $q^{r-1-n_{s+1}} G(\chi)$ in the oriented frames. Its dependence on the last frequency is the stated inverse character. This calculation also covers $s = 0$ with its empty set of interior edges.

The character factors in $\prod C_j$ telescope to $\eta$, since the common $\delta$ cancels at every intermediate determinant. The absolute determinant factors give $\prod_{j=1}^{s} |\det h_j|^t$, since $D_{j+1} - D_{j-1} = 2t$. Their power of $q$ is $t \sum_{j=1}^{s} j(s+1-j)$. It cancels the sum of $-(n_{j+1} - a_{j+1})c$ in the interior integrals. Adding the remaining interior exponents gives

$$\binom{s+1}{3} + \sum_{j=0}^{s-1} (s-1-2j) n_{j+1} + t \sum_{j=0}^{s-1} (s-j-1)(j+1) = 0.$$

Indeed the last sum without $t$ is $\binom{s+1}{3}$. In the middle one substitute $n_{j+1} = k_0 + (j+1)(t+1)$; the constant coefficient sums to zero and the linear term gives $-(t+1)\binom{s+1}{3}$. This leaves exactly the last power of $q$ in the theorem. The units $u_j$ and any fixed basis signs give $u$. Nonvanishing of $G(\chi)$, including $G(1) = -1$, was proved in Theorem 123.6. By definition a successor double coset is supported if the interior entry is nonzero for some core position. The computed entries establish this at each step starting with the required identity. □

Products over the distinct split places give compatible nonzero local entries for a fixed tuple. The global period also integrates the core positions, sums over paths, and projects to its prescribed row. The calculation retains a single path before those operations. Its affine matrices, phase and relative volume are the ones in `Development/AbelianVarietyTheory/Definitions/Foundations/SimpleSum.lean`, line 53–68, 91–93 and `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/StandardOrderExtendedBasisApplyLowBlock.lean`, line 718–732.

## 123.10 Integration over the flag columns at a ramified twist

Fix a split twist place of one of the tuples in Section 6.6. We use the plain frames, Haar measures and conductor of Theorem 123.9. Assume that the residue character $\chi : K^\times \to \mathbb{C}^\times$ is nontrivial. Let $0 \le j < s$, $c = s - j$, $n = n_{j+1}$ and $a = a_{j+1} = r - c$. Use the representative $h_j$ of that theorem at the source. At the target consider a matrix

$$h = Hk_1, \qquad H = \begin{pmatrix} I & \pi^{-e_0} z f_0^t \\ 0 & \operatorname{diag}(\pi^{-e_l})_{0 \le l < c} \end{pmatrix}, \qquad k_1 \in \mathrm{GL}_{n+c}(O).$$

Here $z \in k^n$, the $e_l$ are arbitrary integers and $f_l$ are the flag basis vectors. In the coordinates of $H$, let $W$ be the image under $\bar{k}_1$ of the marked target plane. Give $W$ its transported determinant form. Put $H_i^0 = K^n \oplus \langle f_0, \dots, f_{i-1} \rangle$ for $0 \le i \le c$. The entry below is the integral over the free columns of the affine matrix $A$ in Theorem 123.9, before the Weil scalar and the reciprocal unipotent volume.

**Theorem 123.10** (Support and value of the flag integral)**.** *At a core matrix $X$ the entry vanishes unless all $e_l = j + 1$, $X$ and $v = \pi^{-j}(Xz_j - z)$ are integral, and $W$ projects onto the full flag space $K^c$. A further necessary condition is that the reduced marked columns of $X$, followed by $\bar{v}$, form a basis of $W \cap K^n$. These conditions together are sufficient. Choose $w_i \in W$ lifting $f_{i-1}$ under projection, for $1 \le i \le c$. Let $u$ be the determinant of the specified core basis followed by the $w_i$, in the transported frames and with the fixed marked column ordering. The integral equals*

$$\chi(u)\, G(\chi)^c\, q^{\binom{c}{2} - jnc - (n-a)c}.$$

*Proof.* Integrality of $k_1^{-1}H^{-1}Ah_j$ is equivalent to integrality before multiplication by $k_1^{-1}$. The core columns there are $(X,0)$. The first extra column is $(v,\pi^{d_0}f_0)$, where $d_l=e_l-j$. In extra column $i\ge 1$ the upper vector is $\pi^{-j}(Y_i-zb_{0i})$. Its free lower entries are $v_{li}=\pi^{d_l}b_{li}$ for $l<i$. The fixed entry in row $i<c$ is $\pi^{d_i}$. Thus integrality requires $d_l\ge 0$ for every $l$. The new free variables range independently over integral vectors. The marked test now depends only on their residues and uses the determinant form on $W$. If $d_l>1$, integration in $v_{l,l+1}$ over any coset of $\pi O$ gives zero. Indeed the phase has factor $\psi(-\pi^{-d_l}v_{l,l+1})$, which is nontrivial on this subgroup. Hence every $d_l$ is zero or one.

Integrate the remaining columns in the order $c,c-1,\dots,1$. Write $W_i=W\cap H_i^0$. Initially the last column ranges over all of $H_c^0$ after reduction. At step $i$, the integrand from the later columns is a constant times the character of the determinant of the columns still present, required to give a basis of $W_i$. The current column ranges over all of $H_i^0$. This description holds at $i=c$ by definition. Fix the earlier columns. If they do not span a hyperplane in $W_i$, their contribution vanishes. Otherwise the determinant, as a function of the current column in $W_i$, is a nonzero linear functional with this hyperplane as kernel.

If $d_{i-1}=0$, the additive factor of this column is one. Summing over its allowable vectors gives zero, since each nonzero determinant occurs equally often and $\sum_{t\in K^\times}\chi(t)=0$. Consequently a surviving step has $d_{i-1}=1$. The earlier columns then belong to $H_{i-1}^0$, since the only possible fixed entry in row $i-1$ has zero reduction. If $W_i$ projects trivially on that row the same character sum is zero. Otherwise $W_{i-1}$ has codimension one in $W_i$. The earlier columns must form a basis of $W_{i-1}$. Choose an element of $W_i$ with last coordinate one and use it to induce the determinant form on $W_{i-1}$. At coordinate $t\neq 0$ there are $|W_{i-1}|$ choices for the current column, the determinant acquires a factor $t$, and the phase is $\theta(t)=\psi(-\pi^{-1}\widetilde{t})$. The normalized residue integral therefore multiplies the remaining determinant character by $|W_{i-1}|\,G(\chi)/|H_i^0|$. If $i>1$, the next column ranges over all of $H_{i-1}^0$ because its fixed entry $\pi^{d_{i-1}}$ reduces to zero. This proves the induction assertion.

All projections in the flag have rank one at their successive quotients exactly when $W$ projects onto $K^c$. In that event $\dim W_{i-1}=a+i-1$. The remaining determinant is the one in the statement. Adding an element of $W\cap K^n$ to any chosen lift leaves it unchanged. The product of the residue factors without this determinant is $(q^{a-1-n}G(\chi))^c$. The additive change of variables back to the original Haar measure has Jacobian $q^{c(c+1)/2-jnc}$, since there are $c(c+1)/2$ lower free entries, each divided by $\pi$, and $c$ upper vectors multiplied by $\pi^j$. This proves the formula. The Gauss sum has absolute value $\sqrt{q}$ by Theorem 123.6, proving sufficiency. □

## 123.11 Integral cost of the prescribed scalar orbit projections

Use the common split prime $T$, the words $c$, and the refined finite column spaces of Corollary 9.14. Write $f$ for the number of places above $T$ and $b=T-1$. At each of those places the scalar group acts freely on the nonzero residue covectors. On a product orbit the integral coefficient arrays are therefore copies of $(\mathcal{O}^b)^{\otimes f}$ with the same underlying class coefficient module. This follows also by transporting a base covector as in Theorem 110.4. Tensoring with the class module counts one copy per coefficient rank. Let $B_c=R_c/b^f$ denote the total number of copies counted with these ranks. On these columns the class trace and projection arguments of Theorems 122.17 and 123.14 apply to their definite group. Keep the normalized pairings with their specified level volumes.

**Theorem 123.11** (Elementary divisors of the zero-sum column pairing)**.** *For the projection*

*onto arrays of zero sum in each scalar coordinate, the paired lattices have elementary divisors*

$$b^{j} \quad \textit{with multiplicity} \quad B_c\binom{f}{j}(b-2)^{f-j}, \qquad 0 \le j \le f.$$

*At a coefficient place these give exponents $jv_{\mathfrak{p}}(b)$. The globally normalized logarithmic index per retained rank is exactly*

$$\frac{\log |C_P|}{[K:\mathbb{Q}]\, R_c^{\circ}} = \frac{f \log(T-1)}{T-2} \le \exp(-x^{1/4})$$

*for all sufficiently large $x$ on the common intervals. Here $C_P$ is the finite integral quotient over $\mathcal{O}_K$. A direct summand of relative rank $(1-1/(T-2))^f$ has unit elementary divisors at every finite prime simultaneously. For the separate projection onto one specified scalar character, each of the $B_c$ retained directions has elementary divisor $b^f$ over a coefficient ring containing the character values.*

*Proof.* For one scalar orbit put $M = \mathcal{O}^b$, use its coordinate basis $e_1, \dots, e_b$, and let $A$ be its augmentation kernel, with basis $a_i = e_i - e_b$, $1 \le i < b$. Equal class measures on this free orbit give the orthogonal projection $P_1 z = z - (\sum_i z_i)\mathbf{1}/b$. Set $v = P_1 e_b$. Then $P_1 M = A + \mathcal{O}v$ and $bv = -\sum_{i<b} a_i$. It follows that $a_1, \dots, a_{b-2}, v$ form a basis of $P_1 M$ and that replacing $v$ by $bv$ gives a basis of $A$. Both choices work over the rational integers.

On the product orbit $P = P_1^{\otimes f}$. Splitting $M = A \oplus \mathcal{O}e_b$ in each coordinate shows that the saturated zero-sum module is $A^{\otimes f}$, since the coordinate-sum kernel omits all direct summands using $e_b$ in that coordinate. The projection of the full lattice is $(P_1 M)^{\otimes f}$. Tensor the bases just constructed. A tensor uses the last basis element in exactly $j$ coordinates in $\binom{f}{j}(b-2)^{f-j}$ ways. Its inclusion multiplier is $b^j$. Together with the class module this gives the stated elementary divisors by Theorem 123.14. The tensors using only the $a_i$ for $i \le b-2$ give the claimed direct summand, including for a projective base coefficient lattice.

There are $B_c(b-1)^f = R_c^{\circ}$ retained directions. Differentiating the binomial identity gives $\sum_j j\binom{f}{j}(b-2)^{f-j} = f(b-1)^{f-1}$. The absolute norm of $b\mathcal{O}_K$ is $b^{[K:\mathbb{Q}]}$. These equalities prove the mean index formula. In Theorem 9.13 the common prime satisfies $\log T \ge (16 - o(1))mx^{\varepsilon}$, with $m \asymp x^{1/2-\varepsilon/16}$, and the field degree $f$ is polynomially bounded in $x$. Since $\log y/y$ is decreasing for $y > e$, these estimates give the displayed inequality uniformly for the selected words.

Finally the character projector on the regular product orbit sends a coordinate vector to $b^{-f}$ times its character vector, up to a unit. That vector has unit identity coordinate. It spans the saturated integral character line, while the projected full lattice is the same line divided by $b^f$. This yields the last assertion. □

The exponential bound concerns the actual scalar-orbit projection on the column side. It also bounds the loss per row in any exterior degree $k \ge R_c^{\circ} \exp(-x^{\varepsilon/8})$ by $\exp(-x^{1/4}/2)$ for all sufficiently large $x$, after summing the finite valuations with their norm weights and dividing by $[K:\mathbb{Q}]$. Indeed the largest $k$ nonnegative exponents at each prime have sum at most the total, and $R_c^{\circ}/k \le \exp(x^{\varepsilon/8})$. The size of the projection onto the designated automorphic row is a separate term in the period comparison.

## 123.12 The two adjacent level volumes

We compute the last two split local factors from the tests of Theorem 123.6. Work first at one auxiliary twist place $v$. Let $J \subset K_0 = \mathrm{GL}_d(O)$ be the middle parahoric. In the marked basis it

preserves the subspace $U$ modulo $\pi$. Its character is the tame determinant character $\delta$ on the lower diagonal block. We use Haar volume one on $K_0$.

Write $\Psi(X,y) = c_h\Phi([X \mid y]h)$ with the fixed frame convention of that theorem. Take the two scalar lattices $M \subset S \subset k^d$ used to record the last variable. Thus $\Psi$ is supported in $S$ as a function of $y$, and is invariant by translations by $M$. Such bounds exist for each finite collection of the marked tests. Indeed their common compact support is contained in a scalar box and compactness gives a common neighborhood of translation invariance for the finitely many locally constant functions. Both lattices are preserved by $K_0$. Write $Y = S/M$ and define $L^\perp = \{z : \psi(zy) = 1 \text{ for all } y \in L\}$ for a lattice $L$. For $a \in Y$ put

$$P_a(X) = \Psi(X,a), \qquad T_a(B,z) = \Phi^+(B)\,\mathrm{vol}(M)\mathbf{1}_{M^\perp}(z)\psi(-za).$$

The character in this formula is used only on $M^\perp$, where it is independent of the lift of $a$.

**Theorem 123.12** (Coset weights and original Haar measures)**.** *The coupled last kernel equals $\sum_{a\in Y} P_aT_a$ pointwise. Its contraction over the middle definite group can be formed with one summand for each $J$-orbit, with coefficient $|Ja|$. The other variables, including the two outer tests, may be kept fixed. The same formula holds at all auxiliary twist places together, with products of the orbit sizes.*

*At each vertex $i$ let $J_{i,v} \subset K_{i,v}$ be the tuple level inside the base level at its auxiliary places. In the transported frames these are the parahoric and the integral general linear group. Put*

$$I_i(t) = \prod_{v\in Q(t)} [K_{i,v} : J_{i,v}], \qquad a_j(t) = H\kappa_{j+1}\,\mathrm{vol}(K_{j+1,t}).$$

*Here $K_{i,t}$ is the global finite level at vertex $i$, $H$ is the order of the common finite character group, and $\kappa_i$ is the fixed measure scalar of that vertex. Keep the same Haar measure at every tuple, including the base tuple $0$ with $Q(0) = \varnothing$. For an adjacent pair let $q_j(t)$ be the product of the two local level volumes over $Q(t)$, each measured relative to its base level. Then*

$$\frac{q_j(t)}{a_j(t)a_{j+1}(t)} = \frac{1}{a_j(0)a_{j+1}(0)}.$$

*The equality holds also at coefficient primes dividing the orders of the residue general linear groups.*

*Proof.* Partition the defining integral in $y$ into the disjoint sets $a + M$. On each such set $\Psi$ is constant and the integral of its character is $\mathrm{vol}(M)\psi(-za)\mathbf{1}_{M^\perp}(z)$, by translation. This proves the pointwise identity with the full top test. In particular the support and the phases of a coset are kept in both factors. The unnormalized character table between $Y$ and $M^\perp/S^\perp$ has inverse its conjugate transpose divided by $|Y|$. Indeed the pairing is perfect by the scalar lattice description, and distinct columns are orthogonal by character summation. Thus refining the additive cosets before summing introduces only powers of the residue characteristic into the denominators of this Fourier inversion.

For $g \in J$, let $g_U$ act on $U$ after reduction and put $c(g) = \delta(\det g_U)^{-1}$. The parahoric test at the first edge satisfies $\Psi(gX, gy) = c(g)\Psi(X,y)$, since $g$ preserves the integral target lattice and multiplies the marked determinant by $\det g_U$. On the next edge it is the source which acts by $g$, whence $\Phi^+(Bg) = c(g)\Phi^+(B)$. The two statements use the same inverse tame character. Unramified splitting factors are one on these determinant units. From $zga = z(ga)$ and preservation of $M$ we get

$$P_{ga}(X) = c(g)P_a(g^{-1}X), \qquad T_{ga}(B,z) = c(g)^{-1}T_a(Bg, zg).$$

These identities apply to arbitrary $h$, since left multiplication on the first test commutes with its right source translation. The scalar phases cancel in the product.

Give $J$ probability measure $dk$. The middle average is

$$\int_J P_a(kX)T_a(Bk^{-1}, zk^{-1})\,dk.$$

The last two identities leave this average unchanged when $a$ is replaced by $ga$, by left translation of $k$. In the adelic definite contraction one may insert this probability average by right translation of the intermediate integration variable. The two adjacent Weil actions on that variable have inverse changes of coordinates. Their scalar splittings on $J$ are inverse there because the common tame character is the same and the absolute determinant is one. Thus the inserted average is exactly the compact average just computed in these frames. All other local factors are fixed under this translation. Compactness of the definite quotient and finiteness of the displayed sum justify the interchange. It follows that the contraction is constant on each orbit. Grouping its terms gives $|Ja|$. Distinct places act on distinct coordinates. Taking their product gives the asserted simultaneous grouping, including the empty product for the base tuple.

Let $I = [K_0 : J]$. Multiplicative Haar measure gives $\mathrm{vol}(J) = 1/I$ by the coset partition. In the convention where this volume is placed outside the middle sum, the weight of the representative is accordingly $I|Ja|$. At the last vertex its singleton index has weight $I'$ and local volume $1/I'$. More explicitly a split local index for a distinguished block of rank $r$ in rank $d$ is the number of $r$-planes in $K^d$. Reduction maps $\mathrm{GL}_d(O)$ onto $\mathrm{GL}_d(K)$, by lifting a matrix with unit determinant, and $J$ is the inverse image of a plane stabilizer. Counting ordered bases gives the integer $\prod_{i=0}^{r-1}(q^d - q^i)/(q^r - q^i)$.

Globally the levels agree outside $Q(t)$, and the local coordinates in its distinct factors can be chosen independently. Hence $\mathrm{vol}(K_{i,t}) = \mathrm{vol}(K_{i,0})/I_i(t)$. Transport by a common frame preserves the index and this volume ratio. Rational stabilizers cause no correction in passing to definite classes. Over a parent class with finite stabilizer $\Gamma$, the fine classes are the $\Gamma$-orbits in the $I_i(t)$-element set of level cosets. Summing the reciprocal orders of their stabilizers gives $I_i(t)/|\Gamma|$ by orbit–stabilizer, which is exactly offset by the new level volume. Finally

$$a_j(t) = a_j(0)/I_{j+1}(t), \qquad q_j(t) = \big(I_{j+1}(t)I_{j+2}(t)\big)^{-1}.$$

Substitution gives the claimed equality over the rationals for the index factors and with the original measure scalars. □

The last identity concerns the two volume factors in the seesaw quotient. The orbit sizes and the signed values $P_aT_a$ inside a period still enter its global integral. The relevant normalizations are recorded in `Development/AbelianVarietyTheory/Definitions/LatticeFrames/CosInj.lean`, line 110–150 and `Development/AbelianVarietyTheory/Definitions/LatticeFrames/NotBadPrimeBase.lean`, line 103–107.

## 123.13 All intermediate double cosets at a ramified twist

At a split twist place use the frames and kernels of Theorem 123.9. At a vertex with core dimension $n$ and $c \geq 1$ flag coordinates write $J \subset \mathrm{GL}_{n+c}(O)$ for the inverse image of the marked plane stabilizer. The group on the left of the double quotient consists of the matrices

$$R = \left\{ \begin{pmatrix} g & B \\ 0 & T \end{pmatrix} : g \in \mathrm{GL}_n(k), \quad Bf_0 = 0, \quad T \text{ upper unitriangular} \right\}.$$

For a fixed predecessor, a double coset is supported if some representative gives a nonzero affine integral for some core matrix. The initial representative is the identity.

**Theorem 123.13** (Uniqueness of all intermediate double cosets)**.** *If the tame splitting character is nontrivial on residue units, there is exactly one supported successor double coset at every interior step, beginning at the identity. The matrices $h_1, \dots, h_s$ of Theorem 123.9 represent them. Thus the sum over all interior double cosets at this place, in these representatives, has the value of that single path as a function of its core positions and terminal variables. Its integral over the common position set and the terminal frequency ball of Theorem 126.4, with the compact averages there, is exactly the value in that theorem. For an admissible global twist tuple this applies at every place of its auxiliary twist set $Q(t)$. Its whole contribution from these places on the product of the stated sets is the product of the local values, with empty product one for the base tuple.*

*Proof.* Right target multiplication by $J$ changes the marked test by a nonzero constant from its determinant character. Left target multiplication by an element of $R$ changes the affine variables and the core position. Indeed, acting on $A$ on the left by its inverse fixes extra column zero, sends $X$ to $g^{-1}X$, and makes an invertible affine change of the other free columns. In the first lower superdiagonal each changed entry differs from its old value by a constant coming from the fixed diagonal. The change of measure has positive constant Jacobian and the additive phase changes by a constant of modulus one. Acting on $A$ on the right by the predecessor's corresponding group changes $X$ to $Xg$, fixes extra column zero and has the same properties. Right multiplication of a predecessor representative by its parahoric again gives a determinant character. The Weil factors never vanish. Thus the existence of a nonzero entry is independent of representatives on either side.

We describe representatives of the target quotient. For a matrix with column lattice $\Lambda$, choose a basis of $\Lambda \cap k^n$ and send it by the core group to the standard basis. For $0 \le l < c$, intersect $\Lambda$ with the core and the first $l+1$ flag lines. Its image in the last of these lines is a fractional ideal $\pi^{-e_l}O$, since $\Lambda$ is a full lattice. Lift its generator. Successive subtraction of such lifts expresses every lattice vector in the resulting basis. Each lift, divided by its pivot, has the form $f_l$ plus a vector in the preceding coordinates. The upper unitriangular flag operations and the free core columns of $R$ remove these earlier entries for $l \ge 1$: working in increasing $l$, subtract multiples of coordinate $l$ from the earlier coordinates. This fixes all preceding basis columns. In column zero the normalized lift is $(z, f_0)$. Hence the representative has the form $Hk_1$ of Theorem 123.10. For this fixed column lattice the remaining right cosets modulo $J$ correspond exactly to the residue planes $W = \bar{k}_1 E$, where $E$ is the marked plane in lattice coordinates. Indeed two integral bases give the same plane precisely when their ratio belongs to $J$.

Suppose the predecessor is $h_j$. The preceding theorem forces $e_l = j+1$ for every $l$. It also forces $W$ to project onto the flags, with $W \cap K^n$ spanned by the marked columns of $\bar{X}$ and $\bar{v} = \overline{\pi^{-j}(Xz_j - z)}$. For $j = 0$ we have $v = -z$, so $z$ is primitive and $\bar{z} \in W \cap K^n$. For $j > 0$ integrality gives $\bar{z} = \bar{X}\bar{z}_j$. The marked columns of $\bar{X}$ are independent, and $\bar{z}_j$ is a marked basis vector. This again makes $z$ primitive and puts its reduction in $W \cap K^n$.

An integral core basis change sends $z$ to the chosen $z_{j+1}$, since a primitive vector extends to an integral basis. Call this vector $z_*$. For the resulting lattice with $e_l = j+1$ take $T = I$. In lattice coordinates a core action $g$ has upper flag column zero equal to $\pi^{-j-1}(gz_* - z_*)$; the other upper flag columns are $\pi^{-j-1}Bf_i$. The reductions of these columns can be prescribed arbitrarily, independently of any prescribed $\bar{g}$ fixing $\bar{z}_*$. To see this, lift the other columns of $\bar{g}$ in a basis starting with $z_*$ and choose $gz_* = z_* + \pi^{j+1}u$ with the desired $\bar{u}$. This gives an integral invertible $g$. Choose $Bf_i$ for $i > 0$ by multiplying the desired lifts by $\pi^{j+1}$. The resulting element of $R$ stabilizes the lattice.

The stabilizer of $\bar{z}_*$ in the residue core group is transitive on $a_{j+1}$-planes containing it, by extending bases beginning with that vector. Send $W \cap K^n$ to the standard marked core plane. Since $W$ projects onto every flag coordinate together, the independent upper flag columns just constructed can then cancel the core coordinates of lifts of every $f_i$ in $W$. The resulting plane is the standard marked plane. There is therefore at most one supported double coset. The nonzero entry constructed in Theorem 123.9 supplies one at each step. Induction gives uniqueness along the entire chain, including the empty list when $s = 0$. The terminal orbit and its Haar weight are evaluated in Theorem 123.7. This proves the assertions at one place.

For the last assertion, recall how the fields of the tuple specify its auxiliary twists. At a finite place $v$ of the real subfield the tuple character kills the principal units, and away from the allowed split places it kills all local units. The support used for $Q(t)$ consists of the $v$ where the restriction to local units is nontrivial, choosing one prime of the CM field over each such $v$. At a split $v$ the character is trivial on the diagonal copy of the base field's multiplicative group. Its two local characters there are therefore inverses. Nontriviality on the product of the two unit groups implies nontriviality on the units of either chosen factor. Since principal units are in the kernel, the corresponding residue character and its inverse $\chi$ are nontrivial. This checks the hypothesis simultaneously at every place of $Q(t)$. The integrals over their product are finite products under product Haar measure. The base character is unramified everywhere and has empty auxiliary support. □

The definitions of the auxiliary support and the affine successor integral are in `Formalism/LocalToGlobal/Divisibility/Tuples.lean`, line 50–67, 110–128, `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/StandardOrderExtendedBasisApplyLowBlock.lean`, line 718–732, 922–938. The projection onto the prescribed global automorphic row still involves the global form and its arithmetic normalization.

## 123.14 The exact integral loss in a projected terminal period

At a finite coefficient place localize the terminal table of Theorem 122.17 over the discrete valuation ring $O$ with uniformizer $\varpi$ and fraction field $K$. Write $S, T$ for its integral row and its lattice of trace tests, paired in the original measure divided by the finite level volume. These are mutual duals. Fix a $K$-rational idempotent $P$ on $S_K$. This includes an orthogonal projection when its matrix is defined over $K$, as for the theta images of Theorem 122.12. Write $P^t$ for the transpose on covectors. Define

$$S_P = S \cap PS_K, \qquad S_Q = S \cap (1-P)S_K, \qquad T_P = T \cap P^t T_K, \qquad C_P = S/(S_P + S_Q).$$

**Theorem 123.14** (Elementary divisors of the projected period)**.** *The nonunit elementary divisors of the normalized global pairing on $S_P \times T_P$ are exactly those of the finite module $C_P$. Write the exponents as $c_1 \geq \cdots \geq c_d \geq 0$, padding by zeros to $d = \dim_K PS_K$. Then $T_P$ is the full dual of $PS$ and $PS/S_P \simeq C_P$. In particular $\varpi^{c_1}$ clears the projection on the original lattice with the least possible valuation when $d > 0$.*

*Let an operator with values in $PS_K$ be tested on an integral source lattice. If all its $k$-row determinants against projected integral trace tests have valuation at least $b_k$, its exterior coefficients on the saturated target $S_P$ have valuations at least $b_k - \sum_{i=1}^{k} c_i$, for $1 \leq k \leq d$. This is the sharp universal loss for that projection. For the original Haar measure restore* $\mathrm{vol}(J)^k$ *in the tested determinant.*

*Proof.* Projection induces an isomorphism $S/S_Q \simeq PS$. Its restriction to $S_P$ is the identity. Both lattices span the same space, and the quotient is $C_P$. A covector is fixed by $P^t$ precisely when it annihilates the complementary space. Such a covector is integral on $S$ precisely when

it is integral on $PS$, by the identity $\ell(s) = \ell(Ps)$. Theorem 122.17 identifies all of them with actual integral trace tests in $T_P$.

Choose a basis $e_i$ of $PS$ with $\varpi^{c_i} e_i$ a basis of $S_P$, by the elementary divisor theorem over $O$. The dual basis to $e_i$ is a basis of $T_P$. The tested pairing matrix is therefore diagonal with entries $\varpi^{c_i}$. These formulas also give the least scalar that sends $PS$ into $S_P$, equivalently into $S$.

For a wedge indexed by $i_1, \dots, i_k$ the tested coefficient is its coefficient in the saturated basis multiplied by $\varpi^{c_{i_1}+\cdots+c_{i_k}}$. The largest such sum is $c_1 + \cdots + c_k$. The coordinate wedge on the first $k$ basis vectors attains the loss. It can itself be realized as the wedge of images of $k$ source basis vectors. Finally each unnormalized scalar test has one factor of the level volume by the class trace formula, proving the assertion for determinants. □

The result applies to any arithmetic or analytic column known in the stated rational projected row. At a prescribed projection, the size of $C_P$ measures a genuine integral comparison that remains after the class and level factors have been evaluated.

## 123.15 The two finite type projections on the actual terminal classes

Fix the standard definite terminal group on $E_x$, the coefficient lattice and the embedding of Theorem 123.8. At each of the two completely split primes $g_1, g_2$, distinguish the $r$-dimensional block in each marked frame. Write $f = [E_x^+ : \mathbb{Q}]$ and let

$$\mathcal{G} = \prod_{i=1}^{2} \prod_{v \mid g_i,\ v \text{ in } E_x^+} \mathrm{GL}_r(\mathbf{F}_{g_i}), \qquad m_G = |\mathcal{G}|.$$

Use the original levels on the other blocks and at all other places. On each distinguished block first fix its full residue marking, then allow the prescribed subgroup $P \subseteq \mathcal{G}$ with its finite character $\omega$. Write $S$ for the original integral table at the latter level. A splitting number field for the finite group and its characters may be used throughout. Let $\mathcal{X}$ be the set of irreducible characters in the prescribed central idempotent at the two primes. The argument works for any such set, including the one obtained by summing the selected products of finite series idempotents. Put

$$k_{\mathcal{X}}(a) = \sum_{\chi \in \mathcal{X}} \chi(1)\chi(a^{-1}).$$

**Theorem 123.15** (Class matrix and uniform integral loss for the finite types)**.** *The harmonic projection $P_{\mathcal{X}}$ onto these finite types, on the $P, \omega$ level, is the same endomorphism in decorated class coordinates and in the standard adelic row. It is computed on each integral table $w$ by*

$$(m_G P_{\mathcal{X}} w)(x_i) = \sum_{a \in \mathcal{G}} k_{\mathcal{X}}(a)\, w(x_i \widehat{a}),$$

*where $x_i$ ranges over the original class representatives and $\widehat{a}$ changes their distinguished markings by $a$. In particular $m_G P_{\mathcal{X}} S \subseteq S$.*

*At a finite coefficient place choose bases for the integral class components and let $M$ be the matrix of this displayed sum. For $1 \le l \le \operatorname{rank} M$, let $d_l(M)$ be the least valuation of an $l$-rowed minor. The largest possible loss for an $l$-row determinant of the projected integral trace pairing of Theorem 123.14 is exactly $l\, v(m_G) - d_l(M)$ in the level-volume normalization. Over all finite coefficient places the normalized logarithmic loss per exterior degree is at most*

$$\log m_G \ \le\ f r^2 \log(g_1 g_2) \ll_{\psi,\varepsilon} f r^2 x^{\varepsilon}.$$

*This bound holds at every $x$ on the same intervals, with the same scalar for every word and twist tuple at that $x$. For every fixed $A > 0$ it is $o_{\psi,\varepsilon,A}(b x^{-A})$ there, where $b = \exp(x^{\varepsilon})$.*

*Proof.* At either split prime a matrix in the distinguished general linear group lifts to an integral invertible matrix there: lift its entries, observing that its determinant is a unit. Extend the lift by the identity on the other blocks. It normalizes the level with the distinguished residue marking fixed. In the block parabolic description the unipotent rectangle is preserved by the change of basis, the distinguished diagonal block is the identity modulo the prime, and the characters of the other diagonal blocks are unchanged. Two lifts induce the same change of marking and hence the same translation on these tables. Right translations $R_a w(x) = w(x\widehat{a})$ give a representation of $\mathcal{G}$; they preserve the integral table on this finer level. Indeed the underlying lattice and its coefficient fiber are unchanged when its residue marking varies. They are unitary in the adelic norm by right invariance of Haar.

The central sum $\sum_a k_{\mathcal{X}}(a) R_a / m_G$ acts by a scalar in an irreducible representation of character $\eta$, by Schur's lemma. Its trace is $\sum_{\chi \in \mathcal{X}} \chi(1) m_G^{-1} \sum_a \chi(a^{-1}) \eta(a)$. Character orthogonality gives $\eta(1)$ if $\eta \in \mathcal{X}$, and zero otherwise. Thus this central sum is the orthogonal type projector on the fine space. It commutes with the $P, \omega$ projection, so it restricts to the given coarse level. Each $k_{\mathcal{X}}(a)$ is integral because eigenvalues of a finite order matrix are roots of unity. The unnormalized sum belongs to the fine integral table and once again satisfies the coarse transformation law. That law cuts out the coarse integral table by equality on the same marked lattices. No factor of $|P|$ occurs here.

For a class representative $x_i$, write $x_i \widehat{a} = \lambda x_j u$ with $\lambda$ rational and $u$ in the original level. The summand is read as $k_{\mathcal{X}}(a) \omega_\xi(u) \rho_K(\lambda) w_j$, with $\omega_\xi$ the whole level character of the tuple. The class stabilizer condition makes it independent of that decomposition. Apply the compact coefficient functional of Theorem 123.8 to each summand. The result is the same finite sum of right translates of the standard adelic form at every class and every archimedean argument, with no adjustment of Haar measure.

There is also an explicit verification for the two coordinate lists used at this step. Let $L$ and $V$ have as columns any bases of the coarse character space and its retained type space, respectively, in fine table coordinates. If $H_0$ is the positive definite fine Gram matrix, set

$$R = (L^{*} H_0 L)^{-1} L^{*} H_0 V.$$

The larger space contains the smaller. Since $(L^{*} H_0 L)^{-1} L^{*} H_0$ is a left inverse to $L$, $LR = V$. This holds also for a zero space by the unique empty map. Thus coefficients $F$ in the retained basis and $RF$ in the coarse basis give the identical table on every marked class. Inverse restriction of either list gives the same standard adelic form, using the same coefficient map at every tuple. This proves the coefficient equality without imposing integrality on either chosen basis.

Finally the saturation of the image of $M = m_G P_{\mathcal{X}}$ is $S \cap P_{\mathcal{X}} S_K$, locally over the coefficient ring. In a basis of $P_{\mathcal{X}} S$ the saturated submodule is generated by multiples of valuations $c_1 \ge \cdots \ge c_d \ge 0$, as in Theorem 123.14. Hence the nonzero invariant factors of $M$, measured in its saturated target, have valuations $v(m_G) - c_i$ in increasing order. The saturation is a direct summand locally, so the minimum valuation in the full matrix of $l$-rowed minors is $l\, v(m_G) - \sum_{i \le l} c_i$. The class trace theorem restores the original level volume once per row. Summing $v_w(m_G) \log \mathrm{N} w / [K : \mathbb{Q}]$ gives $\log m_G$. There are $f$ factors above each split prime, and $|\mathrm{GL}_r(\mathbf{F}_g)| \le g^{r^2}$ by counting all matrices. Proposition 94.6 gives $g_i \le 2(kb)^8$, with $\log k = O_{\psi,\varepsilon}(\log x)$. The ranks and $f$ grow at most polynomially in $x$. This proves the uniform bound and the last assertion. □

The two coordinate maps occur in `Formalism/LocalToGlobal/Divisibility/RowSideTie.lean`, line 76 and the finite type selector in `Formalism/SharpRestriction/LevelLetterCocycles/DataN.lean`, line 1001. The equality above concerns both realizations of this finite type

projection. Comparing the full signed iterated theta column with the independently specified Bessel period still requires identifying their values on the common target.

### 123.16 Integral separation of the marked finite types

Use the same field $E_x$, pair of marked primes, coefficient maps and Haar measure as in Theorem 123.15. Let $G = \mathcal{G}$ be its product of finite general linear groups, and let $S$ be the integral table at the prescribed level and character. At a finite place of a splitting field for $G$ and the coefficients, let $O$ be the local ring, $K$ its field of fractions and $v$ the additive valuation normalized by $v(\pi) = 1$. Take $S$ over $O$. For a conjugacy class $D$ write $z_D = \sum_{g \in D} g$. On an irreducible character $\chi$, the class sum acts by the scalar

$$\lambda_\chi(D) = \frac{|D|\chi(g)}{\chi(1)} \qquad (g \in D).$$

This follows by taking the trace of a central operator. The scalar lies in $O$ because it is an eigenvalue of the integral matrix of multiplication by $z_D$ on the regular representation, and $O$ is integrally closed. Group the characters together when all their $\lambda_\chi(D)$ have the same residues modulo the maximal ideal. Denote one such class of characters by $B$. For a subset $A$ of characters let $e_A$ be the sum of its ordinary central character idempotents.

**Theorem 123.16** (The integral cost within each residue class of characters)**.** *Every $e_B$ belongs to $O[G]$. A sum $e_A$ belongs to $O[G]$ exactly when $A$ is a union of the classes $B$. In particular $S = \bigoplus_B S_B$, where $S_B = e_B S$, also at coefficient primes dividing $|G|$.*

*Let $\mathcal{X}$ be the actual set of retained finite types, and write $S_K = S \otimes_O K$. The discrepancy of its two integral lattices has the decomposition*

$$\frac{e_{\mathcal{X}} S}{S \cap e_{\mathcal{X}} S_K} \simeq \bigoplus_{B:\ \varnothing \neq B \cap \mathcal{X} \neq B} \frac{e_{B \cap \mathcal{X}} S_B}{S_B \cap e_{B \cap \mathcal{X}}(S_B \otimes_O K)}. \tag{123.1}$$

*For each class in the sum, every elementary divisor exponent is at most*

$$d_B = \left( v(|G|) - \min_{g \in G} v\left( \sum_{\chi \in B \cap \mathcal{X}} \chi(1)\chi(g^{-1}) \right) \right)_+ .$$

*Here $v(0) = +\infty$, and $z_+ = \max(z, 0)$. The same $d_B$ is obtained using the omitted characters in $B$. In particular, complete residue classes contribute zero loss in every exterior degree of the projected class pairing in the original measure. This statement applies to every $x$ and every auxiliary tuple on the common interval.*

*Proof.* For each $\eta \notin B$, choose a conjugacy class $D_\eta$ on which its residue differs from that of $B$. Such a class exists by the definition of $B$. In the center of $O[G]$ form

$$T = \prod_{\eta \notin B} (z_{D_\eta} - \lambda_\eta(D_\eta)\, 1).$$

It vanishes on all irreducibles outside $B$. On each $\chi \in B$ it acts by a unit $t_\chi$, since each factor has nonzero residue there. Thus

$$1 - \prod_{\chi \in B} (1 - t_\chi^{-1} T)$$

is integral. It acts by one on $B$ and by zero outside. The regular representation in characteristic zero contains all the irreducibles and is faithful, so this element is $e_B$. Conversely an integral central element has an expansion $\sum_D c_D z_D$, $c_D \in O$, because centrality makes its coefficients

constant on conjugacy classes. Its eigenvalues on two characters in $B$ must be congruent. For an idempotent these eigenvalues are zero or one. The residues distinguish those two values, proving the asserted converse for $O[G]$.

Apply these elements first on the tables with full residue markings, where each right translation is integral. Their centrality makes them commute with every element of the level subgroup. Hence they preserve its prescribed character. The integral coarse tables are exactly those fine tables satisfying this equivariance, by restriction to the original class representatives in Theorem 123.15. Thus $e_B S \subseteq S$. Their sum is the identity and distinct idempotents have product zero. This gives the integral direct sum, with no division by the order of the level subgroup. For any idempotent $e$, each element of $S \cap eS_K$ equals its own image under $e$, so belongs to $eS$. Taking images and intersections on the direct summands gives (123.1). When a whole $B$ is retained its map on $S_B$ is the identity. When it is omitted its image is zero.

For $C = B \cap \mathcal{X}$, the coefficient of $g$ in $e_C$ is $|G|^{-1} \sum_{\chi \in C} \chi(1)\chi(g^{-1})$, by character orthogonality. Consequently $d_B$ is precisely the least nonnegative exponent with $\pi^{d_B} e_C \in O[G]$. Since $e_B$ is integral and $e_{B \setminus C} = e_B - e_C$, the same test, at each exponent, works for $B \setminus C$. The integral multiple preserves the coarse table by the same equivariance argument and sends it into $S_B \cap e_C S_K$. It therefore annihilates the quotient in the corresponding summand. Finally the elementary divisors of a direct sum over a discrete valuation ring are obtained by putting the two lists together and ordering them. The largest loss in exterior degree $l$ is the sum of the largest $l$ exponents, padded with zeros, by Theorem 123.14. The original Haar and stabilizer weights enter that pairing just as in Theorem 122.17. □

This identifies the finite types that can contribute an integral denominator to the prescribed projection. The equality with the independently constructed Bessel functional still requires the comparison of their coefficients on the surviving classes.

## 124 References

[1] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts 35, Cambridge University Press, 1997, Chapters 1 and 8. Monograph.

[2] S. S. Kudla, *Notes on the local theta correspondence*, European School of Group Theory, 1996. Chapter III, Section 6, pp. 53–54.

[3] J.-H. Evertse, *Notes on Diophantine Approximation*, Chapter 6, Theorem 6.3.3. Geometry of numbers.

[4] Y. Liu, *Theta correspondence for almost unramified representations of unitary groups*, 2021. Preprint.

[5] A. Ichino, *Theta lifting for discrete series representations of real unitary groups*, 2020. Preprint.