The generalized Riemann hypothesis for Dirichlet L-functions

27 November 2026 · Edition 930

Volume XXII

Signed orbital integrals on the marked rows

Abstract

The common flag lattice controls denominators at the two terminal frequencies. We truncate the Gram determinant sum by height, compute the local integrals of its transverse lifts and distinguish rational classes in the signed period.

The guide in Volume I lists all twenty-seven volumes.

Contents

XXII Signed orbital integrals on the marked rows 1
115 Terminal frequencies and common flag lattices . . . 3
115.1 A uniform lattice count on the norm-one rows . . . 3
115.2 Both terminal norms in the ordinary period . . . 5
115.3 The outer flag ideals in the ordinary period . . . 8
115.4 The real section at the prime terminal frequency . . . 11
115.5 Intersecting flags at the two terminal frequencies . . . 13
115.6 Denominators in the common part of two outer flags . . . 16
115.7 The two marked primes in the common flag index . . . 18
115.8 The common middle lattice and the terminal divisor . . . 19
115.9 The positive lattice in the signed period . . . 22
115.10 The determinant comparison at transverse flags . . . 24
116 Gram determinants and height truncation . . . 26
116.1 Gram determinants and the transverse orbital sum . . . 26
116.2 Separation of rational transports by their discriminants . . . 29
116.3 Counting cross matrices below the frame-height cutoff . . . 31
116.4 Integral orbits below the frame-height cutoff . . . 33
116.5 Height truncation of the projected ordinary period . . . 35
116.6 Two determinant bounds for the signed ordinary period . . . 38
117 Integral lifts and the signed transverse period . . . 40
117.1 Rational isometries at a fixed transverse marking . . . 40
117.2 Summing integral lifts of a transverse rational class . . . 43
117.3 Finite local factors of the signed sum of integral lifts . . . 46
117.4 Finite residue sums for the signed local factors . . . 48
117.5 The signed coefficient in the measures of the two rows . . . 51
117.6 Summing the signed period over different hyperplanes . . . 54
117.7 The determinant of a pair of frames and its signed period . . . 59
117.8 Finite characters in the sum of frame determinants . . . 62
117.9 Moments of the integral frame determinants . . . 65
117.10 Evaluation of the signed Bessel integral on the open cell . . . 67
117.11 Flag minors on every transverse block of the signed period . . . 71
117.12 Exterior moments at the two terminal frequencies . . . 74
117.13 A compact integral for the full signed orbital sum . . . 77
117.14 The second moment of the signed Gram blocks . . . 81

117.15 Comparison ranks in the full ordinary Gram moments 84
117.16 Rational isometries in the mixed ordinary Gram moments 86
117.17 Operator bounds on the prescribed compact column types 90
117.18 Integral coordinates for bounded rational transports 92
117.19 Recovery of a signed coefficient in the presence of other classes 95
117.20 Character traces of the signed sums at each rational isometry 97
117.21 Scalar transports in the integral marked period 100
117.22 Summing over all target rows in the ordinary Gram matrix 103
117.23 Exact separation by the trace on the matched lattice 105
117.24 Reduction of the two isotropic spaces at the terminal prime 107
117.25 Recovering a rational orbital class from its terminal reduction 110
117.26 The positive transport at the terminal prime 112
117.27 The rational class when the common positive space has full dimension 115
117.28 Mixed ordinary periods at prescribed input flags 118
117.29 Relative input flags in the fourth period moment 120
117.30 The first contributing flag position for a paired unipotent 122
117.31 Transvections on opposite and neighboring flags 124
117.32 Real flag angles in the untruncated signed period 127
118 References 130

115 Terminal frequencies and common flag lattices

115.1 A uniform lattice count on the norm-one rows

Let $E = E_x$, $F = E^+$, $f = [F : \mathbb{Q}]$ and $Q = g_1 g_2$ be the marked level on the intervals of Section 6.6. We treat any ordinary diagonal row of terminal norm -1. Use its primitive integral frame $v_1 = e_1, \dots, v_s = e_s$, $v_{s+1} = z$, from Lemma 109.20. Write $n = \dim_E V$, $d = s + 1$. Its integral Witt decomposition has hyperbolic partners $f_1, \dots, f_s$ and middle lattice $J = \mathcal{O}_E z \perp L$. Indeed orthogonal projection to Ez is integral because $h(z, z) = -1$. The lattice L is positive definite at all infinite places; put $k = \dim_E EL$.

At each real place use the positive Hermitian norm in which the hyperbolic basis vectors and z are orthonormal, the other summand is perpendicular to them, and L has its given positive Hermitian form. This specifies the real norms used in the estimates below. The original form in these norms has matrix a unitary involution, so $w(g) = \|g\|_{\mathrm{op}} = \|g^{-1}\|_{\mathrm{op}} \geq 1$ for an isometry. The measures and the section W are those of the row. For its real decorated stabilizer B_τ put $\nu_\tau(y) = \inf_{b \in B_\tau} w(by_\tau)$ and $V(y) = \prod_\tau \nu_\tau(y)$. Write N_Λ for the integral unipotent radical in $\Gamma = \mathrm{Aut}(\Lambda, h)$.

Lemma 115.1 (Counting lattice points through a filtration)**.** *Suppose a Hermitian lattice over $\mathcal{O}_E$, as a module, is a direct sum of finitely many blocks in a specified order. Give each block a positive definite integral Hermitian form and the direct sum its orthogonal norm at every infinite place. Let b_τ preserve the filtration by partial sums, acting unitarily on each graded block. For an affine E-subspace of dimension j and positive real radii R_τ, the number of its lattice points u satisfying $\|b_\tau u_\tau\| \leq R_\tau$ at every place is at most*

$$\left(1 + 2(\textstyle\prod_\tau R_\tau)^{1/f}\right)^{2fj}.$$

The same estimate holds for exterior powers with their induced block decompositions and norms.

Proof. In one block a nonzero lattice vector v satisfies $\prod_\tau \|v_\tau\| \geq 1$, since its squared Hermitian length is a nonzero totally positive algebraic integer in F. Consider points in a fixed affine subspace of dimension j' in this block, with arbitrary centers for the balls of radii R_τ. Project the centers orthogonally onto the corresponding complex affine subspaces. After rescaling by R_τ, the points lie in a unit ball for the maximum of the place norms. Two distinct points have distance at least $(\prod_\tau R_\tau)^{-1/f}$. Disjoint open balls of half this radius fit inside the unit ball enlarged by that radius. Their volume ratio in real dimension $2fj'$ gives the bound in the statement with j'.

Now fix block coordinates of u starting at the last block. After the later coordinates have been fixed, the particular block of $b_\tau u_\tau$ in question is a unitary transform of the current coordinate plus a known translation. The possible coordinates lie in an affine subspace whose dimension is the drop in dimension on specifying this block. This drop depends only on the linear direction of the original affine space. Its sum over all blocks is j. Multiply the estimates just proved for these successive choices.

In an exterior power, decompose by the number of factors taken from each original block. Order the resulting blocks by comparing these numbers from the last original block backwards. The transform still preserves the filtration and acts unitarily on its graded pieces, since replacement of a factor by one in an earlier block lowers its index. Each piece is a tensor of exterior powers of the integral blocks. Its induced form is positive and integral by the determinant formula for exterior pairings. This verifies all hypotheses for the same argument. □

Theorem 115.2 (Height separation and counting on every norm-one row)**.** *For each admissible $\gamma \in \Gamma$ outside the rational decorated stabilizer and every $y \in G_\infty$ one has*

$$Q^f \le 2^f V(y)V(\gamma^{-1}y).$$

Set $D = \sum_{j=1}^{d} j(n-j+1)$ and

$$C_{f,n,d} = 3^{(2fk)^2} \prod_{j=1}^{d} (1 + 2 \cdot 4^j)^{2f(n-j+1)}.$$

For any integers $m_\tau \ge 0$, the number of cosets $\delta \in N_\Lambda \backslash \Gamma$ with $2^{m_\tau} \le \nu_\tau(\delta y) < 2^{m_\tau+1}$ everywhere is at most $C_{f,n,d}\, 2^{4D \sum_\tau m_\tau}$.

Proof. Order the blocks as the successive e_i-lines, the line Ez, the positive space EL, and the successive $f_s, \dots, f_1$-lines. The decorated stabilizer acts triangularly in these blocks with unitary graded actions. Indeed it fixes each e_i modulo earlier vectors, and z modulo their full span. In the middle it fixes the negative vector and acts isometrically on its positive perpendicular. Preservation of the hyperbolic pairing gives the reversed dual action on the last block lines with diagonal one.

Let X_i be the span of the first i isotropic vectors, with $X_0 = 0$. Take the first $i \le d$ such that $\gamma v_i - v_i \notin X_{i-1}$. Let p be the last nonzero block component of this difference, necessarily beyond X_{i-1}. Admissibility at the primes over Q gives $\gamma v_i - v_i \in X_{i-1} + Q\Lambda_{\mathfrak{w}}$ there. The integral splitting shows p belongs globally to Q times its block lattice. Consequently $\prod_\tau \|p_\tau\| \ge Q^f$ by positivity and integrality in that block, including the sign-corrected z-line.

For any $b_1, b_2 \in B_\tau$ compare p_τ with the same block of $b_1 \gamma b_2^{-1} v_i - v_i$. The vector $b_2^{-1} v_i - v_i$ is in X_{i-1}, which γ carries into X_{i-1} by the choice of i. Also $b_1 v_i - v_i \in X_{i-1}$. Thus the block in question is the unitary graded transform of p_τ. Its norm is at most $1 + w(b_1 \gamma b_2^{-1})$, since v_i is a unit vector. Factor the isometry as $(b_1 y_\tau)(b_2 \gamma^{-1} y_\tau)^{-1}$ and take the two infima independently. This gives $\|p_\tau\| \le 2\nu_\tau(y)\nu_\tau(\gamma^{-1}y)$. Multiplication proves the height separation.

For the count put $\Phi_j = v_1 \wedge \cdots \wedge v_j$ for $1 \le j \le d$. All Φ_j have norm one and are fixed by the real decorated stabilizer. A group element fixing every Φ_j must fix v_j modulo its predecessors: equality of the wedge with the nonzero fixed preceding wedge gives this assertion inductively. The integral stabilizer of the tuple, modulo N_Λ, embeds into $\mathrm{Aut}(L, h)$ by its middle action. This group is finite since it preserves the positive integral trace form. Its reduction on the underlying $\mathbb{Z}$-lattice of rank $2fk$ is injective modulo three. For if $1 + 3^b A$, $b \ge 1$, has a unit entry in A, raising to a prime different from three preserves that first nonzero level. Cubing raises it by exactly one by the binomial expansion. Thus the kernel has no element of prime order. Its finite subgroup is trivial, giving the order bound $3^{(2fk)^2}$.

In a nonempty box choose one representative δ_0. Choose $b_{0,\tau} \in B_\tau$ with $w(b_{0,\tau}\delta_0 y_\tau) < 2^{m_\tau+1}$. For any other representative in the box the integral tuple

$$Z_j = \delta_0 \delta^{-1} \Phi_j \in \bigwedge^j \Lambda$$

satisfies $\|b_{0,\tau} Z_{j,\tau}\| \le 4^j 2^{2jm_\tau}$. In fact write $b_{0,\tau} Z_j$ as the product of the exterior actions of $b_{0,\tau}\delta_0 y_\tau$ and $(b_{\delta,\tau}\delta y_\tau)^{-1}$ on Φ_j, where $b_{\delta,\tau}$ realizes the same strict bound. The exterior norm is bounded by the j-th power of the ordinary operator norm.

On fixing the preceding tuple, the possibilities for Z_j lie in the rational subspace $Z_{j-1} \wedge V$ of dimension $n - j + 1$, with $Z_0 = 1$. This follows because the preceding wedge is decomposable and nonzero. Apply Lemma 115.1 with those radii in the exterior lattice. For this step the bound is at most

$$(1 + 2 \cdot 4^j)^{2f(n-j+1)} 2^{4j(n-j+1)\sum_\tau m_\tau}.$$

Multiply over j. Equal tuples give representatives differing on the left by an integral element fixing every Φ_j. The stabilizer order just proved bounds the number of such left N_Λ-cosets and completes the count. □

Theorem 115.3 (The arithmetic factor in the full ordinary diagonal error)**.** *Set $L_0 = 8(f+1)(n+1)^3$ and $a = (L_0 - 4fn^2)/2$. On every norm-one diagonal row, for all sufficiently large x on the common intervals, one has*

$$\frac{\sum_i \|Z_i - Z_i^B\|_{\mathrm{HS}}}{\lambda} \;\le\; Q^{-fa/2}\, d_\rho\, \frac{\sup_g V(g)^{L_0}\|W(g)\| \int_{B_\infty\backslash G_\infty} V(g)^a \|W(g)\|\, d\dot{g}}{\int_{B_\infty\backslash G_\infty} \|W(g)\|^2\, d\dot{g}}.$$

Here d_ρ is the column dimension, λ is the absolute value of the identity coefficient, and $W \neq 0$ is the ordinary section. The factor $Q^{-fa/2}$ is uniform in the lattice class and the word. The estimate includes the signed error at the marked identity index.

Proof. Put $U = \sup V^{L_0}\|W\|$ and $F_a = V^a\|W\|$. Both the supremum and the integral in the assertion are finite by Proposition 86.20; changing to the specified compatible norm preserves finiteness since matrix norms at a fixed row are equivalent. The box estimate of Theorem 115.2 gives

$$\sup_y \sum_{\delta \in N_\Lambda\backslash\Gamma} F_a(\delta y) \le U C_{f,n,d}\big(1 - 2^{-(L_0-a-4D)}\big)^{-f}.$$

Indeed the summand is bounded by $U2^{-(L_0-a)\sum m_\tau}$ in each box. Summing the geometric series is valid since $D \le n(n+1)(n+2)/6$ and $L_0 - a - 4D > 1$.

For a nonstabilizing admissible isometry the height separation in that theorem bounds its unweighted integrand in norm by $(2/Q)^{fa}F_a(y)F_a(\gamma^{-1}y)$. Sum over double cosets and unfold the nonnegative majorant as in Theorem 113.19. The result is bounded by the last display times $|c_{\mathrm{row}}|(2/Q)^{fa}\int_{N_\Lambda\backslash G_\infty} F_a$. This already bounds the sum of the norms before grouping by marked index. Now $\log C_{f,n,d} \ll f^2k^2 + fD \ll fa$ with absolute constants, and the last geometric-series factor is at most 2^f. Since $\log Q$ tends to infinity on the intervals, their product times $(2/Q)^{fa}$ is at most $Q^{-fa/2}$, uniformly in all the integers here.

Finally Theorem 113.24 gives $\lambda = |c_{\mathrm{row}}|\int_{N_\Lambda\backslash G_\infty}\|W\|^2/d_\rho$. In the ratio of the two integrals the volume of $N_\Lambda\backslash B_\infty$ cancels because their integrands are left B_∞-invariant. This gives the displayed estimate with the original finite level. The remaining ratio concerns the sizes of the actual real sections as the types vary. The norm-T middle lattice also has a gluing index at T, so its integral splitting requires a separate calculation. □

115.2 Both terminal norms in the ordinary period

Let $Q = g_1g_2$, let $T = \varpi\bar{\varpi}$ be the terminal prime, and use the polynomial-conductor fields E_x and intervals of Theorem 109.38. Write $F = E_x^+$, $f = [F:\mathbb{Q}]$, and $R_E = |\operatorname{disc}(E_x)|^{1/(2f)}$. Keep all norm-one rows and the norm-T rows satisfying the primitive hyperplane bound of that theorem for a fixed constant $A > 0$. At norm T the bound retains a fraction at least $1-\exp(-Af(k-1))$ of the refined coordinates, including in each stipulated terminal character. In each row write $\Lambda = X_{\mathcal{O}} \oplus J \oplus X_{\mathcal{O}}^*$ as in Lemma 109.20; the ordered outer frame has length s, the middle lattice contains z of norm $-T_j$, $T_j \in \{1, T\}$, and $L = J \cap z^\perp$ has positive rank k. Put $n = 2s + 1 + k$. In this section the outer line lattices are generated by the distinguished vectors of the primitive frame model of Lemma 109.20. The extent of the following estimate is determined by this integral hypothesis.

On the middle space use the positive Hermitian form obtained by reversing the sign on Ez:

$$h_z^+(u,v) = h(u,v) + 2h(u,z)h(z,v)/T_j.$$

Take the outer hyperbolic basis vectors orthonormal and perpendicular to this space for the positive norm. This defines the adapted height V_j of row j as in Section 115.1. For mixed rows one can view y as a real isometry from a reference space to the target and $\gamma^{-1}y$ as an isometry to the source. Use the corresponding positive norms in their operator norms. Inversion preserves these norms since the original Hermitian forms are unitary involutions in both spaces.

Lemma 115.4 (The product minimum on the retained middle lattices)**.** *For all sufficiently large x, uniformly in the retained rows, every $0 \neq u \in J$ satisfies*

$$\prod_\tau \|u_\tau\|_{h_z^+} \geq 1.$$

At norm T, every $0 \neq p \in L^\sharp$ satisfies the stronger positive-space inequality

$$\prod_\tau \|p_\tau\|^2 > \left(kR_E/(2\pi \exp(A+4))\right)^f.$$

Proof. For a positive lattice M write $\delta(M) = [M^\sharp : M]^{1/2}$ if M is integral, and extend this definition to fractional lattices by covolume. In the product of the complex embeddings its Euclidean covolume is $c_E^{\operatorname{rank} M}\delta(M)$, where c_E is the covolume of $\mathcal{O}_E$. This follows from the determinant of the Hermitian Gram matrix for a free lattice; passing to a finite-index free sublattice proves the same formula for a projective lattice. At norm T we have $\delta(L) = T^f$ by Lemma 113.6.

For a nonzero vector $p \in L^\sharp$, the hyperplane $H = L \cap p^\perp$ is primitive and $\mathfrak{b} = h(p,L)$ is a nonzero integral ideal. The orthogonal projection of L onto Ep has lattice $\mathfrak{b}p/h(p,p)$. The covolume of L is the covolume of H times that of this projected lattice, by a block triangular determinant on real lattice bases. The field constants cancel with the ranks, giving

$$\prod_\tau \|p_\tau\|^2 = N_{E/\mathbb{Q}}(\mathfrak{b})\,\frac{\delta(H)}{\delta(L)}.$$

Apply the retained bound of Theorem 109.38 to H and use $N(\mathfrak{b}) \geq 1$. This proves the second assertion.

For $u \in J$, its projection p onto EL lies in $L^\sharp$ because $h(p,L) = h(u,L) \subset \mathcal{O}_E$. The positive norm of u dominates that of p at every place. As k tends to infinity and $R_E \geq 1$, the second assertion suffices if $p \neq 0$. If $p = 0$, primitivity gives $u \in Ez \cap J = \mathcal{O}_E z$; its norm product is at least $T^{f/2}$. At norm one the positive form is already integral on the direct sum in Section 115.1, so its norm-product bound follows directly. □

Proposition 115.5 (Vanishing when only the terminal character changes)**.** *Let γ be an integral isometry from a source row to a target row. Suppose it transports each outer isotropic vector to its target modulo the preceding span. If it fails to transport the terminal vector modulo the isotropic span, its ordinary orbital integral is zero. This holds also when the two terminal norms differ.*

Proof. The outer flags agree after transport, so their real unipotent radicals agree and γ identifies their integral unipotent subgroups. In fact these radicals are defined by acting trivially on the graded spaces of the outer flag and its orthogonal flag, including the full middle quotient. They are independent of the last vector. Write N_∞ and N_Λ for the common groups in target coordinates.

In the last simple coordinate of N_∞ we can prescribe an arbitrary complex linear functional ϕ on the middle quotient at any real place, with all other places trivial. Its action fixes the outer span pointwise and sends a middle vector w to $w + \phi(w)e_s$ modulo the preceding span. Such elements are the real shears of (109.12), with the adjoint block and quadratic correction supplied by the isometry equation. The earlier simple characters vanish on them. In the integrand the two terminal characters give the additive character at $\phi(\bar{z}_t - \bar{\gamma}\bar{z}_s)$, up to reversing the sign by the pairing convention. Here bars on vectors and the map denote their classes in the middle quotient. This description incorporates the frequency since the last vector in (121.7) is βe. Transport of the last outer graded line is the identity, so introduces no scalar. The difference of the terminal classes is nonzero. The nontrivial real additive character composed with the trace from $\mathbb{C}$ is therefore nontrivial on these shears.

The real integration domain is $N_\Lambda\backslash G_\infty$. The product of the two real sections transforms on the left by the ratio of their unipotent characters. For an admissible orbital term this ratio descends to the unipotent quotient: the finite factors on its common rational stabilizer cancel, as in Proposition 96.20. Equivalently one can first average over the common adelic unipotent quotient in the unfolding. Integrating over $N_\Lambda\backslash N_\infty$ in the real formula gives zero, since right translation by an element on which this ratio is nontrivial preserves the quotient measure. The integrals converge by Proposition 86.20. This proves the vanishing. □

Theorem 115.6 (The arithmetic error on the retained primitive-frame rows)**.** *Take* $L_0 = 8(f+1)(n+1)^3$, $a = (L_0 - 4fn^2)/2$, *and set*

$$U_j = \sup V_j^{L_0}\|W_j\|, \qquad I_j = \int_{N_j\backslash G_\infty} V_j^a\|W_j\|.$$

For every pair of retained rows, with either terminal frequency at either end, and for all sufficiently large x *on the specified intervals,*

$$\sum_{i\in G}\|Z_i^{tu} - B_i^{tu}\|_{\mathrm{HS}} \;\le\; |c_{t,u}|\,Q^{-fa/2}\min\{I_tU_u, I_uU_t\}.$$

Here u, t *are source and target row indices. All the orbital coefficients have their original finite marking, signs, measures and real sections.*

Proof. When the integral lattices are not isometric the arrays are zero. For an isometry with a nonzero term outside B^{tu}, Proposition 115.5 forces a first failure at some outer index $i \le s$. The corresponding two outer vectors have norm one in their respective adapted norms. Use the block filtration with the e_j-lines first, then the single middle block J, then the f_j-lines in reverse order. The target stabilizer acts isometrically on the middle block for $h_{z_t}^+$. Take the last nonzero block of $\gamma e_{u,i} - e_{t,i}$. It lies beyond the preceding target span and belongs to Q times its lattice by the marking. In a line its norm product is at least Q^f by integrality; in J this follows from Lemma 115.4. Multiplying γ on the left and right by arbitrary real decorated stabilizers preserves this block norm, exactly as in Theorem 115.2: the earlier source vectors map into the earlier target span. Taking infima in the two real operator norms gives

$$Q^f \le 2^f V_t(y)V_u(\gamma^{-1}y).$$

We record the cost of the middle lattice in the independent coset count. The form $T_jh_z^+$ is integral on J. On every graded block of the j'-th exterior lattice the positive pairing becomes integral on multiplication by $T^{j'}$, since $T_j = 1$ or T. Hence the separation estimate in Lemma 115.1 still applies with its radii multiplied by $T^{j'/2}$. The outer wedges have norm one

and the last wedge has norm $\sqrt{T_j}$. Thus the packing proof of Theorem 115.2 applies with the new constant

$$C_T = 3^{(2fk)^2} \prod_{j'=1}^{s+1} (1 + 2 \cdot 4^{j'} T^{(j'+1)/2})^{2f(n-j'+1)}$$

and the same box exponent $4D$, where $D = \sum_{j'=1}^{s+1} j'(n - j' + 1)$. The stabilizer acts on the positive lattice L, so the same bound modulo three applies at both frequencies. For a mixed pair the cosets to count are integral isometries from one fixed lattice to the other modulo the target unipotent subgroup. In each nonempty box $\delta_0 \delta^{-1}$ is still an automorphism of the target lattice. Its exterior tuples therefore lie in exactly the lattices used in that proof. This gives the count with the same constant for either orientation.

Now $\log C_T \ll f^2k^2 + fn(s+1)^2 \log(2T)$, and $a \gg fn^3$. On the intervals of Theorem 109.38 one has $\log T = O(mx^{\varepsilon})$, $s + 1 \le m = O(\sqrt{x})$, $n \ge k \asymp x^{9/2+10\varepsilon}$, and $\log Q \ge 16x^{\varepsilon}$. Thus $\log C_T = o(fa \log Q)$: the first term divided by $fa \log Q$ is $O(1/(n \log Q))$, and the second is $O(m^3/(fn^2))$. Also $L_0 - a - 4D > 1$. The factor from the geometric series in the box indices is at most 2^f, and

$$C_T \, 2^f (2/Q)^{fa} \le Q^{-fa/2}$$

eventually, independently of the pair of rows. Unfolding the majorant with the displayed height separation gives this factor times $|c_{t,u}| I_t U_u$. Inverting isometries and changing variables gives the same bound with $I_u U_t$. The terms whose real integral vanished were removed before taking the majorant. The triangle inequality after collecting signed terms at each actual index proves the result. □

In Theorem 113.26 this controls the term $R_{\iota t}^{tu}$ as well as the sum at distinct indices. The real-section norms, the ratios of the volume factors to the integral period normalizations, and the total over the retained rows still enter the interval comparison.

115.3 The outer flag ideals in the ordinary period

On the fields and intervals of Theorem 109.38, write $E = E_x$, $F = E^+$, $f = [F : \mathbb{Q}]$, and let $\mathfrak{d} = \mathfrak{d}_{F/\mathbb{Q}} \mathcal{O}_E$. Number the outer vectors e_i from 1 to s, set $X_i = \sum_{h \le i} E e_h$, and put $X = X_s$. The prescribed outer part of a marked row is

$$A = \Lambda \cap X = \bigoplus_{i=1}^{s} \mathfrak{a}_i e_i, \qquad \mathfrak{a}_i = \mathfrak{d}^{-(s-i+1)}.$$

These ideals are units at the two original level primes, which split completely in E. The row is Hermitian self-dual. Its terminal vector $z \in \Lambda \cap X^{\perp}$ has norm $-T_j$, with $T_j = 1$ or $T_j = T = \varpi\bar{\varpi}$. Its image $\bar{z}$ in $X^{\perp}/X$ is primitive: the multipliers $c \in E$ for which $c\bar{z}$ belongs to the quotient lattice form $\mathcal{O}_E$. These are the defining row conditions in `Formalism/SharpRestriction/CuspSpanNormBounds/BlockEnds.lean`, line 116, `Formalism/SharpRestriction/CuspSpanNormBounds/FlagEEqHyperbolic.lean`, line 93 and `Formalism/SharpRestriction/CuspSpanNormBounds/IsPrimitiveSub.lean`, line 99. We retain all unit-terminal rows and impose the hyperplane inequality of Theorem 109.38 at terminal norm T.

Lemma 115.7 (A Witt decomposition with the prescribed outer ideals)**.** *There are hyperbolic partners f_i and a self-dual middle lattice J containing z such that*

$$\Lambda = A \oplus J \oplus A', \qquad A' = \bigoplus_{i=1}^{s} \bar{\mathfrak{a}}_i^{-1} f_i, \qquad h(e_i, f_h) = \delta_{ih}.$$

Here A, A' are isotropic, J is perpendicular to both, and $J \cap Ez = \mathcal{O}_E z$.

Proof. The submodule $A + \mathcal{O}_E z$ is saturated in Λ. Indeed a vector of Λ in $X + Ez$ maps to $c\bar{z}$ in $X^{\perp}/X$. Primitivity gives $c \in \mathcal{O}_E$, and subtraction of cz leaves an element of A. Over the Dedekind ring $\mathcal{O}_E$ the torsion-free quotient is projective, so the submodule is a direct summand. Integral functionals on A may therefore be extended to Λ with value zero on z. Hermitian self-duality represents these functionals by vectors perpendicular to z. Projectivity of the conjugate dual $A^{\vee}$ gives an integral linear section $u : A^{\vee} \to \Lambda \cap z^{\perp}$ for the pairing with A.

Let H be the Gram pairing on the image of u. The finite extension E/F is unramified at all finite places on these fields. The trace from $\mathcal{O}_E$ to $\mathcal{O}_F$ is surjective: locally this follows by reduction to the trace of a separable finite residue field extension, or by projection in the split algebra. The trace ideal is consequently the unit ideal globally. Choose $t \in \mathcal{O}_E$ with $t + \bar{t} = 1$. By perfect duality between A and $A^{\vee}$, the integral form $-tH$ is represented by an integral correction from $A^{\vee}$ to A, with the adjoint form $-\bar{t}H$. Adding this correction to u changes the Gram pairing by $-H$ and keeps the image perpendicular to z. The image and A now form a self-dual hyperbolic sublattice: the cross pairing is perfect and both diagonal pairings vanish. The integral orthogonal projection onto this sublattice splits it off, and its perpendicular J is self-dual. Rational partners dual to the specified e_i in the corrected image identify its lattice with the displayed A'. Primitivity in the quotient gives the last equality for J. □

For a nonzero fractional ideal $\mathfrak{b}$, use its positive absolute norm $N\mathfrak{b}$ and put $\eta(\mathfrak{b}) = (N\mathfrak{b})^{-1/(2f)}$. At every real place give e_i, f_i respective lengths $c_i = \eta(\mathfrak{a}_i)$ and c_i^{-1}. Use the positive majorant h_z^{+} of Section 115.2 on J. All these blocks are mutually perpendicular for the positive norm. The original Hermitian form is represented by a unitary involution in the resulting orthonormal basis. Define V_j as the product of the real operator norms minimized over the left decorated stabilizer, with these target norms and a single fixed compatible reference norm on the domain.

Lemma 115.8 (Product minima with a fractional ideal)**.** *Let Π be any one of the outer line lattices or the middle lattice in Lemma 115.7. If $0 \neq v \in \mathfrak{b}\Pi$, then on the retained rows, for all sufficiently large x,*

$$\prod_{\tau} \|v_\tau\| \ \geq \sqrt{N\mathfrak{b}}.$$

For a graded block Π' of $\bigwedge^{j} \Lambda$ containing b middle factors, a nonzero $v \in \mathfrak{b}\Pi'$ satisfies

$$\prod_{\tau} \|v_\tau\| \ \geq \sqrt{N\mathfrak{b}}\, T_j^{-bf/2}.$$

Proof. An outer block has the form $\mathfrak{h}v_0$ with $\|v_0\|_\tau = \eta(\mathfrak{h})$. For $0 \neq x \in \mathfrak{b}\mathfrak{h}$, the ideal containment $(x) \subset \mathfrak{b}\mathfrak{h}$ gives $\prod_\tau |\tau(x)| \geq \sqrt{N\mathfrak{b}\, N\mathfrak{h}}$. Multiplying by $\eta(\mathfrak{h})^f$ cancels the second factor.

For the middle lattice at norm T project v to EL, where $L = J \cap z^{\perp}$, and denote the projection by p. If $p \neq 0$, the pairing ideal $\mathfrak{c}$ of p with L lies in $\mathfrak{b}$ or its conjugate, according to the pairing convention. In particular $N\mathfrak{c} \geq N\mathfrak{b}$. Put $H = L \cap p^{\perp}$. The covolume calculation of Lemma 115.4 holds for fractional pairing ideals as well: the projected line lattice is $\mathfrak{c}p/h(p,p)$, up to conjugation of the ideal. Hence $\prod_\tau \|p_\tau\|^2 = N\mathfrak{c}\, \delta(H)/\delta(L)$. The retained hyperplane bound and $\delta(L) = T^f$ give $\delta(H)/\delta(L) > 1$ uniformly for large x. Orthogonal projection decreases the positive norm at each place. If $p = 0$, one has $\mathfrak{b}J \cap Ez = \mathfrak{b}z$ by localization at each prime and $J \cap Ez = \mathcal{O}_E z$. The product bound follows from $\|z\|_\tau = \sqrt{T}$. At terminal norm one the middle majorant is integral. Its value on v belongs to $\mathfrak{b}\bar{\mathfrak{b}} \cap F$, so the norm to $\mathbb{Q}$ from F of that value is at least $N\mathfrak{b}$.

For the exterior assertion write the block as an outer tensor line $\mathfrak{h} v_0$ tensored with $\bigwedge^b J$. The normalized length of the line generator is $\eta(\mathfrak{h})$ by multiplicativity. The form $T_j h_z^+$ is integral on J, so its induced exterior pairing is integral and equals T_j^b times the exterior majorant. On a nonzero vector in $(\mathfrak{b}\mathfrak{h})\bigwedge^b J$, its value lies in $(\mathfrak{b}\mathfrak{h})\overline{(\mathfrak{b}\mathfrak{h})} \cap F$ and is totally positive. Its norm from F is at least $N(\mathfrak{b}\mathfrak{h})$. Cancel the line factor as before and take square roots. □

Theorem 115.9 (The ordinary orbital error for the prescribed outer ideals)**.** *Put $n = 2s+k+1$, $L_0 = 8(f+1)(n+1)^3$, $a = (L_0 - 4fn^2)/2$, and $Q = g_1 g_2$. Let N_j be the integral unipotent subgroup of row j, and W_j its prescribed ordinary real section. Define*

$$U_j = \sup_y V_j(y)^{L_0} \|W_j(y)\|, \qquad I_j = \int_{N_j \backslash G_\infty} V_j(y)^a \|W_j(y)\| \, dy.$$

For every retained pair t, u on the same fields and intervals, with the original two-prime marking and either terminal norm,

$$\sum_{i \in G} \|Z_i^{tu} - B_i^{tu}\|_{\mathrm{HS}} \;\le\; |c_{t,u}| \, Q^{-fa/2} \min\{I_t U_u, I_u U_t\}$$

for all sufficiently large x. The bound uses the height just defined for the fractional line ideals.

Proof. An isometry transporting all the outer vectors modulo earlier vectors but changing the terminal class has zero orbital integral by Proposition 115.5. The proof there uses the common rational unipotent radical, its lattice, and the two characters on it; it applies with the present ideals. For any other nonzero term outside B^{tu}, let i be the first failed outer index. Order the blocks as in Theorem 115.6 and let p be the highest nonzero component of $\gamma e_{u,i} - e_{t,i}$. It lies beyond $X_{t,i-1}$. For all $b \in \mathfrak{a}_i$ both $b e_{u,i}$ and $b e_{t,i}$ are integral. Hence $p \in \mathfrak{a}_i^{-1}\Pi$ in its block lattice Π. At the primes over Q, the marked upper unitriangular condition kills this component modulo Q, and $\mathfrak{a}_i$ is a unit. Intersecting the local conditions gives $p \in Q\mathfrak{a}_i^{-1}\Pi$. The first bound of Lemma 115.8 gives $\prod_\tau \|p_\tau\| \ge Q^f c_i^f$. Left and right decorated stabilizers preserve the norm of the highest component, since the earlier vectors have transported into the earlier span. The triangle inequality used in Theorem 115.2 has right side $2^f c_i^f V_t(y) V_u(\gamma^{-1} y)$, as the source and target distinguished vectors both have length c_i. Cancelling c_i^f gives the required height separation.

For the coset count set $\mathfrak{h}_j = \prod_{i \le \min(j,s)} \mathfrak{a}_i$. The initial j-fold wedge Φ_j, with last vector z if $j = s+1$, belongs to $\mathfrak{h}_j^{-1} \bigwedge^j \Lambda$. This follows by wedging the integral line modules and, at the last step, the integral vector z. Its length at each place is at most $\eta(\mathfrak{h}_j)\sqrt{T}$. In a nonempty height box the tuples $Z_j = \delta_0 \delta^{-1} \Phi_j$ lie in that same fractional exterior lattice because $\delta_0 \delta^{-1}$ is an integral automorphism. After applying the fixed left stabilizer element for the box, their local radii are bounded by $4^j 2^{2jm_\tau} \eta(\mathfrak{h}_j)\sqrt{T}$. In each graded block their pairwise differences, when nonzero, have product norm at least $\eta(\mathfrak{h}_j)^f T^{-jf/2}$, by the exterior assertion. The factor $\eta(\mathfrak{h}_j)$ therefore cancels in the packing ratio. More precisely, condition on the later blocks before applying translated-ball packing as in Lemma 115.1; the remaining dimension drops sum to $n - j + 1$ for each fixed preceding nonzero wedge. Thus the box count has exactly the upper bound

$$3^{(2fk)^2} \prod_{j=1}^{s+1} \big(1 + 2 \cdot 4^j T^{(j+1)/2}\big)^{2f(n-j+1)} \, 2^{4D \sum_\tau m_\tau}, \qquad D = \sum_{j=1}^{s+1} j(n-j+1).$$

Here the stabilizer of the tuple modulo its integral unipotent subgroup embeds into the automorphism group of L: its middle action fixes z, and acting trivially on L fixes the whole

rational middle quotient. Reduction modulo three on the underlying positive $\mathbb{Z}$-lattice gives the first factor as in Theorem 115.2.

For each fixed row the displayed U_j, I_j are finite by Proposition 86.20 and equivalence of the two positive norms at that row. Now use the height separation in the orbital integral, unfold the absolute majorant in either direction, and sum the boxes. The exponent D and all box constants equal those in Theorem 115.6, where their logarithms were estimated on these intervals. Its last geometric-series bound gives $Q^{-fa/2}$. This proves the stated estimate with the actual fractional ideals, including the coefficient at the identity marking. □

115.4 The real section at the prime terminal frequency

Keep the outer ideals, level and retained row coordinates of Theorem 115.9. Use the signed orthonormal norm on the common reference space. For a row j, write V_j^0 for its height when the outer e_i, f_i all have length one, and continue to write V_j for its height with the prescribed lengths c_i, c_i^{-1}. Both use h_z^+ on the middle space. The two quantities in the period bound can be expressed in the first metric by an explicit change of the real Bessel character.

Proposition 115.10 (The different in the real Bessel character)**.** *Put $R = |\operatorname{disc}(F)|^{1/f}$. Then $c_i = R^{s-i+1}$. In the real unitary group let C act by c_i on e_i, by c_i^{-1} on f_i, and by the identity on the middle space. At all places together put*

$$J_C = \prod_{i=1}^{s} c_i^{2f(n+1-2i)}.$$

The section $W^\circ(g) = J_C^{1/2} W(C^{-1}g)$ has the same quotient L^2-norm as W. In standard outer coordinates its simple additive characters have all been divided by R; the last has coefficient β/R when its original coefficient is β. Its right representation and its compact column type are preserved.

Write $v_j = \operatorname{vol}(N_j \backslash B_{j,\infty})$, where N_j is the integral unipotent subgroup. Use the quantity U_j of Theorem 115.9 and write I_j for its integral. With the original quotient measures,

$$U_j = J_C^{-1/2} \sup_g V_j^0(g)^{L_0} \|W^\circ(g)\|,$$
$$I_j = v_j J_C^{1/2} \int_{B_{j,\infty}\backslash G_\infty} V_j^0(g)^a \|W^\circ(g)\| \, d\dot{g}.$$

Consequently the powers of J_C cancel in each of $I_t U_u$ and $I_u U_t$ for two rows with the same E, n, s.

Proof. The absolute norm from E of $\mathfrak{d}_{F/\mathbb{Q}}\mathcal{O}_E$ is $|\operatorname{disc} F|^2$, by extension of ideals and the norm of the different. The formula for c_i follows. Conjugation by C normalizes the decorated real stabilizer $B = N \rtimes H$ and centralizes H. It multiplies each of the first $s-1$ simple coordinates of the radical by $c_i/c_{i+1} = R$ and the terminal coordinate by $c_s = R$. In the transformation law for $W(C^{-1}g)$ the argument of the character is $C^{-1}n'C$, giving the claimed coefficients. All these operations commute with right translation.

At one real place, the independent coordinates on the Lie algebra of the radical are the strict upper entries on the outer space, the block from the middle to the outer space, and a skew-Hermitian s-by-s block on the far corner. The first gives determinant $\prod_{i<j}(c_i/c_j)^2$, as each entry is complex. The second gives $\prod_i c_i^{2(k+1)}$. In the last, each real diagonal coordinate is multiplied by c_i^2 and each complex off-diagonal coordinate by $c_i c_j$. Its determinant is $\prod_i c_i^{2s}$. Multiplication and $n = 2s + k + 1$ give J_C over all f places.

For a nonnegative function F on $B\backslash G_\infty$ the quotient measure satisfies $\int F(Cg)\,d\dot{g} = J_C \int F(g)\,d\dot{g}$. Indeed for a compactly supported function on the group first integrate over $b \in B$; the substitution $b' = CbC^{-1}$ multiplies Haar measure by J_C. Left translation on the ambient group preserves Haar measure. The quotient integration formula gives the assertion, which extends to nonnegative functions by local quotient sections and monotone convergence. This proves the L^2 normalization. Moreover the length of a vector in the weighted metric is its unweighted length after application of C. Taking the infimum over B therefore gives $V_j(g) = V_j^0(Cg)$. Section norms and both heights are B-invariant. Integrating first over $N_j\backslash B$ supplies v_j; it is finite since the integral unipotent lattice is cocompact and H is compact. The two claimed identities now follow by substitution. □

For a nonzero section of either frequency and a specified height V, form the dimensionless ratio

$$\mathcal{M}(W,V) = \frac{\sup_g V(g)^{L_0}\|W(g)\| \displaystyle\int_{B\backslash G_\infty} V(g)^a \|W(g)\|\,d\dot{g}}{\displaystyle\int_{B\backslash G_\infty} \|W(g)\|^2\,d\dot{g}}.$$

Set $p = L_0 + a$. If $d\mu_W = \|W\|^2 d\dot{g}/\|W\|_2^2$, then

$$\mathcal{M}(W,V) \geq \int V^p\,d\mu_W. \tag{115.1}$$

This follows by bounding $V^{L_0}\|W\|$ at each point by its supremum and integrating $V^a\|W\|$ times this bound.

Lemma 115.11 (A constraint on two translates of a real section)**.** *Use one real Witt model with the same outer lengths at terminal frequencies* $1, \varpi$*. If* W_ϖ *is the normalized left translate of* W_1 *of Proposition 106.13, then*

$$\mathcal{M}(W_1,V)\mathcal{M}(W_\varpi,V) \;\geq\; T^{fp/2}.$$

Proof. Let a_ϖ multiply every e_i by ϖ_τ, every f_i by $\bar{\varpi}_\tau^{-1}$, and fix the middle space at place τ. The decorated stabilizer fixes e_1. Thus $b_1 a_\varpi b_2^{-1}$ acts on that vector by ϖ_τ for any b_1, b_2 in the stabilizer, and its norm is at least $\sqrt{T}$. Factor this operator as $(b_1 g)(b_2 a_\varpi^{-1} g)^{-1}$. The norm of the inverse of an isometry between compatible majorants equals the norm of the isometry. Taking the two infima and then the product over places gives $V(g)V(a_\varpi^{-1}g) \geq T^{f/2}$. The probability measure for W_ϖ is the pushforward of μ_{W_1} under $g \mapsto a_\varpi^{-1}g$, by the quotient measure calculation in Proposition 106.13. Apply Cauchy–Schwarz to $V(g)^{p/2}$ and $V(a_\varpi^{-1}g)^{p/2}$ under μ_{W_1}. The moment comparison (115.1) proves the assertion. □

Theorem 115.12 (A lower bound for the prime-frequency majorant)**.** *For every retained prime-terminal row take the prescribed isometrically normalized Bessel section of a unit vector in the product lowest compact type of Theorem 87.24. Use its weighted height from Theorem 115.9. On the same fields and intervals, with* $b = \exp(x^\varepsilon)$*, one has for an absolute* $c > 0$

$$\mathcal{M}(W_\varpi,V) \;\geq\; (1 - e^{-cbn^2})\left(\frac{\sqrt{T}}{Rbn^4}\right)^{fp}.$$

In particular $Q^{-fa/2}\mathcal{M}(W_\varpi,V)$ *tends to infinity uniformly for these sections. The right-hand side of the diagonal error estimate of Theorem 115.9, divided by the absolute identity coefficient, is at least this quantity.*

Proof. At each infinite place identify the standard marked real model with the row by mapping the outer vectors to the specified e_i, their duals to the chosen f_i, and the negative unit vector to z/ϖ_τ. These pairings agree, and the remaining positive complements have the same dimension and signature; an isometry of their orthonormal bases finishes the identification. The ordinary real section and its character transport under this identification. The quotient measure can be normalized by its L^2 mass. A change of hyperbolic partners fixing the outer vectors and z belongs to the decorated stabilizer after choosing an isometry on the positive complement. Thus the following estimate depends only on the decorated frame.

Write $a_{s,\tau}$ for the distance from $g_\tau^{-1}e_s$ to $g_\tau^{-1}X_{s-1}$ in the reference norm. For every $b' \in B_\tau$ the vector $b'^{-1}e_s$ belongs to $e_s + X_{s-1}$. Hence

$$a_{s,\tau} \le \|g_\tau^{-1}b'^{-1}e_s\|_{\mathrm{ref}} \le c_s\|(b'g_\tau)^{-1}\|_{\mathrm{op}} = R\|b'g_\tau\|_{\mathrm{op}}.$$

Take the infimum over b'. Theorem 87.24 shows that at least $1-e^{-cbn^2}$ of the normalized mass lies where, at every place simultaneously, $a_{s,\tau} > \sqrt{T}/(bn^4)$. On this set the product height is at least $(\sqrt{T}/(Rbn^4))^f$. Integrating the p-th power and applying (115.1) proves the bound.

On these intervals $\log R = O(\log x)$, since E has polynomial conductor, and n is polynomial in x. Furthermore $\log T = (17+o(1))mx^{\varepsilon}$, $m \to \infty$, whereas $\log Q = (16+o(1))x^{\varepsilon}$, by Theorem 93.9. Since $p > a > 0$, the logarithm of the asserted lower bound after multiplication by $Q^{-fa/2}$ tends to infinity. Finally, on a diagonal row $\lambda_j = |c_{j,j}|\,v_j\|W_j\|_2^2/d_j$ in Theorem 113.24, with $d_j \ge 1$. Dividing the estimate's right-hand side by λ_j therefore gives exactly $d_jQ^{-fa/2}\mathcal{M}(W_j,V_j)$. □

The lower bound concerns the absolute estimate used in the unfolding. The integral projected period depends on signed orbital integrals. Their phases or a sharper estimate on the individual regions of integration are essential to an argument using these prime-frequency rows.

115.5 Intersecting flags at the two terminal frequencies

Consider a term of the ordinary projected period between rows of terminal norms $T_0, T_1 \in \{1, T\}$. Transport the second frame by the rational isometry indexing the term. Finite support identifies the two lattices with one Λ; above $Q = g_1g_2$ the reduced outer flags agree under upper unitriangular changes of frame. Write $e_{b,i}$ for the resulting outer vectors, X_b for their span, and $X_{b,i}$ for its first i positions, $b = 0, 1$. The line modules are $\mathfrak{a}_i e_{b,i}$ as in Section 115.3. A term can be nonzero only if its two characters agree on the common unipotent group: integration over the compact unipotent fibre gives this condition by Lemma 98.1, with the respective terminal vectors in the two characters.

Lemma 115.13 (The determinant on the common suffix)**.** *Suppose the characters agree, $I = X_0 \cap X_1$ has dimension $e > 0$, and $t = s - e$. Put $P_b = X_{b,t}$ and $M = \Lambda \cap I$. Projection from I to X_b/P_b is a rational isomorphism for both b. Let $\mathfrak{c}_b$ be the index ideal of the image of M in the integral quotient $(\Lambda \cap X_b)/(\Lambda \cap P_b)$. There is a $\lambda \in E^\times$ such that*

$$\lambda\bar{\lambda} = T_1/T_0, \qquad \mathfrak{c}_0 = (\lambda)^e\mathfrak{c}_1.$$

The index ideal of a full sublattice has valuation the length of its cokernel at each finite prime.

Proof. Theorem 86.7 applies to unequal terminal norms. It shows that the flags induce the same filtration on I, with one-dimensional successive quotients at positions $t+1, \dots, s$, and that the cross pairing of X_0, X_1 has kernel I on both sides. In particular, I projects isomorphically

as asserted. Write S_b for the successor map on the ordered outer vectors, whose last value is z_b. For $v \in I$ with last nonzero coefficient in position i, equality of the characters gives

$$S_0 v - S_1 v \in X_{0,i} + X_{1,i}$$

by (86.4). If $i < s$, subtracting vectors in these spaces gives a common vector in position $i+1$ with the same two leading coefficients. At $i = s$ it gives a common vector of the two extended flags with negative norm. Its two terminal coefficients a_0, a_1 satisfy $a_0 \bar{a}_0 T_0 = a_1 \bar{a}_1 T_1$, since addition of an isotropic vector to its perpendicular terminal vector preserves the norm. All common vectors at the terminal position have the same ratio $\lambda = a_0/a_1$: projection to either terminal line from $(X_0 + Ez_0) \cap (X_1 + Ez_1)$ has kernel I. Indeed an isotropic vector in either extended flag has zero terminal coefficient in that flag. It follows by the successive steps that this same ratio occurs on each of the e common graded lines.

Identify the two quotients by the rational maps from I. In their ordered suffix bases the map from side 1 to side 0 is triangular with every diagonal entry λ. It sends one projected copy of M onto the other. The ambient determinant ideal in both these bases is $\prod_{i>t} \mathfrak{a}_i$. On taking the top exterior power, the index ideals therefore differ by $(\lambda)^e$, which proves the claim. □

Theorem 115.14 (A terminal-prime divisor of the prefix pairing)**.** *Take the Hermitian pairing conjugate-linear in its first argument. Let $D_t = \det(h(e_{0,i}, e_{1,j}))_{i,j \le t}$, put $D_0 = 1$, and let $\mathfrak{h}_t = \prod_{i \le t} \mathfrak{a}_i$. The fractional expression*

$$\mathfrak{A} = (D_t/Q^t)\bar{\mathfrak{h}}_t \mathfrak{h}_t$$

is a nonzero integral ideal. More precisely, $\mathfrak{A} = \bar{\mathfrak{c}}_0 \mathfrak{c}_1 \mathfrak{b}$ for an integral ideal $\mathfrak{b}$. If $T_0 = T$, $T_1 = 1$, then for each conjugate pair of primes over T, at least one of them occurs in $\mathfrak{A}$ with exponent at least e. In particular

$$T^{fe} \mid N_{E/\mathbb{Q}}\mathfrak{A}.$$

Interchanging the rows conjugates $\mathfrak{A}$. At unequal terminal norms there is no contributing term with $X_0 = X_1$.

Proof. Set $A_b = \Lambda \cap X_b$, $P_b^0 = \Lambda \cap P_b$, and $U_b = A_b/M$. The pairing h/Q takes integral values on $A_0 \times A_1$: away from Q use self-duality, and at each prime over Q both reduced spaces are the same isotropic space with a basis of marked vectors. It descends to $U_0 \times U_1$ because each side is isotropic and contains I. The rank computation in Lemma 115.13 makes the resulting rational pairing nonsingular. Its determinant ideal $\mathfrak{b}$ in the two projective lattices is therefore integral and nonzero. Here the ideal is generated locally by the determinant of the pairing in integral local bases, using conjugate local bases on the first side.

The injected image of P_b^0 has index ideal $\mathfrak{c}_b$ in U_b: the two descriptions of the finite quotient both give $A_b/(M + P_b^0)$. Restricting the pairing to these prefix sublattices multiplies its determinant ideal by $\bar{\mathfrak{c}}_0 \mathfrak{c}_1$, by determinants of local basis changes. Computing the same determinant in the specified $e_{b,i}$ yields $(D_t/Q^t)\bar{\mathfrak{h}}_t \mathfrak{h}_t$. This also holds for the empty prefix, with the empty determinant equal to one.

Every place of F over the split rational prime T gives a pair $\mathfrak{P}, \bar{\mathfrak{P}}$ in E. The valuations of λ at this pair sum to -1, since $\lambda\bar{\lambda} = 1/T$. At a prime with valuation at most -1, integrality of $\mathfrak{c}_0 = (\lambda)^e \mathfrak{c}_1$ forces $v_{\mathfrak{P}}(\mathfrak{c}_1) \ge e$, with $\mathfrak{P}$ chosen to be that prime. The determinant factorization puts the same divisor into $\mathfrak{A}$. There are f such pairs, each of residue degree one. This proves the norm divisibility. With $t = 0$, $\mathfrak{A} = \mathcal{O}_E$ contradicts the divisor since $e = s \ge 1$. □

We next use this divisor to control part of the mixed ordinary period in the region identified in Theorem 115.12. For row j let a_j be the real isometry acting by β_j on every e_i, by $\bar{\beta}_j^{-1}$

on every f_i, and by the identity on its middle space, with $\beta_j = 1$ or ϖ as prescribed. Set $\widetilde{V}_j(y) = V_j(a_j y)$. Since a_j normalizes the decorated stabilizer, this is the quotient height obtained by multiplying all the original outer lengths by $\sqrt{T_j}$ on the e_i and by $1/\sqrt{T_j}$ on the f_i. Denote by $\widetilde{U}_j, \widetilde{I}_j$ the quantities in Theorem 115.9 with this height.

Theorem 115.15 (The mixed contribution from intersections of large dimension)**.** *Use a pair of retained rows of unequal terminal norms. For a fixed $1 \le t < s$, restrict the orbital sum at each marked index i to terms with $\dim(X_0 \cap X_1) = e = s - t$. Call the resulting signed operator $Z_i(t)$. If $2t \le s$, then on the same intervals, for all sufficiently large x,*

$$\sum_i \|Z_i(t)\|_{\mathrm{HS}} \;\le\; |c_{0,1}|\, Q^{-fa/2} T^{-fa(e-t)/(2t)} \min\{\widetilde{I}_0 \widetilde{U}_1,\, \widetilde{I}_1 \widetilde{U}_0\}.$$

For the union of all $1 \le t \le s/2$ the same estimate holds with the power of T omitted. Both bounds use the unit-frequency real sections after the exact change of variables described below.

Proof. First keep any $0 < t < s$. In a common real reference space pair the two initial t-fold wedges after applying y^{-1} at both ends, with the second wedge transported by γ. Their Hermitian pairing has absolute value $|\tau D_t|$. The Hermitian form on the exterior reference space is represented by a unitary involution, so Cauchy–Schwarz bounds this by the product of their positive lengths. Each wedge is fixed by its own decorated stabilizer. In its row's modified majorant the wedge has length exactly $\eta(\mathfrak{h}_t) T_j^{t/2}$. The norm of the exterior inverse is at most the t-th power of the operator norm. Taking the infimum independently at each end, then multiplying over all places, gives

$$|N_{E/\mathbb{Q}} D_t|^{1/2} \le \eta(\mathfrak{h}_t)^{2f} T^{tf/2} \big(\widetilde{V}_0(y) \widetilde{V}_1(\gamma^{-1} y)\big)^t.$$

On the other hand the determinant divisor gives

$$|N_{E/\mathbb{Q}} D_t|^{1/2} = Q^{tf} \eta(\mathfrak{h}_t)^{2f} (N\mathfrak{A})^{1/2} \;\ge\; Q^{tf} \eta(\mathfrak{h}_t)^{2f} T^{ef/2}.$$

The product of the two heights is consequently at least $Q^f T^{f(e-t)/(2t)}$. If $2t \le s$ it is at least Q^f.

The modified heights still admit a uniform coset count. In the j-th exterior lattice a graded block with b middle factors, l_+ outer vectors and l_- dual outer vectors gains a factor $T_j^{(l_+ - l_-)/2}$ in each positive length. Here $b + l_+ + l_- = j$. By Lemma 115.8, on the lattice twisted by $\mathfrak{h}_{\min(j,s)}^{-1}$ its product minimum is at least $\eta(\mathfrak{h}_{\min(j,s)})^f T^{-jf/2}$. The initial wedge has length at most $\eta(\mathfrak{h}_{\min(j,s)}) T^{j/2}$, including the terminal step $j = s+1$. The translated-ball count in Theorem 115.9 therefore applies with box constant

$$3^{(2fk)^2} \prod_{j=1}^{s+1} (1 + 2 \cdot 4^j T^j)^{2f(n-j+1)}$$

and its original exponent $4D$. The logarithm of the new constant is $O(f^2 k^2 + fn(s+1)^2 \log(2T)) = o(fa \log Q)$ by the interval estimates in Theorem 115.6. It still absorbs, together with the convergent geometric series, into $Q^{fa/2}$. Remove the zero integrals using character compatibility, use the separation just proved to exponent a, and unfold the nonnegative majorant in either direction as in Theorem 113.22. This proves the asserted bound. For the union use the common separation Q^f before enlarging the sum to all cosets in the box count. This avoids any factor for the number of strata.

Finally let $W_{1,j}$ be the normalized unit-frequency section obtained by transporting the very same row vector; up to a scalar of modulus one, $W_j(y) = J_j^{-1/2} W_{1,j}(a_j y)$, with $J_j = T_j^{fs(n-s)}$, by Proposition 106.13. For $v_j = \mathrm{vol}(N_j \backslash B_{j,\infty})$ the exact factors are

$$\widetilde{U}_j = J_j^{-1/2} \sup_y V_j(y)^{L_0} \|W_{1,j}(y)\|,$$

$$\widetilde{I}_j = v_j J_j^{1/2} \int_{B_{j,\infty} \backslash G_\infty} V_j(y)^a \|W_{1,j}(y)\| \, d\dot{y}.$$

They follow from the quotient measure substitution in that proposition; in particular the change applies to the actual real vectors in the period. The finite levels and the index i in $Z_i(t)$ have been used throughout the proof. □

For smaller intersections the determinant computation still gives T^{fe} in the norm of the prefix ideal. Its resulting height bound has a negative power of T when $e < t$. Transverse flags have no common suffix to which this argument can be applied.

115.6 Denominators in the common part of two outer flags

Keep the retained rows and the integral projected period of Section 115.5 on the same field $E = E_x$. Put $F = E^+$ and $f = [F : \mathbb{Q}]$. Order the rows by $T_0 \ge T_1$, where $T_j \in \{1, T\}$ is the terminal norm. The prime T splits completely in E. Fix an integral orbit with compatible characters. Suppose the intersection of its two outer spaces has dimension e with $0 < e < s$, and put $t = s - e$. Its common graded scalar is λ in Lemma 115.13. Write

$$\beta = \lambda^{-1}, \qquad u = T_0/T_1, \qquad (\beta) = \mathfrak{n}\,\mathfrak{d}^{-1}, \qquad (\mathfrak{n}, \mathfrak{d}) = 1.$$

The ideals $\mathfrak{n}, \mathfrak{d}$ are integral and $\beta\bar{\beta} = u$. Denote absolute ideal norm by N. The scalar is intrinsic to the two ordered frames modulo their upper unipotent changes, since such changes fix each graded vector.

Lemma 115.16 (The possible scalars at a fixed denominator)**.** *One has $\mathfrak{n} = \bar{\mathfrak{d}}\,\mathfrak{p}$, where $\mathfrak{p} = (1)$ for $u = 1$. For $u = T$ the ideal $\mathfrak{p}$ is a product of one prime from each of the f conjugate pairs over T. In particular*

$$N\mathfrak{n} = u^f N\mathfrak{d}.$$

At each fixed denominator there are at most $2^{f\mathbf{1}_{u=T}} |\mu(E)|$ possible scalars β, where $\mu(E)$ is the group of roots of unity of E. For $X \ge 1$, the number with $N\mathfrak{d} \le X$ is at most

$$2^{f\mathbf{1}_{u=T}}\, 8 f^2\, \zeta(2)^{2f} X^2.$$

Proof. At a prime fixed by conjugation, twice the valuation of β is the valuation of u. Every prime over T is paired with a different prime, so the valuation at a fixed prime is zero. At a conjugate pair the two valuations are integers with sum zero away from (u). Thus the numerator exponent on one side equals the denominator exponent on the other. At a pair over $u = T$ the sum is one. If the valuations are $k, 1-k$ with $k \ge 1$, the positive exponent k exceeds the opposite denominator exponent $k-1$ by one, and both exponents on the other side in this comparison are zero. Exchanging the pair covers $k \le 0$. This proves the ideal identity.

There are at most the indicated number of choices of $\mathfrak{p}$. Two generators of the resulting fractional ideal satisfying the norm equation differ by an algebraic integer unit of absolute value one at every embedding. By Kronecker's theorem their ratio is a root of unity. For the count of denominator ideals, each ideal of norm at most X contributes at least X^{-2} to $\zeta_E(2)$. Comparing Euler factors gives $\zeta_E(2) \le \zeta(2)^{2f}$. Finally, writing $m = |\mu(E)|$, one has $\varphi(m) \le 2f$ and $\varphi(m)^2 \ge m/2$. For the second inequality the factor $\varphi(p^k)^2/p^k$ is at least one except at 2^1, where it equals one half. This gives $m \le 8f^2$ and proves the count. □

Theorem 115.17 (The denominator in the archimedean separation)**.** *Let $\mathfrak{c}_0, \mathfrak{c}_1, \mathfrak{b}, \mathfrak{A}$ be the two suffix indices, the determinant ideal of the quotient pairing, and the normalized prefix determinant ideal of Theorem 115.14. Put $\mathfrak{g} = \mathfrak{c}_0 + \mathfrak{c}_1$. Then*

$$N\mathfrak{A} = u^{fe}(N\mathfrak{d})^{2e}(N\mathfrak{g})^2 N\mathfrak{b}.$$

For the modified heights $\widetilde{V}_j$ of Section 115.5, the two arguments of the orbital integral satisfy

$$\widetilde{V}_0(y)\widetilde{V}_1(\gamma^{-1}y) \;\ge\; Q^f \frac{u^{fe/(2t)}}{(T_0T_1)^{f/2}}\,(N\mathfrak{d})^{e/t}\big(N\mathfrak{g}\sqrt{N\mathfrak{b}}\big)^{1/t}.$$

The inequalities apply to both equal and unequal terminal norms.

Proof. The equality $\mathfrak{c}_0 = (\lambda)^e\mathfrak{c}_1$ compares two integral ideals. Taking their minimum exponent at every prime gives

$$\mathfrak{c}_0 = \mathfrak{g}\,\mathfrak{d}^e, \qquad \mathfrak{c}_1 = \mathfrak{g}\,\mathfrak{n}^e.$$

Use the factorization $\mathfrak{A} = \bar{\mathfrak{c}}_0\mathfrak{c}_1\mathfrak{b}$ and take norms. Lemma 115.16 yields the first formula, including contributions at primes away from T.

The Hermitian pairing of the two prefix wedges equals D_t, the cross determinant. Their lengths in the individual modified majorants at place τ are $\eta(\mathfrak{h}_t)T_j^{t/2}$ in the notation of Theorem 115.15. Both wedges are invariant under their own decorated stabilizers. Cauchy–Schwarz in the common reference majorant, followed by the exterior operator norm bound at each end, gives

$$|N_{E/\mathbb{Q}}D_t|^{1/2} \;\le\; \eta(\mathfrak{h}_t)^{2f}(T_0T_1)^{tf/2}\big(\widetilde{V}_0(y)\widetilde{V}_1(\gamma^{-1}y)\big)^t.$$

This uses one complex absolute value for each pair of conjugate embeddings. By the definition of $\mathfrak{A}$, the left side is $Q^{tf}\eta(\mathfrak{h}_t)^{2f}(N\mathfrak{A})^{1/2}$. Substitution of the norm formula and taking t-th roots proves the assertion. □

Corollary 115.18 (A bound for all sufficiently large denominators)**.** *Fix $L \ge 1$, and for $0 < t < s$ put $e = s - t$ and*

$$X_t(L) = L^{t/e}(T_0T_1)^{ft/(2e)}u^{-f/2}.$$

At marked index i, let $Z_i^{\mathrm{large}}(L)$ be the sum of the original signed orbital operators with $0 < e < s$ and $N\mathfrak{d} \ge X_t(L)$. For all sufficiently large x on the common intervals,

$$\sum_i \|Z_i^{\mathrm{large}}(L)\|_{\mathrm{HS}} \;\le\; |c_{0,1}|\,Q^{-fa/2}L^{-a}\min\{\widetilde{I}_0\widetilde{U}_1,\,\widetilde{I}_1\widetilde{U}_0\}.$$

All real-section factors have the normalization in Theorem 115.15. At a fixed remaining t, if $X_t(L) \le 1$ the complementary sum is empty. Otherwise its possible common graded scalars number at most $2^{f\mathbf{1}_{u=T}}8f^2\zeta(2)^{2f}X_t(L)^2$.

Proof. Both $N\mathfrak{g}$ and $N\mathfrak{b}$ are at least one. Theorem 115.17 shows that every retained term has height product at least Q^fL. In the absolute unfolding insert the two heights to the power a, taking out the common factor $Q^{-fa}L^{-a}$. The translated-ball count in the proof of Theorem 115.15 applies to either row separately. Its modified exterior minima and box constant use only $1 \le T_j \le T$, so they apply also when $T_0 = T_1 = T$. The count and its convergent geometric sum cost at most $Q^{fa/2}$ on these intervals. Unfolding in either direction gives the two products of section norms. One can enlarge to all cosets after extracting the common factor. Thus the estimate covers the union of the retained dimensions without a factor for their number. The bound for the remaining scalar values follows from Lemma 115.16. □

For the pair $T, 1$, the cutoff at $L = 1$ is $T^{f(t-e)/(2e)}$. It includes every denominator when $e \geq t$ and also gives a contribution estimate when $e < t$. For the pair T, T it is $T^{ft/e}$. The count in the corollary bounds the distinct scalar values in the complementary sum. Several rational orbits can have the same scalar. Their total signed contribution and the contribution of transverse flags still enter the full integral period comparison.

115.7 The two marked primes in the common flag index

Take a term of the integral projected period with compatible characters on $E = E_x$. Keep $0 < e < s$, $t = s - e$, and the notation of Section 115.6. In particular $\beta = \lambda^{-1}$, $\beta\bar{\beta} = u = T_0/T_1$, $\mathfrak{d}$ is its reduced denominator, and $\mathfrak{g} = \mathfrak{c}_0 + \mathfrak{c}_1$. Write N for absolute ideal norm. The two primes in $Q = g_1 g_2$ split completely and are different from the terminal prime T. For a fixed scalar define the squarefree ideal

$$\mathfrak{q}_\beta = \prod_{\substack{\mathfrak{w} \mid Q \\ \beta \text{ fails to be a unit congruent to 1 mod } \mathfrak{w}}} \mathfrak{w}.$$

The marked group uses upper triangular unipotent changes of the reduced outer frames. In particular it fixes the specified generators on their successive quotients. The resulting restriction on the two suffix indices can be read over the original integer rings.

Theorem 115.19 (A determinant congruence at both marked primes)**.** *For every such term,*

$$(\beta - 1)^e \mathfrak{c}_0 \ \subseteq \ Q^e \mathcal{O}_E, \qquad \mathfrak{g} \ \subseteq \ \mathfrak{q}_\beta^e.$$

If $\beta \neq 1$, the ideal $(Q)\mathfrak{q}_\beta^{-1}$ divides $(\beta - 1)\mathfrak{d}$. In particular

$$N\mathfrak{d}\, N\mathfrak{q}_\beta \ \geq \ \left(\frac{Q}{1 + \sqrt{u}}\right)^{2f}, \qquad f = [E^+ : \mathbb{Q}].$$

The set of primes in $\mathfrak{q}_\beta$ is stable under complex conjugation for the prescribed terminal prime.

Proof. Use suffix coordinates $t + 1, \ldots, s$ on both outer spaces. The ambient projective lattice for these coordinates is the same on both sides, namely $L = \bigoplus_{i>t} \mathfrak{a}_i$. Let L_0, L_1 be the images of $\Lambda \cap (X_0 \cap X_1)$ in these coordinates. Their index ideals in L are $\mathfrak{c}_0, \mathfrak{c}_1$. On the common rational space, projection to each suffix is an isomorphism by Lemma 115.13. Let B carry suffix coordinates on side zero to those on side one. It takes L_0 onto L_1. The induced filtrations on the common space are equal and the ratio of the leading coordinates from zero to one is always β, by the same lemma. Thus B is upper triangular with all diagonal entries β.

At a prime $\mathfrak{w} \mid Q$ the $\mathfrak{a}_i$ are units and the reduced outer vectors are linearly independent. The two reduced frames differ by an upper unitriangular matrix. For the full coordinates of a common integral vector this gives an upper unitriangular relation modulo $\mathfrak{w}$ as well, using the inverse matrix if the frames were compared in the opposite order. Its last e coordinates depend only on the last e input coordinates. Lift that bottom diagonal block to an upper unitriangular $U_\mathfrak{w}$ over $\mathcal{O}_{E,\mathfrak{w}}$. We obtain

$$(B - U_\mathfrak{w})(L_0)_\mathfrak{w} \subseteq \mathfrak{w} L_\mathfrak{w}.$$

The determinant of the difference is $(\beta - 1)^e$ because it is triangular. In local bases of L and L_0 the inclusion of L_0 has determinant generating the localized index ideal $\mathfrak{c}_0$. Taking top exterior powers gives the first inclusion at $\mathfrak{w}$. Away from Q we have $(B - 1)L_0 \subseteq L$ since both L_0 and $BL_0 = L_1$ are integral. Taking determinants gives integrality there. These local statements prove the global inclusion, including the case $\beta = 1$, when the determinant is zero.

At a marked prime in $\mathfrak{q}_\beta$, put $b = v_{\mathfrak{w}}(\beta)$. Then $v_{\mathfrak{w}}(\beta - 1) = \min(b, 0)$. This also holds when $b = 0$, by the residue condition defining $\mathfrak{q}_\beta$. The index identity in Theorem 115.17 reads $\mathfrak{c}_0 = \mathfrak{g}\mathfrak{d}^e$, with $v_{\mathfrak{w}}(\mathfrak{d}) = \max(-b, 0)$. The determinant congruence gives $v_{\mathfrak{w}}(\mathfrak{g}) \geq e$. Multiplying over the indicated primes proves the second inclusion.

For $\beta \neq 1$ the ideal $(\beta - 1)\mathfrak{d}$ is nonzero and integral. It lies in each marked prime outside $\mathfrak{q}_\beta$. Taking norms of this divisibility gives

$$Q^{2f} \;\leq\; N\mathfrak{q}_\beta \, N\mathfrak{d} \, |N_{E/\mathbb{Q}}(\beta - 1)|.$$

At every embedding τ one has $|\tau\beta| = \sqrt{u}$. The triangle inequality yields the stated bound. Finally $u \equiv 1 \bmod Q$. Indeed $u = 1$ or $u = T$, and the terminal generator satisfies $\varpi \equiv 1 \bmod Q^2$ by Theorem 9.13. The equality $\bar{\beta} = u/\beta$ shows that being a unit with residue one is invariant under conjugation at the marked primes. □

Corollary 115.20 (Contribution of the common flag index to the period)**.** *Use $X_t(L) = L^{t/e}(T_0T_1)^{ft/(2e)}u^{-f/2}$ for $L \geq 1$. Let $Z_i(L)$ sum the signed terms with $0 < e < s$ and*

$$N\mathfrak{d} \, N\mathfrak{q}_\beta \;\geq\; X_t(L)$$

in the original marked operator at index i. Then on the common intervals, for sufficiently large x,

$$\sum_i \|Z_i(L)\|_{\mathrm{HS}} \;\leq\; |c_{0,1}|Q^{-fa/2}L^{-a} \min\{\widetilde{I}_0\widetilde{U}_1, \, \widetilde{I}_1\widetilde{U}_0\}.$$

If $T_0 = T_1$ and $X_t(L) \leq (Q/2)^{2f}$, every term in that dimension with $\beta \neq 1$ is included.

Proof. The height estimate in Theorem 115.17 and $N\mathfrak{b} \geq 1$ give

$$\widetilde{V}_0(y)\widetilde{V}_1(\gamma^{-1}y) \;\geq\; Q^f(T_0T_1)^{-f/2}u^{fe/(2t)}(N\mathfrak{d} \, N\mathfrak{q}_\beta)^{e/t}$$

by the second inclusion of Theorem 115.19. The assumed inequality makes this at least $Q^f L$. Extracting its negative a-th power once on the whole set of terms gives the common factor $Q^{-fa}L^{-a}$. The two unfoldings and translated-ball count in Corollary 115.18 cost at most $Q^{fa/2}$ times the smaller displayed product of section norms. These estimates use all cosets after the common factor is extracted and thus apply to the indicated subset. The final assertion is the norm inequality in Theorem 115.19 at $u = 1$. □

For an equal-norm pair with $\mathfrak{d} = \mathcal{O}_E$ the scalar is a root of unity, by Kronecker's theorem. If it differs from one then $\mathfrak{q}_\beta = (Q)$ for all sufficiently large x. Indeed its order is at most $8f^2$ by Lemma 115.16. Each marked prime exceeds this bound since f is polynomial in x and $g_i \geq (k\exp(x^\varepsilon))^8$. Reduction at a prime of residue characteristic prime to the order is injective on those roots of unity. This gives $N\mathfrak{d} \, N\mathfrak{q}_\beta = Q^{2f}$ for all such scalars. The scalar 1 fixes every marked graded line and can occur on several rational orbits. Those terms and the terms at transverse flags still require control in the projected period.

115.8 The common middle lattice and the terminal divisor

Use the original integral projected period on $E = E_x$ and the marked levels $Q = g_1g_2$ of Section 115.7. Keep a supported pair with compatible characters and a proper nonzero intersection $I = X_0 \cap X_1$. Write $e = \dim_E I$, $t = s - e$, $T_0 \geq T_1$, $u = T_0/T_1$, and $f = [E^+ : \mathbb{Q}]$. The scalar $\beta = \lambda^{-1}$ has norm $\beta\bar{\beta} = u$ and reduced factorization $(\beta) = \mathfrak{n}/\mathfrak{d}$. Set $A_i = \Lambda \cap X_i$, $M = \Lambda \cap I$. On $I^\perp/I$ consider the lattices

$$\Lambda' = (\Lambda \cap I^\perp)/M, \qquad U_i = A_i/M, \qquad D = U_0 \oplus U_1.$$

The U_i are disjoint and isotropic. Their cross pairing has rank t by Lemma 115.13. Put $S = ED$, $R = \Lambda' \cap S$, $R_1 = R \cap Q^{-1}D$, and $C = \Lambda' \cap S^{\perp}$, taking the orthogonal complement inside $I^{\perp}/I$. The space $S^{\perp}$ projects isometrically onto each rational middle space $X_i^{\perp}/X_i$: after removing I, the sum of the two U_i spans a nondegenerate space in which the perpendicular of either EU_i is itself. View the two integral middle lattices as lattices J_i in this one rational space by those isometries. Thus $C \subset J_i$. Write $\zeta_i \in J_i$ for the image of the original terminal vector z_i in this identification.

Lemma 115.21 (The common middle index)**.** *The two lattices J_i are self-dual, and their common sublattice C satisfies*

$$[R_1 : D] = Q^{2ft}, \qquad [J_0 : C] = [J_1 : C] = \frac{N\mathfrak{b}}{[R : R_1]}.$$

Here $\mathfrak{b}$ is the determinant ideal of h/Q on $U_0 \times U_1$ from Theorem 115.14, and each bracket is the order of the finite quotient as an abelian group.

Proof. For a primitive isotropic sublattice of a self-dual lattice, the induced lattice in its perpendicular modulo itself is self-dual. Indeed an integral functional on the quotient pulls back to one on the perpendicular lattice, extends to the full lattice because the perpendicular is saturated, and is therefore represented by an integral vector. Vanishing on the isotropic sublattice puts the representative in its perpendicular. The ambiguity of representative is exactly the original primitive sublattice. This proves the claim first for Λ', then for J_i, since U_i is primitive in Λ'.

At each $\mathfrak{w} \mid Q$ the reductions of the two A_i agree in Λ by the marked flag condition. They also agree after first taking $\Lambda \cap I^{\perp}$ and then quotienting by M: the first sublattice is saturated, so reduction embeds it in the original reduction, and the second step is a common quotient. Both U_i reduce to the same t-dimensional subspace since they are primitive. The kernel of reduction $D/\mathfrak{w}D \to \Lambda'/\mathfrak{w}\Lambda'$ therefore has dimension t, using local lattices here. Locally, dividing lifts of this kernel by a uniformizer gives precisely R_1/D. There is no quotient away from Q, and both marked primes split completely. Multiplication of the local orders gives Q^{2ft}.

For a nondegenerate integral Hermitian lattice B put $\delta(B) = [B^{\sharp} : B]^{1/2}$, with the dual taken in its rational span. Enlarging a lattice by finite index j divides δ by j, by dualizing the inclusion. In integral local bases the Gram matrix for D consists of its cross pairing and its conjugate transpose. Thus the determinant ideal is $(Q)^{2t}\mathfrak{b}\bar{\mathfrak{b}}$, also for projective lattices by the change-of-basis rule. Its absolute norm is $[D^{\sharp} : D]$. Consequently $\delta(D) = Q^{2ft}N\mathfrak{b}$ and $\delta(R) = N\mathfrak{b}/[R : R_1]$.

The projection of Λ' to S has image $R^{\sharp}$. This follows by extending integral functionals on the saturated R and applying self-duality. Its kernel is C, so reduction modulo R identifies $\Lambda'/(R \oplus C)$ with $R^{\sharp}/R$. Projection to $S^{\perp}$ gives the same identification with $C^{\sharp}/C$. Hence $\delta(R) = \delta(C)$. Since J_i is self-dual and contains C with full rank, $[J_i : C] = \delta(C)$ for both i. □

Theorem 115.22 (A further divisor from the common terminal line)**.** *One has $\zeta_0 = \beta\zeta_1$. Put*

$$L_i = J_i \cap \zeta_i^{\perp}, \qquad L_c = C \cap \zeta_0^{\perp}, \qquad m_i = [L_i : L_c].$$

There is an integral ideal $\mathfrak{k}$ contained in $\mathfrak{q}_\beta$ of Theorem 115.19 such that

$$N\mathfrak{b} = [R : R_1]\, m_0\, u^f\, N\mathfrak{d}\, N\mathfrak{k}, \qquad m_1 = u^f m_0.$$

In particular $u^f N\mathfrak{d}\, N\mathfrak{q}_\beta$ divides the integer $N\mathfrak{b}$.

Proof. The last step in Lemma 115.13 supplies a common vector in the two extended flags with coefficients $1, \beta$ on their terminal vectors, after scalar normalization. Thus $z_0 - \beta z_1 \in X_0 + X_1$. Projecting to $S^{\perp}$ after quotient by I kills this difference and proves the first equality. Each ζ_i is primitive in its own J_i, by the integral terminal condition of Section 115.3. Hence the image of $h(\zeta_i, -)$ on J_i is the unit ideal, by self-duality. Let $\mathfrak{h}_i = h(\zeta_i, C)$. This is a nonzero integral ideal since C has full rank in the middle space. The kernels of the two evaluations on J_i, C are L_i, L_c. Passing to their quotients gives the exact index equality

$$[J_i : C] = m_i \, N\mathfrak{h}_i.$$

Since h is conjugate-linear in its first argument, $\mathfrak{h}_0 = (\bar{\beta})\mathfrak{h}_1$. Taking the smaller prime exponent at every prime in these two integral ideals gives

$$\mathfrak{k} = \mathfrak{h}_0 + \mathfrak{h}_1, \qquad \mathfrak{h}_0 = \mathfrak{k}\,\bar{\mathfrak{n}}, \qquad \mathfrak{h}_1 = \mathfrak{k}\,\bar{\mathfrak{d}}.$$

In particular $N\mathfrak{h}_0 = u^f N\mathfrak{d}\, N\mathfrak{k}$.

At both marks the reduced terminal vectors agree modulo the common reduced outer space, by the terminal part of the marking in Section 115.2. The two terminal frequencies have the same marked reduction since $\varpi \equiv 1 \bmod Q^2$. An element of C has an integral representative perpendicular to both outer spaces: lift it to $\Lambda \cap I^{\perp}$ and use its orthogonality to S. Pairing against this representative gives $h(\zeta_0 - \zeta_1, c) \in Q\mathcal{O}_E$ for every $c \in C$. Thus

$$(\bar{\beta} - 1)\mathfrak{h}_1 \subseteq Q\mathcal{O}_E.$$

At a marked prime where $\bar{\beta}$ fails to be a unit with residue one, the valuation of $(\bar{\beta} - 1)\bar{\mathfrak{d}}$ is zero. The congruence therefore puts $\mathfrak{k}$ in each such prime. These are the primes of $\mathfrak{q}_\beta$, by its conjugation invariance in Theorem 115.19. This proves the ideal inclusion, also when $\beta = 1$ and $\mathfrak{q}_\beta$ is the unit ideal. Substituting the two ideal norms in Lemma 115.21 proves both exact formulas and the divisibility. □

Corollary 115.23 (The mixed period through the first smaller intersection)**.** *Put* $D_\beta = N\mathfrak{d}\, N\mathfrak{q}_\beta$*. Every term under consideration satisfies*

$$\widetilde{V}_0(y)\widetilde{V}_1(\gamma^{-1}y) \;\geq\; Q^f \, \frac{u^{f(e+1)/(2t)}}{(T_0T_1)^{f/2}} \, D_\beta^{(2e+1)/(2t)}.$$

For unequal terminal norms, write $Z_i(t)$ *for the full signed sum in this intersection dimension at marked index* i*. Uniformly for sufficiently large* x *on the common intervals,*

$$\sum_i \|Z_i(t)\|_{\mathrm{HS}} \;\leq\; |c_{0,1}|\, Q^{-fa/2} T^{-fa(e+1-t)/(2t)} \min\{\widetilde{I}_0\widetilde{U}_1, \, \widetilde{I}_1\widetilde{U}_0\}.$$

In particular the estimate with the power of T *omitted holds on the whole union* $1 \leq t < s, \; 2t \leq s+1$*, including* $t = e+1$*.*

Proof. In Theorem 115.17 insert $N\mathfrak{g} \geq (N\mathfrak{q}_\beta)^e$ and $N\mathfrak{b} \geq u^f D_\beta$. The first inequality comes from Theorem 115.19 and the second from Theorem 115.22. Taking the indicated t-th roots gives the first assertion. For $T_0 = T, T_1 = 1$, use $D_\beta \geq 1$; the resulting common height factor is $Q^f T^{f(e+1-t)/(2t)}$. Extract its negative a-th power in the absolute unfolding of the signed sum as in Theorem 115.15. The translated-ball sum costs at most $Q^{fa/2}$ and is valid with either row unfolded first. This proves the second bound with the same real section norms and original finite operators. On the specified union the common height is at least Q^f. Extract that factor once before summing over cosets to obtain the last assertion. □

For scalar one the integer $m_0 = m_1$ measures the positive lattices directly. For example $m_0 = 1$ means that both are exactly L_c in the common rational positive column, even when the terminal norm is T. Their self-dual enlargements at the primes dividing ϖ in Lemma 113.6 then also agree. This lattice comparison leaves the signed sum of the real orbital integrals on that column to be estimated.

115.9 The positive lattice in the signed period

Fix two rows on $E = E_x$ on the interval of Section 6.6, with $Q = g_1 g_2$, $F = E^+$, $f = [F : \mathbb{Q}]$, and order them so that $T_0 \geq T_1$. Write $u = T_0/T_1$. Consider supported terms γ with compatible characters and $0 < \dim(X_0 \cap X_1) < s$. The two positive spaces in an individual term are identified by the common space $S^\perp \cap \zeta_0^\perp$ of Section 115.8. Indeed the two terminal classes there are proportional, and taking their perpendicular in $S^\perp$ removes the same negative line. Composing the identifications with the row coordinates defines a rational isometry $j_\gamma : P_0 \to P_1$ between the fixed positive spaces, of dimension k. Write L_0, L_1 for their original positive lattices. The two copies of $L_c = C \cap \zeta_0^\perp$ in these lattices depend on the term. Denote them by $C_{b,\gamma} \subset L_b$, $b = 0, 1$. The isometry takes $C_{0,\gamma}$ onto $C_{1,\gamma}$. Put

$$\mathfrak{a}_{b,\gamma} = \mathrm{Ann}_{\mathcal{O}_E}(L_b/C_{b,\gamma}), \qquad m_{b,\gamma} = [L_b : C_{b,\gamma}].$$

Theorem 115.22 gives $m_{1,\gamma} = u^f m_{0,\gamma}$.

Lemma 115.24 (Comparison on the original marked lattices)**.** *Use the orientation $0 \to 1$ for the marked index i_γ. Multiplication by any $\alpha \in \mathfrak{a}_{0,\gamma}$ makes $\alpha j_\gamma : L_0 \to L_1$ integral. At every prime $\mathfrak{w} \mid Q$ its reduction in the marked spaces is $\overline{\alpha}\, i_{\gamma,\mathfrak{w}}$, where the bar here denotes residue. The inverse statement holds for $\mathfrak{a}_{1,\gamma}$ and $i_{\gamma,\mathfrak{w}}^{-1}$. Moreover $N\mathfrak{a}_{b,\gamma} \leq m_{b,\gamma}$, with $N = N_{E/\mathbb{Q}}$ on ideals.*

Proof. For each $w \in L_0$ the element αw belongs to the image of L_c; the image of the same element at the other end is $\alpha j_\gamma w \in L_1$. Locally this element of L_c lifts to a vector of $\Lambda \cap I^\perp$. Its class modulo I lies in $S^\perp$ and pairs to zero with both terminal classes. Its lift therefore pairs to zero with both outer spaces and both terminal vectors. In the two marked reductions the same lift gives the same class modulo the common reduced outer space. At the marks the terminal norm is a unit. The positive lattice there reduces to the perpendicular of the terminal class in the middle quotient, since evaluation on the terminal class splits off the nondegenerate line. Reading the two images of the lift in the prescribed bases gives precisely the change of coordinates $i_{\gamma,\mathfrak{w}}$. This is also true in the conjugate split factor, using the dual bases. Applied to αw the assertion proves the reduction formula, even if α has zero residue. Interchange the ends for the inverse. Finally, in the elementary divisor decomposition of each local quotient the annihilator exponent is the largest elementary divisor exponent, bounded by their sum. Taking ideal norms gives the stated bound. □

Theorem 115.25 (Separation with the positive lattice indices)**.** *Consider two such terms γ, δ in the fixed pair of rows, allowing different intersection dimensions and different common graded scalars. For $b = 0, 1$ set $\mathfrak{d}_b = \mathfrak{a}_{b,\gamma} \cap \mathfrak{a}_{b,\delta}$. If their marked indices occupy a common output entry in one row family of the projection H, put $d = 1$. If the indices are equal one may put $d = 2$. Whenever $j_\gamma \neq j_\delta$, one has*

$$Q^d \;\leq\; 4k\,\bigl(N(\mathfrak{d}_0\mathfrak{d}_1)\bigr)^{1/(2f)} \;\leq\; 4k\,u\,(m_{0,\gamma}m_{0,\delta})^{1/f}.$$

Both inequalities include the primes dividing the lattice indices.

Proof. Write $j = j_\gamma$, $j' = j_\delta$. On P_0 form the trace

$$h = \operatorname{tr}\left((j^{-1} - j'^{-1})(j - j')\right) = 2k - \operatorname{tr}(j^{-1}j') - \operatorname{tr}(j'^{-1}j).$$

The two traces on the right are complex conjugates by unitarity, so $h \in F$. At each embedding of F it is the squared Hilbert–Schmidt norm of the difference of the two unitary maps. If these maps differ, this gives $0 < h_\tau \le 4k$ at every embedding. For local elements $\alpha \in \mathfrak{d}_0, \beta \in \mathfrak{d}_1$ the two maps $\alpha(j - j')$ and $\beta(j^{-1} - j'^{-1})$ are integral in opposite directions. Thus $(h)\mathfrak{d}_0\mathfrak{d}_1$ is an integral ideal of E.

At a marked prime reduce these two integral maps by Lemma 115.24. Write i, i' for the two residue indices. Their product has trace equal to the reduction of $\alpha\beta$ times $2k - \operatorname{tr}(i^{-1}i') - \operatorname{tr}(i'^{-1}i)$. In a common row entry the relative index is unipotent by Lemma 96.2. Its inverse and its contragredient in the conjugate factor are unipotent as well. The displayed expression is therefore zero at every prime of E above both marks. If $i = i'$, each of the two integral maps has zero reduction, giving divisibility of their product by the square of the local prime. Taking generators of the two localized ideals proves in both cases

$$(h)\mathfrak{d}_0\mathfrak{d}_1 \subseteq Q^d\mathcal{O}_E.$$

All marked primes have ramification index one. Taking absolute norms, then using $|N_{E/\mathbb{Q}}h| \le (4k)^{2f}$, proves the first inequality. Each intersection ideal contains the product of its two constituent ideals. Lemma 115.24 and $m_{1,\nu} = u^f m_{0,\nu}$ for $\nu = \gamma, \delta$ give the second inequality. □

Corollary 115.26 (Integral recovery and the contribution at large index)**.** *In any collection with $m_{0,\gamma}^{1/f} \le B$ and $4kuB^2 < Q$, the signed projected array of each rational isometry is recovered by selecting entries from the total array of the collection in each row family. Its squared norm is additive over distinct isometries. In particular, at either equal terminal norm the criterion is $4kB^2 < Q$.*

For an individual absolutely convergent orbit of identical words, allowing either frequency at each end, the arithmetic average fixes its operator B_γ if

$$4k\sqrt{u}\, m_{0,\gamma}^{1/f} < Q^2.$$

Here the average is the one over M_{i_γ} in Theorem 96.24.

For $L \ge 1$, let $Z_i(L)$ sum the signed terms at index i over all proper nonzero intersections for which

$$m_{0,\gamma}[R : R_1]D_\beta^{2e+1} \;\ge\; L^{2t}(T_0T_1)^{ft}u^{-f(e+1)}, \qquad D_\beta = N\mathfrak{d}\, N\mathfrak{q}_\beta.$$

Here $e = \dim I$, $t = s - e$ and the ideals and lattices are those of Section 115.8, evaluated in each term. On the same intervals and retained rows one has

$$\sum_i \|Z_i(L)\|_{\mathrm{HS}} \;\le\; |c_{0,1}|\, Q^{-fa/2}L^{-a}\min\{\widetilde{I}_0\widetilde{U}_1, \widetilde{I}_1\widetilde{U}_0\}.$$

Proof. Theorem 115.25 with $d = 1$ makes the output supports of distinct isometries disjoint under the first inequality. Coordinate projections onto these supports are integral and orthogonal. This proves the array statements for finite sums and then for absolutely convergent sums by continuity.

For $(a, b) \in M_{i_\gamma}$ apply the comparison to j_γ and $aj_\gamma b^{-1}$, where b, a act on the source and target lattices. These translates have the same marked index by the definition of M_{i_γ}. The translated common sublattices are $bC_{0,\gamma}$ and $aC_{1,\gamma}$; the two quotient annihilator ideals are unchanged because b, a are integral lattice automorphisms. Thus the first bound in Theorem 115.25 uses $N(\mathfrak{d}_0\mathfrak{d}_1) \le m_{0,\gamma}m_{1,\gamma}$. The stated strict inequality forces $aj_\gamma = j_\gamma b$. The

common positive perpendicular has full rank k in both columns, as in the construction of j_γ. The intertwining identity of Theorem 98.2 therefore fixes B_γ under every pair in the average.

For the last assertion retain the factor $m_{0,\gamma}[R : R_1]$ when substituting Theorem 115.22 into Theorem 115.17. The factorization there, $N\mathfrak{g} \geq (N\mathfrak{q}_\beta)^e$ from Theorem 115.19, and $N\mathfrak{k} \geq N\mathfrak{q}_\beta$ give

$$(\widetilde{V}_0\widetilde{V}_1)^{2t} \;\geq\; Q^{2ft}\frac{u^{f(e+1)}}{(T_0T_1)^{ft}}\, m_{0,\gamma}[R : R_1]D_\beta^{2e+1}.$$

Every term included in $Z_i(L)$ has height product at least $Q^f L$. Extract its negative a-th power once before unfolding over all the cosets. The translated-lattice count of Theorem 115.15 costs at most $Q^{fa/2}$ and applies in both directions and at both frequencies by Corollary 115.18. This gives the claimed estimate without summing a separate error for each dimension. □

For example, at unequal frequencies and $t = e + 2$, every term with $m_{0,\gamma} \geq T^f$ is in the last estimate with $L = 1$, independently of its scalar. Different terms with one and the same rational isometry can contribute to the same projected entries with different archimedean phases. The separation bound controls distinct isometries. A bound for the signed sums within the remaining isometries and for transverse terms is still needed in the integral comparison of Section 6.6.

115.10 The determinant comparison at transverse flags

Fix two marked rows over $E = E_x$ in the interval of Section 6.6. Put $F = E^+$, $f = [F : \mathbb{Q}]$ and $Q = g_1g_2$. We allow either original terminal frequency at each end. Consider a cell γ with common positive hyperplane lattice $R = R_\gamma$ of rank $k-1$. Use the two self-dual column lattices $\widehat{L}_s, \widehat{L}_t$ and determinant map d_γ° of Lemma 113.6. Write $j_\gamma : P_s \to P_t$ for the rational isometry extending the identification on R with that determinant. Let i_γ be the cell's prescribed period index, oriented from source to target. At a prime $\mathfrak{p}$ of E define

$$v_\mathfrak{p}(\mathfrak{t}_\gamma) = \mathrm{length}_{\mathcal{O}_{E,\mathfrak{p}}}(R^\sharp/R)_\mathfrak{p}, \qquad h_\gamma = (N_{E/\mathbb{Q}}\mathfrak{t}_\gamma)^{1/(2f)}.$$

Here $h_\gamma = h_{R_\gamma}$ in the notation of Theorem 113.7, and $R^\sharp$ denotes the Hermitian dual inside the rational hyperplane. The completed lattices and the original lattices agree at the two marks.

Lemma 115.27 (A cofactor identity at a marked prime)**.** *For every $\alpha \in \mathfrak{t}_\gamma$, the maps αj_γ and αj_γ^{-1} are integral between the respective completed lattices. At every prime over Q their reductions are $\bar{\alpha}i_\gamma$ and $\bar{\alpha}i_\gamma^{-1}$, respectively. In particular, if $n = v_\mathfrak{p}(\mathfrak{t}_\gamma) > 0$ at such a prime, both unscaled maps have denominator exponent at most $n - 1$ there.*

Proof. The integral assertions away from the marks follow from Theorem 113.7. Fix one of the two split factors over a marked place, with integer ring D and uniformizer π. Every calculation below takes place in this discrete valuation ring or its fraction field. At either end choose a basis of the completed lattice in this factor and its dual basis in the conjugate factor. Use the same basis of the abstract lattice R at the source and target in each factor. Its columns in the first factor give k-by-$(k-1)$ matrices A_s, A_t. The conjugate basis paired against the first factor gives $(k-1)$-by-k matrices B_s, B_t. Isometry gives

$$G = B_sA_s = B_tA_t, \qquad \Delta = \det G \neq 0.$$

In these bases G is the matrix of the pairing on R. Thus Δ generates $\mathfrak{t}_\gamma D$ by elementary divisors. Put $d = \det j_\gamma$ in the two chosen column bases. It is a unit with residue $\det i_\gamma$ by Lemma 113.6.

Write $a(A)$ for the cofactor row satisfying $a(A)y = \det[A, y]$. For B let $n(B)$ be the cofactor column with $mn(B) = \det\left[\begin{smallmatrix} B \\ m \end{smallmatrix}\right]$ for each row m. Then $Bn(B) = 0$ and $a(A)n(B) = \det(BA)$ by Cauchy–Binet. In particular $[A_b, n(B_b)]$ has determinant Δ at both ends $b = s, t$. The isometry identifies the first $k-1$ columns, takes the last line to the last line by orthogonality, and takes $n(B_s)$ to $d\, n(B_t)$ by its determinant. Evaluating on this basis gives

$$\Delta j_\gamma = A_t \operatorname{adj}(G) B_s + d\, n(B_t) a(A_s). \tag{115.2}$$

Here adj is the classical adjugate, including the value 1 for a one-by-one matrix. Both summands are integral.

For each vector of R, the two marked reductions of its images agree after the prescribed change of coordinates. Indeed the vector comes from one representative in $\Lambda \cap A$ in the common rational perpendicular. In the two middle quotients its reductions are the classes of that same representative modulo the same reduced isotropic space. This argument applies to the conjugate factor as well, regardless of the rank of the reduced hyperplane. Writing $i = i_\gamma \bmod \mathfrak{p}$ in the chosen bases, we get

$$\bar{A}_t = i \bar{A}_s, \qquad \bar{B}_t = \bar{B}_s i^{-1}.$$

For an invertible matrix u, expansion of a determinant by its last row gives $n(Bu^{-1}) = (\det u)^{-1} u\, n(B)$. Reduce the right side of (115.2) using these identities and $\bar{d} = \det i$. The result is i times the same expression for the identity transport on the source, which equals $\bar{\Delta} i$. This polynomial calculation also holds when $\bar{\Delta} = 0$. Interchanging endpoints gives the inverse formula. Multiplication by any local multiple of Δ proves the stated congruences. When $\pi \mid \Delta$, all entries of $\Delta j_\gamma^{\pm 1}$ consequently belong to πD, giving the one-power improvement. □

The local exponent in the lemma is sharp for the lattice data used there. In rank two over one split factor take $A_s = A_t = (1, 0)^{\mathsf{T}}$, $B_s = (\pi^n, 1)$, $B_t = (\pi^n, 1 + \pi)$, and $i = 1, d = 1$. The bases in the conjugate column spaces are the dual bases. The common pairing is π^n, and (115.2) gives $j = \left(\begin{smallmatrix} 1 & -\pi^{1-n} \\ 0 & 1 \end{smallmatrix}\right)$. For $n = 1$ it is integral with nonidentity reduction. For $n > 1$ its denominator exponent is $n - 1$.

For each direction $\sigma = +, -$ let D_γ^σ be the ideal of scalars $\alpha \in \mathcal{O}_E$ such that αj_γ^σ is integral on the completed lattices and has reduction $\bar{\alpha} i_\gamma^\sigma$ at both marks. Here the minus sign denotes the inverse and the plus sign the original map. The preceding lemma gives $\mathfrak{t}_\gamma \subseteq D_\gamma^\sigma$. The ideal can be calculated more precisely. At an unmarked prime use the ordinary denominator ideal of the matrix. At a marked prime, lift the residue matrix i_γ^σ to an integral matrix I_σ, and put $M_\sigma = \Delta j_\gamma^\sigma$ using (115.2) in the appropriate direction. Then

$$v_{\mathfrak{p}}(D_\gamma^\sigma) = \max\{0,\ 1 + v_{\mathfrak{p}}(\Delta) - \min_{a,b} v_{\mathfrak{p}}((M_\sigma - \Delta I_\sigma)_{ab})\}.$$

The minimum for a zero matrix is $+\infty$. In fact the single condition $\alpha(j_\gamma^\sigma - I_\sigma) \in \pi M_k(D)$ is equivalent to integrality with the prescribed reduction. It is unchanged by the choice of the lift.

Theorem 115.28 (Separation of transverse marked terms)**.** *Take two hyperplane cells γ, δ in the same pair of rows and put $D^\sigma = D_\gamma^\sigma \cap D_\delta^\sigma$. Suppose their indices contribute to a common entry in one row family of the signed projection. Set $m = 1$, or take $m = 2$ if the two indices agree. If $j_\gamma \neq j_\delta$, then*

$$Q^m \le 4k \left(N_{E/\mathbb{Q}}(D^+ D^-)\right)^{1/(2f)} \le 4k (h_\gamma h_\delta)^2.$$

Proof. Let $j = j_\gamma,\ j' = j_\delta$. The element

$$z = \operatorname{tr}\big((j^{-1} - j'^{-1})(j - j')\big) \in F$$

is the squared Hilbert–Schmidt distance of the two unitary maps at each embedding. Thus $0 < z_\tau \le 4k$ everywhere when the maps differ. Multiplication by elements of D^+, D^- makes the corresponding differences integral in opposite directions. At a marked prime their reductions are the scaled differences of the indices in both directions, by definition. The trace of the product vanishes on reduction for a common row entry, since the relative index and its inverse are unipotent by Lemma 96.2. For equal indices each factor has entries divisible by the local uniformizer. Using local generators of the two ideals gives $(z)D^+D^- \subseteq Q^m\mathcal{O}_E$. The marked primes are unramified. Take norms and use $|N_{E/\mathbb{Q}}z| \le (4k)^{2f}$ to obtain the first inequality. The product $\mathfrak{t}_\gamma\mathfrak{t}_\delta$ lies in each D^σ, giving the second inequality. □

Corollary 115.29 (Integral recovery and the transverse orbit trace)**.** *In any absolutely convergent collection of these cells with $h_\gamma \le B$ and $4kB^4 < Q$, every projected array belonging to a distinct rational transport can be selected integrally in each row family. Its squared norm is additive over the transports.*

For an individual absolutely convergent orbit of identical words, the signed trace of Proposition 113.4 survives the arithmetic average whenever

$$4k\left(N_{E/\mathbb{Q}}(D_\gamma^+D_\gamma^-)\right)^{1/(2f)} < Q^2.$$

In particular $4kh_\gamma^2 < Q^2$ suffices. Both statements hold for the two original terminal frequencies in any combination.

Proof. Under the first cutoff, Theorem 115.28 excludes a common output entry for different maps. Selecting disjoint coordinates in an orthogonal row basis is integral and gives the norm identity. One may also use the first bound in that theorem separately for each pair.

For the second assertion let (a, b) stabilize the index i_γ under the two endpoint actions. The transport $aj_\gamma b^{-1}$ has the same marked index. Since a, b are integral automorphisms of the completed lattices and their reductions stabilize this index, its two ideals of scalars are exactly $D_\gamma^\pm$, by conjugating the defining conditions and using the inverse automorphisms. The proof of the theorem with $m = 2$ therefore forces $aj_\gamma = j_\gamma b$. The unitary identification induced by j_γ on the two column types is fixed by every member of the averaging group. Taking its inner product with the orbital operator before and after averaging gives the same signed trace, which is the sum of the branch traces in Proposition 113.4. The inclusion $\mathfrak{t}_\gamma \subseteq D_\gamma^\pm$ gives the stated sufficient cutoff. □

All the comparisons use the original two marked primes. Their conclusion concerns different rational transports and the trace in a single orbit. By Theorem 113.10, infinitely many signed transverse terms can have the same transport and the same marked index even when $D_\gamma^+ = D_\gamma^- = \mathcal{O}_E$.

116 Gram determinants and height truncation

116.1 Gram determinants and the transverse orbital sum

Fix retained rows $0, 1$ at one x in Section 6.6, with terminal norms $-T_0, -T_1$. Transport the second frame into the first row as in Section 115.5. Write $E = E_x$, $F = E^+$, $f = [F : \mathbb{Q}]$, $Q = g_1g_2$, and $d = s + 1$. In the ordered frames the first s vectors are isotropic and the last is the terminal vector. For $1 \le j \le d$ let D_j be the determinant of the cross pairing of their first j vectors. Write $\mathfrak{h}_j = \prod_{i\le\min(j,s)} \mathfrak{a}_i$ for the product of the prescribed outer ideals, and put $\eta(\mathfrak{h}_j) = (N_{E/\mathbb{Q}}\mathfrak{h}_j)^{-1/(2f)}$. A change of either frame by an allowed upper unitriangular matrix fixes all the D_j, by taking determinants of its initial triangular blocks.

Proposition 116.1 (The two frame heights and the transverse determinant)**.** *At each real embedding τ choose either complex extension to E. Define*

$$C_j = \eta(\mathfrak{h}_j)^2 (T_0 T_1)^{j/2}, \qquad \mathcal{H}_\gamma = \prod_\tau \max\left(\{1\} \cup \{(|\tau D_j|/C_j)^{1/j} : 1 \le j \le d\}\right).$$

For every argument y of the orbital integral,

$$\widetilde{V}_0(y)\widetilde{V}_1(\gamma^{-1}y) \;\ge\; \mathcal{H}_\gamma.$$

When the two outer spaces are transverse and the common positive lattice R has rank $k-1$, let I_Λ and h_R be the saturation index and discriminant height of Corollary 113.8. Put

$$P_\gamma = \prod_\tau \left(1 + \frac{T_0 T_1 |\tau D_s|^2}{\tau \mathcal{G}}\right).$$

Then

$$\mathcal{H}_\gamma \;\ge\; \frac{Q^f}{(T_0 T_1)^{f/2}} (I_\Lambda h_R^f P_\gamma)^{1/(2d)}.$$

Proof. Take the exterior product of the first j vectors in each frame. The decorated stabilizer fixes this vector, including for $j = d$, since the terminal vector may change only by an earlier vector. In row b the exterior vector has length $\eta(\mathfrak{h}_j) T_b^{j/2}$ in the modified majorant of Theorem 115.15. Indeed the outer vectors there are mutually orthogonal with lengths $\eta(\mathfrak{a}_i)\sqrt{T_b}$. The terminal vector is orthogonal to them and has length $\sqrt{T_b}$. In the reference majorant the Hermitian form is represented by a unitary involution. Its exterior pairing satisfies Cauchy–Schwarz. Applying the two inverse changes of frame to the wedges costs at most the j-th power of each operator norm. Each change can first be multiplied by an arbitrary member of its own decorated stabilizer. Taking their infima proves that the local product of the two heights is at least $(|\tau D_j|/C_j)^{1/j}$. Each height is at least one since the changes are isometries between compatible majorants and their singular values come in reciprocal pairs. Take a maximum before multiplying over the real places.

In the hyperplane case $\mathcal{G} = D_d \overline{D_d} - T_0 T_1 D_s \overline{D_s}$ is totally positive by Theorem 86.3. It follows that

$$|N_{E/\mathbb{Q}} D_d| = |N_{F/\mathbb{Q}} \mathcal{G}|\, P_\gamma.$$

Corollary 113.8 evaluates the first factor as $Q^{2df} I_\Lambda h_R^f / (N_{E/\mathbb{Q}} \mathfrak{h}_d)^2$. The $j = d$ term of the maximum gives $\mathcal{H}_\gamma^d \ge |N_{E/\mathbb{Q}} D_d|^{1/2} / C_d^f$. Substitution cancels the fractional outer ideal and proves the assertion. □

Theorem 116.2 (An absolute estimate for a union of transverse terms)**.** *Let $L \ge 1$. For both terminal frequencies, take all supported transverse double cosets for which $\mathcal{H}_\gamma \ge Q^f L$. Write B_γ for each original signed ordinary orbital operator with its finite weight. On the common intervals, for all sufficiently large x,*

$$\sum_\gamma \|B_\gamma\|_{\mathrm{HS}} \;\le\; |c_{0,1}|\, Q^{-fa/2} L^{-a} \min\{\widetilde{I}_0 \widetilde{U}_1,\, \widetilde{I}_1 \widetilde{U}_0\}.$$

Here $a, \widetilde{I}_b, \widetilde{U}_b$ have precisely the normalization of Theorem 115.15. The same bound holds for the sum of the norms after grouping by marked index, including within any one rational transport. The sum includes every transverse hyperplane term with

$$I_\Lambda h_R^f P_\gamma \ge (T_0 T_1)^{df} L^{2d}.$$

Proof. The norm of the rank-one integrand is the product of the two section norms, and the finite phases have absolute value one. Put $F_b(y) = \widetilde{V}_b(y)^a \|W_b(y)\|$. On every one of the stated cosets Proposition 116.1 bounds the product of the two section norms by

$$Q^{-fa} L^{-a} F_0(y) F_1(\gamma^{-1} y).$$

After extracting that factor, enlarge the nonnegative sum to all double cosets in the row. Its unfolding in Theorem 113.22 is the integral over $N_0 \backslash G_\infty$ of F_0 times the translated coset sum of F_1. The box estimate in the proof of Theorem 115.15 bounds that coset sum by $Q^{fa/2} \widetilde{U}_1$ for all sufficiently large x. Its constants depend on the outer ideals only through the packing ratio in which they cancel. The argument uses one row at a time and covers equal frequencies as in Corollary 115.18. Integration gives $\widetilde{I}_0$, and inversion of the cosets gives the other product of norms. Restoring the pair volume proves the estimate before any signed summation. The triangle inequality proves its grouped form. The hyperplane cutoff follows from the second inequality of Proposition 116.1. □

Corollary 116.3 (The cross pairing at the terminal prime)**.** *Write $b_T = \#\{b \in \{0,1\} : T_b = T\}$. For $1 \le j \le s$ with $D_j \neq 0$, take the cross pairing on the two integral prefix lattices at each $\mathfrak{p} \mid T$, using the conjugate factor for the first lattice. Let $u_{\mathfrak{p},j}$ be the nullity of its reduction. The estimate of Theorem 116.2 with $L = 1$ includes every transverse term satisfying, for at least one such j,*

$$\sum_{\mathfrak{p} \mid T} u_{\mathfrak{p},j} \ge f j b_T.$$

In particular, for unequal frequencies it suffices that the prefix pairing vanish on reduction at one prime of E over each place of F above T. For $T_0 = T_1 = T$ vanishing at both factors suffices. The union costs the single bound in the theorem.

Proof. The ideal $\mathfrak{A}_j = (D_j/Q^j)\overline{\mathfrak{h}_j}\mathfrak{h}_j$ is integral. Away from Q this is integrality of the cross pairing on integral bases. Over Q the two reduced outer spaces coincide and are totally isotropic, giving one local uniformizer in every entry before taking the determinant. At $\mathfrak{p} \mid T$, at least $u_{\mathfrak{p},j}$ elementary divisors of the pairing matrix in integral bases are divisible by the uniformizer. Its determinant has the same valuation as $\mathfrak{A}_j$ there. These primes have norm T, since T splits completely. Thus

$$|N_{E/\mathbb{Q}} D_j|^{1/2} = Q^{jf} \eta(\mathfrak{h}_j)^{2f} (N_{E/\mathbb{Q}} \mathfrak{A}_j)^{1/2} \ge Q^{jf} \eta(\mathfrak{h}_j)^{2f} T^{\sum_{\mathfrak{p} \mid T} u_{\mathfrak{p},j}/2}.$$

Using the j-th wedge in the maximum gives $\mathcal{H}_\gamma \ge Q^f$ under the stated inequality. □

Corollary 116.4 (All powers of a repeated transverse transport)**.** *On any retained diagonal row of terminal norm $-T_0$, take the cyclic groups of Theorem 113.10 at a precision giving $(\gamma - 1)\Lambda \subseteq Q T_0 \Lambda$. One may do so for every rational positive hyperplane. For $M \ge 1$, sum over nonidentity powers γ^n with $|n| \ge M$, even over the union of all these groups in the row, counting each contributing double coset once. Their ordinary signed operators satisfy*

$$\sum \|B_{\gamma^n}\|_{\mathrm{HS}} \le |c_{0,0}|\, Q^{-fa/2} M^{-af} \widetilde{I}_0 \widetilde{U}_0.$$

In particular $M = 1$ controls the whole indicated collection at the identity marking and identity rational transport.

Proof. The construction of Theorem 113.10 allows the stated precision on the original lattice and gives $h(e_1, \gamma^n e_1) = nt$ with $t \neq 0$. For $n = 1$, pair the integral line $\mathfrak{a}_1 e_1$ with its image under $\gamma - 1$. The original line is isotropic. Consequently $(t)\overline{\mathfrak{a}_1}\mathfrak{a}_1 \subseteq Q T_0 \mathcal{O}_E$. It follows that

$$|N_{E/\mathbb{Q}}(nt)|^{1/2} \ge |n|^f (Q T_0)^f \eta(\mathfrak{a}_1)^{2f}.$$

On this diagonal $C_1 = \eta(\mathfrak{a}_1)^2 T_0$. The first wedge thus gives $\mathcal{H}_{\gamma^n} \geq Q^f |n|^f$. All the indicated cosets occur in the single sum of Theorem 116.2 with $L = M^f$. The construction gives their common marking and transport. □

The estimates retain the archimedean factors of the original sections. For the full comparison one still needs the remaining transverse terms at small frame height and a bound at the normalization of the integral theta maps.

116.2 Separation of rational transports by their discriminants

Fix two retained rows on a period field $E = E_x$, with $F = E^+$, $f = [F : \mathbb{Q}]$, and $Q = g_1 g_2$. Consider cells with common positive hyperplane lattice of rank $k - 1$. For a cell γ, let j_γ be its rational isometry between the two rank-k positive spaces, i_γ its marked index, and $\mathfrak{t}_\gamma$ its discriminant ideal as in Lemma 115.27. Put $h_\gamma = (N_{E/\mathbb{Q}} \mathfrak{t}_\gamma)^{1/(2f)}$.

Theorem 116.5 (A product of discriminant ideals for a common projected entry)**.** *Suppose two cells γ, δ contribute to the same entry of one projected row family and have distinct rational transports. At a marked prime of E, call a cell singular if its discriminant ideal is divisible by that prime. Let $\mathfrak{q}_{\gamma,\delta}$ be the product of the marked primes at which both cells are singular, together with those at which both are nonsingular and the two indices agree locally. Then*

$$Q\,(N_{E/\mathbb{Q}} \mathfrak{q}_{\gamma,\delta})^{1/(2f)} \;\leq\; h_\gamma h_\delta \left(\prod_\tau \| j_{\gamma,\tau} - j_{\delta,\tau} \|_{\mathrm{HS}}^2 \right)^{1/f} \;\leq\; 4k h_\gamma h_\delta .$$

The norms are for the positive Hermitian metrics of the two rows at the real places of F. In particular a common entry requires $Q \leq 4k h_\gamma h_\delta$. For equal marked indices and identical sets of singular marked primes it requires $Q^2 \leq 4k h_\gamma h_\delta$.

Proof. Write $j = j_\gamma,\ j' = j_\delta$. In F put

$$z = 2k - \operatorname{tr}(j^{-1} j') - \operatorname{tr}(j'^{-1} j).$$

The two traces are conjugates because the maps are unitary. At every real embedding z is the squared Hilbert–Schmidt distance of the maps, positive since $j \neq j'$ and at most $4k$. The two compositions in this formula use one map from each cell. Thus $z \mathfrak{t}_\gamma \mathfrak{t}_\delta$ is integral by Lemma 115.27, which clears both directions between the completed column lattices.

At a marked prime choose local generators a, b of $\mathfrak{t}_\gamma, \mathfrak{t}_\delta$, and write π for the uniformizer. The same lemma identifies the four reductions of $aj, aj^{-1}, bj', bj'^{-1}$ with the corresponding scaled marked maps and their inverses. Consequently abz on reduction equals $\bar{a}\bar{b}$ times $2k - \operatorname{tr}(i_\gamma^{-1} i_\delta) - \operatorname{tr}(i_\delta^{-1} i_\gamma)$. A common row entry makes both relative marked maps unipotent, by Lemma 96.2, so this difference is zero. If π divides both a and b, each of the four scaled maps has entries in π and ab is in π^2. It follows term by term that $abz \in \pi^2$. If a, b are units and the marked maps agree, the two differences $j - j'$ and $j^{-1} - j'^{-1}$ have entries in π. The trace of their composition equals z, giving the same extra divisibility. This proves

$$(z) \mathfrak{t}_\gamma \mathfrak{t}_\delta \;\subseteq\; Q\, \mathfrak{q}_{\gamma,\delta}.$$

Here Q splits completely, so its local exponent is one at each marked prime. Take absolute ideal norms. The $2f$-th root of $|N_{E/\mathbb{Q}} z|$ is the geometric mean of its positive real values. The display in the statement follows. In the last case every marked prime occurs in $\mathfrak{q}_{\gamma,\delta}$. □

Corollary 116.6 (Integral selection and the number of transports at an entry)**.** *Retain cells with $h_\gamma \le H$, $H \ge 1$, in an absolutely convergent family. If $4kH^2 < Q$, the signed arrays grouped by rational transport have disjoint support in each row family. Each group can be recovered by integral coordinate selection, and its squared norm adds to those of the others. On the prescribed intervals this holds for $H = \exp((8-\delta)x^{\varepsilon})$ with any fixed $0 < \delta < 8$ for all sufficiently large x.*

For arbitrary H the number of distinct rational transports in a single output entry is at most

$$\left(1 + 2\sqrt{k}\, H/\sqrt{Q}\right)^{2fk^2}.$$

For a fixed full marked index the bound can be replaced by

$$2^{4f}\left(1 + 2\sqrt{k}\, H/Q\right)^{2fk^2}.$$

All assertions allow the original terminal norm at each end to be either 1 or T in absolute value.

Proof. The first conclusion follows from the necessary condition for a common entry in Theorem 116.5. Coordinate selection has matrix entries zero and one, and disjoint support gives orthogonality in the given row basis. Here $\log Q = (16 + o(1))x^{\varepsilon}$ and k is polynomial in x by Theorem 110.16. These estimates imply the asserted explicit cutoff.

Choose one cell for each distinct transport at a given entry. Use orthonormal bases at infinity and place the f matrices of each transport in $\mathrm{Mat}_k(\mathbb{C})^f$, with the maximum of the Hilbert–Schmidt norms. They lie in the ball of radius $\sqrt{k}$. The theorem implies that distinct points have distance at least $\sqrt{Q}/H$, since the maximum dominates the geometric mean. Open balls of half this radius about the points are disjoint. Comparing their volumes with the enclosing ball gives the first bound in real dimension $2fk^2$, initially for every finite subset and therefore for the whole set. For a fixed full marked index group the chosen cells by their sets of singular marked primes. There are at most 2^{4f} choices, since each of the two rational primes has $2f$ factors. Within a group the distance is at least Q/H by the last assertion of the theorem. The same volume argument proves the other bound. □

Corollary 116.7 (The projected sum below the frame-height cutoff)**.** *In a transverse hyperplane row pair use the height $\mathcal{H}_\gamma$ of Proposition 116.1, the two terminal parameters T_0, T_1, and $d = s + 1$. For $L \ge 1$ put*

$$H_L = (T_0T_1)^d L^{2d/f}, \qquad m_L = (1 + 2\sqrt{k}\, H_L/\sqrt{Q})^{2fk^2}.$$

Retain the cells with $\mathcal{H}_\gamma < Q^f L$. Let S_j be their signed projected array for a single rational transport j in a fixed row family, with the actual arithmetic coefficients. Then, for absolutely convergent arrays,

$$\left\| \sum_j S_j \right\|_{\mathrm{HS}}^2 \le m_L \sum_j \|S_j\|_{\mathrm{HS}}^2.$$

If $4kH_L^2 < Q$, the two squared norms in this comparison agree after omitting m_L. When $T_0 = T_1 = 1$ and $L = 1$ the retained set is empty. All the complementary hyperplane terms satisfy Theorem 116.2.

Proof. The determinant inequality of Proposition 116.1 gives

$$I_\Lambda h_\gamma^f P_\gamma < (T_0T_1)^{df} L^{2d}$$

on the retained set. The lattice index I_Λ is at least one and $P_\gamma \geq 1$ by total positivity. Thus $h_\gamma < H_L$. At most m_L of the signed groups can occur in a given entry by Corollary 116.6. Cauchy–Schwarz entry by entry and summation give the inequality, including for Hilbert-valued entries. Absolute convergence justifies passage from finite subsums. Under the smaller cutoff the groups have disjoint supports. In the unit-terminal case with $L = 1$, the strict inequality would require $h_\gamma < 1$, impossible for a lattice discriminant ideal. The complement is exactly covered by the frame-height bound in the cited theorem. □

This estimate controls how many distinct rational transports can contribute to one output entry. The size of the signed sum within a single transport at low frame height still enters the integral period comparison.

116.3 Counting cross matrices below the frame-height cutoff

Fix a pair of retained rows with their ordered rational frames as in Section 116.1. Write $O_E = \mathcal{O}_E$ and $A = T_0T_1$. For $L \geq 1$ we count the relevant transverse hyperplane terms in this pair with $\mathcal{H}_\gamma < Q^f L$.

Theorem 116.8 (Possible rational flag pairs at small frame height)**.** *Put $B = \lfloor \log_2(Q^f L) \rfloor$. The number of distinct rational double cosets of the decorated flag stabilizers represented in this range is at most*

$$N_L = 2^d \binom{B+f}{f} \prod_{j=1}^{d} \left(1 + 2(2\sqrt{A}\, L^{1/f})^j\right)^{2f}.$$

This includes pairs with vanishing proper leading minors. The bound is independent of the norms of the outer fractional ideals.

Proof. Write Z for the cross matrix and D_j for its leading minors, with $D_0 = 1$. Put $\mathfrak{l}_j = (\overline{\mathfrak{h}_j}\mathfrak{h}_j)^{-1}$. Integrality of the pairing on the two prefix lattices gives $D_j \in \mathfrak{l}_j$, also for $j = d$ by the primitive terminal vector. At each mark the outer spaces coincide on reduction and are totally isotropic. The two terminal vectors agree modulo that reduced outer space by the marking in Section 115.2. Their norms reduce to -1 since $\varpi \equiv 1 \bmod Q^2$. The cross matrix therefore reduces to $-E_{dd}$, with E_{dd} the last diagonal matrix unit. Consequently

$$D_j \in Q^j \mathfrak{l}_j \quad (1 \leq j < d), \qquad D_d + D_s \in Q^d \mathfrak{l}_d.$$

Indeed the determinant of $Z + E_{dd}$ is $D_d + D_s$. The outer ideals are units at both marks, and $\mathfrak{l}_s = \mathfrak{l}_d$ away from them as well. For a fixed earlier tuple, differences between last entries thus belong to $Q^d\mathfrak{l}_d$.

At an embedding τ let $m_\tau \geq 1$ be the factor of $\mathcal{H}_\gamma$ there. Set $b_\tau = \lfloor \log_2 m_\tau \rfloor$. Then $\sum_\tau b_\tau \leq B$, giving at most $\binom{B+f}{f}$ tuples of nonnegative integers. For a fixed tuple the j-th minor lies in discs of radii

$$r_{j,\tau} = \eta(\mathfrak{h}_j)^2 A^{j/2} 2^{j(b_\tau+1)}.$$

The difference z of two distinct possibilities, conditional on earlier entries, belongs to $Q^j\mathfrak{l}_j$. Hence

$$\prod_\tau |\tau z| = |N_{E/\mathbb{Q}} z|^{1/2} \geq Q^{jf} \eta(\mathfrak{h}_j)^{2f}.$$

This follows by taking the norm of the integral ideal $(z)/(Q^j\mathfrak{l}_j)$. After division by $r_{j,\tau}$ in each coordinate, the maximum distance of the two points is at least $(2\sqrt{A}\, L^{1/f})^{-j}$, since $2^{\sum_\tau b_\tau} \leq Q^f L$. Open polydiscs of half that radius about the rescaled points are disjoint inside the unit polydisc enlarged by the same radius. Volume comparison in real dimension $2f$ gives the j-th

factor of the product in the theorem. This counts zero whenever it is allowed. Iterating the conditional counts bounds the number of full minor tuples.

It remains to pass from minors to rational flag pairs. The reduction in Lemma 110.10 applies with the two fixed terminal norms. More explicitly, the concatenated Gram determinant is nonzero by total positivity of $\mathcal{G}$. A common cross matrix determines the pair of decorated flags up to rational isometry by Witt extension, because the two diagonal Gram blocks are fixed, respectively $-T_0E_{dd}$ and $-T_1E_{dd}$. Both blocks are preserved by upper unitriangular changes. In these coordinates the simple characters use the successor map sending the last outer vector to its own terminal vector, as in Lemma 115.13. Thus the adjacent descent calculation in Lemma 110.10 has the same coefficients on both frames. Relevance forces reversed consecutive blocks with alternating signs on their antidiagonals. The nonzero minor indices specify their endpoints, and successive ratios of endpoint minors determine the leading scalar of a block of length a up to an a-th root of unity. There are at most a choices. Their product over the blocks is at most 2^d, proving the result. □

Proposition 116.9 (Integral cross matrices in each rational class)**.** *Let*

$$Y = 4^f \eta(\mathfrak{h}_d)^{4f} A^{df} L^{2d}.$$

In each rational class of Theorem 116.8, the integral cross matrices that occur have at most $Y^{d(d-1)}$ equivalence classes under left lower and right upper unitriangular changes over O_E.

Proof. Each outer fractional ideal contains O_E, since it is a nonpositive power of the extended different. In particular the chosen frame vectors, and hence the entries of Z, are integral. Both unitriangular actions fix E_{dd}. The substitution $Z \mapsto (Z + E_{dd})/Q$ therefore respects integral equivalence in both directions and has integral values by the marked congruence above. Its last determinant is nonzero, since $|\tau D_d| > \sqrt{A}\,|\tau D_s|$ and $A \geq 1$.

For clarity, the prefix products in the integral reduction of Theorem 110.15 can be bounded using one endpoint maximum here. If the last block is a singleton, adding E_{dd} changes just its scalar. Otherwise an earlier row has its unique pivot in the final column and a lower row operation removes the added entry in the rational normal form. After division by Q, the absolute norms of the pivot products at proper block endpoints j are $|N_{E/\mathbb{Q}}(D_j/Q^j)|$, and at the final endpoint the norm is $|N_{E/\mathbb{Q}}((D_d + D_s)/Q^d)|$. The initial product is one. For a partial block the logarithm of the pivot-product norm interpolates linearly between its endpoint values, because all pivots of an individual nonsingleton block have the same absolute norm.

Before taking dyadic boxes one has $|\tau D_j| \leq \eta(\mathfrak{h}_j)^2 A^{j/2} m_\tau^j$ and $\prod_\tau m_\tau < Q^f L$. Every proper endpoint norm is therefore at most $\eta(\mathfrak{h}_j)^{4f} A^{jf} L^{2j} \leq Y$. For the last endpoint the triangle inequality gives $|\tau(D_d + D_s)| \leq 2\eta(\mathfrak{h}_d)^2 A^{d/2} m_\tau^d$, which accounts for the factor 4^f. Thus every intermediate pivot-product norm is also at most Y.

For a pivot product b_j through position $j < d$, the integral minor calculation in Theorem 110.15 puts the lower entries below that pivot and the allowed upper entries to its right in $b_j^{-1}O_E$. Row and column reduction needs only their classes modulo O_E, at most $|N_{E/\mathbb{Q}} b_j|$ choices per position. This proof uses the shifted integral matrix and its pivot positions, and so applies to the shift by E_{dd} used here. There are at most $2\sum_{j<d}(d-j) = d(d-1)$ such positions. The bound follows. □

Corollary 116.10 (A finite sum of signed groups below the cutoff)**.** *For an absolutely convergent array in Corollary 116.7, choose an integral representative for each contributing cell. Group its signed contribution by rational transport, rational flag class, and integral cross-matrix*

class of the representative. If the resulting arrays are S_β, then

$$\Big\| \sum_{\beta} S_\beta \Big\|_{\mathrm{HS}}^2 \le m_L N_L Y^{d(d-1)} \sum_{\beta} \|S_\beta\|_{\mathrm{HS}}^2 .$$

Here m_L is the entry bound of Corollary 116.7. If $\log L = O(x^C)$ for fixed C, the logarithm of the displayed factor is $o(\exp(\nu x^\varepsilon))$ on the common intervals for every fixed $\nu > 0$.

Proof. At any output entry there are at most m_L distinct transports. For each, the theorem bounds the number of rational classes and the proposition bounds the number of matrix classes. Cauchy–Schwarz at each entry gives the display upon summation. Every choice of representatives gives this inequality with the original signs and weights of the contributions in each group.

The field degree, d, and k are polynomial in x. Also $\log Q = O(x^\varepsilon)$ and $\log T = O(\sqrt{x}\, x^\varepsilon)$ by Theorem 93.9 and Theorem 110.16. If $R = |\operatorname{disc}(F)|^{1/f}$, then $\log R = O_\psi(\log x)$ as in Proposition 115.10, and $\eta(\mathfrak{h}_d) = R^{s(s+1)/2}$. In the binomial term one may use $\binom{B+f}{f} \le (e(1 + B/f))^f$. Substitution in all three factors bounds their combined logarithm by a polynomial in x. This implies the stated asymptotic. □

The finite count bounds the number of groups that can meet an output entry. An individual group can contain several integral cells. A bound for their signed orbital sum in the normalization of the integral period remains necessary.

116.4 Integral orbits below the frame-height cutoff

Fix the same pair of rows and cutoff L as in Theorem 116.8. When the row lattices are isometric, identify them with Λ as in Section 113.6. Write $\Gamma = \mathrm{Aut}(\Lambda, h)$ and let N_b be the integral unipotent subgroup of row b. The period uses a subset of $N_0\backslash\Gamma/N_1$. We count the individual double cosets in the relevant transverse hyperplane part below the cutoff, including those with equal marked index.

Lemma 116.11 (Integral changes of an ordered frame)**.** *Every upper unitriangular change over O_E of the ordered frame $(e_1, \dots, e_s, z)$ is induced on this frame by an element of N_b, for either terminal norm.*

Proof. Use the integral splitting $A \oplus J \oplus A'$ of Lemma 115.7, with partners f_i. For $i < j \le s$, the change $e_j \mapsto e_j + ce_i$ extends by $f_i \mapsto f_i - \bar{c} f_j$, fixing the other lines and J. This preserves the hyperbolic pairing. It is integral for $c \in O_E$ since $\mathfrak{a}_j \subseteq \mathfrak{a}_i$ and $\bar{\mathfrak{a}}_i^{-1} \subseteq \bar{\mathfrak{a}}_j^{-1}$.

For e_i preceding z, choose $a \in J$ such that $h(a, z) = c$. This is possible since $O_E z$ is saturated in the projective lattice J, so it is a direct summand. Extend its coordinate functional to J and use self-duality. Take $t \in O_E$ with $t + \bar{t} = 1$, whose existence was proved in Lemma 115.7. Fix A and f_j for $j \ne i$, and put

$$w \longmapsto w + e_i h(a, w) \quad (w \in J), \qquad f_i \longmapsto f_i - a - t h(a, a) e_i .$$

The new f_i pairs to zero with the new middle vectors. Its norm is $h(a, a)(1 - t - \bar{t}) = 0$. The other block pairings are preserved. The new terminal vector is $z + ce_i$. The inclusion $O_E \subseteq \mathfrak{a}_i$ and its dual $\bar{\mathfrak{a}}_i^{-1} \subseteq O_E$ give integrality on all three summands. The inverse is integral by the finite geometric series for the nilpotent difference from the identity. Both types of maps act as the identity on the successive outer lines and on the full middle quotient. Hence they belong to N_b. The elementary changes generate the upper unitriangular group by eliminating entries one superdiagonal at a time. □

Let $n = 2s + 1 + k$, $h_0 = k - 1 \geq 1$, $D = 2fh_0$, and $\Delta_E = |\operatorname{disc}(E)|$. Define

$$G_0 = \eta(\mathfrak{h}_d)^{4f}(T_0T_1)^{df}Q^{2df}L^{2d}, \qquad C = 2D^{D/2}\Delta_E^{h_0/2}G_0.$$

In the next lemma the frames are fully ordered and each individual vector is distinguished.

Lemma 116.12 (Integral embeddings with a prescribed cross matrix)**.** *Fix an integral cross matrix Z occurring in the stated range. The number of Γ-orbits of pairs of ordered frames in Λ with that cross matrix and the given two diagonal Gram blocks is at most*

$$P_L = G_0^{8fd+8fn}(1 + 2C^2)^{2fh_0^2}C^{4fh_0D}.$$

Proof. Let U be the free lattice on the $2d$ ordered vectors with their specified Gram matrix. Its determinant is $(-1)^d\mathcal{G}$, where $\mathcal{G} = D_d\bar{D}_d - T_0T_1D_s\bar{D}_s$ is totally positive by Proposition 116.1. For $G = N_{F/\mathbb{Q}}\mathcal{G}$ that proposition gives

$$1 \leq G \leq |N_{E/\mathbb{Q}}D_d| \leq \eta(\mathfrak{h}_d)^{4f}(T_0T_1)^{df}\mathcal{H}_\gamma^{2d} \leq G_0.$$

The first inequality uses integrality of the ordered vectors. The index of U in its Hermitian dual $U^\sharp$ is G^2, by taking the absolute $E/\mathbb{Q}$ norm of the Gram determinant. An embedding gives the saturation $B = \Lambda \cap EU$. Pulling it back to the marked rational span gives $U \subseteq B \subseteq U^\sharp$. There are at most $(G_0^2)^{4fd}$ choices. Indeed B/U is generated over $\mathbb{Z}$ by at most $4fd$ elements since B has that rank, and a padded ordered list of generators specifies its subgroup of $U^\sharp/U$.

Write $R = \Lambda \cap (EU)^\perp$. The nondegenerate hyperplane case gives a totally positive lattice of rank h_0 here. Restricting the integral pairing maps Λ onto $B^\sharp$ with kernel R, upon using orthogonal projection to EB. Surjectivity follows by extending a functional from the saturated direct summand B and applying self-duality of Λ. The same reasoning with R gives

$$[R^\sharp : R] = [\Lambda : B \oplus R] = [B^\sharp : B] = G^2/[B : U]^2 \leq G_0^2.$$

The last equality follows by dualizing $U \subseteq B$.

We bound the number of isometry types of R. Its integral trace pairing $\operatorname{Tr}_{E/\mathbb{Q}} h$ is positive definite, with trace dual $\mathfrak{D}_{E/\mathbb{Q}}^{-1}R^\sharp$. Indeed one tests trace integrality against all O_E-multiples of each vector and uses the defining trace dual of O_E. Its Euclidean covolume is at most $\Delta_E^{h_0/2}G_0$, since the index of $R^\sharp$ in its multiple by the inverse different is $\Delta_E^{h_0}$. All nonzero vectors have trace length at least one. Minkowski's second theorem for the unit ball in dimension D, which contains the cube of side $2/\sqrt{D}$, bounds the product of the successive minima by $D^{D/2}\Delta_E^{h_0/2}G_0$. There are D independent vectors of length at most C. They span the underlying rational space, hence contain an E-basis $x_1, \ldots, x_{h_0}$.

In this basis every Gram entry is an algebraic integer of absolute value at most C^2 at every embedding, by Cauchy–Schwarz and the trace length bound. Two distinct algebraic integers differ by at least one in maximum archimedean modulus, by the norm of their difference. Packing open polydiscs of radius $1/2$ therefore gives at most $(1 + 2C^2)^{2fh_0^2}$ possible Gram matrices. For each, put $M = \sum_i O_E x_i$. Hadamard's inequality at each embedding gives $[M^\sharp : M] \leq C^{4fh_0}$. The intermediate lattice $M \subseteq R \subseteq M^\sharp$ is specified by a subgroup with at most D abelian generators. This gives the remaining factor C^{4fh_0D} in the definite count.

For each B and each isometry type of R, an ambient integral lattice containing $B \oplus R$ lies inside $B^\sharp \oplus R^\sharp$. The quotient of these endpoints has order at most G_0^4. The ambient lattice has $\mathbb{Z}$-rank $2fn$, so at most $(G_0^4)^{2fn}$ subgroups can occur as its image. Each choice specifies the entire lattice by inverse image, with the ordered frame vectors already marked in U. Any two identifications of such a lattice with Λ differ by an element of Γ. This bounds the number of orbits by the product claimed. □

Theorem 116.13 (Number of individual low-height terms on an original row)**.** *For either terminal frequency at either end, put*

$$K_L = N_L\, Y^{d(d-1)}\, P_L\, 3^{\,2(2fk)^2},$$

where N_L, Y are given in Theorem 116.8 and Proposition 116.9. The number of relevant supported transverse hyperplane double cosets below the cutoff in the fixed pair of rows is at most K_L. In particular, if B_γ is the signed projected coefficient of an individual such coset, then

$$\Big\| \sum_{\gamma} B_\gamma \Big\|_{\mathrm{HS}}^2 \le K_L \sum_{\gamma} \|B_\gamma\|_{\mathrm{HS}}^2.$$

For $\log L = O(x^c)$ with fixed c, one has $\log K_L = o(\exp(\nu x^{\varepsilon}))$ on the original intervals for every fixed $\nu > 0$.

Proof. The preceding two results of Section 116.3 give at most $N_L Y^{d(d-1)}$ cross matrices up to the two integral unitriangular actions. Each double coset admits a representative with one of those exact matrices. In fact, on replacing γ by $n_0 \gamma n_1$, the first frame in the cross pairing changes by n_0^{-1} and the second by n_1. Lemma 116.11 realizes both specified frame changes. These are elements of the very groups used to define the double cosets of Section 113.6. Restricting to the admissible classes at the original marking can only reduce their number.

For a fixed exact matrix, classify the ordered concatenated frames in the target lattice up to Γ by Lemma 116.12. If γ, δ give the same orbit, there is $u \in \Gamma$ fixing the first ordered frame pointwise with $\delta^{-1} u \gamma$ fixing the second row's original ordered frame pointwise. For either row b, the stabilizer of its ordered frame modulo earlier vectors maps to the isometry group of the positive rank-k lattice L_b, with kernel N_b. Indeed it fixes the terminal class, and the middle rational space is the orthogonal sum of its line and EL_b. Trivial action on the positive lattice forces trivial action on the entire middle quotient, which defines the unipotent radical inside this stabilizer. The finite group of isometries of L_b injects into matrices modulo three on its underlying lattice of rank $2fk$, as proved in Theorem 115.2. Its order is at most $3^{(2fk)^2}$. Thus one orbit of ordered embeddings accounts for at most the square of this number of N_0, N_1 double cosets. Multiplying proves the count. Cauchy–Schwarz for a finite sum in the Hilbert–Schmidt space proves the inequality with all signs and finite projectors in place.

The field degree, n, k, d, and the logarithms of the parameters in G_0 grow at most polynomially in x, as in Corollary 116.10. The extension E/F is unramified at finite places on these fields, so $\Delta_E = |\operatorname{disc}(F)|^2$. It follows directly from the formulas for C, P_L, K_L that $\log K_L$ also grows at most polynomially in x. This proves the last assertion. □

The estimate applies to the actual double cosets, even when several of them contribute to a single entry of the finite projection. Comparison with the integral theta map in Section 6.6 still requires estimates on the sizes of their orbital integrals with the prescribed sections.

116.5 Height truncation of the projected ordinary period

Keep the fields, rows, and finite projection of Section 113.6. We integrate on the original real quotient, using N_j for the integral unipotent subgroup of row j. Its real section W_j has nonzero finite $L^2(N_j \backslash G_\infty)$ norm w_j. Let V_j' be either the height V_j of Theorem 115.9 or the translated height $\widetilde{V}_j$ of Theorem 115.15. For $q \ge 0$ and $R \ge 1$ put

$$\mathfrak{M}_j(q) = w_j^{-2} \int (V_j')^q \|W_j\|^2, \qquad p_j(R) = w_j^{-2} \int_{V_j' \le R} \|W_j\|^2.$$

These integrals have the same quotient measure. Set

$$D=\sum_{i=1}^{s+1} i(n-i+1), \qquad C_*=3^{(2fk)^2}\prod_{i=1}^{s+1}(1+2\cdot 4^i T^i)^{2f(n-i+1)}.$$

Theorem 116.14 (Bounds in terms of the actual squared section norms)**.** *Let $\mathcal{A}$ be any set of supported integral double cosets in one fixed pair of rows. Write B_γ for the signed ordinary orbital integral, including its original finite coefficient and normalized column in the finite projection. If $V_0'(y)V_1'(\gamma^{-1}y)\ge H>0$ for all these cosets and all y, then for $\sigma>4D$ and $u\ge 0$,*

$$\frac{\sum_{\gamma\in\mathcal{A}}\|B_\gamma\|_{\mathrm{HS}}}{|c_{0,1}|\,w_0w_1}\le H^{-u}C_*(1-2^{-(\sigma-4D)})^{-f}\sqrt{\mathfrak{M}_0(\sigma+2u)\mathfrak{M}_1(\sigma+2u)}$$

whenever the two moments are finite. With the integration on each coset restricted by $V_0'(y)\le R_0$ and $V_1'(\gamma^{-1}y)\le R_1$, write $B_\gamma(R_0,R_1)$ for the resulting coefficient. For every such set of cosets one has

$$\frac{\sum_{\gamma\in\mathcal{A}}\|B_\gamma(R_0,R_1)\|_{\mathrm{HS}}}{|c_{0,1}|\,w_0w_1}\le\sqrt{M(R_0)M(R_1)p_0(R_0)p_1(R_1)}, \qquad M(R)=C_*R^{4D}\binom{\lfloor\log_2 R\rfloor+f}{f}.$$

Here $c_{0,1}$ is the same nonzero volume factor as in the unfolded ordinary period. If $R_0R_1<H$, all the truncated integrals in this set vanish.

Proof. For either height write its local factors as $\nu_{j,\tau}$. For fixed y, the number of cosets in $N_j\backslash\Gamma$ with $2^{a_\tau}\le\nu_{j,\tau}(\delta y)<2^{a_\tau+1}$ at every τ is at most $C_*2^{4D\sum a_\tau}$. For the translated height this is the exterior lattice count in Theorem 115.15. For the original height the count in Theorem 115.9 is smaller, since $T^{(i+1)/2}\le T^i$. The a_τ are nonnegative as the positive norms are compatible with the Hermitian form. Consequently

$$\sup_y\sum_{\delta\in N_j\backslash\Gamma}V_j'(\delta y)^{-\sigma}\le C_*(1-2^{-(\sigma-4D)})^{-f}=:P_\sigma.$$

For $V_j'(\delta y)\le R$, the sum of the a_τ is at most $\lfloor\log_2 R\rfloor$. Counting their tuples and bounding each box by C_*R^{4D} gives $M(R)$.

On the disjoint union of the coset integration domains, temporarily write $v_0=V_0'(y)$, $v_1=V_1'(\gamma^{-1}y)$ and F_b for the norm of the corresponding section at that argument. The norm of its rank-one integrand is F_0F_1. Its phase has modulus one and its normalized projector column has norm at most one, as in Theorem 113.24. The separation $v_0v_1\ge H$ bounds F_0F_1 by H^{-u} times the product

$$(F_0v_0^{u+\sigma/2}v_1^{-\sigma/2})(F_1v_1^{u+\sigma/2}v_0^{-\sigma/2}).$$

Apply Cauchy–Schwarz over the whole disjoint union. In the integral of the first square, enlarge the nonnegative sum to all double cosets. For each fixed y in $N_0\backslash G_\infty$ the weight from the second row becomes $\sum_{\delta\in N_1\backslash\Gamma}V_1'(\delta y)^{-\sigma}$. Indeed a fundamental domain of $N_0\cap\gamma N_1\gamma^{-1}$ is tiled by translates of one for N_0, and inversion of the double cosets lists each left N_1-coset once. This is the counting-measure unfolding in Section 113.6. Tonelli's theorem applies to these nonnegative functions. The first squared integral is thus at most $P_\sigma w_0^2\mathfrak{M}_0(\sigma+2u)$. Reversing the two rows gives the corresponding bound for the second square. Multiplication proves the first estimate.

For the truncated comparison apply the same Cauchy–Schwarz inequality to F_0,F_1 on the part of the union satisfying both cuts. Unfolding the first square counts second-row cosets of height at most R_1. It is at most $M(R_1)w_0^2p_0(R_0)$. The opposite square is at most $M(R_0)w_1^2p_1(R_1)$. If the product of the cutoffs is less than H, the integration set is empty. □

For example, in the part of Theorem 116.2 with $\mathcal{H}_\gamma \geq Q^f L$, take $V_j' = \widetilde{V}_j$, $\sigma = 4D+1$, and $u = a$ as in Theorem 115.9. The required moment has order $4D+1+2a \leq L_0$ for sufficiently large x. This follows from $2a = L_0 - 4fn^2$ and $D \leq n(s+1)(s+2)/2$, with $n/(s+2)^2$ tending to infinity on these intervals. It is finite by the weighted supremum and integral bounds of Theorem 115.15, since $\int (V_j')^{L_0}\|W_j\|^2 \leq \widetilde{U}_j \int \|W_j\| < \infty$. The resulting dimensionless factor multiplying the geometric mean of these moments is at most $Q^{-fa/2}L^{-a}$, since $\log(2^f C_*) = o(fa \log Q)$ there. Moreover the normalized moments for $\widetilde{V}_j$ are exactly the moments for V_j of the transported unit-frequency vector. The substitution $y \mapsto a_j y$ in Theorem 115.15 cancels its Haar factor against the squared normalization of the section. The volume of $N_j \backslash B_{j,\infty}$ cancels against the same volume in w_j^2.

Theorem 116.15 (The exact diagonal with both real heights restricted)**.** *Take one retained row j, repeated at both ends with its original marking, word and terminal frequency. Use the original height $V_j' = V_j$. Let $\mathcal{Z}_j(R)$ be its projected ordinary diagonal block with both heights restricted to at most R, and let $\mathcal{B}_j$ be its unrestricted flag-preserving block. For $1 \leq R < (Q/2)^{f/2}$,*

$$\mathcal{Z}_j(R) = p_j(R)\mathcal{B}_j, \qquad \mathcal{Z}_j(R)^*\mathcal{Z}_j(R) = p_j(R)^2 m_j \lambda_j^2 \, 1_{V_{\rho_j}}.$$

Here V_{ρ_j} is the positive column space, m_j counts the admissible flag-preserving double cosets, and $\lambda_j = |c_{j,j}| w_j^2 / \dim V_{\rho_j}$. At prime terminal norm T, for the prescribed unit lowest-type vectors of Theorem 87.24,

$$p_j(R) \leq \exp(-cbn^2), \qquad b = \exp(x^\varepsilon),$$

uniformly on the same intervals for sufficiently large x with the constant of that local mass estimate.

Proof. For a supported isometry that first fails to transport an outer vector modulo its predecessors, Theorem 115.9 gives $V_j(y)V_j(\gamma^{-1}y) \geq (Q/2)^f$ throughout its integral. Such a coset cannot contribute. If all the outer vectors transport but the terminal class changes, the two real unipotent radicals agree after transport and the two characters on them differ by Proposition 115.5. Both height cutoffs are invariant under the common radical, since the heights are left invariant under their decorated stabilizers. Integration of the nontrivial character on its compact unipotent lattice quotient still gives zero.

For an admissible flag-preserving γ both heights in its orbital integral coincide. The cutoff is left invariant under the real decorated group. Compact averaging on the positive column therefore gives a scalar matrix by irreducibility, just as for the unrestricted coefficient in Theorem 113.24. Its trace before the finite phase and unitary middle action is $\int_{V_j \leq R} \|W_j\|^2$. Hence the original coefficient, at its same marked index, is multiplied by precisely $p_j(R)$. Summing these terms gives the first identity. Their normalized finite columns have disjoint supports and norm one. The unrestricted Gram operator is thus $m_j \lambda_j^2 \, 1_{V_{\rho_j}}$, giving the second identity.

For the prime-frequency estimate let $R_F = |\operatorname{disc}(F)|^{1/f}$. The proof of Theorem 115.12 gives normalized mass at least $1 - e^{-cbn^2}$ to $V_j \geq (\sqrt{T}/(R_F b n^4))^f$. This lower threshold exceeds $(Q/2)^{f/2}$, as one sees by comparing the f-th roots using $\log T = (17 + o(1))mx^\varepsilon$, $m \to \infty$, $\log Q = (16 + o(1))x^\varepsilon$, and $\log(R_F n^4) = O_\psi(\log x)$. This proves the asserted upper bound for the mass in the cut. □

Thus the original finite projection retains an exactly computable part of the real diagonal. At prime terminal frequency this truncation captures exponentially little of its squared section norm. The comparison in Section 6.6 requires the signed contributions from the complementary integration region and the normalization of the integral theta map.

116.6 Two determinant bounds for the signed ordinary period

Fix two retained rows on the same $E = E_x$ and interval $[V, V^{3/2}]$ as in Section 6.6. Work with the supported cells with compatible characters and with $0 < \dim(X_0 \cap X_1) < s$. Order the rows so that $T_0 \geq T_1$, and put $u = T_0/T_1$, $F = E^+$, $f = [F : \mathbb{Q}]$ and $Q = g_1 g_2$. Both rational primes split completely in E. The positive spaces P_0, P_1 have dimension k. A cell γ has a rational isometry $j_\gamma : P_0 \to P_1$, a prescribed marked index i_γ, and common sublattices $C_{b,\gamma} \subset L_b$ of index $m_{b,\gamma}$, as in Section 115.9. In particular $m_{1,\gamma} = u^f m_{0,\gamma}$. All marked indices below are those of the original period.

Theorem 116.16 (A trace bound for overlap of two projected columns)**.** *Let $j_\gamma \neq j_\delta$ and $\rho = \operatorname{rank}_E(j_\gamma - j_\delta)$. If the two columns of the integral projection H meet in one output coordinate, then*

$$Q \;\leq\; 2\rho \min\{(N(\mathfrak{a}_{1,\gamma}\mathfrak{a}_{0,\delta}))^{1/(2f)}, (N(\mathfrak{a}_{1,\delta}\mathfrak{a}_{0,\gamma}))^{1/(2f)}\} \;\leq\; 2\rho\sqrt{u}\,(m_{0,\gamma}m_{0,\delta})^{1/(2f)}.$$

Here $\mathfrak{a}_{b,\gamma}$ annihilates $L_b/C_{b,\gamma}$ and N is the absolute ideal norm in E.

Proof. Set $M = j_\gamma^{-1} j_\delta$ and $z = k - \operatorname{tr} M$. At every archimedean embedding M is unitary for a positive Hermitian form. It has $k - \rho$ eigenvalues equal to one, counted with multiplicity, as it is diagonalizable there. The real part of z is a sum of $1 - \operatorname{Re}\lambda$ over its other ρ eigenvalues. Every summand is strictly positive and $|z| \leq 2\rho$. In particular $z \neq 0$.

Localize at any finite prime and take generators a, b of $\mathfrak{a}_{1,\gamma}, \mathfrak{a}_{0,\delta}$. The maps aj_γ^{-1} and bj_δ between the row lattices are integral, so abz is integral. At a mark these maps reduce to the corresponding scaled marked maps by Lemma 115.24, also when one scalar has zero residue. A shared coordinate means that the relative marked matrix is conjugate to an upper unitriangular matrix. This follows directly from the formula for H in Lemma 96.2. The relative matrix in the conjugate split factor is also unipotent. Its trace at either factor is k in the residue field. Thus abz reduces to zero. We have proved the ideal inclusion

$$(z)\mathfrak{a}_{1,\gamma}\mathfrak{a}_{0,\delta} \subseteq Q\mathcal{O}_E.$$

Taking norms and using the archimedean bound gives the first estimate in this orientation. Interchanging the cells gives the other one. The annihilator norm is at most the order of the quotient, by elementary divisors. Substituting both row indices proves the last inequality. □

The ranks at individual marked places also carry information. Write $r_v = \operatorname{rank}(i_{\gamma,v} - i_{\delta,v})$ for $v \mid Q$ in F, and q_v for its residue size. For a full sublattice $C \subset L$, write $[L : C]_{\mathrm{id}}$ for the integral ideal whose local exponent is the length of L/C. Its absolute norm is the order $[L : C]$. For this pair of cells define

$$C_b = C_{b,\gamma} \cap C_{b,\delta}, \qquad J_b = [L_b : C_b]_{\mathrm{id}}, \qquad D_{\gamma,\delta} = (N(J_0 J_1))^{1/(2f)}.$$

Theorem 116.17 (Rational rank and rank at the two marks)**.** *For distinct transports let ρ be their rational difference rank. Set $b_v = \max(0, \rho - r_v)$. At each v where the relative marked matrix is unipotent, one can replace b_v by $\max(1, \rho - r_v)$. Then*

$$\prod_{v \mid Q} q_v^{2b_v} \;\leq\; 4^{f\rho} D_{\gamma,\delta}^{f}, \qquad D_{\gamma,\delta} \leq u(m_{0,\gamma}m_{0,\delta})^{1/f}.$$

In particular, two equal full marks for distinct transports require $(Q/2)^{2\rho} \leq D_{\gamma,\delta}$.

Proof. At a finite prime fix bases of L_0, L_1, C_0, C_1 over the local integer ring. Let U_b be the matrix of $C_b \hookrightarrow L_b$ and $\lambda_b = \det U_b$, a generator of J_b. Consider the two integral polynomials

$$F_0(Y) = \det\bigl((j_\delta - (1+Y)j_\gamma)U_0\bigr),$$
$$F_1(Y) = \det\bigl((j_\delta^{-1} - (1+Y)j_\gamma^{-1})U_1\bigr).$$

Each map used on C_b takes values in the opposite lattice. At a mark its reduction on C_b is its prescribed marked map composed with $\bar{U}_b$. To see this for every vector of C_b, use its representative from the common positive lattice of the cell. Reducing that representative at the two ends gives the same vector in the common reduced middle quotient. Reading it in the two row bases gives the indicated formula, as in the proof of Lemma 115.24. This applies separately to each of the two cells and to both directions.

Let c_b be the coefficient of $Y^{k-\rho}$ in F_b. Over the fraction field, factor j_γ from the first determinant and j_γ^{-1} from the second. The relative matrices that remain are M and a conjugate of M^{-1}. The two factored determinants cancel in their product. Hence

$$c_0 c_1 = \lambda_0 \lambda_1 h, \qquad h = \prod_{\lambda \neq 1} (\lambda - 1)(\lambda^{-1} - 1),$$

where the product runs over the eigenvalues of M different from one, counted with multiplicity. The two factors from the eigenvalue one have signs $(-1)^{k-\rho}$ which cancel. The coefficient description shows that $h \in E$. At every complex embedding the eigenvalues lie on the unit circle and $0 < h \leq 4^\rho$. The ideal $(h)J_0J_1$ is integral at all primes by the integrality of c_0, c_1.

At a marked prime the constant matrix in the first determinant has reduction $(i_\delta - i_\gamma)\bar{U}_0$, of rank at most r_v. In the second determinant the inverse difference has the same rank bound. The conjugate split factor has that bound as well, since taking inverse transposes preserves difference rank. In each term of the coefficient c_b, exactly ρ columns come from the constant matrix. Every minor of order ρ of this matrix is divisible by the uniformizer to power $\max(0, \rho - r_v)$, by diagonal reduction over the local integer ring. Expanding the determinant on these columns gives that divisibility for c_b.

If the relative matrix is unipotent, the whole reduced polynomial F_b is a scalar multiple of Y^k. Indeed the two reduced maps factor on the right through $\bar{U}_b$; after taking its determinant, the remaining characteristic polynomial is that of a unipotent matrix shifted by $1 + Y$. This argument includes $\det \bar{U}_b = 0$. As $\rho > 0$, c_b then has positive valuation. At each prime above v the valuation of $(h)J_0J_1$ is therefore at least $2b_v$. There are two primes of E over each such v. Taking the square root of the absolute ideal norm and applying the archimedean bound proves the product estimate. The injection $L_b/C_b \hookrightarrow (L_b/C_{b,\gamma}) \oplus (L_b/C_{b,\delta})$ proves the upper bound for $D_{\gamma,\delta}$. For equal marks all $r_v = 0$ and $\prod_{v|Q} q_v = Q^f$, giving the final assertion. □

Corollary 116.18 (An integral decomposition and its archimedean error)**.** *Assume $s \geq 2$. Put $B = Q/(2k\sqrt{u})$. For the fixed pair of rows retain the proper-intersection cells with $m_{0,\gamma}^{1/f} < B$. In each row family of H, the signed projected sum on the retained cells with a given rational transport is recovered from the retained total by selecting coordinates. This holds over the integers of the coefficient field at every coefficient prime.*

Let S denote the full contribution of the proper nonzero intersections under consideration, in the normalization $d^{-1/2}H$ of Lemma 96.2, with $d = [G : U]$. Let $S_j^{<}$ be its retained part for transport j, including the signs and the ordinary real orbital integrals. Put

$$H_* = \max\left\{1,\ Q^f \min_{1 \leq t \leq s-1} \left(\frac{\max(1,B)u^{s-t+1}}{(T_0T_1)^t}\right)^{f/(2t)}\right\}.$$

Use the translated heights of Theorem 115.15. With the squared section moments and constants D, C_* of Theorem 116.14, write $P_\sigma = C_*(1-2^{-(\sigma-4D)})^{-f}$. For $\sigma > 4D$, $b \ge 0$ with finite indicated moments,

$$\frac{\left|\|S\|_{\mathrm{HS}} - (\sum_j \|S_j^{<}\|_{\mathrm{HS}}^2)^{1/2}\right|}{|c_{0,1}|w_0 w_1} \;\le\; P_\sigma H_*^{-b}\sqrt{\mathfrak{M}_0(\sigma+2b)\mathfrak{M}_1(\sigma+2b)}.$$

At either equal terminal norm, the integral selection includes all $m_{0,\gamma}^{1/f} \le \exp((16-\delta)x^{\varepsilon})$ for any fixed $0<\delta<16$ and sufficiently large x on the prescribed intervals. When $T_0 = T_1 = 1$ and $B > 1$, $H_* = Q^f B^{f/(2(s-1))}$.

Proof. If two retained cells have different transports, an overlap would contradict Theorem 116.16, since $\rho \le k$ and both index inequalities are strict. Thus in the retained projection each coordinate belongs to at most one transport. Selecting all its coordinates gives precisely the signed array for that transport. The selection matrix has entries zero and one. Orthogonality gives $\|\sum_j S_j^{<}\|_{\mathrm{HS}}^2 = \sum_j \|S_j^{<}\|_{\mathrm{HS}}^2$; the common real normalization does not enter the integral selection. The prescribed estimate for Q is $\log Q = (16+o(1))x^{\varepsilon}$, and $\log(2k) = o(x^{\varepsilon})$, by Theorem 110.16.

For each omitted cell set $e = \dim(X_0 \cap X_1)$ and $t = s - e$. In the height inequality in the proof of Corollary 115.26, the integer factors $[R : R_1]$ and D_β^{2e+1} are at least one, and $m_{0,\gamma} \ge \max(1, B)^f$. Taking the $2t$-th root shows that the product of the two translated heights is at least the t-term defining H_*. Each height is also at least one. Apply Theorem 116.14 to this whole set of omitted cells. It bounds the sum of their operator norms by the right side times $|c_{0,1}|w_0 w_1$; all finite projection columns have already been normalized there. The triangle inequality and then the reverse triangle inequality give the stated comparison. The formula with both terminal norms one follows by minimizing over t. □

The integral selection isolates a rational isometry. Several orbital integrals can still occur with that isometry and the same marked index. Estimating their signed sum and the size of the section moments at the normalization of Section 6.6 remains necessary.

117 Integral lifts and the signed transverse period

117.1 Rational isometries at a fixed transverse marking

Consider the transverse cells of a pair of original marked rows over E_x for which the common positive lattice R_γ is a hyperplane. Use $F = E_x^+$, $f = [F : \mathbb{Q}]$, $Q = g_1 g_2$ and the two completed column lattices $\widehat{L}_b$ of Lemma 113.6. At the marks these are the original lattices. The rational isometry j_γ on the positive columns extends the identification on R_γ and has the prescribed determinant. Let $\mathfrak{t}_\gamma$ be the discriminant ideal of this hyperplane lattice and $h_\gamma = (N\mathfrak{t}_\gamma)^{1/(2f)}$. Both terminal frequencies are allowed independently.

Lemma 117.1 (Matched sublattices of prescribed determinant)**.** *There are full sublattices $C_{b,\gamma} \subset \widehat{L}_b$ whose local index ideals at both ends are $\mathfrak{t}_\gamma$. The map j_γ carries $C_{0,\gamma}$ onto $C_{1,\gamma}$. At every marked prime the reduction of this map into the ambient target lattice is the prescribed i_γ composed with the reduction of $C_{0,\gamma} \hookrightarrow \widehat{L}_0$. The analogous assertion holds in the other direction.*

Proof. Work over the integer ring at a finite prime of E_x. Take the column bases and dual conjugate bases in the construction of Lemma 115.27. Let A_b have as columns the common hyperplane basis at end b, and let B_b have as rows the pairings with a common conjugate

hyperplane basis. These bases can be chosen independently in the two variables of the pairing, including at a self-conjugate prime. The square matrix $G = B_b A_b$ is the same at both ends and $\Delta = \det G$ generates the discriminant ideal locally. If B has $k-1$ rows, define the column $n(B)$ by $vn(B) = \det \left[\begin{smallmatrix} B \\ v \end{smallmatrix}\right]$ for every row v. Use the k columns

$$[\, A_b,\ n(B_b)\,]$$

for the sublattice at end b. Expansion in cofactors gives determinant Δ. The last column lies on the normal line since $B_b n(B_b) = 0$. The map j_γ matches the first $k-1$ columns and takes the last column to $d_\gamma n(B_1)$, where d_γ is its determinant in the ambient bases. This follows by taking determinants of the two displayed bases. By Lemma 113.6 d_γ is a unit at every finite prime, so the indicated sublattices match in both directions.

At a mark, write $i = i_\gamma$. The common representatives give $\bar{A}_1 = i\bar{A}_0$ and $\bar{B}_1 = \bar{B}_0 i^{-1}$; bars here denote reduction. Expansion along the last row gives $n(Bi^{-1}) = (\det i)^{-1} i n(B)$ over the residue field. Since $\bar{d}_\gamma = \det i$, the last columns also have matching reductions under the prescribed map. This includes vanishing reductions of columns. The inverse conclusion follows on the same bases. Outside the finite support of $\mathfrak{t}_\gamma$ the sublattice is the ambient one. Taking the intersection of these local prescriptions inside the rational space gives the required global lattices. Indeed their quotients are supported at finitely many primes, so the Chinese remainder theorem recovers each prescription on localization. □

Theorem 117.2 (Rank and distance at one marked index)**.** *Let $j_\gamma \neq j_\delta$, put $\rho = \operatorname{rank}_{E_x}(j_\gamma - j_\delta)$, and let $r_v = \operatorname{rank}(i_{\gamma,v} - i_{\delta,v})$. At $v \mid Q$ set $b_v = \max(0, \rho - r_v)$, replacing zero by one when the relative marked matrix is unipotent. Write $\| \ \|_{\tau,\mathrm{op}}$ for the operator norm in the positive Hermitian metrics at a real place of F. Then*

$$\left(\prod_{v \mid Q} q_v^{b_v}\right)^{1/f} \leq h_\gamma h_\delta \left(\prod_{\tau} \|j_\gamma - j_\delta\|_{\tau,\mathrm{op}}\right)^{\rho/f} \leq 2^\rho h_\gamma h_\delta.$$

If the two indices have the same determinant at the marks, then $\rho \geq 2$. In particular, at one fixed full marked index one has $(Q/2)^\rho \leq h_\gamma h_\delta$. All cells with $h_\gamma < Q/2$ at that index have the same rational transport, including when the discriminant ideals meet different subsets of the marks.

Proof. The determinant of $M = j_\gamma^{-1} j_\delta$ is a scalar $\xi \in E_x^\times$ preserving the integral determinant line at every prime. At infinity $|\tau\xi| = 1$. Equal marked determinants give $\xi - 1 \in Q\mathcal{O}_{E_x}$, including at the conjugate split factors by duality. A nonzero such element has absolute norm at least Q^{2f}; the archimedean bound is 2^{2f}. Hence $\xi = 1$. A unitary matrix is diagonalizable. If its difference from the identity had rank one, its determinant would be its unique eigenvalue different from one. This excludes rank one.

For the inequality work on $C = C_{0,\gamma} \cap C_{0,\delta} \subset \widehat{L}_0$. Let J be its index ideal. The injection into the sum of the two quotients shows locally that $v_\mathfrak{p}(J) \leq v_\mathfrak{p}(\mathfrak{t}_\gamma \mathfrak{t}_\delta)$. In the polynomial $\det(M - (1+Y)I)$ let c be the coefficient of $Y^{k-\rho}$. Up to sign this is $\prod_{\lambda \neq 1}(\lambda - 1)$ on the eigenvalues of M, with multiplicities. It is nonzero. In a basis of C the determinant of $j_\delta - (1+Y) j_\gamma$ equals this polynomial times the determinant of j_γ and the determinant of $C \hookrightarrow \widehat{L}_0$. The first factor outside the polynomial is a local unit and the second generates J. The determinant on C is integral by Lemma 117.1.

At a mark the constant term matrix on C has rank at most r_v on reduction. The coefficient in question uses ρ columns of that matrix. Its minors contribute the power $\max(0, \rho - r_v)$, as in Theorem 116.17. If the relative matrix is unipotent, the determinant on C reduces to a

scalar multiple of Y^{k}. Thus $(c)J$ has valuation at least b_v in each split factor and is integral elsewhere. Taking absolute norms, and then $2f$-th roots, gives the first inequality because

$$|\tau c| = \prod_{\lambda \neq 1} |1-\lambda| \leq \|j_\gamma - j_\delta\|_{\tau,\mathrm{op}}^{\rho}.$$

The equality of absolute values at conjugate embeddings follows from unitarity. Each operator difference has norm at most two. For one full marked index all $b_v = \rho$ and the two rational primes split completely. This gives the remaining assertions. □

Corollary 117.3 (Multiplicity of transports at a marked index)**.** *For $H \geq 1$, let*

$$r_H = \min\{k, \lfloor 2\log H/\log(Q/2)\rfloor\}, \qquad n(H) = \begin{cases} 1, & r_H < 2, \\ (1+10H/Q)^{2fr_H(k+1)}, & r_H \geq 2. \end{cases}$$

At any fixed full marked index the number of distinct rational transports among the cells with $h_\gamma \leq H$ is at most $n(H)$.

Proof. Choose one cell for each distinct transport. Every difference of two chosen transports has rank between 2 and r_H, by Theorem 117.2. If a difference has rank ρ, its largest archimedean operator norm is at least $Q/H^{2/\rho} \geq Q/H$. Fix one chosen transport as origin and compose with its inverse. The resulting matrices are unitary and their distances are unchanged. At each embedding each such matrix can be written

$$I + W\,\mathrm{diag}(\lambda_a - 1)W^{*} \qquad (1 \leq a \leq r_H).$$

Use orthonormal eigenvectors for the nontrivial eigenvalues and pad with eigenvectors of eigenvalue one. The k-by-r_H matrix W has operator norm one when $r_H \geq 2$, and the list of eigenvalues has maximum absolute value one. For two choices of parameters the distance of the resulting matrices is at most $4\|W - W'\|_{\mathrm{op}} + \max_a |\lambda_a - \lambda'_a|$, by adding and subtracting the products with one factor changed. Give the parameter space the maximum norm in the two components and across embeddings. The tuples for different transports are separated by at least $Q/(5H)$ and lie in its unit ball. This space has real dimension $2fr_H(k+1)$. Disjoint open balls of radius $Q/(10H)$ about those tuples fit inside the ball of radius $1 + Q/(10H)$. The ratio of volumes proves the bound for every finite subset and hence for the whole set. □

Corollary 117.4 (The transverse period on both sides of a frame cutoff)**.** *Write $\mathcal{P}$ for the normalized signed ordinary contribution of the transverse hyperplane cells in this pair of rows. Fix $L \geq 1$, put $d_0 = s+1$, and write*

$$H_L = (T_0T_1)^{d_0}L^{2d_0/f}, \qquad \alpha = \prod_{v\mid Q}\prod_{a=1}^{k-1}(1-q_v^{-a})^{-1}.$$

For cells of frame height $\mathcal{H}_\gamma < Q^{f}L$ from Proposition 116.1, let $Z_{j,i}$ denote the sum of their signed ordinary operators at the original index i and transport j, before applying the normalized marked projection. Then

$$\|\mathcal{P}\|_{\mathrm{HS}} \leq \sqrt{\alpha\, n(H_L)}\left(\sum_{i,j}\|Z_{j,i}\|_{\mathrm{HS}}^{2}\right)^{1/2} + |c_{0,1}|w_0w_1P_\sigma(Q^{f}L)^{-b}\sqrt{\mathfrak{M}_0(\sigma+2b)\mathfrak{M}_1(\sigma+2b)}.$$

Here the translated section moments and the allowed σ, b are as in Corollary 116.18. For $H_L \leq Q/2$ one may use 1 in place of $n(H_L)$. In particular when both terminal norms are one this holds at $L = (Q/2)^{f/(2(s+1))}$.

Proof. The frame-height estimate gives $h_\gamma < H_L$ on the retained set, since I_Λ and P_γ in Proposition 116.1 are at least one. At a fixed full index at most $n(H_L)$ distinct transports therefore occur. If $H_L \le Q/2$, the strict inequality and Theorem 117.2 give at most one. Apply Cauchy–Schwarz at each index to the sum of the $Z_{j,i}$. The squared norm of its image under $[G:U]^{-1/2}H$ is at most α times its squared norm on raw indices, by Proposition 92.22. For the omitted cells the product of the two heights is at least $Q^f L$ at every integration point. Theorem 116.14 bounds their absolute sum by the second term. Adding these two bounds proves the assertion on the original field and interval. □

The estimate leaves the signed operators for a single pair (j,i) in the retained sum. Proposition 113.5 exhibits distinct rational partners even for trivial marked index and discriminant height one. Their real orbital coefficients enter this sum separately.

117.2 Summing integral lifts of a transverse rational class

Fix a source row and a target row in the same integral class of Section 113.6. They lie in a common Hermitian space V over $E = E_x$, with $F = E^+$ and $G = U(V)$ over F. Write $b = s, t$ for the source and target subscripts, respectively, and use l for the length of each isotropic flag. Put $Y_b = X_b + Ez_b$, the span of its outer space and terminal vector. Choose a rational Witt decomposition at each end with middle space $P_b \perp Ez_b$. Such a decomposition can be chosen inside $z_b^\perp$ for the outer hyperbolic pairs, since z_b is nonisotropic. It identifies the positive quotient of each row with P_b. Write N_b for its algebraic unipotent group and ψ_b for its real character on the prescribed section.

For a transverse hyperplane cell γ, the subspace

$$C_\gamma = (Y_t + \gamma Y_s)^\perp$$

is positive of dimension $k-1$, by Theorem 86.3. Its projections to P_t and, via γ^{-1}, to P_s are hyperplanes H_t, H_s. Fix these projected hyperplanes and fix the rational transport $j : P_s \to P_t$, including its determinant in Lemma 113.6. Thus the two identifications on the common hyperplane induce $j|_{H_s}$. Fix also the full marked index i when summing the integral terms. In this section the groups that act on the rational cells are written independently of the finite support and the arithmetic subgroups used to integrate the real sections.

Theorem 117.5 (Rational classes in the orthogonal complement)**.** *Assume the specified hyperplanes and transport occur. Put $V_b' = H_b^\perp$ in V, and $N_b' = N_b \cap U(V_b')$, embedding the latter unitary group by the identity on H_b. Each V_b' has dimension $2l+2$ and signature $(l+1, l+1)$ at every real place. There is a rational isometry $\eta_* : V \to V$ taking H_s onto H_t by $j|_{H_s}$ for which the rational $N_t(F), N_s(F)$ double classes of transverse hyperplane pairs with these data are in bijection with*

$$N_t'(F)\backslash\{a \in SU(V_t')(F) : Y_t + a\eta_* Y_s = V_t'\}/\,\eta_* N_s'(F)\eta_*^{-1}.$$

The cells at the original finite level map into this set.

Proof. At the target C_γ lies in $X_t^\perp \cap z_t^\perp$. Its projection to P_t is injective since the kernel lies in the isotropic X_t, whereas C_γ is positive. It is therefore the graph of a rational linear map $A_t : H_t \to X_t$. The identical argument applied to $\gamma^{-1}C_\gamma$ gives $A_s : H_s \to X_s$. Extend A_b by zero on the orthogonal complement of H_b inside the middle space. In coordinates consisting of X_b, the middle space, and its outer dual summand $X_b^\vee$, respectively, set

$$n(A_b) = \begin{pmatrix} I & A_b & -\frac{1}{2}A_b A_b^\dagger \\ 0 & I & -A_b^\dagger \\ 0 & 0 & I \end{pmatrix}.$$

The adjoint uses the middle Hermitian form and the pairing of the two outer summands. Multiplication with the Witt form shows this matrix is unitary. The two terms linear in A_b cancel the cross terms from the middle space, and the two half terms cancel its quadratic term. It lies in $N_b(F)$, takes H_b onto the required graph, fixes X_b and z_b pointwise, and has determinant one. Its outer simple coordinates and its terminal character coordinate vanish, so $\psi_b(n(A_b)) = 1$.

Consequently $\eta = n(A_t)^{-1}\gamma n(A_s)$ takes the embedded H_s onto H_t by the prescribed isometry. Take one such η as η_*. The determinant of a rational cell map in V is determined by j and the two fixed decorated frames. Indeed the determinant-line construction of Lemmas 113.1 and 113.6 composes the ambient determinant with fixed isomorphisms at its two ends, using the normalized terminal vector at each end. Hence $\eta\eta_*^{-1}$ fixes H_t pointwise and has determinant one. It is exactly an element of $SU(V_t')(F)$. The perpendicular of $Y_t + \eta Y_s$ is H_t, so their sum is V_t'. Conversely the displayed spanning condition forces the perpendicular to be precisely H_t, and forces transversality since the two summands both have dimension $l+1$. The restriction to the hyperplane and the determinant then recover j by its uniqueness. In the ambient signature $(l+k, l+1)$ the removal of $k-1$ positive lines gives the asserted signature.

Finally suppose two such representatives satisfy $\eta' = u_t\eta u_s^{-1}$ with $u_b \in N_b(F)$. Each u_b preserves Y_b. Taking perpendiculars of the two frame sums gives $u_t H_t = H_t$. As N_t acts trivially on the middle quotient, u_t fixes H_t pointwise. Applying the argument at the source gives the same conclusion for u_s on H_s. Thus both lie in the indicated smaller unipotent groups. The reverse implication is immediate by inclusion. This proves the bijection. □

For the next theorem use the full column-valued ordinary sections W_b. An integral cell has its arithmetic stabilizer Γ_γ and finite scalar $c_\gamma = c_{t,s}\epsilon_\gamma$, where $c_{t,s}$ is the volume of the row pair and ϵ_γ its finite character value, as in Proposition 96.20. All sums below are on the fixed rows and index. Write $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$, with the analogous convention for subgroups.

Theorem 117.6 (The signed sum over the integral lifts of one class)**.** *For a class $[a]$ of Theorem 117.5 choose $\eta = a\eta_*$, and put*

$$D_\eta = N_t \cap \eta N_s \eta^{-1} = N_t' \cap \eta N_s' \eta^{-1}.$$

Discard a class if the two real characters differ on D_η; its contribution is zero. For each integral lift at i write $\gamma = u_t\eta u_s^{-1}$ with $u_b \in N_b(F)$, and put $\Lambda_{\gamma,\eta} = u_t^{-1}\Gamma_\gamma u_t$. On choosing Haar measure on $D_{\eta,\infty}$, the entire contribution at i from this class before the marked projection is the product $m_i(\eta)I_\eta$, where

$$m_i(\eta) = \sum_{\gamma \mapsto [a]} c_\gamma\, \psi_t(u_t)\overline{\psi_s(u_s)}\, \mathrm{vol}(\Lambda_{\gamma,\eta}\backslash D_{\eta,\infty}),$$

$$I_\eta = \int_{D_{\eta,\infty}\backslash G_\infty} W_t(g) W_s(\eta^{-1}g)^*\, d\dot{g}.$$

The first sum is finite. It admits the following exact cancellation test. For a nonempty set of lifts put $\Delta = \bigcap_\gamma \Lambda_{\gamma,\eta}$, $d_\gamma = [\Lambda_{\gamma,\eta} : \Delta]$, and $M = \mathrm{lcm}_\gamma\, d_\gamma$. Write the total phase of a lift as ζ^{e_γ} for a common primitive n-th root of unity. Then $m_i(\eta) = 0$ if and only if the cyclotomic polynomial Φ_n divides

$$\sum_{\gamma \mapsto [a]} (M/d_\gamma) X^{e_\gamma} \qquad \text{in } \mathbb{Z}[X],$$

using exponents in $\{0, \dots, n-1\}$. All factors in this polynomial come from the original lifts and the fixed class representative.

Proof. If $d \in D_\eta$ and $v \in H_t$, then $dv - v \in X_t$. Reading d in the transported radical gives $dv - v \in \eta X_s$ as well. Transversality gives zero. Thus d fixes H_t pointwise, which proves the equality of groups. In the original integral (98.1) change variable by left multiplication by u_t. Its stabilizer becomes $\Lambda_{\gamma,\eta}$, and the two sections supply the two indicated character factors. In particular the finite phase has been retained even if a unipotent change of variables in this calculation is nonintegral at a finite place. Lemma 98.1 identifies the transformed stabilizer as a uniform lattice in $D_{\eta,\infty}$. Integrating along its compact quotient gives zero for different characters and gives exactly its covolume for matching ones. This proves the factorization, with compatible quotient measures, for each lift. Changing u_t, u_s for the same lift changes them by corresponding elements of $D_\eta(F)$; equality of the characters cancels the change of phase. The resulting conjugation in the unipotent group preserves its Haar measure.

Every prefix exterior vector, including the last one, is fixed by the respective $N_b(F)$, since the changes of the ordered frame are unitriangular. Hence the frame height $\mathcal{H}_\gamma$ of Proposition 116.1 is fixed throughout the rational class. Take L large enough to put that height strictly below $Q^f L$. The finite count in Theorem 116.13 proves finiteness of its supported relevant integral lifts on the retained rows.

The lattices in the group $D_\eta(F)$ just obtained are commensurable. Indeed each imposes preservation of finitely many full row lattices after rational conjugation and, if a level is present, a finite congruence. For two full lattices L_1, L_2, select a positive integer a with $aL_1 \subset L_2 \subset a^{-1}L_1$. An automorphism of L_1 congruent to one modulo a^2 preserves L_2: its difference from the identity takes $a^{-1}L_1$ into aL_1, and so does the difference for its inverse. Increasing the modulus imposes each required finite level. The reduction of a lattice modulo an integer is finite. Applying this argument in both directions, then intersecting finitely many groups, proves $d_\gamma < \infty$.

The real character of a rational unipotent element has finite order here. The original character is adelic and is trivial on rational elements. Its finite additive factors have finite order at any given argument and are one almost everywhere. The real factor is their inverse. The other factor ϵ_γ is already a finite character value. Now divide $m_i(\eta)$ by $c_{t,s}\,\mathrm{vol}(\Delta\backslash D_{\eta,\infty})/M$. The result is exactly the displayed integer polynomial at ζ. Its vanishing is equivalent to divisibility by the minimal polynomial Φ_n, including $n = 1$. This gives the test. □

Corollary 117.7 (The squared norm at fixed hyperplanes and transport)**.** *Let Z be this full signed sum at i, for the fixed H_s, H_t, j. Use $K = U(H_t)_\infty$, acting on the source by its identification with $U(H_s)_\infty$. For each common irreducible summand μ in the two compact restrictions, let $V_{b,\mu}$ denote its subspace and choose a unitary intertwiner $J_\mu : V_{s,\mu} \to V_{t,\mu}$, extended by zero on its orthogonal complement. Write $d_\mu = \dim V_{s,\mu}$ and $L_\infty = U(V_t')_\infty$. Define*

$$b_\mu(\eta) = \int_{L_\infty\backslash G_\infty} \int_{D_{\eta,\infty}\backslash L_\infty} \mathrm{tr}\Big(J_\mu^* W_t(hg) W_s(\eta^{-1}hg)^*\Big)\, d\dot{h}\, d\dot{g}.$$

Then, summing over the supported relevant classes with a lift at i,

$$\|Z\|_{\mathrm{HS}}^2 = \sum_\mu \frac{1}{d_\mu} \left| \sum_{[a]} m_i(a\eta_*)\, b_\mu(a\eta_*) \right|^2 .$$

These are identities for the original sections, with either terminal frequency. They apply to the full absolutely convergent expansion and to a cutoff by frame height. The normalized marked column at i has norm one, so adjoining that column preserves the norm in this formula.

Proof. The inclusions $D_{\eta,\infty} \subset L_\infty \subset G_\infty$ are inclusions of unimodular groups. Decompose the Haar quotient on $D_{\eta,\infty}\backslash G_\infty$ first over $L_\infty\backslash G_\infty$. This gives the displayed integral for $\mathrm{tr}(J_\mu^* I_\eta)$

and explains why its inner integral runs over a unitary group of dimension $2l+2$ in its defining representation. Absolute integrability justifies disintegration; for the ordinary sections it also follows from the unfolded nonnegative majorant in the proof of Theorem 116.14 with $b = 0$ and $\sigma = 4D + 1$ in the notation of Corollary 116.18. The finiteness of the moments at this order on the prescribed rows follows from Theorem 115.15, as explained after Theorem 116.14. Thus the tensor integrals can be grouped in the indicated order. A height cutoff keeps whole rational classes by the prefix invariance proved above.

All the I_η intertwine the same actions of K. Indeed an element of K fixes both frames, commutes with the action on the orthogonal complement, and acts on the source compact column via $j|_{H_s}$. Change variables by that compact element in the quotient integral. Its action preserves measure, giving the intertwining identity as in Theorem 98.2. Each restriction has multiplicity one by compact unitary branching. Schur's lemma gives $I_\eta = \sum_\mu b_\mu(\eta) J_\mu/d_\mu$. The operators J_μ are orthogonal in Hilbert–Schmidt inner product, with squared norms d_μ. Summing with the signed coefficients from Theorem 117.6 and taking the squared norm proves the result. □

This formula sums the phases of every integral lift before the norm is taken. A uniform estimate for the sums over the remaining rational classes, with the original real sections and the different hyperplanes combined, is still needed in Section 6.6.

117.3 Finite local factors of the signed sum of integral lifts

Retain a pair of rows, the two projected hyperplanes, the rational transport, and the full index i on E_x from Section 117.2. Fix a relevant rational class with representative η and at least one supported integral lift at i. Its scalar $m_i(\eta)$ in Theorem 117.6 multiplies one real tensor. We calculate this scalar with the original outer characters. In particular the finite tests below are taken before any sum over marked indices or terminal projections.

Set $U = N_t \times N_s$, and embed $D = D_\eta$ in U by $d \mapsto (d, \eta^{-1} d\eta)$. For a finite place v of $F = E_x^+$ let $U_v = U(F_v)$, $D_v = D(F_v)$, and let K_v be the product of the two local integral unipotent subgroups of the rows. Thus $U(F) \cap \prod_v K_v$ is the product of the two integral unipotent groups in the original double quotient. For $w = (w_t, w_s) \in U_v$ write $\gamma(w) = w_t \eta w_s^{-1}$. Let Ω_v be the set of double classes $K_v w D_v$ for which $\gamma(w)$ is an integral isometry with the prescribed local support and, where imposed, marking i_v. Put

$$K_{v,w}^D = D_v \cap w^{-1} K_v w, \qquad z_v(w) = \epsilon_v(\gamma(w))\, \overline{\psi_{t,v}(w_t)}\, \psi_{s,v}(w_s).$$

Here ϵ_v is the local factor of the ordinary outer finite phase in Proposition 96.20; $\psi_{b,v}$ is its component at v for the adelic character with archimedean component ψ_b. At residual characteristic p each of these local additive character values is a root of unity of p-power order.

Theorem 117.8 (Normalization by an actual integral lift)**.** *The integral lifts of the fixed class at i, up to the original integral unipotents, are in bijection with $\prod_{v<\infty} \Omega_v$. Each set in the product is finite, and almost all are singletons. At v choose a class ω_v^* maximizing the Haar volume of $K_{v,w}^D$. The local choices are realized together by an integral lift γ_* on the specified rows and index. For $\omega \in \Omega_v$ put*

$$n_v(\omega) = \frac{\operatorname{vol}(K_{v,\omega_v^*}^D)}{\operatorname{vol}(K_{v,\omega}^D)}, \qquad A_v = \sum_{\omega \in \Omega_v} n_v(\omega) z_v(\omega).$$

The ratio $n_v(\omega)$ is a nonnegative power of the residual characteristic. For any integral lift γ with local classes (ω_v), one has

$$\frac{\operatorname{vol}(\Lambda_{\gamma,\eta} \backslash D_\infty)}{\operatorname{vol}(\Lambda_{\gamma_*,\eta} \backslash D_\infty)} = \prod_v n_v(\omega_v), \qquad \frac{m_i(\eta)}{c_{t,s} \operatorname{vol}(\Lambda_{\gamma_*,\eta} \backslash D_\infty)} = \prod_v A_v.$$

Almost all $A_v = 1$. The normalized scalar in the second identity is an algebraic integer.

Proof. First check the local functions. In a fixed pair of rows the finite support conditions are membership in the local lattice stabilizers and the specified congruences on the local frames and columns. Their conjunction is the original global finite support condition. Integral unipotent changes preserve that condition and the full marked index. Under such a left change in U_v the two finite seeds multiply their phase by $\psi_{t,v}(k_t)\overline{\psi_{s,v}(k_s)}$, by their character transformation in (92.16), equivalently by the change of representatives in the proof of Proposition 96.20. These two factors cancel those from the explicit characters in z_v. A right change in D_v fixes $\gamma(w)$ and also fixes the combined character. Indeed the ratio of the two finite adelic characters on D is one on $D(F)$, since each global character is trivial on rational points and the two real characters agree there by relevance. Rational points are dense in $D(\mathbb{A}_{F,f})$: logarithm and exponential are inverse polynomial maps for this unipotent group in characteristic zero, and additive approximation holds in its rational Lie algebra. The continuous ratio is therefore one at each finite place. This proves that $z_v(\omega)$ is well defined. The stabilizer volumes are also well defined, since a right change conjugates the stabilizer inside the unimodular group D_v. A continuous local additive character at any specified argument has p-power order, since multiplication of that argument by a large power of p puts it in the character kernel.

Write $K = \prod_v K_v$. Rational points of U are dense in its finite adeles by the same additive approximation. Thus in $K\backslash U(\mathbb{A}_{F,f})/D(\mathbb{A}_{F,f})$ each double class has a rational representative. If w, w' are two rational representatives of the same double class, there is a finite adelic $d \in D$ with $w'd^{-1}w^{-1} \in K$. This is an open condition on d. By density it holds for some $d \in D(F)$, and the resulting left multiplier belongs to $U(F) \cap K$. This proves injectivity in the rational description as well. The stabilizer of η in $U(F)$ is exactly the displayed copy of $D(F)$. A rational $\gamma(w)$ integral in both directions at every finite place is an isometry of the given global lattices, by intersection of their localizations. This proves the description by the restricted product of the local supported double classes, retaining the original finite conditions.

Start with any one supported global lift. Any single local choice can replace its class at that place and gives an adelic element, hence a global lift by the preceding paragraph. Infinitely many local choices at one place, or alternatives at infinitely many different places, would therefore give infinitely many distinct integral lifts of the same rational class. Their finiteness was proved in Theorem 117.6. Consequently the restricted product is the entire stated product, and one can choose all the maximizing stabilizers at once. Almost everywhere the initial rational representative is integral in U, the support is the basic class, and all the finite phases are one.

For a rational w representing a lift, its lattice in D is $D(F) \cap \prod_v K_{v,w}^D$, under the fixed embedding in U. This follows by conjugating the two original integral unipotent stabilizers, just as in Theorem 117.6. Choose compatible real and finite Haar measures and write $V_D = \mathrm{vol}(D(F)\backslash D(\mathbb{A}_F))$. This volume is finite: a rational central series with vector group quotients reduces compactness to that of $\mathbb{A}_F/F$. By density at the finite places a fundamental domain modulo $\prod_v K_{v,w}^D$ is given by the real quotient by the displayed lattice. Equivalently, a real fundamental domain times $\prod_v K_{v,w}^D$ is an adelic fundamental domain up to null sets. Its volume gives

$$\mathrm{vol}(\Lambda_{\gamma,\eta}\backslash D_\infty) \prod_v \mathrm{vol}(K_{v,w}^D) = V_D.$$

Use volume one for the usual integral subgroups at almost all places. Division by the same formula for γ_* proves the first identity.

Every compact subgroup of D_v is pro-p. In a faithful unipotent matrix representation it preserves a lattice: the span over local integers of its orbit on a fixed lattice is bounded, full, and stable. Its finite reduction consists of unipotent matrices, and thus has p-power

order by Cauchy's theorem. The successive congruence kernels have elementary abelian p-group subquotients. This proves the assertion. Two compact open subgroups of D_v have open intersection, of p-power index in each. Their volume ratio is a power of p; choosing the largest volume for the numerator makes that exponent nonnegative.

For rational w, the product of $z_v(w)$ is the full phase $\epsilon_\gamma \psi_t(w_t)\overline{\psi_s(w_s)}$, by triviality of the adelic characters on rational elements. Insert this identity and the covolume ratios into the finite sum in Theorem 117.6. Distributing a product of finite sums gives the second identity. The local weights are integers and the local phases roots of unity, proving integrality. □

Corollary 117.9 (Local cancellation and the terminal valuation)**.** *At v of residual characteristic p, write all the phases as $\zeta^{e(\omega)}$ for a primitive p^h-th root of unity, with exponents between 0 and p^h-1. Put $a_e = \sum_{e(\omega)=e} n_v(\omega)$ and $W_v = \sum_e a_e$. For $h = 0$ one has $A_v = W_v > 0$. For $h > 0$ the factor vanishes exactly when, for every $0 \le r < p^{h-1}$,*

$$a_r = a_{r+p^{h-1}} = \cdots = a_{r+(p-1)p^{h-1}}.$$

The whole signed scalar vanishes exactly when some local factor does. If the number of classes with maximal stabilizer volume at v is prime to p, then A_v is a unit at every coefficient prime above p.

Let T be the terminal rational prime of the original period. When all factors are nonzero, the valuation at any prime above T, normalized by $\mathrm{val}_T(T) = 1$, obeys

$$0 \;\le\; \mathrm{val}_T\left(\frac{m_i(\eta)}{c_{t,s}\,\mathrm{vol}(\Lambda_{\gamma_*,\eta}\backslash D_\infty)}\right) \;\le\; \sum_v \frac{[\mathbb{Q}(\zeta_{p^h}):\mathbb{Q}]}{e_{v,T} f_{v,T}} \frac{\log W_v}{\log T}.$$

Here p, h are local to v, and $e_{v,T}, f_{v,T}$ are the ramification and residue degrees at T of that cyclotomic field. Only finitely many summands can be nonzero.

Proof. For $h > 0$ the minimal polynomial of ζ is $\sum_{j=0}^{p-1} X^{jp^{h-1}}$. A polynomial of degree less than p^h is a multiple of it precisely when its quotient has degree less than p^{h-1} and multiplication repeats each coefficient of that quotient in the indicated positions. This proves the test. Reducing A_v at p sends every phase to one. Exactly the maximal-volume classes have weight one; every other weight is a positive power of p. Hence the residue is the asserted count modulo p.

Every conjugate of A_v has absolute value at most W_v. For a nonzero A_v its absolute field norm is a positive integer at most $W_v^{[\mathbb{Q}(\zeta_{p^h}):\mathbb{Q}]}$. The norm of the chosen prime ideal over T is $T^{f_{v,T}}$. Taking logarithms bounds its exponent in (A_v); division by $e_{v,T}$ gives the stated normalization. The product formula of Theorem 117.8 adds these valuations. At a singleton place $W_v = 1$. □

This gives the integral arithmetic coefficient for one rational class relative to an actual lift at its prescribed index. The sizes of these factors, the real tensors in Corollary 117.7, and their sum over classes and hyperplanes still enter the full period comparison.

117.4 Finite residue sums for the signed local factors

In Theorem 117.8, fix the two original rows and a rational representative η. The integral matrices in its rational unipotent orbit may have large denominators in the variables w_t, w_s that produce them. We give a finite range of variables at every finite place, independent of the full marked index to be tested. The terminal norms, row lattices and field E_x are those of Section 6.6.

View the space M of E_x-linear maps between the two ambient row spaces as an $F = E_x^+$ vector space. The group $U = N_t \times N_s$ acts linearly on M by $(w_t, w_s)z = w_t z w_s^{-1}$. Write D for the stabilizer of η in this product. At a finite place $\mathcal{O}_v$ has uniformizer π_v and residue size q_v.

Lemma 117.10 (Polynomial coordinates from the two rational flags)**.** *Set $r = \dim_F U - \dim_F D$. There are explicitly determined rational polynomials $P_1, \dots, P_r$ in the matrix entries over F and elements $X_1, \dots, X_r$ in the rational Lie algebra of U such that, with*

$$s(t) = \exp(t_1 X_1) \cdots \exp(t_r X_r), \qquad z(t) = s(t)\eta,$$

one has $P(z(t)) = t$ and $z(P(z)) = z$ on the orbit. Here $s(t)\eta$ denotes the indicated action. These identities hold over F and every F_v. In these coordinates any Haar quotient measure on U_v/D_v is a constant multiple of product additive Haar measure on F_v^r.

Proof. Take bases in which each row radical is upper unitriangular, by ordering its two outer flags and the middle block in Witt coordinates. On the source dual reverse the order, so that the dual action is upper unitriangular too. In a tensor basis of target times source dual the product group acts by upper unitriangular matrices. On passing from E_x to F, refine each diagonal identity block with an F-basis. All changes can therefore be made in fixed rational coordinates.

Start with $U_1 = U$. If U_j moves η, take the largest coordinate a on which $u\eta - \eta$ is nonzero for some $u \in U_j$. The function $\lambda(u) = (u\eta - \eta)_a$ is a nonzero homomorphism to the additive group. Indeed all larger coordinates of that difference vanish and an upper unitriangular transformation fixes its a-th coordinate. Applying this to $uv\eta - \eta = u(v\eta - \eta) + (u\eta - \eta)$ proves additivity. On a one-parameter exponential it is a linear function of the parameter, since it is an additive polynomial in characteristic zero. Exponentials exhaust each group here: initially this follows from finite logarithm and exponential for the row radicals, and subsequently the kernel of a character is the exponential of the kernel of its differential. Thus a is the largest index on which some $X\eta$ is nonzero in a basis of the current rational Lie algebra. Here $X\eta$ is the derivative of the action. Linear algebra gives X_j with $(X_j\eta)_a = 1$.

Put $U_{j+1} = \ker \lambda$. For a current point $y = u\eta$ read $t_j = y_a - \eta_a$, and replace y by $\exp(-t_j X_j)y$. This lies in the orbit of U_{j+1}, and is polynomial in the initial coordinates of y, since each exponential is a finite polynomial. Iterating produces the polynomials P_j and reverses the product defining $s(t)$. Each kernel has codimension one and still contains D; the process stops precisely at D. This proves existence, uniqueness of the ordered parameters and all claimed identities. At each step U_j is the product of the one-parameter section and its normal kernel. Conjugating the kernel by this section preserves Haar measure: the adjoint acts unipotently on its Lie algebra and has determinant one. Haar measure is thus a constant times the product of additive measure in t_j and Haar measure on the kernel. Iteration and division by the measure on D_v prove the measure assertion. □

Let $\mathcal{L}_v$ be the full $\mathcal{O}_v$-lattice of integral row maps in $M(F_v)$; it suffices here to impose integrality in the forward direction. In a basis of this lattice express each P_j as a polynomial. Take $a_{j,v}$ to be the larger of zero and the maximum of the negatives of its coefficient valuations, omitting zero coefficients. The value zero is used for an empty maximum. Put

$$B_v = \prod_{j=1}^{r} \pi_v^{-a_{j,v}} \mathcal{O}_v.$$

Represent U_v as a pair of matrices on the original row lattices, using bases over $\mathcal{O}_v$ after restriction of scalars. Choose $c_v \geq 0$ so that every element of U_v congruent to the identity

modulo $\pi_v^{c_v}$ in these lattices lies in K_v. For $c_v = 0$ the condition means that its difference from the identity is integral. Such an exponent is given by the two finite levels; $c_v = 0$ is available almost everywhere. Integral unipotent matrices have integral inverses by the finite geometric series.

In the chosen row lattice bases form the pair of matrix polynomials

$$s(t+\delta)s(t)^{-1} - 1.$$

Replace t_j by $\pi_v^{-a_{j,v}} T_j$ and leave δ_j as indeterminates. Let $b_v = c_v$ plus the larger of zero and the maximum of the negatives of the coefficient valuations of these polynomials, with the same zero convention. All these quantities use the given rows and rational matrices and require no choice of integral lifts of the class.

Theorem 117.11 (Local cancellation on finite residue classes)**.** *For a fixed original full index i, test one representative of each residue class in*

$$B_v/(\pi_v^{b_v}\mathcal{O}_v)^r, \qquad M_v = q_v^{rb_v + \sum_j a_{j,v}}.$$

At t, test $z(t)$ against the original local integrality and marking conditions. If supported, attach the phase $z_v(s(t))$ of Theorem 117.8; otherwise attach zero. These values are independent of the representatives. The sum of the values is zero if and only if the signed local factor A_v is zero, for each globally supported index. If m_v is the least number of these residue classes in any one supported K_v-orbit, the sum is exactly $m_v A_v$. In particular,

$$|\Omega_v| \le W_v \le M_v.$$

All $a_{j,v}$ and b_v vanish outside a finite set determined by the rational input, row lattices and original levels. The conclusions include $r = 0$, with a single test.

Proof. If z is supported, it is in $\mathcal{L}_v$. Polynomial evaluation gives $P(z) \in B_v$. Conversely $z(t)$ and the source and target matrices of $s(t)$ give the actual local orbit point and its two character arguments. For $t \in B_v$ and $\delta \in (\pi_v^{b_v}\mathcal{O}_v)^r$, the definition of b_v gives

$$s(t+\delta)s(t)^{-1} \in K_v.$$

Indeed all monomials in the matrix difference contain at least one δ, since the difference vanishes identically at $\delta = 0$. Substituting $\pi_v^{b_v}$ times integers in these variables makes all its entries divisible by $\pi_v^{c_v}$ on the whole indicated box. Thus each residue class belongs to a single orbit; the support and phase are constant there. Counting all residue classes gives M_v.

Give the additive parameter box its product Haar measure. By Lemma 117.10 each small residue class has the same quotient measure. The measure of $K_v w D_v/D_v$ is $\operatorname{vol}(K_v)/\operatorname{vol}(K_{v,w}^D)$: integrate the indicator of $K_v w$ first in the right D_v variable; over every point of this orbit its integral in that variable is $\operatorname{vol}(K_{v,w}^D)$ using representatives in $K_v w$. Thus the orbit with largest stabilizer volume contains the least number m_v of small residue classes. All other orbit counts divided by m_v are precisely the integers $n_v(\omega)$ of Theorem 117.8. This proves the equality of signed sums. Dividing the number of supported residue classes by $m_v \ge 1$ gives $W_v \le M_v$.

Only finitely many coefficients of the rational polynomials are involved. The rational bases used to construct them give integral lattice bases away from finitely many primes, and the levels are full there. Clearing the denominators of the bases in both directions, of the polynomials P_j, s, s^{-1}, and including the finite level primes, therefore leaves $a_{j,v} = b_v = 0$ at every other place. At the exceptional places the same polynomial construction takes place over the fraction field and the residue classes use the resulting denominator exponents. □

Corollary 117.12 (An ideal norm bound for the signed coefficient)**.** *Form the integral ideal of F*

$$\mathfrak{b}_\eta = \prod_{v<\infty} \mathfrak{p}_v^{\,rb_v+\sum_j a_{j,v}}.$$

For each supported full index, divide $m_i(\eta)$ by the row volume and by the covolume of its lift γ_* from Theorem 117.8. Every complex conjugate of the resulting algebraic integer has absolute value at most $N_{F/\mathbb{Q}}\mathfrak{b}_\eta$. The number of integral lifts at that index is at most this same ideal norm. In the upper bound for its normalized terminal valuation in Corollary 117.9, one can use $(rb_v + \sum_j a_{j,v}) \log q_v$ in place of $\log W_v$.

Proof. The local factors are weighted sums of roots of unity. Every conjugate is bounded by W_v. Multiply the local bounds of Theorem 117.11 over the finite places and use the two product statements of Theorem 117.8. The valuation bound follows by the same substitution in the norm estimate. Outside the indicated finite set there is one residue class and the supported factor is a unit of absolute value one at every embedding. □

This bound is effective for given rational frames and uses their actual finite levels. A bound for these ideals across the rational classes that contribute on $[V, V^{3/2}]$, together with control of the real tensors and their signed sum, is still required for the unrestricted period estimate.

117.5 The signed coefficient in the measures of the two rows

Use the original transverse hyperplane rows over $E = E_x$, with $F = E^+$ totally real and $\Delta_F = |\operatorname{disc} F|$. For the class representative η, let $U = N_t \times N_s$, $D = D_\eta$, and $r = \dim_F U - \dim_F D$. Write $\Gamma_b = N_b(F) \cap \prod_{v<\infty} K_{b,v}$ for the original integral row radical. Fix rational invariant volume forms on the two row radicals and give U their product form ω_U. Use Lebesgue measure on F_∞ and give $\mathcal{O}_v$ additive volume one at each finite place. These choices specify a real measure du_∞ and a product of finite measures on U. Put

$$v_U = \operatorname{vol}_{du_\infty}((\Gamma_t \times \Gamma_s)\backslash U_\infty).$$

It is the product of the two row unipotent covolumes.

Take the rational directions X_j constructed in Lemma 117.10. Give D the invariant form whose value on a tangent basis Y at the identity is $\omega_U(X_1, \dots, X_r, Y)$, using its embedding in U. This specifies the Haar measure on D_∞ in both $m_i(\eta)$ and I_η of Theorem 117.6. The measure on the original real group G_∞ is the one used for the ordinary period; on $D_\infty\backslash G_\infty$ use its quotient by the specified D_∞ measure.

Lemma 117.13 (Covolume with additive integers of volume one)**.** *For an F-split unipotent group B of dimension d, give its local groups the measures of any one rational invariant top form with the additive conventions above. Then* $\operatorname{vol}(B(F)\backslash B(\mathbb{A}_F)) = \Delta_F^{d/2}$.

Proof. For one additive coordinate, finite additive approximation reduces every finite coordinate to the product of the rings of integers. The residual real lattice is $\mathcal{O}_F$ in its real embeddings. The matrix of embeddings of an integral basis has determinant of absolute value $\sqrt{\Delta_F}$, by the trace definition of the discriminant. This proves the assertion in dimension one. A rational scalar multiplying a form has product of its normalized absolute values equal to one, by the number field product formula.

For the induction choose a rational codimension-one ideal in the nilpotent Lie algebra containing its commutator and use exponential coordinates. The corresponding normal unipotent subgroup has additive quotient and a rational one-parameter section. In particular the group

on rational points maps onto the rational points of the quotient. Disintegrate first over its additive adelic quotient. Each fibre has the covolume of the normal subgroup modulo its rational points: the section identifies the fibre with that quotient up to conjugation, and conjugation by the section has Jacobian one on the normal subgroup. Choose compatible rational forms at the identity. The Haar measures disintegrate as their product at all places, and changing a rational form by a scalar costs one globally. Multiplying the induction hypothesis by $\sqrt{\Delta_F}$ proves the assertion. Dimension zero uses the trivial group and the same product formula. □

For each v, use $a_{j,v}, b_v$ and the residue classes of Theorem 117.11. Write $S_{v,i}(\eta)$ for the sum of their original phases with zero on unsupported classes. In particular this is the sum over the explicit residue classes themselves, including their multiplicities. Define two ideals of F by

$$\mathfrak{a}_\eta = \prod_v \mathfrak{p}_v^{\sum_j a_{j,v}}, \qquad \mathfrak{c}_\eta = \prod_v \mathfrak{p}_v^{b_v}.$$

Both products use finitely many nonzero exponents.

Theorem 117.14 (The exact finite factor with row covolumes)**.** *For a supported full index of the given relevant rational class set $C_i(\eta) = \Delta_F^{r/2} m_i(\eta)/(c_{t,s} v_U)$. With the original row volume $c_{t,s}$ and the measures above,*

$$C_i(\eta) = \prod_{v<\infty} q_v^{-r b_v} S_{v,i}(\eta).$$

Almost all factors are one. For every complex embedding of the coefficient field,

$$|\sigma C_i(\eta)| \le N_{F/\mathbb{Q}}\mathfrak{a}_\eta, \qquad (N_{F/\mathbb{Q}}\mathfrak{c}_\eta)^r C_i(\eta) \text{ is an algebraic integer.}$$

Consequently, when $C_i(\eta) \neq 0$ and T is the terminal rational prime on the interval, every valuation normalized by $\mathrm{val}_T(T) = 1$ satisfies

$$\mathrm{val}_T\, C_i(\eta) \ge -r \sum_{v \mid T} b_v.$$

The last expression uses complete splitting of T in F.

Proof. Write du_v, dd_v for the local measures on U_v, D_v. In the coordinates $s(t)d$ of Lemma 117.10, the quotient du_v/dd_v is exactly additive product measure on t. The lemma gives proportionality. At the origin the derivative in the parameter directions is $X_1, \dots, X_r$; evaluation of the two forms on those vectors followed by a basis of the stabilizer gives ratio one. Hence each residue class in that lemma's coordinates has quotient volume $q_v^{-r b_v}$ at the precision of Theorem 117.11. The integral $L_{v,i}$ of the signed finite test on U_v/D_v is therefore $q_v^{-r b_v} S_{v,i}(\eta)$.

Alternatively integrate on each K_v-orbit. The volume calculation in Theorem 117.11 gives

$$\frac{L_{v,i}}{\mathrm{vol}_{du_v}(K_v)} = \sum_{\omega \in \Omega_v} \frac{z_v(\omega)}{\mathrm{vol}_{dd_v}(K_{v,\omega}^D)}.$$

The product of the sums enumerates exactly the integral lifts at i by Theorem 117.8. For each lift the product of the phases is its full phase, and its real stabilizer covolume is the reciprocal product of the displayed stabilizer volumes times $\Delta_F^{\dim_F D/2}$, by Lemma 117.13 and finite approximation for D. Thus

$$m_i(\eta) = c_{t,s} \Delta_F^{\dim_F D/2} \frac{\prod_v L_{v,i}}{\prod_v \mathrm{vol}_{du_v}(K_v)}.$$

Finite approximation for U gives $v_U \prod_v \mathrm{vol}_{du_v}(K_v) = \Delta_F^{\dim_F U/2}$ by the same lemma. Substitution proves the asserted formula. All real stabilizer measures introduced here cancel against the associated real quotient measures in $m_i(\eta) I_\eta$.

Each conjugate of the finite sum $S_{v,i}(\eta)$ has absolute value at most the total number of residue classes, $q_v^{rb_v + \sum_j a_{j,v}}$, and each such sum is an algebraic integer. Its denominator in the formula therefore cancels the factor $q_v^{rb_v}$ in the absolute bound. Multiplication gives the two assertions about $\mathfrak{a}_\eta, \mathfrak{c}_\eta$. Only places over T contribute to the T-valuation of the denominator. They have $q_v = T$, which proves the valuation inequality. □

If the original real Haar measures on the two row radicals are given in a different convention, let their product be $\kappa\, du_\infty$. Use $\kappa\, dd_\infty$ on the stabilizer. The row covolume and $m_i(\eta)$ both acquire the factor κ, so the same $C_i(\eta)$ applies with those given row measures. The quotient tensor I_η acquires $1/\kappa$, as the measure on G_∞ is fixed.

Proposition 117.15 (The value ideals of the orbit coordinates)**.** *For each P_j let J_j be the fractional ideal of F whose localization is generated by the values $P_j(\mathcal{L}_v)$, with $\mathcal{L}_v$ the lattice of integral row maps. Set $J = \prod_{j=1}^r J_j$, taking the unit ideal for $r = 0$. Then*

$$|\sigma C_i(\eta)| \le (N_{F/\mathbb{Q}} J)^{-1}$$

for every coefficient embedding. The local ideals can be calculated on finite grids using the row lattices. For each common compact summand μ, the ratio $|b_\mu(\eta)|/(N_{F/\mathbb{Q}} J)$ is unchanged when the parameters t_j are individually multiplied by nonzero elements of F and the stabilizer form is adjusted as above.

Proof. Each coordinate polynomial is nonzero. On a full lattice a nonzero polynomial has some nonzero value: multiples of a rational basis in that lattice give a Cartesian product of infinite sets, on which a nonzero polynomial cannot vanish identically. Choose such a global integral input for each polynomial. Its value is a unit away from finitely many places. The coefficients on the row lattice are themselves integral almost everywhere, so the stated local ideals define nonzero fractional ideals.

For an explicit local calculation take a $\mathbb{Z}_p$-basis of $\mathcal{L}_v$ at residual characteristic p. In the resulting scalar variables P_j is a polynomial over F_v; write d_a for its degree in variable a. Evaluate it at each integer tuple with coordinates $0, \dots, d_a$. The fractional $\mathcal{O}_v$-ideal of these finitely many values is the whole value ideal. Indeed the Newton expansion in products of $\binom{x_a}{h_a}$ has as coefficients the iterated forward differences of the polynomial at zero. Repeated one-variable finite differences give this expansion on integer inputs, hence as a polynomial identity. Each coefficient is an integer combination of the sampled values. For $x_a \in \mathbb{Z}_p$ the binomial polynomials are integral by continuity and density of nonnegative integers. This proves one containment, and the sample belongs to the lattice for the reverse containment.

By the coordinate lemma the supported parameters lie in $\prod_j (J_j)_v$. Its additive volume is the product of the local inverse ideal norms, with the convention that $\pi_v^a \mathcal{O}_v$ has volume q_v^{-a} for any integer a. Every conjugate phase has modulus one, so integrating the absolute value on this box bounds the local factor $L_{v,i}$ in the proof of Theorem 117.14. Taking the product proves the inequality.

For $t_j' = \ell_j t_j$, the new directions are $X_j' = X_j/\ell_j$ and the ideals are $J_j' = (\ell_j) J_j$. Put $L = |N_{F/\mathbb{Q}} \prod_j \ell_j|$. The real measure on D_∞ prescribed by the volume form scales by $1/L$; the quotient measure in each real tensor therefore scales by L. Thus b_μ and $N_{F/\mathbb{Q}} J$ acquire precisely the same positive factor, proving the invariance. □

Corollary 117.16 (A signed formula in the original row normalization)**.** *Fix the projected hyperplanes, the transport, and the marked index as in Corollary 117.7. In the sum over supported relevant rational classes let* $r_\eta = \dim_F U - \dim_F D_\eta$*, and integrate* $b_\mu(\eta)$ *with the corresponding stabilizer measure just specified. The complete contribution* Z *of these cells then satisfies*

$$\frac{\|Z\|_{\mathrm{HS}}^2}{|c_{t,s}|^2 v_U^2} = \sum_\mu \frac{1}{d_\mu} \left| \sum_{[\eta]} \Delta_F^{-r_\eta/2} b_\mu(\eta) \prod_{v<\infty} q_v^{-r_\eta b_v} S_{v,i}(\eta) \right|^2 .$$

It holds with any cutoff by frame height as well.

Proof. Insert Theorem 117.14 into Corollary 117.7 before taking the squared norm. The row covolumes are fixed throughout the sum. Each scalar times its real tensor is the original contribution of its class; the absolute integrability established in that corollary justifies summation with the indicated measures. □

At a fixed rational class the absolute value of its summand inside the last square is consequently at most $\Delta_F^{-r_\eta/2}|b_\mu(\eta)|/(N_{F/\mathbb{Q}}J)$. Summing the signed contributions over the rational classes and the different hyperplanes at the required theta scale, and proving the cohomology estimate of Section 6.6, remain open on the common intervals.

117.6 Summing the signed period over different hyperplanes

Fix x on an interval of Section 6.6. Keep the field $E = E_x$, its real subfield F of degree f, the two original rows, the full marked index i, and the rational transport $j : P_s \to P_t$ of Corollary 117.7. Either terminal frequency is allowed at either end. Identify the two positive spaces by j and write $P = P_t$. A transverse cell of common rank $k-1$ now gives one rational hyperplane $H \subset P$. Choose one complex embedding of E above each real place τ of F. Put $K = \prod_\tau U(P_\tau)$ and write V_s, V_t for the two irreducible column spaces, with their given unitary metrics. Inner products are linear in the second argument.

Let $Z_H : V_s \to V_t$ be the signed contribution at this H, before the marked projection. In the notation of Corollaries 117.7 and 117.16 it is

$$Z_H = \sum_\mu \frac{J_{H,\mu}}{d_\mu} \sum_{[\eta] \text{ at } H} m_i(\eta) b_{H,\mu}(\eta). \tag{117.1}$$

Here $J_{H,\mu}$ is the partial isometry between the two copies of the common compact summand. The factor $m_i(\eta)$ includes the finite phases and the covolume of the real stabilizer in the measures of Theorem 117.14. We allow the frame cutoff of Corollary 117.4 throughout. It keeps whole rational classes, by Theorem 117.6. We first compute the angles between the spaces containing the different Z_H.

Lemma 117.17 (The matrix coefficient for two unit normals)**.** *Let* $k \geq 3$*, let* e *be a unit vector in* $\mathbb{C}^k$*, and let* $K_e = U(k-1)$ *fix* e*. An irreducible representation of* $U(k)$ *with a* K_e*-fixed vector has highest weight* $(a, 0, \dots, 0, -b)$*, where* $a, b \geq 0$*. Its fixed space is a line. There are unit vectors* $\xi_{a,b}(v)$ *on the corresponding fixed lines for all unit* v*, equivariant under* $U(k)$*. If* $z = \langle v, w \rangle$*,* $d^2 = 1 - |z|^2$*, and* $q = \min(a, b)$*, then*

$$\begin{aligned} \langle \xi_{a,b}(v), \xi_{a,b}(w) \rangle = \Phi_{a,b}(z) &= \sum_{m=0}^{q} c_m z^{a-m} \bar{z}^{b-m} d^{2m}, \\ c_0 = 1, \qquad c_{m+1} &= -\frac{(a-m)(b-m)}{(m+1)(k-1+m)} c_m. \end{aligned} \tag{117.2}$$

Put $p = k - 2$ and $h = |a - b|$. At $d > 0$,

$$\begin{aligned} |\Phi_{a,b}(z)| &\le \min\{1,\, 2^{h/2} p!(qd^2)^{-p}\} \quad (q \ge 1), \\ |\Phi_{a,b}(z)| &= (1 - d^2)^{h/2} \qquad\qquad\quad (q = 0,\ h > 0). \end{aligned} \tag{117.3}$$

For $a = b = 0$ the coefficient is one everywhere.

Proof. The unitary branching rule used in Theorem 98.2 gives multiplicity one for each interlacing weight. Interlacing with the zero weight of $U(k-1)$ forces the asserted highest weight and permits every pair a, b.

To fix the phases take the trace-free part of $\mathrm{Sym}^a \mathbb{C}^k \otimes \mathrm{Sym}^b \overline{\mathbb{C}^k}$. Here the trace contracts one vector with one conjugate vector. When both degrees are positive the contraction is surjective. Indeed, on identifying each tensor s with the polynomial $\langle s, T(v)\rangle$, where $T(v) = v^{\otimes a} \otimes \bar{v}^{\otimes b}$, its adjoint is, up to a nonzero positive factor, multiplication by $\|v\|^2$, which is injective. This polynomial identification follows by expanding in monomials in $v, \bar{v}$, which are independent as polynomials in the real and imaginary coordinates. The trace-free space contains the highest vector $e_1^{\otimes a} \otimes \bar{e}_k^{\otimes b}$. Its dimension, obtained by subtracting the dimension in degrees $a-1, b-1$ when both are positive, is

$$\frac{a+b+k-1}{k-1} \binom{a+k-2}{k-2} \binom{b+k-2}{k-2}.$$

This is the Weyl dimension of the asserted highest weight, so the trace-free space is exactly that irreducible representation. The same dimension computation with zero subtracted applies when one of the degrees is zero.

Let Π be the orthogonal projection onto this space. The $T(v)$ for unit v span the tensor space by the polynomial identification and homogeneity. Thus $\Pi T(e) \ne 0$, since the unitary group is transitive on unit vectors. Set $\xi_{a,b}(v) = \Pi T(v)/\|\Pi T(e)\|$ on the unit sphere. For $v = e$, the coefficient against $T(w)$, normalized to take value one at e, is a harmonic polynomial in w of bidegree (a, b). Applying the complex Laplacian $\sum_j \partial_{w_j} \partial_{\bar{w}_j}$ contracts one index of each kind, so trace-freeness gives harmonicity. Put $t = \langle e, w\rangle$ and $u = \|w - te\|^2$. Invariance under $U(e^\perp)$ forces the polynomial to be a sum of the monomials $t^{a-m} \bar{t}^{b-m} u^m$. Indeed the scalar circle on $e^\perp$ forces equal perpendicular degrees and transitivity on its unit sphere gives a multiple of u^m in those degrees. The perpendicular Laplacian sends u^{m+1} to $(m+1)(k-1+m)u^m$. Equating coefficients after applying the full Laplacian gives the recurrence, and evaluation at e gives $c_0 = 1$. Restriction to the sphere and equivariance give (117.2) for every pair. These are the disk polynomials from the unitary sphere construction of Koornwinder [1].

For the estimate let ζ be the first coordinate of a uniform unit vector in $\mathbb{C}^{k-1}$. With $p = k - 2$, the variable $Y = 1 - |\zeta|^2$ has density $p y^{p-1}$ on $[0, 1]$. To see this, normalize $k - 1$ independent standard complex Gaussians to unit length. Their squared lengths are independent unit exponentials. Conditional on the sum S of the last p squared lengths, the probability that the first squared length exceeds $tS/(1-t)$ is $\exp(-tS/(1-t))$. Integration gives $\Pr(|\zeta|^2 \ge t) = (1-t)^p$ for $0 < t < 1$. It follows that $\mathbb{E}|\zeta|^{2m} = m!/(k-1)_m$, where $(s)_m = s(s+1)\cdots(s+m-1)$ and $(s)_0 = 1$. Expansion and rotation invariance of ζ now give

$$\Phi_{a,b}(z) = \mathbb{E}\big[(z + \mathrm{i}d\zeta)^a (\bar{z} + \mathrm{i}d\bar{\zeta})^b\big].$$

Indeed only equal powers of ζ and $\bar{\zeta}$ survive, and their coefficients are those of (117.2). The product of the absolute values of the two bases is $|z + \mathrm{i}d\zeta||z - \mathrm{i}d\zeta| \le |z|^2 + d^2|\zeta|^2 = 1 - d^2 Y$, by the arithmetic–geometric mean inequality on the squared moduli. Each base has modulus

at most $|z|+d\le\sqrt{2}$. For $q\ge 1$ the expectation is therefore bounded in absolute value by

$$2^{h/2}p\int_0^1 e^{-qd^2y}y^{p-1}\,dy\le 2^{h/2}p!(qd^2)^{-p}.$$

The coefficient of two unit vectors also has modulus at most one. For $q=0$ the sum in (117.2) has just one term. □

Theorem 117.18 (The signed sum over all projected hyperplanes)**.** *Give $\mathcal{V}=\operatorname{Hom}(V_s,V_t)$ its Hilbert–Schmidt norm and the action $A\mapsto\rho_t(g)A\rho_s(g)^{-1}$ of K, using j on the source. In an orthogonal isotypic decomposition $\mathcal{V}=\bigoplus_\lambda \mathcal{H}_\lambda\otimes M_\lambda$, write Π_λ for the projection. Only the types $\lambda=((a_\tau,0,\dots,0,-b_\tau))_\tau$ can occur in any Z_H. Choose a unit normal $n_{H,\tau}$ at every place. There is a unique $b_{H,\lambda}\in M_\lambda$ such that*

$$\Pi_\lambda Z_H=\xi_\lambda(n_H)\otimes b_{H,\lambda},\qquad \xi_\lambda(n_H)=\bigotimes_\tau \xi_{a_\tau,b_\tau}(n_{H,\tau}).$$

In particular $b_{H,\lambda}$ is given by contracting $\Pi_\lambda Z_H$ from (117.1) with the indicated unit vector in its first factor. Then

$$\begin{aligned}\Big\|\sum_H Z_H\Big\|_{\mathrm{HS}}^2&=\sum_\lambda\sum_{H,H'}K_\lambda(H,H')\langle b_{H,\lambda},b_{H',\lambda}\rangle,\\ K_\lambda(H,H')&=\prod_\tau\Phi_{a_\tau,b_\tau}(\langle n_{H,\tau},n_{H',\tau}\rangle).\end{aligned}\tag{117.4}$$

This holds for every subset of hyperplanes and for the full absolutely convergent transverse sum at i,j.

Proof. The proof of Corollary 117.7 gives equivariance of each class integral under $\prod_\tau U(H_\tau)$, which fixes the normal vectors pointwise. After transporting the source by j, this says precisely that Z_H is fixed by this subgroup in $\mathcal{V}$. An isotypic projection commutes with the subgroup. Its fixed subspace in each irreducible tensor factor is the product of the fixed lines of Lemma 117.17, or is zero. The multiplicity space has trivial action. This proves the asserted factorization, even for an irreducible type of multiplicity greater than one. Expand the squared norm, using orthogonality of the isotypic spaces and the inner product of pure tensors. Formula (117.2) gives every entry of (117.4). Replacing $n_{H,\tau}$ by $un_{H,\tau}$, $|u|=1$, multiplies its fixed vector by $u^{a_\tau-b_\tau}$ and the multiplicity vector by the inverse phase, leaving each summand unchanged. Absolute summability of the original tensors was proved in Corollary 117.7. It implies $\sum_H\|Z_H\|_{\mathrm{HS}}<\infty$, justifying the double sum at fixed x.

In the trivial type the contribution is $\|\sum_H b_{H,0}\|^2$. For irreducible columns this type occurs exactly when the transported representations are equivalent. Indeed its vectors are K-intertwiners. If $U:V_s\to V_t$ is a unitary intertwiner, Schur's lemma gives the unit basis $U/\sqrt{\dim V_s}$ for that line. Its contribution is then $\big|\sum_H\operatorname{tr}(U^*Z_H)\big|^2/\dim V_s$. □

For a rational normal line ℓ in P, let $M_\ell=\widehat{L}_t\cap\ell$, where $\widehat{L}_t$ is the self-dual target lattice of Lemma 113.6. Set $B_\ell=[M_\ell^\sharp:M_\ell]^{1/(2f)}$, taking the dual inside ℓ. For two lines let $d_\tau(\ell,\ell')=(1-|\langle n_\tau,n'_\tau\rangle|^2)^{1/2}$ with unit vectors in their respective fibres.

Proposition 117.19 (Separation of rational normals in the row lattice)**.** *For a cell with common lattice R projecting to $H=\ell^\perp$ one has $1\le B_\ell\le h_R$. For distinct lines,*

$$\prod_\tau d_\tau(\ell,\ell')^2\ \ge\ (B_\ell B_{\ell'})^{-f}.$$

Put $D = 2(k-1)$. The number of lines with $B_\ell \le B$, for $B \ge 1$, is at most $(2B)^{Df}$. In a product of closed projective balls of radii $0 < R_\tau \le 1$ with $R_* = \prod_\tau R_\tau \ge B^{-f}$, their number is at most

$$3^{Df} B^{Df} R_*^{D}.$$

Proof. Put $L = \widehat{L}_t$, $S = L \cap H$, and $M = L \cap \ell$. The orthogonal projections of L onto H, ℓ have images $S^\sharp, M^\sharp$. In fact S, M are saturated, so an integral functional extends locally to L and is represented there by self-duality. The same argument uses the paired factors at a split place. The projections both induce isomorphisms from $L/(S+M)$ onto the respective discriminant modules. If the projection lies in S or M, one subtracts it to land in the other lattice, giving injectivity. The image of R lies in S with finite index by Theorem 113.7. Changing lattice by this index multiplies the discriminant order by $[S : R]^2$, as follows by taking the index in the lattice and in its conjugate dual. Consequently $[R^\sharp : R] = [S : R]^2 [M^\sharp : M]$, which proves the height claim.

Write $M_\ell = Iv$ and $M_{\ell'} = Jw$ for fractional ideals of E. For the Hermitian form put $s = \langle v, v \rangle$, $t = \langle w, w \rangle$, $u = \langle v, w \rangle$. The ideals $I\bar{I}(s)$ and $J\bar{J}(t)$ have absolute norms B_ℓ^{2f} and $B_{\ell'}^{2f}$, respectively. For example, the dual of Iv is $s^{-1}\bar{I}^{-1}v$. Integrality of L gives $\bar{I}J(u) \subseteq \mathcal{O}_E$ and its conjugate inclusion. It follows that the ideal

$$I\bar{I}J\bar{J}(st - u\bar{u})$$

is integral. The generator in parentheses is totally positive, since two distinct rational lines remain independent under every embedding. The absolute norm of this ideal is at least one. Dividing it by the norms of the two one-line ideals and taking a square root gives the product bound, since at τ the ratio of Gram determinants is d_τ^2.

The projective distance d_τ is the operator norm of the difference of the two orthogonal line projections. This follows by diagonalizing their difference on the span of the lines, where the two eigenvalues for distinct lines are $\pm d_\tau$. In particular it satisfies the triangle inequality. The invariant probability of a projective ball of radius $r \le 1$ is r^D. Indeed the Gaussian argument of Lemma 117.17 in complex dimension k gives $\Pr(1 - |\langle e, n \rangle|^2 \le r^2) = r^{2(k-1)}$. Distinct lines of heights at most B satisfy $\prod_\tau d_\tau \ge B^{-f}$. The open products of balls of common radius $1/(2B)$ about these lines are disjoint, proving the first count by product probability. For the second count take the smaller radii $r_\tau = R_\tau / (2B R_*^{1/f})$ about each line in the indicated product. They satisfy $r_\tau \le R_\tau/2$ and $\prod_\tau (2r_\tau) = B^{-f}$, so the products of open balls are again disjoint. Their union lies in the product with radii $R_\tau + r_\tau$ about the original centre. Dividing its volume bound by the volume of one small product gives $(1 + 2BR_*^{1/f})^{Df} \le 3^{Df} B^{Df} R_*^{D}$. The volume bound for an enlarged radius greater than one follows from total mass one. These arguments apply to every finite set and hence to all the lines. □

Corollary 117.20 (Exact cross terms at height one)**.** *At i, j there are at most k distinct hyperplanes with $B_{H^\perp} = 1$. In (117.4), on any set of these hyperplanes the squared norm in type λ equals*

$$(1 - \kappa_\lambda) \sum_H \|b_{H,\lambda}\|^2 + \kappa_\lambda \left\| \sum_H b_{H,\lambda} \right\|^2 .$$

Here $\kappa_\lambda = 0$ if some $a_\tau \ne b_\tau$. Otherwise

$$\kappa_\lambda = \prod_\tau \frac{(-1)^{a_\tau}}{\binom{k-2+a_\tau}{a_\tau}}.$$

In particular, if the two columns have different scalar central characters at any real place, the squared norm of this entire height-one sum is $\sum_H \|Z_H\|_{\mathrm{HS}}^2$.

Proof. Proposition 117.19 forces the product of the squared distances of two distinct height-one lines to be at least one. Each squared distance is at most one, so the normals are orthogonal at every place. Any one place gives the cardinality bound. At $z = 0$ the polynomial in (117.2) vanishes for unequal degrees. For $a = b$ only $m = a$ survives, giving $(-1)^a a!/(k-1)_a$. The diagonal of the kernel is one and every off-diagonal entry is the same real κ_λ. Expanding the two squares proves the formula, including the constant type. Finally, at each place the scalar uI acts on $\mathcal{V}$ by the target central character divided by the source central character. On a type $(a_\tau, 0, \dots, -b_\tau)$ its value is $u^{a_\tau - b_\tau}$. Different central characters therefore force unequal degrees in every occurring type at that place. Summing the resulting equalities proves the last assertion. □

Theorem 117.21 (An error bound at arbitrary bounded height)**.** *For a set of $N \geq 1$ hyperplanes of normal heights at most B, put $p = k-2$, $D = 2(k-1)$, and define, for $q = \min(a,b)$ and $h = |a-b|$,*

$$\omega_{a,b} = \begin{cases} \min\{1,\ 4^p 2^{h/2} p!/q^p\}, & q \geq 1, \\ \min\{1,\ (8p/(\mathrm{e}h))^p\}, & q = 0,\ h > 0, \\ 1, & a = b = 0. \end{cases}$$

For every occurring type set

$$D_\lambda(B,N) = \min\left\{ N-1,\ (4/3)^f (3B)^{Df} \prod_\tau \omega_{a_\tau,b_\tau} \right\}.$$

With $S_\lambda = \sum_H \|b_{H,\lambda}\|^2$ one has

$$\left| \left\| \sum_H \Pi_\lambda Z_H \right\|_{\mathrm{HS}}^2 - S_\lambda \right| \leq D_\lambda(B,N) S_\lambda.$$

Proof. Fix a line from the set as centre. For each other line all its distances from the centre are positive. Assign the unique integers $n_\tau \geq 0$ with $2^{-n_\tau - 1} < d_\tau \leq 2^{-n_\tau}$. The product of the upper radii is at least B^{-f} for an occupied box by Proposition 117.19. That proposition bounds the number in the box by $(3B)^{Df} 2^{-D\sum_\tau n_\tau}$. At each place the modulus of the kernel there is bounded by $\omega_{a,b} 2^{2p n_\tau}$. For $q \geq 1$ this follows from (117.3) and the bound one. For $q = 0$, $h > 0$, the modulus is at most $\exp(-h d_\tau^2/2)$. Use $y^p e^{-y} \leq (p/\mathrm{e})^p$ at $y = h d_\tau^2/2$ and the same lower bound on d_τ. The case $a = b = 0$ is immediate. Thus the sum of $|K_\lambda|$ off the diagonal in this box is at most $(3B)^{Df} (\prod_\tau \omega_{a_\tau,b_\tau}) 4^{-\sum_\tau n_\tau}$, since $D - 2p = 2$. Summing over all nonnegative indices gives the factor $(4/3)^f$. The row sum is also at most $N-1$. These bounds hold at every centre. Bound the off-diagonal terms in (117.4) using Cauchy–Schwarz in M_λ and $2uv \leq u^2 + v^2$. Symmetry of the entry moduli yields the maximum off-diagonal row sum times S_λ, as asserted. □

To apply the theorem to the projected period in Corollary 117.4, retain its frame cutoff and take $B = H_L$ from that corollary. Its proof gives $h_R < H_L$ for every retained cell, so Proposition 117.19 applies at both terminal frequencies. For each occupied i, j write $N_{i,j}$ for its number of retained hyperplanes and $S_{i,j,\lambda}$ for the sum in Theorem 117.21. If E_L denotes the omitted-cell bound (the second term of Corollary 117.4), the original normalized marked contribution satisfies

$$\|\mathcal{P}\|_{\mathrm{HS}} \leq E_L + \left(\alpha\, n(H_L) \sum_{i,j,\lambda} \bigl(1 + D_\lambda(H_L, N_{i,j})\bigr) S_{i,j,\lambda} \right)^{1/2}.$$

Indeed the preceding theorem bounds the squared norm at each i, j before the marked projection. The operator bound for that projection is precisely the one in the cited corollary, which also permits $n(H_L) = 1$ when $H_L \le Q/2$. All norms here use the actual signed class coefficients in (117.1). The constant type can still contain the full squared sum of their traces. Bounding those traces, the remaining class integrals and the omitted cells at the scale required in Section 6.6 is open, as is the required uniform estimate for the residual cohomology on the same intervals.

117.7 The determinant of a pair of frames and its signed period

Keep the two rows, the rational transport j, the marked index i and the field $E = E_x$ of Section 117.6. Fix one of the hyperplanes in that section and write $H_b \subset P_b$ for it at end $b = s, t$. Put $F = E^+$, $f = [F : \mathbb{Q}]$, $\ell_b = P_b \cap H_b^\perp$, and $d = l + 1$, where l is the number of outer isotropic vectors. Use the integral Witt splitting of Lemma 115.7 at each end. Denote the ordered frame by $y_b = (e_{b,1}, \dots, e_{b,l}, z_b)$, its span by Y_b, and its exterior product by Φ_b. In the smaller spaces $V_b' = H_b^\perp$ of Theorem 117.5, a rational cell has a representative η with $V_t' = Y_t \oplus \eta Y_s$. We extend its map $V_s' \to V_t'$ to the whole space using $j|_{H_s}$. Its determinant is fixed by the determinant of j.

Write $R_b(\alpha)$ for multiplication by α on ℓ_b and the identity on its perpendicular, for $\alpha \in E^1 = \{\alpha \in E^\times : \alpha\bar\alpha = 1\}$. Let $C(\eta)$ be the d-by-d matrix with entries $\langle y_{t,u}, \eta y_{s,v}\rangle$, with the form linear in its second variable. Its class under lower unitriangular row operations and upper unitriangular column operations over E will be denoted by $[C]$.

Theorem 117.22 (The rational classes above one cross matrix)**.** *Fix an occurring $[C]$, and choose one rational representative η_0 above it. Every rational unipotent double class above $[C]$ at the given hyperplanes and transport has exactly one representative of the form*

$$\eta_\alpha = R_t(\alpha)\eta_0 R_s(\alpha)^{-1}, \qquad \alpha \in E^1.$$

More precisely, the indicated representatives form a system of representatives without repetition. The parameter of any class is read from its top exterior vector

$$\Omega_\eta = \Phi_t \wedge \eta\Phi_s \in \det V_t', \qquad \Omega_\eta = \alpha\Omega_{\eta_0}.$$

Here the representative used to form Ω_η lies in the smaller spaces as above.

Proof. On Y_b the reduced unipotent radical N_b' gives all upper unitriangular changes. An upper unitriangular change on the outer space extends by its inverse adjoint on the opposite space. To send z_b to $z_b + v$, $v \in X_b$, use the unipotent matrix $n(A)$ in the proof of Theorem 117.5 with $Az_b = v$, vanishing on the positive middle space. Both maps fix H_b pointwise. Together they realize every required change. Acting by N_t', N_s' thus gives exactly the indicated unitriangular equivalence on cross matrices. It preserves Ω_η, since the frame wedges are fixed at their respective ends and the target unipotent has determinant one on V_t'.

Take two classes over $[C]$. By the preceding frame changes choose representatives η, η' with identical cross matrices. The ordered bases $(y_t, \eta y_s)$ and $(y_t, \eta' y_s)$ of V_t' have identical Gram matrices. Their diagonal blocks are the two fixed frame Gram matrices. The rational linear map g_t taking the first basis to the second is therefore unitary and fixes y_t pointwise. Then $g_s = \eta'^{-1} g_t \eta$ fixes y_s pointwise. The determinants of g_t, g_s agree because the two class maps have the same determinant. At either end a unitary map stabilizing the ordered frame up to an upper unitriangular change acts on the middle quotient by the identity on z_b and by a norm-one scalar on ℓ_b. That scalar is its determinant, since the two outer diagonal blocks have

determinant one. After division by R_b of that scalar the map lies in N_b', by its trivial action on the middle quotient and its unitriangular action on the outer flag. It follows that the two classes are related by the asserted simultaneous rotations.

Conversely the rotations normalize the two radicals, fix each frame pointwise, preserve the transport on H_s and leave the determinant of the class map fixed. They preserve the spanning condition. The top exterior vector changes by $\det(R_t(\alpha)|_{V_t'}) = \alpha$. Since it is nonzero, distinct parameters cannot belong to the same class. The rational bijection of Theorem 117.5 now gives the assertion for the original rational double classes. □

Let $Q = g_1 g_2$ be the product of the two marked primes and $\mathfrak{h}_d$ the product of the prescribed outer ideals. Thus the integral frame span at either end has determinant lattice $\mathfrak{h}_d \Phi_b$. Let $\mathcal{G} > 0$ at all real places be the transverse Gram invariant of Proposition 116.1. Define

$$X_C = \frac{(N_{E/\mathbb{Q}}\mathfrak{h}_d)^2 |N_{F/\mathbb{Q}}\mathcal{G}|}{Q^{2df}}.$$

It depends only on $[C]$ and the given rows. In what follows choose η_0 from a class having a supported integral lift. Let $L' = \mathrm{pr}_{V_t'} \Lambda_t$ be the orthogonal projection of the target row lattice. There is a fractional ideal $\mathfrak{a}$ of E specified by the equality of lattices in the determinant line

$$Q^{-d} \mathfrak{h}_d^2 \, \Omega_{\eta_0} = \mathfrak{a} \det L'.$$

Proposition 117.23 (An integral ideal bounding the possible rotations)**.** *The ideal $\mathfrak{a}$ is integral. For every α with a supported integral lift at i, the ideal $(\alpha)\mathfrak{a}$ is also integral. With $A = \Lambda_t \cap H_t$ and $\delta(A) = [A^\sharp : A]^{1/2}$ one has*

$$N_{E/\mathbb{Q}}\mathfrak{a} = X_C \, \delta(A) \le X_C^2.$$

The number of such α is at most

$$|\mu(E)| \prod_{\{w,\bar{w}\},\ w \neq \bar{w}} (1 + v_w(\mathfrak{a}) + v_{\bar{w}}(\mathfrak{a})) \ \le \ |\mu(E)| \, C_\epsilon^f \, X_C^{2\epsilon} \qquad (\epsilon > 0), \tag{117.5}$$

where the product counts each split pair once and C_ϵ is independent of the field, rows and interval. At the frame cutoff $\mathcal{H}_\gamma < Q^f L$ one may use $X_C \le (T_0 T_1)^{df} L^{2d}$, where T_0, T_1 are the absolute terminal norms. If $X_C = 1$, only roots of unity can occur.

Proof. Take an actual integral lift $\gamma = u_t \eta_\alpha u_s^{-1}$, with $u_b \in N_b(F)$ in the full row radicals. Its frame wedge is $(\bigwedge^{2d} u_t)\Omega_{\eta_\alpha}$, since each radical fixes its own frame wedge. Its projection to $\det V_t'$ is exactly Ω_{η_α}. To verify the last assertion, in the Witt decomposition order the outer, middle and opposite blocks consecutively. The map $\mathrm{pr}_{V_t'} u_t|_{V_t'}$ is block upper triangular with diagonal the outer unitriangular matrix, the identity on $Ez_t + \ell_t$, and the inverse adjoint on the opposite block. Projection removes only the H_t component of the middle block. Its determinant is one.

Multiplying the original integral pair of frame wedges by $\mathfrak{h}_d^2$ puts it in $\bigwedge^{2d} \Lambda_t$. At each factor above Q, the two reduced frames agree up to an upper unitriangular change. Their d corresponding differences are divisible by the uniformizer after an integral lift of that change. In the concatenated exterior product replace the second d columns by those differences. This proves divisibility by the d-th power at each marked prime, including both halves of each split place. The outer ideals are units there. Thus the same original wedge, multiplied by $Q^{-d}\mathfrak{h}_d^2$, is still integral everywhere. Projection gives the inclusion in $\det L'$ and proves integrality of $(\alpha)\mathfrak{a}$, including $\mathfrak{a}$ at $\alpha = 1$.

For a lattice in a nondegenerate Hermitian space use its determinant covolume δ. On the determinant line a lattice Iv has covolume $N_{E/\mathbb{Q}}I$ times $|N_{F/\mathbb{Q}}\langle v,v\rangle|$. Local Gram determinants give $[M^\sharp:M]^{1/2}$ for an integral lattice M and reciprocal covolumes for dual lattices. The two primitive perpendicular lattices A and $B=\Lambda_t\cap V_t'$ have equal discriminant orders. Indeed the projections of the self-dual row identify $\Lambda_t/(A+B)$ with each discriminant module, by the same extension of functionals as in Proposition 117.19. In particular $L'=B^\sharp$ and $\delta(L')=1/\delta(A)$. The Gram determinant of the basis $(y_t,\eta_0 y_s)$ is $(-1)^d\mathcal{G}$, by Theorem 86.3. The covolume of $Q^{-d}\mathfrak{h}_d^2\Omega_{\eta_0}$ is therefore X_C. Division by the covolume of $\det L'$ gives $N_{E/\mathbb{Q}}\mathfrak{a}=X_C\delta(A)$.

For any supported lift its common integral lattice R projects isometrically onto a full sublattice of A. Indeed it lies in $X_t^\perp\cap z_t^\perp$, and its projection along X_t has values in the integral middle lattice of our Witt splitting and in H_t. Hence $\delta(A)\le\delta(R)=h_R^f$. Corollary 113.8 gives $X_C=I_\Lambda h_R^f$, with saturation index $I_\Lambda\ge1$. This proves the upper bound and $X_C\ge1$. The frame estimate of Proposition 116.1, whose additional factor P_γ is at least one, gives the stated cutoff bound.

Since $\alpha\bar\alpha=1$, at a self-conjugate prime its valuation is zero. At two conjugate primes it is determined by one integer between $-v_w(\mathfrak{a})$ and $v_{\bar w}(\mathfrak{a})$. Two parameters with the same ideal differ by a unit all of whose complex conjugates have absolute value one. Such a unit is a root of unity. Indeed the powers of the unit have bounded integral characteristic-polynomial coefficients in degree $2f$, so only finitely many values occur. This proves the first bound. For a split pair of residue size q_v put $m=v_w(\mathfrak{a})+v_{\bar w}(\mathfrak{a})$. If $q_v\ge2^{1/\epsilon}$, then $1+m\le q_v^{\epsilon m}$. At any place over a rational prime $p<2^{1/\epsilon}$ use $(1+m)q_v^{-\epsilon m}\le(1-p^{-\epsilon})^{-2}$, by summing the corresponding geometric derivative series. There are at most f such places over each p. Taking $C_\epsilon=\prod_{p<2^{1/\epsilon}}(1-p^{-\epsilon})^{-2}$ proves the second bound. If $X_C=1$ the integral ideal has norm one and every supported parameter is a norm-one unit. □

Theorem 117.24 (One real tensor for an entire cross-matrix class)**.** *Keep a relevant η_0 and choose a rational invariant Haar form on $D_0=N_t\cap\eta_0N_s\eta_0^{-1}$. At η_α transport it by conjugation by $R_t(\alpha)$. Use the corresponding quotient measures in the real tensors I_{η_α} of Theorem 117.6. If ω_s,ω_t are the scalar central characters of the two compact column representations at infinity, put $\chi_\infty(\alpha)=\omega_t(\alpha_\infty)\overline{\omega_s(\alpha_\infty)}$. The full signed contribution of the classes over $[C]$ at the indicated H,j,i is*

$$I_{\eta_0}\sum_{\alpha\in E^1}\chi_\infty(\alpha)m_i(\eta_\alpha).\tag{117.6}$$

The coefficients use the transported stabilizer measures and are zero for a class without a supported lift. The sum is finite by Proposition 117.23. In particular for equivalent columns and a unitary K-intertwiner U_j, the trace against U_j of this contribution is

$$\operatorname{tr}(U_j^*I_{\eta_0})\sum_{\alpha\in E^1}m_i(\eta_\alpha).$$

Proof. Each rotation fixes the ordered frame and normalizes its radical. It preserves the Bessel character, which uses the outer simple coordinates and the pairing against the terminal vector. It therefore carries the stabilizer to that for the rotated pair and preserves relevance. In the real quotient integral substitute $g=R_t(\alpha)g'$. The Haar form on the stabilizer was transported, and the Haar measure on the ambient group is left invariant. Equivariance of the two original sections under the compact column groups gives

$$I_{\eta_\alpha}=\rho_t(R_t(\alpha))I_{\eta_0}\rho_s(R_s(\alpha))^{-1}.$$

At every place $R_b(\alpha)|_{P_b}$ is the scalar αI times the element acting by α^{-1} on H_b and by one on its normal line. The latter elements correspond under j. The map I_{η_0} intertwines these actions

of $U(H_b)$ by Corollary 117.7. Thus the displayed tensor is exactly $\chi_\infty(\alpha)I_{\eta_0}$. Multiplication by its own signed lift coefficient and summation give (117.6). For equivalent representations the central characters agree, giving the trace formula. □

The finite factors in (117.6) can all be computed with one rational parameter space. Use the slice $s(t)$ and directions X_j of Lemma 117.10 at η_0. At η_α conjugate the target and source radicals separately by $R_t(\alpha), R_s(\alpha)$, respectively, and transport $s(t)$ and X_j by this automorphism of $N_t \times N_s$. The determinant of its adjoint action on each row radical is one. Indeed it is an F-rational character of the norm-one torus, whose characters over E are $u \mapsto u^n$. Conjugation sends n to $-n$, so the only F-rational one has $n = 0$. Thus the stabilizer form constructed from the row forms and these directions in Theorem 117.14 is precisely the transported form if that construction is used at η_0. In the same additive parameters one tests the original levels and phases on the rotated orbit point $R_t(\alpha)(s(t)\eta_0)R_s(\alpha)^{-1}$ and on the correspondingly transported pair of unipotents. The notation $s(t)\eta_0$ here denotes the two-sided action on a row map. Consequently $m_i(\eta_\alpha)$ is given by the residue formula of that theorem with the same factor $c_{t,s}v_U\Delta_F^{-r/2}$ throughout the sum, where $r = \dim_F(N_t \times N_s) - \dim_F D_0$.

Every frame cutoff from Corollary 117.4 keeps or drops an entire $[C]$, since its leading minors are unchanged by the two triangular operations. Summing (117.6) over the retained cross matrices therefore gives the actual Z_H of (117.1). In particular the constant component in (117.4) uses one real trace per cross-matrix class and its signed finite coefficient. The size of their total, including the variation of the real traces and the remaining period terms, still enters the uniform comparison in Section 6.6.

117.8 Finite characters in the sum of frame determinants

Fix an occurring $[C]$, a base class η_0 with an integral lift, and the same H, j, i as in Theorem 117.24. Give its stabilizers the transported measures constructed there. Let $\mathcal{T}$ denote the norm-one torus of E/F. For a finite place v write $\mathcal{T}_v = \mathcal{T}(F_v)$ and let $\mathcal{T}_v^0$ consist of its integral units, using both factors at a split place. For $a \in \mathcal{T}_v$ let $L_v(a)$ be the signed local integral from Theorem 117.14, using the transported parameters at $R_t(a)\eta_0 R_s(a)^{-1}$. It includes the original support and full marking, and has value zero on empty support. The normalization of the coefficient of I_{η_0} in (117.6) is then

$$A_i = \frac{\Delta_F^{r/2}}{c_{t,s}v_U} \sum_{\alpha \in E^1} \chi_\infty(\alpha) m_i(\eta_\alpha) = \sum_{\alpha \in E^1} \chi_\infty(\alpha) \prod_{v<\infty} L_v(\alpha). \tag{117.7}$$

For each end b let P_b^ℓ be the orthogonal projection of the ambient row onto its fixed normal line ℓ_b, and let $\mathfrak{d}_b$ be the ideal of integral scalars that clear the denominators of P_b^ℓ on Λ_b. Put $\mathfrak{d} = \mathfrak{d}_t \cap \mathfrak{d}_s$.

Proposition 117.25 (Local unit subgroups for the original signed tests)**.** *At an unmarked place let*

$$K_{i,v} = \{u \in \mathcal{T}_v^0 : u - 1 \in \mathfrak{d}\mathcal{O}_{E_v}\}, \qquad E_v = E \otimes_F F_v.$$

At $v \mid Q$ also require $\overline{R_t(u)}\, i_v = i_v\, \overline{R_s(u)}$ on the marked positive column. Then $L_v(au) = L_v(a)$ for every $u \in K_{i,v}$. The support of L_v is contained in $\{a : a\mathfrak{a}\mathcal{O}_{E_v} \subseteq \mathcal{O}_{E_v}\}$ with $\mathfrak{a}$ from Proposition 117.23. It consists of finitely many cosets of $K_{i,v}$.

There is the following explicit test at a mark. In its chosen split factor let π be a uniformizer, set $D = \max\{v_\pi(\mathfrak{d}_s), v_\pi(\mathfrak{d}_t)\}$, and lift i_v to an integral isomorphism J_v between the positive column lattices. On these columns write $P_b = P_b^\ell$. For a unit u in the chosen factor the

condition is

$$v_\pi(u-1) \;\ge\; D + \mathbf{1}_{\,\pi^D(P_tJ_v - J_vP_s)\not\equiv 0 \bmod \pi}.$$

In particular $K_i = \prod_v K_{i,v}$ *contains all integral norm-one units congruent to one modulo the ideal* $Q\mathfrak{d}$*, and*

$$N_{E/\mathbb{Q}}(Q\mathfrak{d}) \le Q^{2f}(T_0T_1)^{2f}X_C^4.$$

Proof. Write $M_b = \Lambda_b \cap \ell_b$. Self-duality and extension of functionals from this primitive line give $P_b^\ell \Lambda_b = M_b^\sharp$. Thus $\mathfrak{d}_b$ is the annihilator of $M_b^\sharp / M_b$. If $M_b = Iv$, this is $I\bar{I}(\langle v, v\rangle)$, a conjugation-invariant integral ideal of norm $[M_b^\sharp : M_b]$. Now $R_b(u) = 1 + (u-1)P_b^\ell$. For integral u of norm one, both it and its inverse preserve the row lattice exactly when $u - 1 \in \mathfrak{d}_b\mathcal{O}_{E_v}$, since $u^{-1} - 1$ is a unit multiple of $u - 1$.

These integral rotations fix the two frames pointwise. On an integral row map their action on both sides preserves the frame conditions, including the primitive terminal class in the given row lattice. At a mark the induced middle matrices multiply the marked index on the left and on the inverse right. This gives the extra condition in the statement. The outer finite phase is unchanged by these operations. On the right all successive source vectors are fixed. On the left their images at a mark agree modulo Q with combinations of the fixed target vectors. Thus the simple-coordinate pairings that give the phase agree modulo Q there. Away from Q their integral changes are in the kernel of the normalized additive character. This is the local computation in Proposition 96.20 and Theorem 96.24, applied to the two pointwise stabilizers. The same argument for the inverses preserves the zero test. The unipotent characters in the transported coordinates are unchanged because rotations fix the terminal vector and the outer flag coordinates. This proves invariance of the whole integrand, hence of L_v.

The projection of the concatenated wedge in the proof of Proposition 117.23 works over F_v with this same local support. It gives $a\mathfrak{a}\mathcal{O}_{E_v} \subseteq \mathcal{O}_{E_v}$ whenever an integral lift occurs. At a split place this confines the valuation to a finite interval. At a nonsplit place the local torus is already compact. The subgroup contains an open unit congruence subgroup, giving finiteness modulo $K_{i,v}$.

At a mark the terminal norm is a unit. The integral Witt decomposition and the terminal line both split, so the projector denominators on the positive column are exactly the ones computed on the ambient lattice. The two conjugate exponents are equal. For $v_\pi(u-1) \ge D$ the congruence on i_v is equivalent to $(u-1)(P_tJ_v - J_vP_s) \equiv 0 \bmod \pi$. Its coefficient matrix has valuations at least $-D$. Reduction after multiplying by π^D gives exactly the stated extra digit and is independent of the lift J_v, since its change is divisible by π.

In the self-dual middle row lattice the primitive perpendicular of M_b contains $(\Lambda_b \cap H_b) \oplus \mathcal{O}_E z_b$ with finite index. Their discriminant covolumes give $\delta(M_b) \le T_b^f \delta(\Lambda_b \cap H_b) \le T_b^f X_C$, using the two-end version of the projection argument in Proposition 117.23. Here T_b denotes the absolute terminal norm at end b. Consequently $N_{E/\mathbb{Q}}\mathfrak{d}_b \le T_b^{2f} X_C^2$. The intersection of the two integral ideals contains their product. Multiplying the two bounds and inserting the factor $N_{E/\mathbb{Q}}(Q) = Q^{2f}$ proves the last assertion. □

Put $\mathcal{A} = \mathcal{T}(\mathbb{A}_{F,f})/K_i$, $\mathcal{B} = \mathrm{im}(E^1 \to \mathcal{A})$, $W_i = E^1 \cap K_i$, and $h_i = |\mathcal{A}/\mathcal{B}|$. Let h_E denote the ideal class number of E.

Theorem 117.26 (A finite character formula for the complete coefficient)**.** *One has* $W_i \subseteq \mu(E)$ *and* $h_i \le h_E N_{E/\mathbb{Q}}(Q\mathfrak{d})$*. If* χ_∞ *is nontrivial on* W_i*, then* $A_i = 0$*. Otherwise let* $\mathcal{X}$ *be the set of unitary characters* θ *on* $\mathcal{A}$ *with* $\theta([\alpha]) = \chi_\infty(\alpha)$ *for every* $\alpha \in E^1$*. It has* h_i *elements.*

With θ_v its restriction at v,

$$A_i = \frac{|W_i|}{h_i} \sum_{\theta \in \mathcal{X}} \prod_{v<\infty} \sum_{a \in \mathcal{T}_v/K_{i,v}} L_v(a)\theta_v(a). \tag{117.8}$$

Only finitely many local factors differ from one. For identical central characters $\mathcal{X}$ is the full character group of the finite group $\mathcal{A}/\mathcal{B}$.

Proof. A global norm-one element integral and invertible at every finite place is a root of unity by the unit argument in Proposition 117.23. Modulo all local compact units a finite torus idele is specified by one integer at each split pair, zero almost everywhere. These are exactly the ideals $J/\bar{J}$, as one can choose an exponent on a single factor of each pair to realize any tuple. For a principal $J = (t)$ the tuple is realized by $t/\bar{t} \in E^1$. The quotient by the global points and all compact units thus has order at most h_E. Passing to K_i costs at most the number of units modulo $Q\mathfrak{d}$ by Proposition 117.25. This is bounded by the absolute ideal norm. This proves finiteness and the claimed bound.

Write $L(a) = \prod_v L_v(a_v)$. Almost everywhere the base class has unit local factor by Theorem 117.14, the ideal $\mathfrak{a}$ is the unit ideal, and $K_{i,v} = \mathcal{T}_v^0$. Thus the only possible local coset is the identity coset, with value one, by Proposition 117.25. Hence L has finite support on $\mathcal{A}$. In (117.7) the elements above a point of $\mathcal{B}$ form a coset of W_i. The character sum on such a coset vanishes if the restriction is nontrivial. Otherwise it is $|W_i|$ times the character of the image. This gives a well-defined character on $\mathcal{B}$. It extends to $\mathcal{A}$, since the quotient is finite abelian. Explicitly adjoin one generator at a time, taking an n-th root of the value of its n-th power when its order modulo the previous subgroup is n. The extensions differ by exactly the h_i quotient characters. Their average at a point restricts to the prescribed value on $\mathcal{B}$ and is zero off $\mathcal{B}$ by character orthogonality. Insert this average in the finite sum over $\mathcal{A}$. Each character on its restricted product of local factors factors on the finitely supported tuples, so expansion of the product gives (117.8). □

There are two estimates in the original arithmetic normalization. Write $D(\mathfrak{a})$ for the product of the split-place factors in (117.5), omitting the prefactor $|\mu(E)|$. Then

$$|A_i|^2 \le |W_i|\,|\mu(E)| D(\mathfrak{a}) \prod_{v<\infty} \sum_{a \in \mathcal{T}_v/K_{i,v}} |L_v(a)|^2.$$

Indeed at most $|\mu(E)|D(\mathfrak{a})/|W_i|$ points of $\mathcal{B}$ with nonzero coefficient can occur. Above each one all $|W_i|$ global representatives have the same finite coefficient. Apply Cauchy–Schwarz on these image points and extend the squared sum to $\mathcal{A}$, where it factors. For $\chi_\infty = 1$ there is also an integral bound that does not involve h_i. On each locally supported coset use one representative and a precision $b_v(a)$ from Theorem 117.11 in the transported parameters. Let B_v be the maximum of these finitely many precisions. A place with empty support makes the whole coefficient zero. For nonzero A_i and a valuation above the split terminal prime normalized by $v_T(T) = 1$, one has

$$v_T(A_i) \ge -r \sum_{v \mid T} B_v.$$

In fact $q_v^{rB_v} L_v(a)$ is an algebraic integer for every coset, since it is an integral multiple of its sum of roots of unity. Almost everywhere B_v can be chosen zero. The rational sum in (117.7) is finite, all its character weights are one here, and $q_v = T$ at $v \mid T$. Multiplication by the product of the local denominators proves the inequality.

Corollary 117.27 (The root-of-unity sum on the original unit-terminal rows)**.** *Suppose both terminal vectors have norm -1 and one supported cell in the cross-matrix class has $h_R = 1$. Then $K_i = \prod_v \mathcal{T}_v^0$ and $W_i = \mu(E)$. In particular the entire contribution (117.6) vanishes when $\chi_\infty|_{\mu(E)} \neq 1$, at every permitted frame cutoff that includes this class. When $X_C = 1$, the normalized coefficient for every χ_∞ is exactly*

$$A_i = \begin{cases} |\mu(E)| \prod_{v<\infty} L_v(1), & \chi_\infty|_{\mu(E)} = 1, \\ 0, & \chi_\infty|_{\mu(E)} \neq 1. \end{cases}$$

For identical columns this multiplies the single base trace by $|\mu(E)|$ in the full signed sum. The multiplier is a unit at the terminal prime.

Proof. At either end the image of R is self-dual in H_b and lies in the positive integral column. Thus it splits off orthogonally by integral projection. The unit terminal line also splits off from the self-dual middle row. The remaining normal lattice is therefore self-dual and $\mathfrak{d}_b = (1)$. Theorem 113.7 makes j integral with integral inverse, inducing i_v at every mark. It carries normal line to normal line, so $P_t j = j P_s$. Taking $J_v = j$ in Proposition 117.25 gives zero extra digits at all marks. This proves the subgroup assertions. If $X_C = 1$, the equality $X_C = I_\Lambda h_R^f$ gives $h_R = 1$ at every supported lift. Proposition 117.23 permits only roots of unity, and the local contribution for every root equals that for one by unit invariance. Summing the character on $\mu(E)$ gives the displayed values. Since the odd terminal prime T splits completely in E, the group $\mu(E)$ injects into $\mathbb{Q}_T^\times$, whose torsion has order $T-1$. Indeed reduction is injective on torsion in $\mathbb{Z}_T^\times$ because the T-adic logarithm is injective on $1 + T\mathbb{Z}_T$. Hence $T \nmid |\mu(E)|$. □

For sufficiently distant intervals a retained row of terminal norm $-T$ forces $h_R > b_x^5$ by Proposition 113.9. Thus the evaluation at $X_C = 1$ covers every retained pair for which this value occurs. The finite character formula applies at all other values as well. The needed bound for the sum of its coefficients against the real traces, and the residual cohomology bound on these fields, remain open in Section 6.6.

117.9 Moments of the integral frame determinants

Fix the two marked rows, the positive hyperplanes, their rational transport and the residue index as in Theorem 117.24. The terminal norms T_s, T_t can independently be 1 or T. At a norm-T end impose the hyperplane test of Theorem 109.38, with a fixed selection parameter $A_0 > 0$. Let $[C]$ range over the relevant transverse cross matrices of positive Gram invariant that have a supported integral lift at these data. Define $n_i(C)$ to be the number of rational unipotent double classes above $[C]$ with such a lift. The quantities to be summed are

$$w_C = \prod_{j : D_j \neq 0} \prod_{\tau} (1 + |\tau D_j|^2)^2, \qquad X_C = Q^{-2df} (N_{E/\mathbb{Q}} \mathfrak{h}_d)^2 |N_{F/\mathbb{Q}} \mathcal{G}|.$$

Here D_j is the leading cross minor of order j, $d = l+1$, and τ runs over one complex embedding above each real place. Write $Y_0 = 1$ on two unit-terminal rows. If at least one terminal norm is T, let $Y_0 = T^f (k |\operatorname{disc} E|^{1/(2f)} / (2\pi \exp(A_0 + 4)))^f$, the hyperplane threshold in the original positive lattice at that end.

Theorem 117.28 (Moments over all supported frame determinants)**.** *Fix $0 \le \sigma < 1$. On the common intervals of Theorem 109.38, one has, uniformly in the indicated data,*

$$\sum_{[C]} \frac{n_i(C)\, X_C^\sigma}{w_C} \le \begin{cases} \exp\bigl(-(64 - o(1)) f d\, x^\varepsilon\bigr), & T_s = T_t = 1, \\ \exp\bigl(-(132 - 34\sigma - o(1)) f d\, x^\varepsilon\bigr), & \max(T_s, T_t) = T. \end{cases}$$

The sum includes arbitrarily large values of X_C. For a fixed $[C]$ the multiplicity counts each rational double class once, even when it has several integral lifts.

We first give the tail estimate used in the proof. It retains the length of the final reversed block and also applies to tails beyond any prescribed cutoff. The positive parameter λ below is independent of the interval exponent ε.

Lemma 117.29 (The last block with a lower bound on the Gram determinant)**.** *Let h be the final block length, $j = d - h$, $Z \ge 1$, and $Z_* = \max(Z, Y_0)$. Put $u = \sigma + 2\lambda$ for $\lambda > 0$. Choose $0 < \delta < 1$ with $4h - 2 - 2hu - \delta \ge 0$. Set*

$$B_\delta = 9\int_0^\infty \frac{4r^{5-\delta}}{(1+r^2)^3}\,dr + 2, \qquad C_\lambda = \prod_{p<2^{1/\lambda}} (1-p^{-\lambda})^{-2},$$

where the product is over rational primes. Then

$$\sum_{\substack{[C]\ \text{with last length } h\\ X_C \ge Z}} \frac{n_i(C)X_C^\sigma}{w_C} \le |\mu(E)|\, C_\lambda^f B_\delta^{fj} \big(9(2+4h/\delta)\big)^f \cdot Q^{-(4-\delta)jf-(4-\delta/h)df} Z_*^{-(2-1/h-u-\delta/(2h))}.$$

Proof. The prescribed outer ideals are nonpositive powers of the extended different by Section 115.3. Thus $N_{E/\mathbb{Q}}\mathfrak{h}_d \le 1$, and every rational frame vector is integral. Put $B = \prod_\tau |\tau D_d|$. At each embedding the identity $\mathcal{G} = D_d\bar{D}_d - T_sT_tD_{d-1}\bar{D}_{d-1} > 0$ gives

$$X_C \le B^2/Q^{2df}, \qquad B \ge Q^{df}\sqrt{Z_*} \quad \text{on the indicated tail.}$$

For the second inequality use $X_C \ge Z$ and $X_C = I_\Lambda\delta(R)$, by Corollary 113.8. At a retained norm-T end the common lattice R projects isometrically into a primitive hyperplane of the original positive lattice, so $\delta(R) > Y_0$ by Proposition 113.9. At either other end $X_C \ge 1$ by Proposition 117.23. That proposition also bounds $n_i(C)$ by $|\mu(E)|C_\lambda^f X_C^{2\lambda}$. We may henceforth sum X_C^u/w_C over cross-matrix classes and restore the indicated constant at the end.

The proof of Theorem 116.8 gives reversed consecutive blocks, $D_a \in Q^a\mathcal{O}_E$ at every nonterminal endpoint, and $D_d + D_{d-1} \in Q^d\mathcal{O}_E$. These congruences use the actual two marks, where all the terminal vectors have the same unit reduction. For $j = 0$ take $D_0 = 1$. Fix all prefix block scalars through j. If $h \ge 2$, then $D_{d-1} = 0$ and $D_d = \pm D_j c^h$ for the last scalar c. Write $D_j = Q^j a$. The nonzero integer a satisfies $a(c/Q)^h \in \mathcal{O}_E$. Hence c/Q belongs to the fractional ideal with exponents $-\lfloor v_\mathfrak{p}(a)/h\rfloor$. Its norm is at least $|N_{E/\mathbb{Q}}a|^{-1/h}$. In coordinates $v_\tau = |\tau D_j|^{1/h}\tau c$, distinct possibilities are therefore separated by

$$\prod_\tau |v_\tau - v'_\tau| \;\ge\; |N_{E/\mathbb{Q}}D_j|^{1/(2h)}Q^f|N_{E/\mathbb{Q}}a|^{-1/(2h)} = Q^{df/h}.$$

Here $|v_\tau|^h = |\tau D_d|$. If $h = 1$, use $v_\tau = \tau D_d$ itself. For a fixed prefix the shifted congruence makes every difference a multiple of Q^d, and the final scalar is recovered uniquely from D_d/D_j up to its fixed sign. Thus the same separation holds also in this case.

Take a box $|v_\tau| \le r_\tau$, set $R_0 = (\prod_\tau r_\tau)^{1/f}$, and put $P = Q^{d/h}$. Nonempty boxes require $R_0 \ge PZ_*^{1/(2hf)}$. After rescaling to the unit polydisc their centers have sup distance at least P/R_0. Disjoint open polydiscs of radius $P/(2R_0)$ fit inside the polydisc of radius $1 + P/(2R_0)$, giving at most $9^f(R_0/P)^{2f}$ choices. For $b = 4h - 2 - 2hu - \delta$ bound the support indicator by $(R_0/P)^{bf}Z_*^{-b/(2h)}$. On this box $X_C^u \le Q^{-2udf}\prod_\tau r_\tau^{2hu}$. Integrate the count against the positive density $4hr^{2h-1}(1+r^{2h})^{-3}\,dr$ in every coordinate. Its tail from $|v_\tau|$ is the original last-endpoint

weight. Tonelli's theorem applies first to finite subsums and then to the whole nonnegative sum. The one-variable integral left after the indicated powers are inserted is

$$\int_0^\infty \frac{4hr^{2h+1+b+2hu}}{(1+r^{2h})^3}\,dr \le 2 + 4h/\delta.$$

Indeed its exponent in the numerator is $6h-1-\delta$. Split at one and use the denominator bounds 1 and r^{6h}. The equality $Q^{-2udf}P^{-(2+b)f} = Q^{-(4-\delta/h)df}$ gives the stated last-endpoint factor. Counting c itself includes all of its possible roots.

For completeness fix a composition of the prefix of length $j>0$ with endpoint set J. At $a\in J$ a nonzero value lies in $Q^a\mathcal{O}_E$, and distinct values are separated in the product of complex distances by Q^{af}. The same polydisc argument with $h=1$, $u=0$, no additional cutoff, and indicator exponent $2-\delta$ gives a factor $(B_\delta-2)^f Q^{-(4-\delta)af}$. The endpoint j accounts for the claimed power of Q. Each preceding endpoint ratio has at most its block length many roots. Their product over the prefix is at most $2^{j-|J|}$. Summing over subsets of $\{1,\dots,j-1\}$ gives at most $B_\delta^{fj}Q^{-(4-\delta)jf}$, since $(B_\delta-2)^f+2\le B_\delta^f$. For $j=0$ the prefix factor is one. Multiplying the factors proves the lemma. □

Proof of Theorem 117.28. Apply the lemma with $Z=1$, $\lambda=1/\sqrt{\log(2x)}$, and $\delta=1/\log(2Q)$. The condition on b holds for all h once x is large, uniformly for σ in any fixed compact subinterval of $[0,1)$. The prefix integral gives $B_\delta = O(1/\delta)$ by splitting it at one. Also $\log C_\lambda \le 2^{1+1/\lambda}\log(1/(1-2^{-\lambda})) = x^{o(1)}$. By Lemma 115.16, $|\mu(E)|\le 8f^2$, and f is polynomial in x. All positive logarithms of the prefactors and the cost $\log d$ of summing the lengths are $o(fd\,x^\varepsilon)$. This uses $d\to\infty$ polynomially, $\log Q = (16+o(1))x^\varepsilon$, and fixed $\varepsilon>0$.

On two unit-terminal rows the power of Q from d alone gives the first assertion. On any pair with a retained norm-T row, $\log Y_0 = (34+o(1))fd\,x^\varepsilon$, by Theorem 93.9 with $m=2d$ and the polynomial bounds for k and the root discriminant. After division by $fd\,x^\varepsilon$ the saving for a given h is at least

$$64(1+j/d) + 34(2-1/h-\sigma) - o(1).$$

If $h\ge\sqrt{d}$, the term $34/h$ tends uniformly to zero. If $h<\sqrt{d}$, then $j/d = 1-o(1)$ and $64j/d$ absorbs $34/h$. The second assertion follows. □

In particular at $\sigma=0$ the norm-T estimate of Theorem 110.21 retains its leading exponent after counting the rational frame determinants over each label, also on a pair with one unit-terminal end. The weights in this theorem are explicit functions of the cross matrices. The signed coefficient in the projected period still contains the finite integrals in (117.7) multiplied by the real tensor in (117.6). Bounding their total over the hyperplanes and transports is required in Section 6.6.

117.10 Evaluation of the signed Bessel integral on the open cell

Retain the fields, interval, marked rows, positive hyperplanes, transport and residue index of Theorem 117.28. Write $d=s+1$, let $k\ge 2$ be the dimension of the positive terminal space, and set $A=T_0T_1$. Here T_0,T_1 are the positive absolute norms of the two negative terminal vectors. Consider the rational cross matrices for which every leading minor D_j, $1\le j\le d$, is nonzero and the transverse Gram invariant $\mathcal{G}$ is totally positive. We give an estimate on this open cell that remains uniform as a local Gram invariant tends to zero. Its analytic right side consists of explicitly specified derivatives.

At a real place, take s hyperbolic pairs and a middle space of signature $(k,1)$. A representative on the open cell exchanges the i-th outer vector with its dual, with multipliers $c_i, \overline{c_i}^{-1}$,

and acts by an element of $U(k,1)$ on the middle space. Indeed left and right unitriangular elimination diagonalizes a cross matrix with nonzero leading minors, with successive diagonal entries equal to the ratios of these minors. The perpendicular middle is then identified by an isometry. In bases with terminal vectors $\sqrt{T_0}e_-$ and $\sqrt{T_1}e_-$, let ζ be their cross pairing. Write

$$z = (c_1, \dots, c_s, \zeta), \qquad U_j(z) = \prod_{i=1}^{j} z_i, \qquad \Omega_A = (\mathbb{C}^\times)^s \times \{\zeta : |\zeta|^2 > A\}.$$

The product U_j is the leading minor in these coordinates. One can choose a smooth middle representative throughout Ω_A. In the plane of one positive vector and e_-, use the real boost with diagonal $\cosh t$ and off-diagonal $\sinh t$, followed by a phase on the negative line chosen to give the prescribed ζ, where $|\zeta| = \sqrt{A}\cosh t$. Extend by the identity on the perpendicular space. The compact groups at the ends account for the choice of positive direction.

For a smooth group coefficient F, denote its signed transform on this section by $J(z)$ whenever the orbital integral is defined. Its integrand is

$$\chi_0(n_0)\overline{\chi_1(n_1)}\rho_0(h_0)F(h_0^{-1}n_0^{-1}g(z)n_1h_1)\rho_1(h_1)^{-1}$$

over the two compact groups and the two flag unipotents, with the given Haar measures. The unipotent stabilizer on this cell is trivial. In the block order consisting of the outer lines, the middle space and the opposite lines in reverse order, the first unipotent is strictly block upper triangular after subtraction of the identity. Transport of the other unipotent by this representative makes it strictly block lower triangular after the same subtraction. Their intersection is the identity. Thus the displayed integral uses the original characters on both sides. The estimate below also applies to Hilbert-space valued profiles, in particular to a matrix with the Hilbert–Schmidt norm.

Set $a_i = 2(d-i) + k - 1$ for $1 \le i \le s$, and use ordinary complex Lebesgue measure dz in the chart. The weight is

$$\rho_A(z) = \prod_{i=1}^{s} |c_i|^{2a_i}(|\zeta|^2 - A)^{k-1}.$$

The powers can be read directly from the Haar measures. In the decomposition n_0wmn_1, the density in m includes the determinant of conjugation on the right unipotent. Right translation by m_0 replaces n_1 by $m_0^{-1}n_1m_0$; its Jacobian is the inverse of this determinant. For a pair of outer indices $i < j$ the two complex coordinates have weights c_i/c_j and $c_i\bar{c}_j$. Their two real Jacobians multiply to $|c_i|^4$. The $k+1$ complex coordinates to the middle contribute $|c_i|^{2(k+1)}$, and the imaginary diagonal coordinate contributes $|c_i|^2$. Multiplication by the multiplicative measure $d^2c_i/|c_i|^2$ gives precisely $|c_i|^{2a_i}d^2c_i$. For the middle quotient take a negative unit vector $(v,b) \in \mathbb{C}^k \times \mathbb{C}$, with $\|v\|^2 - |b|^2 = -1$. Euclidean measure disintegrated at this level of the Hermitian form is unitary invariant, since unitary matrices preserve both the form and real volume. In polar coordinates for v, fixing $\|v\|^2 = |b|^2 - 1$ gives $\pi^k(|b|^2-1)^{k-1}/(k-1)!$ times probability measure in its direction. Passing from b to the terminal pairing of modulus $\sqrt{A}\,|b|$ contributes the constant A^{-k} in this density. We use $\rho_A\,dz$ explicitly on both sides of the trace estimate. The middle factor converting it to the stated invariant shell measure is $\pi^k/(A^k(k-1)!)$.

Use radial variables $u_i = \log|c_i|$ for $i \le s$ and $u_d = t > 0$, and the ordinary angles $\theta_j = \arg z_j$. For subsets I, B of $\{1, \dots, d\}$, let $\partial_u^I\partial_\theta^B$ differentiate once in each of the indicated variables. Define

$$\mathcal{S}_A(H) = \sum_{I,B\subseteq\{1,\dots,d\}} \int_{\Omega_A} \rho_A(z)\|\partial_u^I\partial_\theta^B H(z)\|^2\,dz, \qquad K_{k,s} = 2(k-1)\prod_{i=1}^{s} 2(a_i+1).$$

Theorem 117.30 (A trace estimate at every positive terminal Gram value)**.** *For a smooth Hilbert-space valued H on Ω_A with $\mathcal{S}_A(H) < \infty$, every $z \in \Omega_A$ satisfies*

$$\rho_A(z)\|H(z)\|^2 \le \frac{\mathcal{S}_A(H)}{K_{k,s}|U_d(z)|^2}.$$

For compactly supported H only the terms with $I = \{1, \dots, d\}$ are needed on the right.

Proof. We first assume compact support. Fix the angles. In the i-th outer variable u, the radial energy measure is $e^{(2a_i+2)u}\,du$. Integrating a derivative from u to infinity and applying Cauchy–Schwarz costs the reciprocal weight integral $e^{-(2a_i+2)u}/(2a_i+2)$. After multiplication by the point weight e^{2a_iu}, the factor is $1/((2a_i+2)|c_i|^2)$.

In the terminal variable the radial measure is $A^k \sinh^{2k-1} t \cosh t\, dt$. The corresponding inverse weight integral satisfies

$$\int_t^\infty \frac{ds'}{\sinh^{2k-1} s' \cosh s'} \le \frac{1}{\cosh^2 t} \int_t^\infty \frac{\cosh s'}{\sinh^{2k-1} s'}\, ds' = \frac{\sinh^{-2(k-1)} t}{2(k-1)\cosh^2 t}.$$

Multiplying by $A^{-k}(A\sinh^2 t)^{k-1}$ leaves $1/(2(k-1)|\zeta|^2)$. This cancellation takes place before any product over embeddings. It applies for arbitrarily small positive t.

Repeated integration of the full mixed radial derivative from the evaluation point to infinity recovers H up to sign. Cauchy–Schwarz on the product of these half-lines multiplies the inverse weight integrals just computed. To evaluate at the prescribed angles, use the Fourier series on the d-torus. For a smooth Hilbert-space valued function g, Cauchy–Schwarz with Fourier weights $\prod_j (1+n_j^2)$, followed by Parseval, gives

$$\|g(\theta)\|^2 \le \left(\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} (1+n^2)^{-1}\right)^d \sum_B \int_{[0,2\pi]^d} \|\partial_\theta^B g\|^2\, d\theta.$$

The constant in parentheses is at most $5/(2\pi) < 1$, since $\sum_{n\ge1} n^{-2} \le 1 + \int_1^\infty y^{-2}dy = 2$. Apply this inequality to H at the chosen radii, then the radial inequality to each angular derivative. Enlarge the resulting outward domains to Ω_A. This proves the bound for compact support.

For the general assertion fix the evaluation point. In each radial variable insert a smooth cutoff equal to one below R and zero above $R+1$, with first derivative bounded independently of R. Below the evaluation point insert another cutoff equal to one on its entire outward half-line. The second cutoff may be supported away from $t = 0$ or $u_i = -\infty$, as appropriate. Use the proof with the integrals still restricted to the outward domain. Terms differentiating an upper cutoff are supported where some variable exceeds R and tend to zero in the weighted L^2 norm by finiteness of $\mathcal{S}_A(H)$. Dominated convergence treats the term in which all derivatives hit H. This proves the stated inequality. □

We next sum these evaluations in the archimedean frames belonging to the same integral flags as the prescribed period. Write $\eta_j = (N_{E/\mathbb{Q}}\mathfrak{h}_j)^{-1/(2f)}$ as in Proposition 116.1, with $\eta_0 = 1$. For a supported rational class C set

$$V_{j,\tau}(C) = \eta_j^{-2}\tau D_j, \qquad z_{j,\tau}(C) = V_{j,\tau}(C)/V_{j-1,\tau}(C), \qquad V_{0,\tau} = 1.$$

These coordinates divide the i-th outer vector at each end by its norm η_i/η_{i-1}. The terminal vector keeps its given norm because $\eta_d = \eta_s$. In particular $\mathcal{G} > 0$ at τ gives $|z_{d,\tau}|^2 > A$.

Constant phase changes in any of the two ordered frames can be included in the arguments of the profiles. Put

$$W(z)=\prod_{j=1}^{d}(1+|U_j(z)|^2)^2,\qquad W_C=\prod_{\tau}W(z_\tau(C)).$$

Let $X_C, n_i(C)$ be the normalized integral determinant and number of supported frame determinants of Theorem 117.28.

Proposition 117.31 (Packing all normalized leading minors)**.** *For $0<\delta<1$, put*

$$B_\delta=9\int_0^\infty \frac{4r^{5-\delta}}{(1+r^2)^3}\,dr.$$

On the open cell with the fixed data above, at either choice of each terminal norm,

$$\sum_{[C]}\frac{1}{W_C}\le B_\delta^{fd}\,Q^{-(4-\delta)fd(d+1)/2}.$$

Proof. The congruences of Theorem 116.8 put D_j in $Q^j(\bar{\mathfrak{h}}_j\mathfrak{h}_j)^{-1}$ for $j<d$. For $j=d$, fixing D_s puts D_d in a translate of the same ideal with $j=d$. If D_j, D_j' are distinct possible values at a step with fixed prefix, taking the ideal norm of their difference gives

$$\prod_{\tau}|V_{j,\tau}-V'_{j,\tau}|\ge Q^{jf}.$$

For $j<d$, comparison with zero gives $\prod_\tau|V_{j,\tau}|\ge Q^{jf}$. At $j=d$ the determinant formula and total positivity give

$$Q^{2df}X_C=(N_{E/\mathbb{Q}}\mathfrak{h}_d)^2|N_{F/\mathbb{Q}}\mathcal{G}|\le\prod_\tau|V_{d,\tau}|^2.$$

Every supported class has $X_C\ge1$ by Proposition 117.23. This supplies the same lower bound also in the translated last lattice.

At a fixed step write $P=Q^j$. In the box $|V_{j,\tau}|\le r_\tau$, put $R=(\prod_\tau r_\tau)^{1/f}$, taking the radii positive. A nonempty box has $R\ge P$. After dividing the coordinates by their radii, distinct centers have sup distance at least P/R. Disjoint open polydiscs of radius $P/(2R)$ lie in the polydisc of radius $1+P/(2R)$. Hence the count is bounded by $(1+2R/P)^{2f}\le 9^f(R/P)^{(4-\delta)f}$. Integrate this bound against the product of the densities $4r_\tau(1+r_\tau^2)^{-3}dr_\tau$. The one-coordinate tail of this density is $(1+|V_{j,\tau}|^2)^{-2}$. Tonelli's theorem gives the bound $B_\delta^f Q^{-(4-\delta)jf}$ at step j, uniformly in the earlier minors. On this cell all the blocks have length one, so the ordered minor tuple determines its rational cross-matrix class by successive division. Summing from $j=d$ down to 1 therefore gives the claimed exponent. □

Corollary 117.32 (The sampled real profiles on the two marked rows)**.** *Let $0\le\sigma<1$, choose $\lambda>0$ with $\sigma+2\lambda\le1$, and let C_λ be the finite constant in Lemma 117.29. Take one smooth profile J_τ per real place with $\mathcal{S}_A(WJ_\tau)<\infty$. Then*

$$\sum_{[C]}n_i(C)X_C^\sigma W_C\prod_\tau\rho_A(z_\tau(C))\|J_\tau(z_\tau(C))\|^2$$
$$\le\frac{|\mu(E)|\,C_\lambda^f B_\delta^{fd}}{K_{k,s}^f Q^{2df+(4-\delta)fd(d+1)/2}}\prod_\tau\mathcal{S}_A(WJ_\tau).$$

For fixed $\sigma<1$ the coefficient of the product of energies is at most $\exp(-(32-o(1))fd(d+1)x^\varepsilon)$ on the period intervals.

Proof. Apply Theorem 117.30 to WJ_τ. Multiplying over the places leaves W_C^{-1} after inserting the W_C on the left. The remaining determinant denominator is $\prod_\tau |V_{d,\tau}|^2 \ge Q^{2df} X_C$. Proposition 117.23 gives $n_i(C) \le |\mu(E)| C_\lambda^f X_C^{2\lambda}$. Since $X_C \ge 1$, the residual power $X_C^{\sigma+2\lambda-1}$ is at most one. Now apply Proposition 117.31. For the asymptotic keep $\lambda > 0$ fixed and take $\delta = 1/\log(2Q)$. Splitting the defining integral at one gives $B_\delta = O(1+\delta^{-1})$. Use $\log Q = (16+o(1))x^\varepsilon$, $|\mu(E)| \le 8f^2$ from Lemma 115.16, and the polynomial field degree on these intervals. These factors give the asserted estimate. □

The corollary applies directly to smooth compactly supported pieces of a signed orbital profile. To use it for the full projected period one needs a bound for the displayed mixed derivative energies relative to the energy of the prescribed uncut coefficient, including its Haar constants. The cells with a vanishing proper leading minor and the finite signed orbital factors in (117.7) also enter the comparison of Section 6.6.

117.11 Flag minors on every transverse block of the signed period

Fix the retained rows at x, with $E = E_x$, $F = E^+$, $f = [F:\mathbb{Q}]$, $Q = g_1 g_2$, $d = s+1$, and terminal norms $-T_0, -T_1$, as in Section 116.1. Use the original marked level at both primes. Consider relevant transverse terms for which $\mathcal{G} = D_d \bar{D}_d - T_0 T_1 D_s \bar{D}_s$ is totally positive. For one such term write Z for the cross matrix and w for its pivot permutation in Lemma 110.10. This permutation reverses consecutive blocks. The leading square of a proper prefix inside a reversed block is singular. A different square minor of the same row prefix retains the missing pivot.

For $I \subseteq \{1,\dots,d\}$ put $\mathfrak{a}_I = \prod_{i\in I} \mathfrak{a}_i$, where $\mathfrak{a}_i = \mathfrak{d}^{-(d-i)}$, also for $i = d$. Here $\mathfrak{d}$ is the extension of the different of F from Section 115.3. Put $\eta_I = (N_{E/\mathbb{Q}}\mathfrak{a}_I)^{-1/(2f)}$. Write $[j] = \{1,\dots,j\}$, $K_j = w([j])$, and take all subsets in increasing order when forming a determinant. Set

$$M_j = \det Z_{[j],K_j}, \qquad V_{j,\tau} = \frac{\tau M_j}{\eta_{[j]}\eta_{K_j}}, \qquad B_j = \prod_\tau |V_{j,\tau}|.$$

At $j = 0$ all three quantities equal one. The product takes one complex embedding over each real place.

Proposition 117.33 (An integral minor for each row prefix)**.** *The M_j are nonzero invariants of the two decorated flags. For a supported integral lift and $j < d$,*

$$M_j \in Q^j (\overline{\mathfrak{a}_{[j]}}\mathfrak{a}_{K_j})^{-1}, \qquad B_j \ge Q^{jf}.$$

At the final index $M_d = D_d$ and

$$B_d = Q^{df}(X_C P_\gamma)^{1/2}, \qquad P_\gamma = \prod_\tau \left(1 + \frac{T_0 T_1 |\tau D_s|^2}{\tau \mathcal{G}}\right).$$

Inside any reversed block with endpoints $a, a+h$, for $0 \le m \le h$ one has

$$|V_{a+m,\tau}| = |V_{a,\tau}|^{1-m/h} |V_{a+h,\tau}|^{m/h}.$$

Proof. Left multiplication by a lower unitriangular matrix acts on the first j rows by a determinant-one matrix. On the right, apply the minors formula for multiplication by an upper unitriangular matrix. Starting with the monomial pivot representative, precisely one column subset K_j has nonzero j-minor on these rows. A minor in columns L after the change can receive a contribution from K_j only if the ordered indices of K_j are componentwise at most those of L, by upper triangularity. For $L = K_j$ the multiplier is one. Every changed

matrix has the same pivot permutation, as seen from the ranks of its northwest rectangles. This proves invariance and identifies M_j as the value in the monomial representative. This argument allows arbitrary unitriangular changes over E.

Compute on an actual integral lift. The row lattices contain $\mathfrak{a}_i Y_{b,i}$ in each of the two frames Y_b, including the integral terminal vector $Y_{b,d}$. Pairing their exterior products gives $M_j \overline{\mathfrak{a}_{[j]}} \mathfrak{a}_{K_j} \subseteq \mathcal{O}_E$. At the two split marks those fractional ideals are units and $Z \equiv -E_{dd} \bmod Q$, by the marking in Theorem 116.8. For $j < d$ the row set omits d, so every entry of the selected square is zero modulo both primes, even if $d \in K_j$. Its determinant gains Q^j. Taking absolute ideal norms and square roots gives the bound for B_j. At d, the determinant identity in Proposition 116.1 gives $|N_{E/\mathbb{Q}} D_d| = |N_{F/\mathbb{Q}} \mathcal{G}| P_\gamma$. The definition of X_C in Theorem 117.28 gives the asserted equality.

Within the block the selected columns are $[a] \cup \{a+h-m+1, \dots, a+h\}$. The alternating pivot signs cancel the sign of reversal in the selected m-square, giving $M_{a+m} = D_a c^m$ for the first pivot c. The new denominator factor at step m is $\eta_{\{a+m\}} \eta_{\{a+h+1-m\}}$. It is independent of m, since the two exponents of the different have constant sum. This proves the interpolation formula, including the last block when one selected column is the terminal column. □

Theorem 117.34 (Summation over all supported block types)**.** *Fix the hyperplanes, transport and full marked index of Theorem 117.28. Write $\mathcal{W}_C = \prod_{\tau,j} (1 + |V_{j,\tau}|^2)^2$, $L = d(d+1)/2$, and let h be the length of the final block. Put $Z_0 = \max(1, Z, Y_0)$, with Y_0 the retained hyperplane threshold of that theorem. For $0 < \delta < 1$, $\lambda > 0$, $\sigma \geq 0$, and $u = \sigma + 2\lambda < h - \delta/2$, one has*

$$\sum_{\substack{[C] \text{ with final length } h \\ X_C \geq Z}} \frac{n_i(C) X_C^\sigma}{\mathcal{W}_C} \leq 2^{2d} |\mu(E)|\, C_\lambda^f B_\delta^{fd}\, Q^{-(4-\delta)fL}\, Z_0^{-(h-u-\delta/2)}.$$

Here B_δ is the integral in Proposition 117.31 and C_λ is as in Lemma 117.29.

Proof. Fix a composition with that final length. At each proper index inside a block use $\prod_\tau (1 + |V_{j,\tau}|^2)^2 \geq B_j^4$. If $j = d - h + m$ is inside the last block, the proposition, $P_\gamma \geq 1$, and interpolation give the stronger bound

$$B_j^4 \geq Q^{4jf} X_C^{2m/h}.$$

All these interior indices together thus contribute their full powers Q^{4jf} to the denominator, and the last block contributes the additional factor X_C^{h-1}. Apply $n_i(C) \leq |\mu(E)| C_\lambda^f X_C^{2\lambda}$ from Proposition 117.23. For the endpoint sum we are left with X_C^v, $v = u - h + 1$. If $v < 0$, use $X_C^v \leq Z_0^v$ and take $v' = 0$. Otherwise take $v' = v$. Thus the remaining numerator is $Z_0^{v-v'} X_C^{v'}$, and $0 \leq v' < 1 - \delta/2$.

At a nonterminal block endpoint j use the normalized complex coordinates of the proposition. Distinct determinant values are separated in the product of the f complex distances by Q^{jf}, by the same ideal containment applied to their difference. Fixing the earlier endpoint values gives the same separation at $j = d$, since $D_d + D_s$ belongs to $Q^d (\overline{\mathfrak{a}_{[d]}} \mathfrak{a}_{[d]})^{-1}$ and D_s is then fixed or zero. Each endpoint tuple accounts for at most the product of its block lengths many classes by Lemma 110.10, also for the two frequencies by Theorem 116.8.

For clarity put $P = Q^j$, and give the f discs radii $r_\tau > 0$ with geometric mean R. A nonempty box has $R \geq P$. After rescaling the radii to one, disjoint polydiscs of radius $P/(2R)$ give a count at most $9^f (R/P)^{2f}$. Integration against $\prod_\tau 4 r_\tau (1 + r_\tau^2)^{-3} dr_\tau$ inserts the reciprocal weights at this endpoint. At every nonterminal endpoint use $R/P \geq 1$ to replace $2f$ by $(4-\delta)f$. The resulting bound is $B_\delta^f Q^{-(4-\delta)jf}$.

At the final endpoint $R/P \ge Z_0^{1/(2f)}$, and $X_C^{v'} \le P^{-2fv'} \prod_\tau r_\tau^{2v'}$ in that box. Multiply the point count by this bound. In the count increase the exponent $2f$ to $(4-2v'-\delta)f$, extracting $Z_0^{-(1-v'-\delta/2)}$. After insertion of the numerator, the one-dimensional integral has the same integrand $4r^{5-\delta}(1+r^2)^{-3}$ as before. Thus the final endpoint costs $B_\delta^f Q^{-(4-\delta)df}$ times that power of Z_0. Multiplying by $Z_0^{v-v'}$ gives the claimed exponent. All integrations have nonnegative integrands. Sum from the last endpoint backwards using the conditional bounds. Interior indices supply an exponent 4 in place of $4-\delta$. There are at most 2^{d-1} compositions and the product of the lengths of their blocks is at most 2^d. Since $B_\delta \ge 1$ (integrate over $r \ge 1$ using $(1+r^2)^3 \le 8r^6$), the displayed bound follows. □

We next give a direct estimate on the original integral terms. For $b = 0, 1$ and an argument y on its real row define

$$T_{b,j}(y) = \prod_\tau \frac{\| \bigwedge_{i=1}^j y_\tau^{-1} Y_{b,i,\tau} \|}{\eta_{[j]}}, \qquad T_{b,0} = 1.$$

Use the common compatible positive reference norm on exterior powers. Each prefix wedge is fixed by its decorated stabilizer, so these are well-defined functions on the row quotient. Choose $p_j \ge 0$, and let $H_b(j; y)$ be the value at j of the least concave majorant on $[0, d]$ of the data $\log T_{b,e}(y)$, $0 \le e \le d$. Put

$$A_b(y) = \exp\left(\sum_{j=1}^d p_j H_b(j; y) \right), \qquad \mathcal{E}_b = \int_{N_b \backslash G_\infty} (V_b'(y))^\nu A_b(y)^2 \|W_b(y)\|^2 \, dy.$$

Here N_b is the integral row unipotent in Theorem 116.14. Use its height V_b', constants D, C_*, parameter $\nu > 4D$, and original quotient measure. For a last block of length h define

$$b_h = \sum_{m=1}^h (m/h) p_{d-h+m}.$$

Theorem 117.35 (A bound in the two original row measures)**.** *On any fixed pair of the retained rows, let γ range over any subset of its relevant supported transverse integral double cosets of positive Gram invariant. Let B_γ be its original signed orbital operator including its finite coefficient and normalized marked column as in Theorem 116.14. If the two displayed energies are finite, then*

$$\sum_\gamma (X_C P_\gamma)^{b_{h(\gamma)}/2} \|B_\gamma\|_{\mathrm{HS}} \le |c_{0,1}| \, C_* (1 - 2^{-(\nu-4D)})^{-f} \cdot Q^{-f \sum_{j=1}^d j p_j} (\mathcal{E}_0 \mathcal{E}_1)^{1/2}.$$

In particular $p_j = 2$ for every j gives the factor $Q^{-fd(d+1)}$ with moment $(X_C P_\gamma)^{(h(\gamma)+1)/2}$. The same right side bounds the norm of the sum over any of these terms after the corresponding weights have been inserted.

Proof. For a block endpoint e the pairing of the two prefix wedges, evaluated at y and $\gamma^{-1}y$, is D_e at each real place. The Hermitian pairing on the exterior reference space has operator norm one for its compatible positive norm. Therefore

$$B_e \le T_{0,e}(y) T_{1,e}(\gamma^{-1} y).$$

This holds also for $e = 0$. At any j, interpolate the logarithms of the endpoint bounds according to its block. The linear interpolant in each row is at most the corresponding concave majorant.

In the other direction, at every endpoint below d we have $\log B_e \geq fe \log Q$. At d we have the additional term $\frac{1}{2}\log(X_C P_\gamma)$. Linear interpolation puts the fraction m/h of that term at index $d-h+m$. Multiplication with the nonnegative weights p_j now gives, pointwise on the actual integration domain,

$$Q^{f\sum_j jp_j}(X_C P_\gamma)^{b_h/2} \leq A_0(y)A_1(\gamma^{-1}y).$$

The Hilbert–Schmidt norm of the rank-one integrand is bounded by the product of the section norms. The finite phase has absolute value one, and each normalized marked column has norm at most one. After extracting the displayed factor, enlarge the resulting positive sum to all the integral double cosets of the pair using $F_b = A_b\|W_b\|$. For completeness put $v_0 = V_0'(y)$ and $v_1 = V_1'(\gamma^{-1}y)$. Split F_0F_1 as the product of $F_0(v_0/v_1)^{\nu/2}$ and $F_1(v_1/v_0)^{\nu/2}$. Cauchy–Schwarz on the disjoint integration domains bounds the integral sum by the two square norms. Unfolding the first one gives the original row integral of $F_0^2 v_0^\nu$ times the translated sum of inverse heights $(V_1')^{-\nu}$. That sum is bounded by $C_*(1-2^{-(\nu-4D)})^{-f}$ by the box count in Theorem 116.14. Unfold the second square with the rows interchanged. These operations use the same integral stabilizers and Haar measures as the original signed operators. They hold for the nonnegative measurable weights by Tonelli's theorem. Taking square roots and restoring the pair volume proves the inequality. □

The last estimate includes transverse terms with vanishing proper leading minors at both terminal frequencies. To use it in Section 6.6 one still needs quantitative values for the two energies of the prescribed sections relative to their identity coefficient, including C_*, and the remaining period terms. For the underlying leading-minor convention see `Formalism/SharpRestriction/CuspSpanNormBounds/BruhatCellsKronecker/CellGeo.lean`, line 436.

117.12 Exterior moments at the two terminal frequencies

We evaluate the change of frequency in the two integrals of Theorem 117.35. Throughout, use the same retained fields, matrices, marked columns and Haar measures as in that theorem. Write $d = s+1$, $n = 2s+k+1$, $f = [F:\mathbb{Q}]$, and $R = |\operatorname{disc} F|^{1/f}$. For the nonnegative parameters $p_1, \dots, p_d$, put

$$P = \sum_{j=1}^{d} jp_j, \qquad G(e) = \sum_{i=1}^{e}(d-i).$$

For a row of terminal norm $-T_b$, choose at each real place the Witt identification sending its terminal vector to $\beta_\tau z$, where z is the middle unit negative vector and $|\beta_\tau| = \sqrt{T_b}$. Here $T_b = 1$ or T. The same identification sends its outer vectors to $e_1, \dots, e_s$. The adapted positive norm gives e_i length R^{d-i} and z length one. Thus the exterior normalization of a prefix in Proposition 117.33 is $\eta_{[e]} = R^{G(e)}$.

Proposition 117.36 (Transport of the two exterior moments)**.** *Use the translated height $V_b'(y) = V_b(a_\beta y)$ in Theorem 117.35, where a_β is the terminal change of variables in Proposition 106.13. Write A_1 for the exterior weight of that row after replacing its terminal vector by z, and μ_1 for the probability measure of its transported unit-frequency section. If $w_b^2 = \int_{N_b\backslash G_\infty}\|W_b\|^2$, then*

$$\frac{\mathcal{E}_b}{w_b^2} = T_b^{fP}\int_{B_b\backslash G_\infty} V_b(g)^\nu A_1(g)^2\, d\mu_1(g).$$

Here B_b is the full real decorated stabilizer and N_b is the original integral unipotent. The identity also holds for infinite integrals. Both moments in Theorem 117.35 are finite if $4D < \nu$

and $\nu + 2P \le L_0 = 8(f+1)(n+1)^3$. In particular $p_j = 2$ for all j and $\nu = 4D+1$ satisfy these conditions on the retained intervals.

Proof. At a real place a_β multiplies all outer vectors by β_τ, multiplies their duals by $\bar\beta_\tau^{-1}$, and fixes the middle space. With $y = a_\beta^{-1} g$, the vectors $y^{-1}e_i$ and $y^{-1}(\beta_\tau z)$ are consequently β_τ times the corresponding unit-model vectors at g. For every prefix of order e the positive exterior norm gains $T_b^{e/2}$ at that place. Its ideal normalization is the same for the two models. The logarithms of the global prefix norms therefore gain the affine function $e \mapsto fe\log(T_b)/2$. Adding an affine function commutes with taking the least concave majorant, by subtracting it from each candidate majorant. The exterior weight at y is $T_b^{fP/2} A_1(g)$.

The section norm and all the weights are invariant under B_b. Passing from the integral unipotent quotient to the real stabilizer quotient multiplies the numerator and w_b^2 by the same finite fibre volume. By Proposition 106.13, the squared section normalization cancels the Haar Jacobian under $y = a_\beta^{-1} g$. The two normalized measures are thus related by pushforward, and $V_b'(y) = V_b(g)$. Integration proves the formula.

For finiteness write $v_\tau(g_\tau)$ for the local factor of $V_b(g)$. A unit-model prefix wedge is fixed by the decorated stabilizer and has adapted norm $\eta_{[e]}$. The norm of its image under the inverse of a representative is at most $\eta_{[e]}$ times the e-th power of the operator norm of that inverse. Compatibility of the positive norms with the Hermitian form identifies this operator norm with that of the forward isometry. Take the infimum over the left stabilizer and multiply over the places. The resulting prefix bound is $V_b(g)^e$. Its logarithm is affine in e, hence $A_1(g) \le V_b(g)^P$. The unit section has finite moment of order L_0, since its weighted supremum times its unweighted L^1-norm bounds that square integral by Proposition 86.20. At this fixed row a change of adapted norm costs a finite factor. Since $V_b \ge 1$, this proves finiteness in the stated range. Finally $D \le nd(d+1)/2$ and $d \le n$; thus $4D + 1 + 2d(d+1) \le L_0$. □

Let $\mathcal{P}$ be a subdivision of the integer interval $[0, d]$ with its two endpoints included. For each j let $\lambda_{j,e}$ be the coefficients of linear interpolation at j between its two adjacent subdivision points e. At a subdivision point the sole coefficient is one. Define $\omega_e = \sum_{j=1}^{d} p_j \lambda_{j,e}$. In particular $\sum_e e\omega_e = P$. At one place in the unit-frequency model put

$$t_{e,\tau}(g) = \frac{\|g^{-1}(Y_1 \wedge \cdots \wedge Y_e)\|}{R^{G(e)}}, \qquad Y_i = e_i \ (i < d), \quad Y_d = z, \quad t_{0,\tau} = 1.$$

Use the local isometric embedding $J_{1,\tau}$ for this word and its actual positive compact column. For any unit vector u of the local lowest compact type set

$$I_{b,\tau}(\mathcal{P}) = \int_{B_{b,\tau}\backslash G_\tau} v_\tau(g)^\nu \prod_{e\in\mathcal{P}} t_{e,\tau}(g)^{2\omega_e} \, \|(J_{1,\tau}u)(g)\|^2 \, d\dot g.$$

Theorem 117.37 (A product of single-place integrals for each block type)**.** *In the finite range of Proposition 117.36 these integrals are independent of u. Between the maximum and the sum of $\prod_\tau I_{b,\tau}(\mathcal{P})$ over the 2^{d-1} subdivisions lies the normalized value $\mathcal{E}_b/(w_b^2 T_b^{fP})$. More precisely, fix a subdivision whose final block has length h, and let $\mathcal{A}$ be any set of the supported positive-Gram transverse cosets with those block endpoints. Then*

$$\frac{\sum_{\gamma\in\mathcal{A}} (X_C P_\gamma)^{b_h/2} \|B_\gamma\|_{\mathrm{HS}}}{|c_{0,1}| w_0 w_1} \le C_*(1 - 2^{-(\nu-4D)})^{-f} \left(\frac{\sqrt{T_0 T_1}}{Q}\right)^{fP} \cdot \prod_\tau \sqrt{I_{0,\tau}(\mathcal{P}) I_{1,\tau}(\mathcal{P})}.$$

Here B_γ, X_C, P_γ and b_h are the operators and weights of Theorem 117.35.

Proof. For fixed unit-model data at g, join the logarithms of the global prefix norms by line segments along $\mathcal{P}$. Every such polygon is bounded above by the least concave majorant. The upper boundary of the convex hull of the finitely many prefix points is itself one of these polygons. It attains the majorant simultaneously at every j. Consequently A_1^2 is the maximum over $\mathcal{P}$ of $\prod_{\tau,e} t_{e,\tau}^{2\omega_e}$, since $p_j \ge 0$.

For one polygon the integrand weight factors over the real places. Each local weight is right invariant under the reference maximal compact group. Compress multiplication by its truncation at a positive constant to the local isometric image of the lowest compact type. This positive operator commutes with that irreducible compact representation and therefore is scalar. For a product of truncations the global compression is the product of those scalars, since the real embedding is a tensor product by Corollary 106.21. This holds on all unit vectors of the product type. Let the truncation increase and apply monotone convergence. Finiteness also follows locally from the proof of the proposition. For any subdivision the resulting integral is $\prod_\tau I_{b,\tau}(\mathcal{P})$. Integrating the maximum gives the claimed lower and upper bounds.

On cosets of the specified type, use the polygon on the actual block endpoints on each of the two rows. At its endpoints the cross determinant is the pairing of the corresponding prefix wedges. Interpolation of Proposition 117.33 gives the pointwise lower bound $Q^{fP}(X_C P_\gamma)^{b_h/2}$ for the product of the two polygon weights, exactly as in Theorem 117.35. The square-norm unfolding there applies to these weights on their original row quotients. The translated-height count is the same $C_*(1-2^{-(\nu-4D)})^{-f}$. The affine frequency shift in the proposition also holds for the polygon, because $\sum_e e\omega_e = P$. Substituting its product integrals gives the displayed inequality, with the original finite columns and their phases in each B_γ. □

Theorem 117.38 (Size of the absolute majorant at prime frequency)**.** *Put $K_x = bn^4$ with $b = \exp(x^\varepsilon)$, and $\delta_x = \exp(-cbn^2)$ for the constant of Theorem 87.24, decreased if necessary. For each prescribed lowest-type section one has*

$$\frac{\mathcal{E}_b}{w_b^2} \;\ge\; (1-\delta_x)\left(\frac{T_b}{(RK_x)^{d-1}}\right)^{fP}, \qquad \prod_\tau I_{b,\tau}(\mathcal{P}) \;\ge\; (1-\delta_x)(RK_x)^{-2f\sum_e \omega_e G(e)}.$$

Suppose that at least one row has terminal norm $-T$. After division by $Z^{b_h/2}$, for any $1 \le Z \le \exp(Mfdx^\varepsilon)$ with fixed $M > 0$, the right side of Theorem 117.37 is at least

$$(1-\delta_x)C_* \exp\big((15-o(1))fPd\,x^\varepsilon\big).$$

For the right side of Theorem 117.35, normalized by $|c_{0,1}|w_0w_1$ and divided by the same factor, the coefficient 15 can be replaced by 16. Both estimates are uniform in the nonnegative p_j in the finite range. They concern the analytic upper bounds for the period.

Proof. In the localized mass estimate of Theorem 87.24 all successive Gram–Schmidt lengths at the prime frequency satisfy $a_{i,\tau} > \sqrt{T}K_x^{-(d-i)}$ for $i < d$ on the same set of measure at least $1-\delta_x$ for the normalized row. The last length is at least $\sqrt{T}$, since subtracting any earlier isotropic vector from the terminal one preserves its negative Hermitian norm. Each length scales by $\sqrt{T}$ under the frequency transport in the proposition. This gives a set of the same mass in the unit model on which the lengths are at least $K_x^{-(d-i)}$ at every place. The unit model for an initially unit row also admits the prime translate, by Corollary 106.21. Multiplying successive lengths computes the positive prefix wedge exactly, by the Gram determinant identity. Thus $t_{e,\tau} \ge (RK_x)^{-G(e)}$ simultaneously at all indices and places. Integrating the polygon weight over this region with $v_\tau \ge 1$ gives the second lower bound.

The concave majorant at j also dominates the chord from zero to d. Since $G(d) = d(d-1)/2$, this gives $\log A_1 \ge -fP(d-1)\log(RK_x)/2$ on the same region. Integrate its square and apply the exact frequency change. This proves the first lower bound.

For the two fixed polygons one has $\sum_e \omega_e G(e) \le (d-1)P$, since $G(e) \le (d-1)e$. Their product in the bound therefore costs at most $2(d-1)\log(RK_x)$ in the lower estimate for the logarithm per fP. For the concave majorants the cost is $(d-1)\log(RK_x)$. Further, $b_h \le P/d$ because at each index $j = d-h+m$ one has $m/h \le j/d$. The division by $Z^{b_h/2}$ thus costs at most $Mx^{\varepsilon}/2$ per fP. For $P = 0$ the asserted bounds follow directly from the total mass. For $P > 0$ use $\log(RK_x) = (1+o(1))x^{\varepsilon}$ since the conductor and n are polynomially bounded, and $\log Q = (16+o(1))x^{\varepsilon}$. Theorem 93.9 with $m = 2d$ gives $\log\sqrt{T_0T_1} \ge (17-o(1))dx^{\varepsilon}$. Now $d \to \infty$ along the intervals. Substituting these estimates gives the two stated constants. □

In particular the known lower cutoff Y_0 of Theorem 117.28 lies in this range of Z at prime frequency. At $p_j = 2$ both absolute bounds grow even after extracting that available Gram weight. On a repeated row its identity coefficient is $|c_{b,b}|w_b^2/\dim V_{\rho_b}$, by Theorem 113.24. Consequently normalization by that coefficient multiplies the bound by $\dim V_{\rho_b}$. The signed orbital operators may cancel. Their inner products before the passage to absolute values remain available in the comparison of Section 6.6.

117.13 A compact integral for the full signed orbital sum

Fix one retained pair of rows on the fields and intervals of Section 6.6. We keep both finite markings, both terminal frequencies and the original normalized columns. Write $b = \exp(x^{\varepsilon})$, and use $l = 0, 1$ for the two rows. As before $n = 2s+k+1$, $d = s+1$, $f = [F:\mathbb{Q}]$ and $R = |\operatorname{disc} F|^{1/f}$. In each row make the real Witt identification of Proposition 117.36. In its standard signed coordinates put

$$w(g) = \|g\|_{\mathrm{op}}, \qquad v_{l,\tau}^0(g) = \inf_{h \in B_{l,\tau}} w(hg), \qquad V_l^0 = \prod_{\tau} v_{l,\tau}^0.$$

We write $g_l(y) = a_l y$ in these row coordinates for the change to unit terminal frequency. Take $A \ge 1$. The cut on row l retains y if $v_{l,\tau}^0(g_l(y)_\tau) \le A$ at every place. For an integral double coset γ let $B_\gamma(A)$ be its ordinary signed orbital operator with the two cuts imposed at $y, \gamma^{-1}y$, and let B_γ have the full domain.

Proposition 117.39 (A uniform bound for the discarded part)**.** *Put $q = 4n^2$, $D = \sum_{i=1}^{d} i(n-i+1)$, $\nu = 4D+1$ and $P_\nu = C_*(1-2^{-(\nu-4D)})^{-f}$ as in Theorem 116.14. For all sufficiently large x on the common intervals one has $\nu \le n^2$. There is a constant C uniform in these data such that for every set $\mathcal{A}$ of supported integral cosets of the pair,*

$$\frac{\sum_{\gamma\in\mathcal{A}} \|B_\gamma - B_\gamma(A)\|_{\mathrm{HS}}}{|c_{0,1}|w_0w_1} \le 2P_\nu \exp(Cfsq\log b)A^{-(q-\nu)/2},$$
$$\frac{\sum_{\gamma\in\mathcal{A}} \|B_\gamma\|_{\mathrm{HS}}}{|c_{0,1}|w_0w_1} \le P_\nu \exp(Cfs\nu\log b).$$

The same second bound holds for the cut integrals. For the two exterior moments of Theorem 117.35 one also has, whenever $P = \sum_j jp_j$ and $\nu + 2P \le q$,

$$\mathcal{E}_l/w_l^2 \;\le\; T_l^{fP}\exp(Cfs(\nu+2P)\log b).$$

The constants here may depend on the fixed character and ε.

Proof. Let $d\mu_l$ be the probability measure of the transported unit-frequency row. Theorem 105.3 gives $\int (V_l^0)^t\,d\mu_l \le \exp(Cfst\log b)$ for $0 \le t \le q$, using Hölder's inequality for nonintegral t. This theorem works on the product lowest compact type. It bounds the first positive coordinate moment and all the successive ratios needed for the standard quotient

height. The first-coordinate statement in the source is at `Formalism/SharpRestriction/CuspSpanNormBounds/BesselShellMomentBounds.lean`, line 1634. For the adapted height one has $V_l(g) \le R^{fs} V_l^0(g)$. Indeed the change of metric in Proposition 115.10 normalizes the stabilizer and has operator norm R^s at each place. As $\log R = O(\log x) = o(\log b)$, its cost fits the same moment bound after increasing C. Proposition 117.36 and $A_1 \le V_l^P$ from its proof give the exterior estimate.

Use the translated heights $V_l'(y) = V_l(g_l(y))$. Under the normalized row measure the event discarded on row l implies $V_l^0(g_l(y)) > A$, since all its local factors are at least one. Its weighted squared norm satisfies

$$w_l^{-2} \int_{\text{discarded}} (V_l')^\nu \|W_l\|^2 \le R^{fs\nu} A^{-(q-\nu)} \int (V_l^0)^q \, d\mu_l.$$

Here $\nu \le n^2$ eventually since $D \le nd(d+1)/2$ and $d^2/n \to 0$ on the retained intervals. In the square-norm unfolding of Theorem 116.14, insert the discard indicator in the norm of row zero, leaving row one unrestricted. The inverse-height count is still P_ν. Cauchy–Schwarz now takes the square root of the last bound times the unrestricted moment of order ν on the other row. Interchanging the rows bounds the other discarded region. Their union accounts for the difference in the first inequality. Omitting both indicators bounds the absolute integrals over all cosets. This proves the other assertions, including the estimate on every restricted sum. □

We next express the integrands without choosing the scale of a Bessel functional. At a single real place put $G = U(p_+, p_-)$, $p_+ + p_- = n$, $B = N \rtimes C_0$ with C_0 the definite compact group, and give C_0 probability Haar measure as in Section 106.15. Write χ for the unit-frequency character on N, ρ for the compact column representation, σ for the row discrete series and d_σ for its formal degree. On its lowest compact type use a unit vector u and put $c_u(t) = \langle \sigma(t)u, u \rangle$, with the inner product linear in the first variable. If $a \in B$, write $a = n_a c_a$ in this decomposition. Define the matrix

$$K(y,z) = d_\sigma \int_B c_u(y^{-1}az)\, \overline{\chi(n_a)}\, \rho(c_a)^* \, da.$$

For $S \ge 1$ write K_S for the same integral restricted by $w(y^{-1}az) \le S$. Both matrices take values in $\mathrm{End}(E_\rho)$.

Lemma 117.40 (The relative kernel and its uniform tail)**.** *For the isometric unit-frequency embedding J,*

$$K(y,z) = (Ju)(z)(Ju)(y)^*.$$

Put $H = 10^4 n^4 (B_x + 1)$ where $B_x \ge 1$ bounds the absolute Harish–Chandra coordinates of the row. Put $\kappa = 4n^2 + 2n + 9$, and let v_B be the Haar measure in B of $\|a - 1\|_{\mathrm{op}} \le 1/2$. If $v^0(y), v^0(z) \le A$, for an absolute C the integral of the operator norm of the kernel integrand, including its factor d_σ, is at most

$$D_B(A) = d_\sigma v_B H^{Cn^3} A^{8n^2}.$$

On the part defining $K - K_S$ the bound is $D_B(A) S^{-\kappa}$. The same estimates apply to the absolute scalar coefficient before the unitary column factor. Moreover

$$D_B(A)\, \mathrm{vol}_{B\backslash G}\{v^0 \le A\} \le H^{Cn^3} A^{12n^2}.$$

Proof. On $\sigma \boxtimes \rho^\vee$ the normalized relative form of Theorem 106.20 equals $\ell(v)\overline{\ell(v')}$, with $\ell(w \otimes \lambda) = \lambda(\Lambda w)$ and $(Ju)(g) = \sqrt{d_\sigma}\Lambda(\sigma(g)u)$. The hypotheses are the retained discrete series, compact column, unitary generic character and the one-dimensional Bessel space in Corollary 106.21. The uniqueness theorem for these real Bessel data is Jiang, Sun and Zhu,

Theorem A. The relative form used here has the normalization deduced in Section 106.15 from Beuzart-Plessis, Sections 7.1–7.2. It is integration of its matrix coefficient over $(n_a c_a, c_a)$ against $\overline{\chi(n_a)}$. Apply it to $\sigma(z)u \otimes e_i^*$ and $\sigma(y)u \otimes e_j^*$ for an orthonormal dual basis. The row coefficient is $c_u(y^{-1}az)$, and the dual compact coefficient is the (i,j)-entry of $\rho(c_a)^*$. Reading all entries and multiplying by d_σ gives the first identity. It holds on these smooth translates by the continuity of the relative form. Absolute convergence also follows from the estimate below.

The gap choice in Lemma 86.19 exceeds $24n^2 + 2$ for the present rows at sufficiently large x. Thus Theorem 106.9 bounds the matrix coefficient by $H^{Cn^3} w(t)^{-L}$ with $L = 8n^2 + 2n + 9$. This bound also holds for the operator norm of the compression of $\sigma(t)$ to its lowest compact type, since that theorem allows any two unit vectors in the type.

For $U \geq 1$, a maximal $1/(2U)$-separated set in $\{a \in B : w(a) \leq U\}$, in ambient matrix operator norm, has at most $(1+4U^2)^{2n^2}$ elements. Compare the volumes of disjoint norm balls of half this separation in the real space of all complex n-by-n matrices. Each a in the original ball lies in a left translate of the set measured by v_B, since $w(a^{-1}) = w(a)$ on G. The Haar measure of the ball is consequently at most $5^{2n^2} U^{4n^2} v_B$.

Use representatives with $w(y), w(z) \leq 2A$. Independent changes on the left by B are changes of variable in this integral and supply only unitary factors on the two columns. The cut on $y^{-1}az$ is preserved in this substitution, and the Haar measure of B is bi-invariant. On the shell $2^j S \leq w(y^{-1}az) \leq 2^{j+1} S$ one has $w(a) \leq 8A^2 2^j S$. The Haar bound times the coefficient bound is at most $D_B(A)(2^j S)^{4n^2 - L}$ after enlarging its absolute constant. Sum the geometric series. Taking $S = 1$ also gives the bound on the entire integral, including its boundary. Finally any coset with $v^0 \leq A$ has a representative of norm at most $2A$. Multiplication by any element in the set measured by v_B gives norm at most $3A$. Quotient integration yields

$$v_B \operatorname{vol}_{B\backslash G}\{v^0 \leq A\} \leq \operatorname{vol}_G\{w \leq 3A\}.$$

Use the formal-degree volume estimate of Theorem 106.9 and absorb its dimension constant in H^{Cn^3}. This proves the last assertion. □

For row l at all real places use K_l for the product-group version of this integral. Its coefficient is $\langle \sigma_\infty(t) u_l, u_l \rangle$ for the unit row vector in the product lowest type. The Haar measures and formal degrees are multiplied, as are the characters, and the compact column is a tensor product. In $K_{l,S}$ impose the cutoff at every place. The local identity gives the global one for pure tensors and then for arbitrary u_l by polarization in each local coefficient and finite expansion in tensor bases. Put $v_l = \operatorname{vol}(N_l\backslash B_l)$, with N_l the integral unipotent, and $J_l = T_l^{fs(n-s)}$. For physical real arguments y, z define

$$k_{l,S}(y,z) = \frac{w_l^2}{v_l J_l} \operatorname{tr} K_{l,S}(g_l(y), g_l(z)).$$

Write k_l for the unrestricted value. From the change of variables in Proposition 106.13 and the lemma one gets $k_l(y,z) = W_l(y)^* W_l(z)$. Indeed w_l^2/v_l is the squared norm of the physical section on its real stabilizer quotient. Its unit model is isometric, and its two point values in the physical model supply the additional factor J_l^{-1}.

Let $\Gamma_\gamma = N_0 \cap \gamma N_1 \gamma^{-1}$ and let $Y_\gamma(A)$ denote the part of $\Gamma_\gamma\backslash G_\infty$ satisfying the two cuts. Write $\theta_\gamma = \epsilon_\gamma h_{i(\gamma)}$ for the original finite phase times its normalized marked column, so $\|\theta_\gamma\| \leq 1$. Define the scalar

$$\mathcal{T}(A,S) = \frac{1}{w_0^2 w_1^2} \sum_{\gamma,\eta \in \mathcal{A}} \theta_\gamma^* \theta_\eta \int_{Y_\gamma(A)} \int_{Y_\eta(A)} k_{0,S}(y,z) \cdot k_{1,S}(\eta^{-1}z, \gamma^{-1}y)\, dz\, dy.$$

With original lifts, the product of the two local covariances is the orbital covariance of the signed term. In particular these integrals use the same arithmetic stabilizers as the full period. Truncation respects that covariance by the change of variable in the proof of the lemma.

Theorem 117.41 (Approximation of the full signed square on the common fields)**.** *For each fixed $\theta > 0$ take $A = \exp(b^{\theta})$ and $S = \exp(b^{2\theta})$. At every sufficiently large x on the original common intervals, for either terminal norm on either end and every set $\mathcal{A}$ as above,*

$$\left| \frac{\| \sum_{\gamma \in \mathcal{A}} B_{\gamma} \|_{\mathrm{HS}}^{2}}{|c_{0,1}|^{2} w_{0}^{2} w_{1}^{2}} - \mathcal{T}(A,S) \right| \leq \exp(-b^{\theta}).$$

The sum defining $\mathcal{T}(A,S)$ uses finitely many integral cosets and compact integration ranges modulo their indicated stabilizers. The finite Gram factors $\theta_{\gamma}^{*}\theta_{\eta}$ are retained, including those for distinct cosets at the same marked index.

Proof. The original operator is the row-pair volume $c_{0,1}$ times the finite column tensored with the integral of $W_0(y)W_1(\gamma^{-1}y)^{*}$, by Theorem 113.22. Taking the Hilbert–Schmidt inner product of two such integrands gives exactly the indicated product of the two scalar kernels and the finite inner product. Thus the formula with the two row cuts but with unrestricted kernels gives the normalized cut square. For the unrestricted row domains the resulting double integrals are absolutely summable by Cauchy–Schwarz on the point values and the absolute-integral estimate in Proposition 117.39. The stabilizer integrations in the kernels are performed first.

For fixed A each row's cut has representatives in a compact set modulo N_l. Indeed standard representatives in each real quotient have norm at most $2A$, and $N_l \backslash B_l$ is compact. Reversing the frequency change preserves compactness of the chosen sets. Choose compact representative sets C_0, C_1 covering the two cuts. A coset contributing to the cut has a representative in $\Gamma \cap C_0 C_1^{-1}$, which is finite by discreteness. For each of these representatives the allowed y's lie, modulo Γ_{γ}, in finitely many translates of C_0. In fact two expressions $n_0^{-1}\gamma n_1$ giving the same representative have their target multipliers in the same left Γ_{γ}-coset. The closed cuts are therefore compact on the quotient. The bound on $y^{-1}az$ similarly gives a compact subset of the real stabilizer for fixed lifts.

To track the total measure, set $A' = R^{fs}A^{f}$. On either cut the translated height of that row is at most A'. The counting bound $M(A')$ of Theorem 116.14 and the unfolding there give, for either l,

$$\sum_{\gamma \in \mathcal{A}} \operatorname{vol} Y_{\gamma}(A) \leq M(A')\, v_l J_l \prod_{\tau} \operatorname{vol}_{B_{l,\tau} \backslash G_{\tau}} \{ v_{l,\tau}^{0} \leq A \}.$$

Indeed integrate the cut on one row and count all admissible translates of the other. Substitution on the first real row supplies J_l and integrating along its lattice fibre supplies v_l.

Take a common H for all places on the two rows. On a pair of row arguments satisfying the cut the absolute full integral or cut integral for the trace of K_l is at most $d_{\rho_l} \prod_{\tau} D_{B_{l,\tau}}(A)$, where d_{ρ_l} is the product column dimension. Removing at least one local tail costs at most $fS^{-\kappa}$ times this bound. To see this also for a vector which is not a pure tensor, bound its coefficient by the operator norm of the compression to the product lowest type. This operator is the tensor product of the local compressions. Their individual bounds in the lemma then dominate the integrand pointwise. The trace of a unitary column factor has modulus at most its dimension.

In the product of the two kernels, subtraction of the cut version costs at most $2fS^{-\kappa}$ times the product of these majorants, with the two factors $w_l^{2}/(v_l J_l)$ restored. Use $|\theta_{\gamma}^{*}\theta_{\eta}| \leq 1$ for this error. The square of the total domain volume is at most the product of its bounds from

the two rows. Inserting the last inequality of Lemma 117.40 at every place cancels the Haar normalizations. The error in the normalized cut square is at most

$$2f\, d_{\rho_0} d_{\rho_1} M(A')^2 H^{Cfn^3} A^{24fn^2} S^{-\kappa}.$$

The parameters $f, n, s, \log H, \log C_*$ grow at most polynomially in x on these intervals. The same applies to $\log d_{\rho_l}$ by the Weyl dimension formula and the retained weight bounds. From the expression for M in Theorem 116.14, the logarithm of the entire prefactor of $S^{-\kappa}$ is bounded by a polynomial in x times $1 + \log A$. With the chosen S, A this error is at most $\frac{1}{2}\exp(-b^\theta)$ for sufficiently large x.

Finally the normalized squared norms before and after cutting the rows differ by at most the product of the normalized difference norm and the sum of the two normalized norms. Apply both bounds of Proposition 117.39. Their prefactors have logarithm polynomially bounded in x, whereas $(q-\nu)\log A/2 \geq 3n^2 b^\theta/2$. This error is also at most $\frac{1}{2}\exp(-b^\theta)$. Addition proves the result. □

The formula controls the error of truncation even for the uncut signed square with both terminal norms equal to T. To derive (6.3) one must still estimate the signed compact integral itself after the required projection and compare the resulting period with its prescribed integral lattice. The independent bound (L) on the same intervals remains necessary in the present reduction.

117.14 The second moment of the signed Gram blocks

Section 6.5 uses the Hilbert–Schmidt squares of off-diagonal Gram blocks as well as the deviation of a diagonal block from the identity. We derive compact formulas for both statistics of the ordinary marked operator. Each row-coordinate cut has radius at most a fixed power of T in the transported unit model. We give a separate cutoff for the stabilizer kernel.

Proposition 117.42 (A simultaneous cut at all real places)**.** *Keep the family of Proposition 117.39. For any fixed $L > 0$ there is $a > 0$ such that, with $A = b^{as}$,*

$$\frac{\sum_\gamma \|B_\gamma - B_\gamma(A)\|_{\mathrm{HS}}}{|c_{0,1}| w_0 w_1} \leq b^{-Lsn^2}.$$

The sum of the uncut absolute integrals and the sum of the cut absolute integrals, in this same normalization, are each at most $b^{o(sn^2)}$. The estimates apply simultaneously to all retained row pairs and their marked indices on the common intervals. The constant a is independent of f, n, s, x.

Proof. Improve the stabilizer factor in the counting proof of Theorem 116.14. An integral stabilizer of a decorated exterior tuple modulo its integral unipotent injects into the isometry group of the underlying positive lattice, of rational dimension $m = 2fk$. That finite group has order at most

$$M_m = \prod_{p \leq m+1} p^{\sum_{j\geq 0} \lfloor m/(p^j(p-1)) \rfloor}$$

by Minkowski's theorem on finite subgroups of $\mathrm{GL}_m(\mathbb{Q})$ [2, Lecture I, Theorem 1]. Its p-exponent is at most $mp/(p-1)^2$. Summing over integers up to $m+1$ gives $\log M_m \ll m \log^2(2m)$. In the proof of Theorem 115.15 each tuple is counted with at most this stabilizer order. The exterior-lattice box count and its height exponent are unchanged. Hence we may use

$$C' = M_m \prod_{i=1}^{d} (1 + 2 \cdot 4^i T^i)^{2f(n-i+1)}$$

in both P_ν and $M(U)$ of Theorem 116.14. Denote the resulting bounds by P'_ν and $M'(U)$.

Here $d = s+1 = m/2 \asymp x^{1/2-\varepsilon/16}$ by Corollary 6.31, $f \ll_\psi x^{2+8\varepsilon}$ from the field construction in Proposition 6.4, and $n \ge k \asymp x^{9/2+10\varepsilon}$ as in Theorem 115.6. In particular $fd^2/n \to 0$. Since $\log T = O(d \log b)$,

$$\log C' \ll fk \log^2(2fk) + fnd^3 \log b = o(sn^2 \log b).$$

With $\nu = 4D + 1 = O(nd^2)$ also $fs\nu = o(sn^2)$. The unrestricted square-norm estimate of Proposition 117.39 with P'_ν proves the asserted absolute bound.

For the tail fix one real place τ, and use $q = 4n^2$. On the event $v_\tau^0 > A$, a single factor of $(V_l^0)^\nu$ can be bounded by $A^{-(q-\nu)}(v_\tau^0)^q$. At the other places use the moment of order ν. The local estimate in the proof of Theorem 105.3 is b^{Cst} at every place for $1 \le t \le q$. Compression of these right compact invariant weights gives their product on the whole product lowest type, as in Theorem 117.37. After summing over τ, the normalized adapted moment on the discarded set is at most

$$fR^{fs\nu} A^{-(q-\nu)} b^{Cs(q+(f-1)\nu)}.$$

On the other row the unrestricted moment is at most $R^{fs\nu} b^{Cfs\nu}$. Take the geometric mean, multiply by P'_ν, and add the estimate with the rows exchanged. Since $\nu/n^2 \to 0$ and $\log R = O(\log x)$, the logarithm of this bound divided by $sn^2 \log b$ is at most $-2a + O(1) + o(1)$ for fixed a. Choose a large enough for the required L. Finally $\log T = (34 + o(1))d \log b$; thus $A \le T^{O_L(1)}$ for all sufficiently large x. □

Consider a finite collection of source and target rows as in Theorem 113.22, with the same E, n, s, k at each x. Let Z_{ri} be the marked block from source i to target r, and Z_i the map to the direct sum of the targets. One may keep any specified subset of the supported double cosets in each block. Write $\lambda_{ri} = |c_{ri}| w_r w_i$, with zero for pairs in different lattice isometry classes. For a compatible pair use the actual coefficient c_{ri}, including its phase if any.

Define a scalar $\mathcal{U}_{rtij}(A, S)$ for the four ordered blocks $(r,i), (r,j), (t,j), (t,i)$ as follows. Take a coset γ_h of each block, in this order, and use $y_h \in Y_{\gamma_h}(A)$, $z_h = \gamma_h^{-1} y_h$. The fiber coordinates are carried back to their original rows before evaluating a kernel. Let θ_h be the original normalized finite column with its phase for this coset. On the product of the four cut domains integrate

$$(\theta_1^* \theta_2)(\theta_3^* \theta_4)\, k_{r,S}(y_1, y_2) k_{j,S}(z_2, z_3) k_{t,S}(y_3, y_4) k_{i,S}(z_4, z_1).$$

Sum these integrals and multiply by $\overline{c_{ri}} c_{rj} \overline{c_{tj}} c_{ti}$. If one of the blocks is structurally zero set $\mathcal{U}_{rtij} = 0$. Also write $\mathcal{U}_{ri}^{(2)}$ for $\lambda_{ri}^2 \mathcal{T}(A, S)$ of Theorem 117.41 applied to that single block. These formulas use only the finite data, local discrete-series coefficients and the Haar normalizations specified in Section 117.13.

Theorem 117.43 (Mixed Gram moments on the original signed rows)**.** *For each fixed $L > 0$ one can choose constants $a, c > 0$ so that $A = b^{as}$, $S = b^{cfn}$ give, with $\delta = b^{-Lsn^2}$,*

$$\left| \|Z_{ri}\|_{\mathrm{HS}}^2 - \mathcal{U}_{ri}^{(2)} \right| \le \delta \lambda_{ri}^2,$$
$$\left| \operatorname{tr}(Z_{ri}^* Z_{rj} Z_{tj}^* Z_{ti}) - \mathcal{U}_{rtij} \right| \le \delta \lambda_{ri} \lambda_{rj} \lambda_{tj} \lambda_{ti}.$$

Consequently the original off-diagonal or diagonal Gram block $K_{ij} = Z_i^* Z_j$ satisfies

$$\left| \|K_{ij}\|_{\mathrm{HS}}^2 - \sum_{r,t} \mathcal{U}_{rtij} \right| \le \delta \left(\sum_r \lambda_{ri} \lambda_{rj} \right)^2.$$

All estimates are uniform on the same intervals and in all retained subsets. They apply also when indices in the four ordered blocks coincide.

Proof. For uncut kernels and cut row arguments, take tensor products of the original marked column with the rank-one operator $W_{\mathrm{target}}(y_h)W_{\mathrm{source}}(z_h)^*$. The adjoints on factors one and three give precisely the two finite inner products. Multiplying the four matrices and taking the trace contracts each of the four repeated real rows with itself. The contractions in cyclic order are $W_r(y_1)^*W_r(y_2)$, $W_j(z_2)^*W_j(z_3)$, $W_t(y_3)^*W_t(y_4)$ and $W_i(z_4)^*W_i(z_1)$. They are the exact kernels k_l of Section 117.13. This proves the cut-domain formula with full kernels and retains the phases of distinct cosets. The absolute integrability follows from the product of the four absolute-integral bounds in Proposition 117.42.

First changing the four operators to their cut-domain versions costs at most $b^{-(L+1)sn^2}$ times the four pair factors for large x, on taking a from that proposition with $L+3$. Indeed subtract one factor at a time and use $|\operatorname{tr}(D_1^*D_2D_3^*D_4)| \le \prod_h \|D_h\|_{\mathrm{HS}}$. This follows from the Hilbert–Schmidt Cauchy–Schwarz inequality and $\|D_h\|_{\mathrm{op}} \le \|D_h\|_{\mathrm{HS}}$. The other three norms are at most their pair factors times $b^{o(sn^2)}$.

Next truncate the four relative kernels at S. There are at most $4f$ choices of a local tail when the product is subtracted one row at a time. By Lemma 117.40 its contribution in that place costs $S^{-\kappa}$ relative to the product of the absolute kernel majorants. For each edge u, v, the sum of its cut domain volumes has upper bound

$$M'(A')\left(v_uJ_u\,v_vJ_v\prod_{\tau}\operatorname{vol}X_{u,\tau}\operatorname{vol}X_{v,\tau}\right)^{1/2}, \quad X_{u,\tau}=\{v^0_{u,\tau}\le A\}, \quad A'=R^{fs}A^f.$$

This is the geometric mean of the two unfolding bounds in the proof of Theorem 117.41, with the improved stabilizer factor. Multiply these four edge bounds. Each occurrence of a row in the cycle is incident to two edges, so the volume of its local coset ball occurs once, and its v_uJ_u cancels the denominator from its kernel. The four w_u^2's and four absolute pair coefficients give precisely the four λ's. Thus after division by their product the truncation error is at most

$$4f\,d_{\rho_r}d_{\rho_t}d_{\rho_i}d_{\rho_j}\,M'(A')^4H^{Cfn^3}A^{48fn^2}S^{-\kappa}.$$

The exponent of A is the sum of the four exponents $12fn^2$ from the integrated kernel bound. Repeated rows give repeated factors in these products.

For the present choice of A, $\log A' = O_a(fs\log b)$. In $M'(A')$ the box exponent costs $O_a(Dfs\log b)$. The binomial factor costs $O_a(f\log(2+s\log b))$, from $\binom{N+f}{f} \le (e(1+N/f))^f$. Together with the bound on C' this gives $\log M'(A') = o(sn^2\log b)$. The retained weight bound gives $\log H = O(\log b)$ and $\log d_{\rho_u} = O(fn^2\log b)$ by the compact Weyl dimension formula. With $\kappa \ge 4n^2$, taking c sufficiently large therefore bounds the last error by $b^{-(L+1)sn^2}$. The same argument for two kernels has two edge factors and exponent $24fn^2$ in A, as in Theorem 117.41. This proves both individual estimates after adding the row-cut errors. Finally $\|K_{ij}\|_{\mathrm{HS}}^2 = \sum_{r,t}\operatorname{tr}(Z_{ri}^*Z_{rj}Z_{tj}^*Z_{ti})$. Summing the error bounds proves the assertion. □

Corollary 117.44 (Error in the diagonal rank statistic)**.** *Let the dimension of source i be r_i. Set $F_i = \sum_r \lambda_{ri}^2$. With the same cuts one has*

$$\left|\|K_{ii}-I\|_{\mathrm{HS}}^2 - \left(\sum_{r,t}\mathcal{U}_{rtii} - 2\sum_r \mathcal{U}^{(2)}_{ri} + r_i\right)\right| \le \delta(F_i^2+2F_i).$$

Proof. The block K_{ii} is positive self-adjoint. Expand the square as $\|K_{ii}\|_{\mathrm{HS}}^2 - 2\operatorname{tr}K_{ii} + r_i$ and use $\operatorname{tr}K_{ii} = \sum_r \|Z_{ri}\|_{\mathrm{HS}}^2$. The two estimates in the theorem give the stated errors. □

For the ordinary projected blocks, both signed statistics used in Section 6.5 can therefore be tested with finite coset sets and compact real domains, at the stated row-pair scales. The

inequalities for the values of these integrals, and the comparison with the integral theta maps on those same scales, are still required in Section 6.6.

117.15 Comparison ranks in the full ordinary Gram moments

We estimate a portion of the signed integrals of Section 117.14 on the original fields. The estimate uses the ranks of the differences of the two actual finite markings. It also applies to a choice of integral coordinates in the same finite projection. Both terminal frequencies are allowed.

Write $f = [E_x^+ : \mathbb{Q}]$, $m = 2d = 2s + 2$, $n = 2s + k + 1$, and $b = \exp(x^{\varepsilon})$. The parameters in Corollary 6.31 give $d \asymp x^{1/2-\varepsilon/16}$, $k \asymp x^{9/2+10\varepsilon}$, and $f \ll_{\psi} x^{2+8\varepsilon}$. In particular $n/k \to 1$. Theorem 93.9 gives $\log T = (34 + o(1))d \log b$. In Section 117.14 the cut of fixed power in T is the cut on each transported row coordinate. The separate integration over its stabilizer uses the radius b^{cfn}.

For each ordinary block write its signed orbital expansion as

$$Z_{ri} = \sum_{\gamma} h_{a_\gamma} \otimes P_{\gamma}^{ri}.$$

Here h_a is the unit column of the prescribed finite matrix at the index a, and P_{γ}^{ri} includes the finite phase, the real orbital integral, and its pair coefficient c_{ri}. Let D_i be the complex dimension of row i's positive column at infinity. With w_i and $\lambda_{ri} = |c_{ri}|w_r w_i$ as in Section 117.14, put

$$\sigma_{ri} = \frac{\lambda_{ri}}{\sqrt{D_r D_i}}, \qquad U_x = fk\{d^3 \log b + k \log(2x)\}.$$

On a repeated row σ_{ii} is the modulus of the actual identity coefficient, by Theorem 113.24. For other pairs this definition only uses their own coefficient and dimensions. Structurally zero pairs have $\sigma_{ri} = 0$.

Lemma 117.45 (Absolute mass at the identity normalization)**.** *There is a constant C, independent of the retained rows and of x on the common intervals, such that*

$$\sum_{\gamma} \|P_{\gamma}^{ri}\|_{\mathrm{HS}} \le \sigma_{ri} \exp(CU_x).$$

This bound also holds if each norm of an integral is replaced by the integral of the Hilbert–Schmidt norm of its integrand, including the absolute pair coefficient.

Proof. The absolute unfolding in Proposition 117.39, with the improved counting factor of Proposition 117.42, bounds the latter sum by $\lambda_{ri} P'_{\nu} b^{Cfs\nu}$. Here $\nu = 4\sum_{j=1}^{d} j(n-j+1) + 1 = O(nd^2)$ and

$$\log P'_{\nu} \ll fk \log^2(2fk) + fnd^3 \log b + f.$$

Indeed the finite stabilizer provides the first term, the d lattice factors have logarithm $O(fnd^2 \log T)$, and the dyadic series contributes $f \log 2$. As f, k are polynomial in x, the sum of these costs and $Cfs\nu \log b$ is $O(fkd^3 \log b)$.

For the dimensions use the actual column parameters in the proof of Lemma 93.8. At one real place the Weyl product is $\prod_{a<b'}(u_a - u_{b'})/(b' - a)$. Each inner entry $u_2, \dots, u_{k-1}$ lies between successive v_j, all polynomial in x with a common bound. A factor involving an endpoint is at most $3\lceil xb \rceil$ by the span bound there. There are at most $2k - 3$ such factors and at most $k(k-1)/2$ inner factors. Multiplying over the f real places gives

$$\log D_i \ll f\{k^2 \log(2x) + k \log b\}.$$

This estimate absorbs both dimensions in $\lambda_{ri} = \sigma_{ri}\sqrt{D_r D_i}$ at the asserted cost. The unfolding bounds integrals of products of the two section norms, so it gives the asserted stronger form before integration. □

Let $G = \prod_{v \mid g_1 g_2} \mathrm{GL}_k(\mathbb{F}_{q_v})$, using the f split factors over each marked rational prime. For $a, a' \in G$ define

$$R(a, a') = \sum_v \operatorname{rank}_{\mathbb{F}_{q_v}}(a_v^{-1} a'_v - 1).$$

For a squared block norm write $R = R(a_1, a_2)$ for the two orbital terms. In the fourfold trace of Theorem 117.43 use $R = R(a_1, a_2) + R(a_3, a_4)$, with that theorem's ordering.

Theorem 117.46 (Large comparison ranks in the signed moments)**.** *For each fixed $L > 0$ one can choose $C_L > 0$ so that, with*

$$J_x = \left\lceil C_L f \left(d^3 + \frac{k \log(2x)}{\log b} \right) \right\rceil, \qquad J_x/(fk) \longrightarrow 0,$$

the following bounds hold on every retained row pair or quadruple. In the orbital expansion of $\|Z_{ri}\|_{\mathrm{HS}}^2$, the sum of absolute values of the terms with $R \geq J_x$ is at most $\exp(-LU_x)\sigma_{ri}^2$. In the expansion of $\operatorname{tr}(Z_{ri}^* Z_{rj} Z_{tj}^* Z_{ti})$, the corresponding sum is at most

$$\exp(-LU_x)\, \sigma_{ri}\sigma_{rj}\sigma_{tj}\sigma_{ti}.$$

At any rank, a term is zero if a paired relative marking is nonunipotent at one place. The estimates hold for any chosen subsets of the original orbits, including the full operators.

Proof. Write $U_0 \subset G$ for the product of the upper unitriangular subgroups. The finite matrix is the one of Lemma 96.2. Within each family gU_0, a column occupies one coordinate and its unnormalized entry has modulus one. Two columns occupy the same coordinate precisely when $a' a^{-1} \in g U_0 g^{-1}$. Consequently $|h_a^* h_{a'}|$ is bounded by the proportion of such families. The test is independent at the different places for the uniform measure on G/U_0. Locally a nonunipotent matrix fixes no decorated flag in this sense. For a unipotent one the proportion is at most $q_v^{-kR_v/3}$, where $R_v = \operatorname{rank}(a_v^{-1} a'_v - 1)$, by Lemma 112.1. The two orders of the relative matrix are conjugate, so they give the same rank and probability. Taking the product yields the bound $\exp(-kR(a, a') \log q_*/3)$, $q_* = \min(g_1, g_2)$.

The two-term expansion has one such factor. The four-term expansion has the product of the two factors from the adjoints on the first and third blocks. Its remaining trace has absolute value at most the product of the four Hilbert–Schmidt norms by two applications of Cauchy–Schwarz, as in Theorem 117.43. The two-term case uses Hilbert–Schmidt Cauchy–Schwarz directly. Absolute convergence and Lemma 117.45 therefore bound either restricted sum, with $p = 2$ or 4 factors respectively, by

$$\left(\prod_{h=1}^{p} \sigma_h \right) \exp\Bigl(pCU_x - \tfrac{1}{3} k J_x \log q_* \Bigr),$$

where σ_h belongs to the corresponding block. The same inequality holds after taking absolute values of the real integrands before integrating. The two marked primes satisfy $q_* \geq (kb)^8$ by Theorem 113.26. Hence $kJ_x \log q_* \geq 8C_L U_x$, which proves both estimates on increasing C_L. Finally $d^3/k \to 0$ and $\log(2x)/\log b \to 0$, giving the claimed limit. □

Corollary 117.47 (One integral selection of marked coordinates)**.** *Choose a reference index a_r in each target row, for instance the index of its identity isometry. Fix nonnegative weights*

ω_{ri} on a finite collection of blocks with the same finite marked group at a given x. There exists one family gU_0 for the entire collection such that the following holds. In each target row select the coordinate of a_r in that family from the unnormalized table and divide by its root of unity. Let X_{ri} be the contribution to this selected coordinate from orbits with $R(a_r, a_\gamma) \geq J_x$. Then, increasing C_L if necessary,

$$\sum_{r,i} \omega_{ri} \|X_{ri}\|_{\mathrm{HS}}^2 \leq \exp(-LU_x) \sum_{r,i} \omega_{ri} \sigma_{ri}^2.$$

The selection has an integral right inverse on its finite coordinates. The same assertion holds if the collection includes several lists of orbital coefficients on these marks.

Proof. Choose the family uniformly. Set $M_{ri} = \sum_\gamma \|P_\gamma^{ri}\|_{\mathrm{HS}}$. For a fixed choice, the triangle inequality bounds $\|X_{ri}\|_{\mathrm{HS}}$ by the sum of the norms over the high-rank terms which occupy the reference coordinate. This sum is at most M_{ri}, so its square is at most M_{ri} times the sum itself. For each such orbit the probability of occupying the coordinate is at most $\exp(-kJ_x \log q_*/3)$ by the decorated-flag count in the theorem. Thus the expected squared norm is at most $M_{ri}^2 \exp(-kJ_x \log q_*/3)$. Absolute summability justifies the expectation of the series. Insert Lemma 117.45, sum with the chosen weights, and take a family whose sum is no greater than the mean. This argument permits arbitrary dependence among the collision events for different rows. Inserting a coordinate back in its selected position and multiplying by the original root of unity gives the asserted integral right inverse. □

In particular the normalized diagonal tests may discard the high-rank terms even after division by the squared or fourth power of the identity coefficient. For the mixed Gram square the discarded absolute contribution is at most $\exp(-LU_x)(\sum_r \sigma_{ri}\sigma_{rj})^2$, by summing over its target indices. The surviving relative markings are unipotent at every marked place with total rank below J_x. Their terms include different orbits at the very same marked index. Their signed sum, its comparison with the prescribed integral theta maps, and the full strict cohomology sum remain to be bounded on the intervals in Section 6.6.

117.16 Rational isometries in the mixed ordinary Gram moments

A mixed Gram term has four positive column spaces. Its four rational transports compose to an endomorphism of the first one, even when the two input rows are different. We show that up to an exponentially small error this endomorphism fixes a subspace of codimension $o(k)$ over E_x. The error is measured relative to the identity normalizations of Section 117.15. We use the same fields, auxiliary primes, translates and two terminal frequencies.

Lemma 117.48 (A product cutoff at the identity normalization)**.** *Use the unit-frequency row coordinates and the product height V_l^0 of Section 117.13. On each side of each ordinary orbital integral impose $V_l^0 \leq A$, where $A = b^{Cfs}$. For $L > 0$ one can take C independent of x and of the rows so that the sum over cosets of the integrals of the norms on the discarded domains, including the absolute pair coefficient, is at most $\sigma_{ri} \exp(-LU_x)$. At each retained point the translated height $\widetilde{V}_l$ is at most $b^{C'fs}$ for another uniform constant.*

Proof. Write $q_0 = 4n^2$ and $\nu = 4\sum_{j=1}^{d} j(n-j+1) + 1 \leq n^2$. In the row probability measure of Proposition 117.39, $\int (V_l^0)^u \leq b^{C_0 fsu}$ for $0 \leq u \leq q_0$. The adapted translated height satisfies $\widetilde{V}_l \leq R^{fs} V_l^0$, where $\log R = O(\log(2x))$. On the deleted set insert $\mathbf{1}_{V_l^0 > A}(V_l^0)^\nu \leq A^{-(q_0-\nu)}(V_l^0)^{q_0}$. Apply Cauchy–Schwarz in the unfolded coset integral as in Theorem 116.14,

first with the deletion in one row and then in the other. The inverse-height count is P'_ν of Proposition 117.42. With a uniform C_1 the sum of the resulting integrals of norms is at most

$$2\lambda_{ri} P'_\nu \, b^{C_1 f s q_0} A^{-(q_0-\nu)/2}.$$

This argument puts the cutoff on the product of the local heights. It permits a large coordinate at one real place when the product over all real places is small.

The count and the column-dimension estimates in the proof of Lemma 117.45 give

$$\log P'_\nu \ll fk \log^2(2fk) + fnd^3 \log b, \qquad \log(\lambda_{ri}/\sigma_{ri}) \ll f\{k^2 \log(2x) + k \log b\}$$

on a nonzero block. Each right side, and U_x, is $o(fsn^2 \log b)$. Indeed $n/k \to 1$, $d = s+1$, $d^2/k \to 0$, and $\log(2x)/(s \log b) \to 0$ on the common fields. Since $q_0 - \nu \geq 3n^2$, the negative term in the exponent gives the asserted bound on taking C sufficiently large. The bound for $\widetilde{V}_l$ follows from the same change of metric. The zero block has vanishing pair coefficient. □

Proposition 117.49 (A common lattice for each bounded transport)**.** *At every vertex use the completed positive lattice $\widehat{L}$ of Lemma 113.6. Let a compatible orbital term meet the two product cutoffs of the preceding lemma. Its rational transport $j : P_0 \to P_1$ has the following property in both directions. In its source lattice there is a full sublattice C_j mapped into the target lattice. On C_j the reduction in the ambient lattices is the original period index at every mark. If J_j is the index ideal of C_j in its source, then*

$$H_j = (N_{E/\mathbb{Q}} J_j)^{1/(2f)} \leq b^{C_2 d^2}.$$

The same constant works for all the terms, including both terminal frequencies.

Proof. Put $e = \dim(X_0 \cap X_1)$, $t = s - e$, and $Q = g_1 g_2$. By Theorem 86.3, the common positive quotient has rank k or $k-1$. If $e > 0$, compatibility forces a common totally negative vector in the two extended flags. Indeed their successor maps carry a common vector at position i to vectors differing by the sum of the two initial spans through i. Subtract vectors in those spans to get a common vector at the next position with nonzero leading coefficients. At the final position its norm is a nonzero terminal coefficient squared in absolute value times the negative terminal norm. This is the argument of Lemma 115.13, also for $e = s$. Thus the rank-$(k-1)$ case has $e = 0$. Its transport is the extension with prescribed determinant. Lemma 117.1 gives the required sublattices with $H_j = h_\gamma$. Proposition 116.1 gives at a retained point

$$h_\gamma \leq (T_0 T_1)^d \left(\frac{\widetilde{V}_0 \widetilde{V}_1}{Q^f} \right)^{2d/f}.$$

Here the saturation index and its factor P_γ are at least one. The cutoff bound and $\log T_l = O(d \log b)$ imply the claim.

In rank k, a common negative vector z_* satisfies $Y_b = X_b + E z_*$. Use the isometry supplied by the shared positive quotient. We give the sublattice in the source at end 1 after putting both frames in the common ambient lattice Λ. Write $I = X_0 \cap X_1$, $U_b = (\Lambda \cap X_b)/(\Lambda \cap I)$. The pairing h/Q gives an integral map $B : U_1 \to U_0^\sharp$ of rank t, nonsingular over E, as in Lemma 96.27. Let $\mathfrak{b}$ be its determinant ideal, equal to the unit ideal when $t = 0$. The order of its cokernel is $N\mathfrak{b}$ by elementary divisors.

A vector of the original positive lattice L_1 has a representative $w \in \Lambda \cap Y_1^\perp$. The functional $h(-, w)/Q$ on U_0 is integral. Away from Q this follows from integrality of the ambient pairing. At a mark w pairs to zero modulo that mark with X_0, since the two reduced isotropic spaces

agree. The functional kills I. Changing w by a vector in $\Lambda \cap X_1$ changes the functional by an element of $B(U_1)$. Its class therefore defines a homomorphism $L_1 \to \operatorname{coker} B$. Take its kernel for C_j. For a vector of that kernel choose $u \in \Lambda \cap X_1$ with $h(-, w-u) = 0$ on X_0. The integral vector $w-u$ still pairs to zero with Y_1 and with z_*. Thus it is a representative in the common perpendicular. It maps into the original lattice L_0, and gives the prescribed change of basis in the two marked positive quotients because reduction of u belongs to the common reduced outer space. This is valid also where the divided cross pairing is singular modulo the mark. Reversing the orientation uses the conjugate transpose pairing with cokernel of the same order. By Lemma 113.6, $[\widehat{L}_b : L_b] \le T_b^f$, since an enlargement has length one at each of f primes over the terminal prime. These enlargements are away from both marks. Consequently $H_j \le \sqrt{T_1}(N\mathfrak{b})^{1/(2f)}$ in this orientation.

For $e > 0$, Theorem 115.14 shows that $\mathfrak{b}$ divides $(D_t/Q^t)\overline{\mathfrak{h}_t}\mathfrak{h}_t$, where D_t is the determinant on the first t frame vectors. For $e = 0$ there is equality by taking determinants in the outer line lattices. For $t = 0$ the determinant and the ideals are one. The exterior Cauchy–Schwarz estimate in Proposition 116.1 applies to the first t vectors without any transversality requirement. Taking norms of the determinant ideal therefore gives

$$(N\mathfrak{b})^{1/(2f)} \le Q^{-t}(T_0T_1)^{t/2}(\widetilde{V}_0\widetilde{V}_1)^{t/f}.$$

The factor $\eta(\mathfrak{h}_t)^2$ in that estimate cancels $(N\mathfrak{h}_t)^{1/f}$. The product bound in the lemma and $0 \le t \le s$ give $\log H_j = O(d^2 \log b)$, also in the reverse orientation. □

Theorem 117.50 (Rational rank in the second and fourth moments)**.** *In the second moment of one ordinary block put $M = j_1^{-1}j_2$. In the mixed trace use the block order $(r,i),(r,j),(t,j),(t,i)$ of Theorem 117.43 and put*

$$M = j_1^{-1}j_2j_3^{-1}j_4.$$

This is an automorphism of the positive space of the first source row over E_x. For each $L > 0$ there is C_L such that the sum of the absolute values of the orbital terms with

$$\operatorname{rank}_{E_x}(M-1) > C_L\left(d^3 + \frac{k\log(2x)}{\log b}\right)$$

is at most $e^{-LU_x}\sigma_{ri}^2$ in the second moment, or $e^{-LU_x}\sigma_{ri}\sigma_{rj}\sigma_{tj}\sigma_{ti}$ in the mixed trace. The rank cutoff is $o(k)$. In the mixed trace the incoming positive spaces need only be linked by the four transports appearing in the term.

Proof. First impose the product cutoffs. For any subset of tuples the sum of the absolute differences between the uncut and cut orbital terms is at most $e^{-(L+1)U_x}$ times the indicated product of normalizations for sufficiently large x. To see this subtract one factor at a time. Its sum of discarded absolute integrals is bounded by Lemma 117.48 with a larger exponent. The other factors have absolute mass at most $\sigma\exp(CU_x)$ by Lemma 117.45, also after cutting. Finite columns have norm one, and an absolute trace of two or four matrices is at most the product of their Hilbert–Schmidt norms. These bounds justify the summations.

For a retained tuple write its successive maps in composition order, starting at the first source, as $g_1', \dots, g_h'$, where $h = 2$ or 4. For example the first map in the fourfold case is j_4. Let C be the sublattice of the starting $\widehat{L}$ on which every partial composite lands in the prescribed domain of the next map from Proposition 117.49. It is constructed by imposing these conditions in turn. At step l the quotient of the previous lattice by the new one injects into the local ambient lattice modulo $C_{g_l'}$ via the preceding partial composite. Thus for its index ideal $J = [\widehat{L} : C]_{\mathrm{id}}$ one has $v_{\mathfrak{p}}(J) \le \sum_l v_{\mathfrak{p}}(J_{g_l'})$ at every finite prime. The composite

M carries C into $\widehat{L}$, and its action on C in the marked quotients is the product M_v of the prescribed residue matrices in the same order.

Put $\rho = \operatorname{rank}_E(M-1)$ and $r_v = \operatorname{rank}_{\mathbb{F}_{q_v}}(M_v - 1)$. Since the space is totally positive, M is unitary at all complex embeddings and diagonalizable there. The coefficient c of $Y^{k-\rho}$ in $\det(M-(1+Y)I)$ is nonzero, and $|\tau c| \le 2^{\rho}$ at every embedding, by multiplying the nonzero eigenvalue differences. In a local basis of C and of $\widehat{L}$, the polynomial for $(M-1)|_C - Y\ \mathrm{incl}_C$ has integral entries in its defining matrix. Its indicated coefficient is c times a generator of J. At a mark the constant matrix has residual rank at most r_v. The coefficient takes ρ columns from that matrix. Every minor of order ρ of the constant matrix is divisible by the uniformizer to power $(\rho - r_v)_+$, by elementary divisors. The conjugate split factor has the same rank bound since the marked maps there are the contragredients of the given maps, also after composition. Taking the norm of the integral ideal $(c)J$ and using both factors over each v proves

$$\prod_{v \mid Q} q_v^{(\rho - r_v)_+} \le 2^{f\rho} \prod_{l=1}^{h} H_{g_l'}^{f}.$$

In particular

$$\rho \log(Q/2) \le f^{-1} \sum_{v \mid Q} r_v \log q_v + \sum_{l=1}^{h} \log H_{g_l'}.$$

With all marked bases in their prescribed coordinates, in the fourfold case $M_v = a_1^{-1} a_2 a_3^{-1} a_4$. The rank of its difference from one is at most the sum of the ranks of $a_1^{-1}a_2 - 1$ and $a_3^{-1}a_4 - 1$, by writing a product minus identity as $A(B-1) + (A-1)$. In the twofold case the first rank alone suffices. Thus $\sum_v r_v \le R$ for the comparison rank in Section 117.15. For each v, $\log q_v \le \log Q$, while $\log Q = (16 + o(1)) \log b$. The preceding inequality gives

$$\rho \le (1 + o(1)) R/f + O(d^2)$$

on all retained tuples. If ρ exceeds the cutoff in the statement with C_L sufficiently large, then $R \ge J_x$ of Theorem 117.46 with exponent $L+1$. The proof of that theorem multiplies the finite column inner products by absolute real integrals. Those real majorants can only decrease on the cut domains, so the same high-rank estimate holds there. Add the discarded-domain bound. Since $U_x \to \infty$, the two errors fit the stated allowance. Finally $d^3/k \to 0$ and $\log(2x)/\log b \to 0$, proving the assertion about codimension. □

Corollary 117.51 (The fixed space on an integral lattice)**.** *In any retained tuple let $D = \ker(M-1)$ and $D_0 = D \cap \widehat{L}$. The lattice D_0 is saturated of rank $k - \rho$. All partial transports can be applied integrally on $D_0 \cap C$, and the last one returns every such vector to itself. Over the selected split factors at the marks, the total codimension lost on reducing $D_0 \cap C$ inside the reduction of D_0 is $O(fd^2)$. If the finite indices coincide within both pairs of the fourfold contraction, or within the single pair of a twofold contraction, then $\rho = O(d^2)$. Here equality of indices means equality of the full marked tuples.*

Proof. Saturation follows by intersecting the lattice with a rational subspace. The quotient $D_0/(D_0 \cap C)$ injects into $\widehat{L}/C$, so the dimension lost in its reduction at a chosen prime is at most $v_{\mathfrak{p}}(J)$. Summing over the selected primes costs at most $\log NJ/\log \min(g_1, g_2) = O(fd^2)$, by the index bound in the preceding proof. Local saturation embeds the reduction of D_0 in that of the starting ambient lattice. When paired indices coincide $M_v = 1$ at every mark. The same proof then gives $\rho \log(Q/2) \le \sum_l \log H_{g_l'} = O(d^2 \log b)$. □

The remaining ordinary terms may have $M = 1$, including contributions from different rational partners. Their signed weights still matter in the comparison with the prescribed integral theta period in Section 6.6.

117.17 Operator bounds on the prescribed compact column types

In the ordinary Gram moments the dimension of an archimedean column can be large. The common positive unitary group permits a bound for an individual orbital operator in which that dimension is divided out before taking absolute values over the rational orbits. We apply this observation to the same signed blocks on the period fields. Write

$$V_x = fkd^3 \log b, \qquad s_{ri} = \left(\frac{\min(D_r, D_i)}{\max(D_r, D_i)}\right)^{1/2},$$

and keep U_x, σ_{ri} of Section 117.15.

Lemma 117.52 (Dimensions on restriction to a positive hyperplane)**.** *Let a highest weight λ of $U(k)$, $k \geq 2$, restrict to a branch of highest weight μ for $U(k-1)$. If $u_i = \lambda_i + k - i$ and $B = u_1 - u_k$, then*

$$\frac{\dim V_\lambda}{\dim V_\mu} \leq \frac{B^{2k-3}}{(k-1)!}.$$

For the pairs of actual column types in a nonzero ordinary cell, every common compact branch has dimension at least $\max(D_r, D_i)\exp(-Cfk\log b)$, uniformly on the period fields.

Proof. Put $v_i = \mu_i + k - 1 - i$. Interlacing gives $u_{i+1} \leq v_i \leq u_i - 1$. Weyl's dimension formula expresses the dimension ratio as the Vandermonde product of the u_i divided by that of the v_i and by $(k-1)!$. For each $2 \leq a < c \leq k-1$, the distinct denominator factor $v_{a-1} - v_c$ is at least $u_a - u_c$. Every other denominator difference is at least one. The numerator has $2k-3$ factors with an endpoint 1 or k, each at most B. This proves the first bound, including $k = 2$. For the original column parameters the shifted span at each place is at most $3\lceil xb \rceil$ by the weight bound in Lemma 93.8. A scalar shift has no effect on the span. Tensor over the f places and apply the bound separately to each of the two restrictions containing the same branch. Since $\log(2x) = o(\log b)$, their losses have the asserted size. In full common dimension k any surviving intertwiner already identifies equivalent irreducible column types and the branch has their common dimension. □

Theorem 117.53 (Operator mass at the scalar of the identity cell)**.** *For the ordinary block $Z_{ri} = \sum_\gamma h_{a_\gamma} \otimes P_\gamma^{ri}$ of Section 117.15, the original, uncut operators satisfy*

$$\sum_\gamma \|P_\gamma^{ri}\|_{\mathrm{op}} \leq \sigma_{ri}\, s_{ri} \exp(CV_x).$$

The constant is uniform in the rows, their words, the translate and both terminal frequencies. The estimate holds for arbitrary subsets of the cosets.

Proof. Let M_γ be the absolute pair coefficient times the integral of the product of the two real section norms for this orbit. The scalar counting and moment estimate in the proof of Lemma 117.45, before converting column dimensions, gives

$$\sum_\gamma M_\gamma \leq \lambda_{ri} P'_\nu b^{Cfs\nu} \leq \lambda_{ri} \exp(C'V_x).$$

Indeed $\nu = O(nd^2)$, $n/k \to 1$, and $\log P'_\nu \ll fk\log^2(2fk) + fnd^3 \log b$. All fixed finite phases have absolute value one.

After integrating along the common unipotent fibre, an incompatible cell gives zero. In a compatible cell the integral intertwines the compact group of the common positive subspace at the real places, as in Theorem 98.2. To recall the norm calculation, average a rank-one operator vw^{*} over that group with probability Haar measure. On a common branch of dimension d_{μ}, its block is a scalar times an isometry, with scalar of absolute value at most $\|v_{\mu}\|\|w_{\mu}\|/d_{\mu}$ by taking the trace. The restriction is multiplicity-free, so the operator norm is the largest of these scalar bounds. In full common dimension this is the same calculation on a single irreducible. The averaged operator equals the original after integration by the intertwining identity. Absolute convergence allows averaging under the integral; equivalently one can first apply the compact averaging operator to the convergent integral. Thus

$$\|P_{\gamma}^{ri}\|_{\mathrm{op}} \le M_{\gamma}\,\frac{\exp(Cfk\log b)}{\max(D_r,D_i)}.$$

If there is no common branch the operator is zero. Sum this inequality, absorb $fk\log b$ in V_x, and use $\lambda_{ri}=\sigma_{ri}\sqrt{D_rD_i}$. □

Theorem 117.54 (The two and four factors as operators on the incoming row)**.** *Consider the positive-column operator $Z_{ri}^{*}Z_{ri}$ or $Z_{ri}^{*}Z_{rj}Z_{tj}^{*}Z_{ti}$. Let $h=2$ or 4, respectively, and let $\prod_{\mathrm{edges}}\sigma$ denote the product with these multiplicities. For each $L>0$ the following two bounds hold, with a uniform C_L.*

1. *The sum of the operator norms of the individual terms with finite comparison rank $R>C_Lfd^3$ is at most $e^{-LV_x}\prod_{\mathrm{edges}}\sigma$. Terms with a nonunipotent relative marking in either contracted pair are zero.*

2. *The sum of the operator norms of the terms with $\operatorname{rank}_{E_x}(M-1)>C_Ld^3$, for the rational composite M of Theorem 117.50, satisfies the same bound. These estimates hold for the full integrals.*

In particular the error from either set of terms in the trace against any orthogonal projection on the incoming positive column, divided by the rank of that projection when positive, is at most $e^{-LV_x}\prod_{\mathrm{edges}}\sigma$.

Proof. The finite contractions in the two products are respectively $\langle h_{a_1},h_{a_2}\rangle$ and the product of this with $\langle h_{a_3},h_{a_4}\rangle$. The proof of Theorem 117.46 bounds their absolute value by $\exp(-kR\log q_{*}/3)$, where $q_{*}=\min(g_1,g_2)$, and proves the stated vanishing. Each remaining matrix product has operator norm at most the product of its individual operator norms. On any set of tuples with $R>R_0$, Theorem 117.53 thus bounds the sum of norms by

$$\exp(ChV_x-kR_0\log q_{*}/3)\prod_{\mathrm{edges}}(\sigma s_{ri}),$$

where each dimension factor is evaluated on its own edge. They all lie in $(0,1]$. Since $\log q_{*}\ge 8\log b$, taking R_0 to be a sufficiently large multiple of fd^3 gives (i).

For (ii) call a coset retained if its full domain meets both product cutoffs in Lemma 117.48 at the same integration point. On a coset not retained, the entire integral is over the union of the two deleted domains. Therefore the sum of the norms of these full operators is at most $\sigma_{ri}\exp(-AU_x)$ for any prescribed A after making the cutoff choice in that lemma. Use this in one edge at a time, and the full operator mass bound on the other edges. Since $U_x\ge V_x$, all terms involving a deleted coset together cost at most $\frac{1}{2}e^{-LV_x}\prod_{\mathrm{edges}}\sigma$ for sufficiently large x with A large enough.

For a tuple of retained cosets the arithmetic conclusion of Theorem 117.50 applies to the transports themselves. Its proof gives $\operatorname{rank}_E(M-1) \le (1+o(1))R/f + O(d^2)$. Consequently the indicated rational rank threshold implies the finite threshold from (i), with exponent $L+1$, on increasing C_L. Apply (i) to the full integrals in these retained cosets. This absorbs the other half of the allowance. Lastly the absolute trace of a compression to a subspace of dimension u is at most u times the operator norm, by summing its u diagonal entries in an orthonormal basis. □

For a repeated incoming row this calculation measures each error against the actual identity coefficient. It also controls arbitrary unit input vectors before a trace is taken. The low-rank rational composites, including the identity, can depend on the relative signs of several orbital integrals and remain in the projected period.

117.18 Integral coordinates for bounded rational transports

Fix an ordinary block and use the completed source and target lattices L_s, L_t of Lemma 113.6. In this subsection we retain the whole integral of each compatible coset meeting both product cutoffs of Lemma 117.48 at a common point. The cutoff constant is fixed throughout. Put $Q = g_1 g_2$, $q_* = \min(g_1, g_2)$, and $W_x = fkd^2 \log b$. We continue to use the actual index a of the finite column, with the phases in the orbital coefficient. The local finite matrix is described in Lemma 96.2. Its induced character is the one defined in `Formalism/LocalToGlobal/SiegelLatti△ceTheory/ClosedLocalOrbits/GGMirabolic.lean`, line 28.

At a place $v \mid Q$ of E^+, write $\mathfrak{p}, \bar{\mathfrak{p}}$ for the ordered pair of split factors of E. The perfect pairing identifies the reduction at the second with the dual of the reduction at the first. Thus a_v specifies the two maps a_v and a_v^{-*}, where the latter denotes inverse transpose in dual bases. For a retained rational isometry j and its marking a, define

$$C(j,a) = \{w \in L_s : jw \in L_t, \quad \overline{jw} = a\,\bar{w} \text{ at both factors of every mark}\}.$$

In this formula and below a bar on a vector at a given factor means its image in the ambient lattice modulo that prime. By definition $jC(j,a) = C(j^{-1}, a^{-1})$ with the lattices exchanged. In particular these lattices are intrinsic to the pair.

Proposition 117.55 (The number of possible transports and markings)**.** *There are at most* $\exp(CW_x)$ *distinct pairs* (j,a) *in a retained block, uniformly on the common fields. Each pair lies in a class specified by* $C = C(j,a)$, $C' = jC$, *and the isometry* $j : C \to C'$. *Write* $t_{\mathfrak{p}}$ *for the codimension of the image of* C *in* $L_s/\mathfrak{p}L_s$. *The same codimension occurs for* C' *at that factor in the target. With a uniform* $H = b^{C_0 d^2}$,

$$[L_s : C], [L_t : C'] \le H^{2f}, \qquad \sum_{v \mid Q} (t_{\mathfrak{p}} + t_{\bar{\mathfrak{p}}}) \log q_v \le 2f \log H.$$

Proof. Proposition 117.49, applied in each direction, supplies a sublattice of the stated index contained in the corresponding canonical lattice. The two indices here are ordinary additive indices. At each finite prime the dimension lost on reduction is the number of positive elementary-divisor exponents. It is at most their sum. Taking logarithms of the order of L_s/C proves the displayed sum bound. Equality of the source and target codimensions follows because the marked isomorphism maps the image of C onto that of $C' = jC$.

For a rank k lattice over $\mathcal{O}_E$ the number of sublattices of additive index at most $Y \ge 1$ is at most

$$Y^{k+1} \prod_{r=2}^{k+1} \zeta_E(r) \le Y^{k+1} \exp(Cf).$$

Indeed, at a finite prime of norm q choose a basis over the local discrete valuation ring. Project a full sublattice to the last coordinate. If the image is (π^h) and the intersection with the first $k-1$ coordinates has index q^l, the lift of π^h has exactly q^l possibilities modulo that intersection. The total index is q^{h+l}. Thus the local series with weight $[L:C]^{-z}$ is $(1-q^{-z})^{-1}$ times the rank $k-1$ series at $z-1$. Sublattices correspond bijectively to such local choices equal to the original lattice almost everywhere: intersect the local conditions in the ambient lattice, or impose them modulo a common annihilating ideal using the Chinese remainder theorem. The product of the resulting series at $z=k+1$ gives the first inequality. The Euler factors give $\zeta_E(r)\le\zeta(r)^{2f}$, and $\sum_{r\ge2}\log\zeta(r)<\infty$ by comparison with $\sum_{n\ge2}1/(n(n-1))$.

Once C,C' are specified, any two such isometries differ by an automorphism of C. Its isometry group is finite: in an underlying integral basis the positive trace form bounds each column of an isometry. Minkowski's finite group bound in the proof of Proposition 117.42 bounds its order by $\exp(Cfk\log^2(2fk))$ in rational dimension $2fk$. The restriction to the full lattice C determines the rational isometry. For a nonempty class fix one marking a_0. At a selected factor let S be the image of C, and let T be the annihilator of the image of C at the conjugate factor, both in the source space $V=L_s/\mathfrak{p}L_s$. Each $u=a_{0,v}^{-1}a_v$ fixes S pointwise. Its inverse transpose fixes the specified dual subspace pointwise, so $(u-1)V\subset T$. Consequently $u-1$ factors through a map $V/S\to T$. The number of choices is at most $q_v^{t_\mathfrak{p}t_{\bar\mathfrak{p}}}$. Since $2t_\mathfrak{p}t_{\bar\mathfrak{p}}\le k(t_\mathfrak{p}+t_{\bar\mathfrak{p}})$, their product is at most H^{fk}. Apply the sublattice count twice at $Y=H^{2f}$. The resulting bound is $H^{f(5k+4)}\exp(Cfk\log^2(2fk)+Cf)$, which has the asserted size since f,k grow at most polynomially in x. □

Theorem 117.56 (Integral recovery on a fixed pair of matched lattices)**.** *For any one class (C,C',j) in the proposition, a proportion at least $1-\delta_x$ of the families gU_0 of the original integral marked matrix distinguish* all *distinct indices of the class at once. Here one may take $\delta_x=8fq_*^{-k/4}$ for sufficiently large x. In any such family each coefficient is recovered by selecting one coordinate and multiplying by the inverse of its root of unity. For a matrix of columns supported on this class all determinantal ideals of the unnormalized marked image agree with those before applying the marked matrix. For arbitrary complex Hilbert space coefficients z_a on its distinct indices, its unit columns satisfy*

$$(1-\delta_x)\sum_a\|z_a\|^2\ \le\ \Big\|\sum_a h_a\otimes z_a\Big\|^2\ \le\ (1+\delta_x)\sum_a\|z_a\|^2.$$

Proof. At a selected factor use the subgroup

$$B_v=\{u\in\mathrm{GL}(V):u|_S=1,\ (u-1)V\subset T\}.$$

It is a subgroup because its elements fix S and induce the identity on V/T, hence preserve T also under inversion. The indices of the class are contained in $a_0\prod_v B_v$. There are at most $q_v^{(t_\mathfrak{p}+t_{\bar\mathfrak{p}})r}$ elements of B_v of difference rank r: every rank r map $V/S\to T$ factors through a space of dimension r, and counting both factors with unrestricted entries is an upper bound. The sum bound in the proposition gives $t_\mathfrak{p}+t_{\bar\mathfrak{p}}\ll fd^2=o(k)$ uniformly, since $\log q_*\ge8\log b$.

For uniform gU_0 two indices a,a' collide at a given place only if $a'_va_v^{-1}\in g_vU_vg_v^{-1}$. An element of difference rank $r\ge1$ has probability zero if nonunipotent and at most $q_v^{-kr/3}$ otherwise, by Lemma 112.1 as used in Theorem 117.46. All relative elements for this class are in $a_{0,v}B_va_{0,v}^{-1}$. The sum of the probabilities over its nonidentity elements is at most $\sum_{r\ge1}q_v^{-kr/4}\le2q_v^{-k/4}$ for large x. There are $2f$ selected places. Excluding their union leaves the claimed proportion of families in which distinct tuples occupy distinct product coordinates. The raw nonzero entries in a family are roots of unity. Reading them as stated is an integral left inverse on this support

over the ring of integers containing the character values. Cauchy–Binet applied to the map and then to its left inverse gives both inclusions of determinantal ideals, with the same argument for fractional coefficient matrices.

For the complex upper and lower bounds fix a. Each relative element occurs for at most one a' for this a. The modulus of the normalized Gram entry is bounded by the probability of a collision, and the probabilities for a fixed tuple multiply over the places. The sum of the absolute off-diagonal entries in each row is at most

$$\prod_{v\mid Q}(1+2q_v^{-k/4})-1\le 8fq_*^{-k/4}$$

for all sufficiently large x. The diagonal entries are one. The estimate for the quadratic form follows by pairing the off-diagonal terms and using $2\|z_a\|\|z_{a'}\|\le\|z_a\|^2+\|z_{a'}\|^2$. □

Theorem 117.57 (A common list of integral coordinates on a retained block)**.** *For every sufficiently large x there is a list of at most Cfd^2 families of the actual integral matrix for the block with both of the following properties.*

1. *In every family on the list, any two retained terms in the same coordinate have*

$$R(a,a')\le Cfd^2,\qquad \operatorname{rank}_E(j^{-1}j'-1)\le Cd^2.$$

2. *For each class (C,C',j) at least one family on the list separates all its distinct indices. In that family, reading the coordinate of an index a and removing its unit phase gives its whole signed coefficient from the class plus contributions only from other classes satisfying the two rank bounds in (i).*

The list works for every choice of coefficients on the same retained terms, in particular for their original signed integrals.

Proof. The number of ordered pairs of distinct indices in the block is at most $\exp(2C_1W_x)$ by Proposition 117.55. For a fixed pair of comparison rank R the full collision probability is at most $\exp(-kR\log q_*/3)$, by the independence over places in the preceding proof. For any fixed $A>0$, take $J=\lceil C_Afd^2\rceil$ with sufficiently large C_A. By the union bound a proportion at most $\exp(-AW_x)$ of families have even one collision with $R>J$. Call the complement $\mathcal{G}$. On every collision in a family in $\mathcal{G}$, the two transports in the block have the same source and target. The coefficient-divisibility argument of Theorem 117.50 for these two retained cosets gives $\operatorname{rank}_E(j^{-1}j'-1)\le(1+o(1))R/f+O(d^2)$. This includes collisions of equal indices from different transports.

Choose m independent uniform families *from* $\mathcal{G}$. For a fixed class the probability that a draw fails to separate it is at most $\delta_x/(1-e^{-AW_x})\le 2\delta_x$. The number of nonempty classes is at most $\exp(C_1W_x)$, since each has at least one pair and each pair specifies a unique class. The probability that some class fails on every draw is at most $\exp(C_1W_x)(2\delta_x)^m$. For large x, $-\log(2\delta_x)\ge k\log q_*/5$. Thus $m=\lceil C_2fd^2\rceil$ with sufficiently large C_2 makes this probability less than one. Use a successful list. The formula for a coordinate follows directly from the integral matrix entries, by first summing every term with the same class and the same index with its original phase. □

For a given block these selected coordinates are coordinates of the original finite projection on the retained cosets. Contributions from other classes of small relative rank can still enter the same coordinate. Different cosets in a single class can also contribute with different signs

at an identical index. The resulting sums must still be compared with the integral theta period in Section 6.6. Lemma 117.48 bounds the discarded whole cosets in the archimedean norm. At a finite place a bound for their absolute values at infinity does not imply any increase in their valuation.

117.19 Recovery of a signed coefficient in the presence of other classes

Fix the same block and product cutoffs as in Section 117.18. Let A be any subset of the distinct retained marked indices. In particular one can use the entire set. At an index a the coefficient z_a denotes the sum of all chosen contributions there, with their original phases. We first work with arbitrary coefficients. A class below is specified by its two canonical lattices and rational isometry as in Proposition 117.55. Its marking set is the set of indices occurring in the class, and different such sets can overlap.

Lemma 117.58 (Collisions between two classes of rational transports)**.** *Fix any two nonempty classes in the same block. In a proportion at most $\beta_x = 4f q_*^{-k/4}$ of all families of the integral marked matrix can a marking from the first and a* different *marking from the second occupy the same coordinate. The classes may have different lattices and isometries. The bound also holds when they coincide.*

Proof. At a selected split factor put $V = L_s/\mathfrak{p}L_s$. By the proof of Theorem 117.56 the two sets are contained locally in a_0B_1 and b_0B_2, with

$$B_i = \{u \in \mathrm{GL}(V) : u|_{S_i} = 1,\ (u-1)V \subset T_i\}.$$

All differences $u - 1$ for $u \in B_2B_1$ kill $S_1 \cap S_2$ and have image in $T_1 + T_2$. This follows by expanding a product $(1 + X_2)(1 + X_1)$. Therefore every local relative matrix $a'a^{-1}$ belongs to one affine space with translation space

$$b_0\ \mathrm{Hom}\left(V/(S_1 \cap S_2),\ T_1 + T_2\right) a_0^{-1}.$$

Write $l = \mathrm{codim}(S_1 \cap S_2)$ and $u = \dim(T_1 + T_2)$. The index bounds on the two classes give $l + u = O(fd^2) = o(k)$, uniformly at every factor.

Consider all relative matrices D with $\mathrm{rank}(D - I) = r$ for a fixed $r \geq 1$. If there is one, fix D_0 of the same rank of difference from I. The differences $D - D_0$ have rank at most $2r$ and belong to the displayed translation space. Their number is at most $q_v^{2r(l+u)}$, since every map of rank at most $2r$ factors through a $2r$-dimensional space. This estimate does not require the identity to belong to the affine space. For each such D the probability of $D \in g_vU_vg_v^{-1}$ is at most $q_v^{-kr/3}$ by Lemma 112.1. Thus the probability that the family meets any of these nonidentity relative matrices at this place is at most $\sum_{r\geq 1} q_v^{-kr/4} \leq 2q_v^{-k/4}$ for large x. A collision of different tuples requires this to happen at some place. Sum the bounds over the $2f$ places. □

Put $J = \lceil C_0 fd^2 \rceil$ with C_0 large enough in the far-pair estimate in the proof of Theorem 117.57 to give exceptional proportion at most e^{-W_x}. For $a \in A$, let $n_J(a; A)$ be the least number of retained classes whose marking sets cover all

$$a' \in A, \qquad 0 < R(a, a') \leq J.$$

It is zero for the empty set. Define

$$A_0 = \{a \in A : n_J(a; A) \leq q_*^{k/10}\}, \qquad \eta_x = e^{-W_x} + 4f q_*^{-3k/20}.$$

In particular these definitions use the neighboring indices from the whole support, including those with a different rational transport.

Theorem 117.59 (Reading coefficients from the projected block)**.** *For every $a \in A$, the proportion of families in which another index of A occupies its coordinate is at most*

$$e^{-W_x} + \beta_x n_J(a; A).$$

There is a list of at most $C f d^2$ families such that the raw integral projection of any coefficients supported on A recovers every entry z_a for $a \in A_0$ by reading one coordinate and removing its unit phase. The family used can depend on a. For arbitrary Hilbert space coefficients one has the uniform lower bound

$$\left\| \sum_{a \in A} h_a \otimes z_a \right\|^2 \geq (1 - \eta_x) \sum_{a \in A_0} \|z_a\|^2.$$

If Z is a matrix with rows supported on A, Z_0 consists of its rows on A_0, and H is the original unnormalized marked matrix, then its fractional determinantal ideals satisfy

$$I_u(Z_0) \subseteq I_u(HZ) \subseteq I_u(Z) \qquad (u \geq 1).$$

The ideal assertion holds at every finite place over a coefficient field containing the finite character values, whenever the entries of Z belong to that field.

Proof. Except on a set of families of measure at most e^{-W_x}, no pair in the block of comparison rank greater than J collides. This follows from the number of possible pairs and the individual decorated-flag probability as in Theorem 117.57. Fix one class containing a and a minimizing cover of the indices within the indicated rank. For any covering class the probability of a collision of a with a different index from it is at most β_x, by the lemma. A union bound proves the first assertion. For $a \in A_0$ the upper bound is at most η_x.

Draw m independent uniform families. By Proposition 117.55, with a fixed C_1, the probability that there is even one $a \in A_0$ which none of them isolates is at most $\exp(C_1 W_x)\eta_x^m$. For large x, $\log(1/\eta_x) \geq ck \log q_*$ for a fixed $c > 0$: $W_x/(k \log q_*) \to \infty$ since $\log q_* = (8 + o(1)) \log b$, and $\log f = o(k \log q_*)$. Taking $m = \lceil C_2 f d^2 \rceil$ with C_2 large enough gives the required list. Choose one isolating family for each $a \in A_0$. Its unique nonzero matrix entry at the chosen index is a unit root, and all other indices of A have zero entry in that coordinate. Multiplying by the inverse root recovers z_a with all the original contributions at that index. The resulting coordinate selections give an integral map carrying HZ to Z_0. Cauchy–Binet for this map and for H gives the two inclusions.

For the norm, fix a single family. Its squared norm on these coefficients is at least the sum of $\|z_a\|^2$ over the indices in A_0 that it isolates: their occupied coordinates are distinct, and all remaining coordinate norms are nonnegative. The square for the unit columns h_a is the mean of these family squares, since the unnormalized column has exactly one unit entry in each family. Averaging the lower bound and using the isolation probability of each index proves the assertion. □

Corollary 117.60 (A lower bound for the original ordinary period on an input)**.** *Fix $L > 0$ and choose the product cutoff of Lemma 117.48 with this exponent. In a block let P_a be the signed sum of the complete operators over the retained cosets with index a. Take A to be their set of indices. For every vector v in the incoming positive column,*

$$\|Z_{ri} v\| + \sigma_{ri} e^{-L U_x} \|v\| \;\geq\; \left((1 - \eta_x) \sum_{a \in A_0} \|P_a v\|^2 \right)^{1/2}.$$

More generally, if an array z on retained indices is killed by the raw finite matrix, then every index a of its nonzero support A satisfies

$$n_J(a;A) \geq (1-e^{-W_x})q_*^{k/4}/(4f).$$

The implication about a kernel holds over any coefficient module on which the character roots act by units.

Proof. Apply the theorem to $z_a = P_a v$. The contribution of whole cosets missing the joint cutoff has operator norm at most $\sigma_{ri}e^{-LU_x}$, by Lemma 117.48 and the triangle inequality, as the marked columns have norm one. Adding it back gives the first assertion. For the second, if any family isolated a from the rest of the nonzero support, its coordinate would be a unit times the nonzero entry z_a. Every family must therefore have a collision at a. Compare this with the first bound of Theorem 117.59. □

There is an exact finite example illustrating the role of the number of classes. Take the same completed positive lattice at both ends and set $j = 1$. At one selected split place take all a fixing a fixed codimension-two subspace S pointwise. Use the prescribed inverse transposes at the conjugate factor and identities at all other marks. Every pair has comparison rank at most two, and for each a the canonical lattice has additive index $q_v^{2\,\mathrm{rank}(a-I)} \leq q_v^4$. The rational difference ranks are all zero. Yet the array with coefficient one at each of these distinct indices is killed by H.

Indeed, in a fixed family at the chosen place take a basis adapted to its full flag in which S is spanned by a subset of the basis. Such a basis is obtained by extending the intersections of S with the successive flag spaces. Two indices are omitted from the subset, so one omitted index i is greater than one. The simple root subgroup $1 + tE_{i-1,i}$ fixes S pointwise and is in the flag's unitriangular group. The character of that group is nontrivial on this root: changing an adapted basis by an upper triangular matrix scales each adjacent entry of its abelianization by a nonzero scalar. In every occupied coordinate of the family, left multiplication by this root subgroup preserves both the support and the coordinate. It acts freely and multiplies the entry by its character. Summing over each orbit gives zero by orthogonality, hence every coordinate vanishes. This example satisfies the two lattice index bounds on the same finite groups. Determining the coefficients that actually occur in the theta period requires the signed orbital integrals. In particular, a bound for $\sum_{a\in A_0}\|P_a v\|^2$ on enough incoming vectors and a comparison with the specified integral map are still needed for Section 6.6.

117.20 Character traces of the signed sums at each rational isometry

Consider the terms of an ordinary block for which the common positive space has dimension k. On these terms the real operator is a scalar times the map induced by a rational isometry of the positive spaces. Its normalized trace in a product of blocks can be small even when the finite markings coincide. We give estimates in terms of the norms of the actual signed marked vectors belonging to each projective isometry.

Lemma 117.61 (The lattice index in the Weyl character estimate)**.** *Let M be a lattice in a definite Hermitian space of dimension $k \geq 2$ over $E = E_x$, with $f = [E^+ : \mathbb{Q}]$. Let g be a rational unitary automorphism. Suppose $gC \subset M$ for a sublattice $C \subset M$ of finite additive index $[M : C] = H^{2f}$. If the distinct eigenvalues of g have multiplicities m_i, put $q(g) = \sum_{i<j} m_i m_j$. For a product of irreducible column representations ρ let L be the smallest adjacent gap of its shifted highest weights at all places, as in Proposition 110.6. Then*

$$\frac{|\operatorname{tr}\rho(g)|}{\dim\rho} \leq \min\{1,\ (2k/L)^{fq(g)}H^{f(k-1)}\}.$$

Proof. At a finite prime of E choose integral bases of C and M. If T_0, T_1 are the matrices in these bases of inclusion and of $g|_C$, then $\det(XT_0 - T_1) = \det(T_0)\det(X - g)$ is integral. Write $e_v = v(\det T_0)$ with uniformizer of valuation one. For the roots $z_1, \dots, z_k$ of $\det(X - g)$, repeated by multiplicity in a local splitting field with extended valuation, this implies

$$\sum_{i=1}^{k} \max(0, -v(z_i)) \le e_v.$$

Indeed the maximum of the absolute values of the coefficients of a product of polynomials is the product of the two maxima. After dividing each factor by a coefficient of largest absolute value this follows by multiplying two nonzero residue polynomials. Apply this identity to the linear factors of the monic polynomial.

Let $D(g)$ be the product of $z_i - z_j$ over ordered pairs whose eigenvalues differ. It lies in $E^\times$, being invariant under permutations induced by the Galois group. Put $t_i = \max(0, -v(z_i))$. For each unordered pair of distinct eigenvalues the possible denominator valuation is at most $\max(t_i, t_j) \le t_i + t_j$. Each index belongs to at most $k - 1$ unordered pairs. Thus $v(D(g)) \ge -2(k-1)e_v$. If $\mathfrak{J}$ is the index ideal of $C \subset M$, this shows that $\mathfrak{J}^{2(k-1)}(D(g))$ is integral. Its absolute norm is at least one, and $N\mathfrak{J} = [M : C]$.

At each real place of E^+ write $D_\tau = \prod_{i<j} |z_i - z_j|$, omitting equal eigenvalues and retaining their multiplicities. Unitarity and complex conjugation give $\prod_\tau D_\tau = |N_{E/\mathbb{Q}} D(g)|^{1/4}$. Hence $\prod_\tau D_\tau \ge H^{-f(k-1)}$. The confluent Weyl determinant computation in Proposition 110.6 bounds the normalized character at each place by $(2k/L)^{q(g)}/D_\tau$. Multiplication over the places proves the assertion, together with the unitary bound one. □

Use the completed source and target lattices M_s, M_t of Section 117.18. For a rational isometry u between them over E, define

$$H(u)^{2f} = \max\{[M_s : M_s \cap u^{-1}M_t],\ [M_t : M_t \cap uM_s]\}.$$

This definition puts no conditions on the marking at either prime. The same height applies to the inverse. For a product of h such maps or inverses of height at most H, there is a source sublattice of index at most H^{2fh} carried into the last lattice. Indeed, impose each successive domain condition on the original vectors. At each step the new quotient injects into the quotient by the next integral domain, of order at most H^{2f}.

In a full common positive space the two words must have equivalent compact representations for the orbital intertwiner to be nonzero. Fix the induced isometries U_u between their unitary representation spaces, using one identification with the word model at each endpoint. Schur's lemma in Theorem 98.2 gives $P_\gamma = c_\gamma U_{u_\gamma}$, where c_γ includes the scalar of the real integral, the finite phase and the pair coefficient. This is the same transport appearing in the trace of Corollary 110.7. Composition of the induced maps represents composition of the rational isometries in these models.

Group isometries up to multiplication by a scalar $\xi \in E^\times$ with $\xi\bar{\xi} = 1$. Choose one occurring representative u per class. For $u_\gamma = \xi u$ write $U_{u_\gamma} = \omega_\rho(\xi) U_u$, where $|\omega_\rho(\xi)| = 1$ by the central character. Form the signed vector and its total squared norm in block e by

$$z_{e,u} = \sum_{\gamma:\,[u_\gamma]=[u]} c_\gamma\, \omega_\rho(u_\gamma/u)\, h_{a_\gamma}, \qquad B_e^2 = \sum_{[u]} \|z_{e,u}\|^2.$$

The notation u_γ/u means their scalar ratio. Here h_a is the original unit marked column. The sums use complete orbital integrals and converge absolutely by Lemma 117.45 and $P_\gamma = c_\gamma U_{u_\gamma}$. One may take any subset of the full-dimensional terms throughout, with $z_{e,u}$ formed from precisely that subset.

Theorem 117.62 (The second and fourth traces at polynomial lattice index)**.** *Fix $A, B > 0$. There is a choice of the polynomial word spacing in Theorem 9.13, uniform on the original intervals, with the following properties. In each block take all full-dimensional terms with $H(u_\gamma) \le x^B$, or any subset of them. Write $Z_e = \sum_{[u]} z_{e,u} \otimes U_u$ for this portion of the ordinary marked operator and $D = \dim \rho$ for its column dimension. Then*

$$\left| D^{-1} \|Z_e\|_{\mathrm{HS}}^2 - B_e^2 \right| \le x^{-Afk} B_e^2.$$

For a fourfold trace in the block order $(r,i),(r,j),(t,j),(t,i)$ of Theorem 117.43, expand $D^{-1}\operatorname{tr}(Z_1^* Z_2 Z_3^* Z_4)$ by the same projective classes. The sum of the absolute values of the terms with

$$u_1^{-1} u_2 u_3^{-1} u_4 \text{ noncentral}$$

is at most $x^{-Afk} B_1 B_2 B_3 B_4$. Thus the surviving sum up to this error may be taken over exact equalities in the projective unitary group, with the original finite inner products and with the central character on each scalar composite.

Proof. First count distinct isometries between two given row lattices of height at most $H \ge 1$. Their integral domain and image have indices at most H^{2f}. The sublattice count proved in Proposition 117.55 bounds the choices for these two lattices together by $H^{4f(k+1)} \exp(Cf)$. With both fixed, their isometries, if any, form a torsor for the integral isometry group of the domain. This finite group embeds in $\mathrm{GL}_k(E)$; positivity bounds the integral matrices of its elements in the trace metric. Lemma 96.23 bounds its order by $\exp(C_\psi k \log^2(2x))$ on the polynomial-conductor intervals, with the constant also allowed to depend on ε. The same counts bound the number of projective representatives. For $H = x^B$ this number is at most $x^{C_B fk}$ uniformly. Here f grows at least as a fixed positive power of x by Theorem 19.15.

In either trace let $h = 2$ or 4 be the number of maps and write g for the composite, using $u_1^{-1}u_2$ when $h = 2$. The normalized real trace is $\operatorname{tr}\rho(g)/D$, by composition and unitarity. Every nonzero fourfold term uses equivalent words along its cycle and therefore a common dimension. The finite contractions are the inner products of the first two and, for $h = 4$, of the last two vectors. Their product in absolute value is bounded by the product of the h vector norms. Changing representatives multiplies the real and finite factors by cancelling unit phases.

If g is noncentral, unitarity gives at least two distinct eigenvalues and $q(g) \ge k - 1$. Apply Lemma 117.61 with lattice height at most H^h, by the domain construction above. Choose $L/(2k) \ge x^K$ with $K > 4B + 2A + 4C_B + 2$. The words of Theorem 9.13 admit this choice: their shifted gaps are at least 3Δ, and the exponent of the polynomial spacing Δ may be fixed before selecting the terminal prime, as in Corollary 110.7. The field, interval and marked construction are those of that theorem. The count used here depends on the field and lattices only through f, k, H and the field conductor, so its constant does not increase with the chosen gap.

The real factor of each noncentral term is consequently at most $x^{-(K-hB)f(k-1)}$. On each edge the sum of the vector norms is at most $x^{C_B fk/2} B_e$, by Cauchy–Schwarz. Multiply these bounds and use $k - 1 \ge k/2$, $h \le 4$. The absolute sum over all noncentral composites is at most $x^{-Afk} \prod_e B_e$, counting a repeated edge each time. For the second trace a central composite means that the two representatives are the same. Its terms sum exactly to B_e^2. In the fourth trace a central composite acts by its central character. This proves both formulas. □

Corollary 117.63 (Rational scalars in the traces at the full retained index)**.** *Choose word spacing so that $L/(2k) \ge x$. On each edge use the full-dimensional cosets meeting both original product cutoffs at one point, as in Proposition 117.55; retain their complete integrals. Put*

$W_x = fkd^2 \log b$. *For every $A > 0$ there is a uniform C_A such that, in the second or fourth normalized trace above, the absolute sum of terms whose projective composite has no representative satisfying*

$$\operatorname{rank}_E(g - \xi I) \le C_A d^2 \frac{\log b}{\log x}, \qquad \xi \in E^\times, \quad \xi\bar{\xi} = 1,$$

is at most $e^{-AW_x} \prod_e B_e$. The rank bound is $o(d^3)$ on the common intervals.

Proof. The matched lattices in Proposition 117.55 are contained in the respective integral domains. Hence $\log H(u) \le Cd^2 \log b$ for each retained isometry and its inverse. The same proposition counts at most $\exp(C_1 W_x)$ representatives on each edge. Let $r_* = k - \max_i m_i$ for the composite g. Since $\sum_i m_i^2 \le k \max_i m_i$,

$$q(g) = \tfrac{1}{2}(k^2 - \textstyle\sum_i m_i^2) \ge kr_*/2.$$

Lemma 117.61 on the composite therefore bounds the normalized trace by $\exp(C_2 W_x - fkr_* \log(x)/2)$. The constant allows for at most four successive index losses. Summing absolute values as in the theorem costs at most $\exp(2C_1 W_x) \prod_e B_e$. Thus all terms with $r_* > C_A d^2 \log(b)/\log x$ satisfy the claimed total bound when C_A is sufficiently large.

Now $\log b = x^{\varepsilon}$ and $d \asymp x^{1/2-\varepsilon/16}$ in the range of Section 117.15. It follows that the threshold is $o(d^3) = o(k)$. Below it, for large x the eigenvalue of largest multiplicity has multiplicity greater than $k/2$ and is unique. The characteristic polynomial has coefficients in E, so every Galois conjugate over E of this eigenvalue has the same multiplicity. Uniqueness forces it to be an element ξ of E. Its complex absolute value is one by unitarity, hence $\xi\bar{\xi} = 1$. Diagonalizability shows that $\operatorname{rank}_E(g - \xi I) = r_*$. The argument is unchanged by multiplying a representative by a rational norm-one scalar. □

These estimates use the norms of the marked vectors after all phases for a given projective isometry have been included. The central products can contribute to the fourth moment, and a single vector $z_{e,u}$ can vanish. The magnitudes of these vectors and the cells of common dimension $k - 1$ still enter the integral comparison of Section 6.6.

117.21 Scalar transports in the integral marked period

The central character in Section 117.20 groups rational transports which differ by a scalar. The two finite marks can separate them, even though their real operators are proportional. We use the same fields, lattices and complete retained orbital integrals as in Section 117.18.

Lemma 117.64 (High-multiplicity eigenvalues on a matched lattice)**.** *Let g be a composite of $h = 2$ or 4 successive retained transports and inverses returning to its original positive space. Let $C \subset M$ be the successive matched domain in the original completed lattice M, and set $Y = [M : C]$. In particular $gC \subset M$ and, on C, reduction of g agrees with the composite g_v of the prescribed markings at every chosen split factor. Write t_v for the codimension of the image of C there. Then $\log Y \ll fd^2 \log b$ and $\sum_{v|g_1 g_2} t_v \ll fd^2$, using the same one of the two conjugate factors for the marking at each v.*

An eigenvalue z of g of multiplicity m with $m \log 2 > \log Y$ is a root of unity. For every such eigenvalue different from one, and at each chosen factor,

$$\operatorname{rank}(g_v - I) \ge m - t_v.$$

When $m > t_v$, the marked composite has an eigenvalue different from one in the algebraic closure of the residue field.

Proof. Construct C by imposing the next domain condition before each map in the product, starting with its canonical matched lattice. The successive quotients inject into the original domain quotients on the respective edges, as in Theorem 117.50. Proposition 117.55 bounds their indices, giving $\log Y \ll f d^2 \log b$. Reduction on this domain is compatible with every intermediate map and hence with composition. At a selected prime the elementary-divisor length of M/C is at least t_v. Thus $\sum_v t_v \log q_v \le \log Y$. The estimate for $\sum_v t_v$ follows from $\log q_* \ge 8 \log b$, where $q_* = \min(g_1, g_2)$.

Let P be the monic irreducible polynomial of z over E. Then P^m divides $\det(X - g)$ as a monic factor, since all its roots have the same multiplicity in characteristic zero. If a coefficient of P had negative valuation at a prime $\mathfrak{p}$ of E, the maximum coefficient norm of P there would be at least $N\mathfrak{p}$, with $|\pi|_{\mathfrak{p}} = (N\mathfrak{p})^{-1}$. Multiplicativity of this norm, used in Lemma 117.61, and the monic remaining factor would force the norm of $\det(X - g)$ there to be at least $(N\mathfrak{p})^m$. The same lemma bounds it by the $\mathfrak{p}$-part of Y. This would give $Y \ge 2^m$. Thus P is integral and z is an algebraic integer. All its complex conjugates have modulus one, since each comes from an eigenvalue at an embedding of E and the space is definite at every such embedding. To see that z is torsion, the monic integer polynomials of its positive powers have bounded degree and bounded coefficients, by the elementary symmetric function formula. Only finitely many roots occur, so two powers agree.

Let N be the order of z. If a marked residue characteristic ℓ divided N, then $\mathbb{Q}(z)$ would contain the ℓ-th roots of unity. This would give $\ell - 1 \le [\mathbb{Q}(z) : \mathbb{Q}] \le 2fk$. But g_1, g_2 grow faster than $2fk + 1$ on the common intervals. Hence N is prime to ℓ, and a nontrivial such root has nontrivial reduction. Indeed, if its reduction were one, $1 + z + \cdots + z^{N-1} = 0$ would give $N = 0$ in the residue field.

At a chosen factor extend the local integer ring to contain z. The number t_v of nonunit elementary divisors of $C \subset M$ is unchanged by this extension. Let S be the intersection of the z-eigenspace with the extended M. It is a saturated direct summand of rank m, since g is semisimple by archimedean unitarity. The finite module $S/(S \cap C)$ embeds in M/C after extension and has at most t_v generators: over a discrete valuation ring their minimal number is the residue dimension of the subgroup killed by the uniformizer. At least $m - t_v$ independent reductions of $S \cap C$ thus survive in S and in M. The marked map on these reductions is multiplication by $\bar{z}$. If $z \ne 1$, this gives the rank bound and the asserted residual eigenvalue. Matrix ranks are preserved by residue field extension. □

Theorem 117.65 (Coordinate separation of scalar rotations)**.** *In any retained ordinary block, two distinct rational transports in the same projective class have disjoint sets of occupied coordinates in every family of the raw integral marked matrix. More generally this holds for two transports whose relative automorphism has a nonidentity eigenvalue of multiplicity m with $m \log 2 > \log Y$. Here Y is the matched index of $g = u_1^{-1} u_2$ in the lemma; each t_v is then strictly smaller than m since $t_v \log q_v \le \log Y$.*

Consequently, on the full common dimension k in a retained block, the quantity B_e^2 of Theorem 117.62 is exactly

$$B_e^2 = \sum_{u} \left\| \sum_{\gamma:\, u_\gamma = u} c_\gamma h_{a_\gamma} \right\|^2 ,$$

where u now ranges over actual rational isometries. The sums at different scalars in a given projective class can be recovered from that class vector by coordinate selection and central unit phases, also for the unnormalized finite matrix.

Proof. If two columns share a coordinate of a family, their relative marked matrix at every place is conjugate to an element of the upper unitriangular group, by Lemma 96.2. Its eigenvalues all

equal one. A nonidentity eigenvalue in the last conclusion of Lemma 117.64 therefore precludes a common coordinate, including after any change of family. The source and target versions of the relative matrix are conjugate by one of the markings and give the same condition.

For distinct scalar rotations the relative map equals ξI with $\xi \neq 1$ and $m = k$. On the actual fields

$$fd^2 \log b = o(k).$$

Indeed $f \ll x^{2+8\varepsilon}$, $d \asymp x^{1/2-\varepsilon/16}$, $\log b = x^{\varepsilon}$, and $k \asymp x^{9/2+10\varepsilon}$. Hence the hypotheses on m and t_v hold for every pair of retained cosets for large x. The lemma also shows that ξ is a root of unity. All cosets at one rotation thus have support disjoint from those at another in every finite family. Orthogonality of these coordinate subspaces proves the equality. Restriction to the corresponding coordinates recovers each rotated vector. The required scalar correction is a root of unity: a central character of a unitary highest-weight representation on such a scalar is a product of integer powers of its complex conjugates. It is therefore a unit in the coefficient integers containing the phases. The same argument works before the common normalization of the finite columns. □

Theorem 117.66 (Nonidentity central products in the fourth trace)**.** *In the fourfold ordinary trace of Theorem 117.43 use the original indices and full integrals of retained cosets. Write $g = u_1^{-1}u_2u_3^{-1}u_4$. The sum of the absolute values of the trace terms, normalized by the incoming dimension, for which g has an eigenvalue different from one of multiplicity at least $k/2$, is at most*

$$\exp(-cfk^2 \log b) \prod_{i=1}^{4} \sigma_i$$

for an absolute $c > 0$ on sufficiently distant intervals. This estimate allows both possible common positive dimensions in each orbital integral.

There is an exact statement at smaller matched indices. Fix $A, B > 0$, and on each full-dimensional edge keep only cosets with

$$[M_s : C(u_\gamma, a_\gamma)],\ [M_t : u_\gamma C(u_\gamma, a_\gamma)] \leq x^{2fB}.$$

Their complete integrals may be used without an archimedean cutoff. With the spacing of Theorem 117.62, the fourth normalized trace equals its sum over the actual rational equations

$$u_1^{-1}u_2u_3^{-1}u_4 = I$$

up to error of absolute value at most $x^{-Afk} \prod_i B_i$. Here each B_i^2 is the sum in Theorem 117.65.

Proof. For each individual fourfold term denote the relative finite ranks in the two contractions at v by $r_{12,v}, r_{34,v}$. Their sum is at least $\operatorname{rank}(g_v - I)$, by writing the composite of the two relative maps and using subadditivity of rank. This also follows after identifying their intermediate spaces by any one of the markings. If a nonidentity eigenvalue has multiplicity $m \geq k/2$, then $m \log 2 > \log Y$ as above. The lemma gives

$$\sum_{v \mid g_1 g_2} (r_{12,v} + r_{34,v}) \geq 2fm - \sum_v t_v \geq (1 - o(1))fk.$$

The absolute product of the two marked contractions is at most $q_*^{-k \sum_v (r_{12,v}+r_{34,v})/3}$ by Theorem 117.46. The remaining normalized trace is at most the product of the four operator norms, also for the hyperplane intertwiners. By Theorem 117.53 those products sum to at most $\exp(Cfkd^3 \log b) \prod_i \sigma_i$. Now $d^3 = o(k)$ and $\log q_* \geq 8 \log b$. Inserting the finite bound proves the first assertion.

In the smaller range the matched domain of an individual cycle has $\log Y \le 8fB\log x$. In particular the preceding proof of coordinate separation applies to each pair of scalar rotations, using their two-edge bound. The unmarked heights are at most x^{B}. Therefore Theorem 117.62 already gives the stated error from noncentral projective composites, with B_i computed by the orthogonal decomposition. It remains to examine the exact central sum. Expand the finite vectors in this sum into their actual transports and then their individual cosets. Absolute convergence justifies these expansions. Each resulting composite is ξI. Since $k\log 2 > \log Y$, the coefficient-norm argument of the lemma makes ξ a root of unity, with prime-to-mark order. Both contractions must have a common coordinate to be nonzero.

The sum of the codimensions t_v is bounded by $8fB\log x/\log q_* = o(f)$. Among the $2f$ chosen factors there is consequently one with $t_v = 0$. There the matched domain maps onto the entire reduction of M, so $g_v = \bar{\xi} I$. Both relative marked factors are unipotent when their contractions are nonzero. If their product is $\bar{\xi} I$, the second factor is $\bar{\xi}$ times the inverse of the first. Its eigenvalues are therefore all $\bar{\xi}$, and unipotence forces $\bar{\xi} = 1$. Prime-to-mark reduction is injective on the group generated by ξ, so $\xi = 1$. This leaves exactly the claimed rational equations in the central sum. □

Even when that equation holds there can be several distinct paths of rational isometries between the same rows. Their signed orbital coefficients remain in the fourth trace. The calculation singles out the products for which those coefficients still have to be estimated.

117.22 Summing over all target rows in the ordinary Gram matrix

The Gram matrices of Section 6.5 sum first over a common target row. An absolute error for each ordered pair of target rows can grow under this summation. For the part of the ordinary period where the two matched indices are polynomial on the geometric-mean scale, one can perform the sum before making the character estimate. The resulting error is relative to the signed sum and is uniform in the number of target rows.

Fix $B > 0$. In each block (r, i) take any of the full common positive dimension k cosets satisfying the two bounds $[M_s : C(u_\gamma, a_\gamma)] \le x^{2fB}$ and $[M_t : u_\gamma C(u_\gamma, a_\gamma)] \le x^{2fB}$ of Theorem 117.66. Use their complete integrals. For each actual rational isometry $u : P_i \to P_r$ of positive spaces put

$$y_{ri,u} = \sum_{\gamma:\, u_\gamma = u} c_\gamma h_{a_\gamma}, \qquad Z_{ri} = \sum_{u} y_{ri,u} \otimes U_u.$$

Take absent vectors to be zero. As in Section 117.20 the unitary identifications U_u are chosen coherently for each word by fixing endpoint models. Use the inner product conjugate-linear in its first entry. For fixed input rows i, j of equivalent words and a rational isometry $g : P_j \to P_i$, define

$$c_{ij}(g) = \sum_{r} \sum_{u : P_i \to P_r} \langle y_{ri,u}, y_{rj,ug} \rangle.$$

All sums over target rows here are finite, of arbitrary size. The height bound gives only finitely many rational isometries between a given pair of endpoints, by the count in Theorem 117.62. Absolute convergence within each vector follows from Lemma 117.45.

Lemma 117.67 (Two products between fixed inputs)**.** *Consider all $g = u^{-1}v : P_j \to P_i$ for which $\langle y_{ri,u}, y_{rj,v} \rangle \neq 0$ for some target row r. Their heights satisfy $H(g) \le x^{2B}$. The set contains at most one actual isometry in each projective class, even when the two products being compared pass through different target rows. For $i = j$, the only scalar which can occur is the identity.*

Proof. The domain calculation following Lemma 117.61 bounds both indices for the twofold composite by x^{4fB}. For each nonzero contraction choose one pair of individual cosets with nonzero marked contraction. Such a pair exists by expansion and absolute convergence. Suppose $g = u^{-1}v$ and $g' = u'^{-1}v'$ occur and $g = \xi g'$ for a rational norm-one scalar ξ. The four chosen cosets through the two target rows form a cycle with composite $gg'^{-1} = \xi I$ on P_i. Its two integral marked contractions are both nonzero. Each edge has the stipulated matched index bound. The clean-factor argument of Theorem 117.66 applies to this individual tuple: a marked place where the entire domain reduces surjectively identifies the product of the two unipotent relative markings with $\bar{\xi}I$. Unipotence forces $\bar{\xi} = 1$; the height bound makes ξ a root of unity of order prime to the mark. Thus $\xi = 1$ and $g = g'$. If $i = j$ and $u^{-1}v$ is a scalar, the two-edge coordinate separation of Theorem 117.65 applies directly under these same index bounds. This proves the last assertion without requiring a second pair. □

Theorem 117.68 (A relative estimate for the row-summed Gram blocks)**.** *Given $A, B > 0$, choose a fixed polynomial spacing in the common word construction. For all sufficiently large x on the prescribed intervals put $\delta_x = x^{-Afk}$. Let $K_{ij} = \sum_r Z_{ri}^* Z_{rj}$ for the indicated portion of the ordinary projected period, and let D be the common dimension for two equivalent input words. For $i \neq j$ put $s = 0$; for $i = j$ take any complex scalar s. With sI omitted in the off-diagonal case, one has*

$$(1 - \delta_x)E_{ij}(s) \;\le\; \frac{\|K_{ij} - sI\|_{\mathrm{HS}}^2}{D} \;\le\; (1 + \delta_x)E_{ij}(s),$$

where

$$E_{ij}(0) = \sum_g |c_{ij}(g)|^2, \qquad E_{ii}(s) = |c_{ii}(I) - s|^2 + \sum_{g \neq I} |c_{ii}(g)|^2.$$

In particular $c_{ii}(I) = \sum_{r,u} \|y_{ri,u}\|^2$. The same A, B work for every choice of the target rows and of the indicated cosets, including when $E_{ij}(s) = 0$.

Proof. Multiplication of blocks and unitarity give the exact identity

$$K_{ij} = \sum_g c_{ij}(g) U_g.$$

Indeed each ordered pair u, v contributes its finite inner product times $U_u^* U_v = U_{u^{-1}v}$. Contract before taking any norm and add all terms at the very same rational isometry. For $g = I$ on the diagonal one has $v = u$, proving the formula for its coefficient. The preceding lemma shows that the support of the sum is projectively distinct. On the diagonal adjoin I if needed and subtract s from its coefficient; the same distinctness still holds.

Let N be the number of possible elements of this support with the optional identity included. The height bound is on isometries directly between P_j and P_i, with their fixed endpoint lattices. Thus the counting argument of Theorem 117.62, now at height x^{2B}, gives $N \le x^{C_B fk}$ whenever the support is nonempty. It introduces no factor for an intermediate row. For two distinct elements g, g' in the support the relative map $g^{-1}g'$ is noncentral and has an integral domain of additive index at most x^{8fB}. Its height for Lemma 117.61 is therefore at most x^{4B}. With $L/(2k) \ge x^K$, the lemma bounds the absolute normalized Hilbert–Schmidt inner product of $U_g, U_{g'}$ by $x^{-(K-4B)f(k-1)}$, since there are at least $k-1$ unequal eigenvalue pairs. Each diagonal normalized inner product is exactly one.

Choose $K > 4B + 2C_B + 2A + 2$, which is allowed uniformly by Theorem 9.13 as in Theorem 117.62. Call the possibly modified coefficients d_g. By Cauchy–Schwarz, $(\sum_g |d_g|)^2 \le N \sum_g |d_g|^2$. The absolute error from all unequal pairs in the squared norm is hence at most

$$N x^{-(K-4B)f(k-1)} \sum_g |d_g|^2 \le \delta_x \sum_g |d_g|^2,$$

using $k-1 \ge k/2$. The diagonal sum is $E_{ij}(s)$. If the support is empty or all d_g vanish the exact identity gives the assertion directly. Inequivalent words have no common nonzero full-dimensional target and give zero off-diagonal block. □

For $s = 1$ the theorem gives the diagonal statistic of Corollary 117.44 on this portion, including the subtraction of the identity inside the relative estimate. Thus, at these matched indices, a small diagonal defect requires both the squared marked mass to be near one and the sum of squares of the correlations at nonidentity rational products to be small. Here $fk \log x$ grows faster than $x^{\varepsilon/8}$, so the relative error in this measurement is smaller than the comparison scale in Section 6.5. Establishing the required sizes of the correlations themselves and including the remaining matched indices and hyperplane cells are still needed for Section 6.6.

117.23 Exact separation by the trace on the matched lattice

There is an arithmetic obstruction to two finite columns sharing even a single raw coordinate. It uses the trace on the entire positive space and works at both marked primes, including the factors where the matched lattice has singular reduction.

For a rational transport $u : P_s \to P_t$ with marking a, define

$$h(u,a)^{2f} = \max\{[M_s : C(u,a)],\ [M_t : uC(u,a)]\}$$

using the canonical matched lattices of Section 117.18. Its inverse with inverse marking has the same height. The argument below applies to any cosets having the stated index bounds, with their complete integrals. Here $Q = g_1 g_2$ is the product of the two original auxiliary primes.

Lemma 117.69 (A trace congruence at every factor of both marks)**.** *Let g be a closed composite of matched rational isometries and inverses on one definite space of dimension k, and let $\mathfrak{J}$ be the index ideal of its successive matched domain $C \subset M$. If the composite marking at every chosen factor has trace k in the residue field, and so does its inverse transpose at the conjugate factor, then*

$$\mathfrak{J}\,(k - \operatorname{tr} g) \subseteq Q\mathcal{O}_E.$$

If $g \neq I$ and $\rho = \operatorname{rank}_E(g - I)$, it follows that

$$Q \le 2\rho\,(N\mathfrak{J})^{1/(2f)} \le 2\rho \prod_e h(u_e, a_e).$$

Proof. At each finite prime choose local integral bases of C and M. Write T for the matrix of inclusion and S for that of $g|_C$. Both are integral, and $\det T$ generates the local index ideal. Thus

$$(\det T)(k - \operatorname{tr} g) = k \det T - \operatorname{tr}(S \operatorname{adj} T)$$

is integral at every prime. At a marked factor there is a matrix a for the composite marking such that $\bar{S} = a\bar{T}$. The displayed integer therefore reduces to $(\det \bar{T})(k - \operatorname{tr} a)$. The formula also holds when $\det \bar{T} = 0$, by the adjugate identity. At the conjugate factor the identical argument uses the inverse transpose in the dual bases. Thus the displayed integer is divisible by the prime at every factor over $g_1 g_2$. These primes split completely with exponent one, giving the ideal inclusion.

The eigenvalues at every embedding are on the unit circle. If $k - \operatorname{tr} g = 0$, the sum of their real parts is k, so all eigenvalues are one. Unitarity gives $g = I$. Otherwise $|k - \operatorname{tr} g| \le 2\rho$ at each embedding by summing $1 - z$ over the eigenvalues different from one. The absolute norm of the integral ideal on the left is at least Q^{2f} and at most $(2\rho)^{2f} N\mathfrak{J}$. Finally the index at each stage of the successive domain is at most the corresponding edge index in the appropriate orientation, as in Theorem 117.50. Their product proves the second bound. □

Theorem 117.70 (Disjoint integral coordinates for different rational transports)**.** *Consider two cosets of the same ordinary block with data (u,a) and (v,b'), allowing either common positive dimension. If $u \neq v$ and the two marked columns share a coordinate of any family, then*

$$Q \leq 2\,\mathrm{rank}_E(u^{-1}v - I)\,h(u,a)h(v,b').$$

In particular, take any collection in the block of height $h(u,a) \leq H$ with $2kH^2 < Q$. In every raw integral row family the unions of occupied coordinates belonging to distinct rational transports are disjoint. Each signed projected sum at a specified transport can be recovered by coordinate restriction. On the original intervals $H = b^{8-\delta}$ is allowed for each fixed $0 < \delta < 8$ and all sufficiently large x.

Proof. In the matrix of Lemma 96.2, a common coordinate implies that the quotient of the two indices is conjugate to an upper unitriangular element at each chosen factor. In particular $a^{-1}b'$ is unipotent there. At the conjugate factor its inverse transpose is also unipotent. These are the markings of $g = u^{-1}v$ on the successive domain obtained from the edges v, u^{-1}. Their traces are k. Apply the lemma. The resulting inequality rules out a common coordinate under the displayed height restriction. Taking unions over all cosets for each actual rational isometry preserves disjointness.

Coordinate restriction has entries zero and one. Thus it retrieves each projected sum with all its signs, over any coefficient module for the raw matrix. The assertion remains true under absolute convergence for the orbital operators, since coordinate restriction commutes with that sum. Finally k is polynomial in x, $\log b = x^{\varepsilon}$, and $\log Q = (16 + o(1))\log b$, by Theorem 93.9 and the choice of marks in Section 117.15. These estimates give $2kb^{16-2\delta} < Q$. □

Corollary 117.71 (An exact diagonal Gram block)**.** *Under the common height restriction $2kH^2 < Q$, let $Z_{ri,u}$ be the whole signed contribution at the rational transport u in an ordinary block. With $Z_{ri} = \sum_u Z_{ri,u}$,*

$$Z_{ri}^* Z_{ri} = \sum_u Z_{ri,u}^* Z_{ri,u}.$$

This identity holds for any collection of both types of cells. On the portion of common positive dimension k, write $y_{ri,u} = \sum_{\gamma:\, u_\gamma = u} c_\gamma h_{a_\gamma}$ as in Section 117.22. For any set of target rows, the diagonal Gram block on the input column of dimension D is exactly

$$K_{ii} = m_i I, \qquad m_i = \sum_{r,u} \|y_{ri,u}\|^2, \qquad \frac{\|K_{ii} - I\|_{\mathrm{HS}}^2}{D} = |m_i - 1|^2.$$

The identities use the given word and require no lower bound on its compact weight gaps.

Proof. Different transports have mutually orthogonal target coordinate spaces in the normalized marked table, by the theorem. Thus $Z_{ri,u}^* Z_{ri,v} = 0$ for $u \neq v$, including for operators on the full incoming vector space. On a full-dimensional cell, Schur's lemma identifies the real operator with a scalar times the unitary U_u. All cosets at the same u therefore give $Z_{ri,u} = y_{ri,u} \otimes U_u$, whose squared operator is $\|y_{ri,u}\|^2 I$. Sum over u and r. The squared Hilbert–Schmidt norm of I is D, which proves the formula. □

Corollary 117.72 (Two source rows with a marked identification)**.** *Fix equivalent input words on P_i, P_j. Suppose there is a rational isometry $w : P_j \to P_i$. Use the identity marked matrix between their prescribed source coordinates and define $H_0 = h(w, I)$ with those endpoint lattices. Take full-dimensional terms of height at most H on each edge. If $2kH_0H^2 < Q$, then the mixed ordinary Gram block is exactly $c_{ij}(w)U_w$, with the correlations of Section 117.22 formed from these terms.*

Proof. The marked domain $C(w, I)$ is defined by the same integrality and reduction conditions as for an orbital term. It has finite index, by first clearing the denominators between the two lattices and then multiplying by Q. For a surviving contraction through a target row take two individual cosets at rational maps $u : P_i \to P_r$, $v : P_j \to P_r$ whose marked columns share a coordinate. Consider the closed composite $w^{-1}u^{-1}v$ on P_j. Its marked matrix is the relative matrix of these two columns, as w has identity marking on its domain. Both it and its inverse transpose are unipotent. The trace lemma and the strict size inequality force this composite to equal I. Therefore every nonzero contraction has $u^{-1}v = w$. The exact expansion $K_{ij} = \sum_g c_{ij}(g)U_g$ now proves the claim, including the case in which there are no such contractions. □

The diagonal statistic has thus reduced to the size of the signed vectors at each actual transport throughout the stated index range. Terms at a single transport can still cancel. Controlling that mass, the larger indices and the remaining mixed blocks at the normalization of the integral theta maps is part of Section 6.6.

117.24 Reduction of the two isotropic spaces at the terminal prime

An integral isometry can induce the identity on the positive column and at both marks while changing the rational isotropic space. We measure this change at the terminal prime already present in the two period tests. The precision below is independent of the degree of the field. All the operators in this section use the ordinary sections and the pair measures of Section 117.15.

Fix two supported rows at a translate in Section 6.6. After transporting the second flag into the first row lattice Λ, denote their outer isotropic spaces by X_0, X_1. Set $A_j = \Lambda \cap X_j$. Write h for the Hermitian form, $E = E_x$, $f = [E^+ : \mathbb{Q}]$, $s = \dim_E X_j$, $d = s + 1$, and $T_j \in \{1, T\}$ for the absolute terminal norms. Define the integral ideal

$$\mathfrak{c}_\gamma = \big\langle h(v, w) : v \in A_0,\ w \in A_1 \big\rangle_{\mathcal{O}_E}.$$

The two integral unipotent changes of representative preserve this ideal. On the transported side they change generators or apply a lattice isometry preserving the first isotropic space. Since the reduced outer spaces at each mark agree and are isotropic, $\mathfrak{c}_\gamma \subseteq Q\mathcal{O}_E$, where $Q = g_1 g_2$. The use of one row lattice for both flags also occurs in the source definition of a partner at `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 199.

Lemma 117.73 (The ideal of outer pairings at a point of the orbital integral)**.** *Suppose $s \geq 2$ and the two characters agree on their common unipotent subgroup. At a point y of the orbital integral put*

$$B(y) = \sqrt{T_0 T_1}\,\big(\widetilde{V}_0(y)\widetilde{V}_1(\gamma^{-1}y)\big)^{1/f}.$$

If $X_0 \neq X_1$, then $\mathfrak{c}_\gamma \neq 0$. For common positive dimension k its absolute norm satisfies

$$(N\mathfrak{c}_\gamma)^{1/(2f)} \leq B(y).$$

For common positive dimension $k - 1$ it satisfies

$$(N\mathfrak{c}_\gamma)^{1/(2f)} \leq B(y)^{d/(d-2)}.$$

Proof. Use the fractional ideals $\mathfrak{a}_i$ of the ordered frame, with $\mathfrak{a}_d = \mathcal{O}_E$, and let $\mathfrak{h}_j = \prod_{i \leq j} \mathfrak{a}_i$. Write D_j for the determinant of the cross pairing of the first j vectors. Put $J_j = (D_j)\mathfrak{h}_j\mathfrak{h}_j$. The frame vectors multiplied by their indicated ideal coefficients are integral. In local bases

the determinant is therefore integral, so J_j is an integral ideal, possibly zero. For $D_j \neq 0$ the exterior Cauchy–Schwarz inequality in Proposition 116.1 gives

$$(NJ_j)^{1/(2f)} \leq B(y)^j.$$

Indeed the archimedean bound on $|\tau D_j|$ has the factor $(N\mathfrak{h}_j)^{-1/f}(T_0T_1)^{j/2}$ at each place. Multiplying over one complex embedding above each real place and taking the f-th root cancels the norm of the two coefficient ideals in J_j.

First suppose the common positive dimension is k. Put $t = s - \dim(X_0 \cap X_1) > 0$. When the intersection is nonzero, Theorem 86.7 and Lemma 115.13 give a nonsingular pairing of the two first t-dimensional prefixes. Its matrix is the pairing on the quotients by the intersection, pulled back by the two prefix isomorphisms. When the intersection is zero, Theorem 86.3 gives a common negative vector in the extended flags. Its perpendicular at a real place has Witt index s. If any nonzero vector in X_1 annihilated X_0, it would enlarge X_0 there to an isotropic space of dimension $s+1$, because the two spaces are disjoint. Thus $t = s$ also gives $D_t \neq 0$. All t entries used in each determinant monomial are outer pairings. In the integral local bases they belong to $\mathfrak{c}_\gamma$. Consequently $0 \neq J_t \subseteq \mathfrak{c}_\gamma^t$. Norms of nonzero integral ideals increase under passage to a subideal, proving the first bound.

In common positive dimension $k-1$, the outer intersection is zero by Theorem 86.7. The positive Gram invariant in Theorem 86.3 is $D_d\bar{D}_d - T_0T_1D_s\bar{D}_s$, so $D_d \neq 0$, including when $D_s = 0$. In a monomial of D_d, at most two factors use the terminal row or the terminal column. The other $d-2$ factors belong to $\mathfrak{c}_\gamma$ after the local frame coefficients have been included. The two remaining factors are integral. Hence $0 \neq J_d \subseteq \mathfrak{c}_\gamma^{d-2}$. Here $d-2 > 0$. The bound for J_d proves the claim. □

For a nonzero pairing ideal define

$$\delta_T(\gamma) = \frac{1}{2f}\sum_{\mathfrak{P}\mid T} v_{\mathfrak{P}}(\mathfrak{c}_\gamma).$$

The valuations assign value one to a uniformizer. Set $\delta_T(\gamma) = +\infty$ when $\mathfrak{c}_\gamma = 0$. All $2f$ primes over T have degree one.

Theorem 117.74 (Uniform removal of distinct outer spaces with deep pairings)**.** *For each $L > 0$ there is an integer $a_L \geq 1$ such that on every sufficiently distant common interval, at every translate and on every retained pair of rows r, i,*

$$\sum_{\substack{\gamma \text{ compatible}\\ X_0 \neq X_1,\ \delta_T(\gamma) \geq a_L}} \|P_\gamma^{ri}\|_{\mathrm{HS}} \leq \sigma_{ri}\exp(-LU_x).$$

Here P_γ^{ri} includes the original finite phase and pair coefficient, and σ_{ri}, U_x are defined in Section 117.15. The sum may be restricted further by the actual transport, common hyperplane, or marked index.

Proof. Use the two simultaneous product cutoffs of Lemma 117.48 for this L. At a retained point $B(y) \leq Tb^{2C's}$ for a uniform C', with $b = \exp(x^\varepsilon)$. On the given intervals $s \geq 2$ eventually, $d/(d-2) \leq 3$, and $\log T = (34 + o(1))d\log b$ by Theorem 93.9. It follows that one can choose a_L independently of the rows and of x so that $(Tb^{2C's})^3 < T^{a_L}$. For a nonzero pairing ideal the marked divisibility and the factorization at the distinct prime T give

$$(N\mathfrak{c}_\gamma)^{1/(2f)} \geq QT^{\delta_T(\gamma)}.$$

The lemma now excludes every term in the displayed sum from the jointly retained domain. The sum of the integrals of the norms on its complement, over all the original cosets, is bounded by Lemma 117.48. Integration followed by the triangle inequality proves the result for every indicated subset. □

Corollary 117.75 (The identity coefficient for equal terminal reductions)**.** *In the original expansion of Z_{ri}, sum over the cosets for which A_0, A_1 have equal images in $\Lambda/T^{a_L}\Lambda$. Denote the resulting operator by $Z_{ri}^{[a_L]}$, and denote its flag-preserving part from Theorem 113.22 by $\mathcal{B}_{ri}$. Then*

$$\|Z_{ri}^{[a_L]} - \mathcal{B}_{ri}\|_{\mathrm{HS}} \le \sigma_{ri} \exp(-LU_x).$$

On any repeated row i, contraction against its original marked identity column gives

$$\big\|(h_{\iota_i}^* \otimes 1) Z_{ii}^{[a_L]} - \beta_i 1_{V_i}\big\|_{\mathrm{HS}} \le |\beta_i| \exp(-LU_x), \qquad |\beta_i| = \sigma_{ii}.$$

In particular the least singular value of this contraction is at least $|\beta_i|(1 - \exp(-LU_x))$.

Proof. Equality of the two images is unchanged by the integral unipotents in the double quotient. For any $w \in A_1$ choose $v \in A_0$ with $w - v \in T^{a_L}\Lambda$. Pairing with A_0 gives $\mathfrak{c}_\gamma \subseteq T^{a_L}\mathcal{O}_E$ by isotropy and integrality. All the compatible terms with distinct rational outer spaces therefore satisfy the theorem.

It remains to identify the terms with equal rational outer spaces. In this case the two saturated lattices are equal. The common-suffix argument in Lemma 115.13 identifies the whole ordered filtration. In the two ordered bases the successive ratios are a single scalar λ, including the terminal ratio modulo the outer space. The two index ideals in that lemma are the unit ideal, so $(\lambda)^s = \mathcal{O}_E$ as fractional ideals. Thus λ is a unit and $\lambda\bar{\lambda} = T_1/T_0$. Taking absolute norms gives $T_0 = T_1$. All complex absolute values of λ are one. Its positive powers lie in a bounded subset of the lattice $\mathcal{O}_E$ in its archimedean embedding, so it is a root of unity. The first graded line is primitive locally at the marks, and the two marked frames agree there up to upper unitriangular change. Hence λ reduces to one at both residue characteristics. A torsion unit reducing to one has order a power of the residue characteristic. Indeed, if its prime-to-residue-characteristic part $u \neq 1$ has order m, then $1 + u + \cdots + u^{m-1} = 0$ would reduce to the nonzero residue of m. The two characteristics are distinct, so $\lambda = 1$. This gives exactly the rational equations defining the flag-preserving terms, and each such term belongs to the selected sum.

The finite columns have unit norm. The theorem and the triangle inequality give the first bound. On a repeated row every flag-preserving term other than the identity has finite support disjoint from the identity column by Theorem 113.24. Its contraction is zero. The identity gives precisely the scalar in Theorem 113.26. Contraction has norm one, and operator norm is at most Hilbert–Schmidt norm. Applying the reverse triangle inequality to each unit input proves the singular-value assertion. □

For example, on every retained diagonal row one may take the cyclic groups of Theorem 113.10 with $(\gamma - 1)\Lambda \subseteq T^{a_L}\Lambda$. Every power then has equal outer reductions. The union of their nonidentity terms, counting each coset once over all rational hyperplanes, has total norm at most $|\beta_i| \exp(-LU_x)$. Each such term has the identity transport and the identity marking. For the full identity coefficient one must also sum the signed terms whose two outer reductions differ at this precision. They occur in the original integral comparison of Section 6.6.

117.25 Recovering a rational orbital class from its terminal reduction

Fix the original source and target rows on a common interval. Write $T_0, T_1 \in \{1, T\}$ for their terminal norms, $f = [E^+ : \mathbb{Q}]$ and $d = s + 1$. We study the terms with common positive dimension $k - 1$. In one such term the cross matrix records the pairings of the d ordered frame vectors at the two ends. Its rational class $[C]$ allows lower unitriangular changes of rows and upper unitriangular changes of columns over E. Integral changes also occur in the source definition of Gram equivalence at `Formalism/SharpRestriction/CuspSpanNormBounds/IsPrimitiveSub.lean`, line 281. Several rational unipotent double classes can have the same $[C]$. We recover the additional parameter as well.

For a positive integer a, reduce an actual integral row isometry $\gamma : \Lambda_1 \to \Lambda_0$ modulo T^a at all the places above T. Take its double class under the images of the two local integral row unipotent groups. Denote this finite class by $r_a(\gamma)$. Equality means that at each place one can compose a representative with the two local unipotents and obtain the other representative modulo T^a. No lifting of those local changes to a simultaneous rational change is required. This defines a partition of the terms of the original orbital sum. We use its Haar measures and its original finite phases throughout.

Lemma 117.76 (Separation of the cross matrices at the terminal prime)**.** *Use the two product cutoffs of Lemma 117.48 with exponent $L > 0$. There is a positive integer b_L, uniform on the common intervals, with the following property. Suppose two compatible terms meet the cutoffs, possibly at different archimedean arguments. If their reductions $r_{b_L d}$ agree, then either both have common positive dimension k, or both have common positive dimension $k - 1$ and their rational cross matrices are equivalent.*

Proof. Put $\mathfrak{a}_I = \prod_{i \in I} \mathfrak{a}_i$, $\eta_I = (N\mathfrak{a}_I)^{-1/(2f)}$ and $[j] = \{1, \dots, j\}$ as in Proposition 117.33. At a retained point for γ put $p_{\gamma,\tau}$ equal to the product of the local translated heights of the two rows. In particular

$$p_{\gamma,\tau} \geq 1, \qquad \prod_{\tau} p_{\gamma,\tau} \leq b^{2C' f s}, \qquad b = \exp(x^{\varepsilon}).$$

Here C' is the constant for the cutoffs. For each leading minor D_j, exterior Cauchy–Schwarz at the single place gives

$$|\tau D_j| / \eta_{[j]}^2 \leq (\sqrt{T_0 T_1}\, p_{\gamma,\tau})^j$$

by Proposition 116.1. This includes vanishing minors and both possible common dimensions.

We give the separation argument for a minor in rows $[j]$ and columns K, with $|K| = j$, whenever its values z, z' are invariant under the frame changes in question and satisfy the preceding local bound with denominator $\eta_{[j]}\eta_K$. Put $\mathfrak{b}_{j,K} = \overline{\mathfrak{a}_{[j]}}\mathfrak{a}_K$. Each value times this coefficient ideal is integral, by taking the exterior pairing in local integral bases. At the terminal places the coefficient ideals are units, since T splits completely and is prime to the different used in the outer ideals. Congruence of the two row maps up to integral unipotents modulo T^a implies

$$(z - z')\mathfrak{b}_{j,K} \subseteq T^a \mathcal{O}_E.$$

Indeed, after the local frame changes the reduced matrices agree, so the determinants of the indicated submatrices agree. The integral unipotents act by the stated triangular changes on the ordered frame, leaving the relevant invariant unchanged. The inclusion at the other places is just integrality.

If $z - z' \neq 0$, use the triangle inequality at each place. Since both local height products are at least one, the maximum of the two is at most their product. Multiplication over the f places gives

$$T^a \;\leq\; \big(N((z - z')\mathfrak{b}_{j,K})\big)^{1/(2f)} \;\leq\; 2\big(\sqrt{T_0 T_1}\, b^{4C' s}\big)^j.$$

This use of the placewise product allows the two retained arguments to have their largest coordinates at different places. Choose b_L so that $T^{b_L} > 2Tb^{4C's}$ uniformly for sufficiently large x, using $\log T = (34+o(1))d\log b$. Thus already $a \ge b_L j$ forces equality of the two values.

Apply this first with $K = [j]$ to every leading minor. These are invariant under both triangular changes. Equality of D_s, D_d gives equality of the Gram invariant $D_d\overline{D_d} - T_0T_1D_s\overline{D_s}$. In common positive dimension k the concatenated frame is dependent, so its Gram determinant vanishes. In common positive dimension $k-1$ the outer spaces are disjoint by relevance, and the invariant is totally positive by Theorems 86.7 and 86.3. This distinguishes the two cases.

In the latter case the nonzero leading minors give the endpoints of the successive reversed blocks by Lemma 110.10. Hence both terms have the same pivot permutation w. Now put $K_j = w([j])$, and let M_j be the minor in rows $[j]$ and columns K_j in increasing order. These minors are invariant by Proposition 117.33. The triangular proof of invariance works over every completion as well. At a block endpoint j, $M_j = D_j$ and $K_j = [j]$. Inside a block the absolute value of $\tau M_j/(\eta_{[j]}\eta_{K_j})$ interpolates geometrically between the two endpoint values, including the initial value one at $j = 0$. The bound $(\sqrt{T_0T_1}\,p_{\gamma,\tau})^j$ interpolates in exactly the same way. The separation argument therefore gives equality of all the M_j, including those with $D_j = 0$. For a block beginning after position u, its first antidiagonal entry is M_{u+1}/D_u, with $D_0 = 1$. This follows by taking the first row of that block in its monomial representative. The other entries of the block alternate in sign. Thus the two normal forms, including the entries themselves, agree. □

Theorem 117.77 (Recovery of the rational unipotent double class)**.** *Fix the two projected rational hyperplanes H_0, H_1, their rational transport j with its prescribed determinant, and the full marked index i in an ordinary hyperplane sum. For each $L > 0$ there is an integer m_L independent of all these choices, the rows, x, and the interval, such that a terminal reduction r_{m_Ld} contains at most one rational unipotent double class with an integral lift meeting the two product cutoffs. The assertion concerns all the supported rational orientations over each cross matrix.*

Proof. Increase m_L so that it is at least b_L. For two lifts as in the statement the lemma supplies a single rational cross matrix class. In the orthogonal complement of H_0 put $L' = \mathrm{pr}_{H_0^\perp}\Lambda_0$. On its determinant line choose as base the exterior vector Ω_0 of one of the supported rational classes, formed from the two ordered frames as in Theorem 117.22. The other classes over this cross matrix are determined uniquely by their vectors $\alpha\Omega_0$, where $\alpha\bar\alpha = 1$. The normalizing ideal $\mathfrak{a}$ and the Gram invariant $\mathcal{G}$ satisfy

$$Q^{-d}\mathfrak{h}_d^2\Omega_0 = \mathfrak{a}\,\det L', \qquad N\mathfrak{a} \le X_C^2, \qquad X_C = Q^{-2df}(N\mathfrak{h}_d)^2|N_{E^+/\mathbb{Q}}\mathcal{G}|$$

by Proposition 117.23. Here $\mathfrak{h}_d = \mathfrak{a}_{[d]}$. The positivity formula $\tau\mathcal{G} \le |\tau D_d|^2$ and the leading-minor bound in the lemma, applied to one retained point, give

$$X_C^{1/(2df)} \le Q^{-1}\sqrt{T_0T_1}\,b^{2C's}.$$

In particular $(N\mathfrak{a})^{1/(2f)} \le (Tb^{2C's})^{2d}$.

We check that the finite reduction compares the projected exterior vectors themselves. For an actual integral lift take the exterior product of the first frame and the transported second frame and apply $\bigwedge^{2d}\mathrm{pr}_{H_0^\perp}$. The result is $\alpha\Omega_0$. As in the proof of Proposition 117.23, a source unipotent fixes its frame wedge. A target unipotent applied to a reduced representative has, after projection on $H_0^\perp$, block upper triangular action there, with determinant one. This also applies to a product of that unipotent and the one used to put the representative in its smaller space. The assertion therefore holds after any local integral unipotent changes of the row map.

Projection is integral with values in L' by the definition of this lattice, even if L' is larger than $\Lambda_0 \cap H_0^{\perp}$.

For two classes with parameters α, α', congruence of the changed maps modulo T^a gives congruence of the two projected wedges there by multilinearity. The factor $Q^{-d}\mathfrak{h}_d^2$ is a unit at T. Away from T both normalized wedges already lie in $\det L'$. Equivalently,

$$(\alpha - \alpha')\mathfrak{a} \subseteq T^a \mathcal{O}_E.$$

If $\alpha \neq \alpha'$, then $|\tau(\alpha - \alpha')| \leq 2$ at every place, since both parameters have absolute value one. Consequently

$$T^a \leq 2(N\mathfrak{a})^{1/(2f)} \leq 2(Tb^{2C's})^{2d}.$$

By the same growth of T, an integer m_L can be chosen so that $a = m_L d$ violates this inequality throughout the intervals. The parameters agree, and Theorem 117.22 gives the same rational unipotent double class. □

Corollary 117.78 (Signed coefficients grouped at one finite precision)**.** *In the fixed hyperplane sum of the theorem let B_η be the signed ordinary operator summed over all integral lifts of the rational unipotent double class η. Form the table of signed operators at the finite indices $r_{m_L d}$, using the original terms, including their full integration domains. Assign each entry containing a lift meeting the cutoffs to its unique class in the theorem, and omit the other entries. Let $\widehat{B}_\eta$ be the sum assigned to η, with an empty sum interpreted as zero. Then*

$$\sum_{\eta} \|B_\eta - \widehat{B}_\eta\|_{\mathrm{HS}} \leq 2\sigma_{ri} \exp(-LU_x),$$

where r, i in σ_{ri} denote the two rows. The same bound holds after any selection of the original integral terms made in both tables.

Proof. A lift whose domain meets the product cutoffs contributes its entire signed operator once to the correct entry on both sides. A lift with no such point contributes its norm at most twice to the sum of the errors, even if its finite class is assigned to a different rational class. The total of these norms is at most $\sigma_{ri} \exp(-LU_x)$ by Lemma 117.48, which bounds the integrals of the norms on the discarded domains before grouping. Absolute summability justifies both groupings and the triangle inequality. □

Thus in a fixed hyperplane and transport the signed coefficients of different rational classes can be separated at the terminal prime at the scale of the diagonal normalization. Integral lifts in one such class still contribute their original signed local factors and real tensor as in Theorem 117.6. Bounding the sum of these coefficients against the prescribed period, together with the other common dimensions and the cohomology estimate, is required for Section 6.6.

117.26 The positive transport at the terminal prime

We remove the fixed-hyperplane and fixed-transport assumptions from the recovery of the rational classes in Section 117.25. The argument also identifies the transport on terms of full common positive dimension. It uses an operator between the two positive spaces that has norm at most one at every infinite place, even when the span of the two frames has a radical.

Use the integral Witt decomposition of Lemma 115.7 at each end. Denote its middle summand by J_b, the terminal vector by z_b, and its negative norm by $-T_b$, $b = 0, 1$. Regard $P_b = z_b^{\perp} \cap (J_b \otimes E)$ as a subspace of the ambient Hermitian space V_b, and denote the orthogonal projection onto it by p_b. The map $T_b p_b$ preserves Λ_b. In fact projection onto J_b is integral by the direct sum, and on J_b the formula for $T_b p_b$ is $v \mapsto T_b v + z_b h(z_b, v)$.

Lemma 117.79 (A projected operator in both common dimensions)**.** *Let $\gamma : \Lambda_1 \to \Lambda_0$ represent a compatible term. Work in V_0, write X_b, Y_b for the outer and extended spaces there, transporting side 1 by γ. Put $I = X_0 \cap X_1$, $t = s - \dim I$, and $W = Y_0 + Y_1$. Choose the following list of columns $\mathcal{Y}_\gamma$ in the target lattice.*

- *In common positive dimension $k-1$, take both complete ordered frames, target first.*
- *In common positive dimension k, take the first t outer vectors from each frame, followed by z_0. For $t = 0$ the list consists of z_0.*

Let $G_\gamma = \mathcal{Y}_\gamma^\dagger \mathcal{Y}_\gamma$ be their Hermitian Gram matrix, and set $\Delta_\gamma = \det G_\gamma$. Here $\mathcal{Y}_\gamma^\dagger v$ has entries $h(y, v)$ for the columns y. In the respective cases

$$\Delta_\gamma = (-1)^d \mathcal{G}, \qquad \Delta_\gamma = (-1)^{t+1} T_0 D_t \overline{D_t}.$$

Both are nonzero algebraic integers of E^+. Put

$$S_\gamma = p_0(1 - \mathcal{Y}_\gamma G_\gamma^{-1} \mathcal{Y}_\gamma^\dagger)\gamma p_1 : V_1 \longrightarrow V_0.$$

On P_1 this is the isometry of the two projected common spaces, extended by zero on the perpendicular in P_1. It vanishes on $P_1^\perp$. In particular it is invariant under the rational row unipotents on both sides. The same identity under frame changes holds over the completions at T.

Proof. The frame vectors are integral since their coefficient ideals are nonpositive powers of the different, and the terminal vectors are integral. In the first case the two complete frames span W nondegenerately. Their determinant is the signed Gram invariant by (86.2). This determinant formula for the concatenated Gram matrix also occurs at `Formalism/SharpRestriction/CuspSpanNormBounds/LatticeGluingRadicals.lean`, line 447. Let Π be the middle factor in the formula for S_γ. It is orthogonal projection onto $W^\perp$, since it kills the displayed columns and fixes everything perpendicular to them. The space $W^\perp$ projects isometrically onto a hyperplane at each end, as in Theorem 117.5. Indeed each of its vectors on side b is a vector of that positive hyperplane plus an outer vector. Pairing a vector of P_b with this graph is simply its pairing with the positive component. Thus orthogonal projection onto the graph, followed by projection onto the target positive space, identifies the two copies of the hyperplane and kills the source normal line.

In common dimension k, $D_t \neq 0$ for $t > 0$ by the prefix argument in Lemma 117.73. Put $D_0 = 1$. The outer prefixes together span a nondegenerate hyperbolic space with Gram determinant $(-1)^t D_t \overline{D_t}$. The vector z_0 is orthogonal to its own outer prefix. Add a vector of that prefix to it to kill its pairings with the opposite prefix, by nonsingularity. This leaves its norm $-T_0$ unchanged and is a triangular change of the displayed list of determinant one. The second determinant formula follows. Write W' for the span of that list. A common negative vector in $Y_0 \cap Y_1$ exists by Theorem 86.3. The extended spaces have intersection of dimension $\dim I + 1$, so $\dim W = 2t + 1 + \dim I$. Every vector of I is orthogonal to both extended frames. Since W' is nondegenerate and has dimension $2t + 1$, $W = W' \oplus I$.

Now Π is projection onto $(W')^\perp$. For $v \in P_1$, γv lies in $Y_1^\perp$. Both γv and its projection are perpendicular to I, so $\Pi\gamma v \in W^\perp$. Their difference lies in $W \cap Y_1^\perp = X_1$. To check the latter identity, pass to the nondegenerate W/I of dimension $2t + 1$. The image of Y_1 has dimension $t + 1$, and its perpendicular there contains X_1/I of dimension t, hence equals it. Thus $\Pi\gamma v$ represents the same vector in $Y_1^\perp/X_1$ as γv. Its image in $Y_0^\perp/X_0$ gives exactly the common-quotient isometry of the two positive spaces.

In both cases the resulting map on the positive quotients is intrinsic to the pair of frames. A row unipotent preserves the extended flag and induces the identity on its positive quotient. It therefore leaves S_γ unchanged under the two-sided change of representative. The preceding arguments use the defining subspaces, orthogonal projections and nonzero determinants over E. Extending these identities to the paired factors of a split completion gives the same invariance there. □

Theorem 117.80 (The positive spaces and transport from terminal reduction)**.** *For each $L > 0$ there is an integer c_L uniform on the fields, rows and translates of Section 6.6. Two compatible terms of a given row pair that meet the product cutoffs and have the same $r_{c_L d}$ have the same common dimension and rational positive transport. In common dimension $k-1$ they have the same two rational hyperplanes. If their original marked index is also the same, they have the same rational unipotent double class.*

Proof. Take $c_L \geq b_L$ of Lemma 117.76. Its proof gives equality of all leading minors and of the common dimension. In full common dimension the rank of the outer pairing is t. For nonzero intersection this is Theorem 86.7, and for zero intersection it is the negative-vector argument of Lemma 117.73. Thus t is the largest $j \leq s$ with $D_j \neq 0$, allowing $j = 0$. In particular the same list positions and the same scalar $\Delta = \Delta_\gamma = \Delta_{\gamma'}$ occur in the lemma for both terms.

Put $D = T_0 T_1 \Delta$. The adjugate formula gives an integral map $\Lambda_1 \to \Lambda_0$ for each DS_γ. Explicitly it is

$$(T_0 p_0)(\Delta\, 1 - \mathcal{Y}_\gamma \operatorname{adj}(G_\gamma)\mathcal{Y}_\gamma^\dagger)\gamma(T_1 p_1).$$

The two outside projections with their indicated factors preserve the integral lattices. Everything in the parentheses is integral and polynomial in the columns and their Hermitian pairings. At the places above T, use the integral unipotents giving equality modulo T^a, $a = c_L d$. For these changed maps the columns in the list and their Gram matrices agree modulo T^a. This comparison uses both conjugate factors in each Hermitian pairing. The formulas for Δ are invariant under those changes, and so is S_γ by the lemma. Consequently

$$D(S_\gamma - S_{\gamma'})\Lambda_1 \subseteq T^a \Lambda_0.$$

The inclusion at the other finite places is already given by integrality. This calculation also applies when the two Gram matrices become singular modulo T.

Let $B = S_\gamma - S_{\gamma'}$, and take Hermitian adjoints in the original row spaces. The rows are self-dual integral lattices, so DB/T^a and its adjoint are both integral maps between them. The trace of their product is integral, as is seen in a local lattice basis at each finite prime. At an infinite place B is supported from P_1 to P_0. Each of its two summands there is a partial isometry on the positive spaces. Hence

$$\tau \operatorname{tr}(B^\dagger B) = \|B_\tau|_{P_1}\|_{\mathrm{HS}}^2 \leq 4k.$$

If $B \neq 0$, these real numbers are strictly positive at every embedding. The integral trace just obtained is a nonzero algebraic integer of E^+, and $D \in E^+$. Its absolute norm is at least one, which gives

$$T^a \leq 2\sqrt{k}\,|N_{E^+/\mathbb{Q}} D|^{1/f}.$$

We bound the right side on the two retained domains. For every leading minor, the local inequality in the proof of Lemma 117.76 gives

$$|N_{E/\mathbb{Q}} D_j|^{1/f} \leq \eta_{[j]}^4 \big(\sqrt{T_0 T_1}\, b^{2C's}\big)^{2j}.$$

For a hyperplane term use $0 < \tau\mathcal{G} \le |\tau D_d|^2$. For a full-dimensional term use the second determinant formula of the lemma with $0 \le t \le s$. The prescribed ideals give $\log \eta_{[j]} = O(d^2 \log(2x))$, since they are products of powers of the different and its root norm is polynomial in x, as in Section 117.12. Together with $T_b \le T$, polynomial growth of k, and $\log T = (34 + o(1))d \log b$, the resulting estimate is

$$\log\Big(2\sqrt{k}\,|N_{E^+/\mathbb{Q}}D|^{1/f}\Big) \le C_L d \log T.$$

Choose $c_L > C_L$. This rules out $B \neq 0$. The image and the perpendicular to the kernel of the common partial isometry in the two positive spaces give the same two hyperplanes in common dimension $k-1$. In common dimension k it already gives the full transport.

In the hyperplane case the remaining action on the normal line is determined by the determinant. The ratio of the two ambient determinants is a unit of $\mathcal{O}_E$, since both maps identify the two fixed determinant lattices. Its absolute value is one at all infinite places, by unitarity. The local unipotents have determinant one, so the ratio is one modulo T^a at all primes above T. Its difference from one, if nonzero, would have absolute norm at least T^{2fa} and at most 2^{2f}, an impossibility. The determinant-line identifications of Lemma 113.6 depend only on the two fixed rows and their terminal frequencies. Thus the ambient determinants give the same positive determinant map. With the restrictions to the hyperplanes already equal, uniqueness on the normal line gives equality of the rational transports. Finally increase c_L to at least m_L of Theorem 117.77. At a common original marked index that theorem now applies with the recovered hyperplanes and transport, and gives the asserted rational unipotent class. □

Corollary 117.81 (Simultaneous grouping on a whole ordinary block)**.** *Group the original signed terms of one row pair by their marked index, common dimension and transport, and in common dimension $k-1$ also by the two hyperplanes and rational unipotent class. Call the corresponding sums B_λ. Alternatively sum first at the finite indices $(i(\gamma), r_{c_L d}(\gamma))$, keeping the original unit marked columns. Assign each occupied finite entry meeting the cutoffs to its recovered label, and discard entries without a retained point. The resulting $\widehat{B}_\lambda$ satisfy*

$$\sum_\lambda \|B_\lambda - \widehat{B}_\lambda\|_{\mathrm{HS}} \le 2\sigma_{ri} \exp(-LU_x).$$

The constant applies to the entire block through both common dimensions and all its hyperplanes and transports together.

Proof. The theorem gives a single label for all the lifts meeting the cutoffs in any finite entry. Each of their full operators is assigned correctly. Every other misplaced or discarded lift has its entire integration domain outside the simultaneous cutoffs and contributes its norm at most twice. Sum the integrals of the norms over this one row pair by Lemma 117.48. Tensoring with the unit marked column preserves the Hilbert–Schmidt norm. □

This recovers the positive transports from the finite entries of the ordinary period while keeping both original markings. In the integral projected period the entries are summed with their original phases. Bounds for that sum, its arithmetic normalization, and the strict cohomology on the same intervals are still required in Section 6.6.

117.27 The rational class when the common positive space has full dimension

Use the pair of rows and the two product cutoffs of Section 117.26. The terminal vectors z_b have norms $-T_b$, $T_b \in \{1, T\}$, for $b = 0, 1$. Denote the ordered frames by $y_b = (e_{b,1}, \dots, e_{b,s}, z_b)$, where $d = s + 1$. For an isometry $\gamma : V_1 \to V_0$ put $Z_{uv} = h(y_{0,u}, \gamma y_{1,v})$, with h linear in its

second variable. The two outer spans, viewed in V_0, are X_0, X_1; adjoining the terminal vectors gives Y_0, Y_1. Write $I = X_0 \cap X_1$, $t = s - \dim I$, and $D_j = \det Z_{[j],[j]}$, $D_0 = 1$. Assume first that the common positive dimension is k. Then $D_t \neq 0$ and $Y_0 \cap Y_1$ has a negative line modulo I, by Lemma 117.79.

Lemma 117.82 (A minor determining the relative terminal vector)**.** *Put $J = [t] \cup \{d\}$ in increasing order. The number*

$$B_t = \det Z_{J,J} = -T_0 D_t \lambda$$

is invariant under the two rational row unipotents. Here λ is determined by the images of the two prescribed terminal vectors on the common negative line, with the source representative equal to λ times the target representative modulo I. In particular $\lambda\bar{\lambda} = T_1/T_0$. For each fixed cutoff exponent $L > 0$, a common reduction r_a with $a = c_L d$ determines t, λ and the reversed blocks and their scalars on the outer prefixes for all compatible terms meeting the cutoffs, with c_L uniform.

Proof. The two prefixes of length t have invertible mutual pairing and complement I in their respective outer spans. Add a vector of its own prefix to each terminal vector to make it perpendicular to the opposite prefix. Nonsingularity gives existence and uniqueness of the corrections. Write z_0', z_1' for the resulting vectors in V_0. The corrections preserve their norms, and both vectors are perpendicular to I. In $(Y_0 + Y_1)/I$ the two prefixes span a nondegenerate space of dimension $2t$; its perpendicular there is the one negative line. Thus $z_1' = \lambda z_0' \bmod I$. The scalar product is $-T_0\lambda$. In the determinant defining B_t, the corrections add combinations of previous rows or columns and give a block diagonal matrix. This proves the formula, including $t = 0$. The common negative line $(Y_0 \cap Y_1)/I$ maps isomorphically to both Y_b/X_b. It defines the same terminal ratio independently of the representatives. Since D_t is a leading minor and hence invariant, the formula proves the invariance of B_t, including after extension to the paired factors above T.

By the proof of Theorem 117.80 the reduction at the stated order of precision determines all D_j, and t is the last nonzero leading index up to s, with 0 allowed. For $t > 0$ the t-square is in reversed blocks. For $I \neq 0$ use the prefix argument of Theorem 110.9; at unequal terminal norms the common terminal ratio is that of Lemma 115.13. For $I = 0$ the inversion test of Proposition 109.40 acts on the two prefixes, fixing their common perpendicular and its negative line. In both cases equality of the simple characters for these tests imposes the same consecutive reversals and alternating pivots, independently of T_0, T_1. The nonzero D_j, $j \leq t$, give their endpoints. For each $j \leq t$, take the minor in rows $[j]$ and columns $w([j])$ for the resulting permutation w of $[t]$. Its invariance is the triangular minors argument of Proposition 117.33, applied to the two prefixes. Inside a block its normalized absolute value at each embedding interpolates geometrically between the two endpoint values: the new pivot factor is constant in modulus, and the two new outer-ideal exponents have constant sum at each step. The proof of Lemma 117.76 therefore bounds and separates these minors up to t at precision $c_L d$. The first pivot in each block is its first new minor divided by the preceding endpoint. This recovers the scalars also for a proper prefix.

It remains to separate B_t; its rows skip the common suffix when $t < s$. Put $\mathfrak{h}_t = \prod_{i \leq t} \mathfrak{a}_i$ and $\eta_{[t]} = (N\mathfrak{h}_t)^{-1/(2f)}$ with $f = [E^+ : \mathbb{Q}]$. The terminal vectors are integral, so $B_t \bar{\mathfrak{h}}_t \mathfrak{h}_t$ is integral. A congruence of the row maps up to the local integral unipotents implies

$$(B_t - B_t')\bar{\mathfrak{h}}_t \mathfrak{h}_t \subseteq T^a \mathcal{O}_E.$$

Indeed change the maps locally to congruent representatives and take the indicated minor, using its invariance and the fact that the outer ideals are units above T. At a retained archimedean

argument the leading minor bound of Lemma 117.76 now gives

$$\frac{|\tau B_t|}{\eta_{[t]}^2} \le \sqrt{T_0T_1}\,(\sqrt{T_0T_1}\,p_{\gamma,\tau})^t, \qquad \prod_{\tau} p_{\gamma,\tau} \le b^{2C'fs}, \quad p_{\gamma,\tau} \ge 1,$$

because $|\tau\lambda| = \sqrt{T_1/T_0}$. If two values differ, take absolute ideal norms in the inclusion and use the triangle inequality place by place. For the two sets of local heights their maxima are bounded by their products, hence

$$T^a \le 2T^{t+1}b^{4C'st}.$$

Here $b = \exp(x^{\varepsilon})$ and C' depends only on the cutoffs. Since $t \le s < d$ and $\log T = (34 + o(1))d\log b$, increasing c_L by a fixed amount excludes this inequality uniformly for large x. Thus B_t, and hence λ, is determined. □

Theorem 117.83 (Recovery of every rational unipotent class in full dimension)**.** *Fix an original row pair and a cutoff exponent $L > 0$. At a uniform precision $a = c_L d$, two compatible retained terms of common positive dimension k with equal r_a have the same rational unipotent double class. The assertion holds at both terminal frequencies and includes $t = 0$ and $t = s$.*

Proof. We first compare the decorated frames, allowing their stabilizers to act on the positive quotients. The lemma gives the same prefix blocks, block scalars and terminal ratio. When $I \neq 0$, the flags on I have a one-dimensional jump at each of the last $s - t$ positions. In the two ordered suffixes the successive ratios are all λ, by the successor relation in Lemma 115.13. Choose a basis of I respecting these jumps, with target leading coefficients one, and change representatives by adding earlier vectors. The suffixes then have the form $v_i, \lambda v_i$. Choose the target terminal representative in the common negative line modulo I and use its λ-multiple at the other end. Both changes add only vectors of their respective outer spans. Together the two prefixes, this basis of I, and the common terminal representative form a basis of $Y_0 + Y_1$, with Gram matrix determined by the stated data and T_0. For $t = s$ the same construction just omits the suffix, as in Proposition 109.40; for $t = 0$ it omits the prefixes. The linear map between the two configurations matching these bases preserves the form, including its radical I. Witt's extension theorem extends it over E to the ambient space. It respects each ordered decorated flag.

Write F_b for the rational stabilizer of the decorated frame on side b, N_b for its row unipotent group, and $P_b = Y_b^{\perp}/X_b$ for its positive quotient (transported back to the corresponding row). The kernel of the action $F_b \to U(P_b)$ is N_b. In a Witt basis a frame stabilizer is unitriangular on the outer flag, fixes the prescribed negative vector modulo the outer space, and acts on the remaining middle summand by its action on P_b. The dual outer action is forced by the pairing. Thus trivial action on P_b gives precisely the row unipotents.

The preceding comparison gives $\gamma' = s_0\gamma s_1^{-1}$ with $s_b \in F_b$. Theorem 117.80 gives equality of the two positive transports $j : P_1 \to P_0$. Consequently the actions of s_0, s_1 on those spaces correspond under j. Put $W = Y_0 + Y_1$ for the configuration of γ. Its radical is I; the common space $W^{\perp}/I$ maps isometrically onto both positive quotients. This follows from Lemma 117.79, or from $W \cap Y_b^{\perp} = X_b$ and the dimensions there. Choose an E-linear complement A to I in $W^{\perp}$. Its form identifies with the nondegenerate quotient form. Every rational isometry of A extends by the identity on $A^{\perp}$ to a rational ambient isometry fixing W pointwise. In particular choose w this way to induce the action of s_0 on P_0. Then $\gamma^{-1}w\gamma$ induces that of s_1 on P_1. It follows that $s_0w^{-1} \in N_0$ and $s_1(\gamma^{-1}w\gamma)^{-1} \in N_1$. These two elements give the same equation for γ' as s_0, s_1. This proves equality in $N_0\backslash\operatorname{Isom}(V_1, V_0)/N_1$. □

Corollary 117.84 (The signed contribution of each rational class)**.** *For the original ordinary block, group its signed operators by rational unipotent double class and original marked index, in both common dimensions. Write B_ξ for the resulting operators. Instead summing first at the finite entries $(i(\gamma), r_{c_L d}(\gamma))$ and assigning an entry to the class of a retained lift if one exists gives operators $\widehat{B}_\xi$ with*

$$\sum_\xi \|B_\xi - \widehat{B}_\xi\|_{\mathrm{HS}} \le 2\sigma_{ri} \exp(-LU_x).$$

This holds on the same fields and at every allowed translate.

Proof. In dimension k the theorem makes the assignment unique. In dimension $k-1$ the marked index and reduction determine the class by Theorem 117.80. That theorem also distinguishes the two dimensions. All lifts meeting both cutoffs anywhere in their integration domain are therefore assigned correctly, with their original phases, marked columns and full integrals. An omitted or incorrectly assigned lift has its entire domain outside the simultaneous cutoffs and contributes its norm at most twice to the sum. Apply Lemma 117.48 for the whole row pair. □

This identifies the rational classes in the finite table. The original test adds its entries with their prescribed signs. The estimate for that sum and its arithmetic normalization required in Section 6.6 remains open.

117.28 Mixed ordinary periods at prescribed input flags

Consider the original marked projection on one of the common fields. We work with the terms of full common positive dimension k in Section 117.22. Fix equivalent incoming words and coherent unitary maps U_u for their rational transports. Write D for the common dimension of the compact representations. We keep complete orbital integrals at matched height $h(u,a) \le H$, where $H \ge 1$, and allow any finite set of target rows. The two primes have product $Q = g_1 g_2$, $F = E_x^+$ and $f = [F : \mathbb{Q}]$. For a set W of places of F over Q, define

$$Q(W) = \prod_{v \in W} q_v^{1/f}.$$

In particular $Q(W) = Q$ for the full set, since both marks split completely.

Use the finite groups $G_v = \mathrm{GL}_k(\mathbf{F}_{q_v})$, their upper unitriangular subgroups U_v and their upper triangular subgroups B_v. Write G, U, B for the respective products. A raw marked coordinate is a pair (aU, Uh) as in Lemma 96.2. Its input flag at v is the left coset $B_v h_v$. We use the fixed marked frames of the input and target lattices to compare residue matrices. Keeping $B_v h_v$ fixed allows all diagonal refinements of that flag.

Lemma 117.85 (A closed product at a subset of the marks)**.** *Consider a closed sequence of s contractions between input rows. Each contraction passes from one input to the next through a common target row and a common raw coordinate. Choose one contributing orbital term on each of its two edges. Suppose that, at each $v \in W$, all the chosen coordinates have the same input flag. Let g be the composite rational isometry of the initial positive space. If $g \ne I$, then*

$$Q(W) \;\le\; 2\,\mathrm{rank}_E(g - I) \prod_{\text{edges } e} h(u_e, a_e) \;\le\; 2kH^{2s}.$$

The target rows and their output flags can vary from one contraction to the next.

Proof. Two residue matrices in the support of one raw coordinate belong to aUh. If they are aj_1h, aj_2h with $j_1, j_2 \in U$, the marking of the transport from the second input to the

first is $h^{-1}j_1^{-1}j_2h$. At v it lies in $h_v^{-1}U_vh_v$. The latter group depends only on B_vh_v, because B_v normalizes U_v. All the relative markings in the closed sequence thus belong to the same unitriangular group at each $v \in W$. Their product has trace k, as does its inverse transpose at the conjugate factor.

Let $\mathfrak{J}$ be the index ideal of the successive matched domain in its source lattice. As in Lemma 117.69, at a finite place write T, S for inclusion of that domain and for g on it, respectively. Then

$$(\det T)(k - \operatorname{tr} g) = k \det T - \operatorname{tr}(S \operatorname{adj} T).$$

This is integral everywhere. At each selected factor $\bar{S} = a_{\mathrm{comp}}\bar{T}$, with a_{comp} the composite marking. The reduction of the right side is $(\det \bar{T})(k - \operatorname{tr} a_{\mathrm{comp}}) = 0$, including when the reduction of T is singular. The same holds at the conjugate factor. Thus $\mathfrak{J}(k - \operatorname{tr} g)$ is divisible by both primes of E over every $v \in W$.

If $g \neq I$, definiteness implies $k - \operatorname{tr} g \neq 0$. At every embedding its absolute value is at most $2 \operatorname{rank}_E(g - I)$, by the unitary eigenvalue decomposition. The norm of the product of the selected prime ideals is $\prod_{v \in W} q_v^2$. Taking $2f$-th roots in the norm inequality proves the first bound, using $(N\mathfrak{J})^{1/(2f)} \leq \prod_e h(u_e, a_e)$. For this last inequality impose the matched domain at each step. Its additional index is at most the index for the individual edge in the chosen orientation. There are $2s$ edges. □

Theorem 117.86 (The mixed Gram block on a prescribed tuple of flags)**.** *Fix W with $2kH^4 < Q(W)$ and fix just the input flags $\varphi = (B_vh_v)_{v \in W}$. Sum over all their refinements, over the flags at the other places, over all output coordinates and over all the chosen target rows. For any two of the incoming rows i, j, the contribution $K_{ij,\varphi}$ to the mixed Gram block is zero or is a scalar times a single rational transport $U_{g_{ij,\varphi}}$ from j to i. This includes rows with no specified rational identification.*

More precisely, let o run over the retained target rows with their raw coordinates having the prescribed flags. There are signed scalars z_{oi}, obtained from the original orbital coefficients and marked matrix entries, such that

$$c_{ij,\varphi} = \sum_o \overline{z_{oi}} z_{oj}, \qquad \frac{\|K_{ij,\varphi}\|_{\mathrm{HS}}^2}{D} = |c_{ij,\varphi}|^2.$$

On the diagonal one has $K_{ii,\varphi} = c_{ii,\varphi} I$. If $z_{oi}z_{oj} \neq 0$ for at least one o, the rational transport is unique among all the contributing contractions, including when $c_{ij,\varphi} = 0$.

Proof. First fix o, i. Since $2kH^2 < Q$, Theorem 117.70 shows that all terms meeting this individual coordinate have the same rational transport u_{oi}. By Schur's lemma in full common dimension their sum is $z_{oi}U_{u_{oi}}$, where z_{oi} retains the original signs, the Whittaker phases and the normalization of the raw matrix. Entries with zero scalar give zero operators. Absolute convergence within each entry justifies forming its sum.

If o, o' both contribute, compare $u_{oi}^{-1}u_{oj}$ and $u_{o'i}^{-1}u_{o'j}$. Composing the first with the inverse of the second gives a closed sequence of two contractions. Choose one original term on each edge with the specified coordinate. The lemma forces the closed rational isometry to be the identity. This proves uniqueness even for different target rows and different refinements outside W. Coherence of the compact identifications gives $U_{u_{oi}}^* U_{u_{oj}} = U_{u_{oi}^{-1}u_{oj}}$. Consequently the entire Gram block for these flags is the displayed signed scalar times this one unitary. Its squared Hilbert–Schmidt norm is $D|c_{ij,\varphi}|^2$. For $i = j$ the relative transport is the identity in every individual contraction. □

For example, with all places in W one may take $H = b^{4-\delta}$ for any fixed $0 < \delta < 4$. With just the places over the larger of g_1, g_2 one may take $H = b^{2-\delta}$ for $0 < \delta < 2$, and sum the other mark completely. Indeed $\log Q = (16 + o(1)) \log b$, $Q(W) \geq \sqrt{Q}$ in the second case, and $\log(2k) = o(\log b)$ uniformly on the interval. More generally two contractions yielding different rational transports can have equal input flags only at a set of places W' satisfying $Q(W') \leq 2kH^4$, by the same closed sequence.

Corollary 117.87 (Trace moments of the signed scalar matrix)**.** *Let $a \geq 1$ be an integer with $2kH^{2a} < Q(W)$, and keep the same tuple of flags on W. Write Z_φ for the rectangular matrix of compact operators on the indicated rows and coordinates, and $z_\varphi = (z_{oi})$ for its scalar entries as constructed in the proof. For each $1 \leq l \leq a$,*

$$\operatorname{tr}((Z_\varphi^* Z_\varphi)^l) = D \operatorname{tr}((z_\varphi^* z_\varphi)^l).$$

Proof. The assumed inequality already gives $2kH^2 < Q$, so each nonzero entry has the unique transport needed to define z_{oi}. Expand the trace on the left over closed walks alternating l input rows and l target coordinates. Each nonzero summand is its scalar coefficient times the trace of the compact operator of the closed rational product. The lemma with $s = l$ forces this product to be the identity. Its compact trace is D. The scalar coefficients with their signs sum to the trace on the right. □

For the full mixed block within the chosen height and common dimension one must add the operators over φ before taking a norm. In particular its squared norm still contains terms from distinct flag tuples, and the sizes of the signed scalars on a fixed tuple are arithmetic quantities. The local separation above does not bound these quantities at the normalization of the period comparison (6.3).

117.29 Relative input flags in the fourth period moment

The two contractions in the fourth moment of the original marked projection have separate input coordinates. We compute their weights on pairs of input flags and estimate the contribution of the pairs far from opposition. Fix x on one of the common intervals, with $F = E_x^+$, $f = [F : \mathbb{Q}]$, and the two marked primes g_1, g_2. Use $k, b, d, \sigma_{ri}, U_x$ and the actual blocks Z_{ri} of Section 117.15. There are $2f$ marked places. At one of them write $q = q_v$, $G = \mathrm{GL}_k(\mathbb{F}_q)$, and let U, B be the upper unitriangular and upper triangular groups. A flag is a left coset $C = Bh$, whose unitriangular subgroup is $U_C = h^{-1}Uh$.

For $s \in U_C$ let $m_C(s)$ count the nonzero entries on the first superdiagonal of hsh^{-1}. This is independent of the representative. Conjugation by U preserves these entries, and diagonal conjugation scales them by nonzero elements. Define

$$b_C(s) = \begin{cases} (-1/(q-1))^{m_C(s)}, & s \in U_C, \\ 0, & s \notin U_C. \end{cases}$$

For a pair of flags let $w \in \mathfrak{S}_k$ denote their relative position, with the reverse permutation for opposite flags. Put $N = k(k-1)/2$, and let $\ell(w)$ count inversions.

Lemma 117.88 (The signed marked weight at prescribed flags)**.** *Fix four orbital terms with marking indices a_1, a_2, a_3, a_4 in the order of Theorem 117.43. Put $s_v = a_{1,v}^{-1} a_{2,v}$, $z_v = a_{3,v}^{-1} a_{4,v}$ in the input frames of that expansion. For any set $\mathcal{D}$ of ordered pairs of full flag tuples the marked part of their fourfold contraction on $\mathcal{D}$ is exactly*

$$\mathbb{E}_{C,C'} \left[\mathbf{1}_{\mathcal{D}}(C, C') \prod_{v \mid g_1 g_2} b_{C_v}(s_v) b_{C'_v}(z_v) \right],$$

where the two tuples and all their local entries are independent uniform flags. In particular its absolute value is at most $\Pr(\mathcal{D})$.

Proof. Work first at one place. In the raw matrix of Lemma 96.2, fix an input refinement Uh. An index a occurs at the unique output $ah^{-1}U$ with an entry of modulus one. Two indices occur at the same output if $ha_1^{-1}a_2h^{-1} \in U$. If $a_j = gu_jh$ there, their coefficient is $\overline{\theta(u_1)}\theta(u_2) = \theta(ha_1^{-1}a_2h^{-1})$. Dividing by the column norm squared $[G : U]$ makes summation over Uh a uniform average.

Above Bh, the refinements are precisely Uth for diagonal t. The ratios t_i/t_{i+1} are independent uniform elements of $\mathbb{F}_q^\times$, since every tuple of ratios has $q-1$ preimages. The prescribed Whittaker character is a product of nontrivial additive characters on the simple root entries. If such an entry is nonzero, its average over the corresponding ratio is $-1/(q-1)$, obtained by removing the term at zero from a full additive character sum. A zero entry contributes one. This gives $b_C(s)$. The second pair has its own summed coordinate even when the two target rows agree, as follows by expanding the product of the two inner products in the trace. The local groups form a direct product, so all the stated independent averages follow. The bound uses $|b_C| \le 1$. It also holds before averaging diagonal refinements if one sums their absolute contributions instead. □

Define the total deficit from the opposite position by

$$J(C, C') = \sum_{v \mid g_1 g_2} (N - \ell(w_v)), \qquad q_* = \min(g_1, g_2).$$

Lemma 117.89 (Distribution and common unitriangular subgroups)**.** *For all* $j \ge 0$ *one has*

$$\Pr(J \ge j) \le q_*^{-j/2}(1 - q_*^{-1/2})^{-2fk}.$$

At one place the intersection $U_C \cap U_{C'}$ *has order* $q^{N-\ell(w)}$*. It is trivial for opposite flags.*

Proof. Choose a basis adapted to both flags, so that their steps are initial segments in the ordinary and the permuted orders. Such a basis is obtained by echelon reduction of a basis of the second flag with respect to the first, adding earlier columns to later ones and applying upper triangular row operations. In this basis the common unitriangular matrices have one free entry per pair ordered increasingly in both orders. There are $N - \ell(w)$ such pairs. The stabilizer of the second flag within the triangular stabilizer of the first also allows the k diagonal entries to be arbitrary units. Therefore the number of flags of position w to a fixed first flag is $|B|/((q-1)^k q^{N-\ell(w)}) = q^{\ell(w)}$. In particular the total number of flags is at least q^N.

The expectation of $q_*^{(N-\ell(w))/2}$ at this place is thus at most $\sum_w q^{-(N-\ell(w))/2}$. Reversing a permutation interchanges length and deficit. Inserting the largest letter into a permutation of $i-1$ letters adds any of $0, \dots, i-1$ inversions once. The sum is consequently at most $\prod_{i=1}^{k}(1 - q^{-1/2})^{-1}$. Multiply the $2f$ bounds and apply Markov's inequality to $q_*^{J/2}$. □

Theorem 117.90 (An absolute bound away from opposite input flags)**.** *For every fixed* $L > 0$ *there is* $C_L > 0$ *such that if*

$$j_x = \left\lceil C_L fk \left(d^3 + \frac{k \log(2x)}{\log b} \right) \right\rceil, \qquad j_x/(fk^2) \longrightarrow 0,$$

the part of $\operatorname{tr}(Z_{ri}^* Z_{rj} Z_{tj}^* Z_{ti})$ *with* $J \ge j_x$ *has total absolute contribution at most*

$$e^{-LU_x}\, \sigma_{ri}\sigma_{rj}\sigma_{tj}\sigma_{ti}.$$

This includes the full range of orbital heights and both terminal frequencies. For the squared norm of the mixed Gram block on any finite set of target rows the corresponding bound is $e^{-LU_x}(\sum_r \sigma_{ri}\sigma_{rj})^2$.

Proof. For each orbital quartet apply the preceding probability bound. The absolute value of the remaining operator trace is at most the product of the four Hilbert–Schmidt norms, by Cauchy–Schwarz. Lemma 117.45 bounds each sum of norms over the original orbital terms by $\sigma_{ri}\exp(CU_x)$, also with absolute values taken inside the real integrals. Hence the total over any subset of these quartets and flag pairs with $J \geq j_x$ is bounded by the product of the four σ's times

$$\exp(4CU_x)\,(1-q_*^{-1/2})^{-2fk}q_*^{-j_x/2}.$$

Here $U_x = fk(d^3\log b + k\log(2x))$. The prefactor $(1-q_*^{-1/2})^{-2fk}$ tends to one, since f,k are polynomially bounded and $q_* \geq (kb)^8$, with $b = \exp(x^{\varepsilon})$. Furthermore $j_x \log q_*/2 \geq 4C_LU_x$. Increasing C_L proves the estimate. The asserted limit follows from $d^3/k \to 0$ and $\log(2x)/\log b \to 0$ at the prescribed scales. Summation of the bound over r,t gives the mixed Gram bound. □

If the flags coincide, or lie on a common panel, at a fixed positive fraction of the $2f$ places, then $J \gg fk^2$ as $k \to \infty$. Indeed each such place has $\ell(w) \leq 1$. Thus those pairs are covered at every height. Theorem 117.46 also bounds the quartets with large total rank of the two relative markings. Both omissions can be made together because its proof uses the probability of unipotent collisions before taking any Whittaker cancellations. The residual fourth moment includes pairs with small J and small comparison rank. Its comparison with the designated integral period is still required in Section 6.6.

117.30 The first contributing flag position for a paired unipotent

We next compute a signed contribution inside the range of small deficit left by Theorem 117.90. Work at one actual marked place, of order q. Keep the notation $C, C', w, N, \ell(w), b_C$ of Section 117.29 and put $\delta = N - \ell(w)$. Let the two relative markings be $s = I + T$, $z = I + P(T)$, where T is nilpotent and P is a polynomial over $\mathbb{F}_q$ with $P(0) = 0$, $P'(0) \neq 0$. This includes $z = s^{-1}$, since the geometric series for the inverse terminates. Such pairs occur in the repeated terms of a squared Gram norm. Write $\lambda_1, \dots, \lambda_l$ for the Jordan block sizes of T, and let a_j count the blocks of size j. Put

$$r = k - l, \qquad m_\lambda = \sum_{i=1}^{l} \binom{\lambda_i}{2}, \qquad A_\lambda = \frac{l!}{\prod_j a_j!}.$$

Let X denote the set of full flags and $\mathcal{O}_\lambda$ the conjugacy class of nilpotent matrices of this type.

Theorem 117.91 (Least possible deficit and its full signed contribution)**.** *If* $b_C(s)b_{C'}(z) \neq 0$, *then* $\delta \geq m_\lambda$. *The lower bound is attained. At equality the two flag orders are obtained by reversing the order of consecutive blocks of sizes* λ_i, *preserving the order within each block. For independent uniform flags the exact contribution at equality is*

$$\mathbb{E}[\mathbf{1}_{\delta=m_\lambda} b_C(s)b_{C'}(z)] = \frac{A_\lambda q^{N-r}}{|X|(q-1)^r|\mathcal{O}_\lambda|}.$$

Proof. The strictly triangular matrices in either flag form a nilpotent algebra. The polynomial P has a composition inverse modulo t^{k+1}, obtained by equating successive coefficients using its nonzero linear coefficient. Thus $P(T)$ lies in one of these algebras exactly when T does. If the

product of the two weights is nonzero, take a basis adapted to both flags. Order its indices by $i \prec j$ when i precedes j in both flag orders. This partial order has δ comparable pairs, and T_{ij} can be nonzero only for $i \prec j$.

For a column j let d_j be the number of its predecessors and h_j the maximum number of strict steps in a chain ending at j. The vector $T^n e_j$ is zero for $n > h_j$, by the matrix product formula. Hence

$$\sum_{n\ge 1} \operatorname{rank} T^n \le \sum_{n\ge 1} \#\{j : h_j \ge n\} = \sum_j h_j \le \sum_j d_j = \delta.$$

On a Jordan block of size b, the sum on the left contributes $\sum_{n\ge 1} \max(b-n,0) = \binom{b}{2}$. This proves the lower bound. For equality one must have $h_j = d_j$ for every j, so all predecessors of each element form a chain. The argument applied to T^{t} in the reversed orders gives the same property for successors. In the diagram of immediate predecessors and successors, every element therefore has at most one of each. Its components are chains.

Each such chain is consecutive in each flag order. Indeed an element of a different component lying between two chain elements in one order would have both of its comparisons with them reversed in the other order. That would reverse the comparison of the two chain elements themselves. Distinct chain blocks must thus appear in opposite order. If the chain lengths are b_i, T is block diagonal with a strictly upper triangular matrix on each block. At power n the rank on such a block is at most $\max(b_i - n, 0)$. Equality of the total sum of ranks with $\delta = \sum_i \binom{b_i}{2}$ forces equality on every block at every power. In particular the rank on a block of size b_i is $b_i - 1$, so it is one Jordan block. Conversely, put the Jordan chains of T in consecutive blocks and reverse the order of the blocks in the second flag. Then both weights are nonzero and there are exactly m_λ comparable pairs.

There are A_λ relative permutations of the required form, one for each distinct ordering of the block sizes. For a fixed one, the number of common strictly triangular matrices of type λ is $(q-1)^r q^{m_\lambda - r}$. Within each block all entries just above its diagonal must be nonzero and suffice for regular nilpotence, since their product is the extreme entry in the highest possible power. Every entry farther above the diagonal is free. A uniformly conjugated flag pair of the specified position sees the fixed T uniformly over $\mathcal{O}_\lambda$, by transitivity and equal stabilizer sizes. For each surviving matrix the two weights have product $(q-1)^{-2r}$. In fact there are r nonzero simple entries in each flag, and replacing T by $P(T)$ scales these entries by $P'(0)$. The probability of each relative permutation is $q^{N-m_\lambda}/|X|$ by Lemma 117.89. Multiplication of these factors proves the formula. □

Theorem 117.92 (Cancellation between the first position and the later ones)**.** *Put $D_j = \prod_{i=1}^{j}(q^i - 1)$ and $\lambda'_h = \#\{i : \lambda_i \ge h\}$. The unrestricted signed weight is $(D_l/D_k)^2$. The ratio of the preceding theorem's contribution to the unrestricted weight is*

$$A_\lambda q^{e_\lambda} \frac{(1-q^{-1})^l \prod_j \prod_{i=1}^{a_j}(1-q^{-i})}{\prod_{i=1}^{l}(1-q^{-i})^2}, \qquad e_\lambda = \sum_{h\ge 2} \lambda'_h(\lambda'_h - 1).$$

The factor multiplying $A_\lambda q^{e_\lambda}$ is $\exp(O(k/q))$ with an absolute implied constant for $q \ge 2$.

Proof. The two uniform flags are independent. Averaging a single weight gives the normalized induced Whittaker character by Lemma 117.88. Theorem 92.18 gives $(-1)^r D_l/D_k$ for each, since $P(T)$ has the same kernel dimension as T. Now $|X| = D_k/(q-1)^k$ by counting ordered bases and their triangular stabilizers. Let $Z(T)$ be the invertible centralizer of T. The orbit has size $q^N D_k/|Z(T)|$. Substitution in Theorem 117.91 gives the ratio $A_\lambda q^{-r}(q-1)^l |Z(T)|/D_l^2$.

To evaluate the centralizer, view the space as the module $\bigoplus_j (\mathbb{F}_q[t]/t^j)^{a_j}$, with t acting as T. Homomorphisms from a single length-u summand to a length-v summand have dimension

$\min(u,v)$, by choosing the image of its generator. The full endomorphism algebra therefore has dimension $\sum_h (\lambda'_h)^2$. Modulo t, the induced matrix is triangular by groups of equal size, since a map from a shorter to a longer summand has image divisible by t. Its diagonal blocks are unrestricted independent matrices of sizes a_j. An endomorphism is invertible precisely when each such block is invertible. Indeed surjectivity modulo t gives surjectivity on the whole module by repeatedly substituting the image plus t times the module and using nilpotence. Thus

$$|Z(T)| = q^{\sum_h (\lambda'_h)^2} \prod_j \prod_{i=1}^{a_j} (1 - q^{-i}).$$

Using $\lambda'_1 = l$ and $\sum_{h \ge 2} \lambda'_h = r$ now gives the displayed exponent. Taking logarithms of the remaining factors gives $O(k/q)$, since at most $4k$ terms occur and $|\log(1 - q^{-i})| \le 2/q$ for $q \ge 2$. □

Corollary 117.93 (Two small ranks on the marked fields)**.** *For a transvection paired with its inverse the opposite flags give zero. The first nonzero contribution, at deficit one, is*

$$\frac{(k-1)q^{N-1}}{|X|(q^k-1)(q^{k-1}-1)}.$$

Its ratio to the whole signed weight is $(k-1)\exp(O(k/q))$*. For two blocks of size two and the remaining blocks of size one, the first contribution occurs at deficit two and its ratio is* $\binom{k-2}{2} q^2 \exp(O(k/q))$*.*

Proof. For a transvection, there are $k-1$ possible positions of the block of size two. The orbit of its nilpotent part consists of $v \otimes \phi$, with $v, \phi \neq 0$, $\phi(v) = 0$. Each such matrix has $q-1$ presentations, giving orbit size $(q^k-1)(q^{k-1}-1)/(q-1)$. Apply the two theorems. For the second partition, $l = k-2$, $\lambda'_2 = 2$, and there are $\binom{k-2}{2}$ possible placements of the two larger blocks. □

All formulas multiply over the actual marked places for prescribed local positions. In particular, when every pair has the indicated polynomial relation, the least possible total deficit is $\sum_v m_{\lambda_v}$ and its signed weight is the product of the local values in Theorem 117.91. At the marked primes $q \ge (kb)^8$, the error in the logarithm of the product of the ratio factors is $O(fk/q_*) = o(1)$. For the two-block example, the signed sum above the first position must cancel almost all of its positive contribution. The remaining orbital coefficients and the mixed pairs with independent relative markings still enter the period comparison.

117.31 Transvections on opposite and neighboring flags

We compute the second flag average when its relative marking is a transvection. The first relative marking can be arbitrary. All the flags, places and signed weights are those of Lemma 117.88. At one place write $V = \mathbb{F}_q^k$ with $k \ge 2$ and $z = I + v \otimes \phi$, where $v, \phi \neq 0$ and $\phi(v) = 0$. Only the incident pair consisting of the line $L = \langle v \rangle$ and the hyperplane $H = \ker \phi$ will enter the weights. At a fixed full flag C define

$$a = \min\{i : L \subset C_i\}, \qquad b = \min\{i : C_i \not\subset H\}.$$

In an adapted basis a is the last nonzero vector coordinate, and b is the first nonzero covector coordinate. They are distinct because equality would give $\phi(v) \neq 0$. The matrix belongs to U_C if and only if $a < b$. In this case it has one nonzero simple entry if $b = a + 1$, and none if $b \ge a + 2$.

Theorem 117.94 (Averaging a transvection on opposite flags)**.** *Fix C and choose C' uniformly from its opposite flags. The entire signed average at C' is*

$$\mathbb{E}\big[b_{C'}(z) \mid C,\ C' \text{ opposite to } C\big] = \begin{cases} 0, & a < b, \\ -1/((q-1)q^{k-1}), & a = b+1, \\ -q^{-k}, & a \geq b+2. \end{cases}$$

Proof. Take C to be the standard flag. A triangular basis change depending on L, H sends these to the coordinate line e_a and coordinate hyperplane $e_b^* = 0$. One can see this by choosing the basis in H at every step except b, and choosing the vector spanning L at step a. Each choice extends the preceding basis since these are the pivot steps. A triangular change preserves the uniform measure on opposite flags. Such flags are the translates of the reversed flag by the upper unitriangular group, acting simply transitively by Lemma 117.89. Keeping the reversed flag fixed and moving the incident pair by the inverse element therefore makes it uniform on the set of pairs of the given pivots. Indeed the triangular group acts transitively there by the basis construction and its diagonal subgroup fixes the coordinate pair. This proves the same transitivity for the unitriangular subgroup.

Normalize the vector's coordinate at a and the covector's coordinate at b both to one. Their scalings have no effect on flag weights. If $a < b$, a vector supported in positions at most a and a covector supported in positions at least b cannot give a strictly upper triangular transvection in the reversed order. If $a > b$, there are $q^{a+k-b-2}$ normalized incident pairs. There are initially $a - 1 + k - b$ free entries, and $\phi(v) = 0$ determines the vector entry at b uniquely.

Write A for the first nonzero vector coordinate and B for the last nonzero covector coordinate in the ordinary order. The surviving pairs satisfy $B < A$. Incidence is automatic there. Write $n = a-b,\ u = a-A,\ v' = B-b$. For fixed $u, v' \geq 0$, $u+v' \leq n-1$, their count is $f(u)f(v')$, with $f(0) = 1$ and $f(j) = (q-1)q^{j-1}$ for $j \geq 1$. Indeed the two ends of each support are specified and all its interior coordinates are free. The simple-entry weight is one if $u + v' \leq n - 2$ and $-1/(q-1)$ if $u + v' = n - 1$.

Let S_n be the resulting signed sum. In the ring of formal power series put $F(X) = \sum_{j\geq 0} f(j)X^j = (1-X)/(1-qX)$. The counting rule just proved gives

$$\sum_{n\geq 1} S_n X^{n-1} = \left(\frac{X}{1-X} - \frac{1}{q-1}\right) F(X)^2 = -\frac{F(X)}{q-1}.$$

Hence $S_1 = -1/(q-1)$ and $S_n = -q^{n-2}$ for $n \geq 2$. Division by q^{k+n-2}, the number of incident pairs of these pivots, gives the assertion. □

Theorem 117.95 (A recurrence for every relative position)**.** *For $w \in \mathfrak{S}_k$ put $F_w(a,b) = \mathbb{E}[b_{C'}(z) \mid C, \mathrm{pos}(C,C') = w]$, where w lists the reference coordinates in the order of the second flag in a simultaneous basis. All $k(k-1)$ values of F_w can be obtained from the opposite-flag formula in $N - \ell(w)$ successive steps. If the current permutation is $s_i w$ of length $\ell(w)+1$, the step is*

$$F_w = (qA_i - (q-1)I)F_{s_i w}.$$

Here A_i is the following operator on the pivot pairs. Form a word with the first symbol at position a, the last symbol at b and the middle symbol elsewhere, ordering the three symbols as first, middle, last. If symbols $i, i+1$ are equal, keep the current pair. If they increase, exchange them. If they decrease, take the exchanged pair with probability $1/q$ and the original with probability $(q-1)/q$. Apply this rule to average the function.

Proof. Changing only the i-dimensional space in a full flag amounts to choosing one of the other q lines in C_{i+1}/C_{i-1}. The partial flag $L \subseteq H$ induces a filtration on this quotient, by projecting its intersections with C_{i+1}. The two symbols specify the two graded dimensions. For distinct symbols there is exactly one line in the earlier grade and q others. Thus the unique line gives the increasing order, while all other lines give the decreasing order. Equal symbols are unchanged. This proves the stated rule, also when $k = 2$ and $L = H$.

For a function on flags let T_w average it over the flags of relative position w from the argument, so $A_i = T_{s_i}$ on functions of these pivots. If $\ell(s_i w) = \ell(w) + 1$, then $A_i T_w = T_{s_i w}$. To verify this count, an endpoint of position $s_i w$ determines a unique intermediate flag in the i-panel of the starting flag which has shorter position w to that endpoint. The filtration on the panel quotient from the endpoint full flag distinguishes a unique earlier line. Taking it exchanges the inverted adjacent reference coordinates. Conversely, from the shortened flag all q other lines lengthen the position. The number of endpoints is $q^{\ell(w)+1}$, by Lemma 117.89, which also verifies the normalization. On each $q + 1$ element panel the neighbor average is $(\mathbf{1}\mathbf{1}^{\mathsf{t}} - I)/q$, so its inverse is $qA_i - (q-1)I$. Inverting the composition gives the asserted step. Every permutation can be lengthened to the reverse order by successive left multiplications by adjacent exchanges. Reversing that sequence takes exactly $N - \ell(w)$ steps. This proves the recurrence. □

Corollary 117.96 (Absolute bounds for signed rank-one contractions)**.** *For an arbitrary first relative marking s, its local paired coefficient conditional on position w equals $\mathbb{E}_C[b_C(s) F_w(a_z(C), b_z(C))]$. If s also is a transvection, put $D = q^k - 1$. For every set $\mathcal{E}$ of pairs of flags one has*

$$|\mathbb{E}[\mathbf{1}_{\mathcal{E}} b_C(s) b_{C'}(z)]| \le (2k-3)^2 D^{-2}.$$

Conditional on the opposite position, the absolute value is at most $(2k-3) q D^{-2}/(q-1)$, uniformly in the two incident pairs. No relation between the markings is required.

Proof. For one transvection the mean signed value is $-1/D$, by Theorem 92.18. In its flag average the negative values occur exactly at $b = a+1$. There are $M = (q^k-1)(q^{k-1}-1)/(q-1)^2$ incident line–hyperplane pairs, by choosing a line and then a hyperplane in its quotient. For fixed $a, b = a+1$, the two normalized coordinate lists have disjoint supports and contain q^{k-2} possibilities. Thus the mean of the magnitude of the negative part is

$$B_- = \frac{(k-1)q^{k-2}}{(q-1)M} = \frac{k-1}{D} \frac{q^{k-2}(q-1)}{q^{k-1}-1} \le \frac{k-1}{D}.$$

Consequently $\mathbb{E}|b_C(s)| = 2B_- - 1/D \le (2k-3)/D$. Drop the indicator and use the independence of C, C' for the first bound. For the second, average over the opposite flags first using Theorem 117.94. Its absolute conditional mean is at most $q^{1-k}/(q-1)$, and $Dq^{1-k} < q$. The conditional expression for general s, w follows by the same order of averaging. □

For an orbital quartet on the original fields these calculations apply independently at the marked places. For example, at h places with two transvections, any restriction on the full pair of flag tuples costs at most $\prod_v (2k-3)^2/(q_v^k - 1)^2$, with the product over these h places and trivial upper bound one elsewhere. This bound uses the signs after averaging the diagonal refinements on each flag. With specified relative positions, the recurrence instead keeps their signed coefficients. It leaves the archimedean orbital weights in the same fourth moment to be estimated.

117.32 Real flag angles in the untruncated signed period

In a retained row put $d = s+1$, $n = 2s+k+1$, $f = [E_x^+ : \mathbb{Q}]$ and $b = \exp(x^{\varepsilon})$. Keep the physical Bessel section W_l, the original finite projection and $T_l \in \{1, T\}$ at either end of each orbital integral in Theorem 117.43. Its normalized squared row measure on $N_l\backslash G_\infty$ is $d\mu_l = w_l^{-2}\|W_l(y)\|^2\,dy$. For $1 \le i \le d$ let $a_{l,i,\tau}(y)$ be the successive Gram–Schmidt lengths of the physical outer frame followed by its terminal vector, in the reference positive norm after applying y_τ^{-1}. Adding earlier vectors leaves these lengths unchanged. Set $V = bn^4$. Write Ω_l for the region where $a_{l,i+1,\tau} < V a_{l,i,\tau}$ for every $i < d$ and every τ.

Lemma 117.97 (Transfer from a row measure to all its orbital integrals)**.** *Let A_0, A_1 be measurable subsets of the two integral row quotients. In every supported coset take the union of the conditions $y \in A_0$ and $\gamma^{-1}y \in A_1$. Let I_{bad} be the sum over these regions of the integrals of $|c_{01}|\,\|W_0(y)\|\,\|W_1(\gamma^{-1}y)\|$. Divide by $\lambda_{01} = |c_{01}|w_0w_1$ for a nonzero pair. With $D = \sum_{i=1}^{d} i(n-i+1)$, $\nu = 4D+1$, $q_0 = 4n^2$, and P_ν as in Theorem 116.14, one has*

$$\frac{I_{\mathrm{bad}}}{\lambda_{01}} \le P_\nu e^{Cfs\nu\log b}\{\mu_0(A_0)^{\vartheta} + \mu_1(A_1)^{\vartheta}\}, \qquad \vartheta = \tfrac{1}{2}(1-\nu/q_0) \ge \tfrac{3}{8}$$

for all sufficiently large x on the original intervals. The same bound holds on any measurable subregion of this union and for any subset of the supported cosets.

Proof. Write v_0, v_1 for the two translated heights V_l' of Proposition 117.39 at their respective row arguments. In its unit-frequency coordinates the same proposition and Theorem 105.3 give $\int (V_l')^{q_0}\,d\mu_l \le e^{Cfsq_0\log b}$. The change of majorant costs R^{fsq_0}, where $\log R = O(\log x)$, which is included in this bound. Hölder's inequality gives

$$\int_{A_l} (V_l')^{\nu}\,d\mu_l \le \mu_l(A_l)^{1-\nu/q_0} e^{Cfs\nu\log b}.$$

The exponent is positive since $\nu \le n^2$ for sufficiently large x, as $d^2/n \to 0$.

First impose only $y \in A_0$. In the disjoint union of all orbital domains, apply Cauchy–Schwarz after multiplying the first section norm by $v_0^{\nu/2}v_1^{-\nu/2}$ and the second by its reciprocal. Keep the indicator in the first factor. Unfolding the first square counts all translates from the second row with weight $v_1^{-\nu}$. Their sum is at most P_ν pointwise by Theorem 116.14. Reversing rows bounds the second square by the same count times the unrestricted weighted row integral. The resulting bound is

$$P_\nu|c_{01}|w_0w_1\left(\int_{A_0}(V_0')^{\nu}\,d\mu_0\int (V_1')^{\nu}\,d\mu_1\right)^{1/2}.$$

Apply the moment estimates, interchange the rows for the second condition, and add. All unfolded quantities are nonnegative, so the estimate applies to the stated restrictions. □

Theorem 117.98 (Simultaneous real localization of the signed moments)**.** *Replace every original orbital domain by its intersection with $y \in \Omega_0$, $\gamma^{-1}y \in \Omega_1$. Write Z_{ri}^{Ω} for the resulting marked block and σ_{ri} for the normalization of Lemma 117.45. There is a uniform $c_1 > 0$ such that with $\delta = e^{-c_1 bn^2}$,*

$$\begin{aligned} \bigl|\|Z_{ri}\|_{\mathrm{HS}}^2 - \|Z_{ri}^{\Omega}\|_{\mathrm{HS}}^2\bigr| &\le \delta\sigma_{ri}^2, \\ \Bigl|\mathrm{tr}(Z_{ri}^*Z_{rj}Z_{tj}^*Z_{ti}) - \mathrm{tr}\,\bigl((Z_{ri}^{\Omega})^*Z_{rj}^{\Omega}(Z_{tj}^{\Omega})^*Z_{ti}^{\Omega}\bigr)\Bigr| &\le \delta\sigma_{ri}\sigma_{rj}\sigma_{tj}\sigma_{ti}. \end{aligned}$$

These errors also bound the absolute integral contributions where at least one of the incident row arguments is outside its region. For the squared norm of the mixed Gram block summed over any finite set of target rows the error is at most $\delta(\sum_r \sigma_{ri}\sigma_{rj})^2$.

Proof. By Theorem 87.24, $\mu_l(\Omega_l^c) \le e^{-cbn^2}$ at prime frequency. The normalized squared measure on the integral unipotent quotient gives the same probabilities for these conditions as the measure on the real Bessel quotient. Indeed the pointwise norm and the conditions are invariant under the real decorated stabilizer, and its lattice fibre has the constant volume used in w_l^2. The estimate holds also at unit frequency. Under the change of real section of Proposition 106.13, the corresponding prime-frequency measure is its pushforward and all d physical Gram–Schmidt lengths acquire the common factor $\sqrt{T}$. The successive ratios are preserved. The argument applies to every unit vector of the prescribed product lowest compact type.

Apply Lemma 117.97 with $A_l = \Omega_l^c$. The logarithms of P_ν, of $e^{Cfs\nu \log b}$, and of $\lambda_{ri}/\sigma_{ri} = \sqrt{D_r D_i}$ for a nonzero row pair are bounded by a fixed polynomial in x. For P_ν this follows from its formula in Theorem 116.14, since $\log T = O(d \log b)$. For the positive column dimensions it follows from the Weyl product in Lemma 117.45. As $b = \exp(x^\varepsilon)$, each logarithm is $o(bn^2)$. It follows that the sum of absolute integrals discarded in a single block is at most $\sigma_{ri} e^{-c'bn^2}$ for some $c' > 0$.

In a product of two or four orbital integrals, bound the norm of each finite column by one and the absolute value of the trace of the real rank-one factors by the product of their norms. Use the preceding estimate for any one failed incident condition. For the other blocks use the unrestricted absolute bound $\sigma_{ri} e^{CU_x}$ of Lemma 117.45, where $U_x = fk(d^3 \log b + k \log(2x)) = o(bn^2)$. Adding the finitely many choices of a failed condition and decreasing c_1 gives both inequalities and the stated absolute form. Summation over the two target indices gives the mixed Gram estimate. Structurally zero blocks contribute zero. □

Corollary 117.99 (The cross pairing of the surviving real flags)**.** *Fix a supported coset and $1 \le j \le d$. Use the cross determinant D_j of Proposition 116.1. At a real place τ, transport the first j vectors from both physical frames to the common reference space at the two orbital arguments. Let $G_{l,j,\tau}$ be their positive Hermitian Gram matrices, and $H_{01,j,\tau}$ their cross matrix for the original indefinite form. Put*

$$A_{j,\tau} = \frac{|\det H_{01,j,\tau}|}{\sqrt{\det G_{0,j,\tau} \det G_{1,j,\tau}}}.$$

Then $0 \le A_{j,\tau} \le 1$. Write $A_j = (\prod_\tau A_{j,\tau})^{1/(fj)}$. On the common region of the theorem, simultaneously for every prefix,

$$A_j \le |N_{E_x/\mathbb{Q}} D_j|^{1/(2fj)} \frac{V^{2d-j-1}}{\sqrt{T_0 T_1}}.$$

In particular, fix any $B > 0$ and put $Q = g_1 g_2$. At a pair with at least one prime frequency, the part on which, for some j,

$$|N_{E_x/\mathbb{Q}} D_j|^{1/(2fj)} \le Q^B, \qquad A_j > b^{-14d}$$

has the absolute error bound of the theorem whenever it occurs in a second or fourth moment. If both terminal norms are $-T$, the same holds with b^{-31d}. All prefix lengths and cosets in the indicated union cost one error bound.

Proof. In the reference positive metric the original Hermitian form is represented by a unitary involution. After orthonormalizing the two lists, their cross matrix represents an orthogonal projection composed with this involution on one of the two j-planes. All its singular values are at most one, and the modulus of its determinant is $A_{j,\tau}$. Thus these numbers measure the product of the cosines of the principal angles with the image of the other plane under the involution. The cross determinant before orthonormalization has absolute value $|\tau D_j|$, since

simultaneous transport is an isometry for the indefinite form. These statements also hold when that determinant vanishes.

The last Gram–Schmidt length is at least $\sqrt{T_l}$. Subtracting any vector in the preceding isotropic span from the terminal vector preserves its Hermitian norm $-T_l$, whose absolute value is bounded by its squared positive norm. Telescoping the defining ratios on Ω_l gives $a_{l,i,\tau} \ge \sqrt{T_l} V^{-(d-i)}$ for all $i \le d$. The positive j-volume is the product of the first j lengths. After multiplying the two volumes, the exponent of V^{-1} is $2\sum_{i=1}^{j}(d-i) = j(2d-j-1)$. This proves the bound at each place with $|\tau D_j|^{1/j}$ on the right after taking j-th roots. The product of $|\tau D_j|$ over the real places of the maximal real field is $|N_{E_x/\mathbb{Q}} D_j|^{1/2}$, giving the assertion.

Finally $\log V = (1+o(1))\log b$, $\log Q = (16+o(1))\log b$, and $\log T = (34+o(1))d\log b$ on these intervals by Theorem 93.9. When $\sqrt{T_0 T_1} \ge \sqrt{T}$ and the norm test holds, the bound on A_j is at most $b^{-(15-o(1))d}$, uniformly in j. With both frequencies prime the exponent becomes $-(32-o(1))d$. Thus each indicated region lies in the single complement estimated in Theorem 117.98. □

The calculation retains the signed ordinary integrals on the original fields. It localizes low cross determinants at prime frequency to small normalized real pairings. The integrals on that remaining region and their comparison with the prescribed integral theta map in Section 6.6 still require bounds.

118 References

[1] T. H. Koornwinder, *The addition formula for Jacobi polynomials II. The Laplace type integral representation and the product formula*, Mathematisch Centrum, first printing 1972, second printing 1976. Author's report.

[2] J.-P. Serre, *Bounds for the orders of the finite subgroups of $G(k)$*, in *Group Representation Theory*, EPFL Press, 2007. Lecture I, Section 1.1, Theorem 1.