# The generalized Riemann hypothesis for Dirichlet $L$-functions

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## Volume XVIII

# Terminal characters, measures and Whittaker normalizations

**Abstract**

We fix the marked characters, measures and integral highest lines at the terminal level. The compact projections and the Whittaker pairing determine the normalizations on residue flags and their restriction subspaces.

The guide in Volume I lists all twenty-seven volumes.

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# 92 Weights, terminal lines and finite levels

## 92.1 The prescribed ranks and the final Bessel subgroup

At each $x$ on the intervals of Volume I take $E = E_x$, $F_+ = E^+$, and $n = [E : \mathbb{Q}]/4$. Thus $[F_+ : \mathbb{Q}] = 2n$. The ranks in `Formalism/LocalRingCardinality/Params.lean`, line 6 satisfy $m, k$ even, $t$ odd, $m+3 \le k$, $3m < t$, $k+m+4 \le t$, $m \le 2n \le k$, and $m^2 \le k + m(t+1)$. We write $t_{\mathrm{r}}$ for this odd rank parameter to distinguish it from a Galois translate. The two towers have different ranks, specified in `Formalism/LocalRingCardinality/DataTheta.lean`, line 627. Their signatures and embeddings are set out in `Development/AbelianVarietyTheory/Definitions/AutomorphicMembership/AdelicDataRowMeasureScaleTrivialTuple.lean`, line 260.

**Proposition 92.1** (Simultaneous ranks and signatures)**.** *Put $m = 2\lfloor \sqrt{x}/2 \rfloor$, $k = 2n$, $r = k + m - 1$, $t_{\mathrm{r}} = k + m + 5$, $d_0 = t_{\mathrm{r}} + 1$. For all sufficiently large $x$ these integers obey the rank conditions above and the size inequalities of Proposition 6.3, with the fields and translates of Proposition 6.4. At step $0 \le j \le m$ there are Hermitian spaces $X_j, Y_j$ over $E/F_+$ with dimensions and numbers of negative directions at each real place given by*

$$N_j = k + jd_0, \qquad M_j = r + jt_{\mathrm{r}}, \qquad a(X_j) = 0, \quad a(Y_j) = \lfloor (m-j)/2 \rfloor.$$

*There are isometric inclusions $X_j \subset Y_j$ for $j < m$ and $Y_m \subset X_m$ of codimension one at the endpoint.*

*Proof.* We have $n \asymp_\psi x^{2+8\varepsilon}$, $m \sim \sqrt{x}$ and $t_{\mathrm{r}}$ odd. Thus $k \ge m+3$, $t_{\mathrm{r}} > 3m$, $m \le 2n = k$, and $m^2 \le k$ for large $x$. The stronger inequality $k + m + 4 \le t_{\mathrm{r}}$ holds by construction. The bounds $xm \le k \le n^2 x^{1/2-\varepsilon}$ follow directly from the inequalities for $n, m$. With $h \ge n/2$ and $a_E = xb$, all the other size inequalities apply whenever the chosen distinguished set has density at least $x^{-\varepsilon}$. Also $4t_{\mathrm{r}} r + 1 < \min(q, d)$ on sufficiently distant common intervals. Indeed $x$ there is bounded by a fixed power of the initial selection scale, whereas the lower bounds for $q, d$ in Theorem 5.1, grow faster than every fixed power of that scale.

Take diagonal forms with the indicated numbers of coefficients $1, -1$. For $j < m$ the complement of $X_j$ has dimension $m - j - 1$. If $m - j = 2v$, its signature is $(v-1, v)$, and if $m - j = 2v + 1$ it is $(v, v)$. These are nonnegative pairs, so $X_j$ embeds as the first $N_j$ positive directions. At $m$ the identity $M_m = N_m - 1$ gives the asserted positive coordinate hyperplane. For the definite chain of dual pairs one can multiply alternate Hermitian forms by the same nonzero purely imaginary scalar to obtain skew-Hermitian spaces. This operation preserves their unitary groups. □

Let $P, Q_0 \ge 5$ be the two fixed distinct auxiliary rational primes of Lemma 121.13. At $v \mid P$ write $q_v = |\mathcal{O}_{F_+}/v|$, $R = \prod_{v|P} (\mathcal{O}_{F_+}/v)^\times$, and $U = \widehat{R}$. Use the definite chain $U(X_0), \dots, U(X_m)$, the splitting of (121.2) and one fixed initial function with one central character and full integral invariance at $P$. Take the common vector $\phi^0_{N_j, N_{j+1}}$ of (121.3) on every edge. The other Schwartz functions are fixed across all twists. Project the restriction to $H = U(Y_m)$ to a fixed right-Iwahori-invariant test space, as in Section 121.13.

**Theorem 92.2** (Residue characters at the prescribed endpoint)**.** *There is a choice of $|R|^{m-2}$ tuples of global determinant twists with one common output central character. Their restrictions to $H$ have distinct right residue characters. In particular their projected columns are pairwise orthogonal for every such common test space. Put $\delta_i = \chi_i^{(-1)^i}|_R \in U$ for $1 \le i < m$ and $v_i = \delta_i \cdots \delta_{m-1}$. In the $N_m - 1$ endpoint coordinates the character is constant on successive blocks. The first block has length $N_1$ and value $v_1$. Next come the values $v_2, \dots, v_{m-1}$, each on $d_0$ coordinates, and the final block has length $d_0 - 1$ and value one.*

*Proof.* At a place $v \mid P$, expand each common vector by the exact finite Fourier formula (121.3). In the first edge all $k$ residue labels are trivial in every surviving term, since integration against the initial function is unchanged by independent diagonal units. At vertex $i$, the incoming edge contributes the inverse of its label on the first $N_{i-1}$ coordinates and one on the remaining $d_0$. The internal determinant twist contributes $\delta_i$ in every coordinate. Invariance of integration under the Iwahori forces the outgoing labels to be the incoming labels times $\delta_i^{-1}$ in the first $N_{i-1}$ coordinates and $\delta_i^{-1}$ in the new ones. Failure at any coordinate makes that term vanish by translating by a diagonal unit on which its character differs from one. This argument works independently at every $v \mid P$. On the final output one inverts the last edge labels and appends $d_0$ trivial coordinates. Restricting to $Y_m$ removes the very last coordinate. Iteration gives the blocks claimed, whose consecutive ratios recover all the $\delta_i$, with $v_m = 1$.

To choose the global twists, write $g = \gcd(N_1, \dots, N_{m-1})$ and let $w_i = N_i/g$. The vector $w$ is primitive in $\mathbb{Z}^{m-1}$. Elementary integer column operations take it to $(0, \dots, 0, 1)$: the Euclidean algorithm replaces each successive pair of nonzero entries by its greatest common divisor and zero. The same operations act as automorphisms on tuples in any abelian character group, using integer powers and multiplication. In particular the kernel of $\delta \mapsto \prod \delta_i^{w_i}$ on $U^{m-1}$ is a copy of $U^{m-2}$. Each tuple in this kernel lifts to a tuple of global finite-order characters with the primitive weighted product one. Indeed, in the transformed coordinates lift the first $m-2$ coordinates by Lemma 121.13, take the last coordinate trivial, and undo the integer operations. Taking $(-1)^i$-th powers to return to the $\chi_i$ gives $\prod \chi_i^{(-1)^i N_i} = 1$. The central-character formula of Theorem 70.7 applies at these ranks. Distinct endpoint characters lie in orthogonal subspaces, as Haar measure is translation invariant. The common orthogonal projection commutes with the Iwahori action and preserves this orthogonality. □

**Proposition 92.3** (Local norms and denominators at these ranks)**.** *Let $A_P = R^{N_m - 1}$, let $\Xi \subset \widehat{A}_P$ be the family of endpoint characters in Theorem 92.2, and $Z = \bigcap_{\gamma \in \Xi} \ker \gamma$. Put*

$$S_1 = mk + d_0 m(m-1)/2, \qquad S_2 = \sum_{j=0}^{m-1} \left(N_j N_{j+1} - N_j(N_j - 1)/2\right).$$

*The product of the common local vectors has squared norm $P^{-[F_+:\mathbb{Q}]S_2}$ with the integral matrix lattices of volume one. Its full Fourier expansion has common scalar denominator $|R|^{S_1}$. A single residue character projector is integral after multiplication by $|R|^{N_m - 1}$. The projector onto functions on one level orbit constant on the cosets of $Z$ has exact denominator*

$$|Z| = |R|^{N_m - m + 1} = |R|^{k + m t_{\mathrm{r}} + 1}$$

*on integral scalar functions on that orbit. At the ranks of Proposition 92.1, the logarithms of these denominators and the negative logarithm of the squared norm are $o(b/x)$.*

*Proof.* For an $N_{j+1}$-by-$N_j$ residue matrix in the support of $\phi^0$, exactly $N_j(N_j - 1)/2$ entries are free. All the others are prescribed, including the diagonal ones. Multiply the resulting volumes at all edges and places and use $\prod_{v \mid P} q_v = P^{[F_+:\mathbb{Q}]}$. Formula (121.3) contributes $\prod_{v \mid P}(q_v - 1)^{-N_j}$ at edge $j$. Summing the $N_j$ gives $S_1$. The output projectors are averages over $A_P$ and $Z$, respectively. The character family is the injective image of the primitive kernel in the preceding proof, hence is a subgroup of order $|R|^{m-2}$. Finite character duality gives $[A_P : Z] = |R|^{m-2}$. The principal level at $Q_0$ makes the finite level action free at any rank by the positive-lattice stabilizer argument of Lemma 121.16. On one free orbit the $Z$-average of a point mass is $1/|Z|$ on its coset. This proves exactness of that denominator. Finally $\log |R| \le 2n \log P$, $N_j = O((j+1)n)$, $S_1 = O(m^2 n)$, and $S_2 = O(m^3 n^2)$. All logarithms in question are $O_P(m^3 n^3)$, a fixed power of $x$ for fixed $\psi, \varepsilon$, whereas $b = \exp(x^{\varepsilon})$. □

For example, on a single free level orbit and one unit archimedean coordinate the squared norm of the projected column is still $|p_\gamma(y)|^2/h$, with $h$ the number of orbits at these new ranks. This follows by summing its coordinates $p_\gamma(y)\gamma(a)$ with the Haar probability calculation of Theorem 121.18. No lower bound for this scalar follows from character orthogonality. The representations of the row tower $U(Y_j)$, which can have indefinite signatures before the endpoint, enter the prescribed Bessel period as well. In the supplied source `Formalism/SharpRestriction/Divisibility/FunctionModelRallisFromInputs.lean`, line 534 specifies nonvanishing for vectors in a given chain, and line 548 specifies its continuation into the next representation. The global normalization and comparison of those row and column integrals remain necessary in Section 6.6.

## 92.2 Integral highest weight projections in the definite chain

We compute the norm of the corner minors that occur in the archimedean model and the value of their normalized differential on the polynomial lattice of Section 66.10. The choice of lattice enters the arithmetic assertion.

For a partition $\lambda_1 \ge \cdots \ge \lambda_r \ge 0$, put $\lambda_{r+1} = 0$. Write $\Delta_i(X)$ for the leading $i$-rowed minor and set

$$p_\lambda(X) = \prod_{i=1}^{r} \Delta_i(X)^{\lambda_i - \lambda_{i+1}}, \qquad H(\lambda) = \frac{\prod_{i=1}^{r} (\lambda_i + r - i)!}{\prod_{1 \le i < j \le r} (\lambda_i - \lambda_j + j - i)}.$$

Use the empty product for $r = 0$. Extra rows and columns of $X$ are allowed. The Fischer inner product makes monomials orthogonal with $\|X^\alpha\|_{\mathrm{F}}^2 = \alpha!$.

**Lemma 92.4** (The norm of the two corner factors)**.** *One has $\|p_\lambda\|_{\mathrm{F}}^2 = H(\lambda)$. In an $M$-by-$n$ matrix, use the first $p$ rows and first columns for a partition $\lambda^+$ of length at most $p$. For a partition $\lambda^-$ of length at most $q$, form its $i$-rowed minor from rows $p+1, \dots, p+i$ and the last $i$ columns in their natural order. If $p + q \le M$ and $\max(p, q) \le n$, the squared norm of the product is $H(\lambda^+)H(\lambda^-)$. These squared norms multiply over independent matrices at different embeddings.*

*Proof.* It suffices first to use a square $r$-by-$r$ matrix $G$ of independent standard complex Gaussians. By monomial integration $\|p_\lambda\|_{\mathrm{F}}^2 = \mathbb{E}|p_\lambda(G)|^2$. Write $G = UT$ by complex Gram–Schmidt with positive diagonal on $T$. Left unitary invariance makes the conditional law of $U$ given $T$ Haar on $U(r)$. Indeed left multiplication changes only $U$, and the invariant probability measure is unique. Every leading $i$-rowed minor is multiplied by $\prod_{j \le i} T_{jj}$ under right multiplication by $T$. As in the proof of Theorem 66.30, the moments of the squared residual lengths give

$$\mathbb{E}|p_\lambda(G)|^2 = \left( \int_{U(r)} |p_\lambda(U)|^2 \, dU \right) \prod_{i=1}^{r} (r - i + 1)_{\lambda_i}.$$

To evaluate the integral take the unit tensor made of $\lambda_i - \lambda_{i+1}$ copies of $e_1 \wedge \cdots \wedge e_i$ for each $i$. The matrix coefficient of this vector with itself is $p_\lambda(U)$. The weight $\lambda$ is the highest weight in this tensor product and has multiplicity one. Each factor weight has every initial partial sum bounded by that of its highest weight. Equality in all the sums for the tensor forces the highest weight in each factor. Complete reducibility and the highest weight theorem therefore place the vector in the irreducible module of weight $\lambda$. Its dimension is

$$d_r(\lambda) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

by the Weyl dimension formula. [1, Thm. 5.22.1] Averaging the orthogonal rank-one projection onto the tensor gives a commuting operator on that module. Its trace is one, so Schur's lemma

makes it $1/d_r(\lambda)$ times the identity. Pair with the same vector to obtain the Haar integral. Substituting the dimension formula and cancelling $\prod_i (r-i)! = \prod_{i<j}(j-i)$ proves the first assertion. Reversing the order of the last columns changes the second corner polynomial into a leading-minor polynomial up to sign. The two blocks have disjoint rows, hence use independent Gaussian entries even when their sets of columns intersect. Independence also applies across embeddings. □

**Theorem 92.5** (A normalized differential on the integral lattice)**.** *Let $A$ be an integral domain of characteristic zero. Take an $s$-by-$r$ matrix $X$, $s \ge r$, and let $J$ have the identity in its first $r$ rows and zeros below. Suppose $Q \in A[X]$ has column degrees $\lambda_i$ and is invariant under adding any multiple of an earlier column to a later one, as a polynomial identity. Its component of row degrees $(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$ is $Q(J)p_\lambda(X)$. Consequently*

$$\frac{p_\lambda(\partial)Q(0)}{H(\lambda)} = Q(J) \in A.$$

*The orthogonal projection onto the indicated row weight preserves this lattice. The functional $Q \mapsto Q(J)$ on its complex span has Fischer operator norm $H(\lambda)^{-1/2}$.*

*Proof.* Keeping only the indicated row degrees commutes with the column identities because a column addition preserves the row of every variable. Call the component $Q'$. Only the diagonal monomial $\prod_i X_{ii}^{\lambda_i}$ can contribute at $J$, so $Q'(J) = Q(J)$. Since $Q'$ uses only the top $r$ rows we may now use square matrices. On a lower triangular matrix every surviving monomial with these row and column degrees is diagonal. Indeed $\sum_{ij}(i-j)\alpha_{ij} = 0$ and every summand is nonnegative there. In particular $Q'(L) = Q(J)$ on lower unitriangular matrices. On the dense open set where the leading minors do not vanish, factor $X = LT$ with $L$ lower unitriangular and $T$ upper triangular. The column identities give $Q'(X) = Q(J)\prod_i T_{ii}^{\lambda_i} = Q(J)p_\lambda(X)$. The identity holds over the fraction field by density and hence holds in $A[X]$. The polynomial $p_\lambda$ has real integral coefficients and only the indicated row degrees. Applying its differential at zero pairs it with $Q'$ in the Fischer form. Lemma 92.4 gives the displayed formula. Over $\mathbb{C}$ Cauchy–Schwarz gives the norm bound, with equality on $p_\lambda$. □

**Proposition 92.6** (The exact denominator on an ambient weight space)**.** *On integral polynomials in a 2-by-2 matrix with all row and column degrees $b \ge 0$, the least positive integer clearing the values of $(\det\partial)^b/H((b,b))$ at zero is $\operatorname{lcm}(1, \dots, b+1)$.*

*Proof.* The monomial basis is $(X_{11}X_{22})^{b-a}(X_{12}X_{21})^a$ for $0 \le a \le b$. On this monomial the value equals $(-1)^a/((b+1)\binom{b}{a})$, by the binomial theorem and $H((b,b)) = b!(b+1)!$. Put $n = b+1$. For a prime $\ell \le n$ let $h = \lfloor \log_\ell n \rfloor$ and $e = v_\ell(n)$. The factorial valuation formula counts $v_\ell\binom{n-1}{a}$ as the number of borrows when subtracting $a$ from $n-1$ in base $\ell$. The lowest $e$ digits of $n-1$ are $\ell-1$, and no borrow can leave the highest position. There are at most $h-e$ borrows. At $a = \ell^h - 1$ the identity

$$n\binom{n-1}{\ell^h-1} = \ell^h\binom{n}{\ell^h}$$

attains valuation $h$, since the degree $\ell^h$ coefficient of $(1+z)^n \bmod \ell$ is the nonzero leading base-$\ell$ digit of $n$. This follows by multiplying the digit factors $(1+z^{\ell^j})^{n_j}$. No prime above $n$ occurs. These are exactly the prime powers in the claimed least common multiple. □

**Theorem 92.7** (The definite chain with intermediate projections)**.** *In (66.24) start with $Q_0 = 1$. At every step keep the highest row weight of the output at each embedding, using*

*Theorem 92.5. Use the same masses and terminal identity frame. The simultaneous nonvanishing, stabilizer denominator and finite-place bound of Theorem 66.29 hold for the projected values. The torsion bound* (66.28) *also holds. The polynomial acting on the translated product section now has squared Fischer norm*

$$\|P^{\mathrm{pr}}\|_{\mathrm{F}}^2 = \prod_{j,\tau} H(v_{j,\tau}) = \|P\|_{\mathrm{F}}^2 \Big/ \prod_{j,\tau} d_{R_j}(v_{j,\tau}), \tag{92.1}$$

*where $v_{j,\tau}$ is the unpadded partition in Section 66.11. The field-norm upper estimate* (66.27) *holds with this polynomial and the projected value.*

*Proof.* After projection the output at class $h$ is a scalar times $\prod_\tau p_{\lambda_{j,\tau}}(Y_\tau)$, by Theorem 92.5 applied separately to each matrix. The scalar is its value at $I$. At the next step set the new frame equal to $I$ in (66.22). This replaces the previous frame by the transpose of the leading square block of the derivative matrix. Principal minors are unchanged under transposition. Multiplication by the determinant power adds the same integer to every part. It follows by induction that the scalar is obtained by applying the single polynomial

$$P^{\mathrm{pr}}(D) = \prod_{j,\tau} p_{v_{j,\tau}}(D_{j,\tau,[1:R_j,\,1:R_j]})$$

in independent derivative variables to the very same translated product section $\mathcal{G}_h$ as in the proof of Theorem 66.29. This polynomial takes value one on the rectangular inclusions. Its block degrees $b_j$ agree with those in that theorem. The positive vacuum overlap gives $\mathcal{G}_h \neq 0$. The product-torus norm identity (66.25) and the sampling proof in Section 66.12 use only nonvanishing of the polynomial and these block degrees. They give the asserted common torsion inputs, with algebraic values for the corrected translations.

Projection keeps a subset of the monomials, so after division by each step's previously established prime power the coefficients are still integral at every selected place. Apply Proposition 66.28 inductively, clearing the same masses. Independent square blocks and Lemma 92.4 give the first equality in (92.1). Its second equality is the one-block norm calculation divided into (66.26). All coefficients of $P^{\mathrm{pr}}$ are integers, and the section, degrees and Gaussian envelope in the proof of (66.27) are identical. That proof therefore gives the stated field-norm estimate. □

The two corners used for the indefinite archimedean norm appear in `Formalism/LocalToGlobal/IndefiniteReal/LocalNewformFunction.lean`, line 293. The subtraction involving the indefinite row weights is specified in `Formalism/LocalRingCardinality/KernelPlaceMeanBound.lean`, line 25. Theorem 92.7 computes what the integral projection supplies on the definite polynomials in this manuscript.

## 92.3 Elementary divisors of the finite map at the terminal prime

Let $T$ be any terminal prime furnished by Theorem 7.11. There are $f = [E^+ : \mathbb{Q}]$ split places of $E^+$ above $T$, each with residue field $\mathbb{F}_T$. At one place put $V = \mathbb{F}_T^k$. On functions on $V^* \setminus \{0\}$ use the operator

$$(Dh)(x) = \sum_{\xi(x)=1} h(\xi), \qquad 0 \neq x \in V. \tag{92.2}$$

The same definition is used in `Formalism/LogVandermondeBounds/FinrankFiberSumZero.lean`, line 16. We compute all its elementary divisors over the $T$-adic integers on the summand with zero sum on each scalar orbit. In particular the computation measures both the best common divisor of its image and the largest power needed to invert it.

Let $\omega : \mathbb{F}_T^\times \longrightarrow \mathbb{Z}_T^\times$ be the Teichmüller character, which lifts each residue to the unique root of unity of order dividing $T-1$ in its residue class. For $1 \le j \le T-2$ write $L_j(V^*)$ for functions with $h(c\xi) = \omega(c)^{-j} h(\xi)$. The analogous lattice on $V$ is denoted by $L_j(V)$. Each can be extended by zero at the origin. The multiplicative group has order invertible in $\mathbb{Z}_T$, and its character table has inverse given by orthogonality over $\mathbb{Z}_T$. Thus the common kernel of the orbit sums is exactly the direct sum of the $L_j$ with $j$ in this range.

**Lemma 92.8** (Integral Fourier bases in one variable)**.** *Choose a primitive $T$-th root $\zeta$, put $\varpi = \zeta - 1$ and normalize its valuation to one in $\mathbb{Z}_T[\zeta]$. Put $e_0 = \mathbf{1}_{\{0\}}$, $e_{T-1} = \mathbf{1}_{\mathbb{F}_T}$, and, for $1 \le d \le T-2$, let $e_d(x) = \omega(x)^{-d}$ for $x \neq 0$ and $e_d(0) = 0$. These functions form a $\mathbb{Z}_T$-basis. For the unnormalized Fourier operator $(\mathcal{F}h)(y) = \sum_x \zeta^{xy} h(x)$,*

$$\mathcal{F} e_d = \gamma_d e_{T-1-d}, \qquad v_\varpi(\gamma_d) = d.$$

*Here $\gamma_0 = 1$, $\gamma_{T-1} = T$. For $1 \le d \le T-2$, $\gamma_d = \sum_{a=1}^{T-1} \omega(a)^{-d} \zeta^a$ has squared complex absolute value $T$ at every complex embedding.*

*Proof.* Character orthogonality gives a basis on $\mathbb{F}_T^\times$. Its trivial character extended by zero equals $e_{T-1} - e_0$, so adjoining $e_0$ proves the basis assertion. Fourier transformation interchanges the two endpoint functions with the displayed factors. For the others its value at zero vanishes by orthogonality. For $y \neq 0$ change $x$ to $a/y$ to obtain $\gamma_d \omega(y)^d$.

The shifted cyclotomic polynomial $((1+X)^T - 1)/X$ is Eisenstein, so $v_\varpi(T) = T-1$. Expand $\zeta^a = (1+\varpi)^a$. The coefficient of $\varpi^l$ in the resulting sum is in $\mathbb{Z}_T$ and reduces modulo $T$ to $\sum_{a \neq 0} a^{-d} \binom{a}{l}$. For $0 \le l < d$ this vanishes by summing the monomials over $\mathbb{F}_T^\times$. For $l = d$ the same sum is $-1/d! \bmod T$, since the leading coefficient is $1/d!$. The lower coefficients therefore have $\varpi$-valuation at least $T-1$, and higher powers have order greater than $d$. Thus $v_\varpi(\gamma_d) = d$. At any complex embedding the additive and multiplicative characters in this Gauss sum remain nontrivial. In the squared absolute value write $a = cb$. The inner sum over $b \neq 0$ is $T-1$ for $c = 1$ and $-1$ otherwise. Summing the character of $c$ gives $T$. □

**Theorem 92.9** (All elementary divisors in each scalar character)**.** *On $L_j(V^*)$ the map (92.2) takes values in $L_{T-1-j}(V)$. Up to units its elementary divisors over $\mathbb{Z}_T$, with multiplicities, are*

$$T^{(|d|-j)/(T-1)}, \qquad d \in \{0, 1, \dots, T-1\}^k, \quad |d| \equiv j \pmod{T-1}, \qquad |d| = \sum_i d_i. \tag{92.3}$$

*The minimum exponent is zero and the maximum is $k-1$, both in every nontrivial character. At any complex embedding the squared norm of every vector in such a character is multiplied by $T^{k-1}$ for the counting inner products.*

*Proof.* Use dual coordinate bases for $V, V^*$. The tensor products $e_d = \bigotimes_i e_{d_i}$ form integral bases on the whole spaces. Scalar multiplication acts on a tensor by $\omega^{-|d|}$. Keeping the displayed congruence gives bases of precisely the character lattices, and their elements vanish at the origin. The Fourier transform in $k$ variables sends $e_d$ to $\prod_i \gamma_{d_i}$ times the tensor indexed by $(T-1-d_i)_i$.

For an input $h$ of character $\omega^{-j}$, its sum over each nonzero line is zero. The terms with $\xi(x) = 0$ in the Fourier sum vanish by this cancellation and $h(0) = 0$. Write each remaining covector uniquely as $c\xi_0$ with $\xi_0(x) = 1$. Summing first over $c \neq 0$ gives $\mathcal{F}h = \gamma_j D h$ on $x \neq 0$. At the origin both sides extend by zero. Hence $D$ sends a basis tensor to a single target basis tensor with coefficient $\prod_i \gamma_{d_i}/\gamma_j$. This coefficient lies in $\mathbb{Z}_T$, since (92.2) and both bases are integral over $\mathbb{Z}_T$. Its $T$-valuation is $(|d| - j)/(T-1)$ by Lemma 92.8. The target permutation

is bijective. Independent unit scalings and ordering the diagonal entries give exactly (92.3). Take $d = (j, 0, \dots, 0)$ and $d = (j, T-1, \dots, T-1)$ for the two extremes. Every admissible total lies between their totals. Finally $\mathcal{F}^*\mathcal{F} = T^k I$ by orthogonality of the additive characters and $|\gamma_j|^2 = T$. This yields the asserted complex norm. □

**Corollary 92.10** (The joint finite map on the definite class fibres)**.** *Give the initial definite rank-$k$ Hermitian genus the upper unitriangular levels at the two primes of Theorem 7.11. At each place above $T$, refine the level by a nonzero residue covector. Let $P$ be the number of classes before this refinement and let the free coefficient lattice over $\mathbb{Z}_T$ have rank $a_1$ per class. One may equally extend scalars to the integers of a coefficient field at $T$. In the space with zero sum on every individual scalar orbit the rank is*

$$Pa_1 \left( \frac{T^k - 1}{T - 1} (T - 2) \right)^f . \tag{92.4}$$

*For fixed local characters $1 \le j_v \le T-2$, the elementary divisors of the joint incidence map are the products in (92.3) over the $f$ places, each repeated $Pa_1$ times before collecting equal powers. Its image contains $T^{f(k-1)}$ times the target lattice in those characters. It has an image vector with a unit coordinate. The exponent $f(k-1)$ is the least common exponent for the inclusion.*

*Proof.* An automorphism of a definite integral lattice has finite order. Indeed its integral matrices form a discrete subset of the compact stabilizer of the positive trace form. At $g_1$ the chosen level forces unipotent reduction of each first split component. The other component is its inverse transpose in dual bases. The underlying rational representation therefore has unipotent reduction at $g_1$. The kernel of reduction in $\mathrm{GL}_{2fk}(\mathbb{Z}_{g_1})$ has no torsion. To see this, for a matrix $I + g_1^b A$ with $b \ge 1$ and a unit entry of $A$, raising to a positive exponent prime to $g_1$ preserves $b$. Raising to $g_1$ increases it by exactly one by the binomial formula, since $g_1$ is odd. A nonidentity finite-order stabilizer element would consequently have $g_1$-power order. A power would have order $g_1$, so its rational characteristic polynomial would contain the cyclotomic polynomial of degree $g_1 - 1 > 2fk$. The inequality holds since $g_1 > (kb)^8$ and $f, k$ are polynomially bounded on the common intervals. This contradiction proves freeness.

Above each base class the additional local choices thus form the full product of the nonzero residue covector sets. Indeed a split integral basis identifies each local module with $\mathbb{F}_T^k$, and the only identifications within the fibre would be made by base-class automorphisms. Choosing other dual bases conjugates the incidence maps by integral permutations. Each scalar character on a single factor occurs once on each of its $(T^k - 1)/(T - 1)$ scalar orbits. Summing over the $T - 2$ nontrivial characters independently at each place proves (92.4). Taking tensor products of the integral diagonal bases and repeating for each coefficient coordinate and each class proves the assertions on elementary divisors. The extremal tuples in the one-place theorem can be chosen simultaneously at all places. □

The normalized squared complex norm of this finite incidence map grows by $T^{f(k-1)}$ on the joint character space (with the same counting norm on both sides). Over $\mathbb{Z}_T$ it has a unit elementary divisor even when all the local characters are nontrivial. The formula concerns the full initial class fibres. The valuation of a subsequent global period on a specified vector also depends on the remaining lifts and on the terminal linear functional.

## 92.4 Primitive negative lines at the terminal level

Use the arithmetic row lattice $\Lambda = \mathcal{O}_E^r$ of Theorem 94.9. Its diagonal Hermitian form $h$ has $p_+ = k + m/2 - 1$ entries of sign $+$ and $p_- = m/2$ of sign $-$. At large $x$ these supply at least one

positive and two negative coordinates. Take the form to be linear in its second variable. Write $M = N(g_1 g_2)^2$ and fix a prime from Corollary 94.10, with $T = \pi\bar{\pi}$ and $\pi \equiv 1 \bmod M\mathcal{O}_{K_d}$. The two principal ideals over $T$ in $K_d$ are distinct. Choose three coordinate vectors $e_+, e_-, e$, with the first positive and the last two negative, and set $u = e_+ + e_-$. This isotropic vector is orthogonal to $e$.

**Theorem 92.11** (The norm, complement and metric of a negative line)**.** *For each integer $c$ with $1 \le c < T$ put*

$$z_c = \pi e + Mc\,u.$$

*Then $h(z_c, z_c) = -T$, $z_c \equiv e \bmod M\Lambda$, and $\{t \in E : tz_c \in \Lambda\} = \mathcal{O}_E$. Its orthogonal complement $L_c = z_c^{\perp} \cap \Lambda$ has signature $(p_+, p_- - 1)$ everywhere at infinity. As $\mathcal{O}_E$-modules,*

$$\Lambda/(\mathcal{O}_E z_c \oplus L_c) \simeq \mathcal{O}_E/T\mathcal{O}_E, \qquad L_c^{\#}/L_c \simeq \mathcal{O}_E/T\mathcal{O}_E,$$

*where the dual is taken for the restricted Hermitian form. For the positive norm obtained by replacing every diagonal sign by $+$, at every archimedean embedding one has*

$$\|z_c/\pi\|^2 = 1 + 2M^2c^2/T.$$

*Proof.* The orthogonal and isotropic relations give the Hermitian norm, and the prescribed congruence of $\pi$ gives the level condition. No prime of $E$ can divide both $Mc$ and $\pi$, since $T \nmid Mc$. The coordinates thus generate the unit ideal. Multiplication by a scalar is integral on $z_c$ precisely when that scalar is integral. The functional $\ell(y) = h(z_c, y)$ is onto $\mathcal{O}_E$ by the same coordinate ideal, or by self-duality of $\Lambda$. Its kernel is $L_c$ and $\ell(z_c) = -T$. This proves the first quotient formula and the signature follows by orthogonal decomposition over the real places.

The projection $y \mapsto y + \ell(y)z_c/T$ from $E^r$ onto $z_c^{\perp}$ sends $\Lambda$ into $L_c^{\#}$. It maps onto that dual. Indeed any integral linear functional on $L_c$ extends to $\Lambda$, since $\Lambda/L_c \simeq \mathcal{O}_E$ is free. Represent the extension by an element of the self-dual $\Lambda$ using the first variable of $h$. Projecting this element leaves its pairing with $L_c$ unchanged. The kernel on $\Lambda$ is its intersection with $Ez_c$, which is $\mathcal{O}_E z_c$ by primitivity. Projection is the identity on $L_c$, so passing to the quotient gives the second formula. Finally every embedding of $K_d$ sends $\pi$ to a complex number of squared absolute value $T$, and the three coordinate vectors are orthogonal in the positive norm. This proves the metric formula. □

An integral polynomial $P$ homogeneous of degree $A$ in a terminal vector restricts to a binary form

$$P(su + te) = \sum_{i=0}^{A} b_i s^i t^{A-i}.$$

The same notation applies to a multihomogeneous polynomial of total degree $A$ on several copies of this plane on inserting the same $s, t$ in every copy. When the copies come from a CM type above a fixed embedding of $K_d$, the element $\pi$ is the same scalar at all of them. Coefficients may be in a number field $J$ containing $E$.

**Theorem 92.12** (Integral evaluation in the normalized negative line)**.** *Let $0 \le A < T - 1$ and assume the $b_i$ are integral at a place $w$ above $(\pi)$. The normalized evaluations $P(z_c/\pi)$ are integral at $w$ for all $1 \le c \le A + 1$ if and only if*

$$b_i \in \pi^i \mathcal{O}_{J,w} \quad (0 \le i \le A). \tag{92.5}$$

*In fact, with $v_w(T) = 1$ and $v_w(0) = +\infty$,*

$$\min_{1 \le c \le A+1} v_w(P(z_c/\pi)) = \min_{0 \le i \le A} \big(v_w(b_i) - i\big).$$

*At places above $(\bar\pi)$ the normalization is by a unit. For every nonzero evaluation $y = P(z_c)$ in a normal coefficient field, replacing $y$ by $P(z_c/\pi)$ decreases the normalized logarithm of the absolute field norm of $y\bar y$ by exactly $A\log T$. For integral values before division, (92.5) at all $w \mid (\pi)$ prevents any new finite denominator in any of these evaluations.*

*Proof.* Homogeneity gives $P(z_c/\pi) = \sum_i b_i (M/\pi)^i c^i$. The matrix $(c^i)_{1\le c\le A+1,\ 0\le i\le A}$ has Vandermonde determinant $\prod_{c<d}(d-c)$, a unit at $w$ because all the positive differences are less than $T$. Both this matrix and its inverse preserve the coefficient lattice. Thus its output coordinates generate the same fractional ideal in $\mathcal{O}_{J,w}$ as the input coordinates. Now $M$ and $\bar\pi$ are units at $w$, and $v_w(\pi) = v_w(T) = 1$. Taking valuations proves the criterion and the equality, also if the restricted form is identically zero. At all finite places away from $(\pi)$, division by $\pi^A$ preserves integrality. Finally $P(z_c/\pi) = P(z_c)/\pi^A$. Forming its product with its complex conjugate divides by the rational integer $T^A$, giving the asserted change in the logarithmic norm. □

**Corollary 92.13** (Metric cost at all the high orders)**.** *Let $a = \lceil xb\rceil$ and let $A = ha$ be the sum of the orders at the high embeddings of Theorem 94.2. Then $A+1 < T$ eventually on the common intervals. Uniformly for $1 \le c \le A+1$ and every archimedean embedding,*

$$0 \le A\log\|z_c/\pi\|^2 \ll \exp(-c_0 m x^{\varepsilon})$$

*for an absolute $c_0 > 0$ and all sufficiently large $x$ in the given family.*

*Proof.* Here $h \le f$ is polynomially bounded, $\log M = O_{\psi,\varepsilon}(x^{\varepsilon})$ and $\log(A+1) = O_{\psi,\varepsilon}(x^{\varepsilon})$. Corollary 94.10 gives $\log T \ge (16 - o(1))mx^{\varepsilon}$, where $m \to \infty$. The bound now follows from $A\log(1 + 2M^2c^2/T) \le 2AM^2(A+1)^2/T$. □

The norm and primitivity requirements for the Bessel line are visible in `Development/AbelianVarietyTheory/Proofs/DecoratedLatticeProofs/BesselLevelLineVectorPrimitiveRowLatticeNormStatement.lean`, line 12. The factor $T^{-A}$ in Theorem 92.12 has an exact finite integrality condition in this family of normalized lines. For instance $P(X) = X_+^A$, $A > 0$, has $b_A = 1$, and every tested normalized evaluation has valuation $-A$ at $w$. For the prescribed global period, its restriction to these lines and the required divisibilities of its coefficients still have to be proved. A net saving in its height also requires a bound for the undivided value on $z_c$, whose positive norm grows as $\sqrt{T}$ over the tested range. The example exhibits the role of the coefficients even for primitive lines of the correct norm and congruence.

## 92.5 Terminal levels for the refined column periods

Fix a field, a prime $T = \varpi\bar\varpi$, and a family of column types as in Theorem 9.13 and Corollary 9.15. At every $v \mid T$ of $F_+$ choose a nonzero residue covector $\ell_v$ on the $k$-dimensional standard column lattice. Write $K_\pi(\ell)$ for its stabilizer at every such place, with the previous column level elsewhere. Write $X_\pi(\ell) = [H]/K_\pi(\ell)$. The scalar orbits in each refinement fibre and the multiplicities on their zero-sum subspace are those of Corollary 92.10. At the unrefined row level of Section 86.10 the Bessel coefficient is invariant under the unrefined column level. Hence its pairing with every such zero-sum column vanishes, for every frequency. Indeed the coefficient is constant on each refinement fibre, and averaging a zero-sum column over the covectors in any one place gives zero. A finite translate of the row changes the stabilizer of the coefficient. We give one translate that realizes the full refined column space at both frequencies and preserves the volume of the row level.

**Lemma 92.14** (A split row lattice inducing a covector stabilizer)**.** *There is a finite adelic element $a_f \in G(\mathbb{A}_{F_+,f})$, equal to the identity away from $T$, with the following properties. Put $K'_\sigma = a_f K_\sigma a_f^{-1}$, $K'_B = B(\mathbb{A}_{F_+,f}) \cap K'_\sigma$, and let $K_\Lambda$ be the full original lattice stabilizer. Then*

$$K'_B = (N(\mathbb{A}_{F_+,f}) \cap K'_\sigma) \rtimes K_\pi(\ell), \qquad \chi_\beta(K'_B) = 1 \quad (\beta \in \mathcal{O}_E).$$

*At each $v \mid T$ the index of $K'_{\sigma,v} \cap K_{\Lambda,v}$ in $K_{\Lambda,v}$ equals the number of subspaces of codimension $s+1$ in $\mathbb{F}_T^r$, where $r = 2s+k+1$. The finite volume of $K'_\sigma$ is that of $K_\sigma$.*

*Proof.* At $v \mid T$, take $O = \mathbb{Z}_T$ and write the split Hermitian space in its two factors $V^+, V^-$. The perfect pairing identifies the second factor with the dual of the first. The split unitary group acts by an arbitrary $g \in \mathrm{GL}(V^+)$ on the first and by its contragredient on the second. Use the Witt basis of Section 121.16. It is an integral basis locally, since $T$ is odd. Use a primitive row vector lifting $\ell_v$, also denoted $\ell_v$, on the $W_0^+$ coordinates. In the positive split factor take the lattice

$$L^+ = \bigoplus_{i=1}^{s} TOu_i^+ \;\oplus\; M_\ell \;\oplus \bigoplus_{i=1}^{s} Ov_i^+, \qquad M_\ell = \{(w,z) \in O^k \oplus O : z \equiv \ell_v(w) \bmod T\}.$$

Here $z$ is the coordinate on $Ee$, and superscripts on basis vectors denote their split components. Take $L^-$ to be the lattice dual to $L^+$. Multiplying each $u_i^+$ by $T$, acting on the middle coordinates by $(w,z) \mapsto (w, \ell_v(w) + Tz)$, and fixing each $v_i^+$ takes the original positive-factor lattice onto $L^+$. Together with the dual action this defines $a_v$. Taking the product of these elements gives $a_f$. At $v$ the conjugate level is precisely the stabilizer of $L^+$ and its dual.

Every $nh \in B$ preserves the Witt flag. On the graded middle quotient $n$ acts trivially and $h$ acts by $(w,z) \mapsto (hw, z)$. The induced middle lattice is exactly $M_\ell$. Its projection onto $w$ is $O^k$. Preserving it in both directions therefore requires $h \in \mathrm{GL}_k(O)$ and $\ell_v h = \ell_v \bmod T$. Conversely every such $h$, embedded as the identity on the remaining coordinates, preserves $L^+$ and the dual lattice. Thus for any $nh$ in the stabilizer one has $h$ there, and then $n$ there as well. This proves the local semidirect product formula.

It remains to check the character on this unipotent stabilizer. Inside the flag the $O$-coefficients of $u_i^+$ in $L^+$ form $TO$, for every $i$. In the dual lattice the corresponding coefficients of $u_i^-$ form $O$ because $u_i^-$ pairs with $v_i^+$. Applying $n-1$ to $Tu_{i+1}^+$ and to $u_{i+1}^-$ proves integrality of both split components of $d_i(n)$ for $i < s$. Moreover $Te^+ \in L^+$, and $e^- \in L^-$ because the $z$-coordinate of $M_\ell$ is integral. Applying $n-1$ to these two vectors proves the same assertion for $d_s(n)$. Thus every summand in the finite additive character defining $\chi_\beta$ is integral if $\beta \in \mathcal{O}_E$. The character is trivial on $H$ by its definition. Away from $T$, Lemma 86.30 applies.

The lattice $L^+$ contains $T$ times the original lattice and is contained in the original one. Modulo $T$, the flag conditions give $s$ independent zero coordinates and the middle graph gives one more independent equation. The stabilizer inside $\mathrm{GL}_r(O)$ is the inverse image of the stabilizer of that codimension-$(s+1)$ subspace. Reduction onto $\mathrm{GL}_r(\mathbb{F}_T)$ is surjective by lifting a matrix with invertible determinant. The latter group is transitive on subspaces of the given dimension by extending bases. This proves the index. The volume identity follows from inner invariance of Haar measure on the unimodular group $G(\mathbb{A}_{F_+,f})$. □

**Theorem 92.15** (Prescribing the refined periods by one finite translate)**.** *Let $\mathcal{C}_\ell$ be the sum of the full isotypic spaces of any finite set of archimedean column types on $X_\pi(\ell)$. Let $J$ be any finite set of distinct nonzero integral frequencies in $E$, and let $\eta > 0$. There is an $H_\infty$-equivariant operator from $\mathcal{C}_\ell^J$ to compactly supported smooth, $K_\infty$-finite functions on the original-level quotient $[G]/K_\sigma$, of $L^2$ operator norm at most $\eta$. Its output $F$ for the tuple $(\phi_\alpha)_{\alpha \in J}$ satisfies*

$$\mathcal{F}_\alpha(R(a_f)F) = \phi_\alpha \quad \text{on } X_\pi(\ell) \quad (\alpha \in J).$$

*The same $a_f$ works for all columns, all frequencies and all inputs. For $J = \{1, \varpi\}$, pairing with an orthonormal basis of the refined zero-sum spaces gives the identity matrix at each matching frequency and the zero matrix across frequencies.*

*Proof.* Use the quotient with row level $K'_\sigma$ first. The compact period orbit $B(F_+)\backslash B(\mathbb{A}_{F_+})/K'_B$ embeds there by the argument of Lemma 86.30: a rational identification of two points lies in $B$ at infinity, hence in $B(F_+)$. Its image is an embedded submanifold by compactness and the inclusion of the real Lie groups. Lemma 92.14 ensures that $\sum_\alpha \chi_\alpha(n)\phi_\alpha(h)$ descends to this orbit. Fourier orthogonality on $[N]$ gives precisely the prescribed restrictions. The proof of Theorem 86.31 now applies. Extending along a small normal tube gives the equivariant extension of arbitrarily small $L^2$ norm. The finite compact-group projections followed by inversion of the induced endomorphism preserve the prescribed coefficients exactly. Here finite dimensionality of the complete column isotypic spaces follows as in Proposition 86.29, since the refined definite genus still has finitely many classes.

If $F'$ denotes this output at level $K'_\sigma$, put $F = R(a_f^{-1})F'$. This is at level $K_\sigma$, and right translation is unitary for the adelic quotient measure. It commutes with the archimedean actions. Using the entire column isotypic spaces in the construction allows us afterwards to restrict the input to the zero-sum subspaces. Choosing one basis element in one frequency component proves the matrix assertion by orthonormality. □

**Corollary 92.16** (The analytic cost on a prescribed row type)**.** *For the parameters and terminal primes above, the bound*

$$|\mathcal{P}_\varpi(R(a_f)F, \phi)| \;\le\; \exp(-c\sqrt{T}/r)\,\|F\|_{L^2([G])}\|\phi\|_{L^2([H])}$$

*holds for any row form $F$ at the original level satisfying the archimedean conditions of Theorem 87.2, and for its bounded real translates as in Proposition 87.9, with an absolute $c > 0$. Thus a unit column receiving the scalar value one from a row in this class forces $\|F\|_{L^2([G])} \ge \exp(c\sqrt{T}/r)$.*

*Proof.* Restrict the translated row to the common open subgroup $K'_\sigma \cap K_\Lambda$. At each $v \mid T$ the index from Lemma 92.14 is at most $|\mathrm{GL}_r(\mathbb{F}_T)| \le T^{r^2}$. The added factor in the restriction norm of Theorem 121.24 is at most $T^{fr^2/2}$. Its logarithm is polynomially bounded in $x$ by Lemma 87.1. The finite translation commutes with all the real differentiations and preserves the global $L^2$ norm. In Theorem 87.2, the choice of differentiation order in the $\varpi$-frequency consequently gives the same bound with just this extra factor. Since $\log T \gg mx^\varepsilon$ and $r$ is polynomial in $x$, the additional logarithm is $o(\sqrt{T}/r)$. It is absorbed by decreasing $c$. The real-translation derivative estimate of Proposition 87.9, commutes with the finite action as well. Substituting a scalar value one gives the last claim. □

The finite change here preserves the row volume in the comparison with the refined multiplicities in Corollary 9.14. Theorem 92.15 leaves the decomposition of its smooth row functions into automorphic representations undetermined. To deduce the arithmetic estimate one still needs a nonzero coefficient on the iterated integral theta image, with its arithmetic and analytic normalizations compared.

## 92.6 The Whittaker character at the two auxiliary primes

The unfolded Gram matrix for the projected Poincaré series has a finite factor indexed by changes of basis at $g_1, g_2$. We compute each entry, including its sign and its denominator. Write $F_+ = E^+$, $f = [F_+ : \mathbb{Q}]$, and retain $E, x, k, b, g_1, g_2, T$ from Section 6.6. Both $g_i$ split

completely in $E$. At each place $v$ of $F_+$ over $g_1g_2$ write $q_v = \#\kappa(v)$. Thus $g_1$ and $g_2$ each occur $f$ times in the list of $q_v$.

Over $\mathbb{F}_q$, let $G_n = \mathrm{GL}_n(\mathbb{F}_q)$ and let $U_n$ be its upper unitriangular subgroup. Fix a nontrivial additive character $\psi$ of $\mathbb{F}_q$. The character $\theta_n(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1})$ is multiplicative on $U_n$. Let $\gamma_n$ be the character of the complex representation $\mathrm{Ind}_{U_n}^{G_n} \theta_n$. The formula for an induced character gives

$$\gamma_n(g) = \frac{1}{|U_n|} \sum_{\substack{a \in G_n \\ aga^{-1} \in U_n}} \theta_n(aga^{-1}). \tag{92.6}$$

Indeed, on the lines indexed by left cosets $U_n a$, right translation permutes the lines. Each fixed line contributes the indicated character to the trace, and has $|U_n|$ representatives. In particular $\gamma_n$ vanishes outside the unipotent elements. For $I = \prod_{v \mid g_1 g_2} \mathrm{GL}_k(\kappa(v))$, the normalized matrix in question is

$$A_{ij} = \prod_{v \mid g_1 g_2} \frac{\overline{\gamma_{k,v}(j_v^{-1} i_v)}}{\gamma_{k,v}(1)} \qquad (i, j \in I). \tag{92.7}$$

Here one may choose the nontrivial additive character separately at each place. The source gives the conjugation sum in `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/GGMirabolic.lean`, line 28 and the relative basis change $j^{-1}i$ in `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/TermOpStmt.lean`, line 140. We use $\gamma_n(1)$ to make the diagonal equal to one.

**Lemma 92.17** (Averages on a block parabolic)**.** *If $n = a + b$, $a, b > 0$, and $A_0 \in G_a$, $C_0 \in G_b$, then*

$$q^{-ab} \sum_{X \in \mathrm{Mat}_{a,b}(\mathbb{F}_q)} \gamma_n \begin{pmatrix} A_0 & X \\ 0 & C_0 \end{pmatrix} = \gamma_a(A_0)\gamma_b(C_0). \tag{92.8}$$

*Proof.* Let $P$ stabilize the first $a$ coordinate directions and let $V$ be the normal subgroup with identity diagonal blocks. Averaging over $V$ projects any complex representation onto its $V$-fixed vectors. This projection commutes with block diagonal matrices. The left side of (92.8) is therefore the trace of $(A_0, C_0)$ on these fixed vectors.

Realize the induced representation by functions $h$ on $G_n$ with $h(ug) = \theta_n(u)h(g)$, acting by right translation. Partition their supports into $U_n w P$. The cells are indexed by subsets $J \subseteq \{1, \dots, n\}$ of size $a$. Indeed a subspace in the Grassmannian has a basis in column echelon form with leading (last nonzero) positions $J$. Upper unitriangular row operations take it to the corresponding coordinate subspace. Take $w$ to send the first block to $J$ and the second to its complement, in increasing order within each. If $i \in J$, $i + 1 \notin J$, then $1 + tE_{i,i+1}$ belongs to $U_n \cap wVw^{-1}$. It acts on the left by $\psi(t)$. Invariance on the right by $V$, normal in $P$, forces a fixed function to vanish on $wP$ as soon as $\psi(t) \neq 1$. The only remaining subset is the final segment of size $a$.

For this final segment the intersection $w^{-1}U_n w \cap P$ consists of block diagonal elements in $U_a \times U_b$. The character there is $\theta_a\theta_b$, since consecutive coordinates within either block are still consecutive after the permutation. On the surviving cell a right $V$-fixed function is thus determined by its restriction to $w(G_a \times G_b)$, with precisely these two left transformation laws. Conversely such a function extends by the two transformation laws and right $V$-invariance to the cell, and by zero elsewhere. This identifies the fixed space with the tensor product of the two induced representations. Taking traces proves the formula. □

**Theorem 92.18** (Value at every relative basis change)**.** *Set* $D_q(d) = \prod_{j=1}^{d}(q^j - 1)$*, with* $D_q(0) = 1$*. For a unipotent* $g \in G_n$ *let* $d = \dim\ker(g-1)$*. Then*

$$\gamma_n(g) = (-1)^{n-d} D_q(d).$$

*Proof.* Define $B_n$ by the displayed formula on unipotents and by zero elsewhere. We first prove that the functions $B_n$ satisfy (92.8). If either diagonal block is not unipotent, its characteristic polynomial gives a factor of the block triangular characteristic polynomial, and both sides vanish. Otherwise put $\alpha = \dim\ker(A_0 - 1)$, $\beta = \dim\ker(C_0 - 1)$. A uniformly chosen $X$ induces a uniformly chosen linear map

$$Y : \ker(C_0 - 1) \longrightarrow \mathbb{F}_q^a / \operatorname{im}(A_0 - 1).$$

Restriction followed by quotient is surjective on these spaces of linear maps, by extending bases. Solving the lower equation first and then the upper one gives kernel dimension $\alpha + \beta - \operatorname{rank} Y$ for the block matrix minus the identity.

Expose the columns of $Y$ one at a time. After $j$ columns of rank $h$, write $l = \alpha + j - h$. The next column belongs to their span with probability $q^{h-\alpha}$. The conditional expectation of $(-1)^{\operatorname{rank} Y} D_q(\alpha + j + 1 - \operatorname{rank} Y)$ after that column is

$$(-1)^h \{q^{h-\alpha} D_q(l+1) - (1 - q^{h-\alpha}) D_q(l)\} = (-1)^h D_q(l)(q^{j+1} - 1).$$

Starting with $D_q(\alpha)$ and taking $\beta$ successive expectations gives $D_q(\alpha) D_q(\beta)$. Multiplying by $(-1)^{n-\alpha-\beta}$ proves the block formula for $B_n$.

We prove $B_n = \gamma_n$ by induction. For $n = 1$ both are the character of the regular representation of $\mathbb{F}_q^\times$. At the identity in every rank the induced character has degree $|G_n|/|U_n| = D_q(n)$, by counting ordered independent columns and the free entries in $U_n$. Let $h = \gamma_n - B_n$. Induction and Lemma 92.17 give zero for the average of $h$ over the upper right block with any fixed proper diagonal blocks. We have $h(1) = 0$, and $h$ is supported on unipotents.

For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of $n$ with decreasing parts, let $c_\lambda(g)$ count flags with these successive dimensions such that $(g-1)F_i \subseteq F_{i-1}$ for all $i$, where $F_0 = 0$. Then $\sum_g h(g) c_\lambda(g) = 0$. For $m = 1$ this is $h(1) = 0$. Otherwise fix one flag and choose a basis adapted to it. In the sum over matrices lowering the flag after subtraction of the identity, fix the first diagonal block and all entries in the remaining diagonal block. The upper right rectangle ranges freely, and its sum is zero by the block average. Summing also over flags proves the assertion.

Write $\mu$ for the partition of Jordan block sizes of $g - 1$, and $\mu'$ for its conjugate partition. A counted flag has $F_i \subseteq \ker((g-1)^i)$, so $\sum_{j \le i} \lambda_j \le \sum_{j \le i} \mu'_j$ for every $i$, with missing parts read as zero. When $\lambda = \mu'$ the flag is uniquely $F_i = \ker((g-1)^i)$. Order the partitions by any total order extending these partial-sum inequalities. The matrix of flag counts on unipotent classes is triangular with ones on the diagonal when columns are indexed by $\mu'$. It is invertible. The vector of class sizes times the class values of $h$ is in its kernel. Every such value vanishes. □

**Corollary 92.19** (Content on the integral basis and pointwise decay)**.** *An entry of* (92.7) *vanishes if a local relative matrix is not unipotent. Otherwise set* $t_v = \operatorname{rank}(j_v^{-1} i_v - 1)$*. Then*

$$A_{ij} = \prod_{v \mid g_1 g_2} \frac{(-1)^{t_v}}{\displaystyle\prod_{h=k-t_v+1}^{k} (q_v^h - 1)}. \tag{92.9}$$

*On the lattice with basis* $I$*, the values of this pairing generate the fractional ideal* $D^{-1}\mathbb{Z}$*, where*

$$D = \prod_{v \mid g_1 g_2} \prod_{h=2}^{k} (q_v^h - 1). \tag{92.10}$$

*For $k \ge 2$, the maximum of $|A_{ij}|$ over $i \ne j$ equals* $(\min(g_1, g_2)^k - 1)^{-1}$.

*Proof.* Theorem 92.18 divided by its value at the identity gives (92.9), including the complex conjugation. A unipotent in positive dimension has a fixed vector, so $t_v \le k - 1$. Every denominator divides $D$. Fix any $j$ and at each place let $j_v^{-1} i_v$ have one Jordan block with eigenvalue one. The entry is then $\pm 1/D$. This proves the ideal assertion and the minimality of the common denominator. For distinct indices any nonzero entry has a factor with $t_v \ge 1$, giving the stated upper bound. A single transvection at one place of smallest residue size and identities elsewhere give equality. □

Thus one and the same pair of basis changes gives a primitive entry after clearing the denominator for every word and terminal prime of the prescribed family. The integral basis in this assertion consists of the changes of residue basis at the two auxiliary primes. Forming a coefficient of the projected theta lift also involves summing the global orbital terms with their archimedean weights.

## 92.7 The exact second moment on the residue bases

Let $A$ be the normalized finite matrix of (92.7) on $\mathbb{C}^I$ with its orthonormal basis indexed by $I$. Put $Q = \min(g_1, g_2)$. The number of entries in a row is too large to bound its squared norm by multiplying it by the square of the largest entry. We compute the whole sum by counting the intertwining operators of the finite induced representation.

**Lemma 92.20** (Intertwining operators at one place)**.** *For $\Gamma_n = \operatorname{Ind}_{U_n}^{G_n} \theta_n$ over $\mathbb{F}_q$ one has*

$$\dim_{\mathbb{C}} \operatorname{End}_{G_n}(\Gamma_n) = (q-1) q^{n-1}.$$

*Proof.* An equivariant endomorphism of the induced representation is determined by the image of the function supported on $U_n$ with value one at the identity. Its translates are a basis after one representative per coset has been chosen. Their relations for other representatives impose on the image the right transformation law by $\theta_n$, in addition to its defining left transformation law. Conversely every function with both laws defines an endomorphism on those translates. Such a function's value at $y$ can be chosen freely on its double coset precisely when $\theta_n(u) = \theta_n(yuy^{-1})$ for all $u \in U_n$ with $yuy^{-1} \in U_n$. A failure of this equality forces the value to be zero. Hence the dimension is the number of compatible double cosets.

The Bruhat decomposition writes these cosets uniquely as $U_n t w U_n$ with $w$ a permutation matrix and $t$ invertible diagonal. Indeed the upper triangular double cosets are indexed by the permutation of pivots under elimination. Nonzero diagonal factors on either side can be moved through the permutation and combined. If two diagonal factors represent the same $U_n$-double coset at $w$, their ratio is the diagonal of a conjugate by $w$ of a unitriangular matrix, hence is the identity.

For $i < j$ with $w(i) < w(j)$, conjugation by $tw$ sends $1 + z E_{ij}$ to $1 + z(t_{w(i)}/t_{w(j)}) E_{w(i),w(j)}$. These elementary subgroups generate the intersection in the compatibility condition. Its allowed entries are precisely the pairs increasing in both orders. One can clear them in increasing order of $j - i$, since concatenating two allowed pairs again gives an allowed pair. Testing every $z$ detects equality of the coefficients, as a nonzero multiple of $z$ runs over the field. Every adjacent ascent of $w$ must therefore increase by one. Its increasing runs are disjoint intervals of consecutive values. They occur in decreasing order of intervals, since a descent separates consecutive runs. Conversely these permutations have the required matching of adjacent pairs in both orders. The coefficients then agree exactly when $t$ is constant on the image of each

run. For $l$ runs there are $\binom{n-1}{l-1}$ choices of lengths and $(q-1)^l$ choices of scalars. Summing over $l$ gives $(q-1)q^{n-1}$. □

**Theorem 92.21** (Exact squared deviation and common denominator)**.** *Every row of $A$ has the same squared distance from the corresponding coordinate vector. Write this squared distance as $\delta$. With $D$ the exact denominator of Corollary 92.19, one has*

$$1+\delta=\prod_{v\mid g_1g_2}\prod_{j=2}^{k}(1-q_v^{-j})^{-1},\qquad \|A-\mathrm{Id}\|_{\mathrm{HS}}^2=|I|\delta, \tag{92.11}$$

$$D(1+\delta)=(g_1g_2)^{\,f(k(k+1)/2-1)}. \tag{92.12}$$

*On the fields of Section 92.6 and throughout the specified intervals,*

$$0\le\delta\le\exp\bigl(2f(g_1^{-2}+g_2^{-2})\bigr)-1=O\Bigl(\frac{f}{(kb)^{16}}\Bigr).$$

*All these statements hold for every word using the two auxiliary primes, independently of the choice of terminal prime.*

*Proof.* At a single place the average of conjugation by $G_k$ on $\mathrm{End}_{\mathbb{C}}(\Gamma_k)$ projects onto its invariant endomorphisms. Conjugation by $g$ has trace $\gamma_k(g)\overline{\gamma_k(g)}$, by taking $\Gamma_k\otimes\Gamma_k^{\vee}$ and a unitary metric. Hence

$$\sum_{g\in G_k}|\gamma_k(g)|^2=|G_k|(q-1)q^{k-1}$$

by Lemma 92.20. Divide by $\gamma_k(1)^2=D_q(k)^2$. Since $|G_k|=q^{k(k-1)/2}D_q(k)$, cancellation of $q-1$ leaves $\prod_{j=2}^{k}(1-q^{-j})^{-1}$. In each row of (92.7) the relative elements run over the entire product group. The sum of the squared moduli of the row factors as the product of the local sums. Its diagonal entry is one. Subtracting this entry's squared modulus proves the first line. Multiplication by $D$ replaces each factor $q_v^j-1$ by $q_v^j$, proving the second line.

For $q\ge3$ one has $\sum_{j=2}^{\infty}-\log(1-q^{-j})\le q^{-2}/((1-q^{-1})(1-q^{-2}))\le 2q^{-2}$. Each residue size occurs $f$ times. Furthermore $g_i\ge(kb)^8$, $b=\exp(x^{\varepsilon})$ by Proposition 6.30 and its choice of the two auxiliary primes. Here $f$ grows at most polynomially in $x$, so the exponent in the bound tends uniformly to zero as the left endpoint tends to infinity. The inequality $e^t-1\le 2t$ for sufficiently small nonnegative $t$ proves the stated order. □

**Proposition 92.22** (Quadratic estimates for the orbital amplitudes)**.** *For vectors $z_i$ in a complex Hilbert space set*

$$S=\sum_i\|z_i\|^2,\qquad B=\sum_{i,j}A_{ij}\langle z_i,z_j\rangle.$$

*In addition to positivity, $B$ has the uniform upper bound*

$$0\le B\le\prod_{v\mid g_1g_2}\prod_{j=1}^{k-1}(1-q_v^{-j})^{-1}\,S\le(1+3/g_1)^f(1+3/g_2)^f S. \tag{92.13}$$

*If $s$ of the $z_i$ are nonzero, then*

$$|B-S|\le\frac{(\sum_i\|z_i\|)^2-S}{Q^k-1}\le\frac{s-1}{Q^k-1}S\qquad(s\ge1). \tag{92.14}$$

*In particular the quadratic form is positive definite on any set of fewer than $Q^k$ indices. On the full index set and for $k\ge2$ it annihilates each array constant in any one local index.*

*Proof.* The single-place matrix is a Gram matrix of the operators $\Gamma_k(i)$ in the Hilbert–Schmidt inner product divided by $\gamma_k(1)$, since $\operatorname{tr}(\Gamma_k(j)^*\Gamma_k(i)) = \gamma_k(j^{-1}i)$. The character is real by Theorem 92.18. Products of these matrices are tensor products of positive operators.

To bound the norm at one place, use the regular representation on $\mathbb{C}^{G_k}$. The matrix is convolution by the central function $\overline{\gamma_k}/\gamma_k(1)$. Its centrality gives the same operator in the left and right convolution conventions. On each irreducible constituent $\pi$ of dimension $d_\pi$, Schur's lemma and the trace give the scalar $|U_k|m_\pi/d_\pi$, where $m_\pi = \dim \operatorname{Hom}_{U_k}(\theta_k, \pi)$. Indeed the character inner product with $\gamma_k$ equals this multiplicity by the induced function model, or directly by (92.6) and the character projection on $U_k$.

For completeness, $d_\pi \geq m_\pi \prod_{j=1}^{k-1}(q^j - 1)$. This inequality in rank $n$ holds for any representation of the subgroup $P_n^0$ fixing the last coordinate covector. For $n = 1$ it is equality. In rank $n$ its subgroup of last column translations is the additive group $\mathbb{F}_q^{n-1}$. Its characters are $\psi(\ell(x))$ for row covectors $\ell$. Distinct covectors give distinct characters, and their count is the order of the additive group. The block $G_{n-1}$ acts transitively on the $q^{n-1} - 1$ nonzero covectors, so their character spaces have the same dimension. The space for the last coordinate covector is stable under $P_{n-1}^0$. Inside it the $\theta_{n-1}$-space has dimension $m_\pi$, since adjoining the last column translations generates $U_n$ with the character $\theta_n$. Apply the induction hypothesis there and count all the nonzero covectors. Restricting $\pi$ to $P_k^0$ gives the asserted bound. Since $|U_k| = q^{k(k-1)/2}$, the bound for the scalar is $\prod_{j=1}^{k-1}(1 - q^{-j})^{-1}$.

For $q \geq 3$, the product of $1 - q^{-j}$ is at least $1 - \sum_{j\geq 1} q^{-j} = 1 - 1/(q-1)$, by multiplying the factors successively. Its reciprocal is at most $1 + 1/(q-2) \leq 1 + 3/q$. Tensoring the local bounds, also with the identity on the Hilbert space of amplitudes, proves (92.13). Alternatively, in the off-diagonal part take absolute values and use the sharp entry bound in Corollary 92.19. This gives the first inequality of (92.14). Cauchy–Schwarz on the support gives the second. Finally the local row sum is zero for $k \geq 2$. In (92.6) its sum over $g$ contains $\sum_{u\in U_k} \theta_k(u) = 0$. Tensoring with the other factors proves the last assertion. □

In the source expansion, group the orbital amplitudes by their relative residue bases $i$, and let $z_i$ be the sum in group $i$. After division by the value of the character at the identity, the finite quadratic expression on these amplitudes is $B$. The grouping and its normalization appear in `Formalism/HermitianCurvature/StatementPB.lean`, line 86 and `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/TermOpStmt.lean`, line 135. The estimates above control the effect of the finite matrix on these vectors. In particular a lower bound for $S$ alone allows a zero value for $B$ when the amplitudes are constant in a local index. For the actual projected operator $P_u$, the source uses an estimate for $\|c_u^{-1}P_u^*P_u - \mathrm{Id}\|_{\mathrm{HS}}^2$ on its column space, with $c_u > 0$ the unfolding constant. It combines the finite matrix with estimates for the degenerate and corank-one orbital terms in `Development/AbelianVarietyTheory/Proofs/DecoratedLatticeProofs/DiagonalTheoremOfFourPointBounds.lean`, line 49. Bounding those global sums in this manuscript and comparing the resulting period with the prescribed integral theta pairing remain necessary for the argument in Volume I.

## 92.8 The identity double coset at the common terminal level

We compute the global identity term itself, before estimating the other terms of the Gram matrix. Use $B = N \rtimes H \subset G$ of Section 121.16 over $F_+ = E^+$. The archimedean group $H_\infty$ is compact. The finite row level $K$ is either $K_\sigma$ or $K'_\sigma = a_f K_\sigma a_f^{-1}$ of Section 92.5. Put $K_N = K \cap N_f$, $K_H = K \cap H_f$, where a subscript $f$ denotes finite adeles. We have $B_f \cap K = K_N \rtimes K_H$. The characters $\chi_1, \chi_\varpi$ of Section 121.16 are trivial on $K_N$ and on $H$. At the translated level the column classes for $K_H$ include the full refinement by a nonzero covector at every place over $T$.

Use product Haar measures on $N(\mathbb{A}_{F_+})$, and fix Haar measures on $G_f, G_\infty, H_\infty$. Write $v_N$ for the volume of $N(F_+)\backslash N(\mathbb{A}_{F_+})$. This volume is finite and positive by the compactness of the adelic unipotent quotient. We use the quotient measure $d\dot{g}$ on $B_\infty\backslash G_\infty$ defined by integration first over $N_\infty, H_\infty$. To see that this is an invariant quotient measure, $N_\infty$ is unipotent and $H_\infty$ is compact, so both $B_\infty$ and the reductive group $G_\infty$ are unimodular.

**Lemma 92.23** (A common unipotent covolume for all column classes)**.** *Let $h_i$ run through representatives of $H(F_+)\backslash H_f/K_H$. The finite support $B_fK$ has these same representatives for the action of $B(F_+)$ on its right $K$-cosets. The rational stabilizer of $h_iK$ is*

$$\Lambda_i = N(F_+) \cap h_i K_N h_i^{-1}, \qquad \operatorname{vol}(\Lambda_i\backslash N_\infty) = \frac{v_N}{\operatorname{vol}_f(K_N)}.$$

*Proof.* The group $N(F_+)$ is dense in $N_f$. Indeed the finite exponential and logarithm polynomials for strictly upper block triangular matrices identify this unitary unipotent group with its Lie algebra as varieties in characteristic zero. The logarithm of an isometry is skew adjoint, since taking adjoints commutes with the logarithm and the logarithm of the inverse is its negative. The same polynomials identify their finite adelic spaces, since they are integral outside finitely many primes. Density now follows from additive approximation in a vector space. In each coordinate one may clear the finitely many prescribed denominators and solve the resulting finitely many congruences in a fractional ideal by the Chinese remainder theorem, remaining integral at all other places.

Given $nhK$, first translate $h$ by $H(F_+)$ into $h_iK_H$. Then density supplies a rational unipotent element in the open coset of $h_iK_Nh_i^{-1}$ needed to remove the remaining unipotent component. Conversely a rational identification of two resulting cosets gives a rational identification of the column classes on projection to $H$, so their indices coincide. Each column class has trivial rational stabilizer in $H$. For the unrefined levels this is Proposition 86.29, and passage to a subgroup retains triviality. It follows that its stabilizer in $B(F_+)$ is exactly the subgroup displayed above.

For any compact open $C \subset N_f$, density also gives $N_f = N(F_+)C$. A fundamental region for $N(F_+)$ in the adelic group is given, up to measure zero, by a real fundamental region for $N(F_+)\cap C$ times $C$. This follows by reducing the finite coordinate first and the real coordinate second. Hence $v_N = \operatorname{vol}((N(F_+)\cap C)\backslash N_\infty)\operatorname{vol}_f(C)$. Take $C = h_iK_Nh_i^{-1}$. Conjugation by $H_f$ preserves the Haar measure on $N_f$. In the Witt basis the only independent Lie algebra coordinates on which $h \in H$ acts nontrivially may be taken to be the block from $W_0$ to $E^s$. The opposite block is its negative adjoint. On the independent block $h$ acts by precomposition with $h^{-1}$. Its determinant over $F_+$ is $\mathrm{N}_{E/F_+}(\det_E h)^{-s} = 1$. All other independent coordinates are fixed. This computes the modulus at each place and proves the common covolume formula. □

Let $\rho$ be one of the prescribed irreducible unitary column types on $H_\infty$, with representation space $V_\rho$ of dimension $d_\rho$. A column in class coordinates is a covector $\ell_i \in V_\rho^\vee$ for each $h_i$, with the ordinary sum inner product. Equivalently, write $\ell(\gamma_f h_i k) = \ell_i \circ \rho(\gamma_\infty)^{-1}$ for $\gamma \in H(F_+)$, $k \in K_H$. Triviality of the stabilizers makes this well defined. For $\beta \in \{1, \varpi\}$ take a smooth real-place function $W : G_\infty \to V_\rho$ satisfying

$$W(nhg) = \chi_{\beta,\infty}(n)\rho(h)W(g), \qquad \int_{B_\infty\backslash G_\infty} \|W(g)\|^2 \, d\dot{g} = 1. \tag{92.15}$$

The calculation applies to every such $W$. For existence with these two properties alone, choose a compactly supported smooth section on a coordinate ball of the quotient where the bundle with fibre $V_\rho$ is trivial. Extend it by zero and normalize its norm. For a specified automorphic representation the choice of its real-place function is additional data.

Define the seed $\Phi_{\beta,\rho,\ell}$ on $G(\mathbb{A}_{F_+})$ by zero outside $G_\infty \times B_f K$ and by

$$\Phi_{\beta,\rho,\ell}(g_\infty, n_f h_f k) = \chi_{\beta,f}(n_f)\ell(h_f)(W(g_\infty)) \quad (k \in K) \tag{92.16}$$

on its support. Independence of the finite decomposition follows from $B_f \cap K = K_N \rtimes K_H$, triviality on $K_N$, and $H_f$-invariance of the character of $N_f$. Under left translation by a rational element of $H$ the two occurrences of $\rho$ cancel. Under a rational element of $N$ the finite and infinite characters multiply to one. Thus the seed descends to $B(F_+)\backslash G(\mathbb{A}_{F_+})$. Give a column type and a frequency one fixed $W$ when comparing different columns.

**Theorem 92.24** (The normalized global identity term)**.** *In the integral of $\Phi_{\beta,\rho,\ell}\,\overline{\Phi_{\beta',\rho',\ell'}}$ over $B(F_+)\backslash G(\mathbb{A}_{F_+})$ the answer is zero if $\beta \neq \beta'$ or $\rho \not\simeq \rho'$. For matching data the answer is*

$$c_{\rho,K} \sum_i (\ell_i, \ell_i'), \qquad c_{\rho,K} = \frac{\mathrm{vol}_f(K)\, v_N\, \mathrm{vol}(H_\infty)}{\mathrm{vol}_f(K_N)\, d_\rho}. \tag{92.17}$$

*Here the Hermitian pairing on covectors is linear in its first entry. After division by $c_{\rho,K} > 0$ the operator is the identity also on the refined subspace with zero sum on each scalar orbit. For the common words of Theorem 9.13, distinct words have zero mutual identity term.*

*Proof.* Reduce the finite coordinates using Lemma 92.23. On the coset $h_i K$ the product of seeds is independent of the right coordinate $k$. Its finite integration contributes $\mathrm{vol}_f(K)$ and its real integral is over $\Lambda_i\backslash G_\infty$. Indeed one may choose a fundamental region there in the real variable and the full coset in the finite variable. The residual rational stabilizer acts by measure-preserving permutations of that coset.

Disintegrate the real integral in the order $\Lambda_i\backslash N_\infty,\ H_\infty,\ B_\infty\backslash G_\infty$, using the quotient integration formula twice. If $\beta \neq \beta'$, the ratio of their characters descends to $\Lambda_i\backslash N_\infty$. For a rational point of $\Lambda_i$ its finite value is one by $H_f$-invariance and triviality on $K_N$, and the product of all local values is one. The ratio is nontrivial on $N_\infty$ since its terminal trace coefficient is $\beta - \beta' \neq 0$. Translating the integration variable on the right by an element with nontrivial ratio shows the integral is zero. For equal frequencies the unipotent integral is the covolume in Lemma 92.23.

For fixed $\dot g$, averaging the resulting rank-one operator over $H_\infty$ gives an intertwiner from $\rho'$ to $\rho$. It vanishes for inequivalent irreducibles by Schur's lemma. For equal data it is scalar, and its trace is $\mathrm{vol}(H_\infty)\|W(g)\|^2$. Its scalar is this trace divided by $d_\rho$. Integrating over the last quotient in (92.15) and summing over $i$ proves (92.17). These integrations are absolutely convergent. For the square of one seed this follows by nonnegative integration and the displayed finite factors. For a mixed pair use the two square integrals and Cauchy–Schwarz. Orthogonal compression of the identity to any subspace is its identity. Finally the words have distinct column highest weights, because any differing interior bit changes the corresponding weight by $\Delta > 0$. Their column types are consequently inequivalent. □

The same identity-term formula holds with a common unitary character $\kappa$ on the finite row level whenever $\kappa|_{K_N} = \chi_{\beta,f}|_{K_N}$ for the frequencies in use. In that case include $\kappa(k)$ in (92.16) and the factor $\kappa(k_H)$ in the right transformation law of $\ell$. These factors make the two decompositions agree, and at each $h_i K$ the common factor and its conjugate multiply to one.

**Proposition 92.25** (The change of normalization at $T$)**.** *For $r = 2s + k + 1$ the actual levels of Section 92.5 satisfy*

$$[N_f \cap K_\sigma : N_f \cap K_\sigma'] = T^{\,fs(r-s)}, \qquad c_{\rho,K_\sigma'} = T^{\,fs(r-s)} c_{\rho,K_\sigma}. \tag{92.18}$$

*Both identities apply at every selected $x, T, E$ and to every chosen system of nonzero terminal covectors.*

*Proof.* Away from $T$ the two levels agree. In the first split factor at $v \mid T$ the completed Witt flag has block dimensions $(1, \dots, 1, k+1, 1, \dots, 1)$, with $s$ lines on each side. The chosen split model identifies $N_v$ with the group of upper block unitriangular matrices for this list. In fact the flag and its Hermitian orthogonal flag specify the two strings of lines, and the action on all the successive quotients is the identity. The action on the other split factor is then determined by duality. At the original level its upper blocks are arbitrary integral matrices. The level change $a_v$ multiplies the first $s$ lines by $T$, acts on the middle block of size $d = k+1$ by the matrix $M : (w, z) \mapsto (w, \ell_v(w) + Tz)$, and fixes the last $s$ lines. In particular $\det M = T$, and both $M$ and $TM^{-1}$ are integral.

Conjugating by $a_v$ multiplies an upper block $X_{ij}$ by $a_i$ on the left and by $a_j^{-1}$ on the right. The preceding integrality shows that every resulting block is still integral. Product additive measure in the independent upper blocks is Haar measure, since group translation is triangular in increasing block distance with identity diagonal. For a block of size $d_i \times d_j$, the determinant of its linear change is $(\det a_i)^{d_j} (\det a_j)^{-d_i}$. The blocks with row in the first string and column in the middle contribute valuation $s(d-1)$. The blocks with row in the middle and column in the last string contribute $s$, and the positions between the two strings of lines contribute $s^2$. Within either string the valuation is zero. The total is $s(d+s) = s(r-s)$. With $|T|_v = T^{-1}$ this gives the local subgroup index $T^{s(r-s)}$. All $f$ places have this residue size and their indices multiply. Finally the finite row levels are conjugate in the unimodular group $G_f$, so their volumes agree. Substitution into (92.17) gives the second formula. □

In the Gram matrix of the Poincaré sums over $B(F_+)\backslash G(F_+)$, the terms with identical rational cosets unfold to the integral in Theorem 92.24. For a square this identity contribution unfolds by nonnegative integration, by tiling from a fundamental region for $G(F_+)$. Its mixed version follows by polarization, or by the absolute integrability of the two matching-seed products. This identifies the main term even when estimates for interchanging the other orbital sums have still to be proved. The seed and the normalization occur in `Formalism/LocalToGlobal/SiegelLatticeTheory/HaarTransversals/PoincareInHH.lean`, line 163 and line 257 of the same file. The identity term alone gives no lower bound for the full squared norm. Terms from distinct rational cosets can have either sign in its real expansion. Their control for the prescribed real theta vectors is still required.

# 93 The common terminal prime and its characters

## 93.1 Integral theta test functions at the common split prime

Use the prime $T$, the lattices $\Lambda_1 \subset \Lambda_0$ and the row levels of Proposition 86.17. We work on the first pair $Y_0, Y_1$ of the indefinite theta tower. Its ranks are $r$ and $w = r + t_{\mathrm{r}}$, respectively, by `Formalism/LocalRingCardinality/Params.lean`, line 40. Fix a place $v \mid T$, put $O = \mathbb{Z}_T$, and let $\mathbf{k} = \mathbf{F}_T$. The split Hermitian spaces have two factors, identified with a vector space and its dual by the Hermitian pairing. On the first row factor the rational element $a = a_{\varpi}$ carries $\Lambda_0$ onto $\Lambda_1$. Its determinant has valuation $s$. On the second factor it acts by the contragredient with determinant of valuation $-s$.

Choose one factor at each $v$ and write $\epsilon_v = 1$ for the first and $\epsilon_v = -1$ for the second. In the associated matrix model a position is a map $Y$ from the chosen $r$-dimensional row space to the $w$-dimensional partner. Choose an integral partner frame, identifying its lattice with $O^w$. On the chosen row factor write $g_\epsilon$ for the action of $g$. With the matrix splitting the row acts by

$$\omega_v(g, 1)\phi(Y) = |\det g_\epsilon|_T^{w/2} \phi(Y g_\epsilon). \tag{93.1}$$

Indeed the split symplectic space is the sum of this matrix space and its trace dual. Right multiplication by $g_\epsilon$ on $w$ rows has Jacobian $|\det g_\epsilon|_T^w$, giving the factor for a unitary pullback. This is the split Weil action of Kudla, Chapter III, Remark 6.3 [2]. Another splitting multiplies the formula by a unitary character $\eta_v(g)$, since the two operators lift the same symplectic action. We retain this character when passing to the global kernel.

Let $\mathcal{S}_v$ be the set of $(r-s)$-planes in $\Lambda_0/T\Lambda_0$. For $S \in \mathcal{S}_v$ let $L_S$ be its inverse image in $\Lambda_0$ and put $D_S = L_S$ in the first factor, $D_S = L_S^\vee$ in the second. Choose an ordered basis $b_S$ of $D_S$. Two kinds of Schwartz functions occur in the argument. The first is the lattice vacuum

$$v_S(Y) = \mathbf{1}_{\{Y(D_S) \subseteq O^w\}}.$$

In this position model, invariance under translations by $\mathrm{Hom}(D_S, O^w)$ forces the vacuum to be constant on its cosets. Invariance under multiplication by characters from its dual lattice forces it to vanish off that lattice, since the additive character has conductor $O$. Its value one at zero therefore gives the stated function. For the second kind fix a character $\chi : \mathbf{k}^\times \to \mathbb{C}^\times$. On integral matrices in the frame $b_S$ write

$$\overline{Yb_S} = \begin{pmatrix} C \\ D \end{pmatrix}, \qquad C \in M_r(\mathbf{k}), \quad D \in M_{w-r,r}(\mathbf{k}).$$

Define $p_S(Y) = \chi(\det C)$ if $Yb_S$ is integral, $D = 0$ and $C$ is invertible, and zero otherwise. These are the three residue conditions on a local factor at a place in the extra level set $Q$ in `Formalism/LocalToGlobal/IndefiniteReal/DatumVec.lean`, line 66. The power or inverse of the character there is included in $\chi$. Changing $b_S$ multiplies $p_S$ by a root of unity. The vacuum and this residue function are treated in their indicated frames.

**Theorem 93.1** (An integral left inverse on the orbit of lattices)**.** *Choose either kind of function at each $v \mid T$, keeping that choice throughout its orbit. Write $\phi_{\boldsymbol{S}}$ for their product and $A$ for the ring of integers of a number field containing all the residue character values. The map from the free $A$-module on $\prod_{v\mid T} \mathcal{S}_v$ to the compactly supported locally constant $A$-valued functions given by $[\boldsymbol{S}] \mapsto \phi_{\boldsymbol{S}}$ has an $A$-linear left inverse. The ideal generated by the values of $\sum_{\boldsymbol{S}} c_{\boldsymbol{S}} \phi_{\boldsymbol{S}}$ is exactly $(c_{\boldsymbol{S}} : \boldsymbol{S})$. At a residue place the supports belonging to distinct $S$ are disjoint.*

*Proof.* Take $Y_S$ to send the ordered basis $b_S$ to the first $r$ unit vectors of $O^w$. The inequality $w \geq r$ gives such a linear map, and $Y_S^{-1}(O^w) = D_S$. All the $D_{S'}$ have the same covolume, also when duals are used. If $Y_S(D_{S'}) \subseteq O^w$, containment $D_{S'} \subseteq D_S$ and equality of covolumes imply $D_{S'} = D_S$, hence $S' = S$. Both test functions at $S$ take the value one at $Y_S$. Evaluation at the product of these matrices therefore recovers exactly the corresponding coefficient. Conversely every function value is an $A$-linear combination of the coefficients, proving the assertion on ideals. For a point of the support of $p_S$, the top square block of $Yb_S$ lies in $\mathrm{GL}_r(O)$. Its inverse shows directly that $Yy \in O^w$ implies $y \in D_S$. Thus on that support $Y^{-1}(O^w) = D_S$, proving disjointness. □

**Theorem 93.2** (The Gram matrix of the finite theta vectors)**.** *Give each test vector its unit norm in the complex $L^2$ space of positions with additive Haar measure. If $S, S'$ at a vacuum place meet in codimension $j$ in $S$, their inner product equals $T^{-jw}$. At a residue place the Gram matrix is the identity. Before normalization the ratio of the squared norm of $p_S$ to that of $v_S$ is*

$$T^{-r(w-r)} \prod_{h=1}^{r} (1 - T^{-h}). \tag{93.2}$$

*For the product of all $f = [E^+ : \mathbb{Q}]$ places, the Gram matrix $\mathcal{G}_T$ of the normalized vectors satisfies*

$$\|\mathcal{G}_T - 1\|_{\mathrm{op}} \leq (1 + 32T^{-(t_{\mathrm{r}}+1)})^f - 1.$$

*Proof.* The intersection of two vacuum supports consists of the maps integral on $D_S + D_{S'}$. This sum contains $D_S$ with index $T^{j}$. For the first choice of factor this follows by reducing modulo $T$, and for the dual choice it follows from $L_S^\vee + L_{S'}^\vee = (L_S \cap L_{S'})^\vee$. The dual of the sum on the left is $L_S \cap L_{S'}$, so the identity follows by biduality for lattices over $O$. In each of the $w$ rows, integrality on the larger lattice cuts the volume by $T^{-j}$. The vacuum norms are equal along the orbit, which proves the inner product formula. Theorem 93.1 proves orthogonality at a residue place. On one vacuum support the reduction of the matrix is uniform. The bottom $(w-r)r$ entries must vanish and the first $r$ columns of the square block must be linearly independent. Counting their successive choices gives (93.2).

For fixed $S$, the number of planes at distance $j$ is

$$\begin{bmatrix} r-s \\ j \end{bmatrix}_T \begin{bmatrix} s \\ j \end{bmatrix}_T T^{j^2}.$$

Choose the intersection in $S$, the $j$-dimensional image in the quotient by $S$, and then a linear map from that image to the $j$-dimensional quotient of $S$ by the intersection. Its graph determines the remaining choice. The bound $\begin{bmatrix} a \\ j \end{bmatrix}_T \le 4T^{j(a-j)}$ follows by removing the leading powers in the product of Proposition 86.17, since the resulting denominator is at least $(1-1/2)(1-\sum_{h\ge 2} 2^{-h}) = 1/4$. The off-diagonal row sum is thus bounded by

$$16 \sum_{j \ge 1} T^{-j(w-r+j)} \le 32 T^{-(t_{\mathrm{r}}+1)}.$$

Absolute row sums multiply for the product of local Gram matrices. Removing the diagonal leaves row and column sums bounded by the claimed expression. Cauchy–Schwarz with these two bounds gives the operator norm bound. □

**Proposition 93.3** (The exact denominator after finite averaging)**.** *Let $\boldsymbol{S}_*$ correspond to $\Lambda_1$ at all places. On $\prod_{v\mid T} K_{0,v}$, average the translates of $\phi_{\boldsymbol{S}_*}$ in the character by which its stabilizer acts, extended by the determinant and the chosen splitting. The least positive integer making this function $A$-valued is $I = \begin{bmatrix} r \\ s \end{bmatrix}_T^f$. One has $I \equiv 1 \bmod T$.*

*Proof.* Write $\chi = 1$ at a vacuum place. The local stabilizer is taken here to be the stabilizer of the label $S_*$. It acts on the vector by $\eta_v(k)\chi(\overline{\det k_\epsilon})$. Indeed its determinant is a unit, and changing the domain basis multiplies the top determinant by that unit modulo $T$. This gives the required character on all of $K_{0,v}$. Multiply each translate in the average by the inverse of this character. The corrected translate depends only on the coset of the stabilizer. There is one term for every tuple of planes, because reduction acts transitively, with exactly $I$ cosets by Proposition 86.17. Each term is its indicated test function times a root of unity in $A$ after cancellation of the splitting factor. Hence multiplication by $I$ gives integral values. At a tuple of the matrices in Theorem 93.1 the value is an $A$-unit divided by $I$. If an integer $b$ clears this denominator, $b/I \in A \cap \mathbb{Q} = \mathbb{Z}$. Finally all factors $T^a - 1$ in the Gaussian product are units congruent to $-1 \bmod T$. This proves the stated congruence. □

## 93.2 Integral theta tests for the depth filtration

In the original-level formula (86.13) one may also discard every rational position stabilizing the full isotropic space $A_s = \sum_{i=1}^{s} Eu_i$. Indeed let $Q_s$ be its stabilizer. The sum of all terms indexed by $Q_s(F)$ is the pairing of the two intermediate sums over $B(F)\backslash Q_s(F)$. This follows by the same unfolding as in Theorem 86.22, since the integral of the absolute majorant for this subseries is bounded by that for the full series. At $K^0$ Theorem 86.2 evaluates this pairing as the seed pairing, including on the refined columns. Thus the entire deviation at the original

level comes from pairs with distinct isotropic spaces. For a relevant such pair, Lemma 86.10 also imposes the lower bound there on the product of the two frame heights.

We now compute the averaging index in Proposition 86.25 at the actual terminal lattice. This also gives a realization of its finite subgroup by matrix coefficients on the first theta pair. At $v \mid T$ take the split factor where $\varpi$ has valuation one. The residue size is $T$, and $O = \mathbb{Z}_T$. Choose the basis of the level lattice $\Lambda = O^s \oplus M_v \oplus O^s$ in Section 86.1, so $K = K_v^0 = \mathrm{GL}_r(O)$ in these coordinates. Write $J_j$ for its principal congruence subgroup of depth $j \geq 1$, and put

$$I_j = [K : (B_v \cap K)J_j], \qquad D_B = \frac{r(r+1) - k(k-1)}{2}.$$

Using $\Lambda$ together with its dual in the other split factor gives the same congruence subgroup in the underlying $F_v$-linear action of Section 86.8.

**Theorem 93.4** (The exact terminal index)**.** *For $j \geq 1$,*

$$I_j = T^{(j-1)D_B}(T^k - 1) \prod_{h=k+1}^{r} (T^h - 1).$$

*In particular $v_T(I_j) = (j-1)D_B$, and $0 \leq jD_B \log T - \log I_j \leq 4T^{-k}$.*

*Proof.* In the split factor the parabolic has $s$ line blocks on either side of a middle block of size $k+1$. Its unitriangular radical in the adapted basis has independently chosen entries above the diagonal blocks. There are

$$a = \frac{r^2 - (k+1)^2 - 2s}{2}$$

such entries. Reduction of its integral subgroup modulo $T^j$ therefore has order $T^{ja}$. Theorem 86.1 identifies the rest of $B_v \cap K$ with the integral $k$-by-$k$ matrices $h \in \mathrm{GL}_k(O)$ satisfying $\ell h = \ell \bmod T$, for the primitive row $\ell = \ell_v$. In middle coordinates $(w, z) \mapsto (w, \ell(w) + Tz)$ their matrices are

$$\begin{pmatrix} h & 0 \\ (\ell - \ell h)/T & 1 \end{pmatrix}.$$

To reduce to the identity at depth $j$ therefore requires $h = 1 + T^j X$ and $\ell X = 0 \bmod T$. The map $1 + T^j X \mapsto \ell X \bmod T$ is a surjective homomorphism on the principal congruence subgroup of $\mathrm{GL}_k(O)$, since $j \geq 1$ and $\bar{\ell} \neq 0$. Its kernel has index $T^k$. The original condition on $h$ has index $T^k - 1$ in $\mathrm{GL}_k(O)$, by transitivity on nonzero residue rows. The number of reduced middle matrices is thus

$$|\,\mathrm{GL}_k(O/T^j)|\, T^k/(T^k - 1).$$

The radical and middle images intersect trivially by their diagonal blocks and give the whole image. Finally

$$|\,\mathrm{GL}_n(O/T^j)| = T^{(j-1)n^2 + n(n-1)/2} \prod_{h=1}^{n} (T^h - 1).$$

One obtains this by counting ordered independent columns over the residue field and lifting all entries freely. Dividing the order for $n = r$ by the two image orders gives the formula since $r(r-1)/2 - k(k-1)/2 = a + k$ and $r^2 - k^2 - a = D_B$. Also $D_B = k + \sum_{h=k+1}^{r} h$. The logarithmic difference is $-\sum_{h=k}^{r} \log(1 - T^{-h})$, bounded by $2\sum_{h \geq k} T^{-h} \leq 4T^{-k}$. □

Use a partner lattice basis on the first theta edge, whose partner rank $w = r + t_{\mathrm{r}}$ is at least $r$. Write a position matrix as $Y = \binom{X}{Z}$, with $r$-by-$r$ top block $X$. Let $G_j = \mathrm{GL}_r(O/T^j)$, let $B_j$

be the image of $B_v \cap K$ there, and let $a$ range over the left cosets $B_j\backslash G_j$. The coarse residue support is

$$X \in K, \qquad Z \in TM_{w-r,r}(O).$$

This is the residue condition of Section 93.1 in a basis of $\Lambda$. Let $\kappa$ be the desired smooth unitary level character and let $\zeta$ be the splitting character in the matrix realization (93.1). On the coarse support put $\theta(X) = \kappa(X)/\zeta(X)$ and extend by zero outside to obtain a test $\phi_*$. Without either character this is the characteristic function. The fine test $\phi_{a,j}$ is its restriction to $X \bmod T^j \in a$. Write $\phi_{1,j}$ for the identity coset and let $A$ be the ring of integers of a number field containing the finite character values. The values on $K$ of each smooth character are roots of unity because the open kernel there has finite index.

**Theorem 93.5** (A matrix coefficient and an integral trace)**.** *The $\phi_{a,j}$ have disjoint supports, have equal squared norm $\|\phi_*\|^2/I_j$, and span a free $A$-module of rank $I_j$ with an $A$-linear left inverse given by evaluations. With inner product linear in the first variable,*

$$\frac{\langle \omega(g)\phi_{1,j}, \phi_{1,j}\rangle}{\|\phi_{1,j}\|^2} = \begin{cases} \kappa(g), & g \in (B_v \cap K)J_j, \\ 0, & \text{otherwise} \end{cases} \qquad (g \in \mathrm{GL}_r(F_v)).$$

*On these tests the $\kappa$-average satisfies $I_j e_K \phi_{1,j} = \phi_*$. Its exact denominator over $A$ is $I_j$. The same $\phi_*$ occurs at every depth.*

*Proof.* Choose a lift $x_a \in K$ of one member of each coset. Evaluation at $\binom{x_a}{0}$, followed by multiplication by $\theta(x_a)^{-1} \in A^\times$, extracts precisely the coefficient at $a$. Formula (93.1) gives $\omega(h)\phi_{a,j} = \kappa(h)\phi_{a\bar{h}^{-1},j}$ for $h \in K$, where $\bar{h}$ denotes reduction modulo $T^j$. These unitary translations permute all the supports transitively. Their disjoint union is the entire coarse support, proving the norm formula. If both $Y$ and $Yg$ are in the identity support, their top blocks $X$ and $Xg$ are in $K$. Hence $g \in K$. The two tests then have disjoint supports unless $\bar{g} \in B_j$, and in the latter case the displayed transformation law gives the coefficient. This proves the formula everywhere. Averaging with $\overline{\kappa(h)}$ gives the sum of all $I_j$ translates with equal coefficient $I_j^{-1}$. At the matrix for the identity coset its value is a unit divided by $I_j$. A positive integer $m$ clears this value only if $m/I_j \in A \cap \mathbb{Q} = \mathbb{Z}$. Conversely multiplication by $I_j$ restores the integral coarse test. □

**Corollary 93.6** (A common coarse theta period at all depths)**.** *For each $v \mid T$ choose a depth $j_v \geq 0$, taking the coarse test at depth zero and $\phi_{1,j_v}$ otherwise. Let $I = \prod_{v:j_v>0} I_{j_v}$. Keep all the other theta data fixed. Suppose the theta rows $F_{\mathbf{j}}$ and $F_*$ of the fine and coarse tests are given by convergent lifts in automorphic $L^2$. Let $S_{\mathbf{j}}$ be the Bessel seed periodization of Section 86.8 with the additional indicators at all the places of positive depth. For the product $e$ of their level-character averages one has*

$$IeF_{\mathbf{j}} = F_*, \qquad IeS_{\mathbf{j}} = S_0, \qquad \langle F_*, S_0\xi\rangle = I^2\langle eF_{\mathbf{j}}, eS_{\mathbf{j}}\xi\rangle. \tag{93.3}$$

*Moreover, for fixed arithmetic data,*

$$\|IS_{\mathbf{j}}^* S_{\mathbf{j}} - 1\|_{\mathrm{op}} \leq C_x T^{-\eta\sum_v j_v}, \qquad \eta = L - 4fr^2.$$

*Proof.* Right translation of a theta row by a finite row element $h$ replaces the test by $\omega(h)$ applied to it. This follows by substituting $\omega(g)\omega(h) = \omega(gh)$ in the rational theta sum, which commutes with the independent partner action. The average on each finite orbit is a finite sum after character correction. Linearity of the convergent lift and Theorem 93.5 prove the first equality. For Bessel seeds use Proposition 86.25 at each place. The averages commute

since they act on different factors. Taking the inner product gives (93.3). The same calculation holds after the real row projection, which commutes with the finite averages.

For the last estimate repeat the norm calculation of Lemma 86.23 on its single nonzero affine frame coordinate $z \in F$. At every place $v$ of positive depth the same $z$ has valuation at least $j_v - O_x(1)$. Clearing one common denominator and taking its field norm gives $T^{\sum_v j_v} \le C_x V(g) V(\gamma_\infty g)$ for every nonidentity coset meeting both supports. All constants may be taken over the full finite set of places above $T$, independent of the depths. Disintegration over the finite class cells gives volume fraction $I^{-1}$ on the identity term. The dyadic tail bound and the integrable first height moment in the proof of Theorem 86.24 give the displayed operator bound. □

For the averaging factor in (93.3) one has $\log I^2 \le 2D_B \log T \sum_v j_v$. If the total depth is bounded by a fixed power of $x$, this cost is $o(b/x^A)$ for every fixed $A > 0$ on the intervals of Volume I, since $D_B, \log T$ are polynomially bounded and $b = \exp(x^\varepsilon)$. The theorem gives the denominator of the finite test and the exact trace on convergent theta lifts. It leaves the arithmetic normalization and nonzero value of $\langle F_*, S_0 \xi \rangle$ on the prescribed full chain to be established.

## 93.3 The size of the common terminal prime

Write $m = 2d$, $r = k + m - 1$, $b = \exp(x^\varepsilon)$, and let $Q = g_1 g_2$. Use the actual words, spacings and common terminal prime from Theorem 9.13. Thus $m \asymp x^{1/2-\varepsilon/16}$, $k \asymp x^{9/2+10\varepsilon}$, $\log Q = (16 + o(1)) \log b$, and $f = [E^+ : \mathbb{Q}]$. The field $E$ is abelian and has conductor dividing $N$ with $\log(2N) = o(\log b)$ throughout the selected intervals. Indeed the conductor bound in Lemma 19.12 is $O_{\psi,\varepsilon}(\log(2x) + x^{\varepsilon/5} \log^2(2x))$ already with initial delay one. All limits in this section are uniform in the words and translates.

**Lemma 93.7** (The mass in two adjacent ranges of ranks)**.** *Let $\mathfrak{m}_n$ denote the mass of the standard positive definite Hermitian genus over $E/F$, $F = E^+$, in rank $n$. If $\eta$ is the quadratic character of $E/F$, then*

$$\frac{\mathfrak{m}_r}{\mathfrak{m}_k} = 2^{-f(r-k)} \prod_{j=k+1}^{r} |L_F(1-j, \eta^j)|, \qquad \frac{1}{fk} \log \frac{\mathfrak{m}_r}{\mathfrak{m}_k} = o(m \log b). \tag{93.4}$$

*At an odd prime $p$ the valuation of the ratio is also*

$$\sum_{j=k+1}^{r} \Big( v_p |K_{2j-2}(\mathcal{O}_E)_{\mathrm{odd}}^{c=(-1)^j}| - v_p w_j(J_j) \Big),$$

*where $c$ is complex conjugation, $J_j = F$ for even $j$ and $J_j = E$ for odd $j$, and $w_j(J) = |H^0(J, (\mathbb{Q}/\mathbb{Z})(j))|$.*

*Proof.* Complex conjugation acts nontrivially on the two imaginary quadratic subfields of relatively prime discriminants $-q_0, -d_0$. At every finite place inertia fixes one of these subfields. Thus $E/F$ is unramified at all finite places. The identity Hermitian lattice is maximal for its integral norm condition. In fact if an integral lattice contains it and $y$ belongs to that lattice, polarizing the norm against the multiples of a standard basis vector shows that $\mathrm{Tr}_{E/F}(a y_i)$ is integral for all $a \in \mathcal{O}_E$. The relative different is one, so $y_i \in \mathcal{O}_E$ for every coordinate, also locally above two.

Apply the mass formula of Gan, Hanke and Yu for maximal definite Hermitian lattices, Propositions 4.4 and 4.5 and (4.2)–(4.3) of [3]. Its factor at a split place is one. At an unramified

nonsplit place the signed determinant of the standard form is a unit, hence a norm. Indeed the norm on finite residue fields is surjective and one corrects successive unit digits by the surjective residue trace in the unramified quadratic extension. Consequently all finite correction factors are one, in both parities of the rank. The formula is $\mathfrak{m}_n = 2^{1-fn}\prod_{j=1}^{n}|L_F(1-j,\eta^j)|$. Its mass convention gives the maximal compact group volume one at infinity and each standard lattice stabilizer volume one at the finite places, as in the definition of $\mathfrak{m}_n$. Division proves the product in (93.4).

For even $j$ the factor is $\zeta_F(1-j)$, and for odd $j$ it is the value of $\zeta_E/\zeta_F$ at $1-j$, taken by continuation. In either case it is a product of $f$ primitive Dirichlet values of the parity of $j$ with conductors at most $N$, by factorization for this abelian extension. Each has absolute value at most $C(j-1)!\,N^j$. This follows from the Bernoulli polynomial expression at $1-j$ and the bound $|B_j(y)| \le Cj!$ for $0 \le y \le 1$, obtained by Cauchy's estimate on the unit circle for $te^{yt}/(e^t-1)$. These are nonzero values by the primitive functional equation at $j > 1$. Thus the log of the mass ratio is bounded above by $Cf(r-k)r\log(2rN)$. For a lower bound, append $r-k$ ordered orthonormal vectors to each rank-$k$ lattice. A standard-genus lattice of rank $r$ has at most $rw_E$ choices for a norm-one vector, with $w_E = |\mu(E)|$. Indeed two such vectors are orthogonal or generate the same line with root-of-unity ratio, by integrality and Cauchy–Schwarz at all embeddings. Removing the ordered frame recovers its orthogonal complement. Counting with reciprocal automorphism orders gives $\mathfrak{m}_k \le (rw_E)^{r-k}\mathfrak{m}_r$, as in Lemma 94.7. Here $w_E \le 8f^2$ by the elementary totient bound. Now $\log r = O(\log x)$, $r/k \to 1$ and $\log N = o(\log b)$. Dividing the two bounds by $fk$ proves the second assertion. For the valuation formula, apply the critical-sign higher class-number formula at weight $j$ to each factor of the first identity, as proved in Section 25.2. □

**Lemma 93.8** (The contribution of the extra row parameters)**.** *For the word parameters $u_i$ and row parameters $\lambda_i$ at every real place,*

$$\frac{1}{k}\log\frac{W_r(\lambda)}{W_k(u)} = (1+o(1))\,m\log b.$$

*Proof.* Among the row entries are $k-1$ numbers $v_i = 4\Delta(k-1-i)$, each between $u_i$ and $u_{i+1}$. The other $m$ entries are $-b_j$ with $b_j = B_\tau + 2(j-1)H_0$. In the notation of the word construction $X' = \exp(x^{\varepsilon/3})$ and the smallest base weight is at least $b/(nX')$, where $n$ is bounded by a fixed power of $x$. The largest base weight is $O(xb)$. The two spacings $\Delta, H_0$, the ranks and the parity adjustment all have polynomial size. Hence uniformly for every pair $i, j$,

$$\log(v_i + b_j) = (1+o(1))\log b.$$

There are exactly $m(k-1)$ such factors. The other new factors, between pairs of the $-b_j$, are between one and $2mH_0$.

Write $\Delta(v)$, $\Delta(u)$ for the positive absolute Vandermonde products. The inequality $v_i - v_j \le u_i - u_{j+1}$ for $i < j$ matches each factor of $\Delta(v)$ with a distinct factor of $\Delta(u)$. All unused factors are at least one. Conversely, for $2 \le i < j \le k-1$, $u_i - u_j \le v_{i-1} - v_j$. Only $2k-3$ factors of $\Delta(u)$ involve an endpoint. They are bounded by the full column span, at most $3\lceil xb\rceil$. Thus

$$-(2k-3)\log(3\lceil xb\rceil) \le \log(\Delta(v)/\Delta(u)) \le 0.$$

Finally the quotient of the Weyl factorial denominators contributes $-\sum_{i=k}^{r-1}\log(i!)$, of absolute value $O(km\log k)$. After division by $k$, the contributions from the last three estimates are $O(m^2\log(2mH_0)/k + \log b + m\log k)$. This is $o(m\log b)$, since $k \gg m^2$, $m \to \infty$ and $\log x = o(\log b)$. The $m(k-1)$ mixed factors give the stated leading term. □

**Theorem 93.9** (Asymptotic for the actual common terminal prime)**.** *For all the selected words over the selected fields and intervals, the prime in the original finite theta and Bessel tests satisfies*

$$\log T = (17 + o(1))m \log b. \tag{93.5}$$

*In particular, if* $R = |\Delta_E|^{1/(2f)}$ *and* $\omega = 2\pi/(RQ)$*, then*

$$\log(T\,\omega^{\,2(d-1)}) = (1 + o(1))m \log b.$$

*Proof.* For a selected word $c$, the prime construction gives $0 < fk \log(T/Y_c) < 1$ for large $x$. By the arithmetic volume and formal-degree identities of Volume I, the normalizing target satisfies

$$fk \log Y_c = \log(I_\sigma/I_\pi) + \log(\mathfrak{m}_r/\mathfrak{m}_k) + \sum_\tau \log\{W_r(\lambda_\tau)/W_k(u_\tau)\} - 2\log\Gamma_c.$$

For a split level prime $g$, the index ratio is $(g^k \prod_{i=k+1}^{r}(g^i-1))^f$: it follows by counting independent columns of a general linear matrix and dividing by the sizes of the two prescribed unipotent reductions. The two such primes have $(kb)^8 \le g \le 2(kb)^8$. Consequently the first term after division by $fk$ is $(16+o(1))m \log b$, using $r = k+m-1$ and $m/k \to 0$. Lemma 93.7 bounds the second term and Lemma 93.8 evaluates the third. The size bound $0 \le \log\Gamma_c \le Cfm^5 \log b$ gives an error $o(m \log b)$ after division, since $m^4/k \to 0$. This proves (93.5) for the same prime $T$. The conductor–discriminant formula gives $\log R \le \log N$. Now substitute $\log Q = (16 + o(1)) \log b$, $\log R = o(\log b)$ and $2(d-1) = m-2$ into the second displayed logarithm. □

## 93.4 Integral parameters for the prescribed real character

The frame change with $t = 1$ in (98.13) gives two different scales for the block entries and the character. We now allow the terminal norm in the real model to change and reindex the rational matrices. The finite adelic integral is still the one at its original prime $T$ and level $K^0$. Use indices $1, \dots, d$, and put

$$\alpha = 2d-3, \qquad T^{\#} = Q^{\alpha}T, \qquad L_Q = \operatorname{diag}(Q^{i-1})_{i=1}^{d}, \qquad J_0 = \operatorname{diag}((-1)^{d-j})_{j=1}^{d}.$$

**Lemma 93.10** (An integral cross matrix for each supported label)**.** *A supported relevant pair with* $\mathcal{G} \neq 0$ *has a representative cross matrix* $Z_0$ *with entries in* $\mathcal{O}_E$*. Put* $Z^{\#} = Q^{-1}L_Q Z_0 J_0 L_Q$*. This matrix is integral and has constant antidiagonals in its monomial normal form. Writing* $\beta_\nu$ *for its block scalars and* $D_j^{\#}$ *for its leading minors gives*

$$\begin{aligned} \beta_\nu &= (-1)^{d-j_\nu} Q^{\alpha-e_\nu}\zeta_\nu, \qquad |D_j^{\#}| = Q^{j(j-2)}|D_j|, \\ \mathcal{G}^{\#} &= Q^{2d(d-2)}\mathcal{G}, \qquad w(\beta) \ge Q^{-4f} w_\gamma. \end{aligned} \tag{93.6}$$

*The Gram determinant on the left uses* $T^{\#}$ *on both sides. The map from rational orbit labels to these integral block points is injective, including when the first block is a singleton.*

*Proof.* For every finite place, support gives a representative in $K_v^0$ after two $B_v$ actions. The resulting cross matrix is integral there. The initial rational matrix is integral away from finitely many places. Upper unitriangular changes in its two frames suffice to pass locally to the integral representative. Choose two rational upper unitriangular changes sufficiently close to those local changes at the finitely many places in question and integral everywhere else. For each of their entries this is additive approximation in $E$. Indeed the Chinese remainder theorem in a fractional ideal $\prod_{\mathfrak{p}\in S}\mathfrak{p}^{-n}$ allows arbitrary prescribed residues at $S$ to the required precision once $n$ is sufficiently large. Apply it at both primes of $E$ over a split place of $F$. Integrality of the resulting matrix at the prescribed places is an open condition on the changes since matrix

multiplication and conjugation are continuous. This gives $Z_0$. Its entry $Z_{0,11} = D_1$ lies in $Q\mathcal{O}_E$ by Proposition 110.11.

The power of $Q$ in entry $i, j$ of $Z^{\#}$ is $i + j - 3$. Only the top left entry has negative exponent, and its divisibility was just proved. Diagonal conjugation preserves lower and upper unitriangular groups, so the monomial form of $Z^{\#}$ is $Q^{-1}L_Q M J_0 L_Q$, with $M$ the previous monomial form. On an antidiagonal $i + j = j_{\nu-1} + j_\nu + 1$, this exponent is $\alpha - e_\nu$. The alternating signs down the block cancel against those in $J_0$, giving the first formula. Leading determinants of the diagonal factors acquire $Q^{j(j-2)}$ and the sign of the first $j$ entries of $J_0$. Since $\alpha + (d-1)(d-3) = d(d-2)$, the two Gram terms acquire the same factor. The new Gram invariant is nonzero and integral. All endpoint exponents $j(j-2)$ are nonnegative except at $j = 1$. At that endpoint $1 + |D_1/Q|^2 \geq Q^{-2}(1 + |D_1|^2)$. Squaring and taking the $f$ places gives the weight bound. The block lengths and the scalars $\zeta_\nu$ are recovered from $\beta_\nu$ by (93.6). Lemma 110.10 then recovers the rational double coset. An integral matrix in the cell and its nonzero integral Gram invariant are precisely the arithmetic conditions in `Formalism/SharpRestriction/CuspSpanNormBounds/LeadMinorK.lean`, line 150. □

**Theorem 93.11** (Matching the character and the rotated block point)**.** *Set $t = Q^{\alpha/2}$ and $\kappa = 1/(RQ)$. At each real place scale the $i$-th isotropic vector by $t(-\mathrm{i}\kappa)^{d-i}$ and its dual by the inverse conjugate, and replace each terminal vector by $tz_i$. Use $\gamma^{\#} = A^{-1}\gamma A$, where $A$ is this isometry on the hyperbolic planes and the identity on the middle space. The transported section has character coefficient $c^{\#} = -2\mathrm{i}\,\omega$, with $\omega = 2\pi/(RQ)$. In monomial coordinates its cross matrix is the constant antidiagonal block point $\beta$, embedded at that place and multiplied blockwise by $\mathrm{i}^{e_\nu}R^{-e_\nu}$. In particular these are exactly the rotated local point, the character coefficient, and the two terminal norms $T^{\#}, T^{\#}$ in the source's real orbital integral. They satisfy*

$$\log T^{\#} = (33 + o(1))m \log b,$$
$$\log\{T^{\#}\omega^{\,2(d-1)}\} = (17 + o(1))m \log b.$$

*Proof.* The character transformation in Theorem 98.16 applies to a complex scale $p\kappa$ of phase $p$, since the simple entries of the transformed unipotent action are divided by $p\kappa$, so the coefficient is multiplied by $p\kappa$. Take $p = -\mathrm{i}$ and $c = 4\pi$ to obtain $c^{\#}$. In the cross matrix the old $(i, j)$ entry is multiplied by $t^2\kappa^{2d-i-j}\mathrm{i}^{j-i}$. In a monomial block $2d - i - j = e_\nu$ and

$$\mathrm{i}^{j-i} = \mathrm{i}^{2d-i-j}(-1)^{d-j}.$$

This is just the $J_0$ sign in (93.6) followed by the rotation $\mathrm{i}^{e_\nu}$. The real block factor is $R^{-e_\nu}$ after absorbing $Q^{\alpha-e_\nu}$ into $\beta_\nu$. In zero-based indices the rotation in the source is $(-1)^{d-1}(-\mathrm{i})^{(i-1)+(j-1)} = \mathrm{i}^{2d-i-j}$. One may take group representatives with exactly monomial cross matrix. Each upper unitriangular change of the standard decorated frame is induced by an element of $N$. To see this, prescribe its action on the span of the frame, which preserves the self pairing, and extend over the dual isotropic space. Explicitly the upper isotropic matrix gives a Levi unitriangular action, and a map from the middle space to the isotropic space can send the nonzero terminal vector to any prescribed column. Its negative adjoint and the usual half of its quadratic product complete it to a unipotent isometry, by multiplication in the Witt basis. These changes preserve the simple character and within-block Haar measure as in Section 98.5. They multiply the kernels on either side by unitary column factors, which can equivalently be included in the finite contractions.

For the source conventions see `Formalism/FormalDegreeBessel/StatementG.lean`, line 55, `Formalism/SharpRestriction/CuspSpanNormBounds/FrobeniusCoefficientMemLp/FramesPt.lean`,

line 52 and `Formalism/SharpRestriction/CuspSpanNormBounds/RotatedWeightedBound.lean`, line 11. Finally $\log T^{\#} = \log T + (m-3)\log Q$. Apply Theorem 93.9, and then subtract $(m-2)\log(RQ/(2\pi))$, to get the two asymptotics. These verify the terminal size and positivity requirements in lines 39 and 73 of `Formalism/SharpRestriction/CuspSpanNormBounds/LeadMinorK.lean`, line 35 with a fixed sufficiently large upper size constant. □

**Corollary 93.12** (The arithmetic sum in the matching coordinates)**.** *Let $P_\infty^{\#}$ be the product of point densities in Theorem 93.11. Write $\widehat{K}_{\gamma^\#} = t^{-2kf} K^{\mathrm{root}}_{\infty,\gamma^\#}$ for the kernel of the transported sections including this scalar. With consistent monomial representatives on the two sides,*

$$c_E^a P_\infty^{1/2} K^{\mathrm{area}}_{\infty,\gamma} = R^{-fS} Q^{\,f(d(d-2)-a)} (P_\infty^{\#})^{1/2} \widehat{K}_{\gamma^\#}, \tag{93.7}$$
$$\sum_c |J_{\gamma,c}| \le B_\ell B_{\ell'}\, m_C \Big( \prod_{v\in S(\gamma)} L_{\gamma,v} \Big) R^{-fS} Q^{-f(dk+a)} (P_\infty^{\#})^{1/2} \|\widehat{K}_{\gamma^\#}\|_{\mathrm{op}}.$$

*The squared regular norm of the renormalized local kernel $t^{-2k} K^{\mathrm{root}}_{\gamma^\#}$ equals that of the original local kernel. The equality of these regular integrals allows arbitrary Borel truncations carried by the same map.*

*Proof.* All Jacobians in Theorem 98.16 depend only on the absolute values of the diagonal entries. The added phase has real Jacobian one on every complex entry, leaves the compact column action fixed, and its root phases have absolute value one. Therefore (98.10) and (98.12) hold for the transported kernel, with the rotation included in the change of the regular variables. Passing from area to root measure and multiplying by $c_E^a$ gives $R^{fa}$ before the inverse frame change. After writing the new kernel in terms of $\widehat{K}$, the scalar becomes

$$R^{fa} t^{2df} (RQ)^{-f(S+a)} = R^{-fS} Q^{\,f(d(d-2)-a)}.$$

This proves (93.7). Apply $P_\infty^{1/2} \ge Q^{fd(k+d-2)}$ and the finite covolume formula as in Theorem 98.18. The exponent after division is $d(d-2) - a - d(k+d-2) = -dk - a$. Finally the scalar $t^{-2k}$ cancels the factor $t^{4k}$ in the squared norm from Corollary 98.17. □

**Proposition 93.13** (The spectral parameters of the matched sections)**.** *Fix any positive gap constant $C_g$. At all sufficiently large $x$, the transported two sections of a fixed translate meet the standard local parameter inequalities for the source's averaged sum at $T^{\#}$, with real derivative parameter $\omega$, upper size constant 20, and size parameter $b_{\mathrm{sp}} = b^2$. For pure row inputs, the kernels $\widehat{K}_{\gamma^\#}$ at all the rational labels come from a single compact-finite coefficient at each real place. Its full squared regular norm is the original local regular integral.*

*Proof.* The relevant inequalities are in `Formalism/SharpRestriction/CuspSpanNormBounds/LeadMinorK.lean`, line 35 and lines 68–73 there. The sorted row parameters are integers in odd rank and the columns are half-integers in even rank. The interlacing inequalities $u_i > v_i > u_{i+1}$ and $u_k > -b_1$ hold for both words. All absolute parameters are bounded by $b^2$ for large $x$, since the endpoints and negative base weights in the formulas of Lemma 93.8 are $O(xb)$ and all spacings have polynomial size. The signed chamber stays the same since the representation still acts by right translation. The quantities $\omega, \omega^{-1}$ are between $b_{\mathrm{sp}}^{-20}$ and $b_{\mathrm{sp}}^{20}$, using $\log R = o(\log b)$ and the formula for $Q$. The new terminal norms obey $1 \le T^{\#} \le b_{\mathrm{sp}}^{20m}$ and $T^{\#}\omega^{2(d-1)} \ge 1$ by Theorem 93.11.

The entries strictly between two column entries are exactly the $v_i$. They lie in $[0, B_\tau/8]$ and the others are in $[-2B_\tau, -B_\tau]$ for large $x$, since $B_\tau$ grows faster than every fixed power of $x$. This also gives $B_\tau \ge C_g r^7$. Every row gap is at least $\min(4\Delta, B_\tau, 2H_0) \ge C_g m^3$ by the spacing choice of Section 86.7. For two entries with opposite chamber signs the distance is at

least $\min(B_\tau, 2H_0)$, since all the $v_i$ have the same sign in the chamber. Since $H_0 \ge r^{19} f^5 x^{\varepsilon}$, this exceeds $C_g m^{9/2}(3dr^2 + r^3)$. For equal columns $u_1 - u_2 \ge 4k\Delta$ exceeds $2C_g d^2 + 1$, and the entries $v_1, v_2$ lie on opposite sides of $u_2$. Both are between column entries. For different words an interior coordinate differs by $\Delta$. Inside the interval between the negatives of the two coordinates leave distance two at each end. The remaining closed interval has length $\Delta - 4 \ge C_g d^2$ at large $x$. These prove the separating-gap and cut clauses. The two derivative parameters in the local size test now both equal $\omega$, so their ratio in its terminal norms is one. The fixed nonzero Haar measure on $N$ is a positive finite scalar times its exponential coordinate Haar measure, by uniqueness.

By Theorem 98.13, applied in the new model, one may choose compact-finite test vectors $u_i', \xi_i'$ with $\ell_i' \neq 0$. Use the coefficient

$$F^{\#}(g) = t^{-2k} \frac{\langle v_1, v_2 \rangle}{\overline{\ell_1'} \ell_2'} \langle \sigma(g) u_2', u_1' \rangle \, \xi_1'(\xi_2')^{*}.$$

The isometric transported maps are the maps of that theorem with a choice of phase. The integral of $F^{\#}$ on the group pair quotient equals the local renormalized kernel for every label by the same identity. It is a coefficient $P\sigma(g)Q_0$ with $P^{*}$ and $Q_0$ having compact-finite image, since those images are spanned by $u_1', u_2'$, respectively. In particular these test vectors are independent of the orbit.

There is a single smooth choice of middle isometry in the regular integral. In the plane with signed unit basis $w_0, e$ and $e = z/\sqrt{T^{\#}}$ for the new terminal vector $z$, put $\lambda = -q_d/T^{\#}$, $h = (|\lambda|^2 - 1)^{1/2}$. Take the matrix $\left(\begin{smallmatrix} \bar{\lambda} & h \\ h & \lambda \end{smallmatrix}\right)$ on this plane and the identity on its positive complement. Its columns have norms $1, -1$ and their pairing is zero. It gives the required terminal pairing $q_d$ and varies smoothly for $|q_d| > T^{\#}$. Outside this domain the regular density vanishes almost everywhere. The singleton-block instance of (98.7) identifies the regular orbital transform of $F^{\#}$ with the local renormalized kernel. Its squared integral is therefore the one in Corollary 93.12. This claim also holds in the extended nonnegative sense. □

Put $M_\gamma = m_C \prod_{v \in S(\gamma)} L_{\gamma,v}$. Applying Cauchy–Schwarz on any finite set $\mathcal{A}$ of the relevant rational labels gives

$$\sum_{\gamma \in \mathcal{A}, c} |J_{\gamma,c}| \le B_\ell B_{\ell'} R^{-fS} Q^{-fdk+2f} \times \left( \sum_{\gamma \in \mathcal{A}} w_\gamma^{-1} Q^{-2fa} M_\gamma^2 \right)^{1/2} \left( \sum_{\gamma \in \mathcal{A}} w(\beta_\gamma) P_\infty^{\#} \| \widehat{K}_{\gamma^{\#}} \|_{\mathrm{HS}}^2 \right)^{1/2}.$$

Indeed insert $w(\beta)^{1/2} w(\beta)^{-1/2}$ in the previous bound and use (93.6). The second sum runs over distinct integral points with a common pair of terminal norms. For pure row inputs its summands are bounded above by the corresponding terms of the source's nonnegative sum at $T^{\#}$ and $b_{\mathrm{sp}}$. In fact Proposition 93.13 supplies one compact-finite coefficient at each place and the regular section. The supremum in `Formalism/SharpRestriction/CuspSpanNormBounds/LeadMinorK.lean`, line 103 includes the monomial representative, its normalized root Haar and a disintegration of the group pair measure along that Haar. Such a disintegration exists by the quotient integration formula, using a measurable section of the finite-dimensional homogeneous space. The orbit is relevant by transport. Taking the product over places and then using injectivity in Lemma 93.10 proves the comparison with the sum in `Formalism/SharpRestriction/CuspSpanNormBounds/RotatedWeightedBound.lean`, line 15. An upper estimate for this sum in terms of its regular norm, its normalization against the finite masses, and the arithmetic lower bound for the designated period still require proof here.

**Theorem 93.14** (One definite mass for all rational complements)**.** *The finite Haar measures on $U(C)$ can be chosen in* (98.4) *so that for every relevant supported label*

$$m_C = \mathfrak{m}_{k-1}, \qquad \frac{m_C}{\mathfrak{m}_k} = |\Delta_F|^{1/2-k} \left( \frac{(2\pi)^k}{(k-1)!} \right)^f \zeta_F(k)^{-1}.$$

*All the ordinary factors $L_{\gamma,v} = 1$ retain that value. In the preceding sum one may therefore take the same $\mathfrak{m}_{k-1}$ outside the square root for all labels, leaving the local factors $L_{\gamma,v}$ inside.*

*Proof.* The positive definite complement $C$ has odd rank $n = k - 1$. In any $E$-basis let $\delta \in F^\times$ be its Hermitian determinant. It is totally positive. Scaling the form by $\delta^{-1}$ preserves its unitary group and gives determinant $\delta^{1-n}$, the norm of the element $\delta^{(1-n)/2} \in F^\times \subset E^\times$. The scaled space is isometric to the standard form at every finite place. At a split place this follows by identifying the two factors as mutually dual vector spaces. At an unramified field place diagonalize the form and remove unit factors by surjectivity of the norm on units. Scaling basis vectors reduces each diagonal entry to one or a uniformizer of $F_v$. The number of uniformizers is even by the determinant condition. A pair of equal diagonal entries gives a hyperbolic plane since $-1$ is a unit norm. Indeed it has an isotropic vector, to which one adjoins a vector pairing to one and subtracts half the latter's norm times the isotropic vector to make the new vector isotropic as well. The same construction applies to two diagonal ones, so every pair of uniformizers can be removed.

Choose a self-dual lattice in each of these finite scaled spaces. At an ordinary place start with the self-dual lattice for the original form in Proposition 98.7. There $v(\delta)$ is even at a nonsplit place, by expressing the determinant in a self-dual basis. A scalar rescaling of the lattice then makes it self-dual for the scaled form, with the same stabilizer. At a split place keep one of its two lattices and take the dual for the new pairing in the second factor, again leaving its stabilizer unchanged. These choices agree with integral lattices in fixed global coordinates at almost all places. Their intersection with the diagonally embedded global $E$-vector space $C$ gives a global lattice with the chosen localizations. Indeed it is trapped between two fractional multiples of a basis lattice, and additive approximation realizes every local residue class to arbitrary precision. This lattice is maximal by the unramified trace-dual argument in Lemma 93.7.

Give each of its stabilizers finite volume one. The mass of this positive definite genus is $m_C$, by decomposition of the adelic quotient into its compact open classes. All finite norm-discriminant classes of the scaled form have unit representative, so the same mass formula as in Lemma 93.7 gives $m_C = \mathfrak{m}_{k-1}$. At the exceptional places changing a Haar measure on $U(C_v)$ multiplies $m_C$ by that scalar and $L_{\gamma,v}$ by its inverse. Thus these normalizations give the original cell measure, including at the marked levels.

In the consecutive-rank mass quotient, only $|L_F(1-k, \eta^k)| = |\zeta_F(1-k)|$ occurs because $k$ is even. The completed zeta function for totally real $F$ has factor $|\Delta_F|^{s/2}(\pi^{-s/2}\Gamma(s/2))^f$ and is invariant under $s \mapsto 1-s$. Evaluation at $k \geq 2$, followed by the gamma reflection and duplication identities, therefore gives

$$|\zeta_F(1-k)| = |\Delta_F|^{k-1/2} \left( \frac{2(k-1)!}{(2\pi)^k} \right)^f \zeta_F(k).$$

Division of the two masses is $2^f/|\zeta_F(1-k)|$, which gives the claimed value. □

## 93.5 The rational change of the full Bessel character

Retain the terminal prime $T = \varpi\bar{\varpi}$ and the group of Theorem 94.9. In Section 121.16 the signed row space is written as $s = m/2 - 1$ hyperbolic planes with Witt bases $u_i, v_i$, and

a middle space $W = W_0 \perp Ee$. Write $\mathcal{P}_\beta(F,\phi)$ for its Bessel period with last vector $\beta e$, normalized quotient measures, and coefficient one on each preceding simple root. We include any chosen finite right translate in the row form $F$. The root coordinates of the character are specified in `Formalism/LocalToGlobal/BesselHermitianTheta/TiedBessel.lean`, line 47.

**Theorem 93.15** (Transport of the character and the level)**.** *Let $d$ multiply every $u_i$ by $\varpi^{-1}$, every $v_i$ by $\bar{\varpi}$, and act identically on $W$. Then $d \in G(F_+)$. With $R(d)F(y) = F(yd)$, one has*

$$\mathcal{P}_\varpi(R(d)F,\phi) = \mathcal{P}_1(F,\phi). \tag{93.8}$$

*Among diagonal maps scaling the indicated Witt vectors and fixing $W$ pointwise, this is the unique one for which conjugation $n \mapsto dnd^{-1}$ carries the character with last vector $\varpi e$ to the one with last vector $e$ by pullback.*

*Away from $T$, $d$ preserves the original lattice and is the identity modulo $M$. At $T$ the orbit of $d\widehat{\Lambda}$ under the finite stabilizer of $\widehat{\Lambda}$, also with the levels at $g_1, g_2$ imposed, has cardinality*

$$\left(\prod_{j=0}^{s-1} \frac{T^{r-j}-1}{T^{s-j}-1}\right)^{f}. \tag{93.9}$$

*Proof.* The $u_i, v_i$ retain their pairings, and the other summands are fixed. This gives the claimed rational isometry. For a diagonal unitary $d_0$ with $d_0 u_i = a_i u_i$ and $d_0|_W = 1$, conjugation multiplies the $(i, i+1)$-coordinate by $a_i/a_{i+1}$, and the last coordinate, from $e$ to $u_s$, by $a_s$. Thus the required root coefficients are equivalent to

$$a_i/a_{i+1} = 1 \quad (i < s), \qquad \varpi a_s = 1.$$

For necessity one may vary each coordinate alone. For $i < s$ send $u_{i+1}$ to $u_{i+1} + tu_i$ and $v_i$ to $v_i - \bar{t}v_{i+1}$, fixing the other basis vectors. For the last coordinate use the element $n(t)$ of Lemma 121.20. At an archimedean place, varying the complex scalar $t$ detects any nonzero difference of coefficients by the real trace pairing. The equations give exactly $d_0 = d$.

The map $d$ commutes with $H = U(W_0)$. In the integral defining the left side of (93.8), automorphy under left $d^{-1}$ replaces $F(nh_0 d)$ by $F(d^{-1}nd\,h_0)$. Set $n' = d^{-1}nd$. The map preserves $N(F_+)$. The induced automorphism on $[N]$ carries its invariant probability measure to itself. The old character at $dn'd^{-1}$ now has last coefficient one and all its earlier coefficients one. This proves the identity, including its normalization.

Apply the two-dimensional matrix calculation of Theorem 94.11 to each hyperbolic plane to obtain the congruence away from $T$. At a place above $T$ choose the first split component in which $\varpi$ has valuation one. The enlarged lattice is the inverse image of an $s$-dimensional residue subspace in $T^{-1}\Lambda_w/\Lambda_w$. The second component is its dual. Every change of residue basis lifts to the integral general linear group. Counting ordered independent $s$-tuples in an $r$-space and dividing by the number of bases of an $s$-space gives the factor in (93.9). The $f$ places contribute independently. Right translation by $d$ takes a vector fixed by a level $K$ to one fixed by $dKd^{-1}$. Consequently applying (93.8) to a form already at the original level gives $\mathcal{P}_\varpi(F,\phi) = \mathcal{P}_1(R(d^{-1})F,\phi)$, with the inverse change of level on the right. □

**Proposition 93.16** (An archimedean factor for the undivided period)**.** *At each real place $\tau$ of $F_+$ fix its chosen complex embedding of $E$, and set $\eta_\tau = \overline{\varpi_\tau}/\sqrt{T}$. Write $X_{\tau,\xi}$ for differentiation of right translation along $b \mapsto n_\tau(b\xi)$, $b \in \mathbb{R}$. For integers $a_\tau \geq 0$, put*

$$A = \sum_\tau a_\tau, \qquad D_\varpi = \prod_\tau X_{\tau,\eta_\tau}^{a_\tau}, \qquad D_1 = \prod_\tau X_{\tau,1}^{a_\tau}.$$

*For every smooth row form and column form as in the theorem,*

$$\mathcal{P}_{\varpi}(D_{\varpi}F,\phi) = (4\pi i\sqrt{T})^{A}\mathcal{P}_{\varpi}(F,\phi), \tag{93.10}$$

$$|\mathcal{P}_{\varpi}(F,\phi)|^{2} \le \frac{\|\phi\|^{2}_{L^{2}([H])}}{(16\pi^{2}T)^{A}} \int_{[H]}\int_{[N]} |(D_{\varpi}F)(nh_{0})|^{2}\,dn\,dh_{0}. \tag{93.11}$$

*The transported form satisfies*

$$D_{\varpi}R(d)F = T^{A/2}R(d)D_{1}F. \tag{93.12}$$

*Proof.* On each real line of parameters, the displayed $n_{\tau}(b\xi)$ form a one-parameter subgroup. Multiplication of the matrices in Lemma 121.20 gives the sum of the two real parameters, with the quadratic term of their sum. At the place $\tau$, $\Psi_{E}(\varpi_{\tau}b\eta_{\tau}) = \exp(4\pi ib\sqrt{T})$, since the complex trace is twice the real part. Apply the global translation law of that lemma and differentiate $a_{\tau}$ times. The differentiations under the integral are valid for smooth forms on the compact domain. More explicitly, one may take compact sets of representatives for $[H]$ and $[N]$. Images of their products under translations by a bounded interval of parameters are compact, and all the differentiated integrands are bounded there. Directions from distinct places commute. This proves (93.10), including $A = 0$. Apply Cauchy–Schwarz on the two probability quotients to its left side. The character has absolute value one, yielding (93.11).

Finally $d^{-1}n(t)d = n(\varpi t)$, as one verifies on $u_{s}, e, v_{s}$. Since $R$ is a representation, differentiating after translating by $d$ replaces the direction $\eta_{\tau}$ by $\varpi_{\tau}\eta_{\tau} = \sqrt{T}$. This gives (93.12) after taking all the derivatives. □

In particular (93.11) applies with order $a = \lceil xb \rceil$ at the $h$ high embeddings and zero at the other embeddings on the very intervals of Corollary 94.10. Its $T^{-ha}$ factor is obtained on the undivided global integral. An application to the arithmetic period must estimate the differentiated vector in its specified normalization. Formula (93.12) gives the exact cost when this vector itself undergoes the global change of character. The integral lattice comparison and the strict cohomology estimate of Proposition 7.7 remain open.

## 93.6 The terminal prime and the parameters of the real integral

We continue on the same intervals as Theorem 20.13, with exactly the fields, words, level and terminal prime of Theorem 9.13. Put $F = E^{+}$, $f = [F : \mathbb{Q}]$, $m = 2d$, $r = k + m - 1$, $Q = g_{1}g_{2}$, $b = \exp(x^{\varepsilon})$, and $R = |\Delta_{E}|^{1/(2f)}$. The conductor of $E$ divides a number $N$ with $\log(2N) = o(\log b)$ on these intervals, by the initial delay in Theorem 20.13.

**Theorem 93.17** (The terminal prime at the original level)**.** *Uniformly in the selected words and translates,*

$$\log T = (17 + o(1))m\log b, \qquad \log\{T(2\pi/(RQ))^{m-2}\} = (1 + o(1))m\log b.$$

*Proof.* Use the defining target $Y_{c}$ in Theorem 9.13. Theorem 94.9 expresses $fk\log Y_{c}$ as the sum of the logarithms of the level index ratio, the mass ratio $\mathfrak{m}_{r}/\mathfrak{m}_{k}$, and the product of the local Weyl dimension ratios, minus $2\log\Gamma_{c}$. Proposition 94.6 gives $(16 + o(1))m\log b$ for the normalized level contribution. For these fields $E/F$ is unramified at every finite place. The standard lattice is self-dual with unit determinant, and every unit is a norm at the unramified nonsplit places. Thus the definite unitary mass formula gives

$$\mathfrak{m}_{r}/\mathfrak{m}_{k} = 2^{-f(r-k)} \prod_{j=k+1}^{r} |L_{F}(1-j,\eta^{j})|,$$

where $\eta$ is the quadratic character of $E/F$. Each value is a product of $f$ primitive Dirichlet values of the parity of $j$. Their Bernoulli polynomial expressions bound the logarithm of each Dirichlet value above by $O(j\log(2jN))$. Together with the lower mass estimate of Lemma 94.7, this gives $\log(\mathfrak{m}_r/\mathfrak{m}_k)/(fk) = o(m\log b)$. The local hypotheses and exact odd-primary relation to the critical cohomological sign are proved in Lemma 93.7.

There are $m(k-1)$ differences $v_i + b_j$ between the two parts of the row (87.4). Their logarithms are $(1+o(1))\log b$ uniformly, since the smallest $b_j$ is at least $b/(n\exp(x^{\varepsilon/3}))$, the largest is $O(xb)$, and the $v_i$ have polynomial size. In the quotient of the shared Vandermonde by the column Vandermonde, interlacing matches each shared difference to a larger column difference. In the reverse direction it matches each interior column difference to a larger shared difference. Only the $2k-3$ endpoint factors can cost $O(\log b)$ each. The factorials cost $O(km\log k)$, and differences within the negative row cost $O(m^2\log x)$. Thus the Weyl terms divided by $fk$ contribute $(1+o(1))m\log b$. Finally $\log\Gamma_c/(fk) = o(m\log b)$ and $fk|\log(T/Y_c)| < 1$. This proves the first asymptotic. The other follows from $\log Q = (16+o(1))\log b$ and $\log R \le \log N = o(\log b)$. □

**Theorem 93.18** (Integral labels at the real additive character)**.** *Set $\alpha = 2d-3$, $T^{\#} = Q^{\alpha}T$ and $\omega = 2\pi/(RQ)$. The relevant nondegenerate rational labels at $K^0$ inject into the source's integral block points for terminal norms $(T^{\#}, T^{\#})$. At every real place, the original normalized sections transport to sections with character coefficient $-2\mathrm{i}\omega$ whose cross matrices represent the rotated local points of these very labels. The new terminal parameters satisfy $\log T^{\#} = (33+o(1))m\log b$ and $\log(T^{\#}\omega^{m-2}) = (17+o(1))m\log b$. The finite test still has its original prime $T$.*

*Proof.* Choose an integral cross matrix for the original pair by rational unitriangular changes. This is possible because support supplies an integral representative at all finite places, and additive approximation in the finitely many nonintegral frame entries realizes those open conditions simultaneously. In entry $i,j$ multiply this matrix by $(-1)^{d-j}Q^{i+j-3}$, numbering from one. The only negative exponent occurs at $(1,1)$, where the original entry $D_1$ belongs to $Q\mathcal{O}_E$. The new matrix is integral. The transformation is a diagonal scaling on each side and changes the alternating antidiagonal in each monomial block to a constant one. At an endpoint $j$ the absolute minor acquires $Q^{j(j-2)}$. With the new terminal norms the Gram determinant acquires $Q^{2d(d-2)}$, so it is again nonzero and integral. Each old block scalar can be recovered from its new scalar and length. This proves injectivity.

To match the real variable, scale the $i$-th isotropic vector by $Q^{\alpha/2}(-\mathrm{i}/(RQ))^{d-i}$, its dual by the reciprocal conjugate, and the terminal vector by $Q^{\alpha/2}$. The associated unitary conjugation sends the coefficient $4\pi$ to $-4\pi\mathrm{i}/(RQ)$. In a block with endpoints $j_{\nu-1}, j_\nu$, put $e_\nu = 2d - j_{\nu-1} - j_\nu - 1$. On the antidiagonal $e_\nu = 2d-i-j$, and the phase on entry $i,j$ is $\mathrm{i}^{j-i} = \mathrm{i}^{e_\nu}(-1)^{d-j}$. After the rational sign and power of $Q$ already included in the new point, this leaves precisely the source multiplier $\mathrm{i}^{e_\nu}R^{-e_\nu}$. This proves the simultaneous matching. The size assertions follow by adding $\alpha\log Q$ to the two estimates of Theorem 93.17. Section 93.4 gives the group representatives and checks the remaining real parameter inequalities for each pair of words. □

**Corollary 93.19** (Finite cell contributions in the matching coordinates)**.** *Let $\beta_\gamma$ be the new point, $P_\infty^{\#}$ the product of its local point densities, $t = Q^{\alpha/2}$, and $\widehat{K}$ the transported product quotient kernel multiplied by $t^{-2kf}$. Use $S = d(d-1)$, the block dimension $a$, and $M_\gamma = m_C\prod_{v\in S(\gamma)} L_{\gamma,v}$ as in Theorem 98.39. Then*

$$\sum_c |J_{\gamma,c}| \le B_\ell B_{\ell'}\, M_\gamma R^{-fS} Q^{-f(dk+a)} (P_\infty^{\#})^{1/2} \|\widehat{K}\|_{\mathrm{op}}.$$

*For a pure input the renormalized kernel is the orbital integral of one compact-finite coefficient at each real place, simultaneously for every label. Its regular squared norm at each place equals the original regular integral.*

*Proof.* Apply the general two-parameter identity of Theorem 98.16 with $\kappa = 1/(RQ)$ and the given $t$. Its real Jacobians are unchanged by the added unit-modulus phase. Converting the arithmetic unipotent area to root measure combines with the unipotent covolume to give $R^{fa}$. After the indicated regular norm scalar the multiplier of $(P_\infty^{\#})^{1/2}\widehat{K}$ is

$$R^{fa}t^{2df}(RQ)^{-f(S+a)} = R^{-fS}Q^{f(d(d-2)-a)}.$$

Divide by the physical density lower bound $P_\infty^{1/2} \geq Q^{fd(k+d-2)}$ from the proof of Theorem 98.39 and contract with the finite covectors. This gives the asserted exponent. Equality of the squared regular norms follows by the change of variables on the regular cell, including on every transported truncation. The compact-finite coefficient comes from Theorem 98.37 in the new isometric model. The choices of the two test vectors depend on the words and the place alone. Full details and the comparison of the nonnegative sum over all labels are in Corollary 93.12 and Proposition 93.13. □

The transformed parameter $T^{\#}$ is used only inside the real orbit sum. The source estimate needed there bounds the weighted sum of squared orbital integrals by a uniform multiple of their regular squared norm. Even with that estimate, the finite masses and norm normalizations in Corollary 93.19 must be combined for the original integral row. The lower bound for its designated projected theta pairing in Corollary 83.7 and the complementary cohomology orders in (20.12) remain open on the same intervals.

**Proposition 93.20** (The definite mass of the complement)**.** *At every label in Corollary 93.19, use hyperspecial stabilizers of volume one in the definite complement at finite places. Then*

$$M_\gamma = \mathfrak{m}_{k-1}\prod_{v\in S(\gamma)} L_{\gamma,v}, \qquad \frac{\mathfrak{m}_{k-1}}{\mathfrak{m}_k} = \frac{|\Delta_F|^{1/2-k}}{\zeta_F(k)}\left(\frac{(2\pi)^k}{(k-1)!}\right)^f.$$

*Proof.* The complement has odd dimension $k-1$ and totally positive determinant $\delta$. Multiplying its form by $\delta^{-1}$ preserves its unitary group and makes the determinant a norm. Over an unramified quadratic local extension, diagonalization and unit norms reduce this form to ones and an even number of uniformizers. A pair of uniformizers and a pair of ones both give a hyperbolic plane. The split case is a pair of dual spaces. Thus one can choose self-dual local lattices for this form, with the old stabilizer at ordinary places. They give a global maximal lattice for which every finite factor of the mass formula is one. Its mass is $\mathfrak{m}_{k-1}$. Scaling the exceptional Haar measures changes their local factors inversely, as required in the cell formula. The consecutive-rank quotient is $2^f/|\zeta_F(1-k)|$, since the rank $k$ is even. The functional equation for the completed Dedekind zeta function at $k$ gives the displayed expression. Theorem 93.14 gives the measure and lattice details. □

# 94 Measures, formal degrees and integral weights

## 94.1 A single iterated scalar for each translate

On the fields of Theorem 6.29, use its definite tower of ranks $N_j = k + j(t_{\mathrm{r}}+1)$, $0 \leq j \leq m$. Write $R_j = N_{j-1}$, $s_j = N_j$, $R_\Sigma = \sum_{j=1}^m R_j$. These ranks are all even. Keep $a = \lceil xb \rceil$, the sets $T \subset \Phi_0$, $T_2 + u \subset \Phi_1$ in the notation of Proposition 8.5, and the thresholds $w_\mu = B'$ on

the translated set and $w_\mu = W$ elsewhere in $\Phi_1$. We take $a, W, B'$ to be the integers chosen in Theorem 6.23. We start with the constant one at every initial class. The following choice of determinant powers keeps total degrees under control as the output polynomials pass through the successive substitutions.

**Lemma 94.1** (Rounding the accumulated column degrees)**.** *There are nonnegative integer powers $n_{j,\tau}$ such that the total degrees $B_{j,\tau}$ in the $j$-th derivative matrix in equation 66.22, satisfy*

$$\begin{aligned} R_j a \le B_{j,t} < R_j(a+1) \quad & (t \in T), \\ R_j(w_\mu - 1) < B_{j,\mu} \le R_j w_\mu \quad & (\mu \in \Phi_1), \qquad B_{j,\tau} = 0 \quad (\tau \in \Phi_0 \setminus T). \end{aligned} \tag{94.1}$$

*Proof.* The column-weight rule in Proposition 66.28 gives $B_{0,\tau} = 0$ and $B_{j,\tau} = B_{j-1,\tau} + R_j n_{j,\tau}$. At a high embedding choose the ceiling of $(R_j a - B_{j-1,t})/R_j$. At the first stage the choice is $a$. At every later stage the preceding upper bound gives $B_{j-1,t} < R_{j-1}(a+1) \le R_j(a+1)$. The ceiling is therefore nonnegative, and its rounding error gives the two high inequalities. At a low embedding choose the floor of $(R_j w_\mu - B_{j-1,\mu})/R_j$. The preceding upper bound and monotonicity of the ranks show that this is nonnegative. Rounding down gives the two low inequalities. Use zero at all remaining embeddings and apply induction. □

Let $S$ consist of the unramified primes $5 \le p \le a/W$ inert in $K_d$ and split in $K_q$. Let $\mathcal{P}_x$ be the primes of Lemma 7.8. Take one fixed split support prime for torsion as in Section 66.2, and the positive split representatives there. At each step choose its one basis integral at every prime in $S \cup \mathcal{P}_x$. Fix any terminal class and the identity frame. Put

$$D_x = \prod_{p \in \mathcal{P}_x} p^{2\lceil h/2 \rceil \sum_{j=1}^{m} d_p(R_j a, R_j s_j)}, \qquad d_l(v,s) = \max(0, \lceil (v - s(l-1))/l \rceil).$$

The product, depending on the fields and ranks, is independent of $u$.

**Theorem 94.2** (The centered count from one composed theta scalar)**.** *There is one tuple of rational torsion inputs, one at each step, such that the terminal value for the lists of Lemma 94.1 is nonzero for every $u$. For each $u$ multiply this value by the common stabilizer denominator $a_{\mathrm{cl}}$ of Theorem 66.29, and call the result $y_u$. In one normal coefficient field put $Z_u = y_u \overline{y_u}/D_x$, using complex conjugation at the given embedding. These algebraic numbers have nonnegative valuation everywhere above $S \cup \mathcal{P}_x$, and*

$$\Delta_u \;\le\; \frac{m}{R_\Sigma} \mathcal{H}_S(Z_u) - hmM_{\mathrm{sp}} + O_{\psi,\varepsilon}(hmb/x). \tag{94.2}$$

*Here $\mathcal{H}_S$ and $M_{\mathrm{sp}}$ have the meanings in Proposition 8.5. The common divisor satisfies $\log D_x/R_\Sigma \ge (1/14 - o_{\psi,\varepsilon}(1))ha$.*

*Proof.* The ranks $R_j, s_j$ give the required even products. The bases exist by Lemma 65.14. The split changes of representatives and the torsion support occur away from all the primes under consideration. Apply Theorem 66.29 to the finitely many translates at this $x$. It proves the nonvanishing through all intermediate class sums, algebraicity over a common coefficient field, and the addition of the valuation bounds along this same composition.

At $p \in S$ the permutation $\pi = \sigma_p c_E$ interchanges $\Phi_0, \Phi_1$. Take the halves in its cycles contained in $\Phi_0$. For each $t \in T$ and each odd $k \ge 1$, the second floor bound in equation 66.23, and (94.1) give a valuation contribution at stage $j$ of at least

$$[d_{p^k}(B_{j,t}, R_j s_j) - B_{j,\pi^k t}]_+ \;\ge\; R_j[a/p^k - w_{\pi^k t} - s_j]_+.$$

Indeed $d_l(v,s) \ge v/l - s$. The proof of the cited coefficient bound adds these contributions over all $t, k$ before the next class sum. Let $s_* = \max_j s_j$. At every place above $p \in S$, division by $R_\Sigma$ gives

$$\frac{v_p(y_u\overline{y_u})}{R_\Sigma} \;\ge\; 2\sum_{t\in T}\sum_{k\ \mathrm{odd}} [a/p^k - w_{\pi^k t} - s_*]_+.$$

Conjugation permutes the places above $p$, accounting for the factor two. All exponents $k$ in this proof are positive.

At $p \in \mathcal{P}_x$ every cycle is even and $\pi T$ lies in $\Phi_0 \setminus T$. Choose in each cycle the half meeting $T$ in more points, with either choice in a tie. These halves contain at least $\lceil h/2 \rceil$ points of $T$. For each of them the $k = 1$ bound at stage $j$ is at least $d_p(R_j a, R_j s_j)$, since the accumulated degree at its neighbor is zero. The product $y_u\overline{y_u}$ is therefore divisible at all these places by the stated rational factor. The sets $S, \mathcal{P}_x$ are disjoint, so the quotient $Z_u$ retains the estimate over $S$. Moreover

$$\frac{\log D_x}{R_\Sigma} \;\ge\; 2\lceil h/2 \rceil \sum_{p\in\mathcal{P}_x} (a/p - s_*)\log p.$$

Now use $p \le 7x/4$, $\sum_{\mathcal{P}_x} \log p = x/8 + o(x)$ and $s_* x = o(a)$, as in Theorem 7.9.

At $p \in S$ the desired summand of the prime count for $t$ is $2\log p \sum_{k\ \mathrm{odd}} [a/p^k - w_{\pi^k t}]_+$. Writing $X = a/W$, all contributing prime powers satisfy $p^k \le X$. The total loss from $s_*$ in the positive parts is at most $2hs_* X \log(2X)$, since the sum of $\log p$ over prime powers up to $X$ is at most $X\log(2X)$. This is $o(hb/x)$ on the common intervals, where $X \asymp x\sqrt{b}$ and $s_*$ is polynomially bounded. The missing ramified and small primes contribute $O_{\psi,\varepsilon}(hb/x)$ by the inertia-average argument of Theorem 66.26. For each nonzero field element $\mathcal{H}_S$ bounds above the normalized sum of its signed valuations at $S$, by the product formula. We obtain

$$\sum_{t\in T} C_{\mathrm{sp}}(t,a,w) \le \mathcal{H}_S(Z_u)/R_\Sigma + O_{\psi,\varepsilon}(hb/x).$$

The thresholds in this count are the same for every step. By Proposition 66.27, the omitted quadratic sign differs from its main term by $O_{\psi,\varepsilon}(b/x)$ for each $t$. The main term for the sign retained here is $M_{\mathrm{sp}}$ by Proposition 8.5. Subtract the main terms and multiply by $m$ to obtain (94.2). □

Thus an upper estimate

$$\frac{m}{R_\Sigma}\mathcal{H}_S(Z_u) \le hmM_{\mathrm{sp}} + \tfrac{1}{2}x^{5\varepsilon}\mathcal{B}$$

on the common intervals would give (6.3), since $hmb/x = o(x^{5\varepsilon}\mathcal{B})$. This step requires an upper bound including the denominators outside $S$. The lower divisibilities and the positive translation mean already apply to the same scalar. The source's automorphic period also has a terminal Bessel functional, level constraints, and additional frame and character factors. The transfer to that functional and the uniform bound on $\mathcal{R}_{**}$ in Proposition 7.7 are unresolved.

## 94.2 The size of the archimedean factorial in the same comparison

Keep the scalar of Theorem 94.2 and write $P_u$ for the polynomial which computes its terminal derivative. The full factorial in the product translation mean is known exactly by Theorem 66.30. Here we compute its asymptotic on the prescribed tower. Put $D = t_{\mathrm{r}} + 1$, so $R_j = k + (j-1)D$.

**Theorem 94.3** (Asymptotic for the factorial on the definite tower)**.** *For the particular integer powers in the proof of Lemma 94.1, uniformly in $u$,*

$$\frac{\log \|P_u\|_{\mathrm{F}}^2}{hR_\Sigma} = \log(a!) + (1 - \gamma + o_{\psi,\varepsilon}(1))a. \tag{94.3}$$

*Proof.* Let $v_{j,i}$ be the column weights at a fixed high embedding. They are the same at every high embedding. Write $B_j = R_j a + \delta_j$, $0 \le \delta_j < R_j$, and $\eta = k/D$. The ranks in Theorem 6.29 give $0 < \eta \le 1$ for large $x$. At stage one $n_1 = a$. For $j \ge 2$, the recurrence in Lemma 94.1 gives

$$\frac{n_j}{a} = \frac{1}{j - 1 + \eta} + \frac{\delta_j - \delta_{j-1}}{aR_j}, \qquad |\delta_j - \delta_{j-1}| < R_j.$$

There are $k$ first-stage columns and $D$ new columns at each stage $l \ge 2$. A column first present at $l$ has $v_{j,i} = \sum_{s=l}^{j} n_s$. For $2 \le l \le j$ this yields

$$\frac{v_{j,i}}{a} = \sum_{s=l}^{j} \frac{1}{s - 1 + \eta} + O(m/a).$$

The first-stage columns satisfy $v_{j,i}/a \ll 1 + \log(2j)$, and the error $m/a$ tends to zero uniformly in $j$.

Put $F(z) = z \log z$ for $z > 0$, $F(0) = 0$. We claim that for $x$ tending to infinity and any $j \le m(x)$ tending to infinity,

$$\frac{1}{R_j} \sum_{i=1}^{R_j} F(v_{j,i}/a) \longrightarrow \int_0^1 F(-\log t)\, dt = 1 - \gamma. \tag{94.4}$$

Indeed the $k$ initial terms have proportion $\eta/(j - 1 + \eta)$ and absolute contribution $O((1 + \log(2j))^2/j)$. Each remaining group has proportion $1/(j - 1 + \eta)$. For any fixed $0 < \sigma < 1$ and $\sigma j \le l \le j$, comparison of the harmonic sum with its integral gives $v_{j,i}/a = -\log(l/j) + O(1/(\sigma j) + m/a)$, uniformly in $0 < \eta \le 1$. Uniform continuity of $F$ on the resulting compact interval and Riemann summation give the integral over $[\sigma, 1]$. For $2 \le l < \sigma j$ the absolute summand is bounded by $C(1 + \log(j/(l - 1)))^2$, since $F$ is bounded on $[0, 1]$ and $z \log z \le z^2$ for $z \ge 1$. The decreasing function $(1 + \log(1/t))^2$ is integrable on $(0, 1]$. Comparison with its integral over $(0, \sigma]$ shows that the omitted mean tends to zero as $\sigma \downarrow 0$, uniformly in the indicated limits. Finally $t = e^{-z}$ gives the integral $\int_0^\infty z \log z\, e^{-z} dz = \Gamma'(2) = 1 - \gamma$, by the gamma recurrence and $\Gamma'(1) = -\gamma$.

Use $\log(v!) = v \log v - v + O(\log(v + 2))$, also valid at zero with the preceding convention. Since $v_{j,i} \ll a(1 + \log(2m))$, $\sum_i v_{j,i} = R_j a + O(R_j)$, and $m$ is polynomial in $x$,

$$\frac{\sum_i \log(v_{j,i}!) - R_j \log(a!)}{aR_j} = \frac{1}{R_j} \sum_i F(v_{j,i}/a) + O_\varepsilon(\log(2a)/a).$$

The finite initial stages carry a proportion tending to zero in the $R_j/R_\Sigma$-weighted average since $m \to \infty$. The tail convergence in (94.4) holds along every such sequence $j \to \infty$, and the bounds in its proof give a uniformly bounded mean over all stages. The weighted mean therefore tends to $1 - \gamma$.

To pass to the exact product in equation 66.26, use $(R_j - i + 1)_v = v! \binom{v + R_j - i}{R_j - i}$. The total logarithm of the binomial factors is between zero and

$$|\Phi| \sum_j R_j^2 \log(2 + s_* + R_j(a + B' + 1)), \qquad s_* = \max_j s_j.$$

Indeed each column weight is bounded by the total at its embedding. This expression is polynomially bounded in $x$, hence $o(haR_\Sigma)$. At the low embeddings $\sum_i v_{j,\mu,i} \le R_j w_\mu$. Their factorials have total logarithm divided by $hR_\Sigma$ at most

$$\frac{\sum_{\mu \in \Phi_1} w_\mu}{h} \log(2 + s_*(a + B' + 1)) \ll_{\psi,\varepsilon} bx^{9\varepsilon/8} = o(a).$$

Here $\sum w_\mu \le nB'$, $h \ge nx^{-\varepsilon/8}/2$, and the logarithm is $O_\varepsilon(x^\varepsilon)$ for large $x$. The remaining embeddings have zero weights. Combining the estimates for all embeddings proves (94.3). □

**Proposition 94.4** (Size of the explicit divisor and of the available upper estimate)**.** *For $D_x$ of Theorem 94.2,*

$$\frac{\log D_x}{hR_\Sigma} = (\tfrac{1}{4}\log(7/5) + o_{\psi,\varepsilon}(1))a.$$

*Let $U_u$ be the right side of equation 66.27, less $\log D_x$. It is an upper bound for the normalized logarithmic field norm of $Z_u$. Its distance above the centered main term satisfies*

$$\frac{U_u}{hR_\Sigma} - M_{\mathrm{sp}} \;\geq\; (\tfrac{3}{4} + \tfrac{1}{2}\omega - o_{\psi,\varepsilon}(1))ax^{\varepsilon}, \qquad \omega = h_2/(2n). \tag{94.5}$$

*Proof.* The secant estimate in the proof of Lemma 7.8 applies uniformly for $5x/4 \leq y \leq 7x/4$. Indeed on this compact interior interval it uses the same bound on the remainder over $[x, 2x]$ and a common choice of the relative secant displacement. The prime powers and the excluded residue classes have total weight $o(x)$ on the full interval. Hence, writing $L = 5x/4$, the sum over $p \in \mathcal{P}_x$, $p \leq y$, of $\log p$ is $(y-L)/4+o(x)$ uniformly there. Partial summation against $1/y$ gives $\sum_{\mathcal{P}_x} \log p/p = \log(7/5)/4 + o(1)$ because the endpoint and integral errors are $o(x)/L = o(1)$. For $s \geq 1$ the floor bound satisfies $v/p - s \leq d_p(v,s) \leq v/p$. Use this in the definition of $D_x$. The aggregate error after division by $hR_\Sigma$ is $O(s_* x) = o(a)$, and $2\lceil h/2\rceil/h \to 1$. This gives the first formula.

By (94.3) and $\log a = x^{\varepsilon} + \log x + o(1)$, the contribution of $\log \|P_u\|_{\mathrm{F}}^2/(hR_\Sigma)$ to $U_u/(hR_\Sigma)$ is $(1+o(1))ax^{\varepsilon}$. The product $a_{\mathrm{cl}}M$ in the same estimate is at least one. At each stage it includes a sum of positive integers $A_{N_{j-1}}m_c$ over a nonempty class set. The terms $d_j \log E_j$ and $e(b_j, d_j)$ there are nonnegative. The absolute values of the two elliptic means $L_d, V_d$ are $O_{\psi,\varepsilon}(x^{\varepsilon/4})$ by Theorem 65.22, Corollary 65.23, and Theorem 65.25. Also $\sum_j b_j/(hR_\Sigma) = O(a)$ by (94.1) and $nB'/h \ll bx^{\varepsilon/8}$, while all $d_j$ and $m$ are polynomially bounded. Thus any negative contribution from those means is $o(ax^{\varepsilon})$ after this normalization. The divisor just computed also has size $o(ax^{\varepsilon})$ here.

Finally $\log(a/W) = x^{\varepsilon}/2 + \log x + o(1)$ and $\log(B'/W) = x^{\varepsilon}/2 + o(1)$. Each of the three logarithmic derivatives in $M_{\mathrm{sp}}$ is $O(\log x)$ by Theorem 5.1. Proposition 8.5 gives $M_{\mathrm{sp}} = (1/4 - \omega/2 + o(1))ax^{\varepsilon}$, uniformly since $0 \leq \omega \leq 1/2$. Subtraction proves (94.5). □

Inequality (94.5) measures the excess in the explicit Gaussian upper estimate. The actual period can have a smaller field norm. The allowed error in (94.2), divided by $hm$, is $o(a)$ by Theorem 7.9 and the size of $x^{5\varepsilon}\mathcal{B}$ recorded immediately after it. Consequently controlling the error of the Gaussian integration alone leaves a discrepancy much larger than the permitted one. An upper estimate for the actual period at the required precision and a bound for its negative valuations outside $S$ are still missing on these same fields.

## 94.3 The size of the integral highest weight projections

Keep the fields, intervals, ranks, balanced orders and divisor $D_x$ of Theorem 94.2. We now project onto the highest row weight after every definite step. At the terminal class use the identity frame. Denote the new scalar, with its stabilizer denominator cleared, by $\widetilde{y}_u$ and put $\widetilde{Z}_u = \widetilde{y}_u\overline{\widetilde{y}_u}/D_x$. The degree partition on the $R_j$ input columns is $v_{j,\tau}$ from Theorem 94.3. Define

$$C_u = \sum_{j,\tau} \log d_{R_j}(v_{j,\tau}), \qquad d_R(v) = \prod_{i<l} \frac{v_i - v_l + l - i}{l - i}.$$

**Theorem 94.5** (The projected scalar on the common intervals)**.** *One can choose common rational torsion inputs for which all the $\widetilde{y}_u$ are nonzero, and (94.2) holds with $\widetilde{Z}_u$ in place*

*of $Z_u$. The available Gaussian upper estimate for its normalized logarithmic field norm is $\widetilde{U}_u = U_u - C_u$, with $U_u$ the explicit bound in Proposition 94.4. Uniformly in $u$,*

$$0 \le C_u = o_{\psi,\varepsilon}(b/x), \qquad \frac{\log \|P_u^{\mathrm{pr}}\|_{\mathrm{F}}^2}{hR_\Sigma} = \log(a!) + (1 - \gamma + o_{\psi,\varepsilon}(1))a,$$
$$\frac{\widetilde{U}_u}{hR_\Sigma} - M_{\mathrm{sp}} \ge (\tfrac{3}{4} + \tfrac{1}{2}\omega - o_{\psi,\varepsilon}(1))ax^{\varepsilon}. \tag{94.6}$$

*Proof.* The integral projection, its nonvanishing on the translated product torus, and the coefficientwise prime powers are proved in Theorem 92.7. In particular every valuation bound used in the proof of Theorem 94.2 holds for $\widetilde{y}_u$. Its proof uses only those bounds, the nonzero value, and the product formula. This proves the asserted inequality on the same fields and yields the same divisor $D_x$.

The field-norm calculation for the new polynomial uses the same degree $b_j$ in each matrix block, the same representatives and masses, and the same metric envelopes, uniformly in the torsion choice. The squared Fischer norm is divided by the product of the dimensions by equation 92.1. Consequently the right hand side of the field-norm estimate decreases by exactly $C_u$.

For $i < l$, the factor in $d_R(v)$ is at least one and is at most $1 + v_1$. Here $v_1 \le R_j(a + B' + 1)$ by the balanced total degrees. Consequently, writing $s_* = \max_j N_j$,

$$0 \le C_u \le |\Phi| \sum_j R_j^2 \log(2 + s_*(a + B' + 1)).$$

The ranks, the number of stages, and $|\Phi|$ are bounded by fixed powers of $x$, while the logarithm is $O_{\psi,\varepsilon}(x^{\varepsilon} + \log x)$. The whole upper bound is polynomial in $x$, and $b = \exp(x^{\varepsilon})$. Hence $C_u = o(b/x)$. Subtraction in Theorem 94.3 and in Proposition 94.4 proves both formulas. □

There is a separate integral issue if one applies a corner differential directly to an arbitrary polynomial of the indicated row and column degrees. In rank two, Proposition 92.6 gives the exact common denominator $\operatorname{lcm}(1, \dots, a+1)$ for shape $(a, a)$. Its logarithm is $\sum_{n \le a+1} \Lambda(n) = (1 + o(1))a$ by the ordinary prime number theorem. The normalized differential is integral on the column-invariant lattice by Theorem 92.5. The field-norm calculation above uses that lattice throughout. The additional hook factor in the source uses a second, indefinite row space with its own weights and integral period pairing. Controlling that pairing on these fields and bounding $\mathcal{R}_{**}(x)$ of Proposition 7.7 remain necessary for the proposed argument.

## 94.4 A lower bound for the terminal scale from levels, masses and weights

Write $r = k + m - 1$. At either of the two split primes $g$ from Theorem 7.11, impose the full upper unitriangular group $U_k(g)$ on the column reduction. On the row reduction in $\mathrm{GL}_r(\mathbb{F}_g)$ take $H_r(g)$ upper unitriangular with $k$ specified entries immediately above the diagonal in one column set to zero. These are the consecutive entries ending just above that column's diagonal entry. The column is the last of the middle block of length $k + 1$. This is the level defined in `Formalism/LocalToGlobal/Columns/GenericUnipotentCharacter.lean`, line 526. Let $I_\pi, I_\sigma$ be the products of the respective indices at the $f$ split places for each $g$.

**Proposition 94.6** (The finite level contribution)**.** *Uniformly on the common intervals,*

$$\frac{I_\sigma}{I_\pi} = \prod_{g \in \{g_1, g_2\}} \left( g^k \prod_{i=k+1}^{r} (g^i - 1) \right)^f, \tag{94.7}$$
$$\frac{\log(I_\sigma / I_\pi)}{fk} = (16 + o_{\psi,\varepsilon}(1))\, m x^{\varepsilon}.$$

*Proof.* Choosing independent columns successively gives $|\mathrm{GL}_s(\mathbb{F}_g)| = g^{s(s-1)/2}\prod_{i=1}^{s}(g^i-1)$. The order of the column subgroup is $g^{k(k-1)/2}$. In the row subgroup all entries above the diagonal except the $k$ specified ones are free, so its order is $g^{r(r-1)/2-k}$. The consecutive zero entries are preserved by upper unitriangular multiplication and inversion, as is also seen by preserving the last basis vector in the middle subquotient. Division gives the one-place ratio and hence (94.7). Its logarithm divided by $fk$ is

$$\left(m+\frac{m(m-1)}{2k}\right)(\log g_1+\log g_2)+\frac{1}{k}\sum_{g\in\{g_1,g_2\}}\sum_{i=k+1}^{r}\log(1-g^{-i}).$$

The double sum is $O(\min(g_1,g_2)^{-k})$ by a geometric series. Both $g$ lie between $(kb)^8$ and $2(kb)^8$. Now $\log b = x^{\varepsilon},\ \log k = O(\log x)$, and $m/k \to 0$. □

Let $\mathfrak{m}_s$ be the sum of reciprocal automorphism orders over the isometry classes of positive integral Hermitian lattices in the genus of the standard lattice in $E^s$. These are finite sums by finiteness of the definite class set. Write $w_E$ for the number of roots of unity in $E$. For a strictly decreasing real list $s=(s_1,\dots,s_l)$ put

$$W_l(s)=\frac{\prod_{i<j}(s_i-s_j)}{\prod_{i=0}^{l-1} i!}.$$

For integral differences at least one this is the Weyl dimension for the associated compact highest weight after the usual shift and a central translation.

**Lemma 94.7** (Bounds for the definite masses and interlacing weights)**.** *For $r \ge k$ one has $\mathfrak{m}_r/\mathfrak{m}_k \ge (w_E r)^{-(r-k)}$. Let $u_1 > \cdots > u_k$, $k \ge 2$, and let $u_i \ge v_i \ge u_{i+1}$ for $1 \le i < k$. Suppose a list $\lambda$ of length $r = k+m-1$ is obtained by adjoining $m$ numbers to the $v_i$ and arranging the result in decreasing order. Suppose all entries of $\lambda$ differ pairwise by at least one in absolute value. For $A = \max(1, u_1-u_k)$,*

$$\frac{W_r(\lambda)}{W_k(u)} \;\ge\; A^{-(2k-3)}\left(\prod_{i=k}^{r-1} i!\right)^{-1}.$$

*Proof.* In a positive integral Hermitian lattice, take vectors $v, w$ of Hermitian norm one. Then $\alpha = (v,w)$ is an algebraic integer and every conjugate has absolute value at most one by Cauchy–Schwarz. If $\alpha \neq 0$, its absolute norm is a positive integer, so all these absolute values equal one. It is a root of unity. Indeed the monic integral polynomials whose roots are the conjugates of $\alpha^j$ have bounded degree and bounded coefficients as $j$ varies. Their finite union of roots contains all powers of $\alpha$. Equality in Cauchy–Schwarz gives $w = v\alpha$ in the convention linear in the second argument. Thus there are at most $r$ mutually orthogonal norm-one lines and at most $rw_E$ choices for a norm-one vector.

Count rank-$r$ lattices with an ordered orthonormal frame of length $r-k$, weighting each framed class by the reciprocal of its stabilizer. Over a single unframed lattice the orbit–stabilizer formula gives the number of frames divided by its automorphism order, so the full framed mass is at most $(rw_E)^{r-k}\mathfrak{m}_r$. Every rank-$k$ lattice $L$ in the standard genus gives the framed lattice $L \perp \mathcal{O}_E^{r-k}$ with the standard ordered frame. It belongs to the rank-$r$ genus at every place. The orthogonal complement recovers $L$, and fixing the frame leaves precisely $\mathrm{Aut}(L)$. These classes already contribute $\mathfrak{m}_k$. This proves the mass bound.

For $2 \le i < j \le k-1$, match the column factor $u_i - u_j$ with $v_{i-1} - v_j$, which is at least as large by interlacing. These are distinct factors of the absolute row Vandermonde. All its other factors are at least one. The column factors involving the first or the last entry number $2k-3$,

and each is at most $A$. Division by the two factorial products proves the inequality, including $k = 2$. □

At every embedding choose such regular interlacing lists $u_\tau, \lambda_\tau$ with $u_{\tau,1} - u_{\tau,k} \le 3a$, $a = \lceil xb \rceil$. The bound below is uniform in the lists and in the translate. As a concrete choice, take the common row entries $v_i = 2(k-1-i)$, $1 \le i < k$, and append the distinct negative integers $-j(t_{\mathrm{r}}+4)$, $1 \le j \le m$. One can take $u_i = v_i + 1/2$ for $i < k$, $u_k = v_{k-1} - 1/2$, and increase $u_1$ by $a$ at the prescribed high embeddings. These give regular integral or half-integral parameters, strict interlacing, and the stated span for large $x$. On the lowest compact types these explicit lists give zero Bessel period by Theorem 86.27. The replacement lists of Theorem 87.5 admit all of the column types and their real Bessel models.

**Theorem 94.8** (A terminal prime at the compact dimension scale)**.** *With the two levels in* (94.7)*, put*

$$d_{u,*} = \mathfrak{m}_k I_\pi \prod_\tau W_k(u_\tau), \qquad D^{\mathrm{c}}_* = \mathfrak{m}_r I_\sigma \prod_\tau W_r(\lambda_\tau).$$

*Let* $\Gamma \ge 1$ *satisfy* $\log \Gamma \le C f m^5 x^\varepsilon$ *for a fixed* $C$*. Then the positive target*

$$Y_{\mathrm{c}} = \left( \frac{D^{\mathrm{c}}_*}{d_{u,*}\Gamma^2} \right)^{1/(fk)} \qquad \textit{satisfies} \quad \log Y_{\mathrm{c}} \ge (16 - o_{\psi,\varepsilon,C}(1))\, m x^\varepsilon.$$

*Consequently there are* $Y_{\mathrm{c}}^{1-o(1)}$ *primes as in Theorem 7.11 with* $Y_{\mathrm{c}} < T \le (1 + 1/(fk))Y_{\mathrm{c}}$*. The same conclusion with a positive constant in the logarithmic lower bound holds if the numerator is multiplied by a factor of logarithm at least* $-c_1 f k m x^\varepsilon$ *for a fixed* $c_1 < 16$*.*

*Proof.* Take logarithms and divide by $fk$. The finite index term was evaluated in Proposition 94.6. For the mass term apply Lemma 94.7. The roots of unity generate a cyclotomic subfield, whence $\varphi(w_E) \le 2f$. The prime-power formula for $\varphi$ gives $\varphi(l) \ge \sqrt{l/2}$ for all $l$, since each odd prime-power factor is at least its square root and the power of two loses at most a factor $\sqrt{2}$. Thus $w_E \le 8f^2$. The loss in the logarithm of the mass ratio before division is at most $O(m \log x)$.

For the weight ratio, Lemma 94.7 bounds the loss after division by

$$2\log(3a) + \frac{1}{k}\sum_{i=k}^{r-1} \log(i!) = O_{\psi,\varepsilon}(x^\varepsilon + m \log x).$$

Here $r \le 2k$, $\log(i!) \le i \log i$, and there are $m-1$ terms. Both losses are $o(mx^\varepsilon)$ after the prescribed normalization. Finally $2\log\Gamma/(fk) \le 2Cm^5 x^\varepsilon/k = o(mx^\varepsilon)$ since $m^4/k \to 0$ at the ranks of Theorem 6.29. The lower bound follows. Apply Theorem 7.11 to this target. Multiplying the numerator as indicated loses at most $c_1 m x^\varepsilon$, leaving a positive fixed constant. □

At each of these primes the joint incidence map in Theorem 92.9 and Corollary 92.10 has elementary exponents ranging from zero to $f(k-1)$, with these extremes attained in each fixed tuple of nontrivial scalar characters. In particular a primitive integral input cannot be sent into $T^{f(k-1)+1}$ times the full target lattice. To see this, use its coordinate with a unit coefficient in the integral diagonal basis. Its image coefficient has valuation at most $f(k-1)$. The sum $ha$ of the high orders in Theorem 94.2 exceeds this bound by an unbounded factor because $f, k$ grow polynomially and $a \ge x\exp(x^\varepsilon)$. A subsequent scalar functional can change the valuation by cancellation.

The term $D_*^{\mathrm{c}}$ uses the compact Weyl dimensions. The source uses formal degrees of specified real representations and an arithmetic volume, with their normalizations tied to the indefinite row in `Formalism/LocalRingCardinality/DataTies.lean`, line 17 and `Formalism/SharpRestriction/GlobalSchurBounds/GeneralLinearActions/GapAtLeast.lean`, line 85. Theorem 94.9 below gives the comparison of the numerator for the specified arithmetic unitary group and its discrete series.

## 94.5 The indefinite arithmetic volume and the actual formal degrees

We now identify the two factors in the preceding compact calculation for global unitary groups on the same fields. Put $F_+ = E^+$. Fix $p_+ + p_- = r$, taking $p_+ = k + m/2 - 1$, $p_- = m/2$ for the prescribed signature. In $E^r$ use the lattice $\Lambda = \mathcal{O}_E^r$ with Hermitian matrix $\mathrm{diag}(I_{p_+}, -I_{p_-})$. Write $G$ for its unitary group over $F_+$ and $G_{\mathrm{c}}$ for the unitary group of the positive identity matrix in rank $r$.

At finite places give the local stabilizer of $\Lambda$ volume one. At a real place use the Haar density on $U(p_+, p_-)$ corresponding at the identity to Haar probability on $U(r)$. In the Cartan decompositions the comparison is the identity on $\mathfrak{k}$ and multiplication by $i$ from $\mathfrak{p}$ to $i\mathfrak{p}$. Equivalently take absolute values of the same invariant complex top form on the two real forms and normalize its compact volume to one. The equivalence follows by evaluating the top form on bases of $\mathfrak{k}$ and $\mathfrak{p}$, since the inserted powers of $i$ have absolute value one. This is the compact-dual normalization described in `Formalism/LocalRingCardinality/DataCD.lean`, line 57. Let $\mathcal{V}(G)$ denote the adelic quotient volume for the product of these local measures, before imposing the two finite levels in (94.7).

**Theorem 94.9** (Two normalization identities on the common fields)**.** *For the group and measures just defined, $\mathcal{V}(G) = \mathfrak{m}_r$. At any real place, let $\sigma$ be a discrete series of $U(p_+, p_-)$ with regular integral infinitesimal character, written as a decreasing Harish–Chandra parameter $\lambda$ after permutation. Its formal degree for the indicated Haar density is $W_r(\lambda)$. In this assertion integral means that the shifted parameter is the highest weight of a finite-dimensional representation, with its specified central character.*

*Proof.* The fields $K_q, K_d$ have the distinct prime discriminants $-q, -d$. At a finite prime of $E$, an inertia element restricts to inertia on both quadratic subfields. At least one of those fields is unramified at its underlying rational prime. Complex conjugation acts nontrivially on both, so it lies in no finite inertia group. Inertia in $E/F_+$ is the intersection with $\mathrm{Gal}(E/F_+)$, whose only nonidentity element is conjugation. Thus $E/F_+$ is split or unramified quadratic at every finite place.

In either case $-1$ is the norm of a local integral unit. At a split place take $(-1, 1)$. For an unramified quadratic extension, the norm surjects on nonzero residue classes. Starting with a lift of a solution, one corrects modulo successive powers of a common uniformizer $\pi$ by multiplying by $1 + \pi^j z$. Its norm is $1 + \pi^j \operatorname{Tr}(z) \bmod \pi^{j+1}$ for $j \geq 1$. The trace of the separable quadratic residue extension is onto, also in characteristic two. Completeness gives a unit with the desired norm. Scaling each negative basis vector by such a unit exhibits an integral isometry between the two local Hermitian lattices. The finite stabilizers therefore correspond.

For clarity, compare the product measures through Tamagawa measure. The two groups are inner forms over $F_+$. Indeed after splitting $E/F_+$ both are $\mathrm{GL}_r$, and changing the Hermitian matrix conjugates their descent involutions by an inner automorphism. Transport an invariant rational top form from $G_{\mathrm{c}}$ to $G$ by an inner twisting isomorphism over an algebraic closure. The transported form descends to $F_+$, since inner automorphisms preserve every such form. For $\mathrm{GL}_r$ this last assertion also follows directly from determinant one of conjugation on its matrix

algebra. The ratio of the local absolute-form density to our local Haar density is thus the same at each finite place by the integral isometries, and at each real place by the comparison at the identity above. The convergence factors in the Tamagawa measure are the same as well. They depend on the Galois module of algebraic characters, which in both cases is generated by the determinant with conjugation acting by inversion. The global discriminant and dimension factors are identical. Consequently changing our measure to Tamagawa measure multiplies the two quotient volumes by the same scalar. Neither group has a nontrivial $F_+$-rational character, so these quotients are the full adelic quotients. Kottwitz's invariance of Tamagawa numbers under inner twisting now gives equality of their volumes. [4]

For $G_{\mathrm{c}}$ the archimedean factors are compact of volume one. The finite double cosets with the given stabilizer parametrize the genus in the standard positive Hermitian space. Indeed a finite adelic element transports the completed lattice, and intersecting with $E^r$ recovers a global lattice with these completions. Conversely the defining local isometries of any lattice in the genus give such a tuple, integral almost everywhere. The rational stabilizer of a tuple up to its compact level is precisely the automorphism group of that global lattice. Each double coset contributes the reciprocal of that finite order to the adelic volume. Summing gives $\mathfrak{m}_r$, as claimed.

For the real-place assertion first use $SU(p_+,p_-)$ with both indices positive. This is a connected linear semisimple group without compact factors, with compact Cartan. Fix a symmetric-space metric on $X = SU(p_+,p_-)/S(U(p_+)\times U(p_-))$ and the dual metric on the compact symmetric space $X^{\mathrm{c}}$. For the Haar measure given by Riemannian measure on $X$ and Haar probability on its maximal compact subgroup, the discrete series formal degree is $\dim(\tau)/\operatorname{vol}(X^{\mathrm{c}})$, where $\tau$ has the same infinitesimal character. [5, Cor. 5.2] Here one can take the representation of the complexified group of highest weight $\lambda-\rho_r$, restricted to the derived group, where $(\rho_r)_i=(r+1-2i)/2$. The hypotheses on $\lambda$ are precisely those giving this highest weight. The Weyl dimension formula gives $\dim(\tau)=W_r(\lambda)$. On the compact dual the product of its Riemannian quotient measure with Haar probability on the same maximal compact subgroup has total mass $\operatorname{vol}(X^{\mathrm{c}})$. Dividing by that volume on both real forms gives the measure specified in the theorem. A formal degree scales by the inverse of the factor multiplying the Haar measure, by the matrix coefficient orthogonality integral. This gives $W_r(\lambda)$.

The map $U(1)\times SU(p_+,p_-)\to U(p_+,p_-)$ given by multiplication is onto, by taking an $r$-th root of the determinant, and its kernel is finite and central. Use Haar probability on $U(1)$. The product Haar measure pushes forward with the indicated compact-dual normalization. Its compact counterpart pushes forward to Haar probability on $U(r)$, and the differential of multiplication compares the same central line and the same semisimple tangent densities on both sides. Pullback of the representation is a central character times an irreducible representation of the derived group, since the center acts by scalars. The squared absolute matrix coefficients integrate to the same values under this pushforward. This extends the degree formula to the required unitary group. For a compact signature the formula follows directly from compact Schur orthogonality. □

**Corollary 94.10** (Terminal primes for the indefinite row scale)**.** *Use the regular integral lists from Theorem 94.8 at the real places and take any discrete series in their indicated packets. Such series exist for these parameters by the discrete-series classification for a compact Cartan, including the explicit lists above since $r$ is odd. Put*

$$D_*^{\mathrm{row}}=\mathcal{V}(G)I_\sigma\prod_{\tau}\deg(\sigma_\tau),\qquad T_*=\left(D_*^{\mathrm{row}}/(d_{u,*}\Gamma^2)\right)^{1/(fk)}.$$

*With $\Gamma$ in the range of Theorem 94.8, $\log T_* \geq (16 - o(1))mx^{\varepsilon}$. There are $T_*^{1-o(1)}$ primes in $(T_*, (1 + 1/(fk))T_*]$ with a generator congruent to one modulo $N(g_1g_2)^2\mathcal{O}_{K_d}$. All the finite incidence conclusions of Section 92.3 apply to them.*

*Proof.* Theorem 94.9 identifies the numerator exactly with $D_*^{\mathrm{c}}$. Apply Theorem 94.8 and then the finite-place calculation at each of the chosen primes. □

These are volume and formal-degree identities for specified global groups and local representations. A period evaluated on specified automorphic vectors requires an additional estimate. In particular the negative term from the terminal level in the height of the projected Bessel period, and the uniform bound for the residual length $\mathcal{R}_{**}(x)$ in Proposition 7.7, remain open here.

## 94.6 A self-dual lattice for the normalized terminal lines

Let $K = K_d$, $M = N(g_1g_2)^2$, and use the terminal scale of Corollary 94.10. We may choose its prime with $\pi \equiv 1 \bmod 2M\mathcal{O}_K$. Indeed in Lemma 7.10 replacing $M$ by $2M$ changes $\log(4dM^2)$ by $\log 4$. All interval estimates in Theorem 7.11 still apply. Write $T = \pi\bar{\pi}$. In the signed lattice $\Lambda$ of Theorem 94.9 take pairwise distinct coordinate vectors $e_+, e_-, e$ of signs $+, -, -$. Put

$$u = e_+ + e_-, \qquad v = (e_+ - e_-)/2, \qquad z_c = \pi e + Mc\,u.$$

The Hermitian form pairs $u$ and $v$ to one and vanishes on each of them. The lattice calculations for $z_c$, including its norm $-T$ and the local evaluation criterion, are in Section 92.4.

**Theorem 94.11** (One rational isometry for all the normalized lines)**.** *Define $g$ on the Hermitian space by*

$$gu = \pi^{-1}u, \qquad gv = \bar{\pi}v, \qquad g = 1 \quad \text{on } \langle u, v\rangle^{\perp}.$$

*Then $g \in G(F_+)$ and $\Lambda' = g\Lambda$ is self-dual. At every finite place away from $T$ it is the original lattice, and $g$ acts as the identity modulo $M$ there. All the vectors $z_c/\pi$ for $1 \leq c < T$ split off orthogonally as integral lines of norm $-1$ in $\Lambda'$.*

*The orbit of $\widehat{\Lambda}'$ under the finite adelic stabilizer of $\widehat{\Lambda}$ has cardinality*

$$\left(\frac{T^r - 1}{T - 1}\right)^f. \tag{94.8}$$

*The same count holds if the auxiliary levels at $g_1, g_2$ are imposed on the stabilizer. At every archimedean place the largest and smallest singular values of $g$, for the positive coordinate norm, are $\sqrt{T}$ and $1/\sqrt{T}$.*

*Proof.* The cross pairing after scaling is $\overline{\pi^{-1}}\bar{\pi}h(u, v) = 1$, so the map is a unitary $E$-linear map defined over $F_+$ as a point of the unitary group. Isometry preserves self-duality. To check the integral assertion at all places away from $T$, including those above two, use the original ordered pair $e_+, e_-$. The diagonal entries of its block are $(\pi^{-1} + \bar{\pi})/2$ and both off-diagonal entries are $(\pi^{-1} - \bar{\pi})/2$. Away from $T$, both scalars $\pi^{-1}, \bar{\pi}$ are units congruent to one modulo $2M$. Thus the block and its inverse are integral and equal to the identity modulo $M$. Finally $g^{-1}(z_c/\pi) = e + Mc\,u$ is integral of norm $-1$. A vector of unit Hermitian norm gives an integral orthogonal projection onto its line by dividing the pairing by that norm. This proves the splitting.

At a place of $F_+$ above $T$, identify the split unitary group with $\mathrm{GL}_r$ by choosing the factor of $E$ in which $\pi$ has valuation one. Its residue field is $\mathbb{F}_T$. The hyperbolic basis above is

integral and invertible there since $T$ is odd. In the chosen factor the lattice is enlarged by one inverse uniformizer on the line of $u$, and all other directions keep their integral spans. In the conjugate factor the lattice is its dual, determined uniquely by the first. The possible enlargements are indexed by the residue lines in $T^{-1}\Lambda_w/\Lambda_w$, where $\Lambda_w$ denotes the lattice in the chosen factor. Each line has its full inverse image as the enlargement. The reduction map $\mathrm{GL}_r(\mathcal{O}_{F_+,v}) \to \mathrm{GL}_r(\mathbb{F}_T)$ is onto by lifting entries of an invertible matrix. Its action on lines is transitive. There are $(T^r-1)/(T-1)$ lines, independent at the $f$ places. All other places contribute one orbit element, including the two auxiliary levels. This gives (94.8). In the positive metric $u, v$ are orthogonal and orthogonal to the remaining coordinates. Their eigenvalues have absolute values $T^{-1/2}, T^{1/2}$. This proves the final assertion. □

**Corollary 94.12** (The two polynomial lattices on the evaluation plane)**.** *Let $P$ be homogeneous of degree $A < T-1$, with restricted binary form $P(su+te) = \sum_{i=0}^{A} b_i s^i t^{A-i}$ integral on the original coordinate plane over $\mathcal{O}_{J,w}$, where $w \mid (\pi)$ in a coefficient field containing $E$. The following are equivalent.*

1. *The polynomial on the plane of $\Lambda'$ has integral coefficients in the basis $\pi^{-1}u, e$.*
2. *Every $P(z_c/\pi)$ for $1 \le c \le A+1$ is integral at $w$.*
3. *$b_i \in \pi^i \mathcal{O}_{J,w}$ for every $i$.*

*The operator $P \mapsto P \circ g$ on all complex homogeneous polynomials of degree $A$, with the positive Fischer norm, has norm $T^{A/2}$. On polynomial functions of the plane $\langle u, e\rangle$ with its induced positive norm, the same pullback has norm one.*

*Proof.* At $w$ the original plane lattice has basis $u, e$ and the new one has basis $\pi^{-1}u, e$, by the integral hyperbolic basis in the theorem. In that new basis the coefficients are $\pi^{-i}b_i$. This proves the equivalence of the first and third assertions. Theorem 92.12 proves equivalence with the second one by a Vandermonde determinant that is a unit at $w$. For the norm statement choose an orthonormal positive basis in the directions of $u, v$ and the remaining coordinates. The normalized coordinate monomials form an orthonormal Fischer basis. Pullback scales a monomial by the product of the corresponding eigenvalues, one for each factor of its degree. The largest modulus on the full space is $T^{A/2}$, attained in the direction dual to $v$. On the plane only the eigenvalues $\pi^{-1}, 1$ occur and the largest modulus is one, attained in the coordinate dual to $e$. □

This identifies the precise lattice change underlying the factor $T^{-A}$ in the squared terminal evaluation. A polynomial integral on $\Lambda'$ passes the local test on the plane. For a global theta period one must compare the specified row polynomial and test vector at these two levels, including the normalization in their archimedean pairing. The rational isometry transports a lattice and a vector together. Keeping the prescribed period functional in that transport is a further arithmetic assertion. The uniform height of this period and the residual cohomology bound of Proposition 7.7 are still open.

## 94.7 The mass and the real terminal parameters

Use the common prime of Theorem 9.13, with the original split level primes $g_1, g_2$ and $Q = g_1 g_2$. In this calculation $F = E^+$, $f = [F : \mathbb{Q}]$, $m = 2d$, $s = m-1$ and $r = k+s$. Define

$$R = |\Delta_E|^{1/(2f)}, \qquad \omega = 2\pi/(RQ), \qquad L_j = L_F(j, \eta^j),$$

where $\eta$ is the quadratic character of $E/F$ and $\Delta_E$ the absolute field discriminant. For a list $v$ of distinct real entries write $\mathcal{V}(v)=\prod_{i<j}|v_i-v_j|$. For each selected word $c$, use the shifted highest weights $u_\tau(c)$ and $\lambda_\tau$ that occur in its two Weyl dimensions. Put

$$A_c=\left(\prod_{\tau:F\hookrightarrow\mathbb{R}}\frac{\mathcal{V}(\lambda_\tau)}{\mathcal{V}(u_\tau(c))}\right)^{1/(fk)}.$$

The entries here are the parameters in the Weyl product, including the shifts in Lemma 93.8.

**Lemma 94.13** (The factorials and discriminants in the adjacent masses)**.** *On the period fields one has*

$$\frac{\mathfrak{m}_{k+s}}{\mathfrak{m}_k}=\prod_{j=k+1}^{k+s}\left(\frac{(j-1)!\,R^{j-1/2}}{(2\pi)^j}\right)^f L_j.$$

*For $k\ge 1$ the product of the $L_j$ is positive and the absolute value of its logarithm is $O(f2^{-k})$, with an absolute constant independent of $s$ and the field.*

*Proof.* Inertia at each finite prime of $E/\mathbb{Q}$ fixes at least one of the two imaginary quadratic subfields of relatively prime discriminants $-q,-d_0$. Hence it cannot contain complex conjugation. This proves that $E/F$ is unramified at the finite places, and $|\Delta_E|=|\Delta_F|^2$. For even $j$ the relevant abelian factorization is that of $\zeta_F$, and for odd $j$ it is that of $\zeta_E/\zeta_F$. In each case it consists of $f$ primitive Dirichlet functions with parity $(-1)^j$. The product of their conductors is $|\Delta_F|$ in the even case, by the conductor–discriminant formula, and $|\Delta_E|/|\Delta_F|=|\Delta_F|$ in the odd case.

For a primitive factor of conductor $h$ and this parity, the absolute-value functional equation at $1-j$ supplies a factor

$$h^{j-1/2}\,\frac{2(j-1)!}{(2\pi)^j}$$

times the absolute value of the conjugate character's function at $j$. Indeed the quotient of the two archimedean factors is $|\Gamma_{\mathbb{R}}(j+a)/\Gamma_{\mathbb{R}}(1-j+a)|$, where $a\in\{0,1\}$ is $j \bmod 2$ and $\Gamma_{\mathbb{R}}(t)=\pi^{-t/2}\Gamma(t/2)$. The reflection and duplication formulas give exactly the displayed factor without $h^{j-1/2}$. Taking the product gives $R^{f(j-1/2)}$ from the conductors. Conjugation permutes the primitive factors. The Euler product at $j\ge 2$ is positive, since the quadratic character takes real values. Substitute in the mass formula of Lemma 93.7. Its $2^{-fs}$ cancels the factors $2^{fs}$. Finally $|\log L_j|\le f\log\zeta(j)\le 3f2^{-j}$. The first inequality follows prime by prime from the positive absolute-value series for the logarithm, as there are at most $f$ primes of $F$ above a rational prime. Summing the geometric series proves the bound. □

**Theorem 94.14** (Multiplicative estimates for the two real terminal parameters)**.** *For the selected words on the fields of Theorem 19.15, let $T$ be their actual common terminal prime and $T^{\#}=TQ^{m-3}$ its value after the change of frame. Uniformly for every selected $c$ and every allowed choice of the numbers $\Gamma_c$ in Theorem 9.13,*

$$T\,\omega^{m-2}=\frac{RQ^2}{2\pi}\,A_c(1+O_{\psi,\varepsilon}(x^{-2})),$$
$$T^{\#}\,\omega^{m-2}=\frac{RQ^{m-1}}{2\pi}\,A_c(1+O_{\psi,\varepsilon}(x^{-2})).$$

*These are the parameters of the original and transported real integrals for the same finite tests.*

*Proof.* Let $S=\sum_{j=k+1}^{k+s} j = ks + s(s+1)/2$ and put

$$Z=\prod_{j=k+1}^{k+s}\left(L_j\prod_{g\in\{g_1,g_2\}}(1-g^{-j})^{f}\right).$$

Then $|\log Z|\ll f2^{-k}$ by the lemma and $|\log(1-g^{-j})|\le 2g^{-j}$. At a split level prime the index ratio, per real place, is $g^k\prod_{j=k+1}^{k+s}(g^j-1)$, by the column count in Theorem 93.9. Thus its product contributes $Q^{f(k+S)}$ and the displayed local corrections. In rank $n$ the Weyl dimension is the positive Vandermonde of the shifted entries divided by $\prod_{i=1}^{n-1} i!$. In the ratio between ranks $k+s$ and $k$, the new denominators are $\prod_{j=k+1}^{k+s}(j-1)!$ at each real place. They cancel the factorials in Lemma 94.13 exactly. In the definition of $Y_c$ in Theorem 9.13, this leaves the identity

$$Y_c\omega^s = QA_c\,\Gamma_c^{-2/(fk)}Z^{1/(fk)}\left(\frac{Q}{2\pi}\right)^{s(s+1)/(2k)}R^{s^2/(2k)}.$$

Indeed the mass gives $R^{f(S-s/2)}(2\pi)^{-fS}$. Multiplying the $fk$-th root by $\omega^s$ gives exactly the two powers on the right.

The choice of prime gives $0<fk\log(T/Y_c)\le 3e^{-x^{1/4}}/8$. Also $0\le\log\Gamma_c\ll fm^5x^{\varepsilon}$, $\log Q=O(x^{\varepsilon})$ and $\log R\le\log N_x=O_{\psi,\varepsilon}(\log(2x)(1+\log\log(3x))^2)$ under either conductor bound. It follows that

$$\left|\log\frac{T\omega^s}{QA_c}\right|\ll_{\psi,\varepsilon}\frac{m^5x^{\varepsilon}+m^2\log(2RQ)+1}{k}=O_{\psi,\varepsilon}(x^{-2}).$$

For the last equality use $m\asymp x^{1/2-\varepsilon/16}$ and $k\asymp x^{9/2+10\varepsilon}$. Exponentiation preserves this relative error. Dividing by $\omega$ gives the first assertion, and multiplying by $Q^{m-3}$ gives the second, using the exact frame change in Theorem 93.11. □

The factor $A_c$ is determined by the word and the row. The integral against the designated projected theta vector still requires an estimate for its signed value at these parameters.

## 94.8 Changing the global theta character at the prescribed period

On the fields of Theorem 19.15, use the notation of Theorem 110.18. Thus $F=E^{+}$, $d=[E:\mathbb{Q}]=2f$, $\beta'=(d_0\gamma)^{-1}$, and $u=\beta'/\beta$ is totally positive. Both normalizations use the same skew form. The fractional ideal of $u$ is supported on split places. Write $v$ and $w$ for normalized additive valuations in $F$ and $E$, respectively, and write $Nv, Nw$ for residue field sizes.

**Theorem 94.15** (A norm representative with prescribed split components)**.** *Let $S$ consist of finitely many split places of $F$, including those where $v(u)\neq 0$. Above each $v\in S$ choose a prime $w_v$ of $E$ and a positive integer $a_v$. There are $\xi\in E^{\times}$ and one prime $P$ of $E$ such that*

$$\xi\bar{\xi}=u,\qquad \xi_{w_v}\equiv 1\pmod{w_v^{a_v}},\qquad (\xi)=\left(\prod_{v\in S}\bar{w}_v^{\,v(u)}\right)P\bar{P}^{-1}.$$

*The rational prime below $P$ splits completely in $E$ and can avoid any given finite set $U$ of rational primes. Enlarge $U$ to include the rational primes under $S$ and those ramified in $E$. Put $\mathfrak{m}=\prod_{v\in S}(w_v\bar{w}_v)^{a_v}$. Then*

$$\log NP\ll 1+\log|D_E|+\log N_{E/\mathbb{Q}}\mathfrak{m}+d\sum_{l\in U}\log(2dl)+d\log(2d).$$

*For the split test places and prime levels in Proposition 72.23, the logarithm of $NP$ and $\sum_{w<\infty}|w(\xi)|\log Nw$ are $O_{\psi,\varepsilon}((1+\max_{v\in S}a_v)x^{C})$ for a fixed $C$.*

*Proof.* At a split place every element is a norm. At an inert place $u$ is a unit and $E/F$ is unramified. The residue field norm is onto. Lift an element of the required residue norm to a unit upstairs. The quotient of $u$ by its norm is a principal unit and has a square root by Hensel's lemma, since all dyadic places split. This square is itself the norm of an element of $F_v$. At a real place positivity is exactly the local norm condition. Applying the Hasse–Minkowski theorem to $X^2 + d_0Y^2 - uZ^2$ now gives a nonzero rational zero over $F$. Here $Z \neq 0$ by total reality and $d_0 > 0$. Division by $Z$ gives $\xi_0 \in F(\sqrt{-d_0}) = E$ with norm $u$.

Multiply $\xi_0$ by $t/\bar{t}$ to achieve the required congruences. Indeed weak approximation in $E$ lets $t$ be arbitrarily close to $\xi_{0,w_v}^{-1}$ at $w_v$ and to one at $\bar{w}_v$, simultaneously for all $v \in S$. Call the result $\xi_1$ and put $J = \prod_{v\in S} \bar{w}_v^{v(u)}$. The ideal $(\xi_1)J^{-1}$ is trivial over $S$, and its conjugate is its inverse. It can therefore be written $A/\bar{A}$ for an integral ideal $A$ coprime to $\mathfrak{m}$. Take precisely the positive exponents of the fractional ideal. They occur in opposite pairs away from the conjugation-fixed primes.

Take the ray class of $A$ modulo $\mathfrak{m}$. For each $l \in U$ let $b_l$ be the least positive integer with $\varphi(l^{b_l}) > d$, and put $M = \prod_{l\in U} l^{b_l}$. Then $\log M \leq \sum_{l\in U} \log(2dl)$, by minimality. In the compositum over $E$ of its ray class field and $E(\zeta_M)$, lift the chosen ray class to an element of the abelian Galois group. Every prime above $l \in U$ ramifies in this extension, since the absolute ramification index of $\mathbb{Q}_l(\zeta_{l^{b_l}})$ exceeds $d$. The finite conductor norm of each character is at most $(N_{E/\mathbb{Q}}\mathfrak{m})M^d$. In fact a character extends from the subgroup of the product of the ray and cyclotomic groups to that product. The ray factor has conductor dividing $\mathfrak{m}$. The cyclotomic factor comes by norm from $\mathbb{Q}$, and the determinant of multiplication by $1 + l^{b_l}y$ is one modulo $l^{b_l}$ for integral $y$. This proves the asserted bound on its conductor.

The abelian case of Thorner and Zaman, Theorem 1.1, applied over $E$, gives an unramified prime $P$ of absolute residue degree one with this Frobenius and with the claimed bound on $\log NP$. It avoids $U$. Its rational prime therefore also has absolute ramification index one in the Galois field $E$, so it splits completely. The equality of ray classes gives $P = (s)A$ with $s \equiv 1 \bmod \mathfrak{m}$. Taking $\xi = \xi_1 s/\bar{s}$ proves the three identities.

Finally the disjoint supports give the exact finite sum

$$\sum_{w<\infty} |w(\xi)| \log Nw = \sum_{v\in S} |v(u)| \log Nv + 2\log NP.$$

The first sum is polynomially bounded by Proposition 110.19. For the stated test places the rational primes in $U$ may be taken from the conductor, the two original prime ideals and the fixed levels $g_1, g_2, T$. Their logarithms have polynomial sum by Proposition 72.23 and Lemma 87.1. Also $\sum_{v\in S} \log Nv \leq f \sum_{l\in U} \log l$, $\log|D_E| \leq d \log N_x$, and $d$ is polynomially bounded. These estimates prove the final assertion. At infinity the absolute values are fixed exactly by $|\sigma\xi|^2 = \tau u$ for $\sigma \mid \tau$. □

**Theorem 94.16** (The transported kernel at the same Bessel character)**.** *Start with the global splittings, Schwartz functions and Haar measures of a given convergent theta chain for $\psi_\beta$. Use the same groups and skew form, and use $\psi_{\beta'}$ in every oscillator. There is an explicit transport of the splittings and Schwartz functions for which each theta kernel agrees pointwise with its original kernel. In particular the iterated lift, its prescribed terminal Bessel coefficient and every fixed linear projection of that coefficient agree, including finite signed linear combinations using those transported data. The external character and all group measures in the period have their original values.*

*Proof.* On a tensor space write $B$ for its alternating trace pairing. Multiplication by $\xi$ commutes with both unitary groups and satisfies $B(\xi v, \xi w) = uB(v, w)$. For the Heisenberg group, whose

central term in the multiplication is $B(v,w)/2$, the map $(v,t) \mapsto (\xi v, ut)$ is consequently an automorphism defined over $F$. Pulling the old central character back gives exactly $\psi_\beta(ut) = \psi_{\beta'}(t)$.

Take a rational polarization of the original space with position space $X$. Pull both halves back by $\xi^{-1}$, writing $X' = \xi^{-1}X$, and transport the position measure. In the Schrödinger models the map from old functions to new functions is $D\phi(y) = \phi(\xi y)$, for $y \in X'(\mathbb{A}_F)$. Translation by $v$ in the new coordinates is translation by $\xi v$ in the old ones. Multiplication by a character from the other half gives the same identity by the pairing formula. Thus $D$ intertwines the two Heisenberg actions with the stated automorphism. Conjugating each old unitary Weil operator by $D$ gives a lift of the same group element for the new central character. These operators multiply as before, giving the transported splitting. For every pair $(g,h)$ the theta distribution now gives

$$\sum_{y \in X'(F)} (D\omega_\beta(g,h)\phi)(y) = \sum_{z \in X(F)} (\omega_\beta(g,h)\phi)(z).$$

This is a bijective change of rational summation variables. In particular it respects the rational splittings and automorphy of the kernel. Applying the identity on each edge gives identical functions before each successive integration. Integration against the original Bessel character, and then applying the same projection, preserves equality. If the character is written $\psi_\beta(\ell(n))$ for a linear functional on the unipotent quotient, its expression with the new additive character is $\psi_{\beta'}(\ell(n)/u)$. This specifies the frequency in the new notation even for a character with several nonzero coordinates. □

**Proposition 94.17** (The finite test at the one additional split place)**.** *Use the norm representative in Theorem 94.15 and the split position model at each split place, taking its position half at the chosen prime of $E$. Protect every prescribed nonspherical split test by choosing $a_v$ so scalar units congruent to one to that depth fix its position function. All such position functions can be retained under transport. At every other finite place except $v_* = P \cap F$ the standard lattice vacuum can also be retained. At $v_*$ the only change, up to unitary phase in the local model, is a central translate of the spherical test. On an input with one unitary central character at $v_*$, removing these translates on all edges changes the resulting lift by a scalar of absolute value one. The pointwise identity for a signed sum of different central characters uses the translated tests of the preceding theorem.*

*Proof.* Locally use the same position Haar measure for both split models and the corresponding dual measures in Fourier inversion. On the position half of dimension $n$ the unitary transport can then be taken to be

$$\phi(x) \longmapsto |\xi_w|_v^{n/2}\phi(\xi_w x).$$

On translations this gives the coordinate change by $\xi_w$. On characters the other coordinate scales by $\xi_{\bar{w}} = u/\xi_w$, giving the new additive character. The positive factor is fixed by the change of variable in the squared norm. Unitary identifications with other polarized models differ by a phase for the same Heisenberg intertwining relation, by irreducibility. At $v \in S$ the congruence on $\xi_w$ makes the displayed operator fix the test. Every Schwartz function admits such a depth, by compact support and local constancy. At another split place with $w(\xi) = 0$, scalar multiplication preserves any full integral lattice. At an inert place $\xi$ is a unit in the quadratic completion. It carries the inverse image of the same self-dual lattice in the Heisenberg group onto itself and pulls back its inducing character to the new one. In the lattice model this carries the vacuum normalized to have value one at the identity to the correspondingly normalized vacuum.

At $v_*$ take $w = P$, so $w(\xi) = 1$. In the split unitary group the central norm-one pair $(\xi_w, \xi_w^{-1})$, or its inverse according to the pullback convention, has this very action on the position half. Its Weil operator for the transported splitting is the displayed unitary dilation times a scalar of absolute value one. This follows directly from its linear action and its inverse dual action in the split polarization. Translating the Schwartz vector by that central element translates the source argument of the kernel. Changing variables in the source integral therefore extracts a value of the unitary central character or its inverse. At every next edge the lift still has a single central character there. Indeed a scalar on the source paired with the inverse scalar on the target acts trivially on the tensor space, hence by a scalar in the Weil representation. Moving one scalar through the integral gives the asserted target character. Iteration proves the claim.

For the rational models in Theorem 94.16 the summation formula fixes the global phase. In a separate local model the phase must travel with a summand if coefficients are added before taking an absolute value. At a real place $\xi$ acts by a positive dilation $\sqrt{\tau u}$ followed by a norm-one scalar. The transported real vectors must therefore also be included in that theorem. The size of the fixed-coordinate real test is given by Theorem 110.18. □

**Theorem 94.18** (The original finite tests at both endpoints of an even chain)**.** *Use the tensor convention and the intermediate determinant factors of equation* (70.6)*. Suppose the number $m$ of edges is even, as in Theorem 6.29. Fix one splitting and one set of unitary model identifications per edge. Allow a finite set of columns $c$, with homogeneous real Fock polynomials of degrees $d_{i,\tau,c}$ on edge $i$, and with finite-order intermediate determinant characters $\chi_{i,c}$. Use the inert lattice vacua and the good split lattice indicators of Proposition 94.17. Choose the norm representative to protect the other split test functions of all columns. Write $A_c$ for the original operator and $A'_c$ for the operator at $\psi_{\beta'}$ with the transported splittings, the same finite test shapes and the same real polynomials in fixed coordinates. Use unit-norm real vacua, the split position measures of Proposition 94.17, and the tensor product of unitary model identifications on each side. Then there is a single $C \in \mathbb{C}$ of modulus one, independent of the columns, such that, on every input for which the chain converges,*

$$A_c = C\, a_c \zeta_c A'_c, \qquad a_c = \prod_{\tau} (\tau u)^{D_{\tau,c}/2}, \qquad \zeta_c = \prod_{\substack{1 \le j < m \\ j \text{ odd}}} \chi_{j,c}(z)^{-n_j}, \qquad D_{\tau,c} = \sum_{i=1}^{m} d_{i,\tau,c}.$$

*Here $z$ is a norm-one idele specified below, $n_j$ is the rank of the intermediate group and every positive square root is taken in the chosen real embedding. This is an equality of functions on the original target. It consequently applies to the original terminal Bessel character and projection for arbitrary source vectors.*

*Proof.* Use the convention that a linear map $L$ on a position space acts on functions by pullback by $L^{-1}$, with its unitary modulus factor. At $v_*$ let the first component of $z$ be $\xi_P^{-1}$ and its other component $\xi_P$, and at each real place set $z_\tau = (\sigma\xi/|\sigma\xi|)^{-1}$ in the chosen complex embedding $\sigma \mid \tau$. Take $z_v = 1$ at the other finite places. The finite comparison is now exactly the central translation in Proposition 94.17, up to a phase independent of the test.

At infinity take a compatible complex structure invariant under the compact scalar circle, as in the Fock model of Proposition 104.3. Write multiplication by $\sigma\xi$ as a rotation $U_\tau$ in this circle times $\sqrt{\tau u}$. Pullback sends a homogeneous polynomial $P$ of degree $d'$ to $(\tau u)^{d'/2} P(U_\tau y)$. It is the unitary map between the two Fock norms since the Gaussian parameter is multiplied by $\tau u$ and the normalization of the vacuum cancels the change of real volume. Pullback by the rotation is the action of $z_\tau$ in the source center up to its vacuum phase. This phase concerns the representation and is independent of $P$. These unitary local intertwiners and the rational transport in Theorem 94.16 satisfy the same Heisenberg intertwining relation. Irreducibility

gives a single phase at each place, and at almost all places the identifications agree on the unit vacuum. Thus the transported kernel on edge $i$ equals its fixed-coordinate kernel at $(zg_{i-1}, g_i)$, multiplied by the positive degree factors on that edge and a phase independent of $c$.

For each odd edge $i$ replace the arguments $(zg_{i-1}, g_i)$ by $(g_{i-1}, zg_i)$. The scalar introduced is independent of the column since $(z, z^{-1})$ acts trivially on the symplectic tensor space and its lift is central. This is the kernel identity of Theorem 70.7. On each even edge the translation was already on its odd-indexed source. The translated arguments are now precisely all occurrences of $g_j$ with odd $j$. Since $m$ is even, they are all integration variables. Change variables to $g'_j = zg_j$ there. The quotient measures are invariant. The intermediate factor becomes $\chi_{j,c}(\det g'_j)\chi_{j,c}(z)^{-n_j}$. Neither endpoint is translated. Multiplication gives the displayed identity and the same overall phase $C$ for every column. □

**Corollary 94.19** (Valuations of the projected Gram minors)**.** *Apply the same complex-linear terminal map, for example the specified Bessel coefficient followed by its projection, to both operators in Theorem 94.18. For any finite set of columns and chosen inputs let $G, G'$ be the Gram matrices of the resulting vectors, using an inner product linear in the second argument. Put $D = \operatorname{diag}(a_c\zeta_c)$. Then $G = D^*G'D$. If the entries of one matrix are algebraic, so are those of the other. Choose a number field containing $F$, these entries, the scalars and their conjugates. Let $v_\mathfrak{l}$ be a finite valuation of this field. For every nonzero minor with row and column sets $I, J$ of equal size one has*

$$v_\mathfrak{l}(\det G_{I,J}) - v_\mathfrak{l}(\det G'_{I,J}) = \tfrac{1}{2}\sum_{\tau}\left(\sum_{c\in I} D_{\tau,c} + \sum_{c\in J} D_{\tau,c}\right) v_\mathfrak{l}(\tau u).$$

*The two minors vanish together. Away from the rational primes supporting $(u)$, the corresponding minors have identical valuations. In particular the additional split prime below $P$ has zero correction at every place above it.*

*Proof.* The common phase cancels in the inner product. Each $a_c$ is positive algebraic, since its square is the product of the indicated algebraic conjugates of $u$ to integer powers. Each $\zeta_c$ is a root of unity. Taking a minor of the matrix identity multiplies it by $\prod_{c\in I}\overline{a_c\zeta_c}\prod_{c\in J} a_c\zeta_c$. Its valuation gives the formula. All conjugates of $u$ are units away from the rational support of its ideal, which was excluded in the choice of $P$. □

For example, suppose the determinant tuples range over a finite group of twists and each $D_{\tau,c}$ is independent of $c$. Then $a_c = a$ is constant and $c \mapsto \zeta_c$ is a character $\zeta$ of that group. In a Fourier projection weighted by $\bar\eta(c)$, the identity reads $A_\eta = CaA'_{\eta\bar\zeta}$, since $\bar\eta(c)\zeta_c = \overline{(\eta\bar\zeta)(c)}$. For a general signed sum the individual weights $a_c\zeta_c$ must be kept. The minors and the norm transport above give an exact change of normalization. The required arithmetic lower bound for the projected sum and its comparison with the full global orbital contribution remain open in Section 6.6.

## 94.9 Integral restrictions on the relative scalar

**Proposition 94.20** (The scalar congruence when the isotropic flags meet)**.** *For a relevant, finitely supported pair in Theorem 110.9, put $a = D_t/Q^t$, taking $D_0 = 1$. Then*

$$a \in \mathcal{O}_E \setminus \{0\}, \qquad a\lambda \in \mathcal{O}_E, \qquad a(1-\lambda) \in Q\mathcal{O}_E.$$

*If $\lambda \neq 1$, necessarily $|N_{E/\mathbb{Q}}a|^{1/(2f)} \geq Q/2$. At $t = 0$ the only such label is the identity double coset for $Q > 2$. For every $t$, the ideal $\mathfrak{d}$ in Proposition 110.8 satisfies $N_{E/\mathbb{Q}}\mathfrak{d} \leq |N_{E/\mathbb{Q}}a|$.*

*Proof.* Choose local integral representatives of the two decorated frames at a finite place, as in Lemma 98.30. Their prefixes are primitive. Above $Q$ choose their reductions equal by upper unitriangular changes, as in Proposition 110.11. All prefix minors are unchanged by such choices. When $t > 0$ the integral matrix $B = (h(u_{0,i}, u_{1,j})/Q)_{i,j\le t}$ is nonsingular with determinant $a$. The entries of $y = (h(u_{0,i}, u_{1,t+1})/Q)_{i\le t}$ are integral for the same reason: above $Q$ both isotropic lists have the same reduction and at the other places $Q$ is a unit. Let $z = B^{-1}y$, so that $az$ is integral by the adjugate identity. The vector

$$v = u_{1,t+1} - \sum_{j\le t} z_j u_{1,j}$$

is the unique vector in $I \cap X_{1,t+1}$ of leading coefficient one in the second list. Indeed it is perpendicular to $X_{0,t}$; the prefix pairing is perfect and its kernel on $X_{1,t+1}$ is the one-dimensional intersection with $I$. Its leading coefficient in the first list is $\lambda$ by Theorem 110.9. At $t = 0$ take the empty system, $a = 1$ and $v = u_{1,1}$.

The vector $av$ is integral and belongs to $X_{0,t+1}$. By primitivity its coordinates in the first prefix are integral, so its last coefficient $a\lambda$ is integral. Above $Q$ the expression in the second list, reduced modulo the prime, has last coefficient $a$ in the first reduced prefix, since every $az_j$ is integral. Its reduced vectors are linearly independent; hence $a\lambda \equiv a$ there. This local argument is made in both factors at split places. It proves the global inclusions. If $\lambda \neq 1$, take the norm of $a(1-\lambda)$, whose absolute value is at most $2^{2f}|N_{E/\mathbb{Q}}a|$ and at least $Q^{2f}$. In particular for $t = 0$ one has $\lambda = 1$; the two whole decorated lists then agree by Theorem 110.9.

In the integral map of Proposition 110.8 the images of the first $t$ vectors on either side span sublattices of full rank in $U_i = (\Lambda \cap X_i)/(\Lambda \cap I)$. In these sublattices the determinant of the pairing divided entrywise by $Q$ equals $\bar{a}$ in the order of slots used there. Passing from $U_0, U_1$ to the sublattices adds two nonnegative local determinant valuations (with conjugation on the first slot). Thus the determinant valuation for the full lattices is no larger. The largest elementary divisor is bounded by this determinant valuation at every prime. Taking norms gives the last assertion. □

**Proposition 94.21** (An orbit attaining the scale on every initial column)**.** *For $s \ge 2$ the original finite test on every refined initial column admits one and the same relevant rational label with $t = 1$, $\lambda = 1$ and*

$$w_\gamma^{-1} = (1 + 16Q^2q)^{-2f}.$$

*Here $q$ is the auxiliary rational prime with $\sqrt{-q} \in E$. There is a compatible cell whose two oriented class coordinates agree and whose rational isometry in Corollary 110.7 is the identity.*

*Proof.* Use the first coordinate plane of Section 121.16 with $u = e_+ + e_-$, $v = (e_+ - e_-)/2$, $h(u, v) = 1$. Set $c = 4Q\sqrt{-q}$ and define $\gamma w = w + c\,vh(v, w)$. Since $h(v, v) = 0$ and $c + \bar{c} = 0$, expansion of the pairing shows $\gamma$ is an isometry with inverse obtained by negating $c$. It sends $u$ to $u + cv$ and fixes its orthogonal coordinate-plane complement. In the signed integral coordinate basis both denominators from $v$ divide 4 together, so $\gamma - 1$ has entries in $Q\mathcal{O}_E$. It therefore belongs to the marked level at $g_1, g_2$ and preserves the standard lattice everywhere else. At $T$ the element $a^0$ defining $K^0$ acts only on the middle space of the Witt decomposition and commutes with $\gamma$ by Theorem 86.1. Thus $\gamma_f \in K^0$.

The common isotropic subspace is spanned by $u_2, \dots, u_s$. The two terminal vectors coincide and the only nonzero leading minor is $h(u, u + cv) = c$. Both radicals fix their own first vector pointwise. An element of their intersection fixes both $u, u + cv$, hence their nondegenerate plane pointwise, and preserves its perpendicular. The remaining two lists on that perpendicular coincide. Thus every simple character coordinate agrees: the first is zero on both sides and the

remaining ones are read on the same vectors. The label is relevant and has the asserted weight since $|\tau c|^2 = 16Q^2q$ at every embedding.

Represent any refined class by $h \in H(\mathbb{A}_{F,f})$, where $H = U(W_0)$ fixes the first coordinate plane. Use $h$ as the finite cell representative. Both finite coordinates are in the required support since $\gamma_f^{-1}h = h\gamma_f^{-1}$ and $\gamma_f \in K^0$. They give exactly the same column representative. The common positive complement may be chosen to be $W_0$; both projections are the identity there since $\gamma$ fixes it. Thus the class factors may both be taken to be one, and the rational isometry in the trace is the identity. The assertion concerns finite support and character compatibility. The archimedean coefficient on this cell is still part of the period integral. □

## 94.10 The size of the complete column space

**Lemma 94.22** (Column dimension at the original period level)**.** *Fix a translate on an interval of Theorem 20.13 and use the covector refinements at its original prime $T$. Let $D_c$ be the full finite multiplicity space for word $c$ and $D_{\mathrm{all}} = \bigoplus_c D_c$ over the selected words. Let $D_*$ denote the real comparison dimension called $D$ in Theorem 9.13. For sufficiently large $x$, uniformly over the translates and selected data of that theorem,*

$$\log(2 + \dim D_{\mathrm{all}} + D_*) \le x^{40}.$$

*Proof.* Write $F = E^+$, $f = [F : \mathbb{Q}]$, $Q = g_1g_2$, $b = \exp(x^\varepsilon)$, and let $N$ be the cyclotomic modulus containing $E$. By Corollary 9.14, the dimension for one word is $\mathfrak{m}_k I_\pi (T^k-1)^f \prod_\tau W_k(u_\tau(c))$. Here $\mathfrak{m}_k$ is the mass of the definite genus, $I_\pi$ the column index at the two auxiliary primes, and $W_k$ the compact Weyl dimension. The mass is bounded by the number of full-level classes because each term is the reciprocal of a positive integer. The full-level count in the proof of Lemma 121.17, with underlying integral rank $2fk$, gives $\log^+ \mathfrak{m}_k \ll (2fk)^4 \log(4fkN)$. Counting all residue matrices gives $I_\pi \le Q^{fk^2}$. At every real place the shifted weights have spread at most $3a$ in the construction of Theorem 9.13, where $a = O(xb)$. The product formula for $W_k$ therefore gives $\log W_k \ll k^2 \log(2 + k + xb)$. Together these bounds give

$$\log(2 + \dim D_c) \ll 1 + (2fk)^4 \log(4fkN) + fk^2 \log Q + fk \log T + fk^2 \log(2 + k + xb).$$

The ranges $0 < \varepsilon < 1/52$, $f \ll_\psi x^{2+8\varepsilon}$, $k \ll x^{9/2+10\varepsilon}$ and $m \ll \sqrt{x}$ give $f \ll x^3$, $k \ll x^5$. On the same intervals $\log N = o(x^\varepsilon)$, $\log Q = (16 + o(1))x^\varepsilon$ and $\log T = (17 + o(1))mx^\varepsilon$ by Theorem 93.17. The last display is $O_{\psi,\varepsilon}(x^{33})$. There are at most $2^{k-2}$ words. Finally $\Gamma_c^2 \dim D_c/D_* = 1 + O(\exp(-x^{1/4}))$ by Corollary 9.14, with $\log \Gamma_c \ll fm^5x^\varepsilon$. These bounds imply the assertion after enlarging the initial scale. □

## 94.11 Denominators for compact unitary integration

In the compact period of Theorem 67.9 the Haar average acts on explicit polynomials in unitary matrix entries. The following denominator is independent of their coefficients and of the number of finite classes. Write $(h)_b = h(h+1)\cdots(h+b-1)$ with $(h)_0 = 1$.

**Proposition 94.23** (An integral multiple of every compact moment)**.** *Let $p(g, \bar g)$ be a polynomial in the entries and conjugate entries on $U(n)$, of degree at most $L$ in each set of variables. Let $I$ be the fractional ideal generated by its coefficients in a number field. With Haar probability $dg$, set*

$$\mathcal{D}(n, L) = \prod_{j=1}^{\min(n,L)} (n - j + 1)_{L-j+1}. \qquad \mathcal{D}(n, L) \int_{U(n)} p(g, \bar g)\, dg \ \in I.$$

*The same assertion on a product of unitary groups uses the product of the corresponding integers. Their prime factors on a given group are at most $n + L - 1$.*

*Proof.* It suffices to integrate a monomial. Right multiplication by diagonal unit complex numbers makes its integral zero unless, in each column $j$, the numbers of ordinary and conjugate entries are equal, say $k_j$. Right permutation of columns puts the $s$ occupied columns first. Thus $s \le \min(n, L)$, $k_j \ge 1$ for $j \le s$, and $\sum k_j \le L$.

Conditioned on the first $j-1$ columns $v_a$ of a Haar unitary matrix, column $j$ is a uniform unit vector $v$ in their orthogonal complement. This follows from unitary invariance and transitivity of the stabilizer on that sphere. Set $h = n - j + 1$ and $Q = 1 - \sum_{a<j} v_a v_a^{*}$. If $j$ is the last column left in a monomial and occurs in $b$ matched pairs, then its conditional integral is

$$\mathbb{E}\left[\prod_{\ell=1}^{b} v_{i_\ell}\overline{v_{t_\ell}} \;\middle|\; v_1, \dots, v_{j-1}\right] = \frac{1}{(h)_b} \sum_{\sigma \in S_b} \prod_{\ell=1}^{b} Q_{i_\ell, t_{\sigma(\ell)}}.$$

To prove the identity take a standard complex Gaussian in this $h$-dimensional subspace. Its squared radius is a sum of $h$ independent mean-one exponentials, with $b$-th moment $(h)_b$, by its generating function $(1-t)^{-h}$. Its angle is independent and uniform. The mixed moments of the Gaussian are the sum of products on the right before division. Indeed the coefficient obtained by differentiating its generating function $\exp(\sum_{ij} x_i Q_{ij} y_j)$ once for each labeled occurrence is exactly the sum over the matchings. Dividing by the radial moment proves the formula.

Integrate the occupied columns from $s$ down to 1. Expanding a product of $b$ entries of $Q$ creates at most $b$ matched pairs in preceding columns, with integer coefficients. The earlier original factors are still present. At step $j$, at least $j-1$ original pairs remain in the first $j-1$ columns, so the number $b$ of pairs at column $j$ satisfies $b \le L - j + 1$. Its denominator divides $(n-j+1)_{L-j+1}$. Multiplication of these common denominators proves the assertion, including $L = 0$. Successive integration proves the product assertion. The largest factor in any rising factorial is at most $n + L - 1$. □

For example, on $U(n)$ with $n \ge 2$ the degree-two bound is $n(n-1)(n+1)$. For local computations the exact cost of the displayed denominator at a rational prime $p$ is

$$v_p \mathcal{D}(n, L) = \sum_{j=1}^{\min(n,L)} \sum_{a \ge 1} \left( \left\lfloor \frac{n + L - 2j + 1}{p^a} \right\rfloor - \left\lfloor \frac{n - j}{p^a} \right\rfloor \right).$$

This follows by counting multiples of $p^a$ in each interval of factors. An alternative evaluation of unitary moments uses the character theory of the symmetric group. B. Collins and P. Śniady, *Integration with respect to Haar measure*.

## 94.12 The finite change of level on a theta lift

Keep the fields and intervals of Theorem 20.11 and the levels and terminal characters of Theorem 87.29. On the first edge of the indefinite tower write $w = r + t_{\mathrm{r}}$ for the rank of $Y_1$, so that $Y_0$ has rank $r$. At the $f$ places of $E^{+}$ above $T$, choose either of the two matrix realizations of this split pair. Use the sign $\epsilon_v = 1$ when the row domain is the factor in which $\varpi$ has valuation one and $-1$ when it is its dual. Position matrices $Y$ have $w$ rows and $r$ columns. Put

$$h_\theta = \frac{sw}{2} \sum_{v \mid T} \epsilon_v, \qquad h_B = \frac{f s (r - s)}{2}, \qquad d = T^{h_B + h_\theta}.$$

**Theorem 94.24** (Two projected periods evaluated on theta rows)**.** *Fix a Schwartz test $\Phi_0$ on the first theta pair, with arbitrary real vector and fixed splitting. Form $\Phi_1$ at each $v \mid T$ by the unscaled substitution $Y \mapsto Y a_{v,\epsilon_v}$ in $\Phi_0$, where $a_{v,+}$ is the action of $a_\varpi$ on the first split row space and $a_{v,-}$ is its inverse transpose on the dual. At finite places away from $T$ translate*

*the test by the row Weil action of $a_f$, and keep its real component fixed. These operations act linearly on the tensor factors. At $v \mid T$ the substitution gives a bijection of coefficient-valued Schwartz functions, preserving their ideals of values.*

*Let $F_i$ be the theta rows formed with these two tests and the same automorphic input on $U(Y_1)$. Whenever the lifts are realized by convergent integrals and the rows are in automorphic $L^2$, for every common refined class tuple $\ell$, every selected word and matching real Bessel vectors,*

$$\left|\langle F_1, \mathcal{P}_{1,1}(\ell)\rangle\right| = d\,\left|\langle F_0, \mathcal{P}_{\varpi,0}(\ell)\rangle\right|. \tag{94.9}$$

*This also applies when the input on $U(Y_1)$ is produced by earlier integrations in a convergent iterated lift. The number $d$ is positive rational. For every fixed $A > 0$ it satisfies on these intervals*

$$\log\big(\operatorname{num}(d)\operatorname{den}(d)\big) \le \tfrac{1}{2} f s(w + r - s)\log T = o(b/x^A).$$

*For the same choice of first factor at all places, $d$ is the integer $T^{fs(w+r-s)/2}$.*

*Proof.* In the stated matrix model the determinant of $a_{v,\epsilon_v}$ has valuation $s\epsilon_v$. The unitarily normalized row Weil action therefore multiplies the unscaled substitution by $\eta_v(a_v)T^{-sw\epsilon_v/2}$, with $|\eta_v(a_v)| = 1$, by equation 93.1. The unscaled substitution is a bijective linear change of variables on positions, giving equality of value sets and preservation of local constancy and compact support in both directions. Put $z = \prod_{v|T}\eta_v(a_v)$. In the global Weil representation these definitions give

$$\omega(a_f, 1)\Phi_0 = zT^{-h_\theta}\Phi_1.$$

To see the effect on the rows, write the theta kernel as the rational theta functional applied to $\omega(g,h)\Phi$. In a global position model this functional is the sum over the rational position vectors. For fixed $g, h$ the finite support conditions bound their denominators and the real Schwartz function decreases faster than any inverse power. This proves convergence of the sum. Changing the realization at finitely many places conjugates the action by its local intertwiners in this identity. Since $\omega(g,h)\omega(a_f,1) = \omega(ga_f, h)$, the two theta kernels agree pointwise after the same translation and scalar. Integrating against the given input yields $R(a_f)F_0 = zT^{-h_\theta}F_1$. This argument applies with any fixed input for which the displayed theta integrals exist.

The first identity of Theorem 87.29 gives $\mathcal{P}_{1,1} = T^{h_B}R(a_f)\mathcal{P}_{\varpi,0}$ on the same ordinary column inputs. Taking the indicated inner products and using unitarity of right translation gives (94.9), including zero values on both sides. By Theorem 100.16, $f$ is even. The sum of the signs therefore is even as well, so both exponents are integers. The sum of the logarithms of the numerator and denominator of $d$ in lowest terms is $|h_B + h_\theta|\log T$. The ranks, $f$ and $\log T$ are bounded by fixed powers of $x$ as in Corollary 100.17. This proves the claimed bound because $b = \exp(x^{\varepsilon})$. □

**Corollary 94.25** (Integral first-edge tests on the period fields)**.** *At each $v \mid T$ take either the lattice vacuum or the residue function with invertible top $r$-by-$r$ block and zero bottom $t_{\mathrm{r}}$-by-$r$ block. Use the lattice and its dual of Theorem 87.29 and form the orbit of the level-$K_1$ tests under $K_0$ at these places. In the ring of integers containing the residue character values, their $I = \begin{bmatrix} r \\ s \end{bmatrix}_T^f$ coordinates can all be recovered by evaluation up to roots of unity. The finite average with its determinant character has exact denominator $I$, which is a unit at $T$. For unit vectors in the finite Weil representation, their Gram matrix satisfies*

$$\|\mathcal{G}_T - 1\|_{\mathrm{op}} \le 64 f T^{-(t_{\mathrm{r}}+1)} = o(e^{-x^{1/4}}).$$

*If every place uses the residue function, $\mathcal{G}_T = 1$. The negative logarithm of the ratio of the product squared norm to that for the unnormalized vacua is $o(b/x^A)$ for every fixed $A > 0$.*

*Proof.* Theorem 93.1 constructs at each lattice a matrix that takes its basis to the first $r$ coordinate vectors of the partner. Evaluation on products of those matrices gives the claimed recovery. Proposition 93.3 computes the denominator by the same evaluations. The product estimate of Theorem 93.2 gives the stated norm bound by $(1+u)^f-1\le e^{fu}-1\le 2fu$ for $fu$ sufficiently small, with $u=32T^{-(t_{\mathrm{r}}+1)}$. Here Lemma 87.1 and the prime range of Theorem 9.13 give $\log T\ge(16-o(1))mx^{\varepsilon}$, with $m\asymp x^{1/2-\varepsilon/16}$ and $f$ polynomial in $x$. This yields the asserted decay. At a residue place the test matrix determines its own input lattice as the preimage of the integral partner lattice, which accounts for exact orthogonality. Equation 93.2 gives a cost at most $rt_{\mathrm{r}}\log T+\log 4$ per residue place by $\prod_{h=1}^{r}(1-T^{-h})\ge 1/4$. Multiplication over the places and the polynomial bounds used above give the final estimate. □

The equality in Theorem 94.24 concerns two values of actual theta rows when the tests are transported as stated. Corollary 94.25 recovers the integral coefficients before global summation. A sum over rational positions and projection can annihilate a nonzero test. To deduce (W) from these results one still needs the nonvanishing, arithmetic normalization and uniform estimates for the designated projected theta forms on the full chain, together with the remaining strict orders in (19.15).

## 94.13 Changing the terminal frequency by a rational isometry

Use the terminal space $S_m=C_m\perp U$ of Lemma 88.10, with $H=U(C_m)$ and $G=U(S_m)$. The staircase $U$ consists of $s=m/2-1$ hyperbolic pairs $b_j,b_j^{\vee}$ and a middle vector $z$ of norm $-1$. Let $N$ and $c_j$ denote the flag radical and its additive coordinates in that lemma. Normalize each of $[N]$ and $[H]$ to have mass one. For a character $\chi$ trivial on $N(F)$ put

$$F_{\chi}B(h)=\int_{[N]}B(nh)\overline{\chi(n)}\,dn,\qquad R(g)B(y)=B(yg).$$

Use the additive character $\Psi_E$ of Section 121.16. Write the character $\chi_+$ of the raw period as $\Psi_E(\sigma\sum_j c_j)$, with $\sigma\in E^{\times}$. Indeed the additive characters of $\mathbb{A}_E/E$ are precisely the rational multiples of $\Psi_E$, by adelic additive duality. This absorbs both the choice of sign and the scalar $\xi_0$ in Lemma 88.10.

**Lemma 94.26** (A change of frequency on the whole terminal quotient)**.** *At each point of the common intervals one can choose the common prime $T=\varpi\bar{\varpi}$ of Theorem 9.13 outside all the marked primes of the horizontal grid, for fixed grid data. On $G$ set $\chi_{\varpi}=\Psi_E(\sum_{j<s}c_j+\varpi c_s)$. There is $t\in G(F)$ centralizing $H$ for which*

$$F_{\chi_{\varpi}}(R(t)B)=F_{\chi_+}B$$

*as functions on $[H]$. The last vector defining $\chi_{\varpi}$ is $\varpi z$ and has norm $-T$. For the horizontal lifts of Theorem 89.8, the common function is fixed by $K_v=\mathrm{Stab}_{H(F_v)}(\Lambda_{C_m,v})$ at each $v\mid T$.*

*Proof.* The marked set lies above the two original level primes, the two type primes and a single auxiliary rational prime. This follows by taking the union of the primes of $g_1g_2p_1p_2$ with the one chosen place above $p_0$ in the product defining $\mathfrak{m}$. `Development/AbelianVarietyTheory/Definitions/Divisors/LeastSplitAuxiliaryPrimeThetaChainSpec.lean`, line 250–267. `Development/AbelianVarietyTheory/Definitions/Divisors/BasePlaceNotMemChiGridBadPlaces.lean`, line 70–82. The number of admissible common primes in Theorem 9.13 tends to infinity. Discarding at most five preserves its conclusion. Use this one choice for the comparisons at the point $x$.

Put $a_s=\sigma/\varpi$ and $a_j=\sigma a_{j+1}$ for $j<s$. Let $t$ send $b_j$ to $a_jb_j$ and $b_j^{\vee}$ to $\bar{a}_j^{-1}b_j^{\vee}$, fixing $C_m$ and $z$. It preserves all the pairings of these summands, so is a rational isometry. Conjugating $n$ by

$t$ multiplies $c_j(n)$ by $a_j/a_{j+1}$ for $j < s$, and $c_s(n)$ by $a_s$. It follows that $\chi_{\varpi}(tnt^{-1}) = \chi_+(n)$. By left rational invariance and commutation with $H$, $B(nht) = B(t^{-1}nt\,h)$. Conjugation gives a bijection of the rational unipotent quotient and preserves its probability measure, since it carries invariant measure to a multiple of itself. Changing variables proves the identity. Linearity in the last vector gives the assertion about its norm and character.

At the selected places the last horizontal test is the lattice vacuum by Proposition 88.15. It is fixed under the full target lattice stabilizer with the prescribed splitting. The terminal lattice is the orthogonal sum of the core lattice and the staircase ideals, so $K_v$ embeds into this stabilizer. Right translation by elements of $H(\mathbb{A}_F)$ commutes with the integral $F_{\chi_+}$ by its definition. This proves the last assertion. In particular a nontrivial character of a finite subgroup of $\prod_{v|T} K_v$ has zero projection on this function before changing its finite test. □

## 94.14 Embedding the original graph and its Bessel character

Take the common fields of Theorem 19.15 and use the unit CM ideal in Theorem 64.10. Fix the horizontal grid data and choose the common $T = \varpi\bar{\varpi}$ as in Lemma 94.26. Use the raw group $G = U(S_m)$, its core $H = U(C_m)$ and the simple coefficients $c_j$. Write $\chi_+ = \Psi_E(\sigma \sum_j c_j)$. In the absolute-trace normalization $\sigma = \xi_0$ up to the sign of the base character. It is a unit at $T$, since $\xi_0 = -1/\sqrt{-d_0}$ for the chosen CM ideal and $T$ splits completely in $E$. The staircase lines at $T$ have unit ideals. Indeed $\mathfrak{m}$ is a unit there and the ideal $\mathfrak{A}$ of Proposition 122.7 is supported only on the relative different for this choice. The staircase ideals are the products of these ideals and their conjugates in Proposition 121.4.

**Theorem 94.27** (A rational embedding with the prescribed terminal lattices)**.** *There is a rational isometric embedding of the original row space $S_0$ of Section 121.16, with definite column space $W_0$, into $S_m$. It maps $W_0$ into $C_m$. On the embedded small flag radical $N_0$, the restriction of $\chi_+$ is exactly $\chi_{\varpi}$. At each $v \mid T$ one can choose the core graph change $a_v$ of Proposition 88.6 so that the stabilizer of $a_v\Lambda_{S_m,v}$ intersects $U(S_0)(F_v)$ in the full stabilizer of the prescribed $\Lambda_{\ell_{0,v}}$ of Theorem 86.1. The subgroup $Z_v$ is central in $U(W_0)(F_v)$. It fixes the perpendicular of $W_0$ and permutes the graph lattices through their residue covectors.*

*Proof.* Inside the positive core embed the definite space $W_0 \perp Ev_0$ with $h(v_0, v_0) = T - 1$. To construct it, complete this space to the dimension of $C_m$ by a positive diagonal space of determinant $\det C_m/((T-1)\det W_0)$, using positive Gram representatives. There is room for this last coordinate. The new space and $C_m$ have the same determinant class and all the same signatures. Over a nonarchimedean place these invariants and the dimension classify a Hermitian space, and at the split places the dimension suffices. The local-to-global theorem identifies the two global spaces. B. Gross, Incoherent definite spaces and Shimura varieties, Section 3.

Move the embedding by a rational element of $H$ so that at every $v \mid T$ the prescribed lattice of $W_0$ and the unit-norm-ideal line $Ov_0$ are orthogonal unimodular summands of the core lattice. Such embeddings exist locally. At a split place a unitary isometry is an arbitrary invertible map on one factor and its inverse dual on the other. The Gram matrix on the desired $k+1$ coordinates is invertible over $O = \mathbb{Z}_T$, so coordinate columns for a direct summand and its corresponding dual rows realize this Gram matrix. Completing bases gives a local isometry carrying the rational embedding to this local one. The required integral summand condition is open. Weak approximation holds for $H$ at these places by its rational Cayley chart $X \mapsto (1-X)(1+X)^{-1}$ on skew-adjoint matrices. In each split local group the open chart is dense, since the vanishing of a nonzero determinant polynomial has empty interior. Approximation of the chart coordinates in the underlying $F$-vector space proves the simultaneous assertion.

Let $z$ be the raw middle vector of norm $-1$. Use the Hermitian convention linear in the second argument. Take

$$z' = \bar{\varpi} z + v_0, \qquad u'_j = \sigma^{\,j-s-1} b_j \quad (1 \le j \le s).$$

The norm of $z'$ is $-T + (T-1) = -1$, and $h(z', z) = -\varpi$. Thus $z = \varpi z' + z^{\perp}$ with $z^{\perp}$ perpendicular to the small row. Map its middle vector $e$ to $z'$, its flag generators to $u'_j$ and their duals to the inverse conjugate multiples of the raw duals. This is an isometry. For $j < s$ the small simple coordinate equals $\sigma c_j$. For $n \in N_0$ the extra perpendicular is fixed, so the coefficient of $b_s$ in $(n-1)z$ is $\varpi$ times that in $(n-1)z'$. Since $u'_s = \sigma^{-1} b_s$, the last term $\sigma c_s$ is $\varpi$ times the small last coordinate. This proves the character identity.

At $v \mid T$ use the first split factor, where $\varpi = 0 \bmod T$. The flag generators and their duals have been multiplied by units. The integral line $Oz'$ splits off orthogonally with unit norm. On the perpendicular of $W_0$ in the core use the primitive coordinate $y(X) = -h(v_0, X)$ in the first factor, extended by zero off the core. It is primitive since $h(v_0, v_0) = T - 1$ is a unit. Choose the other coordinates in its kernel and use the given covector $\ell_{0,v}$ on $W_0$ in the graph change $a_v$. On $z'$ this coordinate is $1 - T \equiv 1 \bmod T$. On the integral perpendicular of the whole small row it vanishes modulo $T$, because $h(z', X) = -\varpi X_z + h(v_0, X) = 0$ there. Consequently the new lattice defined by $y = \ell_{0,v}(w) \bmod T$ is precisely the direct sum of the required small graph lattice and the old perpendicular lattice. The conjugate split factor is their dual. Intersecting stabilizers gives the claim in both directions. The scalar action has the asserted block form. □

## 94.15 The raw period with the larger auxiliary spacing

Use Proposition 94.30 on the original Hermitian spaces, with the unit CM ideal and the terminal prime of Theorem 94.27. The bottom vector in the following theorem is any unit lowest-type vector in a given old cuspidal realization as in Theorem 71.10.

**Theorem 94.28** (Nonvanishing through the modified auxiliary core)**.** *There is a compact-finite horizontal lift $B$ of the given vector and an integral class coefficient $e$ of the modified terminal core type such that*

$$P_{\mathrm{raw}} = \int_{[H]} e(h) F_{\chi_+} B(h)\, dh \neq 0.$$

*At every marked place each horizontal test can be a single coset indicator at the original depth. The last test is spherical at every $v \mid T$. The assertion holds on each sufficiently distant common interval for the larger spacing $g$.*

*Proof.* First construct a partner for the bottom vector. The combined signed list of the two real parameters starts with $a'_0, \dots, a'_m$, continues with $v_1, v_1 - 1/2, \dots, v_{k-1}, v_{k-1} - 1/2$, and ends with $-b_1, \dots, -b_m$. The signs alternate in the first and last parts, and the middle consists of positive entries alternating between the two groups. The endpoints have exactly the same signs as in Theorem 71.10. Hence the real corank-one branching law gives the required discrete summand. The least internal gap in each parameter exceeds $24n^2 + 2$ as in that theorem. The added top gaps have only increased. The bound $T_0$ used in Theorems 106.2 and 71.6 is $10^4 n^4$ times one plus an upper bound on the absolute entries. Proposition 94.30 and the original bounds on the $b_j$ give $\log T_0 = O(\log b)$. Those two estimates give a vector $v$ for the branching embedding $j$ such that $d_S d_\sigma \mathcal{T}(v)/|\langle v, jw\rangle|^2 \le \exp(C f n^5 \log b)$. Here $w$ is the chosen bottom vector in the real model and the factor $d_S$ from compact projection satisfies $\log d_S = O(f n^2 \log T_0)$ by the Weyl formula. This is the ratio of the two terms in Theorem 89.3 before their level factors. The old auxiliary depth has $\log N\mathfrak{m} > C_{\mathrm{dep}} f^2 n^6 \log b$ by Theorem 89.2. It

dominates this cost and the lattice return factor $C_n^f$, for which $\log C_n < 6n + 9$. Consequently the nonidentity tail is less than half the identity term at a point above half the supremum of the given bottom vector, as in Theorem 71.10. The resulting Poincaré series is bounded in a cuspidal realization of the new core series, by Proposition 71.8 and Theorem 71.1. The first period of its compact translates is continuous on the real maximal compact group and is nonzero at the identity. Indeed the row is integrable on its finite-volume quotient by Cauchy–Schwarz and the compact translates are uniformly bounded. Some compact matrix coefficient has nonzero integral against this function by Peter–Weyl. Convolution with that coefficient therefore gives a compact-finite partner at the same finite level.

We check the lift through the even core for these data. On edge $i$ center at $c'_{m-i}$. The old entries are at distances at least $\min(g/2 - 1, g - (d - 1)/2)$, exceeding $(d + 3)/2$. Their signs and ordering relative to the center are those in Theorem 71.16. The complete new plus string in the reverse calculation has the $d$ centered entries from $(d-1)/2$ to $-(d-1)/2$. Thus the two real first-occurrence tests in Theorems 71.16 and 71.13 have identical counts and give the successive new parameters in the two directions. In particular any nonzero lift in either direction at these first occurrences is cuspidal. The local real Rallis pairings are nonzero on polynomial tests by the real doubling result used there.

At unmarked finite places the old spherical bottom list is twisted by a unitary character. Its factor at $(d + 1)/2$ is nonzero by the absolute logarithmic bound of Lemma 75.1. Each earlier added string gives the factors $L_E^{S_E}(a, \kappa_h/\kappa_i)$, $1 \le a \le d$, of Theorem 71.2, where $\kappa_i$ is the inverse horizontal character on edge $i$. Their ratios have nonzero angular exponent when $h \neq i$, since consecutive new centers differ by $g$. At one the nontrivial unitary Hecke value is nonzero, and at larger integers the Euler product converges absolutely. The denominators in doubling are at integers $d + j > 1$. The marked local integral is the positive scalar of Theorem 102.7, since the characters agree on finite units and the same rectangular coset is used. These verifications give exactly the local factors of the global inner product in Theorem 71.9. Its argument therefore gives for every nonzero bottom core vector a chain of polynomial tests with nonzero ascent, including vectors in a finite sum of cuspidal constituents by its mixed inner-product identity.

Adjunction now implies that the descents span each compact type at the bottom. In fact a vector orthogonal to all the descents would have zero inner product with its own ascent after pulling the kernels to the other side. The resulting quantity is the positive squared norm of a nonzero ascent. Projection to a bottom compact type preserves the span by equivariance of the first polynomial test. That type has finite multiplicity at the fixed cuspidal level. This gives the span assertion of Theorem 71.13. Apply the nonzero first period to the span. On the terminal compact quotient the integral class coefficients span its prescribed real type, by Proposition 95.4, for arbitrary integral highest weights. Linearity selects one coefficient $e$ and one chain of descents with nonzero first period.

It remains to transport this same coefficient to the horizontal row. In its quotient coordinates all old entries at edge $i$ lie below $c'_{m-i}$ by more than $(r+7)/2$. The cloud lies below $c'_m$, and the preceding centers are separated by $g$. The signed count in Theorem 74.1 gives first occurrence and cuspidality for each nonzero horizontal output. In the first two seesaws the same horizontal character is used on the even core and the horizontal row of each edge. The splitting quotient is therefore the prescribed parity character or its inverse, just as in Proposition 74.9. All finite test lattices and the Fourier expansion of the negative line are the same. The two kernel identities in Theorems 71.3 and 74.5 express the nonzero first period through the second restriction. Expansion in real Hermites converges in Schwartz space, and the marked expansion is finite. It follows that one selection of tests gives a nonzero second restriction with the given $e$.

At each ensuing odd index, Theorem 88.16 takes the nonzero Jacobi period to a nonzero

Bessel period. At an even index Theorem 88.14 takes it back to a Jacobi period on the next space. Their orbital formulas use the same lattices, additive character, rational frames and measures. On the residual line the splitting ratio is $(\gamma_i\eta^{-1})/\gamma_i = \eta^{-1}$ for the new horizontal character $\gamma_i$, as verified in Proposition 74.4. The nonzero value at the base of the orbit persists on a real neighborhood. The supported real test and the finite coset expansion in those theorems thus give one nonzero term at each step. Rapid decrease on cuspidal inputs justifies the integrations as in their proofs. This finite induction reaches $m$ with coefficient $e$. The rational sign change of Lemma 88.10 gives $P_{\mathrm{raw}}$. The tests are unchanged off the marked set up to the indicated unramified splittings. In particular the terminal prime is outside that set and its lattice vacuum is spherical. □

## 94.16 Compact branching and determinant characters

At a real place of $F = E^{+}$ the embedding of the definite column in the large core is conjugate to the coordinate inclusion $U(k) \subset U(N)$. Write $\lambda_1 \geq \cdots \geq \lambda_N$ and $\mu_1 \geq \cdots \geq \mu_k$ for integral highest weights. For the norm-one torus $D = U(1)$ of $E/F$, put $w_E = |\mu(E)|$, the number of roots of unity in $E$.

**Proposition 94.29** (Branching and twists unramified at every finite place)**.** *The type $\mu$ occurs in the restriction of type $\lambda$, followed by multiplication by $\det^{j}$, precisely when*

$$\lambda_{i+N-k} \;\leq\; \mu_i - j \;\leq\; \lambda_i \qquad (1 \leq i \leq k).$$

*For any integers $j_\tau \in w_E\mathbb{Z}$, one for each real place, the twists $\det^{j_\tau}$ are the real parts of one adelic determinant character of $G_0$. This character is unitary, trivial on rational points and unramified at every finite place.*

*Proof.* For adjacent ranks the branching rule follows from the Schur character formula. First shift the highest weight by a determinant power to make it a partition. Removing all boxes of the largest letter from a semistandard tableau leaves a horizontal strip as its complement. The remaining shapes interlace the original one and each of their tableaux extends uniquely for the given strip. Setting the last diagonal entry to one and comparing characters gives the branching rule, including the shift back by the determinant. Iterating the inequalities proves necessity for arbitrary ranks. For sufficiency write $b_i = \mu_i - j$. If $N > k$, make a row of length $N-1$ by putting $l_i = \max(\lambda_{i+1}, b_i)$ for $i \leq k$ and $l_i = \lambda_{i+1}$ otherwise. It is decreasing and lies between adjacent entries of $\lambda$. It satisfies $l_i \geq b_i$ and $l_{i+N-1-k} \leq b_i$. Indeed both the candidate $\lambda_{i+N-k}$ and any candidate $b_{i+N-1-k}$ at that position are at most $b_i$. Induction ends at a row of length $k$ forced to equal $b$.

Write $K_D$ for the product of the maximal finite compact subgroups of $D$. At split places the valuation identifies the quotient by the compact subgroup with $\mathbb{Z}$. At nonsplit places the norm-one group is compact. Hence $D(\mathbb{A}_f)/K_D$ is the group of fractional ideals of the form $J/\bar{J}$. Each principal $J = (a)$ gives the global point $a/\bar{a}$. The quotient by global points is therefore finite, being an image of the ideal class group of $E$. The rational points with finite components in $K_D$ are exactly the norm-one units. All their conjugates have absolute value one. Their powers have bounded integral characteristic-polynomial coefficients, which makes these units roots of unity.

On $D(F)D(F_\infty)K_D$ prescribe trivial values on $D(F)K_D$ and the character $(z_\tau) \mapsto \prod_\tau z_\tau^{j_\tau}$ at infinity. The unit calculation makes the prescription well defined. It is continuous because $D(F_\infty)K_D$ is open and the character is continuous there. Extend it through the finite abelian quotient one cyclic factor at a time. For a new generator whose $l$-th power is already in the subgroup, choose an $l$-th root of the prescribed value in the unit circle. This defines a continuous

unitary extension. Composition with determinant on $G_0$ gives the assertion. In particular it is trivial on the determinants of every finite compact subgroup of $G_0$, including the terminal graph stabilizers and the scalars $Z$. □

### 94.17 A larger spacing for the auxiliary compact type

Use the fields, bottom row, words and terminal prime of Theorem 83.10. Only the intermediate core weights and their splitting characters will vary. Write $g_*$ for the least auxiliary gap, $n = r + 1 = k + m$, $d = n + 4$ and $N = k + m(r + 6)$. Define integers

$$A = 1 + a + b_* + 8k\Delta + 2k, \qquad g = g_* + 2w_E \left\lceil \frac{8(A + N + w_E)}{2w_E} \right\rceil .$$

In the formulas of Theorem 71.10 use $a'_l = v_1 + 1/2 + (m+1-l)g$ and $c'_l = \lfloor (a'_{l-1} + a'_l)/2 \rfloor$. Keep the original $u_i, v_i, b_j, e_j$, with their place dependence. The source distinguishes the inequalities on the gap from the choice of their least solution. `Development/AbelianVarietyTheory/Definitions/Divisors/LeastSplitAuxiliaryPrimeThetaChainSpec.lean`, line 121–152.

**Proposition 94.30** (A common spacing for all real column weights)**.** *The enlarged auxiliary parameters admit horizontal splitting characters with the same restrictions to all finite unit groups as before. They leave the bottom-row parameter and the norm-T terminal frequency intact. After one determinant twist unramified at every finite place, the compact terminal type with the new parameters contains every prescribed word type on restriction to $\prod_\tau U(k)$, including at the high places. The same holds for its dual with the twist chosen in the dual convention. If $d_\rho$ is the dimension of either large type and $d_\mu$ the dimension of a word type, then on the common intervals*

$$\log(d_\rho d_\mu) \ \ll_{\psi,\varepsilon} \ fN^2 \log b, \qquad b = \exp(x^\varepsilon).$$

*Proof.* The difference of the two centers at $l$ is $(m + 3/2 - l)(g - g_*)$, an integer divisible by $w_E$. By the torus construction in Proposition 94.29 is a norm-one character with these differences as real exponents and trivial values on every finite compact subgroup. Pull it back from the norm-one torus by $z \mapsto z/\bar{z}$ and multiply the old horizontal Hecke character by this pullback. Its angular exponent changes by twice the stated difference, its restriction to the real subfield is trivial, and every local finite conductor and character on units is preserved. Use its inverse wherever the old inverse was used. The parity character of Proposition 71.18 also works for these choices. The bottom row uses only the cloud and the entries $-b_j$, and the Fourier character uses the rational Hermitian flag and the fixed additive character. Neither depends on $g$.

Let $\lambda$ be the highest weight of the new terminal compact type at one real place. The same string count as in Theorem 83.10 gives $\lambda_k - \lambda_{N-k+1} \geq (m-1)g - N$, also in dual coordinates. The interval with these endpoints has half-length greater than $A + w_E$ for $m \geq 4$. Every desired highest weight $\mu_i$ has absolute value less than $A$ by (9.1), including both endpoints at every high place. At each place choose a multiple $j_\tau$ of $w_E$ nearest the negative midpoint of the interval. Then all $\mu_i - j_\tau$ belong to it. Proposition 94.29 gives the branching assertion and the global character with these exponents. The same choice works for every word with the given orientation.

The old ranks, $g_*, w_E, k\Delta$ and $f$ are polynomially bounded in $x$. Since $b_* = O(xb)$, we have $g = O(xb)$ on all sufficiently distant intervals. Each new core parameter has absolute value at most $v_1 + 2 + (m+1)g + d$. Thus the logarithm of two plus the largest absolute entry, including the half-root shifts, is $O_{\psi,\varepsilon}(\log b)$ on both groups. In the Weyl dimension formula each factor for $U(l)$ is $1 + (\lambda_i - \lambda_j)/(j - i)$, $i < j$. Applying the logarithmic bound to these factors and summing over all real places proves the estimate. □

## 94.18 Finite levels for the restricted row

Take the fields, intervals, enlarged auxiliary weights and raw period of Theorem 94.28. Use the finite translate of Proposition 90.5. Let $S_{\mathrm{m}}$ be the marked set of the horizontal tests, disjoint from the places above $T$. Let $a_v$ be their prescribed depths, extended by zero outside $S_{\mathrm{m}}$. Choose $v_*$ above the marked odd split prime $g_1$. For a lattice $L$ write $K_L(0) = U(L)$, and let $K_L(b)$ for $b > 0$ be the kernel of reduction modulo $\pi_v^b$ on $U(L)$.

On $H = U(C)$ take the level $J$ of the translated core lattices with depths $a_v$ at the marked places, one at every $v \mid T$, and zero elsewhere. Its finite class number is $h_H$. Every stabilizer is trivial since it is finite and embeds at $v_*$ in a conjugate of a torsion-free odd principal congruence group. All class masses are consequently $1/h_H$, as in Proposition 95.4. Apply Theorem 90.8 with $\omega = 1/h_H$ and $d = d_\rho$ for the real type of the conjugate raw column. For several such types on the same finite data use their largest dimension in the formula for $j$. The bound for $c_Q^j$ then holds for each type separately. Write $j_v = j$ at $v = v_*$ and zero elsewhere.

**Theorem 94.31** (A restricted row at explicit finite levels)**.** *For every prescribed word $\mu$ and terminal character $\lambda$ of Theorem 90.7 is a restricted row $A$ on the original group, fixed by the original terminal graph groups, and fixed at $v \nmid T$ by $K_{\Lambda_{0,v}}(a_v + 2e_v + 2j_v)$. At those same places it is also fixed by the embedded column group $K_{\Lambda_{W,v}}(a_v + 2j_v)$. For $p(h) = F_0 A_\lambda(h, 1_f)$ its real type is $\mu$ and*

$$\|p\|_2 \;\geq\; \frac{\sqrt{|Z|}(1 - c_T)^{f/2}}{2\sqrt{2}} \sqrt{\frac{m_\nu}{d_\rho d_\mu}}\, M_B.$$

*Here $M_B = |P_{\mathrm{raw}}|/\|e\| > 0$ is calculated from the original raw tests for the enlarged spacing. The depths in the assertion are common to the choices of bottom vector, word and terminal character for the fixed grid and type.*

*Proof.* Translate the large row $B$ and its raw column $e$ on the right by the same finite $u$. Equivariance of the large Fourier map under $H$ and right invariance of Haar measure preserve the raw period and the norm of its column. Denote by $\widetilde{B}$ the projection of $R(u)B$ to real type $\rho$ under the definite group. This retains the raw pairing. The unramified vacua and marked coset tests of Theorem 94.28 are fixed by their full lattice groups and principal congruence groups, respectively, by Proposition 88.15 and matrix substitution on an integral marked coset. After translation these are the groups of $L_v$ in Proposition 90.5. The terminal vacua are the original ones.

Set $P = D_\lambda F_{\chi_+} \widetilde{B}$, with the graph change $a$ and the unnormalized finite character sum of Proposition 88.6. At $v \mid T$ each translated core lattice is the inverse image of a hyperplane modulo $T$. Its stabilizer contains the principal subgroup of depth one. Thus $P$ is fixed by $J$. The same compression identity and real nontriviality as in Theorem 90.7 give $\|P\|_2 \geq \sqrt{|Z|}(1 - c_T)^{f/2} M_B$. Let $\nu$ be the occurrence supplied for $\mu$ by Proposition 94.30. Theorem 90.8 gives $g = (s, D(\delta)t)$ such that the $\nu$-character integral of $P(hg)$ has modulus at least $\sqrt{m_\nu d_\mu/(2d_\rho)}\,\|P\|_2$.

Apply Proposition 90.3 at $hg$ to the large row $\sum_z \bar{\lambda}(z) R(za) \widetilde{B}$. Integration against the bounded character on the real compact group commutes with its box limit. Indeed the large unipotent quotient and the compact group together bound all values of the row occurring in the averages at the fixed $g$. The small compact group commutes with $n(x)$ in that formula. Some real $x$ therefore retains at least half the modulus of the character integral. Restrict $y \mapsto \widetilde{B}(yn(x)ga)$ to $U(S_0)$, project its right real translates to $\nu$, and apply the everywhere finitely unramified determinant twist of Proposition 94.30. These operations preserve smoothness and rational left invariance. The terminal scalars commute with $g$ because $D(\delta)$ centralizes the small row, and the other finite translate is supported at $v_*$. Thus the sum and small Fourier

integral of this row give the selected value at one, up to a scalar of modulus one. The twist turns $\nu$ into $\mu$, keeps compact finite levels, and is trivial on the unipotent. The real evaluation bound $d_\mu\|p\|_2$ from Theorem 90.7 now gives the displayed inequality.

At a terminal place conjugation of the small graph group by $g_v$ fixes it pointwise. Conjugation by the inverse local component of the graph change $a$ then places it in the original large lattice group by Theorem 94.27. At $v \nmid T$ put $L = L_v$ and $b = a_v + 2j_v$. If $k$ lies in the asserted small row group, the two inclusions of Proposition 90.5 give $(k-1)L \subset \pi_v^b L$. Indeed the displacement acts on the projection to $S_0$, loses at most $e_v$ on that projection, and loses at most $e_v$ on inclusion back into $L$. For $k \in K_{\Lambda_{W,v}}(b)$ the same inclusion costs no $e_v$, since the column splits off orthogonally in $L$. The inverse satisfies the same inclusion in both cases.

In conjugating $k$ through $g_v$ the determinant section cancels. The remaining $t_v$ and its inverse take $L$ into $\pi_v^{-j_v} L$, since they act in the core summand alone. Hence $t_v^{-1} k t_v$ belongs to $K_L(a_v)$, including when $a_v = 0$. It fixes the translated row there. The real operations do not change that calculation. □

# 95 Compact coefficients and projection norms

## 95.1 The rectangular compact type at the last hyperplane

For $1 \le r < s$ let $V_b$ be the span of ordered products $P_T$ of $b$ maximal minors of an $s$-by-$r$ matrix, with the Fischer inner product of Theorem 65.1. Use the coordinate hyperplane of the terminal inclusion in Proposition 92.1. Let $V_{b,l}$ be the span of the products in which exactly $l$ of the minors use row $s$. In particular their degree in that row is $l$. Write $A_0$ for the minor on rows $1, \dots, r$, and $A_1$ for the minor on rows $1, \dots, r-1, s$, both with rows in increasing order.

**Lemma 95.1** (The integral vectors in the restricted rectangular type)**.** *For $0 \le l \le b$ the spaces $V_{b,l}$ are pairwise orthogonal and irreducible under $U(s-1)$. In the covariant row convention their highest weights are $(b^{r-1}, b-l, 0, \dots, 0)$, where $b^{r-1}$ denotes $r-1$ repeated entries. Dualizing the row action or twisting by its vacuum character transports the whole decomposition. Put $Q_l = A_0^{b-l} A_1^l$. Then*

$$\|Q_l\|_{\mathrm{F}}^2 = \frac{C_{r,b}}{\binom{b}{l}}, \qquad \sum_{T:\,\deg_s P_T = l} |L(P_T)|^2 = C_{r,b}\|L\|^2 \quad (L \in V_{b,l}^\vee), \qquad C_{r,b} = \prod_{j=0}^{r-1} \frac{(b+j)!}{j!}.$$

*Here the dual norm is that of a complex linear functional and the ordered indices retain their multiplicities. On the full group $U(s)$ the corresponding space is $V_b$, with distinguished vector $A_0^b$ of squared norm $C_{r,b}$.*

*Proof.* The unitary translates of $A_0^b$ span $V_b$ by Theorem 65.1. Its diagonal-torus weight has multiplicity one. Indeed an indexed product using any row beyond $r$ has positive degree in some such row, and a product using only the first $r$ rows equals $A_0^b$. A unitary invariant orthogonal projection sends this cyclic vector to a scalar multiple of itself. As the projection is idempotent the scalar is zero or one, and cyclicity gives irreducibility. Additions of an earlier row to a later row in the first minor either leave its determinant fixed or change a row outside the minor. Thus its extremal weight, choosing the Borel for this convention, is $(b^r, 0, \dots, 0)$.

The Schur character of this rectangular representation is the sum of monomials of semistandard tableaux of shape $(b^r)$. In the alphabet $1, \dots, s$ all entries $s$ lie at the right end of the bottom row, since columns strictly increase and rows weakly increase. With $l$ entries $s$, deleting them gives exactly the tableaux of shape $(b^{r-1}, b-l)$ on $1, \dots, s-1$. Extension by those $l$ entries is inverse to deletion. Taking the coefficient of the $l$-th power of the last diagonal

variable in the character therefore gives one irreducible character of $U(s-1)$. The corresponding weight space for the last circle is precisely $V_{b,l}$. Distinct last-row degrees are orthogonal by circle invariance. This also covers $b = 0$.

A real unitary rotation in rows $r, s$ takes $A_0$ to $\cos\theta\, A_0 + \sin\theta\, A_1$. The terms of its $b$-th power with different $l$ are orthogonal. Its squared norm is constantly $C_{r,b}$ by the full-group calculation. Equating coefficients of $u^{b-l}v^l$, with $u = \cos^2\theta$ and $v = \sin^2\theta$, gives the first formula. In detail the homogeneous polynomial for the squared norm agrees with $C_{r,b}(u+v)^b$ on $u + v = 1$ in the positive quadrant, so the two homogeneous polynomials agree identically. For the second formula the full frame operator $\sum_T P_T P_T^*$ is $C_{r,b}$ times identity on $V_b$ by Theorem 65.1. Its restriction to the indicated orthogonal summand uses exactly the terms of degree $l$. □

For a list $b_\tau, l_\tau$ use the tensor products $V, Q$ of these spaces and vectors over the real places. Their frame constant and the squared norm of $Q$ are respectively

$$C_{\boldsymbol{b}} = \prod_\tau C_{r,b_\tau}, \qquad \|Q\|_{\mathrm{F}}^2 = C_{\boldsymbol{b}}/H, \qquad H = \prod_\tau \binom{b_\tau}{l_\tau}.$$

For the full target use $V = \bigotimes_\tau V_{b_\tau}$, $Q = \prod_\tau A_0^{b_\tau}$, and $H = 1$. In either case write $d_V = \dim_{\mathbb{C}} V$. It is the product of the dimensions of the stated highest weights. Vacuum characters have dimension one.

## 95.2 Compact matrix coefficients for an arbitrary partition

Let $n \le s$ and $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$ be integers. Write $e_j = \lambda_j - \lambda_{j+1}$, with $\lambda_{n+1} = 0$, and $d = |\lambda|$. For $m = n, s$ form

$$T_m = \bigotimes_{j=1}^{n} (\textstyle\bigwedge^j \mathbb{C}^m)^{\otimes e_j}.$$

Coordinate wedges are orthonormal. Let $t_m$ be the tensor of the first $j$ coordinate wedges in the indicated slots, and $V_m$ its unitary cyclic span in $T_m$. This is the Cartan component of highest weight $\lambda$, padded by zeros, as in Lemma 9.10. Its integral lattice here is its intersection with the coordinate tensor lattice. For complex inner products in this subsection use linearity in the first entry.

On $T_n \otimes T_s$ define a linear map $M$ to degree-$d$ matrix polynomials as follows. A coordinate tensor has subsets $J_a$ and $I_a$ of the same size in each matched slot. Its image is $\prod_a \det Z_{I_a,J_a}$, with the coordinates in each subset in increasing order. In particular $M(t_n \otimes t_s) = p_\lambda$, the product of leading minors in Lemma 92.4. Use the covariant actions on the tensor factors and on the polynomial coefficients. In an oscillator where matrices or their conjugates act contragrediently, pull back these actions along the corresponding compact group automorphism on both sides. Include the same vacuum character on the two sides.

**Proposition 95.2** (Integration against the two specified coefficients)**.** *Let $\rho_n, \rho_s$ denote the actions on $V_n, V_s$, including the respective vacuum characters, and $\omega$ the action on polynomials. Write $h_\lambda = \|p_\lambda\|_{\mathrm{F}}^2$ and $D_m = \dim V_m$. For $x, y \in V_n$, $x', y' \in V_s$ and a degree-$d$ polynomial $P$, the compact average for Haar probabilities is*

$$\int \overline{\langle \rho_n(g)x, y\rangle \langle \rho_s(h)x', y'\rangle}\, \omega(g,h)P \, dg\, dh$$
$$= \frac{\langle P, M(x \otimes x')\rangle_{\mathrm{F}}}{D_n D_s h_\lambda}\; M(y \otimes y').$$

*For $P = p_\lambda$ the numerator divided by $h_\lambda$ is $\langle t_n, x\rangle\langle t_s, x'\rangle$. In particular it is one on the two coordinate highest vectors. The displayed polynomial $M(y \otimes y')$ has integral coefficients whenever $y, y'$ lie in the stated integral lattices.*

*Proof.* Cauchy–Binet on every slot shows that $M$ intertwines the source and target actions. More explicitly, a change on a wedge acts by its minors, which are precisely the coefficients obtained by expanding that same minor after a matrix substitution. The irreducible tensor product $V_n \otimes V_s$ maps nontrivially since $p_\lambda$ has value one on the top identity frame. By the Cauchy decomposition in Lemma 104.6, its joint type has multiplicity one in the matrix polynomials. Equivalently the degree-$d$ polynomial representation is obtained by taking the simultaneous permutation invariants in the two $d$-fold tensor powers. Schur–Weyl duality and Schur's lemma pair each partition with itself once. The invariant positive form induced by $M$ on the two factors is a scalar times their tensor form, and its value on $t_n \otimes t_s$ gives that scalar $h_\lambda$.

On one irreducible unitary representation of dimension $D$, the integral of $\overline{\langle \rho(g)x, y\rangle}\rho(g)$ maps $z$ to $\langle z, x\rangle y/D$. Indeed averaging $\rho(g)zx^*\rho(g)^*$ gives $\langle z, x\rangle I/D$ by invariance and trace, and evaluation on $y$ gives the formula. Coefficients of inequivalent representations give zero by the same averaging of an intertwiner between them. Apply the two matrix units on the unique joint summand. Converting its tensor inner product to the Fischer inner product divides by $h_\lambda$, giving the assertion for any $P$. The formula for $p_\lambda$ follows by the same isometry. Each coordinate minor product has integer coefficients, so linearity in the coordinate tensor basis gives the integral assertion. □

## 95.3 The integral tests at the first seesaw

Fix one of the period fields $E = E_x$, let $F = E^+$, and use the first step of the ascent from the restricted column period. The two targets there are the initial row space $Y$ and the larger space $X = Y \perp L$, with $L = Ez$ a line. The bottom lattice in $X$ is $\Lambda_X = \iota\Lambda_Y \oplus \mathcal{O}_E z$. The source $S$ of the two oscillator spaces is the first core space. Its map into the first row step is an isometry of lattices: the extra hyperbolic summand at this step has rank zero, so the row lattice is exactly the image of $\Lambda_S$. These are the bottom lattice and the first row lattice in the standard frame. The identification uses no choice of a prime.

Here are the finite tests in coordinates. For $B = X, Y, L$, start with the additive lattice in $\mathrm{Hom}_E(S, B)(\mathbb{A}_{F,f})$ generated by the maps

$$v \longmapsto \alpha a\, t\, h_S(s, v), \qquad \alpha \in \widehat{\mathcal{O}}_F, \quad a \in \mathfrak{a}, \quad s \in \Lambda_S, \quad t \in \Lambda_B.$$

Here $h_S$ is linear in its second argument and $\mathfrak{a}$ is the fixed fractional ideal in the oscillator data. Impose in addition $eu \in (\mathfrak{m}\,\mathrm{Hom}_{\mathcal{O}_E}(\Lambda_S, \Lambda_B))^\wedge$ on a map $u$, for every idempotent $e$ prescribed by the grid. Such an $e$ selects the specified finite components above the split places. The hat denotes finite completion and $\mathfrak{m}$ is the original modulus. Denote the resulting lattice by $\Gamma_B$. On the line put $\Lambda_L = \mathcal{O}_E z$. Each $\Gamma_B$ is compact and open. Indeed the rank-one generators before the extra conditions span a full finite lattice: take a finite set of generators of the two integral lattices and of $\mathfrak{a}$, and use nondegeneracy of $h_S$. The additional congruences affect only finitely many places and contain a full open sublattice there.

Use a rational Schrödinger polarization respecting the sum $Y \perp L$, and the usual symmetric Weyl operators $R_B$ for the fixed additive character. The finite depth tests $\mathcal{V}_B$ are the Schwartz vectors fixed by $R_B(2u)$ for $u \in \Gamma_B$ and by $R_B(u)$ for $u \in \Gamma_B$ supported on one half of a split place. On $-L$ reverse the form and conjugate the oscillator. Let $\mathcal{V}_B^{\mathrm{int}}$ be the functions in this space taking values in $\overline{\mathbb{Z}}$, the ring of algebraic integers in $\mathbb{C}$.

**Proposition 95.3** (Integral factorization at the original depth)**.** *Under restriction to the two target summands, $\Gamma_X$ is $\Gamma_Y \oplus \Gamma_L$. The finite Schrödinger tensor map induces an isomorphism*

$$\mathcal{V}_Y^{\mathrm{int}} \otimes_{\overline{\mathbb{Z}}} \overline{\mathcal{V}_{-L}^{\mathrm{int}}} \;\simeq\; \mathcal{V}_X^{\mathrm{int}}.$$

*It carries product bases to an integral basis, including above* 2. *In each factor the coefficient of a basis function is obtained by evaluation at a finite position.*

*Proof.* The projection $X \to Y$ sends $\iota y + cz$ to $y$ and the projection $X \to L$ sends it to $cz$. They preserve the displayed integral lattices. They split each rank-one generator into the same two generators with projected target vectors. Conversely the two inclusions preserve all generators. The group of integral homomorphisms also splits by composing with the same projections. They commute with scalar multiplication by every $e$, so the congruences split with both inclusions. Precomposition with the isometry from the rank-zero step preserves the source lattice and its Hermitian form. This proves the first assertion for the precise first row and core spaces.

In the direct-sum polarization a Weyl displacement is the product of the two displacements, since their cross pairing is zero. A doubled displacement splits into independent doubled displacements. A displacement supported on one half of a split place projects to two displacements supported on that same half. Each separate displacement is also present by extension by zero. Thus the fixed equations on the two variables can be imposed separately. Complex conjugation reverses the phases on the line and identifies its fixed equations with those for the opposite form.

For integral coefficients one can solve these equations explicitly. In a single factor write the finite position and frequency spaces as $Z$ and $Z^\vee$. Choose compact open lattices $U \subset Z$ and $V \subset Z^\vee$ such that $U \oplus V \subset 2\Gamma_B$, $U$ annihilates the frequency projection of $\Gamma_B$, and $V$ annihilates its position projection. Existence follows by shrinking two compact open lattices inside the indicated open subgroups. The depth equations for $U, V$ give functions supported on $V^\perp$ and constant modulo $U$. The other Weyl operators preserve this finite coordinate space. A position displacement permutes its coordinates, and a frequency displacement multiplies them by roots of unity, including the phase of the symmetric Weyl operator. On each permutation orbit choose a representative. If a word fixing this representative has a nontrivial phase, its coordinate must be zero. On every other orbit propagate the value one at the representative by the fixed equations. This is well defined because two paths differ by a word fixing the representative. The functions so obtained have disjoint supports and root-of-unity entries. Evaluation recovers their coefficients over both $\mathbb{C}$ and $\overline{\mathbb{Z}}$. In the product the two systems of equations act on separate coordinates, so the product functions give exactly the same construction on pairs of surviving orbits. This proves the integral isomorphism. Keeping the doubled and split-half equations throughout covers the dyadic case. □

## 95.4 An integral terminal coefficient on one definite class

On a field of Theorem 71.13, put $G = G_m$, $K_f = K_{m,f}$, $K_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$ and let $h_m = \#(G(F)\backslash G(\mathbb{A}_{F,f})/K_f)$. The real group is compact. Let $\rho$ be the representation of $K_\infty$ of parameter $P_m$, of dimension $d_\rho$. At each selected complex embedding its highest weight is the decreasing Harish-Chandra parameter minus the ordered half sum of roots. Denote by $W$ the resulting algebraic representation at all these embeddings over a number field $J$ containing their images.

**Proposition 95.4** (A terminal form supported on one class)**.** *Fix one representative $t_c$ of each finite class. There are finite lists of terminal forms, depending on the field and the core lattices, which span $\mathcal{A}_m$. Each is supported on a single class $c$, where its real function is $k \mapsto \ell(\rho(k)v)$. Both $v$ and $\ell$ may be taken from generating sets of mutually dual integral coefficient lattices on*

*that class. With probability measure on the quotient its squared norm is*

$$\frac{\|v\|^2\|\ell\|_*^2}{h_m d_\rho}.$$

*For any bottom row vector in Theorem 71.13 one such terminal form gives its nonzero first period after a single descent using the stated monomials.*

*Proof.* The class set is finite by the finiteness theorem for definite Hermitian lattices, or by compactness of the anisotropic adelic quotient and openness of the level in the finite adeles. Let $\Lambda_c = E^{n_m} \cap t_c \widehat{\Lambda}_m$ in rational coordinates. A rational stabilizer of the class representative belongs to the automorphism group of this definite lattice. The integral endomorphisms form a discrete lattice in the product of the archimedean endomorphism spaces, and the isometries lie in a compact group there. Their intersection is finite. At the chosen split auxiliary prime the stabilizer is conjugate into $1 + p_0 \operatorname{Mat}_{n_m}(\mathbb{Z}_{p_0})$. This uses complete splitting and the positive congruence depth. The prime $p_0$ is odd by its defining congruence. This principal congruence group is torsion-free. Indeed, if $s \geq 1$ is the least entry valuation of $g - 1 \neq 0$, raising $g$ to a prime exponent $\ell$ leaves that valuation at $s$ for $\ell \neq p_0$ and raises it to $s + 1$ for $\ell = p_0$, by the binomial expansion. In the latter case all terms beyond the linear one have valuation at least $s + 2$. Iteration precludes finite order. Localization is injective on rational matrices, so the rational stabilizer is trivial.

The quotient at level $K_f$ is consequently a disjoint union of $h_m$ copies of $K_\infty$. In the quotient measure each copy has the same mass, since its finite representative translates the same compact open set and the stabilizers are trivial. Haar probability conditional on a copy is compact Haar probability. On one copy the right $\rho$-isotypic functions are exactly the span of its matrix coefficients by the Peter–Weyl theorem. Compact Schur orthogonality gives the stated squared norm on that copy and division by $h_m$ gives the global probability norm.

At each embedding take the integral Schur module of the lattice $\Lambda_c$ with the indicated highest weight, using the determinant twist and the image of the integral Young symmetrizer as in Theorem 70.1. Tensor these modules over the embeddings to obtain a projective $\mathcal{O}_J$-lattice $W_c$ spanning $W$. Choose finite module generators of $W_c$ and its integral dual. Over $\mathbb{C}$ the resulting coefficients span all matrix coefficients since the two lists span the representation and its full linear dual. The class stabilizer imposes no further conditions. In particular $\ell(v)$ at the identity is integral in these coefficient coordinates. In the descent of Theorem 71.13 expand its terminal form in this finite spanning list. With the tests and the row held fixed the period is linear in the terminal form, so at least one listed term has nonzero period. □

## 95.5 A norm estimate for the tensor projections

Use the tensor projections $P_i = \iota_i p_i$ of Proposition 89.6. The invariant form on the dual defining representation is indefinite. Summing the positive Hilbert norms of the projections therefore requires an estimate. The number of row entries below the interlacing part is $m = 2s + 2$.

**Lemma 95.5** (The sum of the positive Hilbert norms)**.** *On the fields and spacings of Section 86.7 let $U$ be the lowest compact type of the initial row. After $a \leq 4r^2$ unit lowerings, let $w$ be a compact-finite vector in the shifted row whose compact types occur in $(F^*)^{\otimes a} \otimes U$. For every Euclidean unit dual basis vector $\zeta$ one has*

$$\sum_i \|p_i(\zeta \otimes w)\|^2 \leq \exp(Cs \log b)\|w\|^2, \qquad b = \exp(x^\varepsilon).$$

*The constant is uniform in the prescribed word, place and shift. The compact types of the projected vectors occur in $(F^*)^{\otimes(a+1)} \otimes U$.*

*Proof.* We first determine enough of the signs $\epsilon_i$ to control the positive sum. They depend only on the fixed tensor form and the row. For this calculation choose an auxiliary compact column with decreasing half-integral parameters $u'_a$. Place $u'_1$ above $\mu_1$, and place $u'_a$ between $\mu_{a-1}$ and $\mu_a$ for $2 \le a \le k$, at distance greater than one from both adjacent entries. Choose $u'_1$ also at distance greater than one from $\mu_1$. These choices are possible because the row gaps after the allowed shifts tend to infinity. The signs of the row are unchanged. The packet calculation of Theorem 87.7 gives a nonzero Bessel model for this auxiliary column, since the order of the two lists, the chamber signs and their parities agree with those in its proof. This auxiliary choice is used only to determine the sign of the invariant form on a row summand.

For each $i$, the signed scalar of Proposition 105.12 for the auxiliary column is nonzero. The adjoint identity in Proposition 89.6 identifies its sign with $\epsilon_i$. For $i \le k-1$ the numerator product has $k-i$ negative factors and the denominator $r-i$. Consequently the signs of the first $k-1$ summands are the same. The multiplier $\eta$ is common and is real, since the signed scalar for any one summand is nonzero and real while the product quotient is nonzero and real. Call the common sign $\epsilon_0$, and let $J_-$ be the indices whose sign differs from it. Then $|J_-| \le r-k+1 = 2s+2$.

The split Casimir $C_0$ has eigenvalues $\theta_i = -\mu_i - (r-1)/2$, pairwise at distance at least one. Use the real polynomial

$$Q(X) = 2 \prod_{i \in J_-} (X - \theta_i)^2 - 1.$$

It equals $-1$ at the exceptional eigenvalues and is at least one at all other eigenvalues, also when the exceptional set is empty. Orthogonality for the invariant form gives, for $z = \zeta \otimes w$,

$$\sum_i \|p_i z\|^2 \le \epsilon_0 [Q(C_0)z, z] \le \|Q(C_0)z\|_{\mathrm{Euc}} \, \|z\|_{\mathrm{Euc}}.$$

Here $\|\cdot\|_{\mathrm{Euc}}$ is the positive tensor norm from the Euclidean norm on $F^*$ and the shifted Hilbert norm. The matrix of the invariant form in this norm is a unitary involution. The first inequality follows term by term since $\epsilon_0 \epsilon_i Q(\theta_i) \ge 1$.

We estimate the polynomial on $z$ using Lemma 121.25. On the lowest compact type $U$ the norms of all unit compact Lie directions are bounded by the parameter $L_*$ of Theorem 87.24. On the unitary compact tensor power $(F^*)^{\otimes a} \otimes U$ the bound is $L_* + a$, by the tensor differentiation rule. It holds on any compact span having only these types, since each compact type is unitary and compact isotypic summands are orthogonal. The square root of the absolute Casimir eigenvalue of the shifted row is at most $C\sqrt{r}\,(\max_j |\mu_j| + r)$, by the formula in Proposition 105.12. Together these furnish a parameter $L'$ in Lemma 121.25 with $\log(2+L') = O(\log b)$. Here $a, r$ are polynomial in $x$, and $L_* \ll \sqrt{r}(2b + r)$.

In the Euclidean Witt frame $C_0$ has entries $-E_{ij}$ acting on the row. Each complexified matrix unit is a sum of a real unitary Lie direction and $i$ times another one, both with Frobenius norm at most one. Indeed the real structure $X \mapsto -JX^*J$ preserves the Frobenius norm. Expanding $C_0^d z$ uses at most $(2r)^d$ vectors in single dual coordinates when $\zeta$ is a basis vector. Each is a word of $d$ such real derivatives applied to $w$. Thus for $d \le 4s+4$,

$$\|C_0^d z\|_{\mathrm{Euc}} \le \{Cr^2(L' + 4s + 5)\}^d \|w\|.$$

By expanding the product defining $Q$, its norm on $z$ is bounded by

$$\left[1 + 2\{Cr^2(L' + 4s + 5) + \max_i |\theta_i|\}^{\,2|J_-|}\right] \|w\|.$$

The logarithm of the bracket is $O(s \log b)$, giving the claimed estimate. Finally $p_i$ is compact-equivariant. The compact span of $\zeta \otimes w$ has exactly the required containment of compact types, which passes to its image under $p_i$. □

### 95.6 Prime exponents in compact unitary averages

For $h \geq 1$, $L \geq 0$, put

$$R_h(L) = \frac{\operatorname{lcm}(h, h+1, \dots, h+L)}{h}, \qquad R(n, L) = \prod_{h=1}^{n-1} R_h(L)^{\,n-h}.$$

Let $D(n, L) = \gcd(R(n, L), \mathcal{D}(n, L))$, where $\mathcal{D}(n, L)$ is the compact moment denominator of Proposition 94.23. Empty products are one.

**Theorem 95.6** (A logarithmic bound for each prime exponent)**.** *Suppose $P(g, \bar{g})$ has degree at most $L$ in the entries of $g$ and degree at most $L$ in their conjugates, with coefficients in a fractional ideal $I$ of a number field. For Haar probability on $U(n)$ one has*

$$D(n, L) \int P(g, \bar{g})\, dg \;\in I, \qquad v_p(D(n, L)) \leq \binom{n}{2} \log_p(n + L)$$

*at every prime $p$. For $n = 2$ the smallest positive integer clearing every such moment with integer coefficients is* $\operatorname{lcm}(1, \dots, L+1)$.

*Proof.* We give an integration formula in which every angular integration has integer coefficients. For dimension $s \geq 2$ start the first column at $e_1$. Successively act on coordinates $1, k$ for $2 \leq k \leq s$ by the block

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}, \qquad |z|^2 = T, \qquad |w|^2 = 1 - T.$$

In each block choose the phases of $z, w$ independently and uniformly on the circle. Choose $T$ with density $h t^{h-1}\, dt$ on $[0, 1]$, where $h = s - k + 1$. Use independent parameters for different blocks. The squared modulus in coordinate $k$ then receives the fraction $1 - T$ of the mass left in coordinate one.

The resulting first column is uniform on the complex unit sphere. Indeed the coordinate squared moduli of a vector of independent standard complex Gaussians are independent exponentials. Dividing by their sum gives uniform density on the simplex of masses. This follows by the change from the $s$ exponentials to their sum $r$ and their mass fractions, with Jacobian $r^{s-1}$. In a uniform simplex with $h + 1$ coordinates the sum $T$ of the remaining $h$ coordinates has density $h t^{h-1}$, and their normalized fractions are independent of $T$, by the same simplex change of variables. Iteration gives exactly the chosen masses. Given the phases of the $z$'s, each emitted coordinate has the independent uniform phase of its $w$ shifted by a fixed angle. These phases are independent of the $z$'s. The final phase in coordinate one is the product of the $z$-phases and is uniform. Thus the phases have the Gaussian angular law as well.

Multiply this product of blocks on the right by an independent $\operatorname{diag}(1, U)$, with $U$ Haar in dimension $s - 1$. For a fixed first column any two unitary completions differ on the right by its $U(s-1)$ stabilizer in these coordinates. The conditional law of the completion is therefore uniform on the fiber. Since the first column is uniform, the resulting law is left invariant, hence Haar. Recurse down to one uniform phase in dimension one. For the original $n$, the parameter with density $h t^{h-1}$ occurs once in each dimension $s = h + 1, \dots, n$. The independent beta angles in unitary integration also appear in P. Diaconis and P. J. Forrester, *A. Hurwitz and the origins of random matrix theory in mathematics*.

Substitute the product into a monomial of $P$. Each unitary entry has degree at most one in each individual block, so at most $2L$ factors of a given block occur in a term of the expansion.

Averaging its two phases gives zero unless both the $z, \bar{z}$ counts and the $w, \bar{w}$ counts match. Let the numbers of pairs be $a, b$, respectively. Then $a + b \le L$, and the radial average is

$$h \int_0^1 t^{h+a-1}(1-t)^b \, dt = \sum_{r=0}^{b} (-1)^r \binom{b}{r} \frac{h}{h+a+r}.$$

Its denominator divides $R_h(L)$ since $hR_h(L)$ is divisible by every $h + a + r$ in the sum. The final circle phase contributes zero or one. Independence and expansion with integer coefficients prove the assertion with $R(n, L)$. Proposition 94.23 proves it with $\mathcal{D}(n, L)$. An integral Bézout combination gives the assertion with their gcd, also for a fractional ideal. Each $v_p(R_h(L))$ is at most $\log_p(n + L)$, and there are $\binom{n}{2}$ blocks.

For $U(2)$, $R(2, L) = \operatorname{lcm}(1, \dots, L + 1)$. For each $0 \le j \le L$, the polynomial $|g_{11}|^{2j}$ has mean $1/(j+1)$ because $|g_{11}|^2$ is uniform on $[0, 1]$. Every common denominator must therefore contain the whole least common multiple. □

# 96 Whittaker normalizations and terminal restriction spaces

## 96.1 The terminal scalar in the normalized input

Keep the fields, the coded weights and the split generator $\pi \in \mathcal{O}_{K_d}$ of Theorem 9.13. Write $D_0 = t_{\mathrm{r}} + 1$ for the length of each appended block, $n_j = k + jD_0$, and $h$ for the number of high embeddings. Both $m$ and $D_0$ are even. Each embedding in the CM type has the same restriction to $K_d$, as in Section 6.6.1. We write $\pi, \bar{\pi}$ also for the images of its generators in the coefficient field $J$ under that common restriction. The terminal prime is $T = \pi\bar{\pi}$. Normalize $v_w(T) = 1$ at $w \mid T$.

We specify the finite normalization used after the Hecke operation. At a coefficient prime $w$ over $p$ let $e_w$ be the ramification index. If the prescribed total gain minus total error is $s_w$, the source is multiplied locally by $\mathfrak{p}_w^{-\lfloor e_w s_w \rfloor}$, where $\mathfrak{p}_w$ is the maximal ideal of the local integers. The sign comes from the inverse fractional ideal in `Formalism/SizeWindowBrackets/SlopesHypothesisFinitePlaces.lean`, line 27. The data giving $s_w$ are recorded in `Formalism/LocalToGlobal/GlobalTheory/IsResidueClassZero.lean`, line 441, and their use on the input in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1370. The following calculation gives the entire part supported over this word's own $T$.

**Theorem 96.1** (The terminal normalization on the coded family)**.** *The multiplier supported over $T$ is the principal ideal $(\pi^A)$ of $\mathcal{O}_J$, where*

$$A = m \left( 2 \sum_{\tau} e_{m,\tau} + fD_0 + \frac{hD_0(m+1)}{4} \right).$$

*Here $f$ is the number of embeddings in the CM type and $e_{j,\tau}$ are the prescribed count weights of (9.1). The nonnegative integer $A$ is the same for every word and translate at this $x$. If $S'_T$ is the lattice of Theorem 90.15, the input with all the prescribed changes at this $T$ and ordinary input elsewhere is exactly $\pi^A S'_T$.*

*Proof.* For an embedding $\tau$, let $W_\tau$ be the maximum absolute coordinate of its integral weight over all stages $0 \le i \le m$, omitting the first coordinate at each stage when $\tau$ is high. The terminal error per lift in the prescription is zero at $w \nmid \pi$, and at $w \mid \pi$ it is

$$E_T = 2 \sum_{\tau} W_\tau + fn_m + \frac{hD_0(m+1)}{4}.$$

This is the numerical rule in `Formalism/LocalToGlobal/GlobalTheory/IsResidueClassZero.lean`, line 394, lines 394–408 and 441–449 there. To evaluate the maximum, write $u_a - \rho_k(a)$ for an initial coordinate, where $\rho_k(a) = (k+1)/2 - a$. At stage $i$ it gains the shift $-iD_0/2$. The block introduced at step $j \le i$ has constant coordinate

$$-e_{j,\tau} + \frac{k}{2} + \frac{(2j-1-i)D_0}{2}.$$

Indeed at its introduction the unshifted block has value $-e_{j,\tau} + k/2 + (2j-1)D_0/2$ and the shift applies to every coordinate. These are the same lists as in Corollary 127.2 at the last stage. The minimum $B_\tau$ in the coded family grows faster than every fixed polynomial of $x$, by (9.1) and the choices following it. Thus all block coordinates are negative throughout the chosen intervals for large $x$. For fixed $j$ their absolute value is largest at $i = m$. Increasing $j$ there increases the absolute value by $2H_0 - D_0 > 0$. The largest block value in absolute value is consequently

$$e_{m,\tau} - k/2 - (m-1)D_0/2.$$

Every retained initial coordinate at a non-high embedding has absolute value bounded by a fixed polynomial in $x$, even after the stage shift. At a high embedding the same is true of the interior coordinates. The last initial coordinate there has absolute value at most $b_* + mD_0/2$, while $B_\tau - b_* \ge \lfloor b'/x \rfloor$ grows faster than any fixed polynomial. Hence the displayed block gives exactly $W_\tau$ at every embedding. Substitution with $n_m = k + mD_0$ gives $mE_T = A$.

It remains to identify which local terms contribute to $s_w$. The generator gives a prime split completely in $E_x$, different from $d, g_1, g_2$. At such a prime the count gain at each lift is zero since only the inert and signal classes carry this gain. Deleting primes from the gain leaves this value zero. The second arithmetic summand at a lift is also zero. It is supported on the two auxiliary primes, on the prime ramified in $K_d$, and on the specified inert or signal primes below the cutoff. The formulas are given in `Formalism/LocalToGlobal/GlobalTheory/IsResidueClassZero.lean`, line 333 and `Formalism/LocalToGlobal/Divisibility/Numbers.lean`, line 197. The terminal error above is subtracted once for each of the $m$ lifts. It follows that $s_w = -A$ above $\pi$ and $s_w = 0$ above $\bar\pi$.

The first two summands in $A$ are integers. The last is an integer since $mD_0$ is divisible by four. The prime ideals of $\pi, \bar\pi$ are coprime, and their product is $(T)$. Consequently $\operatorname{ord}_w(\pi) = e_w$ at $w \mid \pi$, and zero at every other place. The multiplier has exponent $e_w A$ above $\pi$, with zero at all other places. This proves the principal-ideal formula and the lattice identity. Finally $e_{m,\tau} = B_\tau + (2m-1)H_0$. The three sets of high, distinguished and remaining places have fixed cardinalities. Their preliminary $B_\tau$ values are fixed on each set, and the parity correction adds the same total zero or one at each translate. Therefore $\sum_\tau e_{m,\tau}$ is independent of the word and the translate. □

## 96.2 Whittaker coefficients on a prescribed pair of residue flags

Fix the field, the translate and the terminal marking of Section 6.6. We examine the two auxiliary primes in the finite factor of the projected Poincaré series. At the places of $E^+$ over $g_1 g_2$ put

$$G = \prod_v \mathrm{GL}_k(\mathbb{F}_{q_v}), \qquad U = \prod_v U_k(\mathbb{F}_{q_v}), \qquad \theta = \prod_v \theta_{k,v}, \qquad d = [G : U].$$

Here $U_k$ is upper unitriangular and $\theta_{k,v}$ is the character of its first superdiagonal from Section 92.6. Use counting measure. The matrix $A$ in (92.7) has diagonal one. For arrays $z_i$ on $G$ with values in a complex Hilbert space, write $Q(z) = \sum_{i,j} A_{ij} \langle z_i, z_j \rangle$, with the inner product linear in its second argument. The entries of $A$ are real and symmetric by Theorem 92.18.

**Lemma 96.2** (An integral matrix for the normalized finite pairing)**.** *Choose representatives $a$ for $G/U$ and $h$ for $U\backslash G$. Define the matrix $H$, with entries in the integers of a field containing the values of $\theta$, by*

$$(Hz)_{a,h} = \sum_{u \in U} \theta(u) z_{auh}.$$

*Then $H^* H = dA$. Changing representatives multiplies each output coordinate by a root of unity. These identities hold for any family of columns, including signed sums evaluated at fixed tests.*

*Proof.* For fixed $a$ the sets $aUh$ partition $G$. Two indices $i, j$ occur in the same summand of the squared norm just when $a^{-1} j i^{-1} a \in U$. If $i = auh$ and $j = au'h$, their coefficient is $\overline{\theta(u)}\theta(u') = \theta(a^{-1} j i^{-1} a)$. Sum over $aU$. The result is the induced character $\operatorname{Ind}_U^G \theta$ evaluated at $ji^{-1}$, by the finite induction formula. This element is conjugate to $i^{-1}j$. The character is real, and replacing its argument by the inverse conjugates its value. Dividing by its degree $d$ thus gives the entry of (92.7). Replacing $a, h$ by $au_0, u_1 h$ multiplies the coordinate by $\theta(u_0 u_1)^{-1}$ by a change of variable in $U$. Polarization gives the identity for all pairs of columns. □

Let $B$ be the product of the upper triangular subgroups. Fix at every place an input and an output complete flag. The set $S$ of tuples of matrices sending the former flags to the latter has the form $xBy^{-1}$. There is no restriction on the coefficients at different indices of $S$. Set

$$\beta = \prod_v \frac{(1 - q_v^{-1})^k}{\prod_{j=1}^k (1 - q_v^{-j})}, \qquad \alpha = \prod_v \prod_{j=1}^{k-1} (1 - q_v^{-j})^{-1}.$$

**Theorem 96.3** (Integral recovery and norms on a pair of flags)**.** *For arrays supported on $S$, a proportion $\beta$ of the choices $aU$ have the following property. The single row family $((Hz)_{a,h})_{Uh}$ recovers every $z_i$ by selecting an entry and multiplying it by a root of unity. The same selections work for all columns and all coefficient modules. In particular the map on this support has an integral left inverse at every coefficient prime. Its normalized form satisfies*

$$\beta \sum_{i \in S} \|z_i\|^2 \;\le\; Q(z) \;\le\; \alpha \sum_{i \in S} \|z_i\|^2.$$

*For any matrix of such columns the ideals generated by minors of each order before and after applying $H$ are equal over the indicated coefficient ring. Fractional entries are allowed.*

*Proof.* For the standard flag in one factor let its opposite be the flag of the reversed coordinate basis. Its stabilizer intersects the upper unitriangular group trivially. The flags opposite the standard flag are exactly the translates of this opposite flag by upper unitriangular matrices. Indeed in each successive subspace of an opposite flag there is a unique new basis vector with its reversed pivot normalized to one, its later coordinates zero after elimination, and with arbitrary entries above its pivot. These vectors give exactly one upper unitriangular matrix acting on the reversed basis. Thus their number is $q^{k(k-1)/2}$. All flags number $[\mathrm{GL}_k(\mathbb{F}_q) : B_k]$. Since $|\mathrm{GL}_k(\mathbb{F}_q)| = q^{k^2} \prod_{j=1}^k (1 - q^{-j})$ and $|B_k| = (q-1)^k q^{k(k-1)/2}$, division gives the local factor in $\beta$.

Select $a$ whose flag is opposite the prescribed output flag at every place. Each flag has the same number of lifts to $G/U$. Its unipotent group $aUa^{-1}$ meets $xBx^{-1}$ trivially: in opposite bases the two groups are respectively lower unitriangular and upper triangular, up to a simultaneous change of frame. If $i, j \in S$ occur in the same coordinate for this $a$, then $ji^{-1}$ lies in that intersection. Hence $i = j$. For each $i \in S$ its unique coordinate contributes exactly

$\theta(u)z_i$, and different indices use distinct coordinates. This gives the left inverse. Cauchy–Binet in both directions gives equality of the determinantal ideals.

Each selected $a$ contributes $\sum_{i\in S}\|z_i\|^2$ to the squared norm. Every other contribution is a squared norm as well. Use Lemma 96.2 and divide by $d$. The upper estimate is Proposition 92.22. □

**Corollary 96.4** (Several output flags at the marked tests)**.** *Fix the input flags, and split any signed array on $G$ according to the tuple $b$ of output flags, writing $z=\sum_b z_b$. With $S_2=\sum_b\|z_b\|^2$ one has*

$$0 \;\le\; \alpha S_2 - Q(z) \;\le\; (\alpha-\beta)\left(\sum_b \|z_b\|\right)^2.$$

*In particular for any family of columns supported on the same $h$ tuples of flags its normalized Gram matrix is bounded below by $(\alpha - h(\alpha-\beta))$ times its original Gram matrix. On the prescribed intervals*

$$0\le \alpha-\beta \ll fk(g_1^{-1}+g_2^{-1}), \qquad 1-\beta \ll fk(g_1^{-1}+g_2^{-1}).$$

*These estimates are uniform in the word, the terminal prime and the marking. For instance, for $h\le b_x^4$, where $b_x=\exp(x^{\varepsilon})$, the normalized form on this entire support differs from the original form by relative $O(fk(kb_x)^{-8}b_x^4)$.*

*Proof.* The positive operator $D=\alpha I-A$ has quadratic form on each individual flag support bounded above by $(\alpha-\beta)\|z_b\|^2$. Apply the triangle inequality to $\sum_b D^{1/2}z_b$ and square. Disjoint supports give $\|z\|^2=S_2$. Apply Cauchy–Schwarz to the sum of $h$ norms and test linear combinations of the columns. At a residue size $q\ge 3$, the upper factor has log at most $3/q$ by Section 92.7, and the lower factor has negative log at most $-k\log(1-q^{-1})\le 2k/q$. Each size occurs $f$ times and $g_i\ge (kb_x)^8$. The exponents tend uniformly to zero since $f,k$ have polynomial growth. The stated estimates follow by exponentiation. □

In the orbital expansion the array at index $i$ is obtained by summing the weighted amplitudes with that common relative residue matrix, with a separate component for each ordered pair of lattices. The definition of these three indices is in `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/TermOp.lean`, line 322. One may first hold the terminal data fixed, sum the amplitudes at each index and then apply the corollary. Absolute convergence, when an orbital series is used, permits this finite regrouping. For a single flag support the theorem recovers the resulting coefficients including their cancellations within each index. For the full prescribed period the distribution among output flags, the sums within the individual indices, and their relation to the integral theta lattice still enter the comparison in Section 6.6.

## 96.3 The integral image of the finite Whittaker pairing

We now allow every relative residue matrix at the two auxiliary primes of Section 96.2. Put $N=|G|$. Call an irreducible complex representation of a local factor $G_q=\mathrm{GL}_k(\mathbb{F}_q)$ generic if it contains the character $\theta_q$ on the upper unitriangular subgroup $U_q$. On the regular representation of $G$ let $P$ be the orthogonal projection onto the sum of the isotypic spaces whose factors at every auxiliary place are generic. It acts on the residue index alone, including for vector-valued arrays.

**Theorem 96.5** (The spectrum on all the relative residue matrices)**.** *In the notation of Lemma 96.2,*

$$\ker H=\ker P, \qquad \beta P \;\le\; A \;\le\; \alpha P.$$

*On each surviving isotypic space of $G$, of irreducible degree $d_\pi$, the scalar of $H^*H$ is $N/d_\pi$. In particular the relative norm estimate applies to arbitrary signed amplitudes after projection to $P$, with $\alpha - \beta \ll fk(g_1^{-1} + g_2^{-1})$ on the period interval.*

*Proof.* First work over one finite field. The intertwining algebra of $\mathrm{Ind}_{U_q}^{G_q} \theta_q$ is commutative. To see this, apply the anti-involution $g \mapsto w_0 g^{\mathsf{t}} w_0$, where $w_0$ reverses the basis. It preserves $U_q$ and $\theta_q$. By the double coset argument of Lemma 92.20, the functions in this algebra have the same character law on both sides and are supported on cosets with representatives $tw$ of the following form. The permutation maps consecutive intervals increasingly onto intervals in reverse block order, and the diagonal $t$ is constant on each image interval. Each nonzero block of the monomial matrix is therefore a scalar identity block. Reflection across the antidiagonal preserves each such block and reverses its entries along the block diagonal. It fixes $tw$. The two character laws show that the anti-involution fixes every function in the algebra. It reverses convolution by a change of summation variable. Thus convolution commutes. By complete reducibility the endomorphism algebra has one matrix factor of size the multiplicity of each constituent, so every constituent has multiplicity one.

The central convolution matrix $A_q$ acts on an irreducible $\pi$ by $|U_q| m_\pi / d_\pi$, by character orthogonality as in Proposition 92.22. Here $m_\pi$ is the multiplicity just computed. Conjugating $\theta_q$ leaves the set of constituents fixed since the inverse character is obtained by conjugation with a diagonal matrix. This also covers either regular action convention. The upper bound for this scalar is the local factor of $\alpha$, by that proposition.

For the lower bound we use Green's character degree formula [6]. Its parameters are partitions $\lambda_\phi$ at Frobenius orbits $\phi$ of degrees $e_\phi$, with $\sum_\phi e_\phi |\lambda_\phi| = k$. Use the indexing in which a one-row partition for the trivial degree-one orbit gives the trivial representation. Writing $n(\lambda) = \sum_{i \ge 1} (i-1)\lambda_i$ and $h(c)$ for the hook length of a cell, the degree is

$$d_\pi = \prod_{j=1}^{k} (q^j - 1) \prod_{\phi} \frac{q^{e_\phi n(\lambda_\phi)}}{\prod_{c \in \lambda_\phi} (q^{e_\phi h(c)} - 1)}.$$

In a partition the sum of the hooks is the number of cells plus $n(\lambda) + n(\lambda')$, by counting cells to the right and below. On extracting powers of $q$ we obtain

$$\frac{d_\pi}{|U_q|} = q^{-\sum_\phi e_\phi n(\lambda'_\phi)} \frac{\prod_{j=1}^{k} (1 - q^{-j})}{\prod_{\phi, c} (1 - q^{-e_\phi h(c)})} \;\le\; \frac{\prod_{j=1}^{k} (1 - q^{-j})}{(1 - q^{-1})^k}.$$

There are at most $k$ factors in the denominator. Taking the reciprocal gives the local factor of $\beta$ for each generic representation. Irreducibles, characters and the matrices of the finite pairings tensor over the places. Multiplication of the local bounds and of the nonzero scalars gives the assertion, since $[G : U]|U| = N$. The kernel identity follows by positivity. □

**Proposition 96.6** (The dimension discarded at the two auxiliary primes)**.** *On the same intervals, for all sufficiently large $x$,*

$$1 - \frac{\operatorname{rank} P}{N} \;\le\; 8 f (g_1^{-3} + g_2^{-3}).$$

*The rank here refers to the residue index with scalar coefficients.*

*Proof.* It suffices to work at $q \ge 4k$, as holds for both primes. For $k \ge 2$, the upper estimate on the eigenvalue of $A_q$ shows that a generic representation has degree at least $|U_q| \prod_{j<k} (1 - q^{-j}) \ge |U_q|(1 - 1/(q-1))$. If any $\lambda_\phi$ has two cells in the same row, the exponent $\sum e_\phi n(\lambda'_\phi)$ in the

preceding proof is at least one. Its degree is at most $2|U_q|/q$, since $(1-q^{-1})^{-k} \le \exp(k/(q-1)) \le 2$. Such a parameter is nongeneric.

To count the remaining parameters let $e_d$ be the number of degree-$d$ Frobenius orbits on the multiplicative character parameters. Frobenius to the $a$-th power fixes $q^a-1$ such parameters. Indeed, in the direct limit of the character groups under pullback by norms these are exactly the characters of order dividing $q^a-1$, a cyclic group of that size. It follows that $\sum_{d\mid a} d e_d = q^a - 1$. On taking logarithms of the Euler product, this identity gives

$$F(t) = \prod_{d\ge 1} (1-t^d)^{-e_d} = \frac{1-t}{1-qt}.$$

The parameters with all partitions columns are counted by $F(t)$. Its coefficient of $t^k$ is $(q-1)q^{k-1}$, the dimension of the intertwining algebra by Lemma 92.20. Multiplicity one identifies this dimension with the number of distinct generic representations. Since every generic parameter has columns, all the column parameters are generic.

Allowing arbitrary partitions gives the generating series $F(t)J(t)$, where $J(t) = \prod_{j\ge 2} F(t^j)$, by recording the multiplicities of parts of each size. The number of nongeneric parameters at degree $k$ is at most $q^k(J(1/q)-1)$: the coefficients of $J-1$ are nonnegative and the degree-$a$ coefficient of $F$ is at most $q^a$. The product for $J(1/q)$ telescopes to $(1-q^{-1})^{-1}$. The lost dimension in the regular representation is the sum of squared degrees of at most $q^k/(q-1)$ constituents, each of degree at most $2|U_q|/q$. Since $|G_q| = q^k|U_q|^2 \prod_{j\le k}(1-q^{-j})$, their fraction is at most $4/(q^2(q-2)) \le 8q^{-3}$. Rank fractions multiply in the product group. Subtract their product from one and use the sum of the local deficits. For $k=1$ every character is generic and the deficit is zero. □

**Theorem 96.7** (Integral comparison for arbitrary signed columns)**.** *Let $Z$ be any finite matrix of columns in the residue coordinates, with entries in a number field containing the values of $\theta$. A coefficient space with a fixed basis is allowed, with the identity on this basis understood in all operators. Write $\mathfrak{d}_j$ for the fractional ideal generated by the minors of size $j$, including the zero ideal in deficient rank. Then*

$$N^{2j}\mathfrak{d}_j(PZ) \;\subseteq\; \mathfrak{d}_j(HZ) \;\subseteq\; \mathfrak{d}_j(PZ).$$

*Away from $N$ the two ideals agree for every $j$. For independent projected columns the total logarithmic discrepancy of their maximal-minor ideals, normalized per column and by the field degree, is at most $2\log N$.*

*Proof.* For each complex irreducible use its central idempotent

$$e_\pi = \frac{d_\pi}{N} \sum_{g\in G} \chi_\pi(g^{-1}) g.$$

The sum over the surviving constituents is $P$. Their set is Galois invariant. Indeed a Galois conjugate of the inducing character at a place replaces its nontrivial additive character by a scalar multiple in the additive argument, obtainable by diagonal conjugation on the first superdiagonal. The same argument applies place by place. Character values are algebraic integers because the eigenvalues of finite-order operators are roots of unity. Consequently $NP$ and $N\sum_{\pi\ \text{surviving}} d_\pi e_\pi$ belong to $\mathbb{Z}[G]$, by integrality and Galois invariance of every coefficient. Put $C = N^{-1}\sum_{\pi\ \text{surviving}} d_\pi e_\pi$. The scalar calculation gives $CH^*H = P$ and $HP = H$. Thus $HZ = H(PZ)$ and $N^2 PZ = (N^2 C)H^*(HZ)$. All multiplying matrices on these right sides are integral, including $H^*$ since conjugates of the roots in $H$ are their inverses. Cauchy–Binet

gives the inclusions. This argument also works in local lattice bases with a common coefficient lattice on the residue indices. Taking ideal norms and using $|\mathrm{N}_{J/\mathbb{Q}}(N)| = N^{[J:\mathbb{Q}]}$ proves the logarithmic bound when the minor ideals are nonzero. □

**Corollary 96.8** (The same terminal primes as the strict groups)**.** *At each $x$ on the common intervals one can impose $T \nmid N$ in Corollary 9.34. There are still $Y_x^{1-o(1)}$ eligible choices, with full strict vanishing on all the subfields and in all the weights asserted there. For these choices the Whittaker and projected minor ideals in Theorem 96.7 agree at every place above the actual terminal prime $T$, simultaneously for all sets of columns. Over all coefficient primes the normalized logarithmic discrepancy is $o(b_x/x^A)$ for every fixed $A > 0$.*

*Proof.* Select the two auxiliary primes by Corollary 9.34 before selecting $T$. With $f$ factors of each residue size,

$$\log N \le fk^2 \log(g_1 g_2) \ll_{\psi,\varepsilon} x^C$$

for a fixed $C$, by the polynomial field and rank bounds and the auxiliary intervals. At most $\log N/\log 2$ more primes are excluded. The stated terminal window has $\log Y_x \gg mx^{\varepsilon}$, so this exclusion preserves its count in logarithmic order uniformly for the prescribed data. Apply the two inclusions above at $T$, where $N$ is a unit. At all places use $2\log N$ and $b_x = \exp(x^{\varepsilon})$. □

These results apply to the signed arrays in the original relative residue coordinates at a fixed marking. They give the same rank as the explicit projection $P$. The small dimension fraction lost by $P$ speaks to the full regular representation. A prescribed family of columns may lie in its kernel. The rank and height of the surviving arithmetic amplitudes and the remaining strict groups on the common interval are still required in Section 6.6.

## 96.4 Alternating sums of the signed amplitudes on complete flags

At each of the two auxiliary primes we extract from the pairing in Theorem 96.5 a set of integer linear forms in the signed amplitudes. They can be evaluated with the given terminal marking. Their coefficients are integers of absolute value one on the indicated cosets.

Work first with $G_q = \mathrm{GL}_k(\mathbb{F}_q)$. Write $X_q = G_q/B_q$ for the set of complete flags, with base flag $C_0$. A panel is a flag with one subspace of dimension between 1 and $k-1$ omitted. Call a function on $X_q$ harmonic if its sum over the $q+1$ complete flags on each panel is zero. For $u \in U_q$ write

$$v_u = \sum_{w \in \mathfrak{S}_k} (-1)^{\ell(w)} \delta_{uwC_0}.$$

This is the signed sum on the apartment given by the basis $ue_i$. Here $\ell(w)$ counts inversions. Use counting inner products on all finite sets. The relative permutation of $aC_0, bC_0$ is the unique $w$ with $a^{-1}b \in B_q w B_q$.

**Theorem 96.9** (The finite pairing on the complete flag space)**.** *The $v_u$ form an integral basis of the harmonic functions. In the space of functions on $G_q$ constant on each right $B_q$-coset, the local matrix $A_q$ of Section 92.6 acts as orthogonal projection onto the harmonic functions. In flag coordinates this projector has entries*

$$P_q(C, D) = \frac{|U_q|}{|X_q|} (-q)^{-\ell(w(C,D))}.$$

*In particular $|X_q| P_q$ has integer entries.*

*Proof.* Every apartment containing a chamber on a given panel contains exactly two such chambers, whose permutations differ by an adjacent exchange. Their signs cancel. With $w_0$ the reversing permutation, the opposite chambers are $uw_0C_0$, one per $u \in U_q$, as in Theorem 96.3. The restriction of the displayed family to these chambers is the identity matrix times the sign of $w_0$.

To see why restriction is injective on harmonic functions, take a chamber of position $w \neq w_0$ from $C_0$. Choose an ascent $w(j) < w(j+1)$. On removing dimension $j$, the other $q$ ways of filling that panel have position $ws_j$, of length $\ell(w)+1$. Indeed one may move the chamber by an upper triangular matrix to $wC_0$; in the two-dimensional quotient the other lines have a nonzero coefficient on $e_{w(j+1)}$ and an arbitrary coefficient on $e_{w(j)}$. Adding the latter coordinate to the former pivot is an upper triangular basis change. The zero panel sum therefore determines the value at the original chamber from larger lengths. Descending induction proves injection. Together with the opposite restrictions this proves the basis assertion over the integers, and the same argument works with coefficients in any abelian group.

Over $\mathbb{C}$ the harmonic space is an irreducible $G_q$-representation. For a $B_q$-invariant harmonic vector its values depend only on $w$, and the same panel equation reads $h(w) = -qh(ws_j)$ at an ascent. Thus its fixed space under $B_q$ has dimension at most one. Each irreducible summand of the nonzero harmonic space occurs in $\mathbb{C}[G_q/B_q]$ and has a nonzero $B_q$-fixed vector by finite group reciprocity. Complete reducibility gives the assertion. Its degree is $|U_q|$.

The $\theta_q$-space in all flag functions has dimension one. Indeed every $U_q$-orbit has a representative $wC_0$, by Bruhat decomposition and diagonal conjugation. If $w \neq w_0$ there is an $i$ with $w^{-1}(i) < w^{-1}(i+1)$. The simple root group $1 + tE_{i,i+1}$ fixes this chamber, and $\theta_q$ is nontrivial on it. Equivariance forces zero on its orbit. On the opposite orbit the action is free and transitive. A $\theta_q$-function there is determined by one value and can be extended by zero. This one-dimensional character space already occurs in the harmonic representation. In fact restriction to opposite chambers intertwines $U_q$ and is an isomorphism onto its regular representation by the integral basis just proved. It follows that the harmonic summand is the only generic occurrence in the flag space. The scalar formula of Theorem 96.5 gives one on it and zero on its orthogonal complement. Lifting flag values to all matrices in each coset multiplies their inner product by the fixed number $|B_q|$, so this is also the orthogonal projection for counting measure on flags.

Each diagonal entry of the projector is its rank divided by $|X_q|$, by transitivity. Its column at $C_0$ is invariant under $B_q$ and harmonic. Starting from its diagonal value, the ascent relation gives the displayed factor on every Bruhat cell. Translation proves the general formula; the inverse permutation has the same length. Finally $|U_q| = q^{\ell(w_0)}$ clears all powers of $q$ in this formula after multiplication by $|X_q|$. □

Return to the product of all auxiliary places, and use $X = \prod_v X_{q_v}$, $B = \prod_v B_{q_v}$ and the groups $G, U$ from Section 96.2. Let $J$ copy flag values to their right cosets in $G$. For $u = (u_v) \in U$, put $v_u = \bigotimes_v v_{u_v}$ and let $V$ be the matrix with these columns on $X$. Put $K = V^*V$. For any matrix $Z$ of signed arrays in the original relative residue coordinates define $M = V^*J^*Z$. Thus each coordinate of a column is an explicit alternating sum

$$M_u = \sum_{w \in \prod_v \mathfrak{S}_k} (-1)^{\sum_v \ell(w_v)} \sum_{i \in uwB} Z_i.$$

The inner sums can use the signed coefficients at the prescribed marking in Section 96.2. The formulas act on the residue coordinates and preserve any additional coefficient index. Write $C_u = (u_v w_0 C_{0,v})_v$.

**Theorem 96.10** (A lower bound at the fixed signed amplitudes)**.** *For any such $Z$ the Gram matrices satisfy*

$$Z^{*}AZ \;\geq\; \frac{1}{|B|}M^{*}K^{-1}M, \qquad (K^{-1})_{ut}=\prod_{v}P_{q_v}(C_{u,v},C_{t,v}).$$

*There is equality for columns constant on the right cosets of $B$. In particular for arbitrary columns one has*

$$Z^{*}AZ \;\geq\; \frac{M^{*}M}{|B|\,|U|\,(k!)^{2f}}.$$

*The logarithm of the denominator is polynomially bounded in $x$ on the same fields and intervals, uniformly in the translate.*

*Proof.* The orthogonal projection onto the tensor product of harmonic spaces is $VK^{-1}V^{*}=\bigotimes_{v}P_{q_v}$. On the opposite flags the rows of $V$ form the identity times a common sign. Restricting the projector to those rows and columns gives the claimed entries of $K^{-1}$. The columns of $JV$ lie in the eigenvalue-one space of $A$. Their Gram matrix is $|B|K$ since $J^{*}J=|B|I$. The projection onto their span is dominated by $A$, which is self-adjoint and positive on the orthogonal complement. Its compression by $Z$ is exactly the first right side. For a right $B$-invariant column, the tensor product form of the preceding theorem identifies the action of $A$ with this projection.

Each column of $V$ has $(k!)^{2f}$ entries of modulus one and all its other entries vanish. At each row there are at most $|U|$ nonzero entries. Cauchy–Schwarz row by row, followed by summing over the columns, gives $K\leq|U|(k!)^{2f}I$. Inverting gives the second estimate. The bounds on $f,k$ and the auxiliary primes in Corollary 96.8 give $\log(|B|\,|U|\,(k!)^{2f})\leq 2\log|G|+2fk\log(2k)\ll x^{C}$ for a fixed $C$. □

**Corollary 96.11** (Integral tests at the common terminal prime)**.** *Write $N=|G|$, $d=[G:U]$, and use the minor ideals from Theorem 96.7. For every $j$ one has*

$$d^{j}\mathfrak{d}_{j}(M)\;\subseteq\;\mathfrak{d}_{j}(HZ).$$

*If the columns are right $B$-invariant one also has $N^{j}\mathfrak{d}_{j}(HZ)\subseteq\mathfrak{d}_{j}(M)$. At every place over a terminal prime from Corollary 96.8, a unit minor of the alternating sums gives a unit minor of the same size in the Whittaker matrix of integral columns. For right $B$-invariant columns all the localized minor ideals agree there.*

*Proof.* Since $H^{*}H=dA$ and $A(JV)=JV$, one has $dM=(HJV)^{*}HZ$ with integral coefficients on the left of $HZ$. For invariant columns $H$ kills their component orthogonal to $JV$ in the flag space by Theorem 96.9 and positivity of $H^{*}H$. Hence

$$HZ=(HJV)(|B|K)^{-1}M.$$

The formula for $K^{-1}$ shows $|X|K^{-1}$ is integral. Multiplying by $|B||X|=N$ clears the matrix on the right. Cauchy–Binet proves both inclusions. At the selected $T$, both $N$ and $d$ are units. Over a local ring the ideal generated by a finite list of integral elements can contain a unit only if some element of the list is a unit, which gives the unit-minor assertion. □

The sign test sums all contributions at each residue index and uses the terminal prime at which the strict groups vanish. To apply it to the full period, one still has to prove that the prescribed columns have a nonzero alternating minor of the required size, or establish their independence in other generic representations. The arithmetic sizes of the surviving period coefficients also enter Section 6.6.

### 96.5 Rational isometries at the two auxiliary primes

Let $E = E_x$, $F = E^+$, $f = [F : \mathbb{Q}]$, and retain the two split primes $g_1, g_2$ and the matrix $H$ of Section 96.2. At each $v \mid g_1 g_2$ of $F$ choose the place of $E$ giving the residue coordinate in $G$. We examine the part of the finite pairing in which these coordinates come from maps between two fixed totally definite Hermitian $E$-spaces of dimension $k$. Write $b_x = \exp(x^{\varepsilon})$. The case $k = 1$ has $U = 1$ and each residue coordinate is read directly. We assume $k \geq 2$.

Fix lattices $\Lambda, \Lambda'$ in the two spaces. A rational isometry means an $E$-linear isometry $m$ between the spaces. At the chosen places we consider isometries inducing isomorphisms between the two local lattices. In fixed bases their residues define an index $i(m) \in G$. One can multiply all the residue matrices by the same invertible matrices on the left and on the right at each place. This permits changes of the two marked frames.

**Theorem 96.12** (Exact separation for bounded denominators)**.** *Let $h \geq 1$ be a rational integer prime to $g_1 g_2$. Use any finite set of distinct rational isometries satisfying*

$$h\Lambda' \subseteq m\Lambda \subseteq h^{-1}\Lambda'.$$

*If $4kh^2 < g_1 g_2$, their residue indices are distinct and the restriction of the normalized Gram matrix $A$ to them is the identity. For* every *fixed $aU \in G/U$, the coordinates $(Hz)_{a,t}$, $t \in U\backslash G$, recover every coefficient of an array on these indices by a selection of entries and multiplication by roots of unity. In particular, the ideals of minors of each size agree before and after the finite pairing, over every coefficient ring of integers containing the character values. This holds throughout the period intervals for every $h \leq b_x^8$ prime to $g_1 g_2$.*

*Proof.* Consider distinct maps $m, n$ and put $\gamma = mn^{-1}$ on the target space. Define

$$u = 2k - \mathrm{Tr}_E(\gamma) - \mathrm{Tr}_E(\gamma^{-1}) = \mathrm{Tr}_E\left((1-\gamma)(1-\gamma^{-1})\right),$$

where the trace is the matrix trace over $E$. The adjoint of $\gamma$ is $\gamma^{-1}$, so $u \in F$. At each embedding of $F$ choose an orthonormal complex basis. The image of $u$ is the squared Hilbert–Schmidt norm of $1 - \gamma$, the sum of the squared absolute values of its entries. It is positive because $\gamma \neq 1$ and the embedding is injective. Each eigenvalue of $\gamma$ has absolute value one, so this norm squared is at most $4k$.

The lattice inclusions imply $h^2\gamma\Lambda' \subseteq \Lambda'$ and $h^2\gamma^{-1}\Lambda' \subseteq \Lambda'$. For instance $n^{-1}\Lambda' \subseteq h^{-1}\Lambda$, and $m\Lambda \subseteq h^{-1}\Lambda'$. Traces of lattice endomorphisms are integral, as one sees in a basis at each finite place. Thus $h^2 u$ is a nonzero integer of $F$ and

$$1 \leq |N_{F/\mathbb{Q}}(h^2 u)| \leq (4kh^2)^f.$$

Suppose the relative residue matrix $i(m)i(n)^{-1}$ were unipotent at every marked place. In the chosen bases it is the reduction of $\gamma$ up to conjugacy, including after the two common frame changes. Its trace and the trace of its inverse are both $k$ in the residue field. Hence $u$ reduces to zero at every $v \mid g_1 g_2$ of $F$. Since the primes split completely, $(g_1 g_2)^f$ divides the displayed absolute norm, contrary to the bound.

There is therefore a nonunipotent local relative matrix for every distinct pair. For fixed $a$, two entries of an input array occur in the same output coordinate precisely when their relative matrix belongs to $aUa^{-1}$, by Lemma 96.2. This would require unipotence at all places. Each input entry consequently occurs by itself, multiplied by a root of unity, and distinct inputs have distinct indices. In this row family the squared norm is exactly the sum of the input squared norms. Averaging over $aU$ and polarizing gives the identity Gram matrix. The entry selections give an integral left inverse. Applying Cauchy–Binet to both maps gives the assertion on minors, also for several columns at once. Finally $g_i \geq (kb_x)^8$ on these intervals, and $4kb_x^{16} < k^{16} b_x^{16}$ for $k \geq 2$. □

In particular the full set of isometries of one definite integral lattice onto another is covered with $h = 1$, independently of the orders of their stabilizers. Such a set is finite. Indeed the images of a fixed $\mathbb{Z}$-basis lie in a discrete lattice and in a bounded set for the sum of the positive Hermitian norms at infinity. The same argument applies with target $h^{-1}\Lambda'$. An individual residue index in the period may already be a signed sum. The theorem recovers that sum.

**Lemma 96.13** (The largest probability of a flag collision)**.** *Put $[j]_q = (q^j - 1)/(q - 1)$. For a nonidentity unipotent matrix in $\mathrm{GL}_k(\mathbb{F}_q)$, the proportion of complete flags it fixes is at most*

$$\rho_k(q) = \frac{\sum_{s=0}^{k-2}(s+1)q^s}{[k]_q[k-1]_q} \;\le\; \frac{k-1}{[k]_q}.$$

*Equality in the first bound holds for a transvection.*

*Proof.* Write the matrix as $1 + N$ with $N$ nilpotent. A fixed flag starts with a line in $\ker N$, and its completion is a fixed flag for the induced nilpotent on the quotient by that line. Let $B_j$ be the number of flags fixed by a transvection in dimension $j$, and $D_j$ the total flag count. In dimension two a nonzero nilpotent has rank one and fixes just its kernel line. Assume in dimension $k - 1 \ge 2$ that $B_{k-1}$ is the maximum for nonzero nilpotents. For rank at least two in dimension $k$ there are at most $[k-2]_q$ choices of line and the quotient nilpotent is always nonzero. A rank-one nilpotent allows $[k-1]_q$ lines, including its image line. The quotient is zero at that line and rank one at every other. Thus $B_k = D_{k-1} + ([k-1]_q - 1)B_{k-1}$, which dominates $[k-2]_q B_{k-1}$. This proves maximality by induction.

For rank one put $L = \operatorname{im} N$ and $K = \ker N$. The line lies in the hyperplane. For a fixed flag $F_i$, let $i$ be the first step containing $L$, and $j$ the first step outside $K$. Stability means $i < j$. Indeed $NF_j = L$, and $F_j \cap K = F_{j-1}$. Conversely $L \subseteq F_{j-1}$ makes every step stable. Relative to the fixed flag there are $q^{i-1}$ lines at the indicated step and $q^{k-j}$ hyperplanes first missed at step $j$, by normalizing the first nonzero coefficient of their defining functional. For $i < j$ incidence is automatic. Summing gives $\sum_{s=0}^{k-2}(s+1)q^s$ such pairs $(L, K)$. There are $[k]_q[k-1]_q$ incident pairs altogether. Transitivity on flags and on incident pairs permits counting in either order. Bounding each $s + 1$ by $k - 1$ gives the last inequality. □

**Proposition 96.14** (The number of places distinguishing rational maps)**.** *Retain the two spaces, and require the rational isometries to be integral isomorphisms at the marked places. For a distinct pair, use $u$ from the proof of Theorem 96.12. Let $\mathfrak{d}(u)$ be its denominator ideal in $F$, and put*

$$W = f \log(g_1 g_2), \qquad B_{mn} = f \log(4k) + \log N_{F/\mathbb{Q}}\mathfrak{d}(u).$$

*If the relative residues are unipotent everywhere, then*

$$B_{mn} \ge W, \qquad \sum_{\substack{v \mid g_1 g_2 \\ i(m)_v \ne i(n)_v}} \log q_v \;\ge\; 2W - B_{mn}.$$

*For a uniformly chosen $aU$, the probability that the two indices occur in the same coordinate of the row family $(Hz)_{a,t}$ is zero if some relative residue is nonunipotent. Otherwise it is at most $\prod_{i(m)_v \ne i(n)_v} \rho_k(q_v)$.*

*Proof.* The positive valuations of $u$, with weights $\log Nv$, sum to $\log|N_{F/\mathbb{Q}}u| + \log N_{F/\mathbb{Q}}\mathfrak{d}(u)$ by factorization of its fractional ideal. This is at most $B_{mn}$. At every marked unipotent residue the trace calculation gives $v(u) \ge 1$, with valuation one on a uniformizer. Where the two indices agree, both $1 - \gamma$ and $1 - \gamma^{-1}$ vanish modulo the corresponding place of $E$ in the target

lattice basis. Their product has entries divisible by its square. The place splits above $F$, so $v(u) \geq 2$ in $F$. Summing gives the two inequalities.

At a local unipotent residue, membership in $a_v U_q a_v^{-1}$ is equivalent to preserving the flag of $a_v$, since every diagonal eigenvalue in a triangular form is one. The map from $G_q/U_q$ to the flags has fibers of the same size. Lemma 96.13 bounds the probability when the residue differs from the identity. The choices at distinct places are independent in the product $G/U$. □

**Corollary 96.15** (Integral recovery at larger denominators)**.** *Suppose that $s$ distinct maps obey the lattice inclusions of Theorem 96.12 and that, for some $0 < \xi \leq 1$,*

$$4kh^2 \leq (g_1 g_2)^{2-\xi}, \qquad e = (s-1)(g_1 g_2)^{-\xi f(k-1)/2} < 1.$$

*Their indices are distinct. On these indices $H$ has an integral left inverse, and the eigenvalues of the normalized Gram matrix lie in $[1-e, 1+e]$. The choices are uniform in the common marked frames and work for signed columns at the prescribed terminal prime.*

*Proof.* Here $h^2 u$ is integral, so $B_{mn} \leq f \log(4kh^2)$. Whenever all relative residues are unipotent, the places of disagreement have total log size at least $\xi W > 0$. At every marked prime $q$ one has $\rho_k(q) \leq (k-1)q^{-(k-1)} \leq q^{-(k-1)/2}$, by the prime-size condition above. Proposition 96.14 bounds the probability of collision in a row family by $(g_1 g_2)^{-\xi f(k-1)/2}$ for any distinct pair, also when some relative residue is nonunipotent.

Fix an input index. The fraction of row families where any other input shares its coordinate is at most $e$, by summing over the other $s-1$ choices. Some row therefore isolates this input. Selecting an isolating row separately for each input and undoing the unit character factors gives the left inverse on the entire support. The absolute value of an off-diagonal entry of $A$ is at most the corresponding collision probability, because it is an average of roots of unity on the colliding families and zero on the others. The row sums of the absolute off-diagonal entries are at most $e$. The matrix is Hermitian with unit diagonal, so the eigenvalue bound follows, or equivalently one can apply $2|z_i z_j| \leq |z_i|^2 + |z_j|^2$ to its quadratic form. □

**Theorem 96.16** (Definite theta coefficients with prescribed Gram matrix)**.** *Let $A_0, M$ be integral totally definite Hermitian lattices over $\mathcal{O}_E$, of ranks $k$ and $D \geq k$. Let $A_0^{\#} = \{v : \langle v, A_0 \rangle \subseteq \mathcal{O}_E\}$, and choose positive integers $d_0, h$, prime to $g_1 g_2$, such that $d_0 A_0^{\#} \subseteq A_0$. Consider all isometric embeddings $X : A_0 \to h^{-1}M$. At each marked residue place retain the image subspace of the reduced frame. For each tuple of image subspaces, choose bases of those subspaces and form $H$ on the resulting square frame coordinates. If $4kd_0h^2 < g_1 g_2$, this operation recovers the individual coefficients of all the embeddings over the integers of the character field. Its normalized squared norm is the sum of their squared absolute values.*

*For free $A_0$ with basis and Gram matrix $S$, the assertion applies to the full $S$-coefficient of the definite lattice theta series with these residue data retained. In particular, on each of the period fields take $A_0 = \mathcal{O}_E^k$ with its identity form, $M = A_0 \perp M_0$ for any integral totally definite $M_0$, and $h = 1$. For any assignment of a sign to every embedding, the refined coefficient after $H$ has an entry which is a unit at every coefficient prime.*

*Proof.* Write $X^*$ for the Hermitian adjoint on the rational spaces. One has $X^*X = 1$, and $hX^*M \subseteq A_0^{\#}$, by taking pairings with $A_0$. At each marked place the chosen integers are units. The reductions of $X$ and $X^*$ are therefore defined on their lattices and their composition in the order $X^*X$ is the identity. Thus the frames have rank $k$.

For distinct embeddings $X, Y$ set

$$u_{XY} = \mathrm{Tr}_E\left((X-Y)^*(X-Y)\right) = 2k - \mathrm{Tr}_E(X^*Y + Y^*X) \in F.$$

At an infinite place the two maps are isometric embeddings between ordinary Hermitian spaces. Each has squared Hilbert–Schmidt norm $k$. The triangle inequality gives $0 < u_{XY} \le 4k$ there. At finite places $d_0 h^2 X^* Y A_0 \subseteq A_0$, since $hYA_0 \subseteq M$ and $hX^*M \subseteq A_0^{\#}$. The same inclusion holds for $Y^*X$. Thus $d_0 h^2 u_{XY}$ is integral in $F$.

Different tuples of image subspaces already lie in different rows. Within a tuple, a collision in a row family of $H$ would give, at each chosen place, $\overline{X} = \overline{Y}t$ for some unipotent $k$-by-$k$ matrix $t$. Here the bars mean reduction, and unipotence is invariant under the changes from the chosen subspace bases. Reducing the adjoint identities shows

$$\overline{Y^*X} = t, \qquad \overline{X^*Y}\, t = \overline{X^*X} = 1.$$

These identities use the integral adjoint maps at that place. The traces of $t$ and $t^{-1}$ are both $k$, so every place of $F$ above $g_1 g_2$ divides $u_{XY}$. The positive integer $|N_{F/\mathbb{Q}}(d_0 h^2 u_{XY})|$ would be at least $(g_1 g_2)^f$ and at most $(4k d_0 h^2)^f$. This contradicts the stated inequality. The distinct embeddings therefore occupy separate coordinates in every row family, and entry selections and the norm identity follow as in Theorem 96.12.

In a basis of $A_0$ an embedding is exactly an ordered tuple in $(h^{-1}M)^k$ with Hermitian Gram matrix $S$: the equalities of the pairings extend by sesquilinearity, and definiteness implies injectivity. Grouping the defining theta sum over tuples by this Gram matrix gives its stated Fourier coefficient. At a fixed $S$, all images of a $\mathbb{Z}$-basis are bounded at all infinite places, so only finitely many embeddings occur. Any weights on these tuples can thus be used in the sum. For the last assertion $d_0 = 1$ is admissible, and the inclusion of the first summand gives an embedding. Its signed coefficient is isolated up to a root of unity in every row family. □

**Proposition 96.17** (Cancellation on singular frame orbits)**.** *Over the same $E$, take a full frame $X$ of $k$ vectors. Its first $l \ge 1$ vectors are the first Witt basis vectors of mutually orthogonal integral hyperbolic planes and the remaining $k-l$ form an integral basis of a nondegenerate orthogonal summand. At any given positive precision above $T$ there is a finite family of integral frames congruent to $X$ there, with its column lattice, Gram matrix and residue image subspaces. On this family the coefficient array with all weights one is killed by $H$ if $l \ge 2$. For $l = 1$ the family constructed below has an entry equal to a root of unity after applying $H$.*

*Proof.* Write $\pi$ for the coordinate projection onto the last $k-l$ entries. The determinant-one changes of basis fixing $\pi$ are precisely the matrices with block rows $(A, B)$, $(0, I)$, where $A \in \mathrm{SL}_l$ and $B$ is arbitrary. Since the Gram form factors through $\pi$, integral changes $s$ of this shape preserve both the Gram matrix and the column lattice of $Xs$. At each chosen place use the finite group

$$S_v = \{ s \in \mathrm{SL}_k(\mathbb{F}_{q_v}) : \pi s = \pi \}, \qquad S = \prod_v S_v.$$

Choose one integral lift of this shape per element of $S$, equal to the identity at the specified terminal precision. For the block $B$ use the Chinese remainder theorem, taking it zero at the terminal places. For the block $A$, over a field determinant-one matrices are products of elementary row additions by Gaussian elimination when $l \ge 2$, and $A = 1$ when $l = 1$. The signed interchange and the diagonal scaling needed in elimination follow from

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \qquad (t \neq 0)$$

and the product of two such matrices with parameters $t, -1$. Lift each elementary parameter by the same Chinese remainder argument, making it zero also at the other auxiliary places.

In the bases of the reduced frame $X$ the resulting array is $\mathbf{1}_S$, with a single tuple of image subspaces. Suppose $l \ge 2$. Fix a row $a, t$ of $H$ and one auxiliary place, suppressing its subscript. The first $k-1$ columns of $\pi a$ lie in dimension $k-l < k-1$. Some linear dependence has last nonzero coefficient $b_i$, with $i \le k-1$. Complete its coefficient vector $b$ by zeros after $i$. For $c \in \mathbb{F}_q$, the matrix $u_c = I + cbe_{i+1}^{\mathsf{t}}$ is upper unitriangular and $\pi a u_c = \pi a$. Hence $a u_c a^{-1} \in S_v$. Its character before conjugation is $\theta_v(u_c) = \psi_v(cb_i)$, which differs from one for some $c$. Extend $u_c$ by the identity at the other places. The set of $u \in U$ with $aut \in S$ is stable under left multiplication by $u_c$; its character sum equals that sum times $\theta(u_c)$. Every coordinate of $H\mathbf{1}_S$ is therefore zero. This proof includes totally isotropic frames and frames with a single nonzero Gram entry when $k \ge 3$.

If $l = 1$, each $S_v$ consists of shears into the first coordinate line and fixes that line pointwise. Choose $a_v$ so that this line is last in its ordered basis. Then $a_v U_q a_v^{-1} \cap S_v = 1$, since the shears are lower unitriangular in that basis. Choose the row with $t = a^{-1}$. Its support meets $S$ just in the identity, giving the unit entry. Changing representatives only multiplies entries by roots of unity. Common left and right changes of frame relabel $a, t$ in the formula for $H$, so both conclusions hold in those coordinates as well. □

These statements concern one fixed pair of lattice rows at a time, as in the index of `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/TermOp.lean`, line 322. They can be applied directly to a restricted period sum whose residue labels are the reductions of the indicated rational isometries in common frames. The rectangular theta coefficient in Theorem 96.16 retains its residue image subspaces. The first-plane orbital calculation in Lemma 105.1 also uses singular Gram matrices. Proposition 96.17 exhibits cancellation on its orbit types with radical of dimension at least two, even with fixed terminal congruences. The full orbital formula uses adelic Levi matrices and prescribed path weights. A lower bound for the theta lattice in Section 6.6 requires control of those actual weights on the singular orbits and of the signed sums when residue rows are combined.

## 96.6 Integral lifts of a rational partner flag

Fix two words and an ordered pair of their lattice rows $\Lambda', \Lambda$ on the field $E = E_x$ of Section 6.6. Write $F = E^+$, $f = [F : \mathbb{Q}]$, and $Y' = (e'_1, \dots, e'_s, z')$, $Y = (e_1, \dots, e_s, z)$ for the source and target frames. Let $I'$ be the span of the $e'_j$. These vectors give an ordered isotropic flag. The vector $z'$ lies in $(I')^{\perp}$ and has totally negative norm. Put

$$J' = (I')^{\perp}/I', \qquad C' = (E\bar{z}')^{\perp} \subset J', \qquad L' = C' \cap \operatorname{im}\,(\Lambda' \cap (I')^{\perp} \longrightarrow J').$$

Here and below a bar on $z'$ means its image in $J'$. Removing the $s$ hyperbolic planes leaves signature $(k, 1)$ at every real place. Hence $C'$ is totally positive of rank $k$. The image in this formula spans $J'$ and is finitely generated over $\mathcal{O}_E$. Its intersection with the rational subspace $C'$ is a full lattice there, as one sees by clearing the denominators of a basis of $C'$.

Let $N'$ be the integral source isometries acting trivially on the successive lines of the isotropic flag and on $J'$, and use $N$ for the corresponding target group. Let $S'$ be the integral source isometries fixing every vector of $Y'$ pointwise, and write $D$ for their image in $\mathrm{Aut}(L')$. A lifted term is a rational isometry $\gamma : \Lambda' \longrightarrow \Lambda$ with admissible transported frame, taken modulo $N$ on the left and $N'$ on the right. Its geometric partner is the class of the transported frame $\gamma Y'$. Two transported frames define the same partner if one is obtained from the other by first applying the integral unipotent group of its own isotropic flag and then an element of $N$. This definition includes the terminal vector in both actions. It is the equivalence used in `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/TermDef.lean`, line 18.

At each $v \mid g_1 g_2$ take the chosen split factor, of residue size $q_v$. Use the original marked bases on the two rows and write $i(\gamma)_v$ for the induced matrix on the $k$-dimensional residual complement. There are $f$ places of each residue size. Write $G = \prod_v \mathrm{GL}_k(\mathbb{F}_{q_v})$, $U = \prod_v U_k(\mathbb{F}_{q_v})$, and retain the character $\theta$, the matrix $H$ and $A = H^* H/[G : U]$ of Lemma 96.2. Work at $x$ large enough that $g_1, g_2 > 2fk + 1$, as holds on the common intervals.

**Lemma 96.18** (The stabilizer in the marked middle quotient)**.** *The homomorphism $S' \to D$ has kernel $S' \cap N'$. The group $D$ is finite. In each single marked factor its reduction $d \mapsto d_v$ is injective and only its identity reduces to a unipotent matrix. Writing $\delta(d) = (d_v)_v$, one has*

$$i(n\gamma n') = i(\gamma), \qquad i(\gamma h) = i(\gamma)\delta(d) \quad (n \in N,\ n' \in N',\ h \in S',\ h|_{C'} = d).$$

*Proof.* Each member of $S'$ preserves $J'$, the vector $\bar{z}'$ and the lattice $L'$. Since $J' = E\bar{z}' \perp C'$, trivial action on $C'$ is equivalent to trivial action on $J'$. The pointwise flag condition gives the assertion about the kernel. The sum of the Hermitian norms over all real places is positive definite on the underlying real space of $L'$. The image of each vector of a fixed integral basis under a lattice isometry lies on a compact sphere in a discrete lattice. This gives finitely many possibilities. In particular $D$ has a faithful rational representation of dimension $2fk$.

If $g_i$ divided $|D|$, Cauchy's theorem would give an element of order $g_i$. In characteristic zero its minimal polynomial would contain the cyclotomic polynomial of degree $g_i - 1 > 2fk$, since at least one eigenvalue would have that prime order. Thus $g_i \nmid |D|$. If $d_v$ is unipotent, its order is a power of $g_i$ and divides $|D|$, so it is the identity. Lift to the localization of $L'$ at the chosen place of $E$. If $d^m = 1$ and $d \equiv 1$ there, the sum $1 + d + \cdots + d^{m-1}$ reduces to $mI$ and is invertible for $m = |D|$. Multiplying it by $d - 1$ gives zero. Thus $d = 1$ already on the rational lattice.

For the multiplication formulas, reduce the saturated isotropic lattice and its orthogonal complement at $v$. The latter reduces to the orthogonal of the former. Indeed the pairing map onto the dual of the saturated summand is surjective by self-duality, and its exact sequence splits over the local integers. At a split place this uses the perfect pairing between the two factors. The norm of the marked last vector is a unit at $v$, so taking its perpendicular in the middle quotient commutes with reduction. Admissibility identifies the two reduced isotropic flags up to an upper unitriangular change and identifies their last vectors modulo the isotropic summand. The chosen basis and its dual on the remaining $k$-space therefore read the matrix of the induced map by ordinary pairings. This is exactly the entry definition in `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/TermOp.lean`, line 300. On this quotient $n, n'$ induce the identity, and $h$ induces $d_v$. Composing the induced maps proves both formulas. □

**Theorem 96.19** (All lifts of a fixed geometric partner)**.** *Let $\gamma_0$ be one lift of a geometric partner in the fixed pair of rows. Its lifts are indexed bijectively by $d \in D$ and occupy precisely $i(\gamma_0)\delta(D)$. Each single marked place distinguishes them. For an array supported on this coset, every row family $aU$ of $H$ recovers all its coefficients by entry selection and multiplication by roots of unity. Its normalized squared norm is exactly the sum of the squared norms of those coefficients.*

*Proof.* Each $h \in S'$ fixes the transported frame of $\gamma_0 h$ exactly. Hence the transported frame satisfies every original admissibility condition. If $\gamma Y'$ has the same partner as $\gamma_0 Y'$, write $\gamma Y' = nm\gamma_0 Y'$ with $n \in N$ and $m$ in the integral unipotent group of the transported flag. The latter group is $\gamma_0 N' \gamma_0^{-1}$, by transport of the lattice, the flag and its middle quotient. Thus $\gamma = n\gamma_0 n' h$ for some $n' \in N'$ and $h \in S'$. The group $S'$ normalizes $N'$, since it preserves the lattice and the defining filtration. Move $n'$ to the right of $h$ and pass to the double quotient.

Two choices of $h$ with the same $d$ give the same lift by the kernel assertion. Different $d$'s give different indices at each marked place by Lemma 96.18.

In a row family $aU$ two indices coincide in one entry of $H$ precisely if their relative matrix lies in $aUa^{-1}$. For two distinct lifts the relative matrix at each place is a conjugate of the reduction of a nonidentity element of $D$. It cannot be unipotent. Hence each coefficient occurs in its own entry multiplied by its character value. This proves the integral recovery and the norm identity, including for operator coefficients with their Hilbert–Schmidt norm. □

**Proposition 96.20** (The signed orbital coefficients on a coset)**.** *Let $Z(\gamma)$ include the finite outer unipotent character, the real orbital integral and the common volume factor in the unfolded period. Let $\rho'$ be the unitary source column type on the compact group of $C'$ at infinity. With the choices of the preceding theorem the full operator satisfies*

$$Z(\gamma_0 h) = Z(\gamma_0)\rho'(d).$$

*For finitely many geometric partners, sum their contributions at each actual residue index and denote the resulting operator by $Z_i$. For every coset $C = i_C\delta(D)$ in this sum there is a single operator $Z_{i_C}$ such that*

$$Z_{i_C\delta(d)} = Z_{i_C}\rho'(d), \qquad \sum_{i\in C} \|Z_i\|_{\mathrm{HS}}^2 = |D|\,\|Z_{i_C}\|_{\mathrm{HS}}^2.$$

*The same conclusions hold for an absolutely summable expansion grouped by partner. The assertions use the full source operator before evaluation on any particular test vector.*

*Proof.* Use representatives $\gamma_0 h$ with $h$ fixing $Y'$ pointwise. In the source marked finite basis the matrix of $h$ fixes each reduced $e_j$ and the reduced $z_0$. They are the images of its fixed rational vectors after the prescribed marking. The finite outer character pairs the dual flag vector $f_j$ with the image of the next vector in the list $(e_1, \dots, e_s, z_0)$, sums over $j$, divides by $g_1 g_2$ and applies the finite additive character. Right multiplication by the matrix of $h$ preserves these pairings modulo $g_1 g_2$ at every marked place. At the other finite places the difference is integral and $g_1 g_2$ is a unit. The finite character is trivial on the resulting integral differences, by its conductor normalization. This gives the same finite factor for both terms.

The group determining the real domain is $N \cap \gamma_0 N' \gamma_0^{-1}$. It is unchanged on replacing $\gamma_0$ by $\gamma_0 h$, so take the same fundamental domain in the two unfolded integrals. The real outer character of $h$ is one, since it fixes every next vector and each corresponding simple off-diagonal coordinate is zero. The equivariance in (92.15) consequently replaces $W'(\gamma_0^{-1} y)$ by $\rho'(d)^{-1} W'(\gamma_0^{-1} y)$. In the integrand $W(y) W'(\gamma_0^{-1} y)^*$ this multiplies the operator on the right by $\rho'(d)$, by unitarity. The quotient-integral calculation is independent of the compatible representatives. Indeed in the unfolded adelic integral left multiplication by an integral target unipotent changes the variable by that translation, and right multiplication by an integral source unipotent gives the source equivariance factor. In either case its finite and infinite character values on the rational unipotent multiply to one, by triviality of the global additive character on $E$. In particular translating a fundamental domain by an element of the intersection leaves its integrand unchanged after combining the characters. The volume factor depends only on the rows and words. This proves the operator formula.

For each partner whose coset is $C$, there is a unique lift at $i_C$. Add its operator there, with the sign in the original sum. The same right action $\rho'(d)$ transports each summand to $i_C\delta(d)$. Unitarity and summation give the second formula. Absolute summability allows passage from finite sums by continuity in the finite-dimensional operator space. □

For two distinct cosets $C = i\delta(D)$, $C' = j\delta(D)$ write $J(C, C') = \{v : i_v D_v \neq j_v D_v\}$, where $D_v$ is the image of $D$ at $v$, and let $\mathcal{V}$ denote the set of the $2f$ places. For numbers $0 < t_v \leq 1$ define

$$w_t(C, C') = \begin{cases} 0, & J(C, C') = \varnothing, \\ \prod_{v \in J(C,C')} t_v, & \varnothing \neq J(C, C') \neq \mathcal{V}, \\ \min_{v_0 \in \mathcal{V}} \prod_{v \neq v_0} t_v, & J(C, C') = \mathcal{V}. \end{cases}$$

Set $p_v = \rho_k(q_v)$ as in Lemma 96.13 and $c_v = (q_v^k - 1)^{-1}$. Assume $k \geq 2$. For $k = 1$ the groups $U_v$ are trivial and the entries are read directly.

**Theorem 96.21** (Comparison for several partners in the same two rows)**.** *For an array on any set $\mathcal{C}$ of distinct cosets of $\delta(D)$, let $Z_C$ denote its whole restriction to $C$, with the sum-of-squares norm. Then*

$$\left| \langle Z, AZ \rangle - \sum_{C} \|Z_C\|^2 \right| \leq \sum_{C \neq C'} w_c(C, C') \|Z_C\| \, \|Z_{C'}\|.$$

*The sum counts ordered pairs. Put $\epsilon_p = \max_C \sum_{C' \neq C} w_p(C, C')$, with value zero for an empty family. If $\epsilon_p < 1$, all raw coefficients on the union can be recovered from $HZ$ by integral entry selections. Each index is isolated in at least a proportion $1 - \epsilon_p$ of the row families. For any matrix of columns on this support all minor ideals before and after $H$ agree over the coefficient ring containing the character values.*

*Proof.* Fix a particular index $i'$ in $C$. In one factor and for a fixed row family $a_v U_v$, at most one $d \in D$ can place $j_v d_v$ in the same row as $i'_v$. Two such elements would have unipotent relative matrix, contrary to Lemma 96.18. If the two local cosets agree, this single choice is forced by $j_v d_v = i'_v$ and works for every row family there. If all local cosets agree and these forced choices were consistent, the two global cosets would meet and be equal. This proves the zero case.

When the local cosets agree at some places and differ at others, any agreeing place thus specifies at most one possible global partner. In a differing factor its relative matrix is nonidentity. The fraction of the local row families giving a collision is zero for a nonunipotent and at most $p_v$ for a nonidentity unipotent. Indeed $a_v U_v$ induces a uniformly distributed complete flag, and a unipotent preserving that flag lies in its upper unitriangular group. The choices of local row families are independent. If all local cosets differ, expose any one row family first. Conditional on it there is at most one candidate, to which the product bound at the other places applies. Minimizing over the exposed place proves the bound $w_p$ in all cases. A union bound over $C'$ now gives one isolating row for every $i'$ as soon as $\epsilon_p < 1$. Each chosen output entry, multiplied by the inverse of its character unit, retrieves the input. These selections define a left inverse independent of the columns. Cauchy–Binet applied to $H$ and to this inverse gives both inclusions of minor ideals, also for fractional entries.

For the norm estimate the local normalized Gram entry has absolute value at most the collision probability by the row formula. On a nonidentity relative matrix it is also at most $c_v$. Indeed it vanishes off the unipotents, and for a unipotent with fixed dimension $l < k$ Theorem 92.18 gives absolute value $\prod_{j=l+1}^{k} (q_v^j - 1)^{-1}$. The global entry is the product of the local entries. When some local cosets agree, the forced choices leave at most one nonzero entry in the sum over $j\delta(d)$, and the differing places give the product of the $c_v$. When all differ, keep the collision-probability bound at one chosen place. Its sum over $d \in D$ is at most one since the collision events there are disjoint. At the other places use the bounds $c_v$. This proves that each absolute row sum of the block $A_{C,C'}$ is at most $w_c(C, C')$. The same holds for column sums by symmetry. To bound the operator norm by this common bound, apply weighted Cauchy–Schwarz to each row of a matrix $B$. This gives $\| \sum_j B_{ij} v_j \|^2 \leq (\sum_j |B_{ij}|) \sum_j |B_{ij}| \|v_j\|^2$, and one

sums over $i$. The diagonal blocks are identities by Theorem 96.19. Summing the off-diagonal blocks proves the estimate. □

**Corollary 96.22** (Uniform recovery on the period interval)**.** *On the common fields and intervals, suppose the arrays in each ordered pair of rows occupy at most $b_x^{4(k-1)}$ such cosets, where $b_x = \exp(x^{\varepsilon})$. The integral comparison of Theorem 96.21 then holds, in particular at each terminal prime of Corollary 96.8. The normalized squared norm differs from the raw squared norm by at most $o(b_x^{-2})$ times the latter, uniformly in the arrays and independently of $|D|$.*

*Proof.* Write $q_* = \min(g_1, g_2) \geq (kb_x)^8$. There are at least two marked factors. Every nonzero product in $w_p$ contains at least one factor at most $(k-1)q_*^{-(k-1)}$ by Lemma 96.13. The factors $c_v$ satisfy the same bound. Hence each of the two row sums of weights over the other cosets is at most

$$b_x^{4(k-1)}(k-1)(kb_x)^{-8(k-1)} \leq b_x^{-4(k-1)}.$$

This is $o(b_x^{-2})$ uniformly for $k \geq 2$. Apply the theorem, using $2ab \leq a^2 + b^2$ and symmetry for its norm bound. Distinct ordered pairs of rows are separate coordinates of $H$, so their squared norms add and the integral selections can be made in each pair independently. □

For a single geometric partner the comparison has no support hypothesis. When several partners have the same coset, the operator at its chosen base index is their transported signed sum. A lower bound for the prescribed projected theta periods in Section 6.6 still requires control of these sums, their evaluation on the prescribed tests, and interference across the occupied cosets in the full expansion.

## 96.7 The two frame stabilizers in the signed period

Use the same two words, rows and markings as in Section 96.6. Attach the finite groups $D_t, D_s$ of Lemma 96.18 to the target and source respectively. Denote their embeddings in $G$ by $\delta_t, \delta_s$. The group $P = D_t \times D_s$ acts on relative residue indices by

$$(a, b)i = \delta_t(a)i\delta_s(b)^{-1}.$$

On a source-to-target operator $B$ let $\tau(a, b)B = \rho_t(a)B\rho_s(b)^{-1}$, where $\rho_t, \rho_s$ are the two real column types. This is a unitary action on the Hilbert–Schmidt space. We first bound the orders of both groups in terms of the dimension over $E$.

**Lemma 96.23** (A uniform bound for the column stabilizers)**.** *Suppose $E \subseteq \mathbb{Q}(\zeta_N)$, $4 \mid N$, and $D_0$ is any finite subgroup of $\mathrm{GL}_k(E)$. Then, with an absolute implied constant,*

$$\log |D_0| \leq k \log N + O(k \log^2(2k)).$$

*On the polynomial-conductor intervals of Theorem 19.15 this gives $\log |D_0| = O_{\psi,\varepsilon}(k \log^2(2x))$. The same estimate for $A_*$ applies in Theorems 98.21 and 98.41.*

*Proof.* Use reduction at rational primes that split completely in $\mathbb{Q}(\zeta_N)$, following Minkowski's reduction method [7]. Outside a finite set all matrices in $D_0$ and their inverses are integral, and their reductions remain distinct. For the last assertion choose a nonzero entry of each difference and exclude all rational primes below the prime divisors of these entries. At any remaining split prime $l$,

$$|D_0| \;\mid\; |\mathrm{GL}_k(\mathbb{F}_l)| = l^{k(k-1)/2} \prod_{j=1}^{k} (l^j - 1).$$

Fix an odd prime $p \nmid N$. Choose a generator modulo $p$ and lift it modulo $p^2$ so that its $(p-1)$-st power is different from one there. Among its $p$ lifts precisely one has power one, by binomial expansion and the unit derivative. Impose this residue on $l$, together with $l \equiv 1 \bmod N$. The Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic progressions supply such an $l$ outside the finite excluded set. Then $p \mid l^j - 1$ precisely for $j = (p-1)h$, and $v_p(l^j - 1) = 1 + v_p(h)$. Indeed raising a unit $1 + p^m u$ to a power prime to $p$ preserves $m$, and raising to the $p$-th power increases it by one, by binomial expansion. Thus

$$v_p(|D_0|) \le \sum_{a \ge 0} \left\lfloor \frac{k}{(p-1)p^a} \right\rfloor .$$

For $p \mid N$ put $s = v_p(N)$. Choose $l \equiv 1 + p^s \bmod p^{s+1}$ and $l \equiv 1 \bmod N/p^s$ by the same argument. This splits $l$ completely. Now $v_p(l^j - 1) = s + v_p(j)$, also for $p = 2$ since $s \ge 2$. The bound is $ks + v_p(k!)$. The contributions at all $p \mid N$ are therefore at most $k \log N + \log(k!)$. At the other primes only $p \le k + 1$ contribute, by the preceding floor sum. Their total is at most $\sum_{p \le k+1} kp \log p/(p-1)^2 = O(k \log^2(2k))$, by extending the sum of $\log p/p$ to all integers in that range. This proves the first estimate. On the stated intervals $N$ and $k$ are bounded by fixed powers of $x$. Each coarse column stabilizer acts faithfully on its definite $k$-dimensional $E$-space, so the same argument applies to the maximum of the orders of these stabilizers. □

For a single lifted term choose a representative $\gamma$ and write $i = i(\gamma)$. On the double quotient of lifted terms use $(a, b)[\gamma] = [\tilde{a}\gamma\tilde{b}^{-1}]$, with tildes denoting pointwise integral frame stabilizers. Define two subgroups of $P$ by

$$K_\gamma = \mathrm{Stab}_P([\gamma]), \qquad M_i = \mathrm{Stab}_P(i).$$

For a subgroup $M$ of $P$, let $\Pi_M B = |M|^{-1} \sum_{m \in M} \tau(m) B$.

**Theorem 96.24** (The signed sum above one residue in a two-sided orbit)**.** *The action on the lifted terms is well defined and the residue map is equivariant. Write $B_\gamma$ for the single orbital operator of Proposition 96.20, and sum all distinct terms of $P \cdot [\gamma]$ at their actual residue indices. Then the entries and the squared norm of the resulting array satisfy*

$$Z_{p \cdot i} = [M_i : K_\gamma]\, \tau(p) \Pi_{M_i} B_\gamma \qquad (p \in P),$$
$$\sum_{j \in P \cdot i} \|Z_j\|_{\mathrm{HS}}^2 = [P : M_i][M_i : K_\gamma]^2 \|\Pi_{M_i} B_\gamma\|_{\mathrm{HS}}^2.$$

*The range of $\Pi_{M_i}$ is the space of operators intertwining the two restrictions to $M_i$. These formulas also give the contributions of any absolutely summable collection of such orbits to a common residue by addition at that residue.*

*Proof.* The pointwise integral stabilizer of either frame normalizes its unipotent group. Two lifts of $a$ or $b$ differ by this unipotent group, by Lemma 96.18. Thus the action descends to the lifted terms. Its right half preserves the transported frame. For the left half, $\tilde{a}$ preserves the target lattice and fixes the target frame exactly. Applying it to an admissibility congruence preserves the congruence to the same upper unitriangular combination of target vectors. It carries the transported frame's saturated isotropic lattice, the specified fractional-ideal multiples of its vectors and its integral primitive terminal class to the corresponding data for the new frame. This preserves the remaining row conditions. The induced maps on the reduced middle quotient compose, proving equivariance and $K_\gamma \subseteq M_i$.

For the left action, the reduced images under $\gamma$ of the successive source flag vectors, including the last vector, are linear combinations of the fixed target frame vectors. Hence $\tilde{a}$ fixes these

images at every marked place. In the formula for the finite outer character the simple-coordinate pairings are therefore unchanged modulo $g_1 g_2$. At the other finite places the differences after division are integral. The real domain for $\tilde{a}\gamma$ is obtained from the one for $\gamma$ by left translation by $\tilde{a}$, since the intersection subgroup is conjugated by $\tilde{a}$. Make this change of variable. The target real vector then acquires the factor $\rho_t(a)$ by (92.15), its outer character being one on the pointwise stabilizer. Together with Proposition 96.20 on the source this gives $B_{p[\gamma]} = \tau(p) B_\gamma$. In particular $B_\gamma$ is fixed by $K_\gamma$. The source actions are also displayed in `Formalism/LocalToGlobal/SiegelLatticeTheory/ClosedLocalOrbits/GaugeLatticeShifts/SatGramActR.lean`, line 58.

The orbit is $P/K_\gamma$, its residue set is $P/M_i$, and the fibre above $i$ is $M_i/K_\gamma$. When summing over all elements of $M_i$ each term in that fibre is counted $|K_\gamma|$ times. This gives the first identity at $p = 1$ and equivariance gives it at every index. The group average is self-adjoint since inversion permutes the summands. It fixes every invariant operator and has invariant image by reindexing. It is consequently the orthogonal projection to the operators satisfying $\rho_t(a) B = B \rho_s(b)$ for every $(a, b) \in M_i$. All entries have equal Hilbert–Schmidt norm, giving the second identity. Absolute convergence permits the specified regrouping. □

**Lemma 96.25** (Equality of the compact characters on the residue stabilizer)**.** *At every archimedean embedding the defining $k$-dimensional representations of $M_i$ on the two positive complements are unitarily equivalent. In particular if the two column types have the same highest weight at each real place, there is a unitary intertwiner $U$ of those types on $M_i$. For any such $U$ and every operator $B$ between the two column types,*

$$\|\Pi_{M_i} B\|_{\mathrm{HS}}^2 \;\geq\; \frac{|\operatorname{Tr}(U^* B)|^2}{\dim V_s}.$$

*Proof.* For $(a, b) \in M_i$ the two finite matrices satisfy $a_v = i_v b_v i_v^{-1}$ at every marked place. Their rational traces in the defining spaces belong to $\mathcal{O}_E$ since they preserve lattices. If their difference $z$ were nonzero, each of the $f$ chosen primes of $E$ over $g_1$ would divide $z$. All have norm $g_1$, so that

$$g_1^f \leq |N_{E/\mathbb{Q}}(z)| \leq (2k)^{2f}.$$

The upper bound follows because all eigenvalues of these finite order transformations have modulus one at every embedding. On our intervals $g_1 > (2k)^2$, a contradiction. Hence the traces agree for every element of $M_i$ under each embedding. Character orthogonality gives equal multiplicities for all irreducible complex representations of this finite group. Taking orthonormal bases in their unitary decompositions gives a unitary intertwiner on the defining spaces. Conjugation by such a unitary identifies the two ambient compact unitary groups. Irreducible representations of the same highest weight are equivalent under this identification, which is independent up to inner automorphism of the choices of orthonormal basis. Take the tensor product of the local equivalences for the column types. The resulting $U$ lies in the range of the orthogonal projection $\Pi_{M_i}$ and $\|U\|_{\mathrm{HS}}^2 = \dim V_s$. Taking its inner product with $B$ and applying Cauchy–Schwarz proves the inequality. □

**Theorem 96.26** (Integral comparison on full two-sided residue orbits)**.** *Use the period fields of Theorem 19.15 and the two marked primes of Corollary 96.8. Fix $0 < \lambda < 8$ and $k \geq 2$. On the union of at most $b_x^{(8-\lambda)(k-1)}$ two-sided residue orbits in one ordered pair of lattice rows, $H$ has an integral left inverse. For every array $Z$ on that union one has, uniformly,*

$$\left| \frac{\|HZ\|^2}{[G : U]} - \|Z\|^2 \right| \leq b_x^{-(\lambda - o(1))(k-1)} \|Z\|^2.$$

*For a single full orbit the exponent may be taken to be $-(8 - o(1))(k - 1)$. In particular this compares the full signed array of Theorem 96.24 with its explicit projected operator. At the*

*terminal prime $T$ the averaging and the factor $[M_i : K_\gamma]$ in that formula have unit denominators and the latter factor is a unit.*

*Proof.* The orbit of $i$ is $\delta_t(D_t)i\delta_s(D_s)$, a union of at most $|D_t|$ left cosets of $\delta_s(D_s)$. Lemma 96.23 gives $|D_t| \le b_x^{o(k-1)}$ uniformly, since $k/(k-1) \le 2$ and $\log^2(2x)/x^\varepsilon \to 0$. Every off-diagonal weight of Theorem 96.21 is at most $(k-1)(kb_x)^{-8(k-1)}$, for both its unsigned and its signed estimate. Multiply by the number of cosets just found. The maximum sum of off-diagonal weights is at most the relative error in the statement, which is less than one for large $x$. Entry selection gives the integral left inverse. The quadratic estimate follows by the inequality $2uv \le u^2 + v^2$ on the off-diagonal blocks. For one orbit use only $|D_t|$ in the count. Both arithmetic groups embed in $G$ and $T \nmid |G|$, so all subgroup orders in the average and the orbit indices are units at $T$. □

More explicitly, take several columns of orbital operators on a single $P$-orbit of terms with the same equivariance law. At a place over $T$ where these operators are defined over a coefficient field, use a $P$-stable operator lattice. The minor ideals of the stacked columns in $HZ$ then equal those of the base columns $\Pi_{M_i} B_\gamma$, in local lattice coordinates. Indeed the left inverse in the theorem first recovers all residue blocks. Taking the block at $i$ and dividing by the unit $[M_i : K_\gamma]$ recovers the base columns. Conversely each residue block is an integral translate of the base columns multiplied by this unit. Cauchy–Binet in both directions proves the equality. A stable lattice can be formed by summing the finitely many translates of a lattice in any field of definition of the action.

The formulas determine exactly which component of each orbital operator contributes at the fixed marking. The trace in Lemma 96.25 depends on the real functions and on the rational partner. Its size, and the signed sum of these components over distinct rational orbits at a prescribed residue, still enter the global theta-period estimate.

## 96.8 The residue of the common definite quotient

Retain a fixed pair of lattice rows and the marking of Section 96.6. Assume $k \ge 2$ and put $Q_0 = g_1 g_2$. For a lift $\gamma$, write $Y_0, Y_1$ for the spans of the target frame and the transported source frame in the target space, and $X_0, X_1$ for their isotropic parts. In this section a *full-rank cell* is one for which $Y_0 \cap Y_1$ contains a totally negative vector. The term refers to the dimension of the common positive quotient. In the notation

$$I = X_0 \cap X_1, \qquad A = (Y_0 + Y_1)^\perp, \qquad P_j = Y_j^\perp / X_j,$$

the maps $\pi_j : A/I \to P_j$ are then rational isometries onto spaces of dimension $k$, by Theorem 86.3. Let $P_s, P_t = P_0$ be the positive spaces of the two fixed rows. The rational isometry attached to the cell is

$$j_\gamma = \pi_0 \pi_1^{-1} \bar\gamma : P_s \longrightarrow P_t,$$

where $\bar\gamma : P_s \to P_1$ is induced by $\gamma$. Passage to the lifted double quotient preserves this map, since the unipotent groups act trivially on the two endpoint quotients and carry $A, I$ to the corresponding subspaces.

**Lemma 96.27** (The comparison map at the marked places)**.** *Put $U_j = (\Lambda \cap X_j)/(\Lambda \cap I)$. Let $\mathfrak{a}_\gamma$ be the annihilator of the cokernel of the integral pairing map*

$$U_1 \longrightarrow U_0^\sharp, \qquad x \longmapsto (y \longmapsto h(y, x)/Q_0),$$

*where the dual consists of conjugate-linear integral functionals. For the zero-rank map set* $\mathfrak{a}_\gamma = \mathcal{O}_E$. *If* $L_s, L_t$ *are the positive column lattices, one has*

$$\mathfrak{a}_\gamma j_\gamma L_s \subseteq L_t, \qquad \overline{\mathfrak{a}_\gamma} j_\gamma^{-1} L_t \subseteq L_s.$$

*At every* $v \mid Q_0$ *prime to* $\mathfrak{b}_\gamma = N_{E/F}(\mathfrak{a}_\gamma)$, $j_\gamma$ *induces an isomorphism of local lattices and its matrix in the marked residue bases equals* $i(\gamma)_v$.

*Proof.* The common negative vector $z_*$ gives $Y_j = X_j + Ez_*$. The pairing between $X_0, X_1$ has kernel $I$ on each side. Indeed an isotropic vector in $X_1$ orthogonal to $X_0$ is also orthogonal to $z_*$. Its image in the positive quotient $P_0$ must vanish. Interchanging the indices gives the other kernel. Thus the displayed map is nonsingular over $E$. The two isotropic lattices have the same reduction in the target row at the marked places. A vector of $\Lambda \cap Y_1^\perp$ therefore pairs with $\Lambda \cap X_0$ in $Q_0\mathcal{O}_E$, by reducing in both split factors. Away from $Q_0$ this follows from integrality. In particular the displayed pairing map is integral.

Take $w \in \Lambda \cap Y_1^\perp$ and $\beta \in \mathfrak{a}_\gamma$. The functional $y \mapsto h(y, \beta w)/Q_0$ on $U_0$ belongs to the image. Hence some $u \in \Lambda \cap X_1$ makes $\beta w - u$ orthogonal to $X_0$, and it is already perpendicular to $X_1$ and $z_*$. Thus $\beta w - u \in \Lambda \cap A$. Its image in $P_1$ is $\beta[w]$ and the same vector now represents its image in $P_0$. This proves the first denominator claim after transport by $\gamma$. The reversed pairing map is the conjugate transpose in dual local bases. Transposition preserves the elementary divisors, and conjugation exchanges the conjugate primes. The annihilator of its cokernel is consequently $\overline{\mathfrak{a}_\gamma}$, proving the inverse assertion. This is the local argument of Proposition 110.8 in both directions.

At a marked $v$ prime to $\mathfrak{b}_\gamma$, take $\beta = 1$ locally in each split factor. The vectors $w$ and $w - u$ above define the same class in the common residual middle quotient, since $u$ reduces into the common isotropic space. The projections of the same integral vector $w - u$ therefore identify the positives at both ends by the identity in this common quotient. The negative line splits off modulo $v$ since its norm is a unit. Composing with $\bar{\gamma}$ now gives exactly the matrix read from the marked basis and its dual in Lemma 96.18. □

Define the positive real number

$$h_\gamma = (N_{E/\mathbb{Q}}\mathfrak{a}_\gamma)^{1/(2f)}.$$

It can be read from the largest local elementary divisors of the pairing in the lemma, including when its ideal is nonprincipal.

**Theorem 96.28** (Separation of distinct rational transports)**.** *Consider two full-rank cells* $\gamma_1, \gamma_2$ *in the fixed rows. If their indices share any output entry of* $H$, *then either* $j_{\gamma_1} = j_{\gamma_2}$, *or*

$$Q_0 \le 4k(h_{\gamma_1} h_{\gamma_2})^4.$$

*If their indices agree, the alternative for unequal transports is* $Q_0^2 \le 4k(h_{\gamma_1} h_{\gamma_2})^6$. *In particular, in any collection with* $h_\gamma \le b_x^2$ *the projected arrays belonging to distinct rational transports occupy disjoint output entries in every row family. There is no restriction on the number of cells in this assertion.*

*Proof.* Write $j_l = j_{\gamma_l}$ and let $u = j_2^{-1} j_1$ act on the positive source space. Define $\mathfrak{b} = \mathfrak{b}_{\gamma_1}\mathfrak{b}_{\gamma_2}$ in $F$. Its extension to $E$ is the product of both $\mathfrak{a}_{\gamma_l}\overline{\mathfrak{a}_{\gamma_l}}$, by the ideal norm formula for a quadratic extension. Lemma 96.27 then shows that this extended ideal clears the denominators of both $u$ and $u^{-1}$ on the source lattice. Consider the matrix trace

$$t = 2k - \operatorname{tr}_E u - \operatorname{tr}_E u^{-1} = \operatorname{tr}_E((1-u)(1-u^{-1})) \in F.$$

The two traces are conjugate because $u$ is unitary. If $u \neq 1$, at every embedding of $F$ the number $t$ is the squared Hilbert–Schmidt norm of $1-u$, positive and at most $4k$. Moreover $\mathfrak{b}(t)$ is an integral ideal of $F$. Indeed the traces of integral lattice endomorphisms are locally integral, and integrality of an element of $F$ can be checked in $E$. Thus, putting $B = N_{F/\mathbb{Q}}\mathfrak{b}$, one has $N_{F/\mathbb{Q}}(\mathfrak{b}(t)) \leq B(4k)^f$.

At a marked place prime to $\mathfrak{b}$, a collision in a row of $H$ makes the reduction of $u$ unipotent by Lemma 96.27 and the row criterion of Lemma 96.2. Both traces reduce to $k$, giving $v(t) \geq 1$. If the indices agree, both $1-u$ and $1-u^{-1}$ vanish modulo the chosen split prime. Their product gives $v(t) \geq 2$. Write $e = 1$ or $2$ respectively. The product of norms of the marked primes dividing $\mathfrak{b}$ is at most $B$. The product over all marked primes is $Q_0^f$, as both rational primes split completely. It follows that

$$(Q_0^f/B)^e \leq N_{F/\mathbb{Q}}(\mathfrak{b}(t)) \leq B(4k)^f.$$

Since $B = (h_{\gamma_1}h_{\gamma_2})^{2f}$, this proves the two bounds. For $h_{\gamma_l} \leq b_x^2$ their first right side is at most $4kb_x^{16} < Q_0$ for $k \geq 2$, using $g_i \geq (kb_x)^8$. Comparing each pair of cells proves the disjointness, including after summing any coefficients at repeated indices with the same transport. □

**Corollary 96.29** (The signed array separated by its rational isometry)**.** *For the cutoff $h_\gamma \leq b_x^2$, let $Z^{(j)}$ be the raw contribution of all cells with $j_\gamma = j$. For arbitrary coefficients in finite sums one has*

$$\|HZ\|^2 = \sum_j \|HZ^{(j)}\|^2.$$

*Each $HZ^{(j)}$ is read integrally by entry selection from $HZ$. If all the denominator ideals in the collection are prime to $Q_0$, every row family of $H$ recovers the single raw operator for each $j$. The normalized form equals the raw sum of squares and all corresponding minor ideals agree. These conclusions extend to absolutely summable collections.*

*Proof.* Apply disjointness in Theorem 96.28 and set entries outside the designated output support equal to zero. For denominator ideals prime to $Q_0$, every cell with the same $j$ has the same input index at every marked place by Lemma 96.27. Such an index appears in its own row with coefficient a character unit. Selecting it and undoing the unit gives a left inverse. Averaging the exact single-family norms and applying Cauchy–Binet to the two integral maps gives the remaining assertions. The sets of input and output coordinates are finite, allowing passage to absolutely convergent sums. □

**Theorem 96.30** (Survival of the individual full-rank orbital operator)**.** *For an absolutely convergent full-rank cell of identical words suppose that*

$$Q_0^2 > 4kh_\gamma^6.$$

*Then every $(a,b) \in M_{i(\gamma)}$ satisfies $aj_\gamma = j_\gamma b$ over $E$. In Theorem 96.24 one has $\Pi_{M_i}B_\gamma = B_\gamma$. Thus a nonzero single-cell operator survives the signed sum over its entire two-sided arithmetic orbit, followed by $H$. This applies on the common intervals for every $h_\gamma \leq b_x^5$. At the terminal prime the individual operator is recovered from the orbit array over any stable coefficient lattice on which the operators are defined.*

*Proof.* Write $j = j_\gamma$ and take $u = j^{-1}ajb^{-1}$ on $P_s$. The integral actions of $a, b$ preserve their column lattices. Thus in the trace argument of Theorem 96.28 the single ideal $\mathfrak{b}_\gamma$ clears the denominators of $u, u^{-1}$, since each composition uses $j, j^{-1}$ once. At every marked prime outside this ideal Lemma 96.27 and the definition of $M_i$ give $u \equiv 1$. The same argument with $e = 2$ now has $B = h_\gamma^{2f}$. For $u \neq 1$ it would give $Q_0^2 \leq 4kh_\gamma^6$.

The archimedean cell intertwines the unitary actions on the common $A/I$, by Theorem 98.2. Under its two projections these are the full compact groups of $P_s, P_t$ identified by $j$. In particular $\rho_t(a)B_\gamma = B_\gamma\rho_s(b)$ whenever $aj = jb$. The common finite phase and volume scalar preserve this equality. Therefore the entire average fixes $B_\gamma$. The raw operator at $i$ is exactly $[M_i : K_\gamma]B_\gamma$, whose scalar factor is positive and a unit at the common terminal prime. Theorem 96.26 supplies the integral left inverse for the full orbit. Finally $Q_0^2 \geq (kb_x)^{32} > 4kb_x^{30}$ for $k \geq 2$. □

This calculation permits signed weights and every lift in the chosen arithmetic orbit. Distinct geometric orbits with the same rational transport can contribute different scalars. Their sum, the cells with common positive dimension $k-1$, and the large-denominator contribution are still present in the projected theta periods of Section 6.6.

## 96.9 The marked projection on a codimension-two subspace

The small-rank terms of Theorem 117.46 include collections of matrices with a common restriction to a subspace of codimension two. On each such collection we determine the kernel of the finite projection, the precision of integral recovery, and the loss in its complex norm. All statements concern the finite matrix actually used at the two marked primes.

First work at one place. Let $V = \mathbb{F}_q^k$, $k \geq 2$, let $W \subset V$ have dimension $k-2$, and let $P$ be the pointwise stabilizer of $W$ in $G = \mathrm{GL}(V)$. Put $P^1 = P \cap \mathrm{SL}(V)$. Take $U$ upper unitriangular in a reference basis and $\theta$ a character of $U$ nontrivial on every simple root subgroup, as in Section 92.6. For a table $z$ supported on $P$ use zero extension to $G$. Its finite projection is the unnormalized matrix $H$ of Lemma 96.2. Choose a base point $b_e \in P$ of each determinant $e \in \mathbb{F}_q^\times$. Write

$$(Dz)_y = z_y - z_{b_{\det y}} \quad (y \in P,\ y \neq b_{\det y}).$$

**Theorem 96.31** (The integral differences detected by one marked projection)**.** *Over the integers of a number field containing the values of $\theta$, there are integral matrices $A, B$ with*

$$H|_P = AD, \qquad qD = B(H|_P).$$

*In characteristic zero the kernel on $P$ consists exactly of the functions constant on each determinant fiber. In particular the indicator of $P^1$ is a nonzero integral table killed by the projection. Every two indices in its support differ by rank at most two. The assertions also hold for a left and right translate of $P$, using its transported coordinates.*

*Proof.* Fix a row family $aU$ of $H$, let $F_\bullet$ be its flag, and put $U_F = aUa^{-1}$. A nonempty row entry on $P$ is a sum over $(U_F \cap P)y$ for $y \in P$. Up to a constant root of unity its coefficients are the restriction of the character $\theta_F = \theta(a^{-1}(\,\cdot\,)a)$. Choose a flag-adapted basis in which a subset of the basis spans $W$. To do this, at each step where $\dim(W \cap F_j)$ increases choose the next basis vector there, and at the other steps extend the previous basis in $F_j$. Rescaling the vectors gives the prescribed decorations. Two positions of the basis are outside the chosen subset. At least one such position $j$ is at least two. The simple shear $1 + tE_{j-1,j}$, $t \in \mathbb{F}_q$, lies in $U_F \cap P^1$, and $\theta_F$ is nontrivial on it. Nontriviality is unchanged by the chosen basis lifts, since changing lifts conjugates by a unitriangular matrix. The sum of this nontrivial additive character is zero. Partitioning the entry into cosets of the shear subgroup kills every function of the determinant. Subtracting $z_{b_{\det y}}$ at $y$ and inserting zero at the base points therefore yields the same output. This gives the integral matrix $A$.

Next choose a complement $C$ of $W$, a line $l \subset C$, and flags with $F_1 = l,\ F_2 = C$. Put $S(C, l)$ equal to the unitriangular group of $C$ fixing $l$ pointwise and acting as one on $C/l$, extended by

the identity on $W$. We have $U_F \cap P = S(C,l)$. Indeed such an element preserves $C$ and fixes $W$, and every later flag space is the sum of $C$ and its intersection with $W$. The group $S(C,l)$ is additively isomorphic to $\mathbb{F}_q$. Varying the decorations on the first two quotients of the flag realizes all its nontrivial additive characters, since nonzero scalar multiples of one nontrivial additive character give all the others. Thus for each left orbit $S(C,l)y$ the rows of $H$ contain all its nontrivial Fourier sums, up to known root-of-unity factors. For a parametrization $s_t$ of this group, Fourier orthogonality gives

$$q(z_y - z_{s_t y}) = \sum_{\chi \neq 1} (1 - \overline{\chi(t)}) \sum_{u \in \mathbb{F}_q} \chi(u) z_{s_u y}.$$

All coefficients on the right are integral.

We record a bound on the number of shears needed to join two indices. Every element of $P^1$ is a product of at most $B_0 = 13$ elements of groups $S(C,l)$, allowing the complement and the line to change. Fix one splitting $W \oplus C$. On $C$ every matrix in $\mathrm{SL}_2(\mathbb{F}_q)$ is a product of at most nine upper or lower elementary shears. One row addition makes the upper left entry nonzero if necessary, and two more clear the lower left and upper right entries. For unit $u$ put $w(u) = U(u)L(-u^{-1})U(u)$, using the upper and lower elementary matrices. Then $w(u)w(-1)$ equals $\mathrm{diag}(u, u^{-1})$, which accounts for the other six factors.

The remaining part of $P^1$ consists of the translations $t_X = \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)$, $X : C \to W$. A translation $X = w \otimes \phi$ of rank one is a product of two allowed shears. Choose $0 \neq v \in \ker \phi$ and $Y : C \to W$ with $Yv = w$. With $s = 1 + v \otimes \phi$ on $C$, the product $t_Y s t_Y^{-1} s^{-1}$ is $t_{Y(1 - s^{-1})} = t_X$. Its first shear is taken in the complement $t_Y C$. Every $X$ is a sum of at most two rank-one maps, by expanding on a basis of the two-dimensional domain. This proves the bound of thirteen, including $W = 0$.

For two indices of equal determinant their left quotient belongs to $P^1$. Telescoping the Fourier difference identity along a product of the shears just constructed expresses $q$ times their difference by integral combinations of rows of $H$. This proves the formula for $B$ and the kernel assertion. All indices in $P$ agree on $W$, giving the rank bound. For translated support $xPy$ the substitutions $a \mapsto x^{-1}a$, $h \mapsto hy^{-1}$ relabel the two systems of representatives in the formula for $H$. Changes back to the chosen representatives multiply entries by units. □

**Proposition 96.32** (A uniform lower bound on the detected norm)**.** *For Hilbert-space-valued coefficients on $P$, let $\Pi z$ be the table of their means on the determinant fibers, using counting measure. There is an absolute $c = 3/(8B_0^2) > 0$ such that*

$$c\,\|z - \Pi z\|_2^2 \le \frac{\|Hz\|_2^2}{[G:U]} \le \left( \prod_{j=1}^{k-1} (1 - q^{-j})^{-1} \right) \|z - \Pi z\|_2^2.$$

*Proof.* The kernel in Theorem 96.31 allows subtraction of $\Pi z$. The upper bound then follows from the upper operator bound in Theorem 96.5. For the lower bound take a uniform decorated flag. The probability that $F_2$ is complementary to $W$ is

$$p = \frac{q^{2(k-2)}}{\begin{bmatrix} k \\ 2 \end{bmatrix}_q} = \frac{(1 - q^{-1})(1 - q^{-2})}{(1 - q^{-k})(1 - q^{1-k})}.$$

Indeed each complement is the graph of one linear map from a fixed complement to $W$. Counting ordered independent pairs in $V$ and dividing by their number in a plane gives the Gaussian binomial in the denominator. Conditional on this event, the complement $C$ and the line $l$ in it are uniform and, conditional on the ordinary flag, the restricted character is uniform among the $q - 1$ nontrivial ones.

Write $E_S$ for the average over the left translations of $S = S(C, l)$ on $\ell^2(P)$. On each orbit $Sy$ the sum of the squares of all nontrivial Fourier sums is $q$ times the sum of squared distances to the orbit mean, by the ordinary finite Fourier identity. For a fixed row family the output coordinates meeting $P$ index these orbits once each. Averaging the squared row norms over all families and retaining only the complements bounds the normalized square below by

$$p\,\frac{q}{q-1}\,\mathbb{E}_{C,l}\|(1-E_S)z\|_2^2.$$

For an element $u \in S$ and its left translation $L_u$, $\|L_u z - z\| \le 2\|(1-E_S)z\|$, since $L_u E_S = E_S$. Every $h \in P^1$ is a product of at most $B_0$ such elements by the preceding proof. Conjugate this product by a uniform element of $P$. For each factor its conjugated pair $(C, l)$ is uniform because $P$ is transitive on complements and their lines. The triangle inequality for the product of translations and Cauchy–Schwarz give

$$\mathbb{E}_{a\in P}\|L_{aha^{-1}}z - z\|_2^2 \le 4B_0^2\,\mathbb{E}_{C,l}\|(1-E_S)z\|_2^2.$$

Averaging $h$ over $P^1$ gives $2\|z-\Pi z\|_2^2$ on the left by expanding the square. Finally $p\,q/(q-1) \ge 1 - q^{-2} \ge 3/4$. These inequalities prove the lower bound. □

**Corollary 96.33** (Differences at both auxiliary primes)**.** *Fix one codimension-two source subspace and its prescribed injective restriction at each place over $Q = g_1 g_2$. Consider any matrix $Z$ of signed coefficient tables supported there, with a common choice of restrictions for its columns. These may be the sums of the actual orbital coefficients at every supported index. After translating each local support into $P_v$, write $D_* = \bigotimes_v D_v$, $P_* = \bigotimes_v (1-\Pi_v)$, and $d_* = [G : U]$ for the product marked group. For each complex column $z$,*

$$c^{2f}\|P_* z\|_2^2 \;\le\; d_*^{-1}\|Hz\|_2^2 \;\le\; \prod_{v\mid Q}\prod_{j=1}^{k-1}(1-q_v^{-j})^{-1}\|P_* z\|_2^2.$$

*For algebraic coefficients let $I_h$ denote the fractional ideal generated by the $h$-rowed minors. At every finite place of the coefficient field,*

$$Q^{fh} I_h(D_* Z) \;\subseteq\; I_h(HZ) \;\subseteq\; I_h(D_* Z).$$

*Thus the minor ideals agree away from the two marked rational primes, including at the common terminal prime $T$. After division by $h$, the logarithmic loss from the displayed rational multiplier, averaged over the coefficient field, is at most $f \log Q = o(b/x^M)$ for every fixed $M > 0$ on the original intervals.*

*Proof.* An injective restriction extends to an isomorphism by completing bases. All matrices with this restriction form a translate of the pointwise stabilizer, and missing entries can be filled by zero. The marked matrix is the tensor product of the local matrices by its character-sum formula. The positive local Gram inequalities in Proposition 96.32 tensor in the same order, since the difference of two such products telescopes as a sum of tensor products of positive factors. Thus the bounds hold for arbitrary product-space vectors, also with Hilbert-valued entries. The integral factorizations in Theorem 96.31 tensor with multiplier $\prod_{v\mid Q} q_v = Q^f$. Taking exterior powers, or expanding minors by Cauchy–Binet, proves both inclusions. The logarithmic assertion follows from the norm of a rational principal ideal, $f \ll_\psi x^{2+8\varepsilon}$, and $\log Q = O(\log b)$. □

This result compares the finite projected table with its actual mixed differences whenever the restriction on a codimension-two space is prescribed. A determinant-constant contribution at just one place can vanish even in comparison rank two throughout that place. For the full period, contributions from different restrictions and from different orbits at one index must still be combined with their coefficients in Section 6.6.

### 96.10 The spectrum after fixing a codimension-two subspace

We compute the normalized Gram operator on the support of Section 96.9. This improves its lower norm bound to a factor tending to one over the entire marked field. In particular the finite projection on the detected subspace incurs a vanishing relative error even when the number of archimedean places grows.

At one marked place write $V = W \oplus C$, $\dim_{\mathbb{F}_q} C = 2$, $h = \dim W = k - 2$. The elements of the pointwise stabilizer $P$ of $W$ are

$$g(X, A) = \begin{pmatrix} 1 & X \\ 0 & A \end{pmatrix}, \qquad X \in N = \mathrm{Hom}(C, W), \quad A \in \mathrm{GL}(C).$$

Let $\mathcal{G}$ be the compression of $H^* H / [G : U]$ to functions supported on $P$. At the rank-two group $\mathrm{GL}(C)$ write $\mathcal{G}_2$ for its full normalized finite Gram matrix. For a line $l \subset C$ let $U_l$ be the subgroup fixing $l$ and $C/l$ pointwise. It has order $q$. Fix a nontrivial additive character $\psi$ of $\mathbb{F}_q$, and put

$$\eta_k = \{(1 - q^{-k})(1 - q^{1-k})\}^{-1}, \qquad p_0 = (1 - q^{-1})(1 - q^{-2}).$$

**Theorem 96.34** (Fourier decomposition of the restricted Gram operator)**.** *For each $D \in \mathrm{Hom}(W, C)$, the functions $z(g(X, A)) = \psi(\mathrm{tr}(DX))\, v(A)$ form an invariant space for $\mathcal{G}$. These spaces are mutually orthogonal and exhaust the arrays on $P$. The operator induced on $v$ is as follows.*

(i) *For $D = 0$ it is $\eta_k p_0 \mathcal{G}_2$.*

(ii) *For $\mathrm{rank}\, D = 1$ put $l = \mathrm{im}\, D$. On each set $AU_l$ it is $\eta_k (I - q^{-2} \mathbf{J}_q)$, where $\mathbf{J}_q$ has every entry equal to one.*

(iii) *For $\mathrm{rank}\, D = 2$ it is $\eta_k I$.*

*Only the ranks realizable in $\mathrm{Hom}(W, C)$ occur.*

*Proof.* The trace pairing between $N$ and $\mathrm{Hom}(W, C)$ is perfect, as can be checked on rectangular matrix units. Multiplication by a nonzero field element is surjective on $\mathbb{F}_q$, so the displayed characters are distinct. Finite Fourier orthogonality proves the asserted decomposition of the underlying vector space.

For two indices, their relative matrix in the target has blocks

$$g(X', A') g(X, A)^{-1} = g(Y, M), \qquad Y = (X' - X) A^{-1}, \quad M = A' A^{-1}.$$

By Corollary 92.19 the Gram entry vanishes unless $M$ is unipotent. For unipotent $M$ its value depends on the rank of the stacked map $v \mapsto (Yv, (M - 1)v)$ from $C$. At ranks zero, one and two the values are respectively

$$1, \quad a_1 = -\frac{1}{q^k - 1}, \quad a_2 = \frac{1}{(q^k - 1)(q^{k-1} - 1)}.$$

If rank two is absent, the formulas below only evaluate entries at the ranks that occur. Substitution $X' = X + YA$ in a Fourier mode removes its common factor $\psi(\mathrm{tr}(DX))$. The remaining character on $Y$ is that indexed by $B = AD$. This proves invariance of each Fourier mode.

Set $N_0 = q^h$. For a fixed line $l' \subset C$, the maps $Y$ that kill $l'$ form a group of order $N_0$. Its character sum at $B$ is $N_0$ if $\mathrm{im}\, B \subseteq l'$, and zero otherwise, by the trace pairing on $C/l'$. Every map of rank one is counted once by its kernel line, and the zero map is counted $q + 1$ times.

Hence the Fourier sum of the indicator of $\operatorname{rank} Y \le 1$ is $N_0 n_B - q$, where $n_B$ is the number of lines containing $\operatorname{im} B$. For ranks zero, one and two of $B$ it equals $q+1, 1, 0$, respectively.

At $M = 1$ the entry as a function of $Y$ is $a_2 + (a_1 - a_2)\mathbf{1}_{\operatorname{rank} Y \le 1} + (1 - a_1)\mathbf{1}_{Y=0}$. The diagonal coefficient in the mode $D$ is consequently

$$a_2 N_0^2 \mathbf{1}_{D=0} + (a_1 - a_2)(N_0 n_B - q) + (1 - a_1).$$

For $M \neq 1$ unipotent in dimension two, put $l' = \ker(M - 1)$. The stacked map has rank one exactly when $Y$ kills $l'$. Its Fourier sum is therefore

$$a_2 N_0^2 \mathbf{1}_{D=0} + (a_1 - a_2) N_0 \, \mathbf{1}_{\operatorname{im} B \subseteq l'}.$$

The three scalar simplifications are

$$1 - a_1 - q(a_1 - a_2) = \eta_k, \qquad (a_1 - a_2) N_0 = -\eta_k / q^2, \qquad a_2 N_0^2 = \eta_k / q^3.$$

For rank two this gives the scalar operator in (iii). For rank one, an off-diagonal coefficient occurs exactly when $M$ belongs to $U_{Al} \setminus \{1\}$, or equivalently when $A' \in AU_l \setminus \{A\}$. Together with the diagonal coefficient this gives (ii). In rank zero the diagonal is $\eta_k p_0$, and the value at each nonidentity unipotent is $-\eta_k p_0 / (q^2 - 1)$. All remaining values are zero. These are the entries in (i) by the same character formula in dimension two. □

**Corollary 96.35** (Near-isometry on the determinant differences)**.** *Let* $\Pi$ *denote the determinant-fiber average on* $P$*. In the order on Hermitian forms,*

$$\eta_k (1 - q^{-1})^2 (I - \Pi) \;\le\; \mathcal{G} \;\le\; \eta_k (I - \Pi).$$

*On any product of the prescribed restriction cosets at the two auxiliary primes use* $P_*$ *of Corollary 96.33. The corresponding bounds are* $a_* P_*$ *and* $b_* P_*$*, where*

$$a_* = \prod_{v \mid Q} \eta_k(q_v)(1 - q_v^{-1})^2, \qquad b_* = \prod_{v \mid Q} \eta_k(q_v).$$

*Here* $\eta_k(q_v)$ *denotes the local value defined above. Uniformly on the period intervals,*

$$0 \le 1 - a_* \ll f / q_*, \qquad 0 \le b_* - 1 \ll f q_*^{1-k}, \qquad q_* = \min(g_1, g_2).$$

*Proof.* In the zero Fourier mode the determinant depends only on $A \in \mathrm{GL}_2$. By Theorem 96.31 with $k = 2$ the kernel of $\mathcal{G}_2$ is exactly the space of functions of this determinant. On its orthogonal complement the two bounds from Theorem 96.5 specialize to $(1 - q^{-1})/(1 - q^{-2})$ and $(1 - q^{-1})^{-1}$. Multiplication by $\eta_k p_0$ gives the lower coefficient in the statement and the upper bound $\eta_k(1 - q^{-2})$ on that complement. Each nonzero mode has mean zero on every determinant fiber by summation over $X$. In rank one its two eigenvalues are $\eta_k(1 - q^{-1})$ and $\eta_k$, by the constant and zero-sum decomposition on $AU_l$. Rank two has the single eigenvalue $\eta_k$. This proves the local inequalities.

Tensor the positive Gram inequalities over the $2f$ places, as in Corollary 96.33. Each lower factor is at most one since $k \ge 2$, and is at least $(1 - q_v^{-1})^2$. Each upper factor has logarithm $O(q_v^{1-k})$, by $-\log(1 - t) \le 2t$ for $0 \le t \le 1/2$. The prime bound $q_* \ge (kb)^8$ and polynomial growth of $f$ give the claimed estimates upon summing and exponentiating. □

**Proposition 96.36** (Effect on the Gram blocks of signed inputs)**.** *On each target row prescribe one of the product supports in Corollary 96.33. Allow a distinct support on each target row. Write* $Z_i$ *for any incoming column operator to the direct sum of these tables, including their*

*original signed coefficients. Let $Y_i = P_* Z_i$, using the appropriate projection on each row, and $V_i = [G : U]^{-1/2} H Z_i$. Put $\delta = \max(1 - a_*, b_* - 1)$. Then*

$$\|V_i^* V_j - Y_i^* Y_j\|_{\mathrm{HS}} \le \delta \min\{\|Y_i\|_{\mathrm{op}} \|Y_j\|_{\mathrm{HS}}, \|Y_i\|_{\mathrm{HS}} \|Y_j\|_{\mathrm{op}}\}.$$

*For the full matrix $Y$ of these incoming columns and the corresponding $V$, the two ranks agree. On every subspace of dimension $u$ on which $Y$ is injective the ratio of the Gram determinants for $V$ and $Y$ lies between $a_*^u$ and $b_*^u$.*

*Proof.* On each target support let $B$ be its compressed normalized finite Gram operator. The preceding corollary implies $B = P_* B P_*$ and $\|B - P_*\|_{\mathrm{op}} \le \delta$. The same holds on the direct sum. Thus the mixed difference is $Y_i^*(B - P_*)Y_j$. Applying the operator norm on the left and the Hilbert–Schmidt norm on the right proves the first product bound. Taking adjoints proves the other. The forms for the full maps satisfy $a_* Y^* Y \le V^* V \le b_* Y^* Y$, giving equality of kernels. On the indicated subspace conjugate its restricted Gram form for $V$ by the inverse positive square root of the restricted form for $Y$. All eigenvalues lie in $[a_*, b_*]$. Their product is the claimed determinant ratio. □

These estimates and the integral inclusions of Corollary 96.33 apply on exactly the same supports. The computation controls the contribution of the finite projection to the norm there. The sizes of the signed differences $Y_i$ supplied by the arithmetic periods and the cross terms between different restriction supports on one target row still enter the comparison in Section 6.6.

## 96.11 Mixed periods on different restriction subspaces

Two signed inputs to one target row can have distinct restrictions at a marked place. The dimension on which those restrictions agree controls the size of the finite groups needed to compute their mixed inner product. We first prove the reduction at one place. All inner products use counting measure.

Write $G_j = \mathrm{GL}_j(\mathbb{F}_q)$, with $G_0$ the trivial group. For $0 \le d \le k$ let $P_{k,d} \subset G_k$ fix pointwise a specified subspace $W$ of dimension $w = k - d$. Let $\mathcal{G}_{k,d}$ be the compression to $P_{k,d}$ of the normalized Gram matrix of Lemma 96.2. Thus $\mathcal{G}_{j,j}$ acts on the full group $G_j$, and $\mathcal{G}_{0,0} = 1$. Write

$$\eta_{k,d} = \prod_{i=k-d+1}^{k} (1 - q^{-i})^{-1}, \qquad p_e = \prod_{i=1}^{e} (1 - q^{-i}).$$

Empty products equal one. Translating the support by any fixed element of $G_k$ gives the same operator in the translated coordinates, since its entries are values of a class function on relative matrices.

**Theorem 96.37** (Reduction to the dimension of the moving subspace)**.** *There is an orthogonal decomposition of $\mathcal{G}_{k,d}$ into blocks, each unitarily equivalent to*

$$\eta_{k,d}\, p_e\, \mathcal{G}_{e,e} \qquad (0 \le e \le d).$$

*The following rule specifies the blocks and the transform on the inputs. Choose a complement $C$ of $W$. Fourier transform the coordinate $X \in \mathrm{Hom}(C, W)$ in the matrices $g(X, A) = \left(\begin{smallmatrix} 1 & X \\ 0 & A \end{smallmatrix}\right)$. For frequency $D : W \to C$ put $t = \operatorname{rank} D$. Group the remaining indices $A \in G_d$ by left cosets of the pointwise stabilizer of $\operatorname{im} D$. For $t = 0$ stop with $e = d$. For $t > 0$ continue inside each coset with ambient dimension $d$ and moving dimension $d - t$. A group of dimension zero contributes the identity. On the orthogonal complement of the kernel the eigenvalues of the original operator lie in*

$$[\, \eta_{k,d}(1 - q^{-1})^d,\ \eta_{k,d}\,].$$

*Proof.* Fix a nontrivial additive character $\psi$ of the residue field. The characters $\psi(\operatorname{tr}(DX))$ give every Fourier mode by the perfect pairing of rectangular matrices. For two indices put $M = A'A^{-1}$ and $Y = (X' - X)A^{-1}$. The relative matrix is $g(Y, M)$. By Corollary 92.19, its entry is zero unless $M$ is unipotent. Otherwise put

$$K = \ker(M - 1) \cap \ker Y, \qquad a_k(r) = \frac{(-1)^r}{\prod_{i=k-r+1}^{k}(q^i - 1)}.$$

The entry is $a_k(d - \dim K)$, since the relative matrix minus the identity has rank $d - \dim K$. Acting on one Fourier mode replaces the sum over $X'$ by the sum over $Y$ with character indexed by $B = AD$, times the original character at $X$.

For $0 \le j \le d$ set

$$c_j = a_k(d)(-1)^j q^{wj + j(j+1)/2}.$$

For a subspace $K$ of dimension $s$ one has $a_k(d - s) = \sum_{L \subseteq K} c_{\dim L}$. Indeed, writing $\begin{bmatrix} s \\ j \end{bmatrix}_q$ for the number of $j$-dimensional subspaces of an $s$-space, the sum divided by $a_k(d)$ equals $\prod_{i=0}^{s-1}(1 - q^{w+1+i})$. Here we use the polynomial identity

$$\sum_{j=0}^{s} \begin{bmatrix} s \\ j \end{bmatrix}_q (-1)^j q^{j(j-1)/2} z^j = \prod_{i=0}^{s-1}(1 - zq^i).$$

It follows by induction on $s$, counting subspaces inside a fixed hyperplane and those meeting it in codimension one. The latter number with prescribed intersection of dimension $j - 1$ is $q^{s-j}$. Cancelling the factors of $a_k(d)$ proves the asserted subspace sum.

For $L \subseteq \ker(M - 1)$ of dimension $j$, the sum of the character at $B$ over maps $Y$ killing $L$ is $q^{w(d-j)}$ if $\operatorname{im} B \subseteq L$, and zero otherwise. This follows by applying the trace pairing to $C/L$. Thus the transformed entry can be nonzero only if $M$ fixes $\operatorname{im} B$ pointwise. This is precisely $A' \in A \operatorname{Stab}_{\mathrm{pt}}(\operatorname{im} D)$. In that case let $s = \dim \ker(M - 1)$. Use

$$c_j q^{w(d-j)} = \eta_{k,d}(-1)^{d-j} q^{-d(d+1)/2 + j(j+1)/2}.$$

There are $\begin{bmatrix} s-t \\ j-t \end{bmatrix}_q$ eligible $L$ for each $j \ge t$. The same polynomial identity, now at $z = q^{t+1}$, gives the sum

$$\eta_{k,d}(-1)^{d-s} q^{-d(d+1)/2 + t(t+1)/2} \prod_{i=t+1}^{s} (q^i - 1) = \frac{\eta_{k,d}}{\eta_{d,d-t}}\, a_d(d - s).$$

For nonunipotent $M$ both matrix entries are zero. Conjugating by the representative of the coset identifies the result with $\eta_{k,d}\mathcal{G}_{d,d-t}/\eta_{d,d-t}$. Unitarity follows by normalizing each Fourier transform by the inverse square root of the size of its additive group. The successive factors cancel. At the last step $1/\eta_{e,e} = p_e$, proving the stated formula.

For $e \ge 1$, Theorem 96.5 gives the lower bound $(1 - q^{-1})^e/p_e$ on the positive spectrum of $\mathcal{G}_{e,e}$. Proposition 92.22 gives the upper bound $\prod_{i=1}^{e-1}(1 - q^{-i})^{-1}$. Multiplying both bounds by $\eta_{k,d}p_e$ and using $e \le d$ proves the inequalities. The scalar at $e = 0$ satisfies them as well. □

**Corollary 96.38** (A common transform for mixed signed periods)**.** *At each marked place $v$ suppose the two tables are supported, respectively, on $g_{1,v}P_{W_{1,v}}$ and $g_{2,v}P_{W_{2,v}}$. Here $g_{i,v}$ are invertible and $P_W$ fixes $W$ pointwise. Put*

$$W_v = (W_{1,v} \cap W_{2,v}) \cap \ker(g_{1,v} - g_{2,v}), \qquad d_v = k - \dim W_v.$$

*Apply the theorem on the coset $g_{1,v}P_{W_v}$, tensor over the marked places, and let $\mathcal{F}$ denote the resulting unitary transform. On each final block apply the orthogonal projection onto the*

*generic part of each $G_{e_v}$, as defined in Theorem 96.5, with the identity for $e_v = 0$. Transport the product of these projections back by $\mathcal{F}$ and take their direct sum, denoted by $\Pi$ in the original coordinates. For the two tables $z, z'$, extended by zero to the common support,*

$$\left| \frac{\langle Hz, Hz' \rangle}{[G : U]} - \langle \Pi z, \Pi z' \rangle \right| \ll \frac{\sum_v d_v}{q_*} \, \|\Pi z\| \, \|\Pi z'\|, \qquad q_* = \min(g_1, g_2).$$

*Here $g_1, g_2$ in $q_*$ are the two auxiliary primes. The exact mixed product is obtained by substituting the blocks of the theorem in place of the projections. If each $W_{i,v}$ has codimension two, then $\sum_v d_v \le 8f + \sum_v \operatorname{rank}(g_{1,v} - g_{2,v})$. All statements hold for Hilbert-space-valued signed coefficients on the original period intervals.*

*Proof.* Both supports prescribe the map $g_{1,v}|_{W_v}$ on $W_v$. Translate by $g_{1,v}^{-1}$ and apply the theorem at each place. By Lemma 96.2 the mixed product is exactly the inner product for the tensor of these compressed Gram operators. Its kernel is the orthogonal complement of the indicated surviving blocks. Its positive eigenvalues are bounded below by $\prod_v (1 - q_v^{-1})^{d_v}$ and above by $\prod_v \eta_{k,d_v}(q_v)$. The logarithms of these bounds are $O(\sum_v d_v / q_*)$ in absolute value. This tends to zero uniformly since $\sum_v d_v \le 2fk$, $q_* \ge (kb)^8$, $f, k$ are polynomially bounded in $x$, and $b$ grows faster than every power of $x$. Subtract the identity on the surviving blocks and use Cauchy–Schwarz. Tensoring with the identity on the coefficient space gives the vector-valued assertion. Finally codimension is subadditive under intersection, and imposing the equality of the two maps costs at most the rank of their difference. Summing over the $2f$ places gives the dimension estimate. □

The same construction applies to any collection on a single target row by taking the common subspace of agreement of all its prescribed maps. It retains every sign before the last projections. The size of the resulting generic components of the arithmetic tables still enters the period comparison in Section 6.6.

## 96.12 Integral matrices for mixed periods at the terminal prime

The unitary transform in Section 96.11 computes mixed inner products. Integral periods require the matrix $H$ itself, before division by $[G : U]$. We give a transform for its rows using only character values and Fourier sums. It compares the ideals of minors even at a prime dividing $[G : U]$, provided that it differs from the residue characteristic of the mark.

At one place use $W, C, P_{k,d}$ as in Theorem 96.37, with $w = k - d$. Let $R$ be the ring of integers in a number field containing the additive character values. Let $\Gamma_j$ denote the induced representation $R[G_j] \otimes_{R[U_j]} R_\theta$, where $\theta$ is the nondegenerate upper unitriangular character. Use the same additive character in all dimensions, and set $\Gamma_0 = R$. In its coset basis the matrix $\sum_g z_g \Gamma_j(g)$, with its entries listed as rows of a linear map in $z$, is $H_j z$. Indeed its entry at the basis vectors $h^{-1}U_j, aU_j$, in source and target respectively, is the sum of $\theta(u) z_{auh}$ over $u \in U_j$.

**Theorem 96.39** (Integral reduction at a single mark)**.** *For arrays on $P_{k,d}$ define the following collection of smaller arrays. Take the orbits of $G_d$ acting by postcomposition on the maps $B : W \to C$. In each orbit choose $B$, list its members $B_i = a_i B$ with $a_i \in G_d$, and put $S_B = \{s \in G_d : sB = B\}$. For every ordered pair $i, j$ form*

$$z_{ij}(s) = \sum_{X : C \to W} \psi\big(\operatorname{tr}(B_j X)\big) z\big(g(X, a_i s a_j^{-1})\big), \qquad s \in S_B.$$

*For $B = 0$ retain $H_d z_{11}$. Otherwise repeat the operation on $S_B$, the pointwise stabilizer of a $t$-space in dimension $d$, where $t = \operatorname{rank} B$. A terminal trivial group gives its coefficient itself.*

*Let $L_{k,d}z$ be the stack of all the retained rows. Over $R[1/q]$ the row modules of $H_k|_{P_{k,d}}$ and $L_{k,d}$ coincide inside the dual of the same coefficient lattice. In particular there are matrices over this ring expressing either set of rows in terms of the other. The assertion holds after left translation of the support.*

*Proof.* Work first over the completion $O$ of the coefficient integers at a prime of residue characteristic different from that of $\mathbb{F}_q$. We may extend it to the integers of a finite splitting extension for the finite groups under consideration, in both the ordinary and the residual characteristic. Descent will be justified at the end. Write $n(X) = g(X,1)$, $l(A) = g(0,A)$, so $g(X,A) = l(A)n(X)$. Since $q$ is a unit, the additive group of the $n(X)$ has orthogonal character idempotents

$$e_B = q^{-wd} \sum_X \psi(-\operatorname{tr}(BX))\, n(X), \qquad \sum_B e_B = 1.$$

Conjugation by $l(A)$ sends $e_B$ to $e_{AB}$. Identify all the summands $e_{B_i}\Gamma_k$ in one orbit with $e_B\Gamma_k$ using $l(a_i)$. These summands are free over the discrete valuation ring as direct summands of a free lattice. The block with source $j$ and target $i$ in the matrix of the group algebra element $\sum_g z_g g$ is now precisely

$$\sum_{s \in S_B} z_{ij}(s)\, l(s)|_{e_B\Gamma_k}.$$

Indeed $n(X)$ acts on the source by its character at $B_j$, and $l(A)$ carries this source to $i$ exactly when $A = a_i s a_j^{-1}$. Other orbits give separate blocks.

We identify $e_B\Gamma_k$ as an $O[S_B]$-module. In characteristic zero its character at $s \in S_B$ is the trace of $e_B l(s)$ on $\Gamma_k$. In the expression for $e_B$, $n(X)l(s) = g(Xs, s)$ and $sB = B$. Thus replacing $Xs$ by the summation variable leaves its character factor unchanged. The Fourier sum computed in Theorem 96.37, at frequency $-B$, gives

$$\operatorname{tr}(e_B l(s) \mid \Gamma_k) = \frac{\gamma_k(1)}{q^{wd}\gamma_d(1)} \frac{\eta_{k,d}}{\eta_{d,d-t}}\, \operatorname{tr}(s \mid \Gamma_d), \qquad \gamma_j(1) = [G_j : U_j] = \prod_{a=1}^{j} (q^a - 1).$$

The rank and image used in that sum are unchanged by the sign. The expression is valid at nonunipotent elements as both character values vanish there. The identity for the degree counts ordered bases and divides by $|U_j| = q^{j(j-1)/2}$. Cancelling the factors in the ratio gives the positive integer

$$m = q^{t(t+1)/2} \prod_{a=t+1}^{w} (q^a - 1).$$

Here $t \le w$ because $B$ has domain $W$.

Both representations being compared are projective over $O[S_B]$. In fact $\Gamma_j$ is induced from a rank-one summand of the regular module of $U_j$, using the character average over the group of unit order in $O$. Induction therefore makes it projective over $O[G_j]$, and restriction to a subgroup preserves projectivity because the group algebra is free over the subgroup algebra. Apply this to $S_B \subseteq G_d$ and to $l(S_B) \subseteq G_k$. The idempotent $e_B$ commutes with $l(S_B)$, so its image remains projective.

We use here the usual character theorem for projective modules over a splitting modular system. The reductions of indecomposable projective lattices give the indecomposable projective modules over the residue field, each with a unique projective lift. The ordinary characters of these lifts are linearly independent. Consequently equality of ordinary characters determines all the multiplicities in a direct sum of projectives. Applying this to the character equality just proved yields the integral isomorphism

$$e_B\Gamma_k \simeq (\Gamma_d|_{S_B})^{\oplus m}.$$

These statements about projectives are Proposition 9.4.5 and Corollary 10.2.4 of Webb. [8]

Choose $O$-bases using this isomorphism in each orbit. Vectorizing an integral change of basis on source and target gives an invertible coordinate change on matrix entries. The transformed output consists of zeros and repeated copies of the entries of $\sum_s z_{ij}(s)\Gamma_d(s)$, with at least one copy for every $i, j$. Row span is unchanged on removing these zeros and repetitions. This proves the recursion over the splitting completion. When $t = 0$ the smaller group is all of $G_d$ and the retained matrix is $H_d$. When $t > 0$ its moving dimension $d - t$ is smaller. When the moving dimension is zero the support is a singleton, and the identity matrix has a unit entry. Induction proves equality with the stated final rows, including the case of trivial groups.

All the specified arrays and both row modules are defined over $R[1/q]$. A finite extension of complete discrete valuation rings is faithfully flat, as is completion of a discrete valuation ring. Equality of the two row modules descends along these extensions, by applying faithful flatness to the quotient of their sum by each one. Equality at all maximal ideals of $R[1/q]$ gives equality over the ring itself. Finally left translation multiplies the output matrix by the invertible matrix of the translating group element, which preserves its row module. □

**Corollary 96.40** (The terminal ideals for inputs with different restrictions)**.** *Use the fields, two auxiliary primes $g_1, g_2$, and common supports of Corollary 96.38. Put $Q = g_1 g_2$. On each target row take the tensor of the corresponding $L_{k,d_v}$ and denote the resulting map, summed directly over targets, by $L$. For any matrix $Z$ of integral signed input columns on these supports and each minor size $u$,*

$$I_u(HZ)\, R[1/Q] = I_u(LZ)\, R[1/Q].$$

*In particular the two matrices have exactly the same determinantal valuations and nonzero Smith elementary divisors up to units at every prime over the terminal prime $T$. The factors on the right are computed on groups of dimensions at most $d_v$, for the very same columns and for every $x$ on the period interval.*

*Proof.* Use the two row transformations from the theorem at each place. They still express either tensor product through the other over $R[1/Q]$. Take their direct sums and compose with $Z$ on the right. Cauchy–Binet gives the two inclusions of determinantal ideals. At a prime above $T$ the auxiliary primes are units. Over the resulting discrete valuation ring the valuations of the nonzero determinantal ideals are the successive partial sums of the Smith exponents. Their differences determine the individual exponents. This also preserves the rank. □

Thus the denominators in a normalized complex Gram matrix cause no loss at $T$ in this reduction of the finite projection. The remaining estimates concern the sizes and divisibility of the signed arithmetic coefficients after the indicated Fourier sums, including the other places in the integral theta period.

## 96.13 Global heights of the reduced signed period matrices

We compare the matrices of Section 96.12 at the two auxiliary primes themselves and at every complex embedding. The comparison includes the mixed inner products used in Proposition 6.9. Write $q = p^a$ at one mark, $w = k - d$, $P = P_{k,d}$, and

$$\gamma_j = \prod_{i=1}^{j} (q^i - 1), \qquad b_P = wd + \tbinom{d}{2} = kd - \tbinom{d+1}{2}, \qquad U_{k,d} = \gamma_k / \gamma_w.$$

The symbol $b_P$ here gives the exponent in the $p$-part $q^{b_P}$ of $|P|$. We use the original unnormalized $H_k$ and the integral transform $L = L_{k,d}$ of Theorem 96.39.

**Lemma 96.41** (Multiplicities in the reduced period matrix)**.** *The rows of $L$ are the matrix entries, with zeros omitted if necessary, of a representation $\tau$ of $P$ by monomial matrices with roots of unity as nonzero entries. In characteristic zero $\tau$ is multiplicity-free. Its constituents are exactly the constituents of $\Gamma_k|_P$. Group its output rows $L_\nu$ by the successive choices of character orbits in the reduction. At each stage keep all ordered source and target pairs from that orbit in the same group. For every $z, z'$ on $P$ and at every complex embedding one has*

$$\frac{\langle H_k z, H_k z' \rangle}{\gamma_k} = \sum_\nu \frac{m_\nu}{\gamma_k} \langle L_\nu z, L_\nu z' \rangle, \qquad \gamma_w \le m_\nu \le \gamma_k, \qquad v_p(m_\nu) \le a\tbinom{d+1}{2}.$$

*The inner products on the right are the usual ones on matrix entries. Every constituent in group $\nu$ occurs $m_\nu$ times in $\Gamma_k|_P$.*

*Proof.* If $d = 0$, $P$ is trivial. Use its trivial representation and multiplicity $\gamma_k$, a $p$-adic unit. Otherwise write $N = \mathrm{Hom}(C, W)$ for the normal translation subgroup. For a dual character $B$ with stabilizer $S_B \subseteq G_d$, its character $\chi_B$ extends to $N \rtimes S_B$ by giving it value one on $S_B$. With the notation of Theorem 96.39, the summand of $\Gamma_k|_P$ belonging to the orbit of $B$ is induced from $\chi_B \otimes e_B \Gamma_k$ on $N \rtimes S_B$, where the second factor denotes the action of $S_B$ on that space. Indeed its $N$-eigenspaces are indexed by that orbit and are carried one to another by $l(a_i)$. In characteristic zero the calculation of the trace on $e_B \Gamma_k$ in that theorem gives $m(k, d, t)$ copies of $\Gamma_d|_{S_B}$, where

$$m(k, d, t) = q^{t(t+1)/2} \frac{\gamma_w}{\gamma_t}, \qquad t = \operatorname{rank} B.$$

Define $\tau$ recursively by retaining one copy at each such step. End with $\Gamma_d$ when $t = 0$; in moving dimension zero use the trivial representation. Induction of a representation by monomial matrices and tensor product with $\chi_B$ again give monomial matrices in the coset basis. Their block from $j$ to $i$ is precisely the action of the array $z_{ij}$ in the theorem there. Induction through the steps identifies their entries with the rows asserted in the lemma.

For irreducible representations of $S_B$, the corresponding induced representations are irreducible and are distinguished by their orbit and their original $S_B$-representation. Indeed every invariant subspace splits into its $N$-eigenspaces. Its piece at $B$ is stable under $S_B$, and determines all the other pieces by translation. The same argument applied to an intertwiner shows that inequivalent choices at $B$, or different orbits, cannot give equivalent representations. At the last stage $\Gamma_e$ is multiplicity-free by Theorem 96.5, also in dimension one. This proves the assertion on multiplicities.

Along an orbit path the factors $m(k, d, t)$ multiply, with the factor $\gamma_j$ if the path ends at a singleton in ambient dimension $j$. The first factor is at least $\gamma_w$ because $\gamma_t \le q^{t(t+1)/2}$; all later factors are positive integers. A nonzero representation of multiplicity $m_\nu$ consumes at least $m_\nu$ dimensions of $\Gamma_k$. Finally the moving dimension decreases by $t$ at each positive step, so the successive $t$'s sum to at most $d$. Since all $\gamma_j$ are $p$-adic units and $\sum_i t_i(t_i+1)/2 \le d(d+1)/2$, this proves the last bound.

The action at any embedding is unitary in the original monomial bases. Character equality gives a unitary isomorphism with the indicated orthogonal sum: an intertwining isomorphism $A$ can be made unitary using $A(A^*A)^{-1/2}$. Unitary conjugation preserves the inner product of matrices. Each copy contributes the same mixed product of entries. This gives the identity, including for signed inputs. □

**Theorem 96.42** (The loss at the residue characteristic)**.** *Write $\mathcal{R}_H, \mathcal{R}_L$ for the modules generated by the respective integral rows in the dual of the coefficient lattice on $P$. Over the full coefficient ring of integers one has*

$$q^{2b_P + \binom{d+1}{2}} \mathcal{R}_L \;\subseteq\; \mathcal{R}_H, \qquad q^{2b_P} \mathcal{R}_H \;\subseteq\; \mathcal{R}_L.$$

*Away from $p$ the two modules are equal. The bounds apply to a translated support and to arrays with a common coefficient lattice at the mark.*

*Proof.* Work locally at a prime over $p$, first after a finite splitting extension. Let $\rho$ be either $\Gamma_k|_P$ or $\tau$ and let $T_\rho$ take the entries of $\sum z_g\rho(g)$. The coefficientwise complex conjugates in $T_\rho^*$ are again roots of unity or zero. On functions on $P$, the matrix $T_\rho^*T_\rho$ is central convolution with the character of $\rho$, with inversion according to the convolution convention. In fact its $g,h$ entry is $\operatorname{tr}\rho(g^{-1}h)$. Character orthogonality gives its nonzero scalars $|P|m_\pi/d_\pi$, where $d_\pi$ is the irreducible degree and $m_\pi$ the multiplicity in $\rho$. Index the central projectors in this convention, replacing $\pi$ by its dual if necessary. Each projector $e_\pi$ has coefficients of the form $d_\pi\chi_\pi(g)/|P|$, possibly with $g$ inverted. The character values are algebraic integers since each eigenvalue of a finite-order matrix is a root of unity.

On the common support of the two representations take

$$C_\rho = \sum_{\pi:\, m_\pi>0} \frac{d_\pi}{|P|m_\pi} e_\pi.$$

Thus $C_\rho T_\rho^*T_\rho$ is the identity on those constituents and zero on their complement. If $\rho'$ is the other representation, its matrix of entries satisfies

$$T_{\rho'} = T_{\rho'}C_\rho T_\rho^*T_\rho.$$

The identity first holds over the splitting field in characteristic zero by the scalar calculation. All its entries are algebraic, so it holds also in the chosen completion. Each denominator in $C_\rho$ divides $|P|^2m_\pi$ for its summand. Since $|P| = q^{wd}|G_d|$, its $p$-part is $q^{b_P}$. Lemma 96.41 now clears all denominators at $p$ by $q^{2b_P+\binom{d+1}{2}}$ for the original representation and $q^{2b_P}$ for the reduced one. The displayed factorization gives the two inclusions. Faithful flatness descends them to the original local integers. At primes away from $p$ Theorem 96.39 gives equality. Localizing at all maximal ideals proves the global inclusions. Left translation changes the original output by an invertible monomial matrix, and tensoring row transformations with the identity preserves them. □

**Corollary 96.43** (All-place comparison on the original period fields)**.** *Use the same marked places, translates and fields as in Corollary 96.40. On each target row $\alpha$ choose one common support for all its incoming signed columns, with moving dimensions $d_{\alpha v}$. Form the product of the local transforms on that row and then the direct sum over rows. Denote the full original and reduced maps by $H, L$, and normalize $H$ at infinity by dividing on each row by $(\prod_v \gamma_k(q_v))^{1/2}$; write $H^\circ$ for that metric normalization. In the $L$-coordinates put the squared-norm weight $\prod_v m_{\nu_v}/\gamma_k(q_v)$ on each final block of that row. With this metric, the complex Gram matrices of $H^\circ Z$ and $LZ$ coincide for every family $Z$ of signed input columns and at every embedding.*

*For the unweighted usual metric on $L$, let $h$ be the logarithmic height of the exterior product of $u$ independent output columns. Use the original integral matrices at finite places. Uniformly for every such choice over a coefficient field $J$,*

$$\frac{|h(H^\circ Z) - h(LZ)|}{u} \;\ll\; k \sum_{v\mid g_1g_2} (\max_\alpha d_{\alpha v}) \log q_v \;\ll\; fk^2 \log(g_1g_2).$$

*This bound is $o(b_x/x^A)$ for each fixed $A>0$ at all $x$ on the period intervals, where $b_x = \exp(x^\varepsilon)$. The ranks of the two outputs agree. At the terminal prime their determinantal ideals agree exactly.*

*Proof.* The mixed-product identity of the lemma tensors over the marks and adds over target rows, proving the exact Gram assertion. With the unweighted metric on $L$, put $U = \max_{\alpha} \prod_{v} U_{k,d_{\alpha v}}(q_v)$. The squared norm for $H^{\circ}$ on any input then lies between $U^{-1}$ times the squared norm for $L$ and the squared norm for $L$. This proves equality of kernels, also after taking any subfamily. Restrict the forms to the span of $u$ independent outputs using their common input quotient. Taking determinants there bounds the ratio of the exterior norms between $U^{-u/2}$ and one.

Let $D$ be the product over $v$ of $q_v$ to the largest exponent $2b_P + \binom{d_{\alpha v}+1}{2}$ on a target row at $v$. Theorem 96.42 gives integral row transformations in both directions after multiplication by $D$, by tensoring on each row and taking the direct sum. Cauchy–Binet therefore gives $D^{u} I_u(LZ) \subseteq I_u(HZ)$ and $D^{u} I_u(HZ) \subseteq I_u(LZ)$ for the fractional ideals of the $u$-rowed minors. The bound for the normalized difference of their logarithmic ideal norms is $u \log D$. These ideals agree at $T$ by Corollary 96.40.

Explicitly, the height used here is the sum of the logarithms of the exterior norms over all embeddings of $J$, less the logarithm of the absolute norm of the minor ideal, all divided by $[J : \mathbb{Q}]$. The product formula makes it independent of a common change of basis of the input columns. Combining the finite and infinite estimates costs at most $\log D + \frac{1}{2} \log U$ per rank. Now $2b_P + \binom{d+1}{2} = 2kd - \binom{d+1}{2} \le 2kd$ and $\log U_{k,d} \le kd \log q$. There are $f$ places over each of $g_1, g_2$, with the residue sizes specified in Corollary 96.8. The last bound there is polynomial in $x$ for fixed field data and $\varepsilon$, giving the asserted little-oh estimate. □

In particular one can form the diagonal and off-diagonal second moments in Proposition 6.9 with the weighted reduced matrices and their original signed coefficients. Their numerical values at infinity are unchanged. To establish that proposition’s hypotheses for the theta maps one still needs estimates for the arithmetic coefficients in these matrices and the comparison with the prescribed global period lattice.

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