The generalized Riemann hypothesis for Dirichlet L-functions

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Volume XVI

Integral theta projections and terminal determinants

Abstract

We descend the integral projection through the theta tower, evaluate the critical torus tests and study the terminal coefficient lattices. Finite valuations and archimedean norms give separate bounds on the same projected determinants.

The guide in Volume I lists all twenty-seven volumes.

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81 Integral projection and descent in the theta tower

81.1 An integral map on the definite target and its adjoint on good lifts

Return to the single lift from $G = U(W_j)$, where $s = \dim W_j = k+m-1$, to the definite group of rank $s+t$, where $t = 5s$. Decompose its space as $I_p \oplus V^\circ$ as in Section 73.10. Let $\Theta^{\mathrm{cl}}_{j,\vartheta}$ denote the lift with its original point test at the old marks, its determinant test at the new ones, and the target class-sum metric of Section 73.4. We specify the real normalization by testing the mixed Weyl vector with its Fock polynomial in the flag-minor normalization of Section 77.8. It is a common normalization for all twists. Hermitian pairings in this subsection are linear in the second argument.

Lemma 81.1 (The row test used by the divided lattice)**.** *For an index i with old matrix Λ, stack the rows of Λ over those of ϖI_p to form w_i over $R = \mathcal{O}_E/\mathfrak{a}^2$ in the chosen components. Use a simultaneous residue uniformizer ϖ at the old marks. The first-block datum of Corollary 73.25 equals D_{w_i} of Theorem 77.41 at every finite place. All stabilizers in the refined class sum of that theorem are trivial for this choice of rows. The complementary class set at level J_H of Section 77.1 also has trivial stabilizers.*

Proof. At an old mark the lower rows impose $X = e+\pi U$, with U integral. The upper rows then impose $\bar{\Lambda}\bar{U} = 0$ over the residue field, exactly the condition $X \in e + (\pi K)^s$ in Theorem 73.24. On this set $e^{\mathsf{t}}X$ reduces to the identity modulo π, so its tame determinant character is one. At a new mark the row condition on the old residue ring is empty. The new datum on the integral box is the character of the determinant of the e-rows extended by zero, as required. Elsewhere the test is the lattice vacuum. The upper rows already contain Λ, so the factorization condition on w in Theorem 77.41 holds.

Each definite class has a finite rational stabilizer. The condition $w_i\bar{A} = w_i$ forces $A \equiv I \bmod \pi$ at every old mark by the lower rows. At a mark of rational residue characteristic l, the first principal congruence subgroup is pro-l: the successive congruence quotients embed in the additive groups of residue matrices and their kernels intersect trivially. Projection of a rational matrix into any one completion, followed by conjugation by the class representative, is injective. Thus the order of each stabilizer element is a power of both distinct old rational primes p_0, p_1. Such an element is the identity. The complementary level is principal modulo π^2 at both primes, so the same argument applies there. □

Write T_j for the direct sum over pairs (i, y) of the negative Weyl coefficient spaces on V°, where y runs through representatives of its classes at J_H. Take their direct-sum Hermitian metric. In a tuple ζ, integrality means membership in the transported negative coefficient lattice at each y, with the contragredient action of Lemma 73.26. Denote the good source space by $\mathcal{H}_j$. Use the integral normalizing power M^{E_j} of Theorem 73.27. For each i, choose its γ_i from Corollary 73.25 once for all twists. Let $\mathcal{C}_{i,\vartheta}$ denote the refined classes on I_p in the lemma.

For a point z of the finite definite target and a mixed coefficient vector v, define $\Delta_z(v)$ to be the form supported on its level class with value $\sum_{\gamma\in\Gamma_z} \rho_z(\gamma)v$ at z. Here Γ_z is its finite rational stabilizer and ρ_z is its action on the coefficient fiber, including the fixed level character. Extend the value equivariantly to the class. It is invariant under the stabilizer, so the extension is well defined. With the reciprocal stabilizer weights one has

$$\langle \Delta_z(v), A\rangle = \langle v, A(z)\rangle.$$

Indeed $A(z)$ is invariant and the action is unitary. Each term in the sum contributes the same pairing and their number cancels the stabilizer denominator.

Write $u \odot v$ for the mixed Cartan coefficient obtained on disjoint blocks as in Lemma 73.26. Set

$$z_{i,h,y} = \operatorname{diag}(h, y)\gamma_i,$$
$$G_\vartheta \zeta = M^{E_j} \sum_{i,y} \sum_{h \in \mathcal{C}_{i,\vartheta}} \Delta_{z_{i,h,y}}(\mathcal{Y}_{i,\vartheta}(h) \odot \zeta_{i,y}).$$

The first coefficient is the arithmetic vector of Theorem 77.38 for the indicated i. The formula is linear in the tuple. Repeated target classes are summed in their common fiber.

Theorem 81.2 (Integral tests and a common injective period map)**.** *At every coefficient place above a prime away from the new twist conductor, G_ϑ takes integral tuples to integral definite forms in the original target coefficient lattice. There is a linear injection $P_j : \mathcal{H}_j \longrightarrow T_j$ independent of ϑ such that*

$$b_{j,\vartheta}\, G_\vartheta^* \Theta_{j,\vartheta}^{\mathrm{cl}}|_{\mathcal{H}_j} = P_j.$$

In particular, if $d_j = \dim \mathcal{H}_j$, the determinant of the Gram operator of $G_\vartheta^* \Theta_{j,\vartheta}^{\mathrm{cl}}$ on this space equals

$$|b_{j,\vartheta}|^{-2d_j} \det(P_j^* P_j) > 0.$$

The determinant on a zero-dimensional source is one.

Proof. The positive coefficients are integral in their transported lattices by Theorem 77.36 and Theorem 77.38. Theorem 73.27 makes $M^{E_j}(\mathcal{Y}_{i,\vartheta}(h) \odot \zeta_{i,y})$ integral at $z_{i,h,y}$. Stabilizer actions preserve this lattice: in its finite frame they are integral level elements, and the level character has values in roots of unity. Equivariant transport takes the local lattice to the lattice at the transported point. The definition of Δ_z uses an unnormalized sum and the lemma gives unit weights on the two block class sets. Thus the whole map preserves integrality at the stated prime, including when that prime divides M.

We next compute the adjoint by testing against an arbitrary tuple. The class evaluation formula and the definition of the single lift express $\langle G_\vartheta \zeta, \Theta_{j,\vartheta}^{\mathrm{cl}} F \rangle$ as M^{E_j} times the sum of integrals against F of the scalar theta kernels at $z_{i,h,y}$, tested by the conjugate Cartan coefficient. To check the real test, apply the mixed Cartan inclusion of Lemma 73.26. On the vector $u \odot v$ it gives the tensor of the two included vectors. Each flag polynomial is read by T_c on the variables of its sign. For evaluation at a matrix B, the defining product of Gram minors for T_c shows that including rows by a coordinate matrix I replaces X by $I^{\mathrm{t}} X$. Evaluations span the dual coefficient space, so this substitution holds on all its vectors. The matrix entries of these operations are real. The mixed test consequently equals the product of $\mathcal{L}(\overline{\mathcal{Y}_{i,\vartheta}(h)})$ on the positive block and the flag polynomial of $\overline{\zeta_{i,y}}$ on the negative block. The real Gaussian also factors.

On the block target groups the source character is υ_1^{m-j} on each block determinant. On the shared source the two target characters multiply to the target character of the original pair. Indeed the first includes the splitting power $j + (t + k - 1)/2$ in addition to the lift character, the second has power $m/2 - j$, and their exponents sum to $(s + t)/2$. The new tame character occurs only in the first factor. Thus the splitting and residue choices in Corollary 73.25 apply. Its change of variables in the good source integral replaces translation by γ_i with the nonzero factor $C_{\gamma_i} N_{\mathcal{M}}$ times the two-block test. Both factors are independent of the new twist. Every source integral is absolutely convergent since F is cuspidal and the fixed theta kernels have moderate growth. The class sets and the old congruence group in the change of variables are finite.

Let $\theta^{\circ}_{\bar{\zeta}}(g,y)$ denote the complementary kernel with the polynomial of $\bar{\zeta}$. By the lemma and Theorem 77.41, summing the first-block kernel over h gives $S_j L_{i,\vartheta}(g)$ exactly. Hence

$$b_{j,\vartheta}\langle \zeta, G^{*}_{\vartheta}\Theta^{\mathrm{cl}}_{j,\vartheta}F\rangle = M^{E_j} N_{\mathcal{M}} S_j \sum_{i,y} C_{\gamma_i} \int_{[G]} F(g)\, \mathcal{B}_i(g)\, \theta^{\circ}_{\overline{\zeta_{i,y}}}(g,y)\, dg.$$

Here Theorem 79.20 was used inside each integral. The right side is conjugate-linear in ζ and linear in F. It defines $\langle \zeta, P_j F\rangle$ and contains no twist. This gives the claimed equality of maps.

If $P_j F = 0$, take tuples supported at one (i,y) and vary that entry. Their polynomials include the compact translates used in Corollary 77.4: evaluation at the identity gives the product of leading minors, and equivariance of the Gram-minor polynomial gives each translate. The constants $M^{E_j} N_{\mathcal{M}} S_j C_{\gamma_i}$ are nonzero. Theorem 77.25 now implies $F = 0$. Taking the adjoint followed by the map in the displayed operator identity gives the Gram determinant formula. □

This result gives integral test forms for the projected periods. To use the periods in a covolume estimate one must also control the denominators of projection onto the good lift inside the ambient integral space.

81.2 A positive integral operator and its height at the trivial twist

Use the map G_ϑ of Theorem 81.2. Write $f = [F : \mathbb{Q}]$, $n_0 = s + t$, $N = t - 9m - 1 = \dim V^\circ$ and $H = s^8$. The coefficient metrics below are the Gram-minor metrics of Section 77.8, with the determinant line of unit length in the standard frame. We take the class-sum metric on the target and the sum metric on T_j. Set $B_0 = |\operatorname{GL}_{2fn_0}(\mathbb{F}_3)|$.

Lemma 81.3 (Duals in the transported coefficient lattices)**.** *There is a positive integer h_j^- depending only on the complementary diagram and N with the following properties. At any rational prime, multiplication by h_j^- sends the metric dual of the tuple lattice into the tuple lattice. For the definite target, multiplying the class pairing of two integral forms by B_0 makes it integral. These assertions hold simultaneously at the places of a coefficient field above the rational prime. One may take*

$$\log h_j^- \ \le\ 3L \log \max(1, L), \qquad L = 2fHt^2.$$

Proof. First work in the identity frame. If $D = C^*$ is a flag-minor Weyl lattice, its Gram map is $T_c : D \to C$, obtained by contracting an integral kernel. Testing against the dual basis of C shows that its metric dual amounts to $T_c^{-1}(C)$, with complex conjugation on the tested argument. For the negative diagram at the f embeddings, Lemma 77.27 provides an integer h_j^- clearing this inverse. The number of boxes in its complementary diagram is $\sum_{\tau,h}(N-h)c^-_{\tau,h}$. There are at most $2H$ columns at each of fewer than s positive heights, and $s, N \le t$, by Theorem 73.27. Hence this number is at most L. This gives the asserted logarithmic bound, including the empty diagram. The ordinary mixed target Gram pairing is integral on its standard Weyl lattice as well, directly by Cauchy–Binet on its Gram-minor kernel.

For an algebraic formulation of the metric dual, enlarge the coefficient field to a finite normal field $\mathcal{K}$ containing all entries under consideration. Use its ring of integers tensored with $\mathbb{Z}_\ell$. Complex conjugation permutes the factors of this ring, and the pairing is conjugate-linear in one argument. At a finite class one may bring its frame into the integral group simultaneously above ℓ by a rational unitary change of coordinates. The Cayley chart and weak approximation prove this as in Theorem 77.36. The rational change is an isometry of the algebraic Hermitian pairing and transports the local lattices. The finite level character introduces only units. Thus the assertions about the coefficient pairing and its dual transfer to the moving frames, including the contragredient action on the negative block. Tuple classes have unit mass by Lemma 81.1.

A target class stabilizer acts faithfully on a lattice of $\mathbb{Z}$-rank $2fn_0$. It is finite by positivity. Reduction modulo three embeds each such group in $\operatorname{GL}_{2fn_0}(\mathbb{F}_3)$. Indeed, for an element $I + 3^a U$ with $a \ge 1$, $U \not\equiv 0 \bmod 3$, a power of prime order exponent different from three has first nonzero term at valuation a, whereas cubing has first nonzero term at valuation $a+1$. Thus the kernel has no element of prime order. The stabilizer order divides B_0, so B_0 clears every reciprocal mass in the target class pairing. □

Proposition 81.4 (Uniform height of the untwisted integral map)**.** *At every embedding ρ of the coefficient field one has*

$$\log \max(1, \|\rho(G_1)\|_{\mathrm{op}}) \ll_{q_0, d_{\mathrm{K}}} fmb_x \log b_x$$

on the given sufficiently distant intervals. Here 1 *denotes the trivial new twist. The two fixed old marks are retained.*

Proof. The Cartan product on disjoint blocks has norm $\|u \odot v\| = \|u\|\|v\|$. To see the normalization, the mixed Cartan inclusion of Lemma 73.26 is equivariant for the compact unitary group. The irreducible mixed highest-weight space has an invariant Hermitian metric unique

up to scale. Evaluation at the identity in the mixed flag model maps to the tensor of evaluation at the identity in the positive model and at the reversal permutation in the negative model. Each evaluation has norm one in its Gram-minor kernel, since the defining frame is unitary. This fixes the scale of the inclusion. The two coordinate block inclusions are themselves isometries: their columns are orthonormal, so the matrices of pairwise Gram minors of evaluating frames are preserved. Such evaluations span the dual flag space. This reasoning applies to the tensor over embeddings. All maps between the standard coefficient bases are rational, so the norm identity applies to conjugate vectors as well.

In the class metric, $\|\Delta_z(v)\| \le |\Gamma_z|^{1/2}\|v\|$, by the triangle inequality at z and the weight $1/|\Gamma_z|$. The same bound holds on conjugating the coefficients. Rational unitary matrices remain unitary at every conjugate embedding of the CM field, and finite character values remain roots of unity. The matrices of G_ϑ in class representatives are algebraic: its coefficients are algebraic by Theorem 77.36, and the changes between representatives act through rational isometries and finite characters. For $\vartheta = 1$, that theorem bounds each squared norm $\|\rho(\mathcal{Y}_{i,1}(h))\|^2$ by $\exp(C f m b_x \log b_x)$. Each individual tuple entry has norm at most the total tuple norm.

We bound the number J of triples (i, y, h) in G_1. All three class sets (the short-row class, the complementary class, and the first-block refined class) have ranks at most n_0. At the trivial new twist their levels contain the principal subgroups modulo the squared old marks. By the full-level count in Lemma 121.17 the logarithm of the number of full-level classes in any of these ranks is

$$O\big((2fn_0)^4 \log(2(2fn_0)q_1 d_{\mathrm{K}})\big),$$

since $E \subseteq \mathbb{Q}(\zeta_{q_1 d_{\mathrm{K}}})$. Over a full-level class the number of classes at the finer level is at most the index of the finer subgroup. At a split old mark in rank u this index is at most $q_v^{2u^2}$, by reduction of all matrix entries modulo π_v^2. As $\prod_{v\in\mathcal{M}} q_v \le M^f$, this adds at most $2fn_0^2 \log M$ to each class count in the logarithm.

The number of old residue matrices is at most $|R|^{3mp} \le M^{6fmp}$. Choose once the $6m{+}1$ short rows for each short-row class as in Theorem 77.29. There are at most $(6m+1)^{2mf\Delta'}$ choices of their ordered lists, where $\Delta' = s^8 - (t+1)/2 \le H$. Thus the sum of the three logarithmic class bounds, $6fmp \log M$, and $2mfH \log(6m{+}1)$ bounds $\log J$. On these intervals $f \ll x^{12}$, $s \ll x^5$, $m \ll \sqrt{x}$, $q_1 \ll_{q_0} x^{11}\log(2x)$, $\log M \le 22 \log b_x$, and $b_x \ge x^{400}$. In particular $(fn_0)^4 \ll x^{68}$. It follows that both $\log J$ and $\log B_0 \le (2fn_0)^2 \log 3$ are $O_{q_0,d_{\mathrm{K}}}(f m b_x \log b_x)$. Now sum the bounds for the J class vectors, including M^{E_j}, whose logarithm satisfies the same estimate by Theorem 73.27. This proves the proposition. □

Theorem 81.5 (The positive integral comparison operator)**.** *On the definite target put*

$$A_\vartheta = B_0 h_j^- \, G_\vartheta G_\vartheta^*.$$

This is an algebraic positive semidefinite operator preserving the target lattice at every prime away from the new conductor. Uniformly at every complex embedding,

$$\log \max(1, \|\rho(A_1)\|_{\mathrm{op}}) \ll_{q_0,d_{\mathrm{K}}} f m b_x \log b_x.$$

With the good source and the common map of Theorem 81.2, its pullback is

$$(\Theta_{j,\vartheta}^{\mathrm{cl}})^* A_\vartheta \Theta_{j,\vartheta}^{\mathrm{cl}}\big|_{\mathcal{H}_j} = \frac{B_0 h_j^-}{|b_{j,\vartheta}|^2}\, P_j^* P_j,$$

where the adjoint on the left is taken for the lift restricted to $\mathcal{H}_j$. The right side is positive definite there.

Proof. Fix an allowed ℓ and use the product of its coefficient completions as in the lemma. For any integral Φ in the target and integral ζ in the tuple lattice,

$$B_0\langle \zeta, G_\vartheta^{*}\Phi\rangle = B_0\langle G_\vartheta\zeta, \Phi\rangle$$

is integral by Theorem 81.2 and the lemma. The dual-lattice assertion now gives $B_0 h_j^{-} G_\vartheta^{*}\Phi$ integral in the tuple space. Applying G_ϑ proves the target assertion. All lattices and linear maps split by the idempotents of the coefficient completions, so this also proves preservation at each individual place.

For the archimedean assertion take full class tables in fixed representatives. Their Gram matrices H_T, H_A have rational entries from Gram minors and class masses. The matrix of the adjoint on the target subspace is the restriction of $H_T^{-1}\overline{G_1}^{\mathsf{t}} H_A$ on full tables. The latter matrix has algebraic entries. Write $\rho^c(a) = \overline{\rho(\overline{a})}$. Under ρ it becomes the adjoint of $\rho^c(G_1)$ on the corresponding full tables. It follows that

$$\|\rho(A_1)\|_{\mathrm{op}} \le B_0 h_j^{-}\, \|\rho(G_1)\|_{\mathrm{op}}\, \|\rho^c(G_1)\|_{\mathrm{op}}.$$

The proposition applies to both embeddings. Here $L = 2fHt^2 \ll x^{62}$, so the lemma absorbs $\log h_j^{-}$ in the asserted bound. Finally the quadratic form of $G_\vartheta G_\vartheta^{*}$ on a vector u is $\|G_\vartheta^{*}u\|^2$. Insert the identity of Theorem 81.2 and polarize on the good source. Its injectivity proves positive definiteness. $\square$

81.3 The split Hecke root and first occurrence in the descent tower

The integral comparison in Theorem 81.5 will be applied to the true lift range Z. In Proposition 52.4 a false lift at a good split place w belongs to $eV \cap \ker A_w \cap Z^{\perp}$. The present argument uses the equation $A_w = 0$ on its constituents. It applies even when only one split place with this equation is specified.

Lemma 81.6 (A bound for all entries of a unitary spherical parameter)**.** *Let k be a nonarchimedean local field with residue cardinality q. If an irreducible unitary spherical representation of $\mathrm{GL}_b(k)$ has unitary-normalized Satake entries $\alpha_1, \dots, \alpha_b$, then*

$$q^{-(b-1)/2} \le |\alpha_i| \le q^{(b-1)/2} \quad (1 \le i \le b).$$

Proof. Give the integral group mass one. The indicator of the integral matrices of determinant valuation $d \ge 0$ is a compactly supported spherical Hecke operator on the group. Its right cosets correspond to sublattices of index q^d. For diagonal valuations $a_1, \dots, a_b \ge 0$ of sum d in upper triangular Hermite normal form, their number is $q^{\sum_i (b-i)a_i}$: in row i each of the $b-i$ later entries is specified modulo ϖ^{a_i}. In normalized unramified induction the value of the spherical function used in the Satake transform at such a triangular representative is $\prod_i \alpha_i^{a_i} q^{-(b+1-2i)a_i/2}$. Summing the cosets gives the eigenvalue

$$q^{(b-1)d/2} h_d(\alpha), \qquad h_d(\alpha) = \sum_{\sum a_i = d} \prod_i \alpha_i^{a_i}.$$

This eigenvalue also holds on the spherical constituent by the Satake isomorphism. In a unitary representation convolution by an indicator has norm at most its mass. There are $\binom{d+b-1}{b-1}$ diagonal types, each with mass at most $q^{(b-1)d}$. Therefore $\sum_{d \ge 0} h_d(\alpha) X^d$ converges for $|X| < q^{-(b-1)/2}$. As a rational function it is exactly $\prod_i (1 - \alpha_i X)^{-1}$. Its numerator is one, so none of its poles in the finite plane cancel. This proves the upper bound, and the contragredient gives the lower bound. □

Proposition 81.7 (A gap in the ranks of unitary local theta partners)**.** *Let $n = s + t$, and let π_w be a spherical representation at the split place of the rank n definite group. Suppose A_w kills its spherical line with the value X_w in Section 52.2. At any rank b with $s < b < t$ there is no irreducible unitary local theta partner for π_w with the unramified unitary splittings of the tower. The same conclusion holds for its contragredient.*

Proof. Write β_i for the unitary-normalized spherical entries of π_w. On its line the polynomial before evaluation at X_w is $\prod_i (1 - q^{(n-1)/2} \beta_i X)$. Indeed the i-th minuscule double coset has eigenvalue $q^{i(n-i)/2}$ times the i-th elementary symmetric function, and its multiplier in A_w is $q^{i(i-1)/2}$. Since $|X_w| = q^{-(t-1+s/2)}$, one of the β_i has modulus $q^{(t-1)/2}$.

At a split place the oscillator is the type II matrix correspondence, up to unitary unramified determinant characters and the choice of dual component in Proposition 72.17. We recall why a possibly ramified partner cannot introduce other entries. By Mínguez, *Correspondance de Howe explicite*, Theorem 1(2), the Langlands quotient data on the larger member are the dual essentially square-integrable data on the smaller one together with $n-b$ characters of successive exponents $(1-(n-b))/2, \dots, ((n-b)-1)/2$, reordered. For a spherical larger member the unique Langlands data are all unramified characters of rank one: its Satake multiset gives just this Langlands quotient. Hence all the smaller data must themselves be unramified characters. Its Langlands quotient is spherical. The unitary determinant twists have absolute value one on the uniformizer.

Lemma 81.6 bounds the absolute values of the smaller entries and their inverses by $q^{(b-1)/2}$. The added string and its inverse are bounded by $q^{(n-b-1)/2}$. Both bounds are strictly less than $q^{(t-1)/2}$, since $b < t$ and $n - b < t$. The same two bounds hold on the inverses of all larger entries. This contradicts the entry already found, and also excludes the contragredient for which its reciprocal occurs. □

Theorem 81.8 (First occurrence for representations satisfying the split test)**.** *Let π be an irreducible automorphic representation on the definite group $U(I_n)$ having the local component in Proposition 81.7. In the rational Witt tower through W_j use coherent splittings and let $W^{(a)} = W_j \oplus \mathbb{H}^a$, with $\mathbb{H}$ a hyperbolic plane. If the full global theta lift of π to $U(W^{(a)})$ is nonzero for some $1 \le a \le p = s + 9m + 1$, then its full theta lift to $U(W_j)$ is nonzero. This uses the fields and ranks of the given intervals.*

Proof. Here the full lift allows all Schwartz data and all input vectors in the given automorphic copy. Coherence keeps the Weil splitting characters fixed when two dimensions are added. In the external norm-one convention of the manuscript, adding a plane multiplies the smaller space's external character by υ_1: base change multiplies by μ^2, cancelling the change of two in the dimension exponent. This leaves the splitting on the definite member compatible with passage down the tower.

Recall the tower property in this definite-source case. In the mixed model at the parabolic of an isotropic h-plane, the extra position variables lie in I_n^h. In the constant term, integration over the center of the unipotent radical kills every rational tuple with nonzero Hermitian Gram matrix, by additive character orthogonality. A zero Gram matrix forces the whole tuple to vanish, since I_n is positive definite. On the zero slice the remaining unipotents act trivially, and the kernel is the lower theta kernel with Schwartz datum evaluated at zero in the extra variables, up to the nonzero Levi and measure factors. These evaluations are onto the lower Schwartz space: tensor any lower datum with a Schwartz function in the extra variables of value one at zero. The mixed-model change is an invertible partial Fourier transform. All integrations for these constant terms use a compact definite input quotient and a compact unipotent quotient; at fixed adelic arguments the Schwartz sums converge absolutely and uniformly on compact sets. Rational parabolics are obtained from isotropic flags, and Witt's extension theorem identifies planes of each fixed dimension. Thus a nonzero lower lift persists at the next rank, and all proper constant terms at the first nonzero rank vanish. This is Rallis's tower property in the form used by Gan, Qiu and Takeda, §11.2.

If the lift to W_j were zero, persistence would give vanishing at the smaller members of its tower as well. Take the first nonzero member through the specified $W^{(a)}$. Its rank is $b = s + 2a'$ with $1 \le a' \le p$. Its automorphic forms are cuspidal by the constant-term calculation. The center is anisotropic, so a nonzero such lift pairs nontrivially with a vector in an irreducible cuspidal summand. This summand is unitary. The pairing with the theta kernel and π gives a nonzero local theta pairing at w. More explicitly, test the nonzero global pairing on decomposable data and vectors, fix the factors away from w, and vary the three local factors. The resulting invariant trilinear functional is nonzero. A local Schwartz vector has an open compact stabilizer, so its image on the other two factors is in their smooth dual by admissibility. This supplies the required local joint quotient, with either dual convention covered by Proposition 81.7.

Finally $s + 2p = 3s + 18m + 2 < 5s = t$ for every sufficiently large node, since $s \ge x^5$ and $m \le 2\sqrt{x}$. Thus $s < b < t$. The proposition excludes the local pair at the very place w where the original Hecke test was imposed. This contradiction proves the theorem. □

To apply the tower result to the false-lift tests, one must pass from vanishing against the prescribed datum at the original level to vanishing against the Schwartz data needed for the block period. This passage concerns the marked local tests and the selected real weight.

81.4 Finite tests at the bottom of the descent tower

Fix W_j, I_n of ranks $s, n = s + t$ on one of the intervals of Theorem 73.8. Let S consist of the original marked places and the primes of the actual new conductor, using one component of each split pair. At a new place write χ for its nontrivial tame residue character. The untwisted case has no new places. We use matrix coordinates in which the s selected rows come first. Write O for the local integer ring, π for a uniformizer, $k = O/\pi O$, and $e = \binom{I_s}{0}$. On n-by-s matrices the block function is the product of the following local functions:

$$\phi_{\mathrm{old}}(X) = \mathbf{1}_{e+\pi\,\mathrm{Mat}_{n,s}(O)}(X), \qquad \phi_{\mathrm{new}}(X) = \mathbf{1}_{\mathrm{Mat}_{n,s}(O)}(X)\,\chi(\det \overline{X'}).$$

In the second expression take the value zero on singular top blocks and outside the integral matrices. Here and below X' denotes the square top block and X'' the remaining rows. The old level J in $\mathrm{GL}_n(O)$ fixes $\bar{e}$. At a new place its reduction has zero upper-right block, with invertible diagonal blocks. The character of J on the form is $d(h) = 1$ at an old place and $d(h) = \chi(\det \bar{h}_{11})^{-1}$ at a new one. The operator $U_h\phi(X) = \phi(h^{-1}X)$ acts on the block function by this same character. These are the physical translation conventions of the proof of Theorem 17.33.

Lemma 81.9 (The finite hyperplane average on the chosen types)**.** *For every nonzero row $\lambda \in k^{1\times s}$ put $H_\lambda = \{1+v\lambda : v \in k^{s\times 1},\ 1+\lambda v \neq 0\}$. At each old place, let $R(z)$ be ordinary right translation on the level space by a lift of $\mathrm{diag}(z, I_t)$, for $z \in \mathrm{GL}_s(k)$. For every $\Xi \in e_{\mathrm{mark}}V$, where e_{mark} is the idempotent of Proposition 52.4, one has $\sum_{z\in H_\lambda} R(z)\Xi = 0$.*

Proof. Translation is well defined because any two such lifts differ by an element of the old level. In a basis with λ the last coordinate row, H_λ is the group of matrices $\left(\begin{smallmatrix} I_{s-1} & u \\ 0 & a \end{smallmatrix}\right)$ with $a \in k^\times$. It contains the whole unipotent radical of the hyperplane parabolic and the last GL_1. If an irreducible finite type has an H_λ-fixed vector, its unipotent invariants therefore contain $\tau \boxtimes \mathbf{1}$ on the Levi for some irreducible τ of $\mathrm{GL}_{s-1}(k)$. Indeed the last-factor invariants are stable under the first factor and finite-group representations over $\mathbb{C}$ are semisimple. Frobenius reciprocity and transitivity of parabolic induction put the trivial degree-one character in its finite cuspidal support. For $s = 1$ the assertion just says the character is trivial. By the finite support calculation in Theorem 17.33, the ordinary translation types occurring in the marking have no such character. More explicitly, on each orbit the marked label is tensored by the inverse of the old determinant character, of order dividing Q. A degree-one label has order not dividing Q, and a larger orbit preserves its degree under this twist. Thus the average over H_λ, the projector onto its invariants times its order, vanishes on every summand of $e_{\mathrm{mark}}V$. □

Theorem 81.10 (All finite marked data from the block test)**.** *Let $\Xi \in e_{\mathrm{mark}}V$. Fix a real Schwartz vector and one matrix coefficient of Ξ. Use the lattice vacuum outside S. If its descent from I_n to W_j by the product block function is zero at every adelic argument, then it vanishes for every Schwartz–Bruhat function on the product of the matrix spaces at S, at every argument. The real vector and the coefficient in these two assertions are the same.*

Proof. At argument g denote the integral with arbitrary marked data by $B_g(\phi, \Xi)$, linear in ϕ and conjugate linear in Ξ. Use the split matrix model, with the external character on the W_j argument removed if present; it is a nonzero scalar at g. The intrinsic splitting on integral left translations in these coordinates is unramified. Consequently simultaneous integral left translation of the test and right translation of the form gives

$$B_g(U_h\phi, R(h)\Xi) = B_g(\phi, \Xi).$$

This follows by the substitution $u \mapsto uh$ in the compact definite quotient integral, since the oscillator action on the test is pullback by h^{-1}. Right multiplication of its matrix argument by $A^{-1} \in \mathrm{GL}_s(F_v)$ is the oscillator action of an element on the W_j side up to a nonzero determinant scalar. Such elements at the different split places are independent and can be taken to be the identity elsewhere. Thus the hypothesis also kills all products of these right translates of the block function, for all g.

First consider a point X at a marked place where $A = X'$ is invertible over the local field and $T = X''A^{-1}$ lies in $\pi\,\mathrm{Mat}_{t,s}(O)$ at an old place, or in $\mathrm{Mat}_{t,s}(O)$ at a new one. Then $X = \binom{I_s}{T}A$ belongs to JeA. This orbit is exactly the support of $\phi_{\mathrm{old}}(\,\cdot\,A^{-1})$ or $\phi_{\mathrm{new}}(\,\cdot\,A^{-1})$, respectively. To verify the reverse inclusion, write $YA^{-1} = \binom{C}{D}$ on this support. Completing it by the remaining standard columns gives $\left(\begin{smallmatrix} C & 0 \\ D & I_t \end{smallmatrix}\right) \in J$. The orbit is compact and open. For a ball about X contained in it, average its indicator with the operator $\int_J d(h)^{-1} U_h \, dh$ for probability Haar measure. This leaves the pairing with Ξ unchanged by covariance. The result is supported on one orbit and transforms by d. Such functions form a line, since their values are determined by the value at eA and the translated block function is nonzero there. Thus the average is a scalar multiple of this translate.

Suppose now the preceding conditions fail. There is an integral row $l = (l_1, l_2)$ with $lX = 0$, $\bar{l}_1 \neq 0$ and, at a new place, $\bar{l}_2 = 0$. If A is singular, scale a nonzero left kernel vector to be primitive and take $l_2 = 0$. Otherwise take a row T_i offending the integral condition. Divide $(-T_i, e_i^{\mathrm{t}})$ by an entry of T_i of largest absolute value, where e_i^{t} selects row i of X''. This value is at least one at an old place and strictly greater than one at a new place. The row has all the required properties.

Put $\lambda = \bar{l}_1$. For each $z = 1 + v\lambda \in H_\lambda$ lift v to $\tilde{v} \in O^s$ and put

$$h_z = I_n + \begin{pmatrix} \tilde{v} \\ 0 \end{pmatrix} l.$$

Its determinant $1 + l_1\tilde{v}$ is a unit. It fixes X exactly and preserves every ball about X taken in the supremum norm, since both h_z and its inverse are integral. At an old place h_z has the same first residue columns as $\mathrm{diag}(z, I_t)$. Its translation on Ξ is $R(z)\Xi$. At a new place $h_z \in J$ and its translation is $\chi(\det z)^{-1}\Xi$. Now $1 + \lambda v$ runs over $k^\times$, so some h_z acts by a scalar different from one in the new case. For a product ball with this component, covariance gives the same pairing with $R(h_z)\Xi$ as with Ξ. At an old place sum this equality and apply Lemma 81.9; at a new place use the nontrivial scalar. Both give zero.

Finally give the product of the marked matrix spaces the nested basis of balls obtained by taking the same integer depth at each place. At every point where a compactly supported locally constant test is nonzero, choose a ball on which it is constant. If all components have the indicated integral ratios, shrink the ball into the product of the open orbits. Its pairing vanishes by averaging at every place and applying the block hypothesis. If one component fails, its stabilizers already give zero for the entire ball. Compactness gives a finite cover of the support. Two balls from this nested basis which meet are nested, so the maximal members form a disjoint cover. Linearity proves the assertion. When S is empty it is the given block vanishing itself. □

81.5 The descent tower with the unramified tests fixed

Write S for the finite set from Section 81.4. At the real places and at S allow arbitrary Schwartz data. At every other finite place keep the original lattice vacuum. Call this space of data $\mathcal{S}_S(B)$ for a member B of the coherent Witt tower through W_j. The variable g in a descent on $U(B)$ still ranges over its whole adelic group. Fix a matrix coefficient y of one of the finite-level algebraic forms on $U(I_n)$, with its given real type. Write

$$D_B(\phi, y)(g) = \int_{[U(I_n)]} \theta_B(g, h; \phi)\overline{y(h)}\, dh.$$

The compactness of the integration domain gives continuity in the real Schwartz topology for each g, by uniform convergence of the theta sum on compact sets of group arguments.

Lemma 81.11 (Transfer of the full Hecke polynomial on the lattice test)**.** *At a split $w \notin S$, write $q = Nw$. In ranks $b < n$ put $f = \mathbf{1}_{\mathrm{Mat}_{n,b}(O_w)}$, in lattice coordinates. Give the integral groups mass one. On rank d let $C^{\pm}_{d,i}$ be convolution by the unnormalized elementary double coset with i elementary divisors $\varpi^{\pm 1}$ and the rest one, and set $\mathsf{E}^{\pm}_{d,i} = q^{-i(d-i)/2} C^{\pm}_{d,i}$. The two group actions on f satisfy*

$$\sum_{h=0}^{n} (-Y)^h \mathsf{E}^{+}_{n,h} f = \prod_{i=0}^{n-b-1} (1 - Yuq^{(n-b-1)/2-i}) \sum_{l=0}^{b} (-Yu/v)^l \mathsf{E}^{-}_{b,l} f.$$

Here u is the unitary phase of a positive elementary divisor on the left and v that of a negative one on the right, including the determinant twists of the chosen splitting.

Proof. The changes of variables are $X \mapsto h^{-1}X$ on the left and $X \mapsto Xg$ on the right in the split model. Their square-root Jacobian multipliers in the two cosets are $q^{bh/2}$ and $q^{nl/2}$, respectively, where h in the exponent counts elementary divisors. The coset phases are u^h, v^l. For a matrix of residue rank a, the numbers of contributing right cosets are $\begin{bmatrix} n-a \\ h \end{bmatrix}_q$ and $\begin{bmatrix} b-a \\ l \end{bmatrix}_q$. The first count chooses a subspace containing the column space, the second an l-space in the kernel. Both sides vanish off the integral box. With $D = n - b$, the Grassmannian calculation of Lemma 78.7 gives

$$\begin{bmatrix} n-a \\ h \end{bmatrix}_q = \sum_{l} \begin{bmatrix} b-a \\ l \end{bmatrix}_q \begin{bmatrix} D \\ h-l \end{bmatrix}_q q^{l(D-h+l)}.$$

Divide by the two Hecke normalizations and include the Jacobians. For $j' = h - l$ the coefficient multiplying $\mathsf{E}^{-}_{b,l} f$ is $u^h v^{-l} \begin{bmatrix} D \\ j' \end{bmatrix}_q q^{-j'(D-j')/2}$. Indeed the exponent is $(bh - h(n-h) - nl + l(b-l))/2 + l(D-h+l)$. The normalized Gaussian coefficient is the elementary symmetric function of degree j' in $q^{(D-1)/2-i}$, $0 \le i < D$, by expanding the product successively. Multiplying the two generating polynomials proves the identity. □

Lemma 81.12 (Constant terms and the prescribed vacuum)**.** *Let $B' = B \oplus \mathbb{H}^h$ be two members of the tower, using the diagonal sign lattices. A constant term of $D_{B'}(\phi, y)$ along a parabolic stabilizing a rational isotropic h-plane, at any adelic argument, is a nonzero Levi and measure factor times a lower descent with datum in $\mathcal{S}_S(B)$, if $\phi \in \mathcal{S}_S(B')$. At the identity of the extra Levi and compact factors the zero-slice map onto $\mathcal{S}_S(B)$ is surjective.*

Proof. We first choose an integral rational splitting of the standard tower. The relative extension E/F is unramified at all finite places, including the split ones, by Lemma 77.37. The trace ideal of $\mathcal{O}_E$ is the unit ideal. Locally this follows at a split prime by taking $(1, 0)$, and in an unramified quadratic field by the surjective finite-field trace; an element of unit trace generates the local trace ideal. Hence there is $\lambda \in \mathcal{O}_E$ with $\lambda + \bar{\lambda} = -1$. For a positive and a negative

coordinate with vectors u, v the two columns $u+v,\ u+\lambda(u+v)$ have norms zero, pairing one and determinant -1. They thus identify the diagonal lattice of this pair with the integral hyperbolic lattice at every finite place. The rational anisotropic kernel of $W_j = I_{k+j-1} \oplus (-I_{m-j})$ is the remaining definite summand. Its definiteness bounds the Witt index at a real place, so these pairs account for every rational rank in the tower.

For the standard parabolic write $g = pk$ by local Iwasawa decomposition. Away from S use the lattice stabilizer for the compact factor, which is hyperspecial. In the split case this is the integral basis theorem for flags. In the unramified field case intersect an isotropic plane with the lattice to obtain a primitive isotropic summand. Unimodularity gives integral dual partners. Subtracting linear combinations of the isotropic basis makes their Gram matrix zero: solve the off-diagonal entries in pairs and use the surjectivity of the integral trace on the diagonal. The orthogonal complements are unimodular and integrally isometric by the diagonalization in Section 73.5. Thus the compact group is transitive on the planes, giving the asserted decomposition. The other finite and the real compact factors can act freely on the allowed data.

In the mixed model use h positions in I_n in addition to the lower oscillator. As in Theorem 81.8, integration over the center of the radical selects zero Hermitian Gram matrix, hence only the zero tuple, because I_n is definite. The other radical coordinates act trivially there. The remaining Levi action gives a scalar and translation of the lower argument. At an unmarked finite place the compact factor fixes the vacuum. The integral splitting identifies its self-dual oscillator lattice with an orthogonal sum. Its vacuum in the mixed model is the lower vacuum tensored with the lattice indicator in the extra positions, up to a nonzero scalar. Indeed partial Fourier transform of an integral lattice indicator is its volume times the indicator of the dual lattice, and the chosen measure is self-dual. This argument also works in the scaled position coordinates of Lemma 77.9. The extra indicator takes value one at zero. Almost everywhere the changes of measure have scalar one. At the other places extension from the zero slice is obtained by tensoring a lower test with any Schwartz function of the extra positions having value one at zero, then inverting the mixed-model change. This proves both assertions by taking products and linear combinations. Witt's extension theorem and automorphy reduce any rational plane to the standard plane. □

Theorem 81.13 (Propagation of vanishing with the same finite tests)**.** *Suppose y is spherical at one split $w \notin S$ with $A_w y = 0$ for the test and root of Proposition 52.4. Suppose $D_{W_j}(\phi, y) = 0$ for all $\phi \in \mathcal{S}_S(W_j)$. Then for every $1 \le a \le p = s + 9m + 1$ and every $\phi \in \mathcal{S}_S(W_j \oplus \mathbb{H}^a)$ one has $D_{W_j \oplus \mathbb{H}^a}(\phi, y) = 0$, at every argument.*

Proof. Surjectivity of the zero slice first gives vanishing below W_j: take a constant term of the identically zero lift at a Levi argument giving any prescribed lower argument. If a larger member up to the specified rank has a nonzero lift, choose the first such member B and put $b = \dim B$. All proper maximal constant terms of every allowed lift there vanish by Lemma 81.12. A constant term along a flag is an iterated such integral. By continuity and the density of Hermite polynomials times the Gaussian in the Schwartz space, choose an allowed ϕ with polynomial Fock vector at infinity and $F = D_B(\phi, y) \neq 0$.

This function is cuspidal, of finite level and real compact type. For clarity, it has moderate growth: on a fixed finite level component its theta functional is a tempered lattice sum applied to the moving real test. A fixed finite collection of Schwartz seminorms bounds this sum with h ranging in a compact set of representatives of the definite quotient. These seminorms after oscillator translation grow polynomially in the group norm: in the equivalent oscillator Sobolev norms one commutes finitely many position and momentum operators past the unitary action,

which transforms each by the linear symplectic matrix. Restriction to the real compact input type of y gives a single infinitesimal character on each factor of $U(B)(F_\infty)$ by the compact dual-pair decomposition (Davidson, §2, Theorem 2.7). Thus the classical rapid decay theorem for cusp forms on Siegel sets makes F square integrable; the center is anisotropic. One may apply it on the finitely many arithmetic components at this level (Miller and Schmid, Theorem A and its adelic formulation).

Normalize $Y = q^{(n-1)/2} X_w$, so $|Y| = q^{-(t-1)/2}$. The Hecke polynomial of the hypothesis is $\sum_h (-Y)^h \mathsf{E}^{+}_{n,h}$. It is normal on the definite spherical space since inverse double cosets are its Hecke adjoints and all spherical operators commute. Consequently its adjoint kills y as well. Move this adjoint from the conjugated y through the definite integral. Complex coefficients are conjugated and inverse cosets are reversed by the change of integration variable. This puts the polynomial with coefficients $(-Y)^h$ on the left action of the local matrix vacuum. Lemma 81.11 transfers it to the right action. The latter is precisely adelic right translation of F on the smaller group, because acting on the test before applying the theta functional replaces the argument g by the corresponding right translate. We obtain

$$0 = \prod_{i=0}^{n-b-1} (1 - Y u q^{(n-b-1)/2-i}) \sum_{l=0}^{b} (-Yu/v)^l \mathsf{E}^{-}_{b,l} F.$$

Here $b > s$, so $n - b < t$, and the scalar product is nonzero.

The remaining polynomial is invertible on the spherical L^2-space at these ranks. Its constant term is the identity. Each $\mathsf{E}^{-}_{b,l}$ has norm at most its mass, $\begin{bmatrix} b \\ l \end{bmatrix}_q q^{-l(b-l)/2}$, by unitarity of right translation. The sum of the norms of the positive-degree terms is thus at most

$$\prod_{i=0}^{b-1} (1 + q^{-(t-b)/2-i}) - 1 \;\le\; \exp(q^{-(t-b)/2}/(1-q^{-1})) - 1.$$

Since $b \le s + 2p = 3s + 18m + 2$, $t = 5s$, $m \le 2\sqrt{x}$ and $s \ge x^5$, one has $t - b \ge 4$ throughout the sufficiently large nodes. Already for $q = 2$ the upper bound is at most $\exp(1/2) - 1 < 1$. Taking norms gives $F = 0$, a contradiction. The argument uses the same fixed coefficient throughout, with the lattice test at w on every member of the tower. □

81.6 Vanishing of the descent of a false lift

Fix one of the fields, intervals, lift indices and characters of Theorem 81.5. Set $G = U(W_j)$, $K' = U(I_n)(F_\infty)$, and let V_ρ be the irreducible coefficient module of the definite target at infinity, with its invariant Hermitian metric. For a unit vector $\zeta \in V_\rho$ write

$$y_\zeta(h) = \langle \zeta, \rho(h_\infty^{-1})\Xi(h_f)\rangle.$$

Use an inner product linear in the second argument, and write ζ_0 for the specified highest vector of length one. Include in the standard block function the vacuum at every unmarked place. Write Φ for its tensor product with the single lift's joint highest harmonic at infinity, chosen to match the coefficient y_{ζ_0}. Put $d_\Xi(g) = \overline{D_{W_j}(\Phi, y_{\zeta_0})(g)}$, restoring both external characters of the single lift in this formula. The character on the G argument can subsequently be removed by pointwise multiplication by its inverse.

Proposition 81.14 (The block descent in the marked source space)**.** *If $e_{\mathrm{mark}}\Xi = \Xi$, then d_Ξ belongs to the original good source $\mathcal{H}_j$. With the fixed measures and joint harmonic normalization, it is the Hilbert adjoint of $\Theta^{\mathrm{cl}}_{j,\vartheta}$ on this space up to a common nonzero scalar. In particular $d_\Xi = 0$ for $\Xi \in e_{\mathrm{mark}}V \cap Z^\perp$.*

Proof. On the finite source use the level in Theorem 73.8. Right multiplication of a marked matrix by an old integral element of residue z takes the point indicator to its left translate by $\mathrm{diag}(z^{-1}, 1)$, in the convention U_h of Theorem 81.10. The source splitting supplies the inverse old determinant character in D. Conjugation in d_Ξ changes it to the direct character. Changing variables on the definite group moves the left translate to translation of Ξ by $\mathrm{diag}(z, 1)$. Thus the right action on d_Ξ is the marked action on Ξ. In particular the source principal subgroup fixes d_Ξ. At a new prime the factor $\chi(\det \overline{X'})$ under right multiplication by k acquires $\chi(\det \bar{k})$, cancelled by the source splitting there. All other places have the fixed hyperspecial test. This gives precisely the finite source level, and applying the old central idempotent gives $e_{\mathrm{mark}} d_\Xi = d_\Xi$.

We spell out why the finite assertion already forces cuspidality. Decompose the finite marked action into tuples as in Lemma 73.7. Each tuple has an irreducible finite cuspidal label at some original split place v. For a proper rational parabolic P its localization is a proper parabolic of $\mathrm{GL}_s(F_v)$. The invariants of that finite cuspidal type under the residue unipotent radical vanish by finite Frobenius reciprocity. To evaluate the P-constant term at g, write $g_v = p_v k_v$ with $p_v \in P(F_v)$ and k_v integral. Averaging the input by $k_v^{-1}(N_P(F_v) \cap K_v)k_v$ leaves that constant term unchanged: after multiplication by g_v this is left translation by elements of the adelic unipotent radical, under which the constant term is invariant. The same average is zero on the finite type, since the integral radical maps onto the residue radical. This proves vanishing for each tuple and all proper P. The descent is smooth of moderate growth by the compact integral and seminorm argument of Theorem 81.13.

We verify its real line at one embedding; tensor over the real places. Write $a = k + j - 1$, $b = m - j$, $a + b = s$, $b' = (t-1)/2$, and suppress the embedding in w_j, σ_i from Theorem 73.27. The positive and negative column determinants in the polynomial use disjoint target rows. Every source mixed lowering contraction kills the polynomial. Triangular substitution on the ordered compact blocks makes it a highest vector. The source action on a positive column is dual and on a negative column is direct. The target character has external weight $-w_j + n/2$ after the rank twist and enters inversely on the source. The vacuum further contributes one to each negative column for each of the n target rows. Thus the vacuum exponents are $w_j - n/2, w_j + n/2$ on the two blocks. For a positive column $i = 0, \dots, a-1$, the number of letters is $\sum_{h>i} c_h^+ = \sigma_i - (a-i) - (b' - w_j)$. This telescopes the counts in Theorem 73.27 to the truncated endpoint at

$h = a$. In a negative column $i = s-1-l$, $0 \le l < b$, the count is $\sum_{h>l} c_h^- = -w_j - b' - \sigma_i - b + l$, by the other endpoint calculation of Theorem 77.22. The resulting two compact weights, before conjugating the descent, are

$$-\sigma_i - i + (a-b-1)/2 \quad (i < a), \qquad -\sigma_i + l + (a-b+1)/2 \quad (i = s-1-l).$$

These are the lowest compact type of the physical contragredient discrete series, in its highest-line order, as computed in Theorem 77.22. The case $b = 0$ uses only the positive count.

For completeness, compact dual-pair decomposition identifies the multiplicity space of the matching K'-type with an irreducible real G-module, and the joint harmonic belongs to its lowest compact type. This is the harmonic decomposition of Davidson, §2, Theorems 2.4, 2.5 and 2.7, tensored by the splitting characters. The coefficient of this vector is in L^2 by Lemma 142.12. Its pairing with the indicated discrete-series coefficient is positive by Theorem 142.13, now applied to the compact weight just calculated. Schur orthogonality therefore identifies the irreducible module with that discrete series (with the usual finite-dimensional orthogonality when $b = 0$). Its highest line in the lowest compact type is one dimensional. An equivariant theta map on its algebraic vectors consequently puts the output on that line in every nonzero copy. Applying this observation to D and conjugating puts d_Ξ on the prescribed source line. The cuspidal decay argument of Theorem 81.13 gives square integrability. This proves $d_\Xi \in \mathcal{H}_j$.

For $f \in \mathcal{H}_j$, insert the bilinear kernel of the single lift into $\langle \Theta^{\mathrm{cl}}_{j,\vartheta} f, \Xi \rangle$. On a definite class, integration of its highest coefficient and the conjugate coefficient over the probability measure on K' gives the vector inner product divided by $\dim V_\rho$, by compact Schur orthogonality. Disintegration at the finite level gives the class weight $1/|\Gamma|$ times the level volume. Thus replacing the class pairing by the quotient integral multiplies by a nonzero constant independent of f, Ξ. The two quotient integrations may be interchanged since the definite one is compact, the kernel has moderate growth there uniformly in that variable, and f is rapidly decreasing. The result is this constant times $\int_{[G]} \overline{f(g)}\, d_\Xi(g)\, dg$, including any fixed normalization of the lift. If $\Xi \perp Z$, take $f = d_\Xi$. Its norm is zero and continuity gives pointwise vanishing. □

Theorem 81.15 (The restricted tower of every false lift)**.** *Let w be a place tested in Proposition 52.4, and take $\Xi \in e_{\mathrm{mark}} V \cap \ker A_w \cap Z^\perp$. Then for every $\zeta \in V_\rho$, every $0 \le h \le s+9m+1$, and all $\phi \in \mathcal{S}_S(W_j \oplus \mathbb{H}^h)$, one has*

$$D_{W_j \oplus \mathbb{H}^h}(\phi, y_\zeta) = 0.$$

Proof. The proposition gives zero for the block and highest harmonic at every argument. Fix the finite block for the moment. Averaging at K' in the definite integral kills all real oscillator types except the one matching y_{ζ_0}. In that isotypic space the integral at each g contracts the compact factor against the coordinate of ζ_0. Indeed covariance under simultaneous translation and compact Schur orthogonality give the invariant pairing of the two compact modules, and that pairing has multiplicity one. The chosen joint harmonic has nonzero matched compact coordinate by construction: the mixed flag-minor map sends the highest vector to the product of its disjoint positive and negative flag minors, which has value one at the corresponding identity frames.

In the remaining factor of the oscillator, the compact source translates and real enveloping algebra applied to the harmonic vector span all compact-finite vectors of that irreducible multiplicity module. This uses algebraic cyclicity of the Harish–Chandra module of an irreducible unitary admissible representation. Every polynomial Fock vector is compact-finite; projection to a fixed compact coordinate preserves this property. The argument applies factor by factor at infinity. The theta integral intertwines real source translations with right translations

on G; real differentials may be passed under the compact definite integral by continuity of the Schwartz derivatives. Differentiating or translating the zero function from the proposition gives zero on every one of these polynomial vectors. Finite Hermite expansions are dense for the Schwartz topology, so vanishing extends to all real Schwartz vectors, still with the same finite block.

Apply Theorem 81.10 separately to each real vector. This gives zero for every datum in $\mathcal{S}_S(W_j)$ for the coefficient y_{ζ_0}. For $k' \in K'$ one has $y_{\rho(k')\zeta_0}(h) = y_{\zeta_0}(hk')$. Changing variables in the integral moves this compact translation to the real Schwartz datum. The translates of ζ_0 span V_ρ by irreducibility. Thus the same bottom vanishing holds for every coefficient y_ζ. Restoring or removing an external character of the small argument multiplies each function by a nowhere zero character. Finally the map $\Xi \mapsto y_\zeta$ is linear in Ξ and commutes with the right finite action. Thus $A_w y_\zeta = 0$ at precisely the same w. Theorem 81.13 now proves the assertion at all the required ranks. □

81.7 Fourier coefficients and orthogonality of the comparison

Use the rational orthogonal decomposition $I_n = I_p \oplus I_N$ of Section 73.10, with $p = s + 9m + 1$ and $N = t - 9m - 1$. Write $H = U(I_n)$, $H_1 = U(I_p)$, $H_2 = U(I_N)$. Normalize Haar measures consistently under disintegration; their positive factors will not affect vanishing. Put $e_p = \binom{I_p}{0}$, a frame of Gram matrix I_p. The coherent splitting for $\mathbb{H}^p$ is the one in which H acts by substitution on the p positions in I_n. Indeed each hyperbolic plane contributes the ordinary position representation with unit absolute Jacobian on H; the external power υ_1^p is exactly the one in Theorem 81.8.

Lemma 81.16 (Detecting a complementary period with the fixed vacuum)**.** *Let Y be a continuous function on $[H]$, right invariant by the lattice stabilizers of H outside S at finite places. If its integrals against the theta kernels for $\mathbb{H}^p$ vanish for all small-group arguments and data in $\mathcal{S}_S(\mathbb{H}^p)$, then*

$$\int_{[H_2]} Y(\operatorname{diag}(h_1, h_2))\, dh_2 = 0 \qquad (h_1 \in H_1(\mathbb{A}_F)).$$

The integrals here pair the kernel with Y by multiplication.

Proof. Set $g_0 = \operatorname{diag}(h_1, 1)$ and $J(h) = \int_{[H_2]} Y(\operatorname{diag}(1, h_2)h)\, dh_2$. This continuous function descends to $H_2(\mathbb{A}_F)\backslash H(\mathbb{A}_F)$ by right invariance of quotient measure. The map on this homogeneous space given by $h \mapsto h^{-1}e_p$ is the map to isometric frames. At each local place its stabilizer is exactly H_2, and its image is the whole Gram-I_p orbit by Witt extension.

Take the Fourier coefficient for the character of I_p on the Siegel radical of $U(\mathbb{H}^p)$, at an element of its $\mathrm{GL}_p(\mathbb{A}_E)$ Levi. In position coordinates the radical multiplies the term of $X \in I_n^p(F)$ by the additive character paired with X^*X. This follows from the Weil action of the upper unipotent in the polarization by the two isotropic halves. Any fixed trace-pairing scalar is used both in this character and in taking the coefficient. Orthogonality on the additive quotient therefore retains $X^*X = I_p$. Witt extension over E/F makes these rational frames one $H(F)$-orbit, with stabilizer $H_2(F)$. Unfolding the sum over that orbit and integrating first over $[H_2]$ gives, up to a nonzero Levi scalar,

$$0 = \int_{H_2(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi(h^{-1}e_p c)\, J(h)\, d\dot{h} \qquad (c \in \mathrm{GL}_p(\mathbb{A}_E)).$$

Here ϕ is the position function in the hyperbolic model, and the Levi action pulls its argument back by right multiplication by c, after choosing the corresponding Levi element. The unitary H-action has no remaining factor in this splitting. Absolute convergence before unfolding follows from uniform Schwartz summation on the compact definite and radical quotients, with c fixed. This is the unitary frame computation; the corresponding orthogonal calculation is in Walls, Proposition 2.1.1.

Away from S at finite places the position vacuum is a nonzero scalar times the indicator of $\alpha_v \operatorname{Mat}_{n,p}(\mathcal{O}_{E_v})$, with the finite scale of Lemma 77.9. Choose $c_v = \alpha_v h_{1,v}$ there. The support condition becomes integrality of $x_v = h_v^{-1}e_p h_{1,v}$, whose Gram matrix is I_p. These integral frames are a single orbit of $K_v = H(\mathcal{O}_v)$. Indeed the span of the columns splits orthogonally in the integral lattice by the retraction x_v^*. The complement is unimodular. At an unramified field place it has an orthonormal basis by the unit norm argument of Section 73.5. At a split place choose a basis on the first component and its dual on the other. This proves that $x_v = k_v^{-1}e_p$ for some $k_v \in K_v$. Equivalently the contributing class belongs to $H_2(F_v)g_{0,v}K_v$. On this single orbit J cannot distinguish right translations by K_v.

At infinity and at S take $c_v = h_{1,v}$ and choose smooth nonnegative bumps in the real position spaces and compact-open indicators in the finite ones, positive at e_p and supported

in arbitrarily small neighborhoods of e_p. Restricted to the frame orbit these localize the coset at $g_{0,v}$. In detail, the local frame map has a continuous local section. At a real place complete nearby orthonormal frames by projecting the old complementary basis and orthonormalizing it. At a marked split place the frame is a pair of matrices A, B with $B^{\mathsf{t}}A = I_p$. Complete A by applying $I - AB^{\mathsf{t}}$ to the old complementary basis, and use the inverse transpose of the completed matrix on the other component. Its determinant remains a unit near the standard frame. These sections also show that the chosen bumps are positive on a relatively open neighborhood of the desired coset. The product of the unmarked compact orbits is compact and open in the unmarked restricted product, and has positive finite quotient measure. The function J is invariant on the right by their full product of compact groups. Thus if $J(g_0) \neq 0$, one constant complex rotation makes its real part strictly positive throughout the support of sufficiently small bumps on the quotient. The factors outside these places are positive indicators times a common nonzero scalar; remove that scalar in the integral. The integral must then have positive real part, a contradiction. This proves the lemma. □

Proposition 81.17 (The period with a theta kernel on the second block)**.** *For a false lift Ξ as in Theorem 81.15, every coefficient y_ζ, and every finite $\gamma \in H(\mathbb{A}_{F,f})$ supported on S, one has*

$$\int_{[H_2]} \theta^{\circ}(g, h_2; \phi_-)\, \overline{y_\zeta(\mathrm{diag}(h_1, h_2)\gamma)}\, dh_2 = 0$$

for all h_1, g, with any allowed second-block test ϕ_- and the characters of Section 77.1.

Proof. Replacing $y_\zeta(h)$ by $y_\zeta(h\gamma)$ in the tower integrals moves γ^{-1} to the test by change of variables. Its support only in S preserves the allowed space. Fix a bottom test $\phi_+ \otimes \phi_-$ for the splitting of the target into the two blocks; form the function

$$Y(h) = \theta_{W_j, I_n}(g, h; \phi_+ \otimes \phi_-)\, \overline{y_\zeta(h\gamma)}.$$

It is continuous, rationally invariant and fixed by the unmarked finite compact group. The two blocks use the same source-space splitting on their determinants. Their external characters on the W_j argument may be taken inverse to one another, with the prescribed character on the second block. On $\mathbb{H}^p \oplus W_j$ use the coherent splitting, and omit any overall character depending only on the small-group argument. For a tensor of the hyperbolic test and the bottom test its theta kernel restricted to $U(\mathbb{H}^p) \times U(W_j)$ is the product of the two theta kernels. This follows because the oscillator of an orthogonal sum is a tensor product, and the rational positions in a product polarization sum independently. All these tensors have the fixed vacuum away from the indicated places. Theorem 81.15 at $h = p$ therefore gives the hyperbolic vanishing of Lemma 81.16 for Y.

At the block diagonal point the bottom theta kernel factors again, now in the two target spaces. Its first factor can be made nonzero at any fixed (g, h_1) by choosing ϕ_+. To see this, choose a rational polarization of the underlying symplectic space and any nonzero allowed finite tensor. After action by $(g_f, h_{1,f})$ it is a nonzero Schwartz–Bruhat function. Some rational position lies where this function is nonzero, since a vector space over F is dense in its finite adeles. Rational positions in its compact support belong to a single fractional lattice, discrete at infinity. Choose a real smooth bump supported around the selected position and excluding all the others, with value one at that point. Apply the inverse of the real oscillator action to obtain the real test. The resulting theta value has a single nonzero term. With this ϕ_+, divide the period supplied by the lemma by its nonzero first factor. This gives the required second-block integral. □

Theorem 81.18 (The comparison of true lifts belongs to the tested space)**.** *With $Z = \Theta^{\mathrm{cl}}_{j,\vartheta}\mathcal{H}_j$ and T the simultaneous orthogonal complement of the false spaces of Theorem 52.2, the operator of Theorem 81.5 satisfies*

$$A_\vartheta Z \;\subseteq\; T.$$

In fact $G_\vartheta(P_j\mathcal{H}_j) \subseteq T$.

Proof. At each complementary class y, let $K(g,y)$ be the Riesz vector in the negative coefficient space, with $\langle v, K(g,y)\rangle = \theta^{\circ}_v(g,y)$ from the formula in Theorem 81.2. Thus at an infinite argument $u \in H_2(F_\infty)$ the same coordinate is $\langle r(u)v, K(g,y)\rangle$, where r is the negative coefficient representation. Indeed the compact oscillator substitutes u in its negative row variables; the flag-minor map on the conjugated coefficient gives $\overline{r(u)v}$. This is the contragredient block in Lemma 73.26; the vacuum and source determinant on H_2 have opposite weights, as for the original single lift. For a fixed positive coefficient ξ the Cartan map $C_\xi v = \xi \odot v$ intertwines $r(u)$ with the restriction of $\rho(\mathrm{diag}(1,u))$, by its tensor construction.

Fix i and a first-block finite representative h. Apply Proposition 81.17 with $h_1 = (1,h)$, $\gamma = \gamma_i$, the second-block test of the gluing formula with coefficient $\bar v$, and the coefficient $y_{C_\xi v}$. Take the conjugate of its zero integral and sum over an orthonormal basis v. At $h_2 = (u,y)$ this gives

$$\sum_v \overline{\langle r(u)v, K(g,y)\rangle}\,\langle C_\xi r(u)v, \Xi(z_{i,h,y})\rangle = \langle C_\xi K(g,y), \Xi(z_{i,h,y})\rangle.$$

The equality is expansion in the orthonormal basis $r(u)v$. The resulting integrand no longer depends on u. It descends to the complementary finite level J_H: at the old marks $J_H \equiv 1 \bmod \pi^2$ and conjugation by γ_i costs at most one power of π, since $\pi L \subseteq \gamma_i L \subseteq L$ in Theorem 73.24. At a new mark the second block acts trivially on the top determinant of the level, and at other places the levels are the lattice stabilizers. The kernel itself has at least this complementary level. By Lemma 81.1 all complementary class stabilizers are trivial. Disintegrating the period therefore gives the unweighted sum of the displayed pairing over y, times one positive level and compact volume. The sum is zero.

Set $\xi = \mathcal{Y}_{i,\vartheta}(h)$, sum over h, and use the definition of G_ϑ and the class evaluation identity preceding Theorem 81.2. It gives

$$\sum_y \langle K(g,y), (G^*_\vartheta \Xi)_{i,y}\rangle = 0 \qquad (i,g).$$

The integral formula for $\langle \zeta, P_j F\rangle$ in that theorem now gives zero for $\zeta = G^*_\vartheta \Xi$: its inner sum over y is the conjugate of the last display. Thus $\langle \Xi, G_\vartheta P_j F\rangle = 0$ for every good F and every false lift at each tested place. This proves the second assertion. Since $b_{j,\vartheta} G^*_\vartheta \Theta^{\mathrm{cl}}_{j,\vartheta} = P_j$ on the good space and $b_{j,\vartheta} \neq 0$, multiplication by $B_0 h_j^{-} G_\vartheta$ proves the first. □

81.8 The projected determinants on the common period interval

Fix q_0, d_{K}, an interval starting at V_0 in the shorter family, and the simultaneous twists of Corollaries 52.21 and 73.23. At a node x, write b_x for its weight scale, $s = k + m - 1$, $f = [E^+ : \mathbb{Q}]$ and

$$B_x^{\det} = \frac{f x b_x (\log b_x)^3}{mk}.$$

For a single lift at index j write β for its prescribed twist, Z_β for its true range and P_β for orthogonal projection onto that range. Distinguish this projection from the injective map P_j into the tuple space. Use the comparison operator A_β of Theorem 81.5 and set

$$C_j(\beta) = P_\beta A_\beta|_{Z_\beta}, \qquad \Delta_j(\beta) = \det C_j(\beta), \qquad D_x = \dim Z_\beta.$$

The dimension is common to the indices and twists by Theorems 73.8 and 81.2. In dimension zero take determinants equal to one.

Theorem 81.19 (Finite and archimedean estimates for the same compression)**.** *Each $\Delta_j(\beta)$ is positive and algebraic. With $v_\ell(\ell) = 1$, the simultaneous local estimates are*

$$\sum_{\ell} (\log \ell) \sup_{x,j,\iota} \frac{\max(0, -v_\ell(\iota \Delta_j(\beta_{\ell,\iota})))}{\max(1, D_x)\, s B_x^{\det}} \ll_{q_0, d_{\mathrm{K}}} (\log(2V_0))^{-2}.$$

Here $\beta_{\ell,\iota}$ is the twist at index j of the family chosen for this coefficient embedding. At every complex embedding,

$$\log \max(1, |\sigma \Delta_j(1)|) \ll_{q_0, d_{\mathrm{K}}} D_x f m b_x \log b_x.$$

Proof. Theorem 81.5 gives a positive definite pullback by the single lift from $\mathcal{H}_j$; the lift is injective and onto Z_β by definition of the range. Thus the compression is positive definite there. At a coefficient embedding with prime ℓ outside the new conductor, the same operator preserves the ambient integral lattice L. Theorem 81.18 gives $A_\beta Z_\beta \subseteq T$. Consequently

$$C_j(\beta)(L \cap Z_\beta) \;\subseteq\; P_\beta(L \cap T).$$

Over a finite local field of definition take elementary divisors π^{-a_i} for the larger lattice on the right relative to $L \cap Z_\beta$, with π a uniformizer. The top exterior power of the inclusion shows $v_\ell(\iota \Delta_j(\beta)) \geq -\sum_i a_i / v_\pi(\ell)$. The quotient on the right is the normalized length λ of Theorem 52.2. The twists attached to ℓ have new conductor prime to ℓ. Inserting the bound for the actual λ in Corollary 52.21, which takes the same supremum over nodes, indices and coefficient embeddings, proves the first estimate.

By Proposition 52.22, on the transported tested space the transported P_β is an orthogonal projection in its positive class metric. Since A_β takes the true range into the tested space before and after transport, the norm of the transported compression is at most $\|\sigma(A_\beta)\|_{\mathrm{op}}$. The determinant in dimension D_x has absolute value at most the D_x-th power of this norm by the top exterior power. For $\beta = 1$ apply the estimate of Theorem 81.5. For an arbitrary fixed twist the ambient class tables, their metrics and the entries of A_β are algebraic in fixed frames. Their finitely many conjugates give a finite bound depending on that twist. Thus the orbit of $\Delta_j(\beta)$ under all automorphisms of $\mathbb{C}$ is bounded. A transcendental number z has unbounded orbit. To see this, extend z to a transcendence basis, replace it by az for positive integers a, and extend the resulting rational function field automorphisms to $\mathbb{C}$. This proves algebraicity also when the original analytic source basis contains periods. □

Use the notation of Theorem 79.20. At index j and $c = \chi_M^j \beta$ put

$$\mathcal{L}_j(\beta) = \mathcal{D}_{m/2}(c\upsilon_1^{m/2})\,\mathcal{D}_m(c^{-1}\upsilon_1^m)\,\mathcal{D}_{e'}(\sigma_c), \qquad e' = \frac{k+7m}{2}, \qquad \sigma_c = \alpha^{-1}\upsilon_1^{3m+e'}.$$

Indeed the two ranks in the third exchange are $3m$ and $p = s + 9m + 1$, giving $e' = (p - 3m)/2$. Each $\mathcal{D}$ includes its own conductor power and its squared absolute finite L-values as in Section 78.7. Let $\mathcal{V}_j(\beta)$ denote the Gram determinant of the single lift with the class metric of Theorem 73.5.

Proposition 81.20 (Exact compression identity for a single lift)**.** *On the fixed node one has*

$$\frac{\mathcal{V}_j(1)}{\mathcal{V}_j(\beta)} = \frac{\Delta_j(\beta)}{\Delta_j(1)} \left(\frac{\mathcal{L}_j(1)}{\mathcal{L}_j(\beta)} \right)^{D_x}.$$

Proof. Represent the lift by an invertible matrix U between orthonormal bases of the good source and its true image. Then the determinant of its pullback of A_β is $|\det U|^2 \Delta_j(\beta) = \mathcal{V}_j(\beta)\Delta_j(\beta)$. By Theorem 81.5 that same determinant is $(B_0 h_j^-)^{D_x} |b_{j,\beta}|^{-2D_x} \det(P_j^* P_j)$, which is nonzero. The map P_j and both integer factors are common to the twists. Theorem 79.20 gives $|b_{j,\beta}|^{-2} = \text{const.}\,\mathcal{L}_j(\beta)$ with the same positive constant for β and 1. Division gives the identity. □

Corollary 81.21 (A common multiplier for the terminal determinant)**.** *Write $\mathfrak{Z}_\vartheta$ for the terminal class Gram determinant of Theorem 73.22. At index j use $\beta = \vartheta^j$ and put*

$$y_{\ell,\iota,x} = \frac{\mathfrak{Z}_1}{\mathfrak{Z}_\vartheta} \prod_{j=1}^{m} \left(\frac{\mathcal{L}_j(\vartheta^j)}{\mathcal{L}_j(1)} \right)^{D_x}, \qquad H_x = \prod_{j=1}^{m} \Delta_j(1) > 0.$$

These $y_{\ell,\iota,x}$ and H_x are algebraic. The multiplier is independent of ℓ, ι. Uniformly over the same nodes,

$$\log \max(1, |\sigma H_x|) \ll_{q_0, d_{\mathrm{K}}} D_x f m^2 b_x \log b_x,$$

$$\sum_{\ell} \sup_{\iota} \log \max(1, |\iota(H_x y_{\ell,\iota,x})|_\ell) \le \max(1, D_x)\delta + O_{q_0,d_{\mathrm{K}}}\left(\frac{D_x m s B_x^{\det}}{(\log(2V_0))^2} \right).$$

Here $\delta > 0$ is the tolerance of Corollary 73.23. The two bounds can in particular be added with the same order of error as in the second bound.

Proof. Let $R_{\ell,\iota,x}$ be the left side of (73.1). That identity uses the product of the m auxiliary Gram determinants. Proposition 81.20 at each index consequently gives the exact equality

$$H_x y_{\ell,\iota,x} = R_{\ell,\iota,x}^{-1} \prod_{j=1}^{m} \Delta_j(\vartheta^j).$$

All factors on the right are nonzero and algebraic by Corollary 73.23 and the theorem. The first bound follows by adding the m archimedean bounds. For the second, apply the positive part of the negative valuation to the product. The inverse of R costs its positive valuation times $\log \ell$; the sum of those terms is at most $\max(1, D_x)\delta$ on the chosen interval. For each compression the theorem bounds the corresponding sum by $O(D_x s B_x^{\det}/(\log(2V_0))^2)$ when $D_x > 0$. In dimension zero all the factors are one. Finally, $f m b_x \log b_x / (s B_x^{\det}) = m^2 k/(sx(\log b_x)^2) \le 4/(\log b_x)^2$. Since $b_x \ge x^{400}$, this absorbs the archimedean cost. □

The prime-sum argument uses the ratio $\mathfrak{Z}_1/\mathfrak{Z}_\vartheta$ itself. Its remaining factor in the last corollary is the inverse of the displayed product of the $\mathcal{L}_j$ ratios. In particular $7m/2 < z \le e'$ in the third factor contributes $k/2$ critical integers at each j; the direction-dependent divisibility estimate for these factors enters before one can deduce the required prime-sum inequality.

82 Critical torus periods and their finite tests

82.1 The norm of the torus lift at a critical integer

Let E/F be a field of the shorter family, $T = U(1)$, and $n = 2z$ with $z \geq 2$. Let ν be a unitary character of $[T]$ with weight $-c_\tau$ at each real place and put $d_\tau = c_\tau - z \geq 0$. Its finite part is unramified off the original marks S_0. Let β be a prescribed weight-zero tame twist with new split marks S_β, disjoint from S_0. Take both sets as sets of places of F, distinguishing a member of the pair in E above each one. Write

$$Q_\beta = \prod_{v \in S_\beta} q_v, \qquad M_0 = \prod_{v \in S_0} (q_v - 1), \qquad \lambda_\eta(a) = \eta(a/\bar{a}).$$

A superscript S on $L_{E,f}$ omits the finite places above S. Absorb $\nu\beta$ in the theta kernel from T to $H = U(I_n)$, with external source factor $((\nu\beta)\upsilon_1^z)^{-1}$ and trivial external target factor. At infinity use the one-row highest polynomial of degree d_τ, with a unit highest vector in the coefficient module $V_{\mathbf{d}}$. At the old marks use the indicator of a specified nonzero residue column on the integral box. At a new mark use the tame character on the specified entry of the column, extended by zero, on that box. At other finite places use the lattice vacuum. Let $K_\beta(g)$ be the resulting vector-valued kernel at a source class g. Use the class group C and the target class metric of Section 50.1. In particular

$$\phi_\beta = S_{\nu,z} \sum_{g \in C} K_\beta(g)$$

with the same nonzero scaling $S_{\nu,z}$ before and after twisting. This convention allows the fixed vacuum and derivative scales in an integral normalization. The measure on each compact real group is probability Haar and the original maximal compacts at finite places have mass one.

Theorem 82.1 (Norm and conductor for the torus period)**.** *In the Fock metric with monomial squared norm $\alpha!$, there is a positive constant $a_{E,n}$ depending only on the adelic spaces, the oscillator normalization and the measures such that*

$$\|\phi_\beta\|_{\mathrm{cl}}^2 = a_{E,n}\, |S_{\nu,z}|^2 M_0\, (\dim V_{\mathbf{d}}) \prod_\tau d_\tau!\, Q_\beta^{n-1}\, \frac{L_{E,f}^{S_0 \cup S_\beta}(z, \lambda_{\nu\beta}^{-1})}{\zeta_F(n)}.$$

In particular all these lifts are nonzero. If $R_{\nu,z}(\beta)$ is the ratio of the absolute values of the displayed numerator and its value at $\beta = 1$, with their respective omissions, then

$$R_{\nu,z}(\beta) = \frac{\|\phi_\beta\|_{\mathrm{cl}}^2}{Q_\beta^{n-1} \|\phi_1\|_{\mathrm{cl}}^2}.$$

Proof. First integrate the constant function on $[T]$ with Haar measure. Apply the convergent unitary Siegel–Weil formula to the double of the source line. The doubled group has Siegel rank one, the definite target has Witt index zero and $n > 2$. The section is evaluated at $(n-1)/2 > 1/2$, in the region of absolute convergence. These are the hypotheses of Yamana, Theorem 2. For this double every rational isotropic line is a graph of an isometry of the source line. Indeed a nonzero isotropic vector in the double has both coordinates nonzero and their norms agree. In the graph term the doubled Schwartz function at zero, in difference and Fourier coordinates, is the inner product of the original test and its translate by that isometry. Unfolding the sum over the rational isometries in the two torus integrals gives $\mathrm{vol}([T]) \int_{T(\mathbb{A}_F)} \langle \Phi, \omega(u)\Phi \rangle\, du$, times the positive Siegel–Weil and measure scalar. This is the graph calculation of Proposition 142.14, now with the constant source. The absolute local sum

at a good split prime is bounded by $1 + 2q^{-z}/(1 - q^{-z})$, as calculated below. At a good inert prime it is one. Their product converges absolutely for $z > 1$, which justifies unfolding and multiplying the integrals.

At a real place the circle acts trivially on the polynomial times the vacuum after inserting the specified character. In the positive one-column model each of its d_τ letters has source weight -1, the external factor has weight $c_\tau - z = d_\tau$, and the positive vacuum has source weight zero in this convention. Its integral is therefore its squared Fock norm. For the highest monomial this is $d_\tau!$, by applying the derivative of order d_τ to the monomial at zero. In passing from this one scalar coefficient to the vector class metric, compact Schur orthogonality multiplies by $\dim V_{\mathbf{d}}$. In fact the compact average of the projector onto the unit highest vector is $I/\dim V_{\mathbf{d}}$ by invariance and trace.

At a good split v put $\xi = \lambda_{\nu\beta}(\varpi_w)^{-1}$, where ϖ_w is on the chosen component of E. For a source scalar of valuation h the change of position variable is $X \mapsto X\varpi^h$ on F_v^n, with half-Jacobian $q^{-nh/2}$. The integral of the product of the two lattice indicators has measure 1 for $h \geq 0$ and q^{nh} for $h < 0$. The unitary multiplier is ξ^h. Indeed for a single target coordinate the intrinsic split multiplier at ϖ is the splitting character on the distinguished component, as in the calculation of Theorem 78.1. Its $n = 2z$ powers cancel the external υ_1^{-z}, since on (a, a^{-1}) the restriction of the splitting character is the square of its value on $(a, 1)$ by conjugate duality. The remaining factor is $(\nu\beta)^{-1}$. Reversing the chosen component exchanges ξ with ξ^{-1}. The support of each valuation shell has multiplicative mass one. Summing the coefficients gives

$$\sum_{h\in\mathbb{Z}} \xi^h q^{-z|h|} = \frac{1 - q^{-2z}}{(1 - \xi q^{-z})(1 - \xi^{-1}q^{-z})}.$$

This is the factor of $L_{E,f}(z, \lambda_{\nu\beta}^{-1})/\zeta_F(2z)$. At an inert prime the entire local torus is compact and fixes the vacuum. A uniformizer can be taken in F_v, so its base-change character is one and its norm in E is q^2. The same Euler quotient equals one there. These cases exhaust the unmarked places by Lemma 77.37.

At an old place the point column and its scalar translate overlap only for a source unit congruent to one. The test has squared norm q^{-n} and that subgroup has measure $1/(q - 1)$. At a new place overlap requires a unit, and the tame factor of its action on the entry is cancelled by the source character. The squared norm is $1 - q^{-1}$. Conversion to the target class measure multiplies these two numbers of integration by $q^n - 1$ and $(q^n - 1)/(q - 1)$, respectively. The first is the number of nonzero columns, the second the number of residue hyperplanes, as in Lemma 73.4. Thus the actual factors are $(1 - q^{-n})/(q - 1)$ and $q^{n-1}(1 - q^{-n})$. The factors $1 - q^{-n}$ at these places complete the reciprocal of $\zeta_F(n)$.

Finally the Haar integral on the source equals $1/M_0$ times the class sum. Indeed the source level has that volume, its real group has mass one, its stabilizers are trivial by Proposition 50.1, and the new places require only the full unit group on the source. The passage to ϕ_β multiplies the squared norm by $|S_{\nu,z}|^2 M_0^2$. The remaining measure constant is independent of the character and the weight. At a split place the two numerator Euler factors have positive product, the inverse of a squared modulus. At an inert place the factor is positive as well. Absolute convergence gives a positive global value. Taking the quotient of the two norm identities now proves the assertions. □

Corollary 82.2 (Two tests for the inverse critical ratio)**.** *At a coefficient embedding ι with $\ell \nmid Q_\beta$, use $v_\iota(\ell) = 1$ and the line projection P_β of Section 50.1. Put $c(a) = \bar{a}$. For two integral tested vectors U, V whose pairings with ϕ_β are nonzero, one has*

$$\begin{aligned} -v_\iota(R_{\nu,z}(\beta)^{-1}) = {} & - v_\iota(\|\phi_1\|_{\mathrm{cl}}^2) - v_\iota(\langle P_\beta U, P_\beta V\rangle) \\ & + v_\iota(\langle U, \phi_\beta\rangle) + v_{\iota\circ c}(\langle V, \phi_\beta\rangle). \end{aligned}$$

For a main pair, the positive part of the second term has the uniform projected-period bound in Corollary 52.21 on the same interval. The tests for the last two terms may be chosen individually for ι and $\iota\circ c$ with the same twist β, provided they satisfy the respective integrality conditions of Theorem 50.3.

Proof. For a Hermitian metric linear in its second argument, $P_\beta v=\phi_\beta\langle\phi_\beta,v\rangle/\|\phi_\beta\|_{\mathrm{cl}}^2$. Hence

$$\frac{1}{\|\phi_\beta\|_{\mathrm{cl}}^2}=\frac{\langle P_\beta U,P_\beta V\rangle}{\langle U,\phi_\beta\rangle\,\overline{\langle V,\phi_\beta\rangle}}.$$

By the theorem $R_{\nu,z}(\beta)^{-1}$ is this expression times $Q_\beta^{n-1}\|\phi_1\|_{\mathrm{cl}}^2$. The conductor power is a unit at ℓ. Taking negative valuations proves the equality, including the conjugated embedding in the last summand. The period estimate is uniform over the allowed tests of the given lift. A directional saving requires upper bounds for the two last valuations, as well as the simultaneous size of a multiplier for the untwisted norm. □

82.2 Integral flat torus values at the original level

Keep the fields, old marks and measures of Section 82.1. For an even rank $r = 2a \geq 4$ put $D = fr$, with $f = [F : \mathbb{Q}]$, and $M = p_0 p_1$. A flat pair of rank r has $\nu = \psi \upsilon_1^{-a}$ with ψ of weight zero and unramified off the old marks. We take ψ tame of level at most one there. Thus its external source character in the kernel, before inversion, is $\psi\beta$. The real polynomial is the constant one and the target coefficient space is the scalar line. The mark on the old residue column can be any nonzero column. Use the standard unit box at all unmarked places.

In the notation of Lemma 77.33, write t_E for the integer denoted there by n, chosen with its product coordinates for the pair I_1, I_r. Put

$$b_r = (M t_E)^D, \qquad B_r = |\mathrm{GL}_{2D}(\mathbb{F}_3)|.$$

Write a_r for the scalar that sends the unit oscillator vacuum to its marked elliptic normalization at degree zero. Precisely, it is C_r^{-1} in the identification of a kernel with the value of its marked section at the origin in Theorem 67.8, using the product coordinates just chosen. It is common to all finite data and characters in this rank. Let $\phi_{\psi,\beta} = a_r \sum_{g \in C} K_\beta(g)$. This fixes the freedom in $S_{\nu,a}$ of Theorem 82.1 for the flat pairs. Define

$$J_r(\psi, \beta) = B_r b_r^2 \, \|\phi_{\psi,\beta}\|_{\mathrm{cl}}^2 / Q_\beta^{r-1}.$$

Theorem 82.3 (Flat coefficients with a common normalization)**.** *For every finite target argument h the value $b_r \phi_{\psi,\beta}(h)$ is algebraic and integral at all places above every $\ell \nmid Q_\beta$. The same holds for $J_r(\psi, \beta)$, which is positive. Moreover*

$$J_r(\psi, \beta) / J_r(\psi, 1) = R_{\psi \upsilon_1^{-a}, a}(\beta).$$

For $4 \leq r \leq 2s$ on the shorter interval one has

$$\sup_{\psi, \sigma} \log \max(1, |\sigma J_r(\psi, 1)|) \ll_{q_0, d_{\mathrm{K}}} x^{70}(1 + \log b_x) = o(B_x^{\det}),$$

uniformly as the starting node tends to infinity. Here σ runs through complex embeddings and $s = k + m - 1$.

Proof. All residue functions at an old place are indicators on a group k^r. Fourier inversion writes one point indicator as a sum of additive characters with coefficients roots of unity divided by q^r. Their absolute values sum to one also after any conjugation. At a new place the tame function on the chosen entry, extended by zero, is an integral linear combination of point indicators on the same group. The Fourier coefficients thus have denominator dividing q^r. The product of these denominators at the old places is M^{fr}, and every new denominator is a unit above $\ell \nmid Q_\beta$. Multiplication by each additive character on an integral position box is a translation in the dual coordinates by a torsion point. This is finite Fourier inversion in the split position model as in Proposition 77.30.

We explain the remaining scalar in these coordinates. Since r is even, Lemma 77.33 supplies the product period lattice B with $t_E B \subset L \subset t_E^{-1} B$, for the vacuum lattice L. In transferring the normalized product section to L, one sums unitary torsion translates with a factor $I^{-1/2}$, $I = [L : L \cap B]$, by Lemma 65.8. The group defining I embeds in $t_E^{-1} B / B$. Thus $I \mid t_E^{2D}$ and $t_E^D I^{-1/2}$ is an algebraic integer. For the two finite arguments choose split representatives off ℓ, the level and the discriminant by Theorem 65.11, at compact opens fixing the given data. The change to the moved lattice L' has the same normalized sum, now with $[L' : L \cap L']^{-1/2}$ by Lemma 65.12. This is a unit at ℓ since $L'_\ell = L_\ell$. All phases of the torsion translations are roots of unity. The external character $\psi\beta$ has finite order since it is trivial at infinity and factors

through a finite torus ray class group. The remaining intrinsic split phases are units above ℓ on these representatives by the determinant calculation in Theorem 65.13.

Each summand is now the value at the origin of a translate of the product of marked elliptic sections. Its factors are algebraic and integral at the prime under consideration by the order-zero case of Theorem 66.35. This case applies even when the translation is of ℓ-power order, since its derivative order is zero. The passage to these marked sections accounts exactly for a_r by Theorem 67.8. Source real rotations act trivially after the flat character cancellation, and the target real representation is trivial. Hence rational changes of representatives move no positive-degree polynomial or coefficient lattice. These calculations give the asserted integrality and algebraicity of $b_r a_r K_\beta(g)(h)$ for all finite arguments. The source sum has integer multiplicities and trivial stabilizers by Proposition 50.1.

For a target class representative h its stabilizer is a finite group acting faithfully on its underlying integer lattice of rank $2D$. Reduction modulo three injects this group into $\mathrm{GL}_{2D}(\mathbb{F}_3)$ by the binomial argument in Theorem 77.36. Thus its order divides B_r. The class sum for the norm of $b_r\phi_{\psi,\beta}$ after multiplication by B_r is integral at every such prime, also for the conjugated coefficients, since the coefficient integrality holds at all primes over ℓ. The power of Q_β is a unit there. Positivity and the ratio identity follow from Theorem 82.1 with this single choice of a_r.

For the size bound we only need $\beta = 1$. The old Fourier coefficients have absolute sum one, so the normalized finite vector is a linear combination of unitary translates of the vacuum with that absolute sum in every embedding. After moving the finite arguments, Theorems 65.13 and 65.9 bound its conjugated marked section at the origin by

$$\left[e \max(1, 2r \operatorname{rd}(E)/\pi) \max(1, \nu_\sigma)\right]^{D/2}.$$

Here ν_σ is the squared metric normalization of the marked elliptic section of that theorem. The marked curve and differential for $K = \mathbb{Q}(\sqrt{-d_{\mathrm{K}}})$ are fixed on the interval and have only finitely many conjugates. This is the degree-zero estimate in the proof of Theorem 77.32. Its proof uses the orthogonality of the translated vacuum sections, so the indices of moved lattices introduce no archimedean factor. We have $\operatorname{rd}(E) \le q_1 d_{\mathrm{K}}$ by the cyclotomic discriminant bound, $q_1 \ll_{q_0} x^{11}\log(2x)$, $\log t_E = O_{q_0,d_{\mathrm{K}}}(\log(2x))$ and $\log M \le 22 \log b_x$. It follows that every conjugate of each normalized kernel times b_r has positive logarithmic size at most $O_{q_0,d_{\mathrm{K}}}(D(\log(2x)+\log b_x))$.

It remains to count terms in the norm. At full level the logarithm of the number of target classes is at most $O((2D)^4 \log(2Dq_1 d_{\mathrm{K}}))$ by the definite lattice count used in Theorem 77.36. The same bound applies to the rank-one source since $f \le D$. On passing to marked levels each full class splits into at most the index of the smaller compact subgroup. The target index at $\beta = 1$ is $\prod_{v\in S_0}(q_v^r - 1) \le M^D$, and the source index is at most M^f. The stabilizer weights are at most one, while $\log B_r \le (2D)^2 \log 3$. For any conjugate of a squared absolute value $u\bar{u}$ bound the two factors under σ and $\sigma \circ c$ separately. The triangle inequality, first over source classes and then over target classes, now bounds the logarithm in the statement by

$$O_{q_0,d_{\mathrm{K}}}((2D)^4 \log(2Dq_1 d_{\mathrm{K}}) + D(\log(2x) + \log b_x)).$$

Since $D = fr \ll_{q_0} x^{17}$, this gives the stated estimate. Finally $m \le 2\sqrt{x}$, $k \ll x^5$, $b_x \ge x^{400}$ and $B_x^{\det} = fxb_x(\log b_x)^3/(mk)$. Dividing the bound by this quantity gives a quantity tending to zero uniformly. □

Corollary 82.4 (A multiplier for every factorization of a torus character)**.** *Let $G = \widehat{C}$ be the finite group of characters of the original torus classes. For two even ranks r_1, r_3 in the range of the theorem, choose a root L of*

$$L^{|G|} = \prod_{\eta\in G} J_{r_1}(\eta, 1) J_{r_3}(\eta, 1).$$

It is a nonzero algebraic integer, and its conjugates satisfy the same order of logarithmic upper bound as in the theorem. For every coefficient valuation v and every $\psi \in G$ one can factor $\psi = \psi_1\psi_3$ so that

$$v\big(J_{r_1}(\psi_1, 1)J_{r_3}(\psi_3, 1)\big) \le v(L).$$

Proof. The class characters inflate to the allowed tame weight-zero characters by the definition of C. All factors in this equation are nonzero algebraic integers by the theorem at $Q_\beta = 1$. A root of the monic equation is integral over the algebraic integers and therefore is an algebraic integer. Taking absolute values at any complex embedding and the $|G|$-th root proves the size estimate. For fixed ψ, as η runs over G, so does $\psi\eta^{-1}$. Consequently the valuations of $J_{r_1}(\eta, 1)J_{r_3}(\psi\eta^{-1}, 1)$ have mean $v(L)$. Any pair attaining their minimum supplies the factorization. □

82.3 Four-block tests for the critical torus lift

Take a main critical pair (ν, z) with $7m/2 < z \le (k+7m)/2$ from Section 81.8. We use that ν has nontrivial tame character at each old mark, as in Theorem 50.6; the prescribed β is ramified at each of its new marks. Set $n = 2z$, $t = n-1$, $d_\tau = c_\tau - z$, and use C, K_β of Section 82.1. At these nodes $n \ge 46$ and $z \le s$ since $k \ge 5m+2$ eventually. Set

$$I_n = I_{12} \oplus I_4 \oplus I_4 \oplus I_{n-20}.$$

Call the first summand I_0. Index the remaining summands by $2, 3, 1$ in that order, with ranks $r_2 = r_3 = 4,\ r_1 = n-20$. Write $H_i = U(I_{r_i})$ for these three groups. The character $\psi_i \in \widehat{C}$ specifies flat data on H_i as in Theorem 82.3. Take $\beta_2 = \beta,\ \beta_3 = \beta_1 = 1$. Use one fixed standard vector in each flat block for the old mark, and use its coordinate in block 2 for the new marks.

Lemma 82.5 (The complementary period of a false torus lift)**.** *Let $Y \in \operatorname{im} e \cap \ker A_w \cap \ker D$ be a false form as defined in Section 50.1, at a split w off the marked places. For a coefficient v of Y fixed by $H_1(F_\infty)$, a finite integral translation A supported on the marks, any argument h_+ on the first twenty rows, and any flat kernel $K_1(g,h)$ on the last block,*

$$\int_{[H_1]} K_1(g,h)\, \overline{y_v(\operatorname{diag}(h_+, h)A)}\, dh = 0.$$

Here y_v is the scalar coefficient of the algebraic form as in Section 81.6. The kernel on the last block has the lattice data away from the marks.

Proof. The equation $DY = 0$ gives $\langle K_\beta(g), Y\rangle = 0$ on the full set of source classes C. This includes every finite source argument by its level invariance. Integrating a highest real coefficient of Y against the block theta kernel gives the conjugate of this class pairing up to the positive compact Schur and level factors of Proposition 81.14. In this rank both real groups are compact. The homogeneous polynomials of degree d in the holomorphic Fock model on one column constitute one copy of the degree-d symmetric power, and different degrees have different central weights. Indeed they are the homogeneous summands of the polynomial algebra in n variables, with the compact group acting by substitution, up to the fixed character. Schur orthogonality kills all other summands. On the matching summand simultaneous compact translation carries the highest polynomial and highest coordinate to their translates, with the same invariant contraction. Thus zero for that pair gives zero for every real polynomial and every coefficient, by irreducibility at each real place. Density of the Hermite expansions in Schwartz space extends this to all real data, by continuity of the lattice integral over the compact quotient.

At an old mark the normalized residue action on the form is ordinary right translation by $\operatorname{diag}(u, I_t)$ times the tame character of ν on u. It fixes Y since the idempotent is the group mean. Consequently the ordinary translations over $u \in k^\times$ sum to zero on Y by nontriviality of that character. This is exactly the hyperplane average needed for one column in the proof of Theorem 81.10. More explicitly, on the old open orbit the top entry of the column is nonzero and the ratios of its other entries to that entry belong to the maximal ideal. At a new place the ratios need only be integral. Averaging over the corresponding target level carries each ball inside an open orbit into the line of the right scalar translate of the block test. The descent of every such translate vanishes because all source arguments are allowed. Outside these orbits the annihilating row and shears constructed in that proof fix every sufficiently small ball there. At an old place their mean is the zero translation sum above. At a new place one shear acts by a nontrivial tame scalar. Compact disjoint subdivision therefore proves vanishing for all marked data at the bottom with the other finite vacua fixed.

We need only the tower through the positive line with at most twenty hyperbolic planes adjoined. If the first nonzero restricted lift occurs at rank $b = 1 + 2h$, $1 \le h \le 20$, the zero-slice calculation of Lemma 81.12 kills every proper constant term. Its integral hyperbolic splitting applies to the same unramified extension E/F. One may take a polynomial real test giving a nonzero cusp form in L^2, by the continuity, moderate-growth and cuspidal decay argument of Theorem 81.13. The normalized variable in A_w has modulus $q^{-(n-2)/2} = q^{-(t-1)/2}$, since its specified Satake entry is the top of the length-t string. Applying Lemma 81.11 at the same w as in that theorem, the scalar product of degree $n - b$ is nonzero because $n - b < t$. On the spherical L^2 space of the smaller group the sum of the norms of the positive-degree terms of the remaining polynomial is at most $\exp(q^{-(t-b)/2}/(1 - q^{-1})) - 1 < 1$. Here $b \le 41$ and $t - b \ge 4$. The constant term is the identity, giving a contradiction. Thus all the restricted descents through rank 41 vanish for each coefficient.

Translate the form by A, which can be moved onto the marked tests in these integrals. Restrict the rank-41 kernel to $U(\mathbb{H}^{20}) \times T$ and use a tensor test. It is the product of the hyperbolic and the bottom kernels by the orthogonal-sum oscillator identity. The Fourier coefficient of the former at Gram matrix I_{20} unfolds to the complementary period by Lemma 81.16 with frame length twenty. Its proof uses only integral Gram-one frames, the unmarked lattice vacuum and definiteness of I_n. The bottom kernel factors over the first twenty and the last $n - 20$ target rows. Assign the prescribed flat source character to the latter and the quotient character to the former. Both have trivial external target character. For fixed g, h_+ one can make the first factor nonzero by choosing a real bump isolating one rational position in the finite support, exactly as in Proposition 81.17. Division by this nonzero value proves the stated period. □

Choose a residue column x_0 of length twelve at each old mark and a multi-index $\alpha = (\alpha_{\tau i})$, $1 \le i \le 12$, with $\sum_i \alpha_{\tau i} = d_\tau$. Set

$$m_\alpha = \prod_\tau \frac{d_\tau!}{\prod_i \alpha_{\tau i}!}, \qquad \chi = \frac{\nu \beta v_1^z}{(\psi_2 \beta)\psi_3 \psi_1}.$$

Let $K_{0,\chi,\alpha}(g; x_0)$ be the unscaled twelve-row kernel with external source factor χ^{-1}, old residue column x_0, no new residue condition, polynomial $\prod_{\tau,i} X_{\tau i}^{\alpha_{\tau i}}$, and target identity. In particular the first kernel has the indicated old point test even if x_0 is zero.

Theorem 82.6 (A tested integral form with a prescribed derivative monomial)**.** *For the above choices there is a form Ξ on $U(I_n)$, integral at all coefficient places away from Q_β and orthogonal to every false torus form in the sense of Section 50.1, such that for each g*

$$|C|^3 \left(\prod_{i=1}^{3} a_{r_i} \right) \langle \Xi, K_\beta(g) \rangle = m_\alpha K_{0,\chi,\alpha}(g; x_0) \prod_{i=1}^{3} J_{r_i}(\psi_i, \beta_i).$$

The same Ξ works for every source argument at this twist.

Proof. In the one-row coefficient representation take at each embedding the tensor $v_{\alpha,\tau}$ which is the sum of the distinct words with $\alpha_{\tau i}$ copies of e_i. It belongs to the integral lattice of symmetric tensors. Pairing it with $X^{\otimes d_\tau}$ gives the monomial times the multinomial coefficient, because each word contributes the same monomial with coefficient one. This includes degree zero. The tensor product v_α over the embeddings is fixed by all matrices acting identically on the first twelve columns. The coefficient representation has no determinant twist since the source line is positive and the external target character is trivial.

At an old mark let A carry the originally specified column to $(x_0, e_2, e_3, e_1)^{\mathsf{t}}$, in block notation, where e_i are the standard columns in the flat blocks. This is always a nonzero residue

vector and may be completed to a basis and lifted integrally. At a new mark choose a permutation for which the original coordinate of $A^{-1}X$ reads the prescribed entry of X in block 2. Take $A = 1$ elsewhere. For $h = (h_2, h_3, h_1)$ put $y = \operatorname{diag}(1, h_2, h_3, h_1)A$ and define the point form $F(h)$ by

$$\langle F(h), Y\rangle = \langle v_\alpha, Y(y)\rangle \quad \text{for all target forms } Y.$$

In the class metric this has support on the class of y; its value at y is the sum of the stabilizer translates of v_α, including the level character. Indeed division by the stabilizer order in the metric cancels the size of this sum on taking the pairing with an invariant value. The stabilizer action preserves the moving coefficient lattice at y. The inverse of the block diagonal point fixes v_α, and A is integral with integral inverse at every place. Thus v_α is integral in this lattice and so is $F(h)$, including above primes dividing a stabilizer order.

Pair $F(h)$ with the full unscaled kernel. The translation by A imposes x_0, e_2, e_3, e_1 at the old marks and reads only block 2 at the new marks. The polynomial and source characters also have the prescribed product over the four blocks. Rational positions in the direct-sum polarization sum independently, with absolute convergence, as in Theorem 77.41. We get

$$\langle F(h), K_\beta(g)\rangle = m_\alpha K_{0,\chi,\alpha}(g; x_0) \prod_{i=1}^{3} K_i(g, h_i).$$

For each flat block apply to F, in its h_i variable, the weighted class sum

$$\frac{B_{r_i} b_{r_i}^2}{Q_{\beta_i}^{r_i-1}} \sum_{h_i} \frac{\phi_{\psi_i,\beta_i}(h_i) F(h)}{d(h_i)},$$

where $d(h_i)$ is the stabilizer order at that block level. Use block 1 first. The summands descend to the stated classes. A rational block change fixes the coefficient tensor. The block level fixes its standard old column and, on block 2 at a new mark, preserves the kernel of the selected coordinate. Its action at the evaluation point therefore has exactly the block level character. Since the point vector occupies the conjugate-linear slot, its scalar under a level translation is the inverse of the scalar on the flat form, so their product is invariant. This verification persists when the other two variables are fixed. By Theorem 82.3, $b_{r_i}\phi_{\psi_i,\beta_i}$ is integral away from Q_{β_i}, and $d(h_i) \mid B_{r_i}$. Hence each of the three sums preserves integrality above every $\ell \nmid Q_\beta$.

For fixed h_2, h_3 and a false form Y, apply the lemma with $h_+ = \operatorname{diag}(1, h_2, h_3)$ and $v = v_\alpha$. The integrand is constant in the compact real variable of block 1, because both v_α and the flat coefficient are fixed there. Its adelic integral is the finite class sum with weights $1/d(h_1)$ times the positive level volume. Taking the conjugate and summing the resulting identities over the source classes gives the vanishing of $\sum_{h_1} \overline{\phi_{\psi_1,1}(h_1)}\langle F(h), Y\rangle / d(h_1)$. This is the pairing of the first weighted class sum with Y up to its fixed real scalar. The other two finite weighted sums remain orthogonal to Y. This proves the required testing for the resulting Ξ, simultaneously for every w.

Finally, on block i one has for every g

$$\langle \phi_{\psi_i,\beta_i}, K_i(g)\rangle = \|\phi_{\psi_i,\beta_i}\|_{\mathrm{cl}}^2 / (a_{r_i}|C|).$$

The normalized central translations of Lemma 50.2 permute the kernels transitively and fix their sum. Thus the left side is independent of g. Sum it over C and use $\phi_{\psi_i,\beta_i} = a_{r_i} \sum_g K_i(g)$ to obtain the displayed value. Conjugate-linearity in the first slot now replaces each $K_i(g, h_i)$, under its weighted sum, by $J_{r_i}(\psi_i, \beta_i)/(a_{r_i}|C|)$. The four-block factorization gives the asserted identity. □

82.4 The valuation of the twelve-row derivative table

Fix one of the main pairs from Section 82.3. Write $N = |C|$ for the number of source classes and $h = \sum_\tau d_\tau$ for the total derivative order. Use the marked elliptic differential of Theorem 67.8 and put $u_E = \pi/A$ in the notation of that theorem. For a fixed nonzero algebraic number b, define a table indexed by the old residue data x_0 and the multi-indices α of the previous subsection by

$$T_{\alpha,x_0} = b\, a_{12} u_E^{h/2}\, m_\alpha K_{0,\chi_0,\alpha}(1; x_0), \qquad \chi_0 = \nu \upsilon_1^z/\psi_2.$$

Here ψ_2 is any fixed original tame class character. One may take the ramified order-Q character in the simultaneous critical family for use of its height estimates. The kernel in the display has identity source and target, so the value there is independent of χ_0. Let $G_{x_0}(\mathbf{Z})$ be the marked twelve-row theta section in the matrix coordinates of Theorem 67.8 with old point x_0. Then

$$T_{\alpha,x_0} = b \left(\prod_\tau d_\tau! \right) [\mathbf{Z}^\alpha] G_{x_0}(\mathbf{Z}). \tag{82.1}$$

Indeed the marked oscillator identity sends the polynomial $\mathbf{X}^\alpha$ to its undivided derivative, with scalar $a_{12} u_E^{h/2}$. At zero this derivative is $\prod_{\tau,i} \alpha_{\tau i}!$ times the indicated coefficient. This cancels the denominator of m_α. Finite Fourier expansion of the old indicator and the marked theta expansion in Theorems 65.9 and 67.8 give algebraicity. The matrix variables are obtained from product coordinates by the algebraic trace-basis matrix of Lemma 77.33.

Use compatible product coordinates in the four even blocks and their orthogonal sum. In the notation of Theorem 82.1 take

$$S_{\nu,z} = b\, a_{12} u_E^{h/2} \prod_{i=1}^{3} a_{r_i}.$$

With these coordinates $a_n = a_{12} \prod_i a_{r_i}$. Indeed the finite position boxes, lattice intertwiners, product marked sections and real Gaussian factor on a direct sum. The identity of the two normalized vacuum sections holds as functions of their independent variables. Both are nonzero sections, so their proportionality constant is the product of those for the summands, even if a value at the origin vanishes. The marked oscillator identity and finite Fourier expansion therefore give algebraic entries for the full scaled ϕ_β as well. For general finite arguments move to split representatives by Theorem 65.13, and then move back by rational automorphy. The changes of polynomial coordinates and finite determinant characters are algebraic. This proves algebraicity also for $\|\phi_1\|_{\mathrm{cl}}^2$, by its finite class sum.

Set $L > 0$ equal to the positive root of Corollary 82.4 for $r_1 = n - 20,\ r_3 = 4$. Define

$$c_* = L J_4(\psi_2, 1), \qquad \rho_\beta = J_4(\psi_2, \beta)/J_4(\psi_2, 1), \qquad H_{\nu,z} = \frac{c_*^2}{N^4 \|\phi_1\|_{\mathrm{cl}}^2}.$$

The first and last quantities are independent of the twist. All three are positive and algebraic. In particular c_* is integral and its conjugates have positive logarithmic size $o(B_x^{\det})$ by Theorem 82.3. For a normalized coefficient valuation v_ι above ℓ, put $e_\iota = \min_{\alpha,x_0} v_\iota(T_{\alpha,x_0})$, with $v_\iota(0) = +\infty$. This is the valuation of the fractional ideal generated by the table in any number field containing it, using the zero ideal convention.

Theorem 82.7 (Passing from the untwisted derivative table to the projected period)**.** *For each allowed twist β with $\ell \nmid Q_\beta$ and $e_\iota < +\infty$, there is a vector U tested and integral at all primes above ℓ with*

$$v_\iota(\langle U, \phi_\beta \rangle/\rho_\beta) \le v_\iota(c_*) + e_\iota - 2v_\ell(N).$$

Let $p_{\ell,\iota} \ge 0$ be the bound for the positive part of the negative valuation of the projected pairing for this main pair in Corollary 52.21. For every prescribed twist in that simultaneous family,

$$-v_\iota(H_{\nu,z}\rho_\beta^2 R_{\nu,z}(\beta)^{-1}) \le p_{\ell,\iota} + e_\iota + e_{\iota\circ c}, \qquad c(a) = \bar{a}. \tag{82.2}$$

In particular the two table valuations use identical finite data at the identity, with no conductor primes adjoined.

Proof. Choose an index attaining e_ι, so the corresponding kernel value at the identity is nonzero. On representatives g of C containing the identity representative, put $s(g) = K_{0,\chi_0,\alpha}(g; x_0)$. For $\psi \in \widehat{C}$ form

$$A_\psi = N^{-1} \sum_{g \in C} \psi(g) s(g).$$

Orthogonality of characters gives $\sum_\psi A_\psi = s(1)$: the sum of $\psi(g)$ is zero if $g \neq 1$ in C and is N otherwise. Thus at least one ψ satisfies $v_\iota(A_\psi) \le v_\iota(s(1))$. Equivalently this can be applied after multiplying every term by the common nonzero scalar $ba_{12}u_E^{h/2}m_\alpha$, which makes the quantities algebraic. This step holds also if $\ell \mid N$. Choose $\psi_1\psi_3 = \psi$ by Corollary 82.4. It gives $v_\iota(J_{r_1}(\psi_1, 1)J_4(\psi_3, 1)) \le v_\iota(L)$.

Take U to be the actual vector of Theorem 82.6 for these data. With $\chi = \chi_0/\psi$, its twelve-row kernel at g is $\psi(g)s(g)$ because the source character occurs inversely. Multiply the kernel pairing by $S_{\nu,z}$ and sum on C. The identity of that theorem, in the chosen normalization, is

$$\frac{N^2\langle U, \phi_\beta\rangle}{\rho_\beta} = ba_{12}u_E^{h/2}m_\alpha A_\psi\, J_{r_1}(\psi_1, 1)J_4(\psi_3, 1)J_4(\psi_2, 1).$$

The three factors of N there become two since the unweighted sum giving A_ψ is NA_ψ. Substituting the two valuation estimates just proved gives the first assertion. In particular its pairing is nonzero.

For the second assertion the inequality is automatic if the table is zero. Otherwise perform the first construction at ι and at $\iota \circ c$ with the very same β, obtaining U, V. Both meet the all-embeddings integrality requirement of Theorem 50.3. Use them in Corollary 82.2. The two upper estimates for the single pairings contribute at most $2v_\iota(c_*\rho_\beta) - 4v_\ell(N) + e_\iota + e_{\iota\circ c}$, since c_* and ρ_β are real. The projected term contributes at most $p_{\ell,\iota}$. Moving the common scalars inside the negative valuation gives (82.2). The first construction applied no restrictions on primes beyond $\ell \nmid Q_\beta$, so it applies to both tests without changing the Selmer or critical choices in that corollary. □

The arithmetic quantities still to be bounded in the directional estimate are the upper valuations of the explicit Taylor table in (82.1) and the archimedean size of $1/\|\phi_1\|_{\mathrm{cl}}^2$ in this same normalization. Changing b multiplies the table by the same scalar and the squared norm by its squared modulus. The contributions of this change to the two sides of (82.2) therefore agree at every coefficient embedding.

82.5 The complete linear system of the twelve-row tests

Keep the two original marks p_0, p_1 and the chosen position component at every place above them. Put $M = p_0 p_1$, $D = 12f$. The period module for the twelve-row vacuum is $\Lambda = (\mathfrak{b}^{-1})^{12}$ in the coordinates of Theorem 67.8. Let $\mathfrak{m}$ be the product of the primes of E on the chosen position side at the old places, and put $\Lambda' = \mathfrak{m}\Lambda$. The original marks split completely and are unramified, so $\mathfrak{m}\overline{\mathfrak{m}} = M\mathcal{O}_E$. The inclusion of lattices induces the finite isogeny

$$A' = \mathbb{C}^D/v(\Lambda') \longrightarrow A = \mathbb{C}^D/v(\Lambda), \qquad v(a) = \delta\Phi(a).$$

Let $\mathcal{L}$ be the pullback to A' of the vacuum theta line on A, with its original multipliers. At the origin use the same holomorphic coordinates and the same frame as in (82.1).

Theorem 82.8 (The residue sections and their Taylor coefficients)**.** *The sections G_{x_0} indexed by all choices of the twelve old residue entries at every marked place form a basis of $H^0(A', \mathcal{L})$. For every $a \geq 0$ with $3a + 2 \leq M$, their Taylor polynomials through total degree a span $\mathbb{C}[\mathbf{Z}]/(\mathbf{Z})^{a+1}$.*

Proof. At an old split place the finite test defining G_{x_0} is the indicator of the residue column x_0 inside the unit position lattice. A Weyl translation from Λ' has position in the maximal ideal times this lattice and frequency in its full integral dual. Its position shift preserves the indicator, and its frequency character is one on the unit lattice. Its scalar Weyl factor is the same as on the unit vacuum. At the other places the two finite tests coincide. Thus all the finite tests have the vacuum period character restricted to $\widehat{\Lambda'}$. The translation formula of Theorem 67.8 gives the multipliers of $\mathcal{L}$ for every G_{x_0}.

Suppose a linear combination of these sections vanishes. Take the same linear combination of the finite tests and tensor it with the real vacuum. Its theta function under all real Weyl translations then vanishes by the coherent-state formula in Theorem 67.8. By Lemma 67.7 the integral of its squared absolute value over the real torus for Λ' equals the squared norm of the tensor. The distinct finite tests have disjoint supports, each of positive measure, and the real vacuum has norm one. Every coefficient of the linear combination therefore vanishes.

Write B for the integral alternating form of the vacuum. The trace polarization of Theorem 64.10 makes Λ self-dual for B, with the adjoint of multiplication by $u \in E$ given by multiplication by $\bar{u}$. Consequently v in the rational space pairs integrally with $\mathfrak{m}\Lambda$ precisely when $\overline{\mathfrak{m}}v \subset \Lambda$. The dual of Λ' for B is $\overline{\mathfrak{m}}^{-1}\Lambda = M^{-1}\Lambda'$. It follows that B/M is a positive integral principal Riemann form on A'. Choose a line bundle P_0 with this form by the Appell–Humbert theorem. The quotient $\mathcal{L} \otimes P_0^{-M}$ has trivial first Chern class. Multiplication by M on the dual complex torus $\mathrm{Pic}^0(A')$ is surjective. Twisting P_0 by an M-th root of the quotient gives an ample principal line bundle P with $P^{\otimes M} \simeq \mathcal{L}$. By Riemann–Roch and vanishing of higher cohomology for an ample line bundle on a complex abelian variety,

$$h^0(A', \mathcal{L}) = M^D \frac{c_1(P)^D [A']}{D!} = M^D.$$

The last fraction is one by principality. There are f places of F above each p_i, each with residue field of size p_i. The number of distinct x_0 is thus $p_0^{12f} p_1^{12f} = M^D$, which proves the basis assertion.

Use the very ampleness of P^3 on a complex abelian variety (Milne, *Abelian Varieties*, Chapter I, §6). Choose a section s_0 of P^3 nonzero at the origin. The ratios of global sections to s_0 give coordinates in the affine chart of the projective embedding. After subtracting their values at zero, choose D such ratios s_i/s_0 with independent differentials. They form local holomorphic

coordinates. For $|\gamma| \le a$ the section $s_0^{a-|\gamma|}\prod_i s_i^{\gamma_i}$ of P^{3a} has, in the frame s_0^a, the corresponding monomial in these coordinates. These monomials span every jet of order a.

Put $u = M-3a \ge 2$. There is a section of P^u nonzero at zero. Indeed take a nonzero section of P, which exists by Riemann–Roch, with divisor D_0. Choose $u-1$ translation parameters on A' and take the last to be the negative of their sum. They may all be chosen outside D_0, since the coordinate projections and the negative sum are surjective and finitely many proper closed subsets do not cover $(A')^{u-1}$. The product of the translated sections is then nonzero at zero and belongs to P^u by the theorem of the square. Multiplication by this section identifies the two local jet spaces. A change of holomorphic coordinates with invertible differential and a change of frame by a unit also act invertibly on finite jets. We therefore have the asserted spanning property in the actual frame and coordinates of the derivative table. □

Corollary 82.9 (Nonzero tests at every prescribed derivative)**.** *On the sufficiently distant common intervals of Theorem 82.7, the matrix (T_{α,x_0}), with rows of multidegree $(d_\tau)_\tau$, has full row rank over any number field containing its entries. In particular each such row has a nonzero entry. The valuation e_ι in (82.2) is finite at every coefficient place. For a fixed main pair it is zero outside a finite set of places.*

Proof. We check the order $h = \sum_\tau d_\tau$ from Section 82.4. The original weights $w_{j,\tau} = \kappa_\tau + (2j+1)s^8$ are at most $2b_x$ by the estimates of Theorem 73.27. The exponent of α in the scalar exchange is $m-\mu_\tau$, where $\mu_\tau = w_{j,\tau} - j - (t+k-1)/2$. The exponent of $\sigma_c = \alpha^{-1}\upsilon_1^{3m+e'}$ has absolute value at most $|m-\mu_\tau| + 3m + e'$. Passing to the inverse has the same absolute value, and a tame twist has weight zero. Thus $d_\tau = c_\tau - z \ge 0$ is $O(b_x)$ uniformly, since $z \le e' \le s,\ t = 5s,\ s \le 3x^5$ and $m \le 2\sqrt{x}$. Now $f \ll_{q_0,d} x^{12}$, $b_x \ge x^{400}$ and $M \ge b_x^{20}$ by the original prime choices (Corollary 12.10). Hence $3h+2 \le M$ throughout all sufficiently distant intervals.

Project the space of h-jets onto the coordinates belonging to the required multidegree. This projection is onto, so the corresponding coefficient rows for the G_{x_0} are linearly independent by the theorem. In (82.1) all entries have the same nonzero multiplier $b\prod_\tau d_\tau!$. A maximal row minor remains nonzero after this scaling, and the entries are algebraic there. Finally the table has finitely many entries. Outside the finite set of primes dividing a numerator or denominator of one of its nonzero entries, every such entry is a unit. □

This removes the nonvanishing hypothesis in the construction of the tested vector in Theorem 82.7. The directional prime sum also requires a quantitative bound on the valuations of the table as the field and weights vary.

82.6 Integral leading terms in the twelve-row linear system

Put $D = 12f$, $M = p_0p_1$, and $I = \mathfrak{m}\mathfrak{b}^{-1}$ as in Section 82.5. For a fractional ideal J of E, write $v_\iota(\tau J)$ for the valuation of a local generator of its extension via τ to the coefficient ring at ι. Normalize $v_\iota(\ell) = 1$. The following estimate uses the first nonzero homogeneous term of an integral section. This makes the comparison independent of the higher terms in the change to the marked analytic frame.

Lemma 82.10 (Extracting one residue coefficient over a valuation ring)**.** *Let R be a local integer ring of a sufficiently large number field, and let $\mathcal{L}_R$ be an extension of the line bundle in Theorem 82.8 to the good abelian model of A'. Fix a residue column y with $g = G_y(0) \neq 0$. Let λ_R be the value, in the marked frame, of a generator of the fiber of $\mathcal{L}_R$ at zero. If $S = \sum_x c_x G_x$ is an integral global section, then*

$$M^D c_x g/\lambda_R \in R \qquad \text{for every } x.$$

Proof. The functions are identified with algebraic sections over a common number field. Indeed the marked product theta has algebraic divisor. By the torsion translation formula in Theorem 65.9, the vacuum and its finite translates are a common product theta times algebraic rational functions on a finite isogenous cover. The finite Fourier expansion of a residue indicator has algebraic coefficients. Their ratios therefore descend to algebraic rational functions on A', since on the cover the ratios are invariant under the kernel of the covering isogeny. The common bundle is algebraic as well. Its divisor can be obtained from a nonzero section on this cover and descended: pullback of its complex divisor is algebraic, and pushforward recovers the divisor times the degree of the isogeny. We can use G_y as rational frame near zero. The new frame has value g in the marked frame, an algebraic number by the order-zero torsion formula. This also defines the algebraic comparison at the fiber.

Denote the additive group of residue columns by X, of order M^D. For each $\xi \in \widehat{X}$ use multiplication by ξ on the old unit position box. Its operator on sections satisfies $D_\xi G_x = \xi(x)G_x$. A unit position shift gives an operator carrying G_x to any specified G_y up to a root of unity. These formulas follow on the indicators by multiplication and translation, respectively. They are translations on the analytic theta line by the Weyl formula of Theorem 67.8. More explicitly each corresponding finite Weyl displacement pairs integrally with $\widehat{\Lambda'}$. Choose a rational representative modulo $\widehat{\Lambda'}$ by lattice approximation. It belongs to the B-dual $M^{-1}\Lambda'$, so the translation on A' is by a torsion point. Replacing a representative introduces only the vacuum period character and a rational Weyl phase, both roots of unity. We take the scalar giving the displayed action for D_ξ. A positive power of each chosen lift is the identity up to a root of unity.

We verify integrality of these operators even above M. Extend the field to define the torsion points. Each translation extends to the abelian scheme by properness and the group law. The isomorphism of the translated line bundle with the original one exists algebraically on the generic fiber. An algebraic choice differs from the analytic lift by a scalar, algebraic since a positive power of each lift acts by an algebraic scalar. After multiplication by an element of the fraction field we obtain an isomorphism over R. Indeed the divisor of the generic isomorphism has only vertical terms, and the special fiber is irreducible, reduced and cut out by a uniformizer. A power of this integral isomorphism covering the identity is a unit scalar, since the global invertible functions on the abelian scheme are those of R. The original lift has a root of unity for that scalar after taking a common power. Their ratio thus has valuation zero. In particular the original operators act on integral sections.

The sum $\sum_{\xi\in\widehat{X}}\xi(x)^{-1}D_\xi S$ is integral and equals $M^D c_x G_x$, by character orthogonality. Apply a shift carrying x to y, then evaluate at zero. This gives $M^D c_x g$ in the integral fiber, up to a unit root of unity, as asserted. □

Theorem 82.11 (An upper bound from integral leading terms)**.** *At each sufficiently distant node there is a single nonzero algebraic integer γ_x, depending only on the original field, marks and vacuum, with*

$$\sup_{\sigma}\log^{+}|\sigma\gamma_x| \ll_{q_0,d_{\mathrm{K}}} f\log(2b_x).$$

For each permitted row α, every main pair, every nonzero scalar b of (82.1), and every coefficient place one has

$$\min_{y} v_\iota(T_{\alpha,y}) \le v_\iota(b\gamma_x) + \sum_{\tau}\left(v_\ell(d_\tau!) + d_\tau\big(v_\iota(\tau\mathfrak{b}) - v_\iota(\tau\mathfrak{m})\big)\right). \tag{82.3}$$

Proof. Use the good model $(I^{12})\otimes_{\mathcal{O}_K} C$ for A', where C is the marked elliptic curve. The construction by a projective summand and its complex uniformization are those of Theorem 64.10. After enlarging the number field choose the root $P^M \simeq \mathcal{L}$ constructed in Theorem 82.8 algebraically. Indeed the fiber of the M-th power map in the Picard scheme has a complex point and hence a point over $\overline{\mathbb{Q}}$. Locally at a finite place extend P by closure of a divisor on the regular model. Its extension P_R is fiberwise ample. To see this, the Hermitian form for P is $1/M$ times the positive trace form. Diagonalize this form over K and clear denominators in both its diagonal entries and its independent coordinate forms on I^{12}. A positive integer multiple is a sum of positive integral multiples of norms of $\mathcal{O}_K$-linear coordinates $I^{12}\to\mathcal{O}_K$. Together these coordinates give an isogeny to a product of copies of C on every fiber, since an integer multiple of the inverse also preserves the lattices. The corresponding product of positive powers of the elliptic polarization pulls back to an ample bundle fiberwise. Its polarization homomorphism agrees with the appropriate power of P_R on the generic fiber, hence on the whole model by separatedness, as in the proof of Theorem 64.10. Fiberwise their quotient lies in Pic^0, proving ampleness.

Take $\mathcal{L}_R = P_R^M$. Its sections map onto all order-h jets at the zero section over R, with $h=\sum_\tau d_\tau$. On a geometric special fiber the cube of an ample bundle is very ample and its powers of degree at least two are generated by sections. The coordinate monomials in the proof of Theorem 82.8 therefore give surjectivity there as $3h+2\le M$. These statements on powers, and vanishing of higher cohomology of an ample bundle on an abelian variety, hold in every characteristic (Mumford, *Abelian Varieties*, Chapter III, §§16–17). Cohomology and base change lifts sections of the special fiber. Jets form a finite free module over the local ring by smoothness, so Nakayama's lemma gives the asserted surjectivity over R.

The marked elliptic differential generates the invariant differentials in good reduction by the unit-discriminant calculation in Theorem 67.12. In the i-th copy of $I\otimes C$, the differential $dZ_{\tau i}$ in matrix coordinates is the linear form $a\mapsto\tau(a)$ on the module I, in that elliptic differential. This follows by pulling back along $z\mapsto(\tau(a)z)_\tau$ for $a\in I$. If u_τ generates the inverse of the fractional ideal $\tau(I)R$, each $u_\tau dZ_{\tau i}$ is thus an integral cotangent vector. The top term of the jet module with all lower terms zero is the h-th symmetric power of the cotangent module tensored with the fiber of the bundle. Prescribe there the monomial $\prod_{\tau,i}(u_\tau dZ_{\tau i})^{\alpha_{\tau i}}$ times a fiber generator, and lift it to an integral section S. In marked coordinates it satisfies

$$[\mathbf{Z}^{\alpha}]S = \lambda_R\prod_{\tau}u_\tau^{d_\tau}.$$

Only the differential of the change of parameters and the value of the frame at zero enter this homogeneous term. This also holds at $h=0$ by evaluation.

Fix once and for all a column y_* with $g = G_{y_*}(0) \neq 0$; it exists by Theorem 82.8 at order zero. Expand $S = \sum_y c_y G_y$. By the lemma each $M^D c_y g/\lambda_R$ belongs to R. Taking the indicated Taylor coefficient in this expansion shows that $M^D g \prod_\tau u_\tau^{d_\tau}$ is an integral linear combination of the numbers $[\mathbf{Z}^\alpha]G_y$. Multiply by $b\prod_\tau d_\tau!$ and use (82.1). The valuation of the ideal on the right is at most the valuation of this nonzero combination. Here $v_\iota(u_\tau) = v_\iota(\tau\mathfrak{b}) - v_\iota(\tau\mathfrak{m})$.

Finally take $\gamma_x = (Mt_E)^D g$, with t_E from Lemma 77.33. We check its size and integrality in this same marked frame. The finite Fourier expansion of the old point indicator has denominator M^D and its coefficients have absolute sum one at each embedding. The passage to the product theta has normalization $I_0^{-1/2}$ with $I_0 \mid t_E^{2D}$ by that lemma. Each remaining summand at zero is a product of torsion values with a root of unity factor and is integral by Theorem 66.35. It follows that γ_x is integral. At any complex embedding the squared magnitude of each translated vacuum at zero is at most $[e\max(1, 24\,\mathrm{rd}(E)/\pi)\max(1,\nu_\sigma)]^D$ by Theorem 65.9. The finitely many elliptic norms are bounded for fixed d_{K}. The bounds $\log M \le 22\log b_x$, $\log t_E \ll_{q_0,d_{\mathrm{K}}} \log(2x)$ and $\mathrm{rd}(E) \le q_1 d_{\mathrm{K}}$ from the proof of Theorem 82.3 give the asserted estimate for every conjugate. Since t_E is an integer, replacing $M^D g$ by γ_x preserves the valuation upper bound. □

Corollary 82.12 (The projected period above the derivative weights)**.** *Let c denote complex conjugation on the coefficient numbers. For an allowed twist with $\ell \nmid Q_\beta$, suppose that $\ell > \max_\tau d_\tau$ and that ℓ is prime to the absolute norm of the relative different of E/K. Then on the same field and interval as Corollary 52.21,*

$$-v_\iota\Big(H_{\nu,z}\, b\bar{b}\,\gamma_x\bar{\gamma}_x\,\rho_\beta^2 R_{\nu,z}(\beta)^{-1}\Big) \le p_{\ell,\iota}.$$

This holds in particular at all primes larger than Cb_x for a fixed sufficiently large $C = C(q_0, d_{\mathrm{K}})$ and all sufficiently distant nodes, subject to $\ell \nmid Q_\beta$.

Proof. All factorials in (82.3) are units at ℓ. The relative different is $\mathfrak{b}\bar{\mathfrak{b}}$, and $\mathfrak{m}$ is integral. Hence $e_\iota \le v_\iota(b\gamma_x)$, also at $\iota\circ c$ with the same γ_x. Substitution into (82.2) proves the display. The weights are $O(b_x)$ by Corollary 82.9. The cyclotomic extension defining E/K ramifies only above the primes dividing its conductor $q_1 \ll_{q_0} x^{11}\log(2x)$, which is smaller than b_x on these intervals. This gives the last assertion. □

At primes within the derivative weights the factorials in (82.3) can contribute. The directional estimate requires finer control of these contributions together with the archimedean size of the inverse untwisted norm.

82.7 Formal theta frames and the split primes

Write $B = I \otimes_{\mathcal{O}_K} C$ for one of the twelve factors of the abelian scheme of Section 82.6. Use its marked matrix coordinates $Z = (Z_\tau)_\tau$ on the complex universal cover and, by extension of the coefficients, as formal logarithm coordinates at a finite place. In these coordinates the group law is addition. Let R be a completed local integer ring as in that section. We may make a further extension to a complete discrete valuation ring, keeping $v_\iota(\ell) = 1$.

Lemma 82.13 (Cancellation of the theta frame along a formal diagonal)**.** *There are $\delta_1, \dots, \delta_6 \in \mathbb{Z}_\ell$, with $\delta_1 = 1$ and $\sum_i \delta_i^2 = 0$, with the following property. Let L be the one-factor theta line whose external twelfth power on B^{12} is $\mathcal{L}$. Take an integral local frame for an extension L_R along the zero section. Write $F(Z)$ for its scalar in the marked analytic chart divided by its value at zero, so $F(0) = 1$. Then $\prod_{i=1}^{6} F(\delta_i Z)$ and its inverse belong to the integral completed local ring of B along zero.*

Proof. The standard trace Hermitian form is the orthogonal sum over the twelve entries. The period sign is $(-1)^{\mathrm{Tr}_{E/K} \sum_i a_i \bar{a}_i}$ on I^{12} by Lemma 65.8, so the factor of automorphy is the product of the identical one-entry factors. This identifies L by restriction to one factor and gives the asserted external product in the actual analytic chart. The algebraicity argument in Lemma 82.10 applies to the restriction. Concretely a residue section nonzero at zero restricts to an algebraic local frame, and its Taylor series after division by its value there has algebraic coefficients. The ratio of another algebraic local frame to this one is an algebraic function. Hence all the finite jets of F can be read in the local coefficient field.

For each nonnegative integer n, pullback by $[n]$ gives the same analytic multipliers as L^{n^2}. The exponential part of the factor is quadratic under simultaneous scaling of the period and variable, and its sign is unchanged on replacing the exponent n by n^2. The resulting identification of rigidified bundles is algebraic by the theorem of the cube. Normalize it to send the chosen fiber generator to its n^2-th power at zero. It extends integrally: as in the proof of Lemma 82.10 a generic isomorphism extends up to a power of the uniformizer, and normalization on the fiber at the zero section fixes that power. Therefore, in integral parameters T at zero,

$$F(nZ)/F(Z)^{n^2} \ \in R[[T]]^\times.$$

Indeed this is the ratio of the pulled-back integral frame and its tensor power under that isomorphism. The normalization at zero cancels their constant factors.

For each fixed jet order, the coefficients of this ratio in T depend continuously on $n \in \mathbb{Z}_\ell$. In the logarithm coordinates substitution multiplies a term of degree j by n^j. For the power $F(Z)^{-n^2}$ use the finite binomial expansion in $F - 1$ through this order. Both calculations have coefficients polynomial in n over the fraction field. The change from logarithm coordinates to T is fixed and invertible in characteristic zero. Consequently $F(\delta Z)/F(Z)^{\delta^2}$ has integral coefficients and constant one for every $\delta \in \mathbb{Z}_\ell$, by closure of R.

For each $N \geq 1$ write the nonnegative integer $\ell^N - 1$ as four squares by Lagrange's theorem. Compactness of $\mathbb{Z}_\ell^4$ gives a convergent subsequence of the four coordinates as N tends to infinity. Their limiting squares sum to -1. Prepend 1 and append 0 to obtain the stated six scalars. Multiplying the six integral ratios cancels the powers of $F(Z)$, by the binomial law for a formal unit of constant one. The product has constant one, so its inverse is integral as well. □

Theorem 82.14 (Two scalar formal groups in the twelve-row tests)**.** *Let H be an integral scalar function on the formal completion of $B \times B$ at the identity. In the two sets of logarithm coordinates U, V, put $q = [U^\mu V^\eta]H$, where $\mu_\tau + \eta_\tau = d_\tau$ for every τ. For $q \neq 0$ the derivative*

table of the same field and marks satisfies

$$e_\iota \le v_\iota(b\gamma_x) + v_\iota\left(q\prod_\tau d_\tau!\right). \tag{82.4}$$

The common algebraic integer γ_x is precisely the one in Theorem 82.11.

Proof. Choose L_R as in the lemma, and on B^{12} use its external product, with the identification on the generic fiber just proved. Over a local integer ring this model is isomorphic to the model of $\mathcal{L}$ used in Theorem 82.11: any discrepancy between extensions over the same regular total space is a vertical divisor, hence principal. The integral surjection onto h-jets established there thus applies to this model, also after flat extension of the local integer ring.

Identify the first six logarithm variables with $(\delta_1 U,\dots,\delta_6 U)$ and the last six with $(\delta_1 V,\dots,\delta_6 V)$. Write $J(U) = \prod_i F(\delta_i U)$. The function $H(U,V)/(J(U)J(V))$ is integral in the formal group parameters of B^2 by the lemma. Pull that function back along the projection onto factors $1,7$ of B^{12}. In the external product frame use its Taylor jet through order $h = \sum d_\tau$. Lift this integral jet to a global section S by the surjection. Let λ_R be the value of this product frame at zero in the marked analytic chart. After the indicated linear substitutions, the marked expansion of S/λ_R agrees through degree h with $H(U,V)$, because $\delta_1 = 1$ on both selected factors and the other frame factors restore $J(U)J(V)$.

Now expand $S = \sum_y c_y G_y$. By Lemma 82.10, $M^D c_y g/\lambda_R$ is integral for the single nonzero $g = G_{y_*}(0)$ chosen in the previous theorem. This assertion commutes with the flat base extension, or follows directly by the same integral translations. The coefficient at $U^\mu V^\eta$ of a substituted G_y is a sum of coefficients of that section in the original twelve variables, with integral multipliers given by products of powers of δ_i. Each index in this sum has total d_τ at τ. Thus $M^D g\, b(\prod_\tau d_\tau!)q$ is an integral linear combination of the actual $T_{\alpha,y}$. Since $\gamma_x = t_E^D M^D g$ with t_E integral, taking valuations proves (82.4). □

Corollary 82.15 (The projected theta period at every split prime)**.** *Suppose ℓ splits in K. For every main pair, with no restriction on its derivative weights, one has*

$$e_\iota \le v_\iota(b\gamma_x) - \sum_\tau d_\tau\, v_\iota(\tau(\bar{\mathfrak{b}}\mathfrak{m})).$$

In particular at every such $\ell \nmid Q_\beta$, for the same twists and integral local cohomological bound as in Corollary 52.21,

$$-v_\iota\Big(H_{\nu,z}\, b\bar{b}\,\gamma_x\bar{\gamma}_x\,\rho_\beta^2 R_{\nu,z}(\beta)^{-1}\Big) \le p_{\ell,\iota}.$$

Proof. We first produce a multiplicative character of the integral formal group of C whose logarithmic derivative at zero is a unit. At a split prime $\mathcal{O}_K \otimes \mathbb{Z}_\ell = \mathbb{Z}_\ell \oplus \mathbb{Z}_\ell$ splits the ℓ-divisible group into two height-one factors. The heights follow on the generic Tate module from the lattice of C, which has rank one over $\mathcal{O}_K$. Exactly one idempotent acts on the tangent line of the good model. The other factor is étale: its finite torsion levels have zero tangent space on the special fiber and are finite flat. The elliptic principal polarization interchanges the two idempotents under Cartier duality, since its Rosati involution on K is conjugation. After passage to the completed strict henselization the étale factors are constant, with compatible cyclic generators. Their Cartier duals identify the formal torsion of C at every level with μ_{ℓ^n}.

For completeness take an integral parameter T at zero. The compatible multiplicative coordinates on these torsion groups define a unit $X(T)$ of constant one over the completed ring. Indeed at level n the coordinate is defined modulo $[\ell^n](T)$. These ideals are nested and lie in $(\pi,T)^{n+1}$, where π is a uniformizer, because $[\ell](T) = \ell T + O(T^2)$. Compatible polynomial

or power series lifts therefore converge in the completed local ring. They give the prescribed coordinates modulo every such ideal. The group law for X follows by the same argument in two variables from the laws on all torsion levels. Its linear coefficient is a unit, since on the special fiber the isomorphism at level one induces an isomorphism on tangent lines, and the differential of $[\ell]$ there is zero. The form dX/X is invariant and is $u\,dz$ for a unit u, by the integral differential calculation in Theorem 67.12. Consequently in the marked characteristic-zero logarithm coordinate $X(z) = \exp(uz)$, by the differential equation and $X(0) = 1$.

The trace dual of $I = \mathfrak{m}\mathfrak{b}^{-1}$ over $\mathcal{O}_K$ is $\mathfrak{D}_{E/K}^{-1} I^{-1} = (\overline{\mathfrak{b}}\mathfrak{m})^{-1}$. Choose w in this ideal generating it at all primes above ℓ. For example, modulo each such prime times the ideal choose a nonzero class, and use the Chinese remainder theorem on the ideal. The trace map $a \mapsto \mathrm{Tr}_{E/K}(wa)$ then takes I into $\mathcal{O}_K$ and defines a morphism $B \to C$ over the good model. Its logarithm coordinate is $\sum_\tau \tau(w) U_\tau$ by the trace formula. Pull back X along this map on the first factor of B^2, taking the second variable constant. The result is an integral function with

$$\left(\prod_{\tau} d_\tau ! \right) [U^{\mathbf{d}} V^0] H = u^h \prod_{\tau} \tau(w)^{d_\tau}.$$

The first inequality follows from (82.4) and the local generator property of w. Both $\mathfrak{b}$ and $\mathfrak{m}$ are integral, so their displayed correction can be discarded for an upper bound, at ι and at $\iota \circ c$. Substitution into (82.2) gives the period inequality. □

The large-prime bound of Corollary 82.12 and this split-prime estimate use the same scalar. For nonsplit primes within the weights, (82.4) reduces the remaining valuation problem to integral functions on the two indicated formal groups.

82.8 The ramified prime and the pairs of larger weights

At a prime ℓ nonsplit in K, let e_0 and Q be the ramification index and the residue cardinality of K_v. Write

$$\lambda_\ell = \frac{1}{\ell - 1} - \frac{1}{e_0(Q-1)}.$$

We first bound the Taylor table using the marked elliptic formal group alone. All valuations have $v_\iota(\ell) = 1$.

Theorem 82.16 (An elliptic derivative bound at a nonsplit prime)**.** *For $h = \sum_\tau d_\tau$ and the same γ_x as in Theorem 82.11,*

$$e_\iota \le v_\iota(b\gamma_x) + h\lambda_\ell - \sum_\tau d_\tau\, v_\iota(\tau(\overline{\mathfrak{b}}\mathfrak{m})).$$

Proof. Take an integral parameter T of the good elliptic formal group as in Lemma 64.5. Write ∂_z for differentiation in the marked logarithm coordinate. Define a_n as the minimum valuation of $(\partial_z^n T^j)(0)$ for $0 \le j \le n$. This is the minimum over all integral functions, because their Taylor series in T can be truncated through degree n for this derivative. It is finite by $j = n$ and nonnegative since ∂_z is an integral invariant derivation by Theorem 67.12. Also $a_0 = 0$.

We have $a_{n+m} \ge a_n + a_m$. To see this, pull an integral function back by formal group addition to a function integral in the two separate parameters. Its derivative of bidegree (n, m) has valuation at least $a_n + a_m$, by expanding in monomials of the two parameters. In the two logarithm coordinates this is exactly the derivative of order $n + m$ of the original function at zero.

For each $r \ge 0$, the coefficient of z^{Q^r} in $T(z)$ has valuation $-(Q^r - 1)/(e_0(Q - 1))$. Indeed in Lemma 64.5 the parameter T, after the rescaling by λ with λ^{Q-1} a uniformizer of K_v, has a nonzero reduced coefficient at every Q^r. The linear change from that logarithm coordinate to the marked elliptic one is a unit because both differentials generate. Since Q^r is an ℓ-power, Legendre's formula gives $v_\ell((Q^r)!) = (Q^r - 1)/(\ell - 1)$. Hence $a_{Q^r} \le (Q^r - 1)\lambda_\ell$. Fix $n \ge 1$, divide Q^r by n with quotient j_r, and use superadditivity and nonnegativity to get $j_r a_n \le (Q^r - 1)\lambda_\ell$. Letting r increase gives $a_n \le n\lambda_\ell$.

Choose j realizing a_h (take the constant for $h = 0$). Pull T^j back to B by the integral trace morphism associated to the generator w of the trace dual at the primes above ℓ, as in Corollary 82.15. This construction of the trace morphism works for every decomposition type of ℓ. In (82.4) use this function on the first factor with $\mu_\tau = d_\tau$ and $\eta_\tau = 0$. The multinomial formula gives

$$\left(\prod_\tau d_\tau!\right)[U^{\mathbf{d}}]H = (\partial_z^h T^j)(0) \prod_\tau \tau(w)^{d_\tau}.$$

Using $v_\iota(\tau(w)) = -v_\iota(\tau(\overline{\mathfrak{b}}\mathfrak{m}))$ proves the assertion. □

For the shorter family put $G = xb_x/x_{1,x}$, and write $\kappa_\tau = b_x$ on the specified set $\mathcal{T}$ of larger-weight embeddings and $\kappa_\tau = G$ elsewhere. Use the choice of $\mathcal{T}$ in Lemma 51.19. Recall that the oriented embeddings are indexed by $(\mathbb{Z}/q_1)^\times$. Every nonsplit ℓ is prime to q_1, since the primes of its conductor split in K by the proof of Theorem 51.20. Write $P_a\tau = -\ell^a\tau \bmod q_1$. Let J_ℓ be the set of positive odd integers at an inert prime and all positive integers at the ramified prime. The directional local sum is

$$S_\ell = \frac{1}{e_0} \sum_\tau \sum_{a \in J_\ell} \left(\kappa_\tau/\ell^a - \kappa_{P_a\tau}\right)_+.$$

All the nonnegative series below are bounded by geometric series.

Theorem 82.17 (The ramified prime and small inert primes)**.** *On the same nodes and intervals as Theorem 82.7, put* $n_x = \lceil 4b_x/k \rceil$, $A_x = (n_x!)^f$. *For a nonsplit* $\ell \le k$ *let* L *be the least positive integer with* $\ell^L \ge k\ell$, *and set*

$$E_\ell = \frac{1}{e_0} \sum_{\tau \in \mathcal{T},\ \substack{a \in J_\ell,\ a \le L \\ P_a\tau \in \mathcal{T}}} \frac{b_x}{\ell^a}.$$

At every coefficient place on sufficiently distant intervals,

$$e_\iota \le v_\iota(b\gamma_x A_x) + S_\ell + E_\ell.$$

The sum E_ℓ *is zero at the ramified prime. At an inert prime its terms come from disjoint pairs* $\{\tau, P_a\tau\}$ *of distinct embeddings. In particular it vanishes if* $-\ell^a \not\equiv 1 \bmod q_0$ *at each of the odd* $a \le L$. *For every prescribed twist at* $\ell \nmid Q_\beta$,

$$-v_\iota\Big(H_{\nu,z}\, b\bar{b}\, \gamma_x\bar{\gamma}_x A_x^2\, \rho_\beta^2 R_{\nu,z}(\beta)^{-1}\Big) \le p_{\ell,\iota} + 2S_\ell + 2E_\ell. \tag{82.5}$$

Proof. For an inert prime $e_0 = 1$, $Q = \ell^2$, and for the ramified prime $e_0 = 2$, $Q = \ell$. Summing the respective geometric series gives $\lambda_\ell = e_0^{-1}\sum_{a\in J_\ell} \ell^{-a}$. The identity $u - (u-v)_+ = \min(u,v)$ for $u, v \ge 0$ therefore gives

$$\lambda_\ell \sum_\tau \kappa_\tau - S_\ell = e_0^{-1} \sum_\tau \sum_{a \in J_\ell} \min(\kappa_\tau/\ell^a, \kappa_{P_a\tau}).$$

For a fixed τ, all terms with $a > L$ have total at most $b_x/((\ell-1)\ell^L)$. At an exponent $a \le L$ the minimum is at most G if either endpoint has weight G. If both endpoints have the larger weight the minimum is b_x/ℓ^a, giving exactly its contribution to E_ℓ.

The derivative orders satisfy $|d_\tau - \kappa_\tau| \le W = C_{q_0,d_{\mathrm{K}}} x^{42}$ uniformly. In the main-pair window of Theorem 15.3, the absolute character weight is $w_{j,\tau} - h_\tau$ with $w_{j,\tau} = \kappa_\tau + (2j+1)s^8$, $0 \le h_\tau \le 4s$. The derivative order further subtracts $z \le s$, so the claimed bound follows from $s \le 3x^5$, $m \le 2\sqrt{x}$. Also $L \le 2 + \log_2 k$. The preceding terms and the error from replacing the weights are at most

$$LG + \frac{b_x}{(\ell-1)\ell^L} + \frac{W}{\ell - 1}$$

per embedding. The first and third terms are each $o(b_x/(k\ell))$ uniformly for $\ell \le k$: for the first the ratio is at most $(2+\log_2 k)xk^2/x_{1,x}$, and for the third it is at most $2Wk/b_x$. Here $x_{1,x} \ge x^{10^6}$ and $b_x \ge x^{400}$. The middle term is at most $b_x/(k\ell)$. Since b_x/k^2 tends to infinity, the sum of the three terms is at most $\lfloor n_x/\ell \rfloor \le v_\ell(n_x!)$ at every such prime on sufficiently distant intervals. Applying Theorem 82.16 and discarding the nonpositive ideal correction gives the claimed bound for e_ι.

The members of $\mathcal{T}$ all have the same unit class modulo q_0. A pair in E_ℓ thus requires $-\ell^a \equiv 1 \bmod q_0$. At the ramified prime $\ell = d_{\mathrm{K}} \equiv q_0/2 - 1 \bmod q_0$, whose square is $1 \bmod q_0$ since $4 \mid q_0$. Neither 1 nor $q_0/2 - 1$ is $-1 \bmod q_0$. This proves the ramified assertion (and $d_{\mathrm{K}} \le k$ eventually for fixed K).

To see disjointness use $2L < l_0$, valid here since $L \le 2 + \log_2 k$, whereas $l_0 \ge 27 s_2$ with $2^{s_2} \ge x^{11}$ in Lemma 51.19. That lemma forbids three visits to $\mathcal{T}$ with positive gaps and total gap less than l_0, including repeated visits, when the ratios are the indicated powers of ℓ up to sign. Two arrows from the same start at distinct exponents would give such a triple. For arrows to the same end with exponents $a < a'$, start at the source of the longer arrow and use times $a' - a, a'$. Equal exponents and equal ends imply equal starts because P_a is a permutation.

Two successive arrows of lengths a, a' give visits at times $a, a+a'$, including when a single arrow is a loop and can be repeated. These alternatives exclude all repeated vertices. Finally S_ℓ, E_ℓ are independent of the coefficient embedding. Insert the bounds at $\iota, \iota \circ c$ in (82.2) to get (82.5). □

The positive scalar A_x has $\log A_x \ll f b_x \log(2b_x)/k$, by $n_x! \le n_x^{n_x}$. Even summing this cost over mk critical pairs and dividing by sB_x^{det} gives $O((\log b_x)^{-2})$: here $B_x^{\mathrm{det}} = f x b_x (\log b_x)^3/(mk)$, $k \le s$ and $m^2 \ll x$. In particular the ramified-prime comparison has the directional bound with an explicitly controlled common multiplier. At a small inert prime the terms E_ℓ require further savings whenever these pairs occur.

82.9 The logarithmic derivative at an inert prime

Let p be inert in K and use the marked elliptic curve, differential dz, and even theta function $\Theta = \Theta_{t_*}$ of Theorem 66.13. In the shorter family 2, 3 split in K, so $p \geq 5$. Work over a completed local integer ring R, extended as needed, with residue field $\mathbf{k}$. Throughout this subsection reduction of an integral series means coefficientwise reduction into $\mathbf{k}$. Take $\lambda^{p^2-1} = p$ and put $c_p = \lambda^{p+1}/p$. Thus $v_p(\lambda) = 1/(p^2-1)$ and $v_p(c_p) < 0$.

Lemma 82.18 (The reduced elliptic logarithmic derivative)**.** *The series $c_p^{-1}(d/dZ)\log\Theta(\lambda Z)$ belongs to $R[[Z]]$. There are $u_j \in \mathbf{k}^\times$, for all $j \geq 0$, such that its reduction is*

$$\sum_{j\geq 0} u_j Z^{p^{2j+1}}.$$

Proof. Let T be the integral elliptic parameter of Lemma 64.5, multiplied by a unit so that its differential at zero is dz. That lemma gives

$$Y(Z) = \lambda^{-1}T(\lambda Z) \in R[[Z]], \qquad \bar{Y}(Z) = \sum_{j\geq 0} h_j Z^{p^{2j}}, \quad h_0 = 1, \quad h_j \neq 0.$$

In fact the linearized parameter makes the first nonlinear term of $[p]$ in degree p^2 a unit. Substitution of λY in its exponential identity gives $\bar{Y} + \bar{b}_{p^2}\bar{Y}^{p^2} = Z$, as in that lemma.

Let $g(z) = (\log\Theta)''(z) = -u_\Lambda - x(z + t_*)$, using Theorem 65.5. Write $g\,dz = \sum_{i\geq 0} a_i T^i\,dT$ at zero. These coefficients lie in R: u_Λ is integral here and the mark stays away from the origin at p. We claim that

$$a_{p-1} \in R^\times, \qquad a_{p^2-1} \in pR.$$

For the first assertion work on the geometric special fiber. If a_{p-1} vanished there, the terms through degree $p-1$ could be integrated in the parameter T. For a polynomial v this would give $g - \partial_z v = O(T^p)$. Consequently $(\partial_z^{p-1}g)(0) = (\partial_z^p v)(0) = 0$ by Lemma 66.12. The left side is the nonzero κ_{t_*} of that lemma.

For the second assertion work over $S = \operatorname{Spec}(R/pR)$. On a smooth relative curve the relative Cartier operator takes $\sum b_i t^i\,dt$ to $\sum_j b_{pj+p-1}(t^{(p)})^j\,dt^{(p)}$ in a parameter and its Frobenius twist. This follows by writing forms over the relative Frobenius in the basis $1, t, \ldots, t^{p-1}$: all powers except $t^{p-1}dt$ have primitives in that basis, and its class corresponds to $dt^{(p)}$. The operation is linear in base scalars and functorial for morphisms of smooth curves. This is the relative Cartier isomorphism over an arbitrary base of characteristic p. N. Katz, *Nilpotent connections and the monodromy theorem*, Theorem 7.2.

The meromorphic differential $g\,dz$ has only the double pole at $-t_*$ and has zero residue there. In characteristic zero this follows from the even Laurent expansion of x in the elliptic logarithm about its pole. On the good model the pole has order at most two, and the residue in an integral parameter is the same zero scalar, also after reduction modulo pR. Cartier therefore takes it to a global regular differential on the Frobenius twist: neither its double-pole term nor any regular term gives a pole by the displayed local rule, and its residue is zero. Cartier kills all global regular differentials of the elliptic curve over S. Indeed a CM endomorphism with tangent scalar a generating the quadratic residue extension acts by a on the invariant differential and by a^p on its twisted counterpart. Naturality and S-linearity give $(a^p - a)C(dz) = 0$, with unit multiplier. The same argument on the twist uses the unit $a^{p^2} - a^p$. Regular differentials on an elliptic scheme are invariant. Thus applying relative Cartier twice to $g\,dz$ gives zero over R/pR. Its constant coefficient in the twice twisted parameter is $a_{p^2-1} \bmod pR$. This proves the claim with the asserted precision for ramified coefficient rings as well.

The marked function is even, so $(\log\Theta)'(0) = 0$. Termwise integration in characteristic zero now gives

$$c_p^{-1}\frac{d}{dZ}\log\Theta(\lambda Z) = \sum_{m\geq 1} a_{m-1}\frac{p}{m}\lambda^{m-p}Y(Z)^m.$$

The scalar in this sum before a_{m-1} has valuation $1 - v_p(m) + (m-p)/(p^2-1)$. It is positive when $p \nmid m$, and for $v_p(m) = 1$ it is positive except at $m = p$, where it is zero. For $v_p(m) = 2$ it is positive once $m \geq 2p^2$. At $m = p^2$ the additional factor $a_{p^2-1} \in pR$ makes it positive. For $v_p(m) = r \geq 3$, use $m \geq p^r$: the bound $1 - r + (p^r - p)/(p^2-1)$ is positive at $r = 3$ and increases thereafter. Thus reduction leaves precisely $\bar{a}_{p-1}\bar{Y}^p$, with the asserted nonzero coefficients. □

Theorem 82.19 (The odd Frobenius powers in matrix coordinates)**.** *Use $B = I \otimes C$, its theta line L, and $P_a\tau = -p^a\tau$ from Sections 82.7–82.8. There is an integral formal frame of L at the origin whose normalized scalar $F(Z)$ in the marked chart has $F(0) = 1$ and, for every label σ,*

$$c_p^{-1}\partial_{Z_\sigma}\log F(\lambda Z) \in R[[Z]], \qquad \overline{c_p^{-1}\partial_{Z_\sigma}\log F(\lambda Z)} = \sum_{j\geq 0} u_j \sum_{\tau:\, P_{2j+1}\tau = \sigma} Z_\tau^{p^{2j+1}}. \tag{82.6}$$

Moreover, every integral scalar formal function on B has integral expansion after $Z \mapsto \lambda Z$.

Proof. At p the different of E/K and the two old marks are units, so $I_p = \mathcal{O}_{E,p}$. The trace Hermitian form $H(a,b) = \mathrm{Tr}_{E/K}(a\bar{b})$ on this lattice is perfect since $p \nmid q_1$. Its reduction is a nondegenerate Hermitian space over $\mathbb{F}_{p^2}$. It has an orthogonal basis of vectors of nonzero norm: if all norms vanished, polarization with multiples of a vector by every residue scalar and nondegeneracy of the finite-field trace pairing would force the form to vanish. Split off a nonzero norm line and use induction. Lift this basis to I and orthogonalize over K. All pivots are p-units by the chosen reductions. Clear the resulting denominators away from p to obtain orthogonal $w_1, \ldots, w_f \in I$ forming a basis at p. Their norms $n_i = H(w_i, w_i)$ are positive rational integers prime to p: $I \subset \mathfrak{b}^{-1}$ and $\mathfrak{b}\bar{\mathfrak{b}} = \mathfrak{D}_{E/K}$ give integrality by trace duality.

The homomorphism $C^f \to B$ with columns w_i is an isogeny of degree prime to p, by the lattice index. It is an isomorphism on formal neighborhoods of zero over R. In logarithm coordinates the inverse linear forms are

$$l_i(Z) = n_i^{-1}\sum_\tau \overline{\tau(w_i)}Z_\tau.$$

Indeed the matrix $V_{\tau i} = \tau(w_i)$ satisfies $V^*V = \mathrm{diag}(n_i)$. All these entries and the inverse entries are integral at p. On the product lattice the trace form and the original period sign split as the elliptic form and sign with multiplicities n_i, by the sign calculation of Lemma 65.8. Thus

$$F(Z) = \prod_i \Theta(l_i(Z))^{n_i}$$

is a normalized scalar of a formal frame for L. In detail, on C^f this product is the section of the external product of the powers of the marked elliptic divisor bundles, and the divisors miss the zero section on the special fiber. It gives an integral frame there. The generic identification of the pulled-back line with this product extends over the local base up to a scalar, since discrepancies are vertical divisors. Normalization by the value at zero removes that scalar. Transport through the formal isomorphism gives the frame on B. An integral scalar germ pulls back to a series in the f integral parameters of C. After the indicated rescaling these become $T(\lambda l_i(Z)) = \lambda Y(l_i(Z))$, which are integral with zero constant term. This proves the last assertion.

Apply the lemma to each factor and differentiate the product logarithm. For $a = 2j + 1$, the coefficient of $u_j Z_\tau^{p^a}$ in the reduced derivative at σ is the reduction of

$$\sum_i \overline{\sigma(w_i)}\big(\overline{\tau(w_i)}/n_i\big)^{p^a}.$$

Frobenius to the odd power a acts on K by conjugation and on μ_{q_1} by the p^a-power map. Composing with conjugation is therefore the oriented label $P_a\tau$. This gives $\overline{\tau(w_i)}^{\,p^a} \equiv (P_a\tau)(w_i)$ in the residue field, with the outer bar on the left denoting complex conjugation before reduction. Since n_i is a rational p-unit, the sum reduces to the $(P_a\tau, \sigma)$ entry of $V\,\mathrm{diag}(n_i^{-1})V^*$, which is the identity matrix. This proves (82.6). $\square$

The formula retains the coefficient of each individual odd power. In particular it distinguishes two ends having different preimages under the same Frobenius power, as required for the pairs in Theorem 82.17.

82.10 Coefficients from prescribed Frobenius pairs

Fix an inert prime p on the same node, and let F, λ, c_p be as in Theorem 82.19. Choose a set A of oriented embeddings. For each $\tau \in A$ choose a positive odd a_τ, and write $t_\tau = P_{a_\tau}\tau$. Assume that for each τ the only solution to

$$P_a\sigma = t_\tau, \qquad \sigma \in A, \quad a \text{ positive odd}, \quad a \le a_\tau$$

is $(\sigma, a) = (\tau, a_\tau)$. In particular the t_τ are distinct, since two coincident ends would violate this condition for the longer exponent. The condition holds for all embeddings with $a_\tau = 1$, by bijectivity of P_1. It also holds for the starts of the pairs occurring in E_p of Theorem 82.17. Indeed any competing start with the specified end at a smaller or equal exponent would itself give a term of E_p, and the pairs there are disjoint with unique exponents.

Attach nonnegative integers j_τ to A, put $J = \sum_{\tau\in A} j_\tau$, and form two multidegrees

$$r_\tau = p^{a_\tau} j_\tau \quad (\tau \in A), \qquad s_{t_\tau} = j_\tau \quad (\tau \in A),$$

with the unspecified entries zero. For a multidegree u write $|u| = \sum_\tau u_\tau$, $u! = \prod_\tau u_\tau!$. In particular $|s| = J$.

Lemma 82.20 (An exact mixed coefficient of the actual frame)**.** *Use $\delta_1 = 1, \dots, \delta_6 \in \mathbb{Z}_p$ of Lemma 82.13. Put*

$$D(U,V) = F(U+V)\prod_{i=2}^{6} F(\delta_i U), \qquad C_{r,s} = [U^r V^s]D(U,V).$$

Then $C_{r,s} \neq 0$ and

$$v_\iota(C_{r,s}) = J v_p(c_p) - v_p(s!) - (|r| + J)/(p^2 - 1). \tag{82.7}$$

Proof. Use $D_\lambda = D(\lambda U, \lambda V)$, a unit in the power series ring over the local field, with constant one. Its restriction $D_\lambda(U, 0)$ is integral and has constant one. Indeed Lemma 82.13 gives an integral function in formal group parameters before rescaling, and Theorem 82.19 makes that function integral in the rescaled logarithm variables. The cancellation lemma applies also to the formal frame used here. Replacing an algebraic frame by an integral formal one multiplies its normalized value by a formal unit of constant one. Pullback of this unit by any δ_i is integral, since integer multiplication is integral and its finite jets converge coefficientwise when the integers tend to δ_i in $\mathbb{Z}_p$. This last convergence follows by linear multiplication in the logarithm coordinates and a fixed change of parameters through each jet order.

For a multi-index v put

$$L_v = c_p^{-|v|}(\partial_V^v D_\lambda)/D_\lambda.$$

Starting with $L_0 = 1$, the product rule gives

$$L_{v+\mathbf{e}_t} = c_p^{-1}\partial_{V_t} L_v + L_v\, c_p^{-1}\partial_{V_t}\log D_\lambda.$$

The last logarithmic derivative is just the one of $F(\lambda(U+V))$, as the other factors are constant in V. It becomes integral after multiplication by c_p^{-1} by (82.6). Since $v_p(c_p^{-1}) > 0$, induction proves integrality of L_v, and the first term of this recurrence disappears after reduction. Consequently the reduction of $L_s(U, 0)$ is

$$\prod_{\tau\in A}\left(\sum_{\substack{a\ge 1 \text{ odd},\ \sigma \\ P_a\sigma = t_\tau}} u_{(a-1)/2} U_\sigma^{p^a}\right)^{j_\tau}.$$

Set $U_\sigma = 0$ for $\sigma \notin A$. In each factor before exponentiation the unique term of smallest total degree is $u_{(a_\tau-1)/2}U_\tau^{p^{a_\tau}}$. Their product through the indicated powers has lowest term exactly in degree r, with nonzero coefficient. Multiplication by the reduction of $D_\lambda(U,0)$ preserves this coefficient, since its constant is one. Thus $[U^r](L_sD_\lambda)(U,0)$ is a unit. Coefficients supported on A are unchanged by the specialization outside A. By definition this last coefficient is $c_p^{-J}s!\lambda^{|r|+J}C_{r,s}$. This proves the identity. □

Theorem 82.21 (Valuation savings in the integral twelve-row tests)**.** *Let $d = (d_\tau)_\tau$ be the actual derivative orders and suppose that $\eta = d - r - s$ is componentwise nonnegative. Write $h = |d|$ and $\lambda_p = 1/(p-1) - 1/(p^2-1)$, as in Theorem 82.16. At the fixed coefficient place the derivative table satisfies both bounds*

$$e_\iota - v_\iota(b\gamma_x) \le v_p(d!) - J, \tag{82.8}$$

$$e_\iota - v_\iota(b\gamma_x) \le \lambda_p h - J + v_p\Big(\frac{d!}{r!s!\eta!}\Big). \tag{82.9}$$

Proof. First record two consequences of (82.7). Since $v_p(c_p) = 1/(p-1) - 1$ and $|r| \ge pJ$, one has $v_\iota(C_{r,s}) \le -J$. More precisely $v_\iota(C_{r,s}) = -J - v_p(s!) - (|r| - pJ)/(p^2-1)$. Also $v_p(r!) \le |r|/(p-1)$, so

$$v_\iota(r!s!C_{r,s}) \le \lambda_p(|r| + J) - J.$$

For each integral formal function $H(W)$ of B, we claim that the actual table satisfies

$$e_\iota \le v_\iota(b\gamma_x) + v_\iota\big(d!\, C_{r,s}[W^\eta]H\big)$$

whenever the last coefficient is nonzero. In the first six factors of B^{12} make the logarithm substitutions $U+V, \delta_2U, \dots, \delta_6U$; in the last six make the substitutions $\delta_1W, \dots, \delta_6W$. Write $J_0(W) = \prod_i F(\delta_iW)$, an integral unit in the formal group parameters by the cancellation lemma. In the external product frame prescribe the integral jet of H/J_0 on the seventh factor, pulled back by projection to that factor. Lift through order h to an integral global section S as in Theorem 82.11. If λ_R denotes the generator value of the integral frame, the substituted marked expansion of S/λ_R through that degree is $D(U,V)H(W)$. The first six frames supply D and the last six restore J_0. Write $S = \sum_y c_yG_y$. Multiplying by $M^{12f}g$ makes each coefficient c_y/λ_R integral by Lemma 82.10, with the g in the definition of γ_x. Substitution in a monomial uses only integer binomial coefficients and powers of the integral δ_i. All monomials contributing to $U^rV^sW^\eta$ have the same per-embedding degrees d_τ in the original variables. Multiplication by $b\,d!$ yields an integral combination of the table entries. As $M^{12f}g$ divides γ_x in the sense that their quotient is the integer t_E^{12f}, the claim follows.

For the first bound take H with leading homogeneous part W^η in marked coordinates. Such an integral function exists here: $I_p = \mathcal{O}_{E,p}$ makes each dW_τ an integral cotangent vector by the differential computation in Theorem 82.11. Lift these vectors to the ideal of zero in the smooth formal group and multiply the lifts to the prescribed powers. The coefficient at η is one. The first bound follows. For the second bound use the scalar trace construction of Theorem 82.16 with total degree $|\eta|$ and the trace morphism $w = 1$. The trace of I is integral and all coefficients of the linear trace coordinate are one. The construction therefore gives $v_\iota(\eta![W^\eta]H) \le \lambda_p|\eta|$. Combining this with the inequality for $r!s!C_{r,s}$ proves the second bound, including $\eta = 0$. □

83 Integral positions in the terminal tests

83.1 Original-level projection of terminal covector translates

Fix one of the fields E/F, $F = E^{+}$, and its terminal prime $T = \varpi\bar{\varpi}$ from Section 86.1. There are $f = [F : \mathbb{Q}]$ places $v \mid T$, all with $F_v = \mathbb{Q}_T$. Put $O = \mathbb{Z}_T$, $q = T$, and $r = 2s + k + 1$. At these places use the first split factor of that section. The row group is $\mathrm{GL}_r(F_v)$. The compact column group $\mathcal{U}_v = \mathrm{GL}_k(O)$ acts on the middle k coordinates and fixes all other coordinates. For a nonzero residue covector $\ell \in X_v = (\mathbb{F}_q^k)^{\vee} \setminus \{0\}$, put

$$\Lambda_\ell = O^s \oplus \{(w, z) \in O^k \oplus O : z = \ell(\bar{w}) \bmod T\} \oplus O^s.$$

Each Λ_ℓ is a sublattice of index q in the same lattice $\Lambda_* = O^r$. The compact row group K_v in the original period is the stabilizer of $\Lambda_{\ell_{0,v}}$, for the prescribed $\ell_{0,v}$. The paired lattice in the other split factor is its dual.

Write R for right translation. If the row transforms by a smooth unitary character κ_v of K_v, use the following convention for translations at v. In a basis of its lattice $K_v = \mathrm{GL}_r(O)$. Every character is trivial on $\mathrm{SL}_r(O)$. Row reduction over the local ring gives generation of that group by elementary matrices, and each $1 + tE_{ij}$ is the commutator of $1 + tE_{il}$ and $1 + E_{lj}$ for a third index l. Thus κ_v factors through the determinant. Extend this unit character to $F_v^{\times}$ by assigning T the value one and write $\widetilde{R}(g) = \kappa_v(\det g)^{-1} R(g)$, using the same letter for the extended character. Under this action the original level space is fixed by K_v. All factors introduced on compact translations are roots of unity. We use the same twist on the column action through its block embedding. For trivial level character, $\widetilde{R} = R$.

Lemma 83.1 (A uniform bound for the neighboring lattices)**.** *In lattice coordinates put* $d = \mathrm{diag}(T, T^{-1}, 1, \dots, 1)$ *and let* e_v *be orthogonal projection to the* K_v*-fixed vectors for* $\widetilde{R}$*. On the cuspidal row space the operator* $M_v = e_v \widetilde{R}(d) e_v$ *is self-adjoint and*

$$\|M_v\| \le c_q, \qquad c_q = \frac{3q - 1}{q(q + 1)}. \tag{83.1}$$

Proof. We first exclude vectors invariant under $\mathrm{SL}_r(F_v)$. The rational row is isotropic. Let N_0 be the unipotent radical of the proper rational parabolic stabilizing the first isotropic line. Its rational points are dense in its adeles away from v, including the real places. Here is a proof of the approximation used in this assertion. For the additive group, a finite list of nonarchimedean conditions away from v can be met by the Chinese remainder theorem after clearing denominators, with integrality at the other unspecified finite places. A fractional ideal $\mathfrak{a}$ preserves all these conditions by addition, as does $\mathfrak{a}\mathfrak{p}_v^{-j}$ for any $j \ge 0$. The union of these ideals is dense at infinity. Indeed, a character of the real torus $F_\infty/\mathfrak{a}$ annihilating the union would have frequency in every trace-dual ideal $\mathfrak{d}_F^{-1}\mathfrak{a}^{-1}\mathfrak{p}_v^{j}$. Their intersection is zero by the valuation at v. Characters separate closed subgroups of a torus, proving density. The finite exponential and logarithm polynomials identify N_0 with its Lie algebra as varieties in characteristic zero. For unitary matrices the logarithm is skew adjoint since taking the adjoint takes the matrix to its inverse. The polynomials preserve integrality at almost all places. Additive approximation in a rational basis of the Lie algebra proves the claim for N_0.

Suppose a smooth cusp vector A is invariant under $\mathrm{SL}_r(F_v)$. For $g \in G(\mathbb{A}_F)$ and $n \in N_0(\mathbb{A}_F)$, approximate n by products γu_v with $\gamma \in N_0(F)$, $u_v \in N_0(F_v)$, correcting the v-component exactly. Then $A(\gamma u_v g) = A(g)$ by automorphy and by $g_v^{-1} u_v g_v \in \mathrm{SL}_r(F_v)$. Continuity gives $A(ng) = A(g)$. Its constant term along N_0 is therefore $A(g)$ times a positive covolume and vanishes by cuspidality. This also excludes invariant vectors in the cuspidal Hilbert space.

The invariant subspace is stable under the whole right action by normality, so smoothing on the group and taking compact types gives a dense set of smooth cuspidal vectors there. The determinant twist is trivial on $\mathrm{SL}_r(F_v)$.

In any unitary representation of $\mathrm{GL}_r(F_v)$, a vector fixed by a full elementary root subgroup is fixed by $\mathrm{SL}_r(F_v)$. To see this, write $x(b), y(t)$ for the upper and lower elementary matrices in the two associated coordinates. For $a \in F_v^{\times}$, take $b = (a-1)/t$, $t \neq 0$. Then

$$x(b)y(t)x(-b/a) = \begin{pmatrix} a & 0 \\ t & a^{-1} \end{pmatrix}.$$

On an x-fixed vector the norm of the change on the left is the norm of its change under $y(t)$, which tends to zero with t. Hence every such diagonal (a, a^{-1}) fixes the vector. For $a = T$, conjugation by powers of this diagonal or its inverse contracts each root subgroup having at least one index in this pair to the identity. Invariance and continuity give invariance under all those subgroups. Taking their commutators gives the remaining elementary subgroups, and row reduction generates $\mathrm{SL}_r(F_v)$.

Let U consist of the two entries in rows $1, 2$ of column 3 added to the identity. In the spectral resolution of the unitary action of $U \simeq F_v^2$, the projection at the zero character vanishes by the preceding argument. We use the spectral theorem for unitary representations of a locally compact abelian group. In these coordinates the projection for a dual ball $\xi_0 + T^j O^2$ is also the Haar probability average on $T^{-j}O^2$ against the conjugate character of ξ_0, using an additive character of conductor O. Project each nonzero frequency to its direction in $\mathbb{P}^1(F_v)$. For every K_v-fixed unit vector, the squared lengths of its spectral projections on directions give the same probability m. Indeed the upper 2-by-2 copy of $\mathrm{SL}_2(O)$ preserves the vector and acts transitively on directions of primitive pairs. A primitive pair extends to a determinant-one basis over O. Transitivity and compact Haar measure give uniqueness of the probability.

Choose the affine coordinate on the dual projective line so that d acts by $t \mapsto T^2 t$. Write the three parts of the line as $A = T^2 O$, $B = TO \setminus T^2 O$, and $C = \mathbb{P}^1(F_v) \setminus TO$. Their masses for m and $d_* m$ are

	A	B	C
m	$1/[q(q+1)]$	$(q-1)/[q(q+1)]$	$q/(q+1)$
$d_* m$	$q/(q+1)$	$(q-1)/[q(q+1)]$	$1/[q(q+1)]$.

Reduction modulo T gives $q+1$ equal direction classes, each splitting into q equal classes modulo T^2, as is seen by lifting a primitive projective pair in either affine chart. In particular $m(O) = q/(q+1)$. Scaling and inversion give the second row. For two fixed unit vectors, project the first and the d-translate of the second onto the three sets. Cauchy–Schwarz on each set bounds their inner product in absolute value by the sum of the three square roots of the products of corresponding masses. The sum is c_q. Finally d^{-1} is conjugate to d by interchanging the first two basis vectors in K_v. The compressed operator is therefore self-adjoint. This proves (83.1). The general root-subgroup bound is due to H. Oh [1]. □

Write $X = \prod_{v \mid T} X_v$, $\ell_0 = (\ell_{0,v})_v$, and give $\mathbb{C}^X$ the inner product in which the point masses are orthonormal. Use the action $h \cdot \ell = \ell h^{-1}$. Choose $h_{\ell,v} \in \mathcal{U}_v$ sending $\ell_{0,v}$ to ℓ, and take $h_{\ell_{0,v},v} = 1$. For an original-level cusp vector A and $a \in \mathbb{C}^X$ write

$$D(a, A) = \sum_{\ell \in X} a(\ell)\, \widetilde{R}((h_{\ell_v,v})_v) A, \qquad e = \prod_{v \mid T} e_v.$$

The definition is independent of the chosen lifts because the stabilizer of $\ell_{0,v}$ preserves $\Lambda_{\ell_{0,v}}$. Let $U_X \subset \mathbb{C}^X$ consist of arrays whose sum over each single coordinate is zero with the others fixed. It contains the refined columns whose sum on each local scalar orbit is zero.

Theorem 83.2 (The Gram matrix and its original-level projection)**.** *On original-level cusp vectors put $L_T = \prod_{v|T}(1 - M_v)$. For $a, b \in U_X$ and such A, A', with inner products conjugate-linear in the first variable, one has*

$$\langle D(a, A), D(b, A')\rangle = \langle a, b\rangle \, \langle A, L_T A'\rangle, \tag{83.2}$$

$$eD(a, A) = a(\ell_0) L_T A, \tag{83.3}$$

$$(1 - c_q)^f \, 1 \le L_T \le (1 + c_q)^f \, 1. \tag{83.4}$$

Set $p_r = (q^r - 1)/(q - 1)$, $n_r = p_r(p_r - 1)$. The operator $n_r^f L_T$ on this level is a sum of finite translations with algebraic integer coefficients.

Proof. Each lattice is the inverse image of its hyperplane in $\Lambda_*/T\Lambda_*$. For two distinct covectors the hyperplanes are distinct, even if one covector is a scalar multiple of the other, since the coefficient of z is fixed. Their sum is Λ_*, and their intersection has index q in each. The elementary divisors of the second lattice relative to the first therefore have one T, one T^{-1}, and $r - 2$ units. Indeed the positive exponents measure the quotient of the first lattice by the intersection and the negative ones measure the corresponding quotient for the second. Smith reduction over O then identifies the relative double coset with $K_v d K_v$, independently of the distinct covectors. For equal covectors it is K_v.

The compressed translation between two lifts is consequently 1 on the diagonal and M_v off it. With J_v the matrix of ones on X_v, the single-place Gram matrix is $1_{X_v} \otimes (1 - M_v) + J_v \otimes M_v$. At different places translations and projections commute. The matrix on the product is the product of these factors acting on the corresponding indices and the common row space. Each J_v kills U_X, giving (83.2). Its row at ℓ_0 evaluates the K_v-compression at all terminal places. On U_X this is (83.3). The commuting M_v are self-adjoint contractions with the bound in Lemma 83.1. The spectral bounds on their product give (83.4).

To count the right K_v-cosets in $K_v d K_v$, start at the original lattice and adjoin an overlattice of index q. There are p_r choices, the lines of its inverse-T quotient by itself. In each overlattice take a sublattice of index q different from the starting lattice. There are $p_r - 1$ choices, by counting hyperplanes modulo T. The intermediate lattice is recovered as the sum of the first and last ones. The elementary-divisor computation shows that this counts the entire double coset exactly once. On fixed vectors M_v is the average over these n_r translates with equal weights, by transitivity of left K_v on the right cosets. Multiplication by n_r clears the denominator. The character twists on the representatives are roots of unity. Multiplying the resulting integral sums over the places proves the last assertion. □

Corollary 83.3 (Integral character sums in the first theta test)**.** *Let $Z = \prod_{v|T} \mathbb{F}_q^\times$ be the scalar subgroup of $\prod_{v|T} \mathrm{GL}_k(\mathbb{F}_q)$. Choose a character $\lambda = \prod_v \lambda_v$ nontrivial in each coordinate. In the permutation action on X form*

$$w_\lambda = \sum_{z \in Z} \overline{\lambda(z)} \, \delta_{z \cdot \ell_0}. \tag{83.5}$$

Its coordinates are algebraic integers. For the iterated complex theta operator $\mathcal{T} = \mathcal{T}_{\mathbf{a},0}$ of Theorem 72.21, using its fixed first-edge test ϕ_* at each $v \mid T$, one has

$$\begin{aligned} \mathcal{T} D(w_\lambda, A) &= \mathcal{T} L_T A, \\ e^{-\Delta_x}(1 - c_T)^{2f} \|A\|^2 \le \frac{\|\mathcal{T} D(w_\lambda, A)\|^2}{C_x P(\mathbf{a})} &\le e^{\Delta_x}(1 + c_T)^{2f} \|A\|^2. \end{aligned} \tag{83.6}$$

Here A belongs to the original-level space of that theorem. On the common intervals, $\|L_T - 1\| = O(f/T)$ and $\log(n_r^f) = O(fr \log T) = o(b/x^B)$ for each fixed $B > 0$, where $b = \exp(x^\varepsilon)$.

Proof. The scalar action on a nonzero covector is free. Its orbit sum has coefficient one at ℓ_0 and sum zero along each coordinate, by nontriviality of that character. Thus (83.3) gives $eD(w_\lambda, A) = L_T A$. The first theta kernel is formed from the vector ϕ_* which transforms by κ_v in its source variable, by Theorem 93.5. In its pairing with the source row, translate the integration variable by $k \in K_v$. The two compact characters cancel on the matched isotypic space. Averaging the source with the projector to that character leaves the pairing unchanged because the kernel belongs to that character space in the integrated variable. This argument also defines the first lift on the finite sum of smooth translates in D. All subsequent tests stay fixed, so $\mathcal{T}D = \mathcal{T}eD$. The operator L_T acts at the terminal places, preserves their original level, and commutes with all other level and real type conditions on A. Apply (72.8) at depth zero, where $I = 1$, to $L_T A$, and use (83.4) to bound its squared norm.

The ranks f, r and $\log T$ are bounded by fixed powers of x. Also $\log T \geq (16 - o(1))mx^{\varepsilon}$ on these intervals, with $m \to \infty$, by Lemma 87.1 and Theorem 9.13. In particular $fc_T \to 0$, and $(1 + c_T)^f - 1 = O(f/T)$. The lower endpoint in (83.4) differs from one by at most fc_T. Finally $p_r \leq 2T^{r-1}$, so $f \log n_r = O(fr \log T)$, which is polynomial in x and has the asserted comparison with b/x^B. □

Corollary 83.4 (Mass of the full zero-sum projection)**.** *At each terminal place let $P_{1,v}$ be averaging under the twisted action of the residue scalars, using their multiplicative unit lifts. Put*

$$P_{\mathrm{nz}} = \prod_{v \mid T} (1 - P_{1,v}), \qquad \delta_T = \left(\frac{T-2}{T-1}\right)^f.$$

On the original-level cuspidal space one has

$$eP_{\mathrm{nz}} A = \delta_T L_T A,$$
$$\|(1 - P_{\mathrm{nz}})A\|^2 \leq \frac{5f}{T} \|A\|^2. \tag{83.7}$$

For the individual sum $D_\lambda = D(w_\lambda, A)$ and $A \neq 0$, the fraction of squared norm retained at the original level lies in the interval

$$\frac{(1 - c_T)^f}{(T-1)^f} \;\leq\; \frac{\|eD_\lambda\|^2}{\|D_\lambda\|^2} \;\leq\; \frac{(1 + c_T)^f}{(T-1)^f}. \tag{83.8}$$

The operator $(T-1)^f P_{\mathrm{nz}}$ is integral over the ring of finite character values. Its coefficients on the scalar translations in the twisted action are ordinary integers.

Proof. For odd T, the $T - 1$ nonzero residues have unique lifts solving $z^{T-1} = 1$ in $\mathbb{Z}_T^\times$, by Hensel's lemma. They form a subgroup. Its finite character projections are pairwise orthogonal, so P_{nz} is an orthogonal projection. On A it is $|Z|^{-1} \sum_\lambda D(w_\lambda, A)$, summed over the $(T-2)^f$ characters nontrivial at every place. Equations (83.5) and (83.3) give its asserted compression. Since $eA = A$,

$$\|(1 - P_{\mathrm{nz}})A\|^2 = \|A\|^2 - \delta_T \langle A, L_T A \rangle.$$

By (83.4) the relative loss is at most $1 - [(1 - 1/(T-1))(1 - c_T)]^f$. This is at most $f(1/(T-1) + c_T) \leq 5f/T$ for $T \geq 3$.

For one character, the squared norm before projection is $|Z|\langle A, L_T A\rangle$ by (83.2). After projection it is $\langle A, L_T^2 A\rangle$. Their quotient gives (83.8) by the spectral bounds for L_T. Finally $(T-1)^f P_{\mathrm{nz}}$ is the product of $(T-1)\,1 - \sum_{z_v} \widetilde{R}(z_v)$, proving integrality. The same fixed first-edge test sends $P_{\mathrm{nz}} A$ to $\delta_T \mathcal{T} L_T A$. Thus the estimate for the combined characters also applies before the remaining theta integrations. □

We next apply the same finite symmetry to the Bessel restriction itself. On a smooth row let

$$\mathcal{F}_\beta A(h) = \int_{N(F)\backslash N(\mathbb{A}_F)} A(nh)\overline{\chi_\beta(n)}\, dn, \qquad \beta = \varpi.$$

One may follow this map by orthogonal projection to any of the chosen real column types in $L^2([H])$, and write $\mathcal{F}$ for the resulting map or for $\mathcal{F}_\beta$ itself. Both quotients in (121.7) are compact, so on these smooth vectors $\mathcal{F}A$ has finite norm. Its pairing with a column in the specified type is precisely the period in that equation. Let $p_{\lambda,H}$ be the projection to scalar character λ on the terminal finite orbit of $\mathcal{F}A$, using the twisted column action. In $\mathbb{C}^X$ denote the corresponding character space by V_λ.

Theorem 83.5 (One Bessel vector for each terminal scalar character)**.** *For $a, b \in V_\lambda$ and smooth original-level cusp rows A, A', there is an exact identity*

$$\begin{aligned}\langle \mathcal{F}D(a, A), \mathcal{F}D(b, A')\rangle = (T-1)^f \langle a, b\rangle \\ \times \langle p_{\lambda,H}\mathcal{F}A,\, p_{\lambda,H}\mathcal{F}A'\rangle.\end{aligned} \tag{83.9}$$

There are $((T^k-1)/(T-1))^f$ independent directions in V_λ. In each tuple of residue lines the vector (83.5) transported to those lines is an integral generator of the corresponding one-dimensional character space over the ring of integers containing the character values.

Proof. Right translation on H commutes with taking the unipotent coefficient, by substitution in the displayed definition of $\mathcal{F}_\beta$. The real projection and the compatible determinant twists preserve this equivariance. The orbit map $\delta_\ell \mapsto \mathcal{F}D(\delta_\ell, A)$ is therefore equivariant for the product of the compact column groups. The principal congruence kernel in each group fixes the base vector since it fixes the initial covector. It is normal in the compact column group, so it fixes the entire orbit. Consequently this is an equivariant map for the finite product group $\prod_{v|T} \mathrm{GL}_k(\mathbb{F}_T)$.

We verify irreducibility on a single character space for one factor. The functions of that character on a fixed residue line form a line supported on its nonzero vectors. Distinct residue lines give orthogonal lines in the function space. For a pair of different residue lines choose an element of $\mathrm{GL}_k(\mathbb{F}_T)$ acting by one on the first and by any specified scalar on the second. This is possible in a basis of the dual space extending representatives of the two lines. Since $\lambda_v \neq 1$, one may choose a scalar whose action on the second character line is nontrivial. An equivariant endomorphism has zero entry between these lines. Its diagonal entries agree by transitivity on residue lines. Hence its commutant consists of the scalars, proving irreducibility by orthogonal projection. The same proof works on tuples. For two different tuples use a differing place to kill the off-diagonal entry, and use product transitivity for the diagonal. It also gives the stated dimension since there are $(T^k-1)/(T-1)$ residue lines at each place.

The Gram form of the orbit map on V_λ, using A in its first variable and A' in its second, thus multiplies the ordinary inner product by a single scalar. To compute it, insert $p_\lambda \delta_{\ell_0} = w_\lambda/|Z|$ in both variables. Its squared norm is $1/|Z|$. Its image for A is precisely $p_{\lambda,H}\mathcal{F}A$, by equivariance and the character average. This gives (83.9), including its normalizing factor $|Z| = (T-1)^f$. On each tuple of lines all coordinates of an eigenvector are determined by one coordinate times roots of unity. Conversely arbitrary integral values of that coordinate give integral eigenvectors, proving the last assertion. □

For the actual integral Fourier sum, (83.9) gives $\mathcal{F}D(w_\lambda, A) = |Z| p_{\lambda,H}\mathcal{F}A$ directly by equivariance. The lower bound (83.6) is relative to the Petersson norm of A. In particular, putting $A = S_0\xi$ leaves its lower bound proportional to $\|S_0\xi\|^2$, the original projected Bessel Gram form. For the arithmetic comparison one still needs a lower estimate on the designated Bessel

pairings, with their integral normalizations. A choice of a nonzero row in the complex theta correspondence does not supply that estimate.

83.2 The original row projection on integral terminal characters

Keep the field E, the interval and the prime T of Theorem 9.13. In this paragraph r is the rank of the indefinite row, k the rank of the definite column and $f = [E^+ : \mathbb{Q}]$. At each $v \mid T$ use the original row lattice $\Lambda_{\ell_{0,v}}$ of Section 86.1. Its middle block consists of pairs (w, z) satisfying $z \equiv \ell_{0,v}(\bar{w}) \pmod{T}$. For $h \in \mathrm{GL}_k(\mathbb{Z}_T)$, the translated lattice uses the residue covector $\ell_{0,v}h^{-1}$.

Theorem 83.6 (A terminal character sum surviving the original row projection)**.** *Let A be a cusp row of the specified real type and original finite level. Write e for projection to its prescribed compact type at all $v \mid T$. Let $Z = \prod_{v|T} \mathbb{F}_T^{\times}$, acting by scalar matrices on the column coordinates. For each $z \in Z$, lift these scalars to units and write $\widetilde{R}(z)A$ for the corresponding row translate, with its compact determinant character removed. For any character λ of Z nontrivial at each place put*

$$A_\lambda = \sum_{z \in Z} \overline{\lambda(z)} \widetilde{R}(z) A.$$

There is one positive operator L_T on the original-level cuspidal row, independent of λ, such that

$$eA_\lambda = L_T A, \qquad \|L_T - 1\| \ll f/T. \tag{83.10}$$

More precisely, if $c_T = (3T-1)/(T(T+1))$, its spectrum lies in $[(1-c_T)^f, (1+c_T)^f]$. Writing $p_r = (T^r-1)/(T-1)$, multiplication of L_T by $(p_r(p_r-1))^f$ gives an integral linear combination of translations over the ring of integers containing the finite character values. The logarithm of this denominator is $o(b/x^B)$ for every fixed $B > 0$.

Proof. At one place, two distinct covectors give distinct hyperplanes in the reduction of the same rank-r lattice. Their intersection has index T in each lifted lattice. The relative elementary divisors are $T, T^{-1}, 1, \dots, 1$. Hence every translate with covector different from the base has the same original-level compression M_v. The unitary matrix coefficient estimate for this double coset on cuspidal rows gives $\|M_v\| \le c_T$. Its proof uses the three sets of directions of an abelian unipotent subgroup in Lemma 83.1. There are $p_r(p_r - 1)$ right cosets in the double coset. The scalar element one gives the identity compression, and the sum of the other character coefficients is -1. Thus the single-place projection is $1 - M_v$. Multiplication over the places gives $L_T = \prod_{v|T}(1 - M_v)$, with the claimed spectrum and denominator by Theorem 83.2. Lemma 87.1 gives $f/T \to 0$ and bounds $f, r, \log T$ by fixed powers of x. The assertions about the norm and $b = \exp(x^{\varepsilon})$ follow. □

Corollary 83.7 (Two period norms for the same terminal character)**.** *Let $\mathcal{T}$ be the iterated complex theta operator in Theorem 72.21 with its original first-edge test at T, and let A satisfy the other real and finite type conditions of that theorem. Write $\mathcal{F}A$ for the Bessel coefficient of frequency ϖ restricted to the definite column group, optionally projected to the specified real column type. Then*

$$\frac{\|\mathcal{T}A_\lambda\|^2}{C_x P(\mathbf{a})} \ge e^{-\Delta_x}(1 - c_T)^{2f}\|A\|^2, \tag{83.11}$$

$$\|\mathcal{F}A_\lambda\|^2 = (T-1)^{2f}\|p_{\lambda,H}\mathcal{F}A\|^2. \tag{83.12}$$

Here $p_{\lambda,H}$ is the orthogonal scalar-character projection on the finite orbit of the Bessel coefficient. On the entire character space of dimension $((T^k-1)/(T-1))^f$, the Bessel Gram form for this row is a scalar times the ordinary inner product on covectors. Its scalar is $(T-1)^f\|p_{\lambda,H}\mathcal{F}A\|^2$.

Proof. The first theta kernel has the prescribed source character at T, so its integral depends only on eA_λ. Apply its norm formula to (83.10), retaining its real constant C_x, its finite factor $P(\mathbf{a})$ and its error Δ_x. For the second identity, right translation on the column group commutes with the unipotent integral defining $\mathcal{F}$. The finite sum therefore gives $|Z|p_{\lambda,H}\mathcal{F}A$. The finite character space is irreducible under $\prod_{v|T}\mathrm{GL}_k(\mathbb{F}_T)$. Its Gram scalar follows by evaluating on the character sum on the chosen scalar orbit, whose squared coordinate norm is $|Z|$. Full proofs, including irreducibility, are in Corollary 83.3 and Theorem 83.5. □

Proposition 83.8 (The mass retained by all nontrivial terminal characters)**.** *Let P_{nz} remove the trivial scalar character at each terminal place, using the twisted translations on the cusp row. Then*

$$eP_{\mathrm{nz}}A = \left(\frac{T-2}{T-1}\right)^f L_T A, \qquad \|(1-P_{\mathrm{nz}})A\|^2 \le \frac{5f}{T}\|A\|^2.$$

Clearing $(T-1)^f$ in P_{nz} gives an integral sum of translations. For an individual character and $A \neq 0$, one has uniformly on the common intervals

$$\frac{\|eA_\lambda\|^2}{\|A_\lambda\|^2} = \frac{1+O(f/T)}{(T-1)^f}.$$

Proof. There are $(T-2)^f$ allowed characters. Their orthogonal projections on A are $A_\lambda/(T-1)^f$. Summing (83.10) gives the first formula and multiplying out the complementary scalar averages proves integrality. The lost squared norm is $\|A\|^2 - ((T-2)/(T-1))^f\langle A, L_T A\rangle$. The spectrum in Theorem 83.6 bounds it as claimed. The unprojected squared norm of a single sum is $(T-1)^f\langle A, L_T A\rangle$, by the Gram identity in Theorem 83.2. Its projected squared norm is $\|L_T A\|^2$, giving the ratio. □

For $A \neq 0$ the theta norm in (83.11) is positive. For the designated Bessel column $A = S_0\xi$, the bound still contains $\|S_0\xi\|^2$. Likewise (83.12) leaves the projected global coefficient $p_{\lambda,H}\mathcal{F}A$ to be estimated on the integral row and paired with the designated integral column. These quantities enter the arithmetic comparison at the original finite level. That comparison and the uniform estimate for the cohomology sum in (20.12) remain open in the manuscript.

83.3 Mixed Gram determinants along the theta lift

Theorem 83.9 (Exterior Gram forms at the original level)**.** *For a fixed translate and the data of Theorem 88.1, sum the maps S_c on the direct sum of their domains to obtain $S : D_{\mathrm{all}} \to \mathcal{H}$. Use one theta operator on their common real line and at their common finite level. Put $\widehat{\Theta} = \alpha^{-1/2}\Theta$. For any two lists $A = (a_1, \dots, a_s)$ and $B = (b_1, \dots, b_s)$ in D_{all}, write*

$$G_{A,B} = (\langle Sa_i, Sb_j\rangle)_{ij}, \qquad H_{A,B} = (\langle \widehat{\Theta}Sa_i, \widehat{\Theta}Sb_j\rangle)_{ij}.$$

The Hermitian forms for one list satisfy $e^{-\Delta_x}G_{A,A} \le H_{A,A} \le e^{\Delta_x}G_{A,A}$ and have the same kernel. Moreover,

$$e^{-s\Delta_x}\det G_{A,A} \le \det H_{A,A} \le e^{s\Delta_x}\det G_{A,A},$$
$$\left|\det H_{A,B} - \det G_{A,B}\right| \le (e^{s\Delta_x}-1)\sqrt{\det G_{A,A}\,\det G_{B,B}}.$$

For $1 \le s \le \dim D_{\mathrm{all}}$ the factor $e^{s\Delta_x}-1$ is at most $\exp(-(\log 2)x^{377}/3)$ on every sufficiently distant common interval. All determinants of higher order vanish.

Proof. Set $F_0 = SD_{\mathrm{all}}$ and restrict $\widehat{\Theta}$ to it. Let L be the product of the adjoint of this restriction with the restriction itself, acting on F_0. By Theorems 88.1 and 72.21 its quadratic form lies between $e^{-\Delta_x}$ and e^{Δ_x} times the squared norm. Applying this to each linear combination of the Sa_i proves the Hermitian inequalities and kernel equality. On $\bigwedge^s F_0$ the eigenvalues of $\bigwedge^s L$ are the s-fold products of distinct entries in an orthonormal eigenbasis of L, with multiplicities. They lie in $[e^{-s\Delta_x}, e^{s\Delta_x}]$.

Use the exterior inner product for which the squared norm of a simple wedge is its Gram determinant. For $v_A = Sa_1 \wedge \cdots \wedge Sa_s$, and likewise v_B, the two mixed determinants are respectively $\langle v_A, (\bigwedge^s L) v_B\rangle$ for H and $\langle v_A, v_B\rangle$ for G. Taking $A = B$ proves the first determinant inequality. Subtracting the identity and applying Cauchy–Schwarz proves the second, including when either list is dependent. If $s > \dim F_0$ every such wedge is zero.

Here $\Delta_x = Cmfr\, 2^{-(t_{\mathrm{r}}+1)/2+10(r+1)^3}$, with $t_{\mathrm{r}} \geq x^{377}$ by the choice of the rank step. Lemma 94.22 gives $\log s \leq x^{40}$, $r = k + m - 1 = O(x^5)$, and mfr is polynomial in x. Thus $\log(s\Delta_x) \leq -(\log 2)x^{377}/2 + O(x^{40})$ uniformly in s. The elementary bound $e^y - 1 \leq 2y$ for $0 \leq y \leq 1$ gives the claimed estimate. □

83.4 The prescribed columns at the ordinary real places

Use the weights in (9.1), the spacing of Theorem 110.2 and the core of Theorem 71.16. Call the real places outside the h designated high places ordinary. Write J for all the ordinary places. The desired local highest weights are the column parameters $u_1, \ldots, u_k$ minus the half sum of positive roots of $U(k)$. The conclusions also hold in the contragredient convention.

Theorem 83.10 (All ordinary places and the obstruction at high places)**.** *On every sufficiently distant common interval, for each given word, bottom row and terminal character of Theorem 88.7, one can impose its prescribed real column type at all the ordinary places simultaneously on a restricted row after a determinant twist unramified at every finite place. Its coefficient on $H_0(F_\infty)$ over the identity finite component has norm at least $\mathcal{C}_w/(2\sqrt{d_\rho})$, with w as in Theorem 88.8. The field, terminal lattice, frequency ϖ and finite scalar character are those of the original word. At each high place the prescribed type has multiplicity zero in the restriction of the compact core type ρ under every determinant twist.*

Proof. Let $n = r + 1 = k + m$, $d = n + 4$, $N = k + m(r + 6)$, and write g_* for the gap of Theorem 71.10. At the terminal step the compact parameter, up to duality, has the $k - 1$ entries $v_i - 1/2$ below all the others. The next lowest entry is $a_m = v_1 + 1/2 + g_*$. Indeed the lowest added string is centered at $\lfloor (a_{m-1} + a_m)/2 \rfloor$ and its radius is $(d-1)/2$, with $g_* > d+2$. The string centered at $\lfloor (a_0 + a_1)/2 \rfloor$ contributes $d \geq k$ entries at least $a_m + (m-1)g_*$, since $a_0 = a_m + mg_*$. Let λ_i be the decreasing list after subtraction of $(N+1)/2 - i$. We obtain

$$\lambda_k - \lambda_{N-k+1} \;\geq\; (m-1)g_* - N.$$

Negating and reversing the list, or shifting it by a constant, preserves this difference. This proves the same lower bound for the actual highest weight of ρ_τ.

At an ordinary place the desired highest weights $\mu_i = u_i - ((k+1)/2 - i)$ have absolute values $O(k\Delta + k)$. In the common family $f \ll_\psi k$ by Theorem 6.29. The chosen $\Delta = 8(f+1)n^3 + 8r^2 + 5$ gives $k\Delta = O_\psi(n^5)$. We have $N = O(n^2)$ and $g_* \geq n^7$. Also $w_E \leq 2[E : \mathbb{Q}]^2 = O_\psi(n^2)$. For the last bound, $\varphi(w_E) \leq [E : \mathbb{Q}]$; the prime-power formula gives $p^a/\varphi(p^a)^2 \leq 1$ at odd primes and at most two at $p = 2$. Multiplication gives $w_E \leq 2\varphi(w_E)^2$.

At each ordinary place choose $j_\tau \in w_E\mathbb{Z}$ nearest the negative of the midpoint of $[\lambda_{N-k+1}, \lambda_k]$. For all sufficiently large x, half the length of this interval exceeds $\max_i |\mu_i| + w_E$, uniformly in

the word. Thus every $\mu_i - j_\tau$ belongs to it and satisfies all the inequalities of Proposition 94.29. Set $j_\tau = 0$ at the high places. The tensor product of these ordinary shifted types occurs in $\rho|_{K_J}$. The ratio q_ν of Theorem 88.8 is at least $1/d_\rho$, since its numerator is a positive integer. Use that theorem and multiply the resulting row by the global determinant character of Proposition 94.29. It has modulus one, is trivial on every unipotent argument, and takes value one on the terminal scalar operators. Consequently the projected norm is unchanged and its ordinary real type is the prescribed one.

At a high place the largest and smallest prescribed highest weights are a and $-b_*$, by (9.1). Their difference is $a+b_*$, with $a = \lceil x \exp(x^{\varepsilon}) \rceil$. Every entry of the terminal core parameter and its half-root shift has polynomial size in x by Theorem 71.10 and the string formulas above. Any restricted highest weight has spread at most $\lambda_1 - \lambda_N$ by Proposition 94.29. Taking a dual or a determinant twist preserves spread, which proves the vanishing. □

83.5 An integral column on the scalar orbit

Set $p(h) = F_0 A_\lambda(h, 1_f)$ for the row of Theorem 90.7. At $v \mid T$ take the original column group $K_{\pi,v}(\ell) = \{u \in \mathrm{GL}_k(\mathbb{Z}_T) : \ell_v u \equiv \ell_v \bmod T\}$. At the remaining places use a sufficiently fine level fixing the Fourier coefficient. Let $h_\circ$ count the finite classes at the combined level $K_\circ$ chosen in the proof. At the identity class take finite integral generating lists v_i and l_j of the word representation lattice and its dual, of sizes I and J. Use a unitary real norm. Denote the smallest eigenvalues of the positive frame operators $\sum_i v_i v_i^*$ and $\sum_j l_j^* l_j$ by b_V and b_L, respectively. They are positive because the lists span over $\mathbb{C}$.

Corollary 83.11 (An arithmetic column with the prescribed real and terminal characters)**.** *There is an integral column Q_λ of the full real word type and terminal character λ, at the original column level above T, satisfying*

$$\left|\langle Q_\lambda, F_0 A_\lambda \rangle_{[H_0]}\right| \;\geq\; \frac{|Z|}{h_\circ} \left(\frac{b_V b_L}{d_\mu I J} \right)^{1/2} \|p\|_2 .$$

It is supported on $|Z|$ distinct classes above the same underlying column lattice. In the displayed pairing the column level away from T may depend on the restricted row. For any finite list of L word and character pairs on the same field, row group and terminal graph, a single integral linear combination of the constructed rows gives every corresponding pairing at least half its individual lower bound. The integer coefficients can all lie between zero and L.

Proof. The initial row is fixed by the terminal graph stabilizers. Their intersection with H_0 is $K_{\pi,v}(\ell)$ by Theorem 86.1. The small Fourier integral commutes with right column translations. Since the terminal scalars are central on H_0, the character sum retains these levels on the column. At every other place intersect its finite level of smoothness with the stabilizer of the original column lattice. At one place above the odd split auxiliary prime g_1 intersect further with the principal congruence subgroup of depth one. These operations yield an open compact $K_\circ$. At every finite class its rational stabilizer at infinity is a finite group of definite lattice isometries. At the chosen place it embeds in a conjugate of the principal congruence group, which is torsion-free by the binomial argument of Proposition 95.4. Thus all stabilizers are trivial. Each class consequently has mass $1/h_\circ$ for quotient probability.

The translates of the identity class by Z are distinct. Indeed, an equality of two such finite classes gives a rational isometry γ relating their representatives up to $K_\circ$. Each representative acts by integral units on the column lattice, so γ preserves that global definite lattice. At the chosen place above g_1 it is in the torsion-free principal congruence group. Hence $\gamma = 1$. Equality now requires the quotient of the two scalars to lie in $K_{\pi,v}(\ell)$ for every $v \mid T$. The

reduction of each covector is nonzero, so each quotient is one modulo T. For the Teichmüller scalars this implies equality. All these classes use the same underlying lattice.

At the identity class take $Q_{ji}(s) = l_j(\mu(s)v_i)$ on the real variable, extended by zero on the other classes. The lattice here is the tensor of the integral Schur models of the original lattice at the given embeddings with their determinant factors. Its matrix coefficients in these lists are integral class coefficients in the sense of Proposition 95.4. Write $p(s) = \operatorname{tr}(M\mu(s))$. Orthogonality gives $\|M\|_{\mathrm{HS}}^2 = d_\mu \|p\|_2^2$. The squared pairings on the real group with the Q_{ji} sum to $d_\mu^{-2} \sum_{ij} |v_i^* M l_j^*|^2$. Applying the two lower frame bounds gives a sum at least $b_V b_L \|p\|_2^2 / d_\mu$. One pair attains at least $1/(IJ)$ times this squared bound. The global pairing with that Q_{ji} is its real pairing divided by $h_\circ$.

Take $Q_\lambda = \sum_{z \in Z} \overline{\lambda(z)} R(z) Q_{ji}$. It belongs to the same finite level since Z is central. On each translated class use z^{-1} as its representative. Its integral lattice and real coefficient are the original ones there, up to the root of unity in the sum. Thus all coefficients are integral after adjoining the character values. The sum over Z is self-adjoint and acts on $F_0 A_\lambda$ by $|Z|$. This proves the inequality. Its right side is positive by Theorem 90.7.

For the simultaneous assertion fix a witness row A_i and column Q_i just constructed for each pair. Take the common refinement of their levels away from T. Their pairings are integrals on the same adelic quotient and are unchanged by this refinement. Let f_i be pairing against Q_i after the small Fourier map and the indicated character sum. This is a complex linear functional of the input row. The real word projector can also be inserted, since it fixes Q_i. Choose independently and uniformly $c_1, \dots, c_L$ from $\{0, \dots, L\}$. With all but c_i fixed, successive values of $f_i(\sum_j c_j A_j)$ on their line have spacing $|f_i(A_i)| > 0$. At most one lies in the open disk of radius $|f_i(A_i)|/2$ about zero. The probability that any of the L lower bounds fails is therefore at most $L/(L+1) < 1$. Some choice satisfies them all. The common terminal row invariance is preserved under the linear combination. □

The comparison for Dirichlet GRH requires control with the prescribed row and the original finite level at every place. The class count $h_\circ$, the integral frame constants and the raw factor M_B in this result enter its normalization. A uniform comparison on the common intervals, the upper bound for the full theta period, and the strict cohomology bound remain open.

83.6 An integral trace to one fixed theta test

At every $v \mid T$ take the split matrix model whose row factor gives positive valuation to ϖ, with the actual terminal lattice Λ of Theorem 86.1. In the first theta edge write $Y = \binom{X}{Z}$, where X is square of size r. Take the coarse test with X integral invertible and with the lower block Z divisible by T. Multiply its values by the level character divided by the Weil splitting character, evaluated on X in the chosen bases. Both take unit values. At depth $j \geq 1$, retain the part where X mod T^j is in the image of $B_v \cap K_v^0$. Depth zero means the whole coarse test. All depths thus use the same coarse theta datum.

Theorem 83.12 (The integral trace and the original period)**.** *Write $D_B = (r(r+1) - k(k-1))/2$, $I_0 = 1$, and set*

$$I_j = T^{(j-1)D_B}(T^k - 1) \prod_{h=k+1}^{r} (T^h - 1) \quad (j \geq 1).$$

For arbitrary depths $\mathbf{j} = (j_v)_{v|T}$ let $I = \prod_{v|T} I_{j_v}$. The normalized average of the fine finite theta test over the original levels has exact integer denominator I. Multiplying by I gives the coarse integral test. Let $F_{\mathbf{j}}$ and F_* denote their rows for the same partner input, whenever both lifts

converge and belong to automorphic L^2. Let e project onto the original finite level characters at the refined places. With $S_{\mathbf{j}}$ the corresponding fine Bessel map, for each input ξ one has

$$\langle F_*, S_0\xi\rangle = I^2\langle eF_{\mathbf{j}}, eS_{\mathbf{j}}\xi\rangle,$$
$$\|IS_{\mathbf{j}}^*S_{\mathbf{j}} - 1\|_{\mathrm{op}} \le C_x T^{-\eta\sum_v j_v}, \qquad \eta = L - 4fr^2.$$

In particular the depth-one denominator is prime to T at every refined place. The exact T-valuation of the total denominator is $D_B\sum_v \max(0, j_v - 1)$.

Proof. The split radical has $a = (r^2 - (k+1)^2 - 2s)/2$ free matrix entries in the basis of Λ. On the middle block an integral column isometry h must preserve the fixed nonzero residue covector ℓ. Its action modulo T^j also retains $(\ell - \ell h)/T \bmod T^j$. Including this last row gives $T^k|\mathrm{GL}_k(\mathbb{Z}/T^j\mathbb{Z})|/(T^k - 1)$ possible middle blocks. Dividing the full row group order by this number and T^{ja} gives exactly I_j, as proved in Theorem 93.4.

The translates of the fine theta test partition the coarse support by left cosets. Their corrected values can be recovered separately by evaluating at a representative top block with zero lower block. Products of these evaluations give a unit divided by I for the average, proving the exact denominator. Theorem 93.5 also realizes the corresponding finite subgroup by a Weil matrix coefficient on the same pair. Right translation of a theta row acts on its test by this representation, so finite averaging gives $eF_{\mathbf{j}} = F_*/I$. For the Bessel seed the same fraction of the original right coset survives, giving $eS_{\mathbf{j}} = S_0/I$. The two identities prove the period equality. The nonidentity rational positions satisfy the product of the congruence divisibilities at all the refined places. Taking the norm of one nonzero frame coordinate gives their height separation by $T^{\sum_v j_v}/C_x$. Applying the integral tail estimate of Theorem 110.3 as in Corollary 93.6 proves the norm bound. Finally every $T^h - 1$ is a unit at T, which gives the asserted valuation. □

For total depth polynomial in x, $\log I^2 \le 2D_B(\sum_v j_v)\log T = o(b/x^A)$ for every fixed $A > 0$. The preceding proof controls this explicit part of the finite normalization. At fixed x, the norm estimate concerns the whole fine image. The period equality selects its projection at the original level and its pairing with a specified theta row. A uniform nonzero, arithmetically normalized pairing of this kind along the full chain, as well as the remaining strict cohomology bound in (19.15), is still required to prove (W).

Proposition 83.13 (The nonvanishing test on an original word)**.** *Fix a word c, a unit real vector $v \in U$, and write $S\ell = S_0(v \otimes \ell)$ on its full class space D_c. Let $R \subset D_c$ be its subspace of ramified columns. For each rational double coset outside the stabilizer of the entire isotropic space let O_γ be its operator on D_c given by the integral in equation 86.13, with this common v. Then*

$$E_c = \sum_{\gamma,\delta} \mathrm{tr}(O_\gamma^* O_\delta) = \|S^*S - 1\|_{\mathrm{HS}}^2 \ge \dim\ker S.$$

The double series is absolutely convergent. If $E_c < \dim R$, some ramified column has nonzero image at the prescribed level. For the closed span $\mathcal{T}$ of any specified collection of convergent theta rows there, let $p_{\mathcal{T}}$ denote orthogonal projection. One has

$$\|p_{\mathcal{T}}S|_R\|_{\mathrm{HS}}^2 = \dim R + \sum_\gamma \mathrm{tr}_R(O_\gamma) - \|(1 - p_{\mathcal{T}})S|_R\|_{\mathrm{HS}}^2,$$

where the trace on R uses orthogonal compression. Strict positivity of this expression is equivalent to a nonzero pairing of a ramified column with one of those rows.

Proof. Use the absolute integral bounds of Theorem 110.2 on every pair of a fixed orthonormal basis of D_c. There are finitely many pairs, so $\sum_\gamma \|O_\gamma\|_{\mathrm{HS}} < \infty$. The terms from its stabilizer give the identity. Indeed their absolutely convergent subseries unfolds to the intermediate pairing evaluated at K^0 in Theorem 86.2. The remaining cosets have two distinct isotropic spaces. Thus $S^*S - 1 = \sum_\gamma O_\gamma$. Squaring the Hilbert–Schmidt norm gives the first identity since each trace product is bounded by the product of the two Hilbert–Schmidt norms. On an orthonormal basis of the kernel the difference from the identity acts by minus one, giving the dimension bound. If all of R vanished its dimension would be at most E_c. This is the dimension argument at `Development/AbelianVarietyTheory/Proofs/DecoratedLatticeProofs/NonvanishingFromTruePack.lean`, line 26. The source asks for a surviving ramified column for every word in `Development/AbelianVarietyTheory/Definitions/LatticeFrames/PairForm.lean`, line 143. For the second identity sum Pythagoras for $S\ell$ over an orthonormal basis of R and use the parabolic expansion for $\|S|_R\|_{\mathrm{HS}}^2$. The projected operator is zero exactly when every such image is orthogonal to the whole collection. This proves the equivalence. On the varying fields the comparison of E_c with $\dim R$, and control of the component orthogonal to the specified theta rows, are additional estimates. □

At `Development/AbelianVarietyTheory/Proofs/DecoratedLatticeProofs/DiagonalRowOfSiegelWeilUnitary.lean`, line 50 the proposed bound is assembled from estimates for degenerate terms, collisions and two corank-one contributions. Absolute convergence in Proposition 83.13 justifies the expansion to which such estimates would apply at the original level.

83.7 A second moment on the terminal integral coefficient lattices

Fix an original field, finite levels and ascents as in Theorem 75.3. Use probability measure on the definite terminal quotient. Denote the whole ascent with the selected tests by T, and its restriction to $H(v)$ by T_v. For a terminal form e, write $T^\dagger e$ for the reverse composition of the adjoint theta integrals with the same tests and measures. Write P_v for orthogonal projection onto the bottom subspace $H(v)$. Fix a complex linear functional ℓ on that finite-dimensional Hilbert space. Its dual norm is $\|\ell\|_*$.

For each terminal class c, in the notation of Proposition 95.4, take a basis $v_{c,b}$ over the coefficient field in the integral lattice W_c and a basis $l_{c,a}$ in its integral dual. Such bases can be extracted from any finite module generating lists because the lattices span their spaces over the field. No freeness over the ring of integers is required. Let $e_{c,a,b}$ be zero off c and on c the function $g \mapsto l_{c,a}(\rho(g)v_{c,b})$. Use a Hermitian inner product linear in its second argument. Let G be the Gram matrix of this ordered family of $M = h_m d_\rho^2$ functions and let $y_{c,a,b} = \overline{\ell(P_v T^\dagger e_{c,a,b})}$.

Theorem 83.14 (A second moment of the descending periods)**.** *With C_{fix} and δ of Theorem 75.3 for the selected measures,*

$$e^{-\delta} C_{\mathrm{fix}} \|\ell\|_*^2 \;\le\; y^* G^{-1} y \;\le\; e^{\delta} C_{\mathrm{fix}} \|\ell\|_*^2.$$

The matrix G is positive definite. It is zero between classes and its within-class entries are

$$G_{(c,a,b),(c,a',b')} = \frac{\langle l_{c,a}, l_{c,a'}\rangle_* \langle v_{c,b}, v_{c,b'}\rangle}{h_m d_\rho}.$$

In particular at least one of the given integral terminal coefficients satisfies

$$|\ell(P_v T^\dagger e_{c,a,b})| \;\ge\; \left(\frac{e^{-\delta} C_{\mathrm{fix}}}{M \|G^{-1}\|_{\mathrm{op}}} \right)^{1/2} \|\ell\|_*.$$

These conclusions apply to the first restriction functional $\ell(A) = \mathcal{B}(A, T_0 w)$ of Theorem 71.11 on $H(v)$. For each prescribed nonzero smooth row vector $T_0 w$, some pure compact-finite v makes this functional nonzero.

Proof. At the prescribed level, Proposition 95.4 identifies the quotient with h_m copies of the compact group, each of mass $1/h_m$. On a copy, the coefficients of two bases as above are a basis for the full right ρ-isotypic functions. This follows by the compact matrix coefficient decomposition. Schur orthogonality applied to a pair of coordinates gives the stated Gram entry, with the dual inner product taken on complex linear functionals. Different classes have disjoint supports. Hence these functions are a basis of the full terminal ρ-space and their Gram matrix is invertible.

Let $I : \mathbb{C}^M \to \mathcal{A}_m$ take coordinates to this basis. Then $I^* I = G$, and $I G^{-1} I^*$ is the identity on its range. Let $a_\ell \in H(v)$ be the Riesz representative, so $\ell(A) = \langle a_\ell, A \rangle$. For a terminal vector e, the adjoint calculation in Theorem 71.13 gives $T_v^* e = P_v T^\dagger e$. Indeed one can test against any element of $H(v)$ and move each kernel across its Petersson pairing. These integrals converge by cuspidal decay on both sides at each noncompact step, as proved there. The last orthogonal projection expresses restriction of the domain to $H(v)$. Consequently $y = I^* T_v a_\ell$. The real type of the ascent is ρ at the terminal group and it has the given finite level, so $T_v a_\ell$ is in the range of I. We obtain the identity

$$y^* G^{-1} y = \|T_v a_\ell\|^2.$$

Apply Theorem 75.3 to a_ℓ, whose norm equals $\|\ell\|_*$. The individual bound follows by using $y^* G^{-1} y \le M \|G^{-1}\|_{\mathrm{op}} \max_\nu |y_\nu|^2$.

For the first restriction, the given row form is smooth and cuspidal and the bottom core forms in $H(v)$ are cuspidal. The restriction integral exists by Theorem 71.11 and its proof. That theorem supplies a compact-finite core partner with nonzero pairing. At the fixed finite level such a partner is a finite sum of pure real compact-finite tensors times finite multiplicity vectors. Indeed only finitely many compact types occur in that vector, their spaces in the real representation are finite-dimensional, and the real Harish-Chandra module is the algebraic tensor product over the real places. At least one tensor has nonzero pairing. Normalizing its real part gives the required v. □

The common real norm factor C_{fix}, the metric of the integral terminal coefficients and the dual norm of the restriction enter the individual lower bound. Section 6.6 requires absolute arithmetic estimates after the specified period projection on these fields.

83.8 The valuation of a prescribed signed period coefficient

The normalized arithmetic column of Theorem 121.3 is a finite signed sum of algebraic coefficients under the hypotheses of that theorem. Keep its levels and its prescribed endpoint. Section 119.3 writes the resulting scalar as $Z = \sum_{t \bmod L} C_t \zeta^t$, where ζ is a primitive L-th root of unity. The same notation applies to the prescribed joint auxiliary row when its fiber sums are algebraic. We give a finite computation at every coefficient prime using these very C_t, allowing ramification of any order.

Theorem 83.15 (Fixed values in a ramified coefficient field)**.** *Choose a finite place above p of a number field containing the data, and write E for its completion. Normalize $v_p(p) = 1$. Let K be the maximal unramified subextension of $E/\mathbb{Q}_p$. Put $e = [E : K]$ and choose a uniformizer ϖ of E. Write $L = p^b m$, with $(m, p) = 1$, and factor $\zeta = \omega\eta$ with ω of order m and η of order p^b. Then $\omega \in K$.*

If $b>0$, put $M=p^{b}$, $D=p^{b-1}$, $n=(p-1)D$, $h=e/n$. Expand uniquely over K

$$C_t\omega^{t}=\sum_{i=0}^{h-1}\varpi^{i}\sum_{j=0}^{n-1}c_{ijt}\eta^{j}.$$

For $v \bmod M$ and $0\le i<h$ set

$$A_{i,v}=\sum_{j+t\equiv v \bmod M} c_{ijt}.$$

Here j,t run over the full ranges in the expansion of Z. For $0\le k<n$, writing $k=v+uD$ with $0\le v<D$, put

$$B_{i,k}=A_{i,k}-A_{i,v+(p-1)D},\qquad D_{i,j}=\sum_{k=j}^{n-1}\binom{k}{j}B_{i,k}.$$

The valuation of the prescribed scalar, with $v_p(0)=+\infty$, is

$$v_p(Z)=\min_{\substack{0\le i<h\\0\le j<n}}\left(v_p(D_{i,j})+\frac{i+hj}{e}\right). \tag{83.13}$$

In particular, for each integer a, $v_p(Z)\ge a$ if and only if $B_{i,k}\in p^{a}\mathcal{O}_K$ for every i,k. Vanishing is equivalent to equality of all p coefficients $A_{i,v+uD}$ as u runs from zero to $p-1$, for every i,v.

If $b=0$, expand $C_t\omega^{t}=\sum_{i=0}^{e-1}c_{it}\varpi^{i}$ over K. Then

$$v_p(Z)=\min_{0\le i<e}\left(v_p\Big(\sum_{t}c_{it}\Big)+i/e\right).$$

Proof. The coprime orders give the factorization of ζ in its cyclic group. Reduction of a prime-to-p root of unity is injective on the group of such roots. Its reduction has a unique lift of that order by Hensel's lemma in K, which has the same residue field as E. This proves $\omega\in K$. For $b>0$, the polynomial $\Phi_{p^{b}}(1+T)$ is Eisenstein over $\mathcal{O}_K$. Its constant coefficient is p, and modulo p it equals T^{n} by $\Phi_{p^{b}}(X)=\sum_{u=0}^{p-1}X^{uD}$. Consequently $K(\eta)/K$ is totally ramified of degree n, $v_p(\eta-1)=1/n$ and $n\mid e$. The e elements $\varpi^{i}(\eta-1)^{j}$ in the indicated ranges have all the distinct valuations $(i+hj)/e$, for $0\le i+hj<e$. The valuation group of $K^{\times}$ is $\mathbb{Z}$, so in a nonzero K-linear combination there is a unique term of least valuation. In particular these elements are linearly independent. Their number is $[E:K]$, so they form a basis. Replacing $(\eta-1)^{j}$ by η^{j} uses an integral triangular basis change with diagonal one and proves existence and uniqueness of the asserted coordinates.

Substitution and collection modulo M gives $\sum_i\varpi^{i}\sum_v A_{i,v}\eta^{v}$. The identity $\sum_{u=0}^{p-1}\eta^{v+uD}=0$ removes the last term in each such progression. The remaining coefficients are exactly $B_{i,k}$. Expanding $\eta^{k}=(1+(\eta-1))^{k}$ gives $D_{i,j}$ in the basis with distinct valuations. The unique least term proves (83.13), including the zero case by the stated convention. Since $0\le(i+hj)/e<1$, an integer threshold a is attained exactly when all the $D_{i,j}$ belong to $p^{a}\mathcal{O}_K$. The integral triangular change of basis has integral inverse, so the same holds for the $B_{i,k}$. It also identifies vanishing with all the displayed progression differences being zero. When $b=0$, powers ϖ^{i} for $0\le i<e$ alone have distinct valuations modulo $\mathbb{Z}$, and the same argument applies. □

Corollary 83.16 (A test before any average over frequencies)**.** *At one auxiliary place v of residue characteristic p let the joint fiber weights in Theorem 99.12 at a prescribed frequency ν belong, under the chosen coefficient embedding, to an unramified extension $K/\mathbb{Q}_p$. For the Haar formulation take the fiber means as weights, removing the common $\mathcal{V}$. Set*

$T_y = \sum_b \chi(b) H_\nu(b,y)$. *After removing all factors independent of the position fibers, the value is* $U = \sum_y T_y \zeta^{\nu y}$ *with* $\zeta = \psi(p^{-s})$. *For any integer* a,

$$v_p(U) \ge a \quad \Longleftrightarrow \quad T_{y+p^{s-1}} - T_y \in p^a \mathcal{O}_K \quad \text{for every } y \bmod p^s.$$

In particular this one frequency gives zero exactly when T *is constant on each coset of* $p^{s-1}\mathbb{Z}/p^s\mathbb{Z}$. *If the* T_y *are integral and at least one of these differences is a unit, then* $0 \le v_p(U) < 1$.

Proof. The character χ takes values in the $(p-1)$-st roots of unity in $\mathbb{Q}_p$, so all T_y belong to K. Apply the theorem in $E = K(\zeta)$, with $L = p^s$, after permuting the indices by multiplication by ν. There is one index i; in the initial expansion only $j = 0$ is used. Multiplication by ν permutes the progressions modulo p^{s-1}, and permutes their members. The theorem tests equality modulo p^a to one designated member of each progression. This is equivalent to equality for every pair of successive members because the progression is cyclic. Equality in K follows from equality modulo every p^a. Integrality and failure of the test with $a = 1$ prove the last assertion. The other factors in the Haar integral are $\mathcal{V}\eta_\chi p^{r-1-(s+1)d} G(\chi)^h \chi(\bar{\nu})^{-1}$ with the notation of Theorem 99.12. □

For the normalized definite column take $Z = D_0(\prod_i A_i^{\mathrm{jet}})\mathcal{A}$, with its original class tests as in Theorem 122.10. There Z is an algebraic integer. Once the rational-array test of Theorem 119.4 detects $Z \neq 0$, the same scalar satisfies

$$\frac{1}{[B:\mathbb{Q}]} \sum_{w \text{ finite}} \operatorname{ord}_w(Z) \log Nw \;\le\; \tfrac{1}{2} \log \left(\frac{1}{\varphi(L)} \sum_a b(a)^* G(a) b(a) \right)$$

in the notation of (119.1). Indeed the finite sum is the logarithm of its absolute norm divided by $[B:\mathbb{Q}]$. That is the mean of $\log|\sigma Z|$, bounded by half the logarithm of the mean of $|\sigma Z|^2$ by concavity. At a place $w \mid p$ the term uses $\operatorname{ord}_w(Z) = e_w v_p(Z)$, with e_w its absolute ramification index. Formula (83.13) and its $b = 0$ case compute these orders from the same inputs. The uniform comparison in Section 6.6 still requires control of the actual signed class coefficients as x varies and their comparison to the prescribed analytic pairing.

83.9 The norm and denominators of the split transfer

Retain the notation and the additive Haar measures of Section 90.9. Write p_m for the measure of the primitive locus in $M_{m,n}(O)$. Choosing the rows over the residue field gives $p_m = \prod_{i=0}^{m-1}(1 - q^{i-n})$. Put $L = e - c \ge 0$ and let $N_b(L)$ be the number of O-lattices J between $\pi^L O^b$ and O^b, including both endpoints.

Theorem 83.17 (Exact norm at a ramified split place)**.** *On the full space of level-*e *tests, with additive* L^2 *norms, the transfer of Theorem 90.10 satisfies*

$$\|\mathcal{T}\|_{\mathrm{op}}^2 = \frac{p_a}{p_b} N_b(e-c), \qquad \frac{p_a}{p_b} = \prod_{j=b+1}^{a} (1 - q^{-j}).$$

On the primitive-supported covariant subspace its squared norm ratio for every nonzero vector is p_a/p_b.

Proof. Let dk denote Haar probability on K_n. The frames X_k, Y_k separately have the normalized additive distributions on the corresponding primitive loci. Indeed K_n is transitive there by basis completion, and preserves additive measure. Thus $\|\mathcal{T}\Psi\|_2^2 = p_a \int_{K_n} |A_\Psi(Y_k)|^2 \, dk$. In

the primitive covariant case $A_\Psi = \Psi$ on these frames. The formula for p_m gives the asserted ratio.

Let Ω consist of invertible integral g with largest Smith exponent at most L. Lemma 90.9 permits us to restrict $A_\Psi(Y_k)$ to Ω. Two such matrices have the same column lattice precisely when they belong to one coset $g_J K_b$, so Ω has multiplicative measure $N_b(L)$. Cauchy–Schwarz gives

$$|A_\Psi(Y_k)|^2 \le N_b(L) \int_\Omega |\Psi(gY_k)|^2 |\det g|^n \, d^\times g.$$

For each lattice J fix g_J with $g_J O^b = J$. A full-rank matrix with column lattice J is uniquely $g_J Y$ with Y primitive. Left multiplication by g_J has additive Jacobian $|\det g_J|^n$. In the double integral over k and $g_J K_b$ the factor from K_b can be absorbed into the invariant distribution of Y_k. Its contribution, including $|\det g_J|^n$, is $p_b^{-1} \int_{XO^n = J} |\Psi(X)|^2 \, dX$. Summing the disjoint sets proves the upper bound.

For sharpness choose a nonzero covariant function f on primitive b-frames modulo π^c. One may give it value one at a chosen basis of a single direct summand, extend by the unit character under changes of basis, and set it to zero on the other summands. The character is trivial on changes of basis congruent to the identity modulo π^c. Define a test on the matrices with allowed column lattice by

$$\Psi(gY) = \nu(\det g)^{-1} |\det g|^{-z} f(Y), \qquad g \in \Omega, \quad Y \text{ primitive},$$

and set it to zero elsewhere. A second decomposition has the form $(gk', k'^{-1}Y)$ with $k' \in K_b$, giving the same value by covariance. Perturbing gY by $\pi^e U$, U integral, changes Y by $g^{-1}\pi^e U \in \pi^c M_{b,n}(O)$, so the new frame is primitive with the same f. The allowed set and its complement are therefore unions of residue classes. This test has level e. Its integrand on Ω equals $f(Y)$, and its support uses all allowed column lattices and no others. Both inequalities in the upper bound are equalities, and the test has positive norm. □

Proposition 83.18 (The lattice count at excess depth)**.** *Set $\gamma_0 = 1$. For integers $0 \le u_1 \le \cdots \le u_b \le L$, let m_j be the multiplicity of j among them. Then*

$$N_b(L) = \sum_{u_1 \le \cdots \le u_b} q^{\sum_{i>j} (u_i - u_j)} \frac{\gamma_b}{\prod_{j=0}^{L} \gamma_{m_j}}.$$

In particular, with $d_b = \lfloor b^2/4 \rfloor$, $q^{L d_b} \le N_b(L) \le (L+1)^b \exp(2b) q^{L d_b}$, and $N_b(0) = 1$.

Proof. Each Smith type forms one K_b-orbit. The stabilizer of $\mathrm{diag}(\pi^{u_i}) O^b$ consists of the integral invertible matrices with ij-entry divisible by $\pi^{\max(0, u_i - u_j)}$. The unconstrained reduction is block upper triangular and is invertible precisely when each equal-exponent diagonal block is invertible. Its additive mass is therefore $q^{-\sum_{i>j}(u_i - u_j)} \prod_j \gamma_{m_j}$. Dividing γ_b by this mass gives the index. The exponent is $\sum_i (2i - b - 1) u_i$, at most $L d_b$. For $L > 0$ the type with $\lfloor b/2 \rfloor$ entries L and all others zero attains this exponent. Its ratio of γ's is at least one by comparing the last $\lfloor b/2 \rfloor$ factors of γ_b with the first ones. For the upper bound there are at most $(L+1)^b$ possible types and at most b nonempty blocks. The inequality $-\log(1 - q^{-i}) \le 2q^{-i}$ gives $\gamma_j \ge \exp(-2)$. For $L = 0$ there is exactly one lattice. □

Theorem 83.19 (The transfer in the two partial Fourier models)**.** *Let P be the set of columns Fourier transformed on the small side, $d = |P|$, and use the complementary columns on the large side. Suppose the additive character ψ has conductor O, meaning its dual lattice to O is O. The linear map between the original level-e tests obtained by matching these two charts has the same squared operator norm $(p_a/p_b)N_b(e-c)$, for every P. If $D_0\Psi$ has algebraic integer values and $D_1\nu(\pi)$ is an algebraic integer, where D_0, D_1 are positive integers, all coefficients of the output divided by $\sigma_P = (\nu(\pi)q^{-z})^{e(a-d)}$ become algebraic integers upon multiplication by*

$$D_0 q^{ebd} H_b(e)\,(D_1 q^{\delta})^{b(e-c)}.$$

Proof. For a level-e test the partial Fourier transform $\widehat{\Psi}$ integrates the entries in P against ψ of the entrywise pairing. It vanishes outside the dual of the translation lattice in those columns, by translating the integral by a period. Inside that dual it is the sum over R_e^{bd} with coefficient q^{-ebd}. Let D' be diagonal with π^{-e} in the columns P and one elsewhere. Then $\Psi_1(Y) = \widehat{\Psi}(YD')$ has level e. Put $D = \pi^{-e}D'^{-t}$, the matrix with π^{-e} on P^c and one elsewhere. For a primitive frame X_k, $k \in K_n$, use $h = kD^{-t}$ in the definition of F. This gives $X_h = X_k D$ and $Y_h = \pi^e Y_k D'$. Since $\det(D^{-t}) = \pi^{e(n-d)}$, small-side covariance gives

$$F_{\widehat{\Psi}}(X_k D) = (\nu(\pi)q^{-z})^{e(n-d-b)}\nu(\det k)\,A_{\Psi_1}(Y_k) = \sigma_P\,(\mathcal{T}\Psi_1)(X_k).$$

Restrict $F_{\widehat{\Psi}}$ to primitive rows relative to $O^n D$, extending it by zero. Every row space again has exactly one K_a-coset of bases primitive in this lattice. The proof of Theorem 90.10 now gives $B = S$ in the transformed charts at every h. Apply the inverse partial Fourier transform for ψ^{-1} on P^c to recover the large-side test. The result is supported on integral matrices and is constant modulo π^e. Integration in each of its dual variables is an unnormalized sum indexed by R_e, since each coset of O has mass one. Finite character orthogonality gives Fourier inversion in the two respective charts.

On the self-dual additive measures each partial Fourier transform preserves the squared norm, by the same orthogonality. Converting the small-side transform to the coordinates of Ψ_1 multiplies its squared norm by q^{-ebd}. Returning from the standard large frame to $O^n D$ multiplies the squared norm by $q^{ea(n-d)}$. Together with $|\sigma_P|^2 = q^{-ne(a-d)}$ these factors have product one, since $a + b = n$. The Fourier transforms and diagonal changes biject the corresponding level spaces, so the extremizer of Theorem 83.17 also gives equality in these charts.

All Fourier phases and the factors $\nu(\det k)$ are roots of unity. In a contributing summand of Lemma 90.9 the uniformizer exponent s satisfies $0 \le s \le b(e-c)$, and its only additional factor is $\nu(\pi)^s q^{-\delta s}$ times a unit character value. The small Fourier coefficient, the group mass $H_b(e)^{-1}$, and these factors give the displayed common integer. Keeping σ_P explicit also covers the case $d > a$. □

Corollary 83.20 (The split factors on the period interval)**.** *For $x \in [V, V^{3/2}]$ as in Section 6.6, fix a complementary-rank square as above on $F = E_x^+$. At its marked split places $v \in T = S_0 \cup Q(\xi)$ use the local characters, Fourier blocks and levels of that square. Write q_v for the residue cardinality and c_v for the conductor exponent of the quotient character. For $e_v \ge c_v$ at every marked place, the squared norm of the tensor transfer on their full level spaces is*

$$\prod_{v\in T}\left(N_{b_v}(e_v - c_v)\prod_{j=b_v+1}^{a_v}(1-q_v^{-j})\right),$$

where the lattice count is taken over O_v. On any prescribed subspace this product is an upper bound. At minimal levels the map on the primitive covariant data in the linear charts has unit

elementary divisors. The negative logarithm of the product of the primitive measure ratios is $O([F:\mathbb{Q}]\,2^{-(b_*+1)})$*, with* $b_* = \min_{v\in T} b_v$ *when* T *is nonempty.*

Proof. The character quotient is ramified at the marked places of this square, and their additive characters give self-dual integral lattices. Apply Theorem 83.19 place by place. Singular values multiply under Hilbert tensor products, or one can apply the bound one factor at a time and tensor the extremizers. At $e_v = c_v$ only invertible residue matrices contribute. The isomorphism on covariant frames in Theorem 90.10 is integral in both directions. If an input is specified at $e_v < c_v$, a unit in U_{e_v} with nontrivial character can multiply one row of the integration matrix without changing its residue. This makes the entire small section and the matched output zero at that level. Thus all levels can be tested on the fields where they occur.

Above a rational prime p there are at most $[F:\mathbb{Q}]$ places and $q_v \ge p$. Summing $-\log(1-q_v^{-j}) \le 2q_v^{-j}$ first in $j \ge b_*+1$ gives at most $4[F:\mathbb{Q}]\sum_{m\ge 2} m^{-(b_*+1)}$. For $u \ge 2$ the latter sum with exponent u is at most $2^{-u}(1+2/(u-1))$, by integration from 2 to infinity. This proves the bound uniformly on the original interval. □

At the other split places, Proposition 90.11 supplies the common normalized spherical value. Almost all its local values on an adelic group element are one, since that element is integral there and the splitting characters are unramified. The resulting products of split sections are therefore equal pointwise. Any common finite linear combination of their translates, in particular the specified finite character average acting on those sections, preserves that equality. For the global period in Section 6.6 one still needs to compare the two global functionals on these matched sections and to bound the signed value on the designated integral column. A norm on the full space above gives no lower bound on that value.

83.10 The kernel on refined terminal columns

Use the class tables of Corollary 9.14 with the determinant character removed as in Theorem 83.6. Thus a local scalar acts by permuting the nonzero residue covectors. In the original action the statements below use the corresponding scalar character obtained by putting the determinant twist back. Write $a = k$ for the column rank of that corollary and $s \ge 1$ for the flag length. Keep both auxiliary primes, the integral lattices away from T, the real components and the field fixed. Use the first Bessel position with the marked halves of Section 86.11.

Theorem 83.21 (Factorization through projective residue covectors)**.** *At every terminal place give the row the graph compact type of Theorem 86.1. On the refined column type of multiplicity* R_c*, the pairing with all such rows factors through summing the column coordinates over scalar orbits at each place. The rank in the column variable is at most*

$$B_c = d_c \left(\frac{T^a - 1}{T - 1} \right)^f .$$

It is zero on the subspace of dimension R_c° *in Corollary 9.14. On the integral class table the map that sums over the fibers is split surjective and every one of its nonzero Smith factors equals one over* $\mathbb{Z}$*.*

Proof. Fix any row of the indicated type. At a terminal place, every scalar unit in $H(O)$ preserves J_ℓ. Theorem 71.21 makes its pairing invariant under these units on the column. The other exchanges and all conditions at other places commute with this action. Thus two basis vectors with covectors in the same product of scalar orbits have equal values for every row. Conjugate linearity on the column gives the conjugate of the sum of its coordinates on each such fiber.

The unrefined classes have trivial stabilizers, by Proposition 86.29. Above each class the refinements are the independent choices of a nonzero covector at the f places. The image of each coordinate under the fiber sum is its tuple of projective covectors. Choose one representative in every fiber and send the corresponding projective basis element to it. This is an integral section. Within each fiber the differences from the representative form an integral basis of the kernel. The same computation applies to each multiplicity coordinate of the unrefined table. There are B_c image coordinates. A function with zero sum separately on every local scalar orbit has zero sum on the whole fiber. This proves all assertions, including the integer Smith calculation before extension of scalars. □

Corollary 83.22 (Loss in the proposed Gram estimate)**.** *Let D_c be the map from this refined column into the dual row space furnished by the first-position pairing on graph rows, using the Hilbert space identification with conjugate duals as needed. The same assertions apply after projecting arbitrary rows to this graph compact type before evaluation. For any scalar u one has*

$$\|(uD_c)^*(uD_c) - I\|_{\mathrm{HS}}^2 \;\geq\; R_c - B_c.$$

On the separately zero-sum subspace the squared discrepancy from the identity is exactly R_c°. For a row A of the original graph type and its signed sum A_λ of Theorem 83.6, the raw first-position pairing $\mathcal{J}_\phi(A_\lambda)$ vanishes for every column whenever λ is nontrivial at at least one terminal place.

Proof. An orthonormal basis beginning with a basis of the kernel contributes at least $\dim\ker D_c$ to the squared Hilbert–Schmidt norm. The preceding theorem gives the stated bound and full vanishing on the indicated subspace. For a scalar lift v the change of variables in Theorem 71.21 gives $\mathcal{J}_\phi(\widetilde{R}(v)A) = \mathcal{J}_{\widetilde{R}(v^{-1})\phi}(A) = \mathcal{J}_\phi(A)$. Summing a nontrivial character over its finite group gives zero. □

Here $B_c/R_c = (T-1)^{-f}$. Thus even an arbitrary scalar normalization cannot give a mean squared Gram error tending to zero on these refined tests. If the prescribed row projection e is applied to A_λ first and λ is nontrivial at every place, equation (83.10) gives $\mathcal{J}_\phi(eA_\lambda) = \mathcal{J}_\phi(L_T A)$. This is again invariant under terminal scalars on the column. Equation (83.11) gives a positive lower bound for the norm of the complex theta image of the raw signed test for every nonzero A under the hypotheses there. The comparison still requires the specified projection of the theta image. Equation (83.12) concerns the symmetric Bessel coefficient. The exchange integral used above occurs only at the first position.

Proposition 83.23 (A translation retaining the residue directions)**.** *In the chosen split half, interchange e and the first flag vector e_1, using the inverse adjoint on the opposite half. For the resulting translated graph lattices, the $q-1$ nonzero scalar multiples of any fixed residue covector give distinct orbits under the full exchange group $\prod_{i=1}^{s} X_i(A_k)$. The scalar stabilizer of one of these orbits inside $O^\times$ is $1+\pi O$.*

Proof. Put $B = A_k \oplus ke_1$. The common exchange group fixes B pointwise. For any invertible P in it, one has $P\Lambda \cap B = \Lambda \cap B$ since P^{-1} fixes B. In the translated graph this intersection is precisely

$$\{w + te_1 : w \in O^a,\ t \in O,\ t \equiv \ell(w) \pmod{\pi}\}.$$

Two such lattices in B coincide only if their covectors agree, as is seen by testing every w. A scalar $v \in O^\times$ changes the covector to $\bar{v}^{-1}\bar{\ell}$; when $v \equiv 1$ it fixes the entire lattice, which is a direct sum with the other coordinate lines. This proves the assertion. □

This translation is available at each of the same terminal places. Distinct exchange orbits distinguish the local lattice tests. Their integrals against a global automorphic row may still agree or vanish. A lower bound on the projected global pairing needs information about those actual values.

83.11 The first theta test at the translated terminal graph

Use the chosen half over T of Theorem 71.24. Write $q = T$, π for its uniformizer and O for its valuation ring. The row lattice $\Lambda = \Lambda_{\ell_0}$ has rank $r = a + 2s + 1$. Here $s \geq 1$. The middle coordinates satisfy $t \equiv \ell_0(w) \bmod \pi$ for $w+te \in O^a \oplus Oe$. Write g for the first-position translate in that theorem. We work in a basis of this particular Λ.

Proposition 83.24 (Elementary divisors in the terminal graph frame)**.** *The elementary-divisor exponents of $g\Lambda$ relative to Λ are -1 with multiplicity $s+1$, 1 with multiplicity one and zero with multiplicity $r-s-2$. Put $h = s+1$, $K = \mathrm{GL}(\Lambda)$, $S = K \cap gKg^{-1}$. Then*

$$[K : S] = q^h \begin{bmatrix} r \\ h \end{bmatrix}_q \begin{bmatrix} r-h \\ 1 \end{bmatrix}_q ,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts k-dimensional subspaces of $\mathbf{F}_q^n$. The h-dimensional subspace $W = (\pi g\Lambda + \pi\Lambda)/\pi\Lambda$ contains the whole first flag $E_s = \langle \bar{e}_1, \dots, \bar{e}_s \rangle$.

Proof. Change the integral basis in the column space to make ℓ_0 a coordinate modulo π. Choose its last basis vector w_0 with coordinate one. Alongside the other column vectors the graph has basis $u = w_0 + e$, $v = \pi e$. The translate interchanges e_1, e and then multiplies the first s flag vectors by π^{-1}. On the ordered triple e_1, u, v its matrix is

$$\begin{pmatrix} 0 & \pi^{-1} & 1 \\ 0 & 1 & 0 \\ \pi^{-1} & -\pi^{-1} & 0 \end{pmatrix} .$$

The least valuation of an entry is -1, the least valuation of a minor of size two is -2, and the determinant has valuation -1. Thus the three exponents are $-1, -1, 1$. The other $s-1$ first flag vectors have exponent -1. All remaining vectors are fixed. The reductions of $\pi g e_1$ and $\pi g u$ span $\bar{v}, \bar{e}_1$, and $\pi g e_i = e_i$ for $2 \leq i \leq s$. This proves the description of W. In a Smith basis the reduction of S stabilizes the flag of dimensions $h, r-1$. Every element of this residue parabolic lifts. Inside its full inverse image the last row of the first h columns must vanish also modulo π^2. There are h independent extra residue conditions because the lifts of the entries are free at the first congruence level. Counting the two successive subspaces in the flag gives the index. □

Let $w \geq r$ be the next rank and $d' = w - r$. Write $Y = \binom{X}{Z}$ in the matrix model of Theorem 93.5. At depth j put $L = (B_v \cap K)K_r(j)$ for $j \geq 1$, and $L = K$ for $j = 0$. The local tests ϕ_i are supported on $X \in Li$, $Z \in \pi M_{d',r}(O)$, where $i \in L\backslash K$. Their nonzero values are $\theta(X)$, with θ the compact character specified there. Use the partner translate $b = \mathrm{diag}(g, I_{d'})$ in the same matrix frame. Let $S = K \cap gKg^{-1}$ as above, and set

$$J = L \cap S, \qquad H = gLg^{-1} \cap S, \qquad C = J \cap H.$$

Use the inner product linear in the first variable. All entries below are divided by the same original squared norm $\|\phi_i\|^2$.

Theorem 83.25 (The mixed Gram matrix on the original finite tests)**.** *Every nonzero entry of the matrix* $\langle \omega(g,b)\phi_k, \phi_i\rangle / \|\phi_i\|^2$ *equals*

$$m = \eta(g,b)\, q^{-d'(s+2)/2} [L : C]^{-1},$$

where η *is the scalar character in the chosen splitting relative to the matrix splitting. The active rows are indexed by* $J\backslash S$ *and the active columns by* $H\backslash S$*. Their entry is nonzero exactly when the two cosets meet. In particular the fractional ideal generated by all these entries is* (m) *in any valuation ring of a coefficient field containing* $\sqrt{q}$ *and the specified splitting value. With the standard Euclidean norms on the coordinate vectors the matrix has operator norm* $|m|\sqrt{[J : C][H : C]}$ *and squared Hilbert–Schmidt norm* $|m|^2[S : C]$*.*

At every depth $j \geq 1$ *its rank divided by* $[K : L]$ *is at most*

$$\frac{\begin{bmatrix} s+1 \\ s \end{bmatrix}_q}{\begin{bmatrix} r \\ s \end{bmatrix}_q}.$$

At depth zero the unique entry has absolute value $q^{-d'(s+2)/2}/[K : S]$*. Translating the row by* g *with the identity on the partner gives a zero matrix at every depth.*

Proof. In these coordinates the matrix action is

$$(\omega(g,b)\phi_k)(Y) = \eta(g,b) |\det g|^{w/2} |\det b|^{-r/2} \phi_k(b^{-1} Y g).$$

In the normalization of (93.1) the Jacobian of $Y \mapsto b^{-1}Yg$ is $|\det g|^w |\det b|^{-r}$, and its square root gives the unitary matrix action. The top block changes by $X \mapsto g^{-1}Xg$ and the lower block by $Z \mapsto Zg$. The character θ on K factors through determinant. Indeed the elementary matrices generate $\mathrm{SL}_r(O)$ by elementary row reduction over the valuation ring. Each elementary matrix is a commutator with an integral diagonal unit whose difference from one is a unit, since $q > 2$. Thus the top character cancels on the common support.

For an exponent t of g, the normalized overlap on a single lower coordinate is $q^{-|t|/2}$. In fact the support intersection has relative measure $q^{-\max(0,-t)}$ and the square root of the Jacobian contributes $q^{-t/2}$. Changes of integral frames preserve the lattice and measure. By the proposition, over all lower rows the absolute factor is $q^{-d'(s+2)/2}$. On the top block the common support is $Li \cap gLkg^{-1}$ inside S. Every nonempty intersection of S with Li is one coset of J. The corresponding intersections for the translated column cosets are exactly those of H. When the two cosets meet they meet in one coset of C. On the matrices of unit determinant absolute value, additive measure is proportional to multiplicative Haar measure. The ratio to the volume of L is therefore $[L : C]^{-1}$. The identity cosets meet, giving the ideal assertion including at a prime dividing $[L : C]$.

Each active row meets $[J : C]$ active columns, and each active column meets $[H : C]$ active rows. Applying Cauchy–Schwarz to each row and then summing bounds the operator norm by the stated expression. The vector constant on the active columns and zero elsewhere attains the bound. The nonzero entries are indexed by $C\backslash S$, which gives the squared Hilbert–Schmidt norm.

The compact Bessel group preserves the first isotropic flag, and $K_r(j)$ fixes its reduction for $j \geq 1$. Consequently $i^{-1}E_s$ is well defined on Li. If Li meets S, represent it by an element of S. It preserves W, so the inclusion in the proposition gives $i^{-1}E_s \subset W$. As Li ranges over the finite table, $i^{-1}E_s$ takes each s-plane equally often by the transitivity of K modulo π. Only $\begin{bmatrix} s+1 \\ s \end{bmatrix}_q$ of the $\begin{bmatrix} r \\ s \end{bmatrix}_q$ planes lie in W. This bounds the number of active rows and hence the rank. For $L = K$ one has $J = H = C = S$. Finally a row translation alone requires both X and Xg to lie in K. Since $g \notin K$, these two support conditions are disjoint. □

For tensor products over the f places above T, the local matrices tensor, their ideals multiply, and the rank fractions multiply. The result concerns the arithmetic matrix of overlaps of the actual prescribed local tests. Estimating the global signed projection in Section 6.6 also requires its values on the automorphic forms obtained by integrating the theta kernels.

83.12 The auxiliary character on the original signed projector

At a half over g_1 or g_2 use the signed subgroup Γ, character λ_Γ and ordinary Bessel character χ of Lemma 99.16. Put $P_\lambda = \int_\Gamma \overline{\lambda_\Gamma(k)} R(k)\, dk$ with probability Haar measure. Assume $s \geq 1$ so that there is a simple string subgroup Z_j as in that lemma.

Theorem 83.26 (Vanishing on returning to the original signed type)**.** *For the unmodified absolute-trace character the ordinary global Bessel period of every row in the image of P_λ is zero at every column. Furthermore, for either auxiliary diagonal t of Theorem 99.17 or Proposition 99.18,*

$$P_\lambda R(t) P_\lambda = 0.$$

This identity holds in every unitary representation of the row group, including on projected automorphic theta forms. It allows arbitrary simultaneous operations commuting with this local group.

Proof. By Lemma 99.16, for any simple string root one can choose $w \in O$ with $u = Z_j(w) \in \Gamma$ and $\lambda_\Gamma(u) \neq 1$. The absolute-trace character has conductor ideal O on this root, so $\chi(u) = 1$. The column group acts on A and commutes with u, which involves only the string. In the ordinary Bessel integral translate the unipotent variable on the right by u at this place. The quotient measure is right invariant and this change multiplies the character by one. Thus replacing a row F by $R(u)F$ leaves its period fixed. For a signed row it also multiplies it by $\lambda_\Gamma(u)$. This proves the first assertion, with the other local and archimedean factors kept fixed.

Both diagonals satisfy $v(l_{j+1}/l_j) = 1$. Consequently $t^{-1}ut = Z_j((l_{j+1}/l_j)w)$ is in the first principal congruence subgroup and has trivial compact character. It follows that $R(u)R(t)P_\lambda = R(t)P_\lambda$. On the other hand $P_\lambda R(u) = \lambda_\Gamma(u)P_\lambda$, by changing the variable in the compact average. Applying these two equalities to the same vector proves the operator identity. A determinant twist has no effect on u or its conjugate. The argument works in the actual automorphic representation before any subsequent linear period is applied. Thus this projector annihilates the translated signed row even when the local principal matrix indicators in Theorem 99.19 have positive overlap. □

83.13 The integral map at the last hyperplane

At one embedding let $N = n + 1$, with $n \geq 1$. Write the two dominant weights as λ and μ, in decreasing order. Suppose $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for $1 \leq i \leq n$ and $s = \lambda_N = \mu_n$. Retain the determinant lines separately. In the resulting polynomial models the partitions are $l_i = \lambda_i - s$ and $m_i = \mu_i - s$. For a partition v let p_v be its product of leading minors and put $H_v = \|p_v\|_{\mathrm{F}}^2$. Use the rational bilinear form

$$B_v(P, Q) = H_v^{-1} \sum_\alpha \alpha! P_\alpha Q_\alpha.$$

Monomials have their usual (undivided) coefficients. The corresponding positive Hermitian norm on complex polynomials uses complex conjugates in the second slot. The lattice in a frame is the intersection of the flag polynomial space with the integral monomial lattice, as in Theorem 122.6.

Let D_i be the leading minor of order i of an N-by-N variable matrix. Let D_i' use the first i columns and rows $1,\dots,i-1,N$, in that order. Put

$$a_i=\mu_i-\lambda_{i+1},\qquad b_i=\lambda_i-\mu_i,\qquad X=\prod_{i=1}^{n}D_i^{a_i}(D_i')^{b_i},\qquad \kappa=B_l(X,X).$$

For an N-square matrix put its last column equal to zero and extract the coefficient of $\prod_{i=1}^{n}X_{Ni}^{b_i}$ in the last row. Denote this linear operation by C_b, using X_{Ni} just for the matrix entries in this expression. Let P_m be Fischer orthogonal projection onto the smaller flag polynomial space of partition m. Define

$$d_0=\sum_i a_i,\quad I_i=[\lambda_{i+1},\min(\lambda_i,\lambda_{i+1}+d_0)]\cap\mathbb{Z},\quad c=\prod_i\prod_{v\in I_i\setminus\{\mu_i\}}\prod_{j=1}^{n}(v-i-\mu_j+j).$$

With the normal line in its standard frame, the prescribed restriction on polynomials is $L=c\kappa P_mC_b$. These are the constants in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 53 and lines 113–138 there. We compute their effect on the lattice.

Lemma 83.27 (The squared length of the interlacing line)**.** *Put $h=\lambda_1-\lambda_N$. The rational number κ is positive, and both c and κ are units in $\mathbb{Z}_{(p)}$ for every rational prime $p>h+n$. Moreover $C_bX=p_m$.*

Proof. In extracting the last-row power, the variable in column n must come from every modified minor of size n. Their cofactors are D_{n-1} with positive sign. Continuing in decreasing order forces the last-column term of each remaining modified minor. The result is $\prod_i D_i^{a_i}D_{i-1}^{b_i}=p_m$, with $D_0=1$. The diagonal monomial of p_m has coefficient one. Consequently X is primitive in the integral polynomial lattice.

The flag minors place X in the large module of partition l. Under row changes on the first n coordinates it is a highest vector. Each factor is unchanged by adding an earlier row to a later one among these coordinates. Its row weight is m, as is also seen by counting the rows in the factors. Restriction has multiplicity one by the interlacing rule of Proposition 94.29. Thus the subspace of weight m in the m-isotypic summand is this one line.

Over $\mathbb{Z}_{(p)}$ the large flag lattice is self-dual for B_l by Theorem 122.6. Indeed every hook of l has length at most $h+n$. The isotypic projector for the row block is integral there. To see this directly on polynomials, use the single block of size n in (70.2). Its matrix generators are sums of operators of the form $X_{ij}\partial_{X_{kj}}$, up to transpose and the separate integral determinant shift. They preserve integral polynomials. The denominator has prime factors at most $h+n-1$. The isotypic projector is orthogonal for the compact unitary action and rational by that formula. Its matrix therefore is self-adjoint also for the rational bilinear form. Extraction of the row weight m is an integral self-adjoint projection by monomials. These projections commute. Their product splits off the highest line orthogonally. An orthogonal direct summand of a self-dual lattice is self-dual, since a functional on it extends by zero on its complement. Its lattice is generated by X, by primitivity and its coefficient one. Its Gram entry κ is consequently a unit at p. Positivity follows from the real Fischer form and $X\neq0$.

For the integer c, the element μ_i belongs to I_i because $a_i\le d_0$. If $j<i$, then $\mu_j\ge\lambda_i\ge v$, and if $j>i$, then $\mu_j\le\lambda_{i+1}\le v$. In either case $v-i\neq\mu_j-j$. The factor with $j=i$ is nonzero by the exclusion of μ_i. All absolute values are at most $h+n-1$, proving the assertion also for the empty product. □

Theorem 83.28 (A split surjection in the prescribed normalization)**.** *Let L^* be the adjoint between the two normalized bilinear spaces. On the smaller space*

$$LL^*=c^2\kappa\,1.$$

The nonzero singular values of L in the two normalized Hermitian norms are all $|c|\sqrt{\kappa}$. At every prime $p > h + n$ the map on the indicated polynomial lattices is a split surjection with integral section $L^*/(c^2\kappa)$. These assertions tensor over embeddings and extend to the local rings of a coefficient number field.

Proof. Write $A = P_m C_b$. Row changes in the first n coordinates commute with coefficient extraction, and the projection commutes with their compact group. Its complexification gives equivariance for GL_n. The two separated determinant twists agree on this block. The adjoint also intertwines, first for the compact group in the Hermitian metrics, then for its complexification. All matrices in the polynomial bases are real and rational, so the bilinear adjoint gives the same matrix. The highest line computed in Lemma 83.27 forces $A^* p_m = tX$. Since $AX = p_m$ and $B_m(p_m, p_m) = 1$, pairing with X gives $t = 1/\kappa$. The endomorphism AA^* is scalar on the irreducible target. Its value on p_m is $\kappa^{-1} p_m$. Multiplication by $c\kappa$ proves the identity and the assertion on singular values.

For the lattice assertion, C_b takes integral coefficients to integral coefficients. Pairing such an output with any integral target polynomial by the Fischer form divided by H_m is integral at p, since H_m is a unit there. Projection preserves these pairings. Self-duality of the target now shows that the projected output is integral. Both source and target are self-dual, so the adjoint preserves their lattices as well. The scalar in the identity is a unit by the lemma. It gives the stated section. Taking products of the sections and extending scalars proves the last assertions. □

83.14 One integer for the terminal normalizations of all selected words

Use the common fields, weights and intervals of Theorems 9.13 and 22.11. At a fixed x, a translate fixes the small terminal row and its weights. A binary word w determines the large weights by (9.1). Denote by U_w the nonzero rational number in Theorem 127.1, and by Z_w its normal-coordinate factor for the chosen π.

Theorem 83.29 (Finite and archimedean variation in the last restriction)**.** *There is an integer $J_x \geq 1$, with $\log J_x \ll x^C$ for a fixed C uniformly on the common intervals, such that for every translate and every two words at that translate both the numerator and denominator of $|U_w/U_{w'}|$, in lowest terms, divide J_x. At the three marked primes the ratio is a unit. In the selected family at its common prime T and generator π, the same assertions hold for $(Z_w U_w)/(Z_{w'} U_{w'})$.*

Proof. First fix one translate and one embedding τ. Suppress that index and write $N = n + 1$ for the terminal rank. In the explicit lists of Corollary 127.2, the repeated tail, the first and last core entries, the rank shifts, and the entire small list μ are fixed. Only $\lambda_2, \dots, \lambda_{k-1}$ change. Each has two possible values differing by Δ, the spacing in (9.1). In particular the common determinant shift $s = \lambda_N = \mu_n$ is fixed.

Form a box of the exponents $a_i = \mu_i - \lambda_{i+1}$, $b_i = \lambda_i - \mu_i$ using their coordinatewise minima and maxima over all words. There are at most $2k$ varying exponents, each of range at most Δ. Consequently the difference of the total degrees of the two extremal products is at most $2kN\Delta$. Their upper degree is at most $N \max_w(\lambda_1 - \lambda_N) + 2kN\Delta$. All of N, k, Δ are polynomial in x, and $\log(2 + \max_w(\lambda_1 - \lambda_N)) \ll x^{\varepsilon} + \log(2x)$ by the explicit terminal lists. Theorem 120.2 now gives one integer of logarithm $O(x^C)$ whose divisors represent every ratio of the squared norms $q(X_w)$. Apply the same theorem to the leading-minor products $p_{l_w} = \prod D_i^{\lambda_i - \lambda_{i+1}}$. Here at most $2k$ exponents change, by at most 2Δ. This provides the same type of estimate for $H_{l_w} = q(p_{l_w})$. Since $\kappa_w = q(X_w)/H_{l_w}$, the product of these two integers clears the numerator and the denominator of every $\kappa_w/\kappa_{w'}$.

It remains to treat the integer c_w. For each i all its intervals $I_i(w)$ from Section 83.13 contain μ_i. Their lower endpoints vary by at most Δ. Their upper endpoints vary by at most $(k+1)\Delta$, since $d_0 = \sum_i(\mu_i - \lambda_{i+1})$ varies by at most $k\Delta$ and taking a minimum cannot increase the largest of the two variations. Delete μ_i from every interval. The factors for its common intersection cancel in all ratios. In the union outside the intersection, at most $(k+2)\Delta+2$ integers v remain per index. For each of these v and each $1 \le j \le n$ form the absolute value of $v - i - \mu_j + j$. They are positive and at most $n + \max_w(\lambda_1 - \lambda_N)$ by the proof of Lemma 83.27. Their product accounts for every remaining factor with its multiplicity and has logarithm $O(x^C)$. Both parts of each reduced ratio of the c_w divide this common integer.

The target hook factor H_{m_τ} is fixed within a translate. Multiply the three integers just obtained, then multiply over the CM embeddings and the translates. Both finite index sets have size at most $[E_x : \mathbb{Q}]$, polynomial in x by Proposition 6.4. If the two parts of each reduced fraction divide J_i, both parts of their product divide $\prod_i J_i$, by adding prime valuations. This proves the claimed bound with a common J_x even when the word family has exponential size. The marked-prime assertion follows from Theorem 127.1. For the selected family $Z_w = Z_{w'}$ by Corollary 127.2. □

Corollary 83.30 (The same factor in every exterior minor)**.** *At one translate fix a selected reference word w_0. Take any $a \ge 1$ selected columns labelled $w_1, \dots, w_a$, and any a tests on their common small row. Let M be the matrix of their prescribed periods, and S the matrix of the weighted coefficient sums of Theorem 127.1, including the same stabilizer weights. Then*

$$\frac{\det M}{(Z_{w_0}U_{w_0})^a} = R \det S, \qquad R = \prod_{j=1}^{a} U_{w_j}/U_{w_0}.$$

The rational scalar obeys

$$\frac{1}{a}\sum_p |v_p(R)| \log p \le \log J_x, \qquad \frac{1}{a}|\log |R|| \le \log J_x.$$

Its valuations at g_1, g_2, T vanish. All these identities apply to the original signed columns and tests.

Proof. The period formula multiplies an entire column by its same scalar for every test. Factoring the determinant gives the identity, including for dependent columns. At any p, every individual valuation difference lies in $[-v_p(J_x), v_p(J_x)]$. Summing this bound over the a factors and then the primes gives the first estimate. A positive fraction whose two reduced parts divide J_x lies between J_x^{-1} and J_x, giving the second estimate. The marked-prime assertion follows from the theorem. In particular these relative logarithmic costs per column are $o(b/x^A)$ for every fixed $A > 0$, since $b = \exp(x^\varepsilon)$. □

The scalar for one reference word and the determinant of the actual coefficient sums still appear in an absolute estimate. The latter depends on the incoming rows and on cancellation between the prescribed classes at the same translate.

84 Local divisors and archimedean scales of the projected determinant

84.1 The directional period estimate at every finite prime

Keep the fields, old marks, larger-weight embeddings and main pairs of Section 82.8. Write $S_p = 0$ at a split prime and use the sum of that section otherwise. For the given $d = (d_\tau)_\tau$, let

$$n = \lceil 4b_x/k \rceil, \quad D_\tau = (d_\tau - n)_+, \qquad B_\tau = \prod_{i=1}^{n} (D_\tau + i), \qquad A_d = \prod_\tau B_\tau^2.$$

Every B_τ is divisible by $n!$, by the binomial formula. On all sufficiently distant intervals

$$\log A_d \ll_{q_0, d_{\mathrm{K}}} f b_x \log(2b_x)/k.$$

Indeed $d_\tau \le 2b_x$ by the derivative window in the proof of Theorem 82.17, $n = O(b_x/k)$ and every factor in the products is at most $2b_x + n$.

Lemma 84.1 (A common normalization at the original marked places)**.** *There is a nonzero algebraic number b with absolute value $M^{h/2}$ at every complex embedding and with*

$$v_\iota(b) = \sum_\tau d_\tau v_\iota(\tau \mathfrak{m})$$

at every finite coefficient embedding, where $h = \sum_\tau d_\tau$.

Proof. View the oriented embeddings as automorphisms of the abelian field E fixing K. In E form $\mathfrak{a} = \prod_\tau \tau(\mathfrak{m})^{d_\tau}$. Since conjugation commutes with these automorphisms, $\mathfrak{a}\bar{\mathfrak{a}} = (M)^h$. Choose $N \ge 1$ and $w \in E^\times$ with $\mathfrak{a}^N = (w)$ by finiteness of the ideal class group. Take any root satisfying $b^{2N} = M^{hN} w/\bar{w}$. The ideal on the right is $\mathfrak{a}^{2N}$, so taking local valuations gives the display. At any complex embedding of E the image of $w/\bar{w}$ has absolute value one. Taking absolute values in the defining equation gives the stated value for every conjugate of the root; in particular $b\bar{b} = M^h$. □

Theorem 84.2 (The derivative table at all finite primes)**.** *Use the b of Lemma 84.1 in (82.1). Put $T_d = \gamma_x A_d$, using the nonzero algebraic integer of Theorem 82.11. For every main pair on the sufficiently distant common intervals and every coefficient embedding above every prime p,*

$$e_\iota \le v_\iota(T_d) + S_p. \tag{84.1}$$

In particular T_d is independent of p, ι and of the prescribed twist.

Proof. At a split prime use Corollary 82.15. Its correction from $\mathfrak{m}$ cancels $v_\iota(b)$ by the lemma. The other ideal correction is nonpositive, so $e_\iota \le v_\iota(\gamma_x)$. At all nonsplit primes b is a unit because its ideal is supported only above the split marks. At the ramified prime use Theorem 82.17, where $E_p = 0$ and $(n!)^f$ divides A_d. Here $p = d_{\mathrm{K}} \le k$ once the interval is sufficiently distant. We treat the two remaining ranges of inert primes.

Suppose first that $p \le k$. In the notation of Theorem 82.17 use every pair $\tau \mapsto t_\tau = P_{a_\tau}\tau$ occurring in E_p and put $j_\tau = \lfloor d_\tau / p^{a_\tau} \rfloor$. The ends and starts are all disjoint, and the smallest-exponent condition in Section 82.10 holds. The power $p^{a_\tau} j_\tau$ fits at its start, and $j_\tau \le (b_x + W)/p \le b_x - W \le d_{t_\tau}$ fits at its end; here $W = O(x^{42})$ bounds $|d_\tau - \kappa_\tau|$, and W/b_x tends to zero. Thus $\eta = d - r - s \ge 0$ componentwise. With $J = \sum j_\tau$ we have $E_p - J \le f(1 + W/p)$, since each start occurs at most once and has weight b_x.

Legendre's digit formula applied to a three-part multinomial at each of the f embeddings gives

$$v_p(d!/(r!s!\eta!)) \le 3f(1+\log_p(2b_x)).$$

In fact each of the three lower integers is at most $2b_x$, and its base-p digit sum is at most $(p-1)(1+\log_p(2b_x))$. The upper integer's digit sum is subtracted. From the geometric-series calculation in the proof of Theorem 82.17 we have, before absorbing any error into a factorial,

$$\lambda_p h - S_p - E_p \le f\left(LG + \frac{b_x}{(p-1)p^L} + \frac{W}{p-1}\right).$$

Here L is as in that theorem. Substitution in (82.9) shows that the excess over $v_\iota(\gamma_x)+S_p$ is bounded per embedding by

$$LG + \frac{b_x}{(p-1)p^L} + \frac{W}{p-1} + 1 + \frac{W}{p} + 3(1+\log_p(2b_x)).$$

The middle geometric tail is at most $b_x/(kp)$. The other terms are together $o(b_x/(kp))$ uniformly here. For the first and third terms this was checked in the preceding theorem on the small primes. For the additional terms use $p \le k$, $Wk/b_x \to 0$ and $k^2\log(2b_x)/b_x \to 0$. The last limit follows from $k \le 3x^5$, $b_x \ge x^{400}$ and monotonicity of $\log(2y)/y$ for large y. Since $b_x/k^2 \to \infty$, the display is at most $\lfloor n/p \rfloor \le v_p(n!)$. This proves the stated bound for all these inert primes, including those at which $E_p \neq 0$.

Finally let $p > k$ be inert. Use every oriented embedding with exponent one, and put

$$i_\tau = \lfloor D_\tau/p \rfloor, \qquad j_\tau = \min(i_\tau, d_{P_1\tau}).$$

Here $i_\tau \le 2b_x/p \le n$. The power entries pj_τ at τ are bounded by $(d_\tau - n)_+$, and the entry arriving at τ is bounded by $\min(n, d_\tau)$, since P_1 is bijective. Their sum is at most d_τ, even when the permutation fixes τ. Apply (82.8). For each τ one has

$$v_p(d_\tau!) \le v_p(B_\tau) + v_p(D_\tau!) = v_p(B_\tau) + i_\tau + v_p(i_\tau!) \le 2v_p(B_\tau) + i_\tau.$$

The first inequality is equality when $d_\tau \ge n$; when $d_\tau < n$, $d_\tau!$ divides $B_\tau = n!$. The middle equality uses the multiples of p in $D_\tau!$. The last inequality uses $i_\tau! \mid n! \mid B_\tau$.

The actual orders satisfy $\kappa_\tau \le d_\tau \le \kappa_\tau + W$. For the lower bound the shift in the main-pair window is $(2j+1)s^8 - h_\tau - z$ with $0 \le h_\tau \le 4s$, $z \le s$; it is positive for these ranks. Also $n \ge W$ eventually by the scale bounds. Hence $i_\tau \le \kappa_\tau/p$ and $d_{P_1\tau} \ge \kappa_{P_1\tau}$. It follows that

$$i_\tau - j_\tau \le (\kappa_\tau/p - \kappa_{P_1\tau})_+.$$

The sum of the right sides is the first-exponent contribution to S_p, and all its other terms are nonnegative. Summing the factorial inequalities after subtracting j_τ proves the required assertion in the last range. □

Corollary 84.3 (The same-field comparison for the projected period)**.** *On these same fields and intervals take the simultaneous local cohomology and critical twists of Corollary 52.21, and use the marked normalization of Section 82.4 with b from Lemma 84.1. For every allowed p, ι, β, with $p \nmid Q_\beta$,*

$$-v_\iota\Big(H_{\nu,z}\, T_d\overline{T_d}\, \rho_\beta^2 R_{\nu,z}(\beta)^{-1}\Big) \le p_{p,\iota} + 2S_p. \tag{84.2}$$

The common multiplier satisfies $\sup_\sigma \log^+|\sigma T_d| \ll_{q_0,d_{\mathrm{K}}} f(1+b_x/k)\log(2b_x)$. Even summed over mk pairs its logarithmic cost divided by $sB_x^{\det}$ is $O((\log b_x)^{-2})$.

Proof. Use (84.1) at ι and $\iota \circ c$, with c complex conjugation. The same T_d and S_p occur at both. Substitute in (82.2); moving $v_\iota(T_d \bar{T}_d)$ inside the negative valuation gives (84.2). The cohomological term $p_{p,\iota}$ is the one for these very tests and twists there. The archimedean bound follows from the estimate for A_d and the conjugate bound for γ_x in Theorem 82.11. The final assertion follows by substituting $B_x^{\mathrm{det}} = f x b_x (\log b_x)^3/(mk)$ and $m^2 \ll x,\ k \le s$. □

Equation (84.2) bounds the finite valuations for the marked high-weight period. To use it in the terminal determinant one also needs the archimedean size of the inverse untwisted norm contained in $H_{\nu,z}$, with the same b.

84.2 The relative different and the archimedean period scale

Fix a node in the shorter family of Section 84.1. Write $f = [E^+ : \mathbb{Q}]$, $d_{\mathrm{K}} = |\operatorname{disc}(K)|$, ξ for the quadratic character of K, and c for complex conjugation at the distinguished complex embedding. Put

$$D_E = N_{E/\mathbb{Q}}(\mathfrak{D}_{E/K})^{1/(2f)}, \qquad \Omega_E = \frac{2\pi}{\operatorname{rd}(E)} \exp\!\left(\gamma - \frac{L'(1,\xi)}{L(1,\xi)}\right).$$

The tower discriminant formula gives $D_E = \operatorname{rd}(E)/\sqrt{d_{\mathrm{K}}}$. Let $J \subset \overline{\mathbb{Q}}$ be a number field of definition for the marked discriminant-one elliptic curve and its differential. We may take J Galois over $\mathbb{Q}$. For an embedding $\rho : J \to \mathbb{C}$, let A_ρ be the area of its period lattice in the given differential and $q_\rho = \pi/A_\rho$. Complex conjugating both the curve and the differential conjugates the period lattice, so $q_{c\rho} = q_\rho$. By Theorem 65.22, with the mean taken over all embeddings,

$$\log D_E + \mathbb{E}_\rho \log q_\rho = -\log \Omega_E. \tag{84.3}$$

Lemma 84.4 (A common unit for the area factors)**.** *For any $H \geq 0$ there is an algebraic unit u such that every embedding $\sigma : J(u) \to \mathbb{C}$, restricting to ρ, satisfies*

$$\left|\log|\sigma u| - \tfrac{1}{2}\left(\mathbb{E}_\eta \log q_\eta - \log q_\rho\right)\right| \leq \frac{1}{2(H+1)}.$$

Proof. The prescribed vector on the embeddings of J is real, is constant on each conjugate pair and has sum zero. Dirichlet's unit theorem says that the logarithms $(\log|\rho v|)_\rho$ of units v of J span this real vector space. Here both members of a complex pair are counted. Express the vector as a real linear combination of finitely many such vectors. Approximate their coefficients by rationals until each coordinate error is at most $1/(2(H+1))$. Write these rationals as m_i/N with $N > 0$. Take any root u of $u^N = \prod_i v_i^{m_i}$. Both u and u^{-1} are integral. At every extension of ρ its log modulus is exactly $\sum_i (m_i/N)\log|\rho v_i|$. This also covers the zero dimensional logarithm space by taking $u = 1$. □

Theorem 84.5 (A simultaneous finite and archimedean normalization)**.** *For all main pairs at this node choose a common unit as in Lemma 84.4 with $H = \max_{\mathbf{d}} \sum_\tau d_\tau$. There are algebraic numbers $b_{\mathbf{d}} \neq 0$, independent of the critical twist, such that the table and projected period inequalities (84.1) and (84.2) hold with $b = b_{\mathbf{d}}$ in the definition of the table, the lift, and $H_{\nu,z}$. The cohomology bound and T_d are those of Corollary 84.3. For every $\sigma \in \operatorname{Aut}(\mathbb{C})$ let $\sigma' = c\sigma c$, $h = \sum_\tau d_\tau$, and use q_σ for $q_{\sigma|J}$. Then*

$$\left|\log\!\left(|\sigma b_{\mathbf{d}}|\,|\sigma' b_{\mathbf{d}}|\,(q_\sigma q_{\sigma'})^{h/2}\right) - h\log(M/\Omega_E)\right| \leq 1. \tag{84.4}$$

Proof. Put $\mathfrak{j} = \mathfrak{m}\bar{\mathfrak{b}}$ using the integral ideals of Corollary 82.15. One has $\mathfrak{j}\bar{\mathfrak{j}} = (M)\mathfrak{D}_{E/K}$. For a rational prime p let r_p be the exponent of the different at a prime above p, and e_p its absolute ramification index. These integers are independent of the choice of prime because $E/\mathbb{Q}$ is Galois and K is stable. The prime ideal norm formula gives $D_E = \prod_p p^{r_p/e_p}$. Choose a positive integer N with $\mathfrak{j}^N = (w)$ and $e_p \mid N$ at every prime in this finite product. This is possible by finiteness of the ideal class group. Then D_E^N is a positive rational number and $(D_E^N)\mathcal{O}_E = \mathfrak{D}_{E/K}^N$.

For each oriented embedding, viewed as an automorphism τ of E fixing K, take any algebraic root

$$a_\tau^{2N} = (MD_E)^N\, \tau(w/\bar{w}).$$

The fractional ideal of the right side is $(\tau\mathfrak{j})^{2N}$, since conjugation commutes with τ. Thus $v_\iota(a_\tau) = v_\iota(\tau\mathfrak{j})$ for all normalized finite valuations, including after extension. The conjugates

of $w/\bar{w}$ have modulus one, also by commutativity in $\mathrm{Gal}(E/\mathbb{Q})$. Hence $|\sigma a_\tau| = (MD_E)^{1/2}$ at every complex embedding. Take $b_{\mathbf{d}} = u^h \prod_\tau a_\tau^{d_\tau}$.

At a split prime the full correction in Corollary 82.15 is $-\sum_\tau d_\tau v_\iota(\tau\mathfrak{j})$. This cancels $v_\iota(b_{\mathbf{d}})$. At a nonsplit prime the old marks are units and the relative extension is unramified there. Indeed the conductor primes of the auxiliary cyclotomic factor split in K, as in Section 82.8, and base change to K preserves unramifiedness elsewhere. Both integral ideals in $\mathfrak{j}$ are then units at the prime. Consequently the proof of Theorem 84.2 gives the identical bound there. Applying the table inequality at ι and ιc as in Corollary 84.3 proves the finite assertion.

For each σ, the preceding modulus calculation and the unit lemma give

$$\left|\log\left(|\sigma b_{\mathbf{d}}|q_\sigma^{h/2}\right) - \tfrac{h}{2}\left(\log(MD_E) + \mathbb{E}_\rho \log q_\rho\right)\right| \le h/(2(H+1)).$$

Apply this at σ and σ', and add. Substitution of (84.3) proves (84.4). □

Corollary 84.6 (The untwisted norm at the distinguished embedding)**.** *Take $n = 2z$ and $z \ge 2$ in the main-pair range. Let $\phi_{\nu,1}$ be the untwisted high-weight lift scaled by $b_{\mathbf{d}} a_n (\pi/A)^{h/2}$. At the same rank and with the same old column take a degree-zero pair $\eta = \psi \upsilon_1^{-z}$ as in Theorem 82.3, scaled by a_n. At the distinguished complex embedding,*

$$\left|\log \frac{\|\phi_{\nu,1}\|_{\mathrm{cl}}^2}{\|\phi_{\eta,1}\|_{\mathrm{cl}}^2} - \log\dim V_{\mathbf{d}} - \sum_\tau \log(d_\tau!) - h\log(M/\Omega_E)\right| \le 1 + 4f\log\zeta(2). \tag{84.5}$$

Moreover $J_n(\psi,1)H_{\nu,z}T_d\overline{T_d}$ in place of $H_{\nu,z}T_d\overline{T_d}$ still satisfies (84.2) for every prescribed twist.

Proof. In the quotient of the two identities in Theorem 82.1 the scalar $a_{E,n}$, the old-level mass, $\zeta_F(n)$ and $|a_n|^2$ cancel. The remaining factors are $|b_{\mathbf{d}}|^2 q_1^h$, the dimension, $\prod_\tau d_\tau!$, and the ratio of the two partial Hecke Euler products at z. Here $q_1 = \pi/A$ denotes the elliptic factor at the identity embedding in this formula only. For either unitary character the absolute value of the logarithm of the positive partial Euler product is bounded by

$$\sum_{\mathfrak{p}} \sum_{j\ge 1} \frac{(N\mathfrak{p})^{-2j}}{j} = \log\zeta_E(2) \le 2f\log\zeta(2).$$

The last inequality holds prime by prime because there are at most $2f$ primes above a rational prime and each has norm at least that prime. Apply (84.4) at the identity, where $\sigma' = \sigma$, to obtain the stated error. Finally $J_n(\psi,1)$ is a nonzero algebraic integer by Theorem 82.3. Multiplying the common multiplier by it decreases each negative valuation.

The class norm in (84.5) is an algebraic number. At a general embedding its image is a pairing of two conjugated vectors. Extending the norm comparison to that image requires identifying both conjugated theta lifts in the same class metric. □

84.3 Galois conjugation of the symmetric Weyl translates

Take $\sigma \in \mathrm{Aut}(\mathbb{C}/K)$ and the elliptic curve $C = \mathbb{C}/(\delta\mathcal{O}_K)$ of Section 65.3. The coordinate on its universal cover is the integral of the marked algebraic differential. On its conjugate C^σ we use the conjugated differential. We first put both period lattices in the same rational alternating space.

Lemma 84.7 (The conjugated torsion in a norm-one ideal)**.** *There is a fractional ideal $\mathfrak{a}$ of K of norm one and a complex scalar $\delta_\sigma \neq 0$ for which the conjugated lattice is $\delta_\sigma\mathfrak{a}$. Under the uniformizations by these lattices, the map on the rational finite Tate modules induced by σ is multiplication by some $b_\sigma \in \mathbb{A}_{K,f}^\times$. One has*

$$b_\sigma\widehat{\mathcal{O}}_K = \widehat{\mathfrak{a}}, \qquad b_\sigma\bar{b}_\sigma = \chi_{\mathrm{cyc}}(\sigma), \qquad \frac{|\delta_\sigma|^2}{A_\sigma} = \frac{|\delta|^2}{A} = \frac{2}{\sqrt{d_{\mathrm{K}}}}.$$

Here $\chi_{\mathrm{cyc}}(\sigma) \in \widehat{\mathbb{Z}}^\times$ describes the action on all roots of unity.

Proof. The maximal-order CM lattice of C^σ is homothetic to a fractional ideal. Every ideal class of K is a square. Indeed a class of order dividing two has a representative $\mathfrak{l}$ with $\bar{\mathfrak{l}}/\mathfrak{l} = (y)$. Then $y\bar{y} = 1$, since this positive rational generates the unit ideal. Write $y = t/\bar{t}$ by Hilbert's theorem 90. The ideal $t\mathfrak{l}$ is stable under conjugation. In its prime factorization the split pairs give rational ideals and the inert primes are rational ideals. The only ramified prime is $(\sqrt{-d_{\mathrm{K}}})$, since $-d_{\mathrm{K}}$ is the discriminant and d_{K} is prime in this family. The ideal is therefore principal. Squaring on the finite class group is now bijective. If $[\mathfrak{t}]^2$ is the class of the conjugated lattice, take $\mathfrak{a} = \mathfrak{t}/\bar{\mathfrak{t}}$. Its norm is one. The area of $\mathcal{O}_K$ is $\sqrt{d_{\mathrm{K}}}/2$, giving the area identity.

The finite Tate modules, in the period coordinates just chosen, are $\widehat{\mathcal{O}}_K$ and $\widehat{\mathfrak{a}}$. The action on torsion commutes with the $\mathcal{O}_K$-action, since a CM endomorphism has its prescribed tangent multiplier in K, fixed by σ. It therefore acts by an idele on the rank-one rational Tate modules and identifies the two integral modules. Both principal polarizations are given in these coordinates by the same alternating form, namely the imaginary part of $x\bar{y}$ times $2/\sqrt{d_{\mathrm{K}}}$ with a common orientation. On p^r-division points the Weil pairing is the exponential of this integral pairing divided by p^r on representatives in the Tate lattice. It commutes with field embeddings, as follows also from its definition by evaluating the functions of the two torsion divisors. Substituting $b_\sigma x, b_\sigma y$ multiplies the alternating form by $b_\sigma\bar{b}_\sigma$. Perfectness modulo p^r therefore gives $b_{\sigma,p}\bar{b}_{\sigma,p} \equiv \chi_{\mathrm{cyc,p}}(\sigma) \bmod p^r$ for every r. This proves the idele identity. For the Tate-module and polarization conventions see B. Conrad, *Main theorem of complex multiplication*, Sections 1 and 6. □

For the even rank n use $D = fn$ product coordinates in Lemma 65.8. Write $W_f = \mathbb{A}_{K,f}^D$. Let $R(x)$ be the symmetric finite Weyl translation of Theorem 67.8. Its multiplication law has scalar $\psi_f(B(x,y)/2)$, where B is the rational alternating form in that theorem and ψ_f is the finite additive character. In the marked model, $R(x)$ at a rational vector $x \in K^D$ acts by $U_{\delta x}$ in the product variables. This follows by canceling the common factor $e^{\pi i h_K(x,x)}$ in the translation identity in the proof of Theorem 67.8. Write Θ for the product of the marked elliptic sections, normalized to have value one at zero. The space of finite vectors is spanned by $R(x)\Theta$. Use $R'(x)$ for the corresponding translation in the model with $\delta_\sigma, \mathfrak{a}, \Theta^\sigma$. Its rational action is $U_{\delta_\sigma x}$. The rational alternating form and the multiplication law are the same by the area identity. To obtain the full finite model from the conjugate product, use one translate per coset modulo $\widehat{\mathfrak{a}}^D$. These translates are independent by their distinct commutator characters on this self-dual lattice. They span the smooth model by finite Fourier expansion, exactly as in the proof of Theorem 67.8. The Weyl action on each finite vector is locally constant.

Theorem 84.8 (Exact conjugation of symmetric Weyl translations)**.** *Coefficientwise conjugation of the Taylor series in the product differentials gives a σ-semilinear bijection L_σ of the two smooth finite models. It sends Θ to Θ^σ, and for every $x \in W_f$ and finite vector G,*

$$L_\sigma(R(x)G) = R'(b_\sigma x)L_\sigma G. \tag{84.6}$$

In the matrix coordinates $(Z_{\tau i})$ this statement uses the permutation $\tau \mapsto \sigma\tau$ of the oriented embeddings. The same map applies to every homogeneous degree.

Proof. For a positive integer N sufficiently divisible for a finite list of vectors, the proof of Theorem 65.9 writes each in the form

$$G(z) = \Theta(z)S([z/N]), \qquad S \in \mathbb{C}(C^D).$$

Indeed for a rational torsion translate $U_w\Theta$ the ratio to Θ, pulled back by $z \mapsto Nz$, is rational by Theorem 65.6. We allow arbitrary complex coefficients in the finite linear span. Substitution gives the Taylor series of $\Theta^\sigma(z)S^\sigma([z/N])$ after coefficientwise conjugation. The local differential parameters are functorial by the formal integration calculation in Theorem 65.6. When the quotient has a pole at zero this identity is an identity in the fraction field of the formal local ring, with the product regular. By that theorem each individual translated product becomes a root of unity times a translate in the conjugate model. Thus the conjugated series represents a finite vector. It determines that vector uniquely by holomorphic continuation. This constructs L_σ, proves its injectivity, and identifies its value on Θ by Theorem 65.5.

We show first that the two sides of (84.6) are proportional by a scalar $\epsilon(x)$ independent of G. Include Θ in the list containing G. By density in the finite adeles choose a rational $a \in K^D$ arbitrarily close to x. Choose a rational a' arbitrarily close to $b_\sigma a$. These choices can be made so that $R(a)$ and $R(x)$ agree on the list, and $R'(a')$ and $R'(b_\sigma x)$ agree on its images. They can also be made to satisfy

$$[\delta_\sigma a'/N] = \sigma([\delta a/N]) \quad \text{on } (C^\sigma)^D.$$

In fact this last condition is exactly $a' - b_\sigma a \in N\widehat{\mathfrak{a}}^D$ by Lemma 84.7, an open condition. In $R(a)G$ the rational function S is evaluated at $[z/N] + [\delta a/N]$. Conjugating it gives precisely the argument in $R'(a')L_\sigma G$. The remaining factor is the conjugate of $U_{\delta a}\Theta$, proportional to $U_{\delta_\sigma a'}\Theta^\sigma$ by Theorem 65.6 in each coordinate. Its proportionality factor is determined by evaluating this comparison as an identity of sections for $G = \Theta$. That section is nonzero. Thus for the given x one scalar works for every G.

By the idele identity of Lemma 84.7,

$$\sigma(\psi_f(B(x,y)/2)) = \psi_f(B(b_\sigma x, b_\sigma y)/2).$$

Applying the two Weyl multiplication laws to x, y therefore gives $\epsilon(x+y) = \epsilon(x)\epsilon(y)$. The reflection $I : G(z) \mapsto G(-z)$ commutes with coefficientwise conjugation. It acts on both finite models: the marked elliptic section is even since its symmetric line has a unique normalized section with value one at zero, and reflection interchanges its translates at opposite points. The formulas for symmetric translation give $IR(x)I = R(-x)$ and the corresponding identity for R'. It follows that $\epsilon(-x) = \epsilon(x)$. Each value has square one. Since every x in the additive finite adelic space is twice $x/2$, $\epsilon(x) = \epsilon(x/2)^2 = 1$. This proves the exact identity, including above 2. Its range contains all translates of Θ^σ, since b_σ is invertible. These span the target.

Finally the matrix taking product to matrix coordinates has columns $(\tau(v_i))_\tau$ for vectors of the K-orthonormal basis in Lemma 65.8. Conjugating its entries permutes the rows as stated. The inverse coordinate change also transports because $E/\mathbb{Q}$ is abelian. Taking homogeneous parts of the same series proves the last assertion. □

84.4 The untwisted theta norm at all complex embeddings

Keep the two lifts and the field of Corollary 84.6. They have the same even rank $n = 2z$ and the same point column at every old mark. Write F_μ for the class sum with $S_{\mu,z} = 1$ in Theorem 82.1, with no new places. Thus F_μ uses the unit real vacuum and the one-row coefficient representation for μ.

For an automorphism σ fixing K define μ^σ at finite ideles of the norm-one torus by applying σ to their character values. This is again a unitary automorphic character, with the weights permuted by $\tau \mapsto \sigma\tau$. Indeed, modulo a compact open kernel and rational torus points the finite class group is finite. A positive power of any finite idele is a rational norm-one element times a member of the kernel. Its character value to this power is a product of integral powers of CM embeddings of that rational element. All conjugates have modulus one. Taking roots proves algebraicity and unit modulus for every conjugate of every finite value. The finite kernel is preserved. The inverse archimedean character on a rational point is permuted in precisely the stated way, which proves automorphy with these infinite weights. It also gives $\mu^{c\sigma c} = \mu^\sigma$: at finite places $\overline{\sigma(\overline{\mu(g)})} = 1/\overline{\sigma(\mu(g))} = \sigma(\mu(g))$, and the two permutations of the CM embeddings agree since $E/\mathbb{Q}$ is abelian.

Lemma 84.9 (Transport of the marked finite test)**.** *Identify the source and target models in Theorem 84.8 with the same abstract finite oscillator representation. For the common old point test Φ there are a scalar $\ell_\sigma \neq 0$ and a finite norm-one idele u such that $L_\sigma\Phi = \ell_\sigma D_u\Phi$. Here D_u is normalized change of position by u at split places and is the identity elsewhere. For split-supported source and target representatives g, y, away from the marks, let A act on positions by $X \mapsto yXg^{-1}$, and let D_A denote normalized change of position without a splitting character. Then*

$$L_\sigma(D_A\Phi) = \ell_\sigma \frac{\sigma(t_y)}{t_y}\, D_u D_A\Phi, \qquad t_y = \prod_{v \text{ moving}} |\det y_v|_v^{-1/2}. \tag{84.7}$$

The constants u, ℓ_σ are independent of the character, the real degree and the representatives.

Proof. The identification with the finite representation can be made by the theta realization in the proof of Theorem 67.8. For the conjugate product use the product-period vector for $\widehat{\mathfrak{a}}^D$ with its symmetric period character. This is a line in the same finite oscillator space: in a lattice model a prescribed period character of a self-dual lattice occurs once, by expanding in the translates on its cosets, or by the normalized lattice intertwiner and its translates. The product section and this vector are identified up to a common nonzero scalar. The action on rational translations is $U_{\delta_\sigma x}$ by rational invariance of the theta sum, and agrees on finite translations by density. This proves the asserted identification in the same symmetric Weyl convention.

Regard b_σ of Lemma 84.7 as an idele of E. Write $b_\sigma = ua$, with $u\bar{u} = 1$ and $a \in \widehat{\mathcal{O}}_E^\times$, choosing $a = 1$ on the position member of every old split pair. Indeed, at each nonsplit place b_σ is a unit because its norm is a unit. Take $u = 1$ there. At an old split pair take for u the position component of b_σ and its conjugate inverse on the other member. Make the same choice at the finitely many other split pairs at which b_σ has nonzero valuation, and set $u = 1$ elsewhere. Then $a\bar{a} = \chi_{\mathrm{cyc}}(\sigma)$. Put $L^0 = D_u^{-1}L_\sigma$. Conjugation of a symmetric Weyl operator by D_u multiplies its argument by u: on positions this is change of variables and on frequencies it is its inverse transpose. Thus $L^0R(x) = R(ax)L^0$.

Let ϕ_0 be the standard unmarked finite vacuum. Its period lattice $\widehat{M}$ is stable under a. In the symmetric convention its period character, on this lattice, is the sign given at 2 by the parity of $h_K(m, m)$, as in Lemmas 65.8 and 65.12. This formula on the completion follows by

continuity from its rational periods. The local trace norm of am at 2 is $\chi_{\mathrm{cyc},2}(\sigma)h_K(m,m)$, with an odd unit as multiplier. Thus the sign is fixed. The period line occurs once and L^0 is injective. Consequently $L^0\phi_0 = \ell_\sigma\phi_0$ with $\ell_\sigma \neq 0$.

Here is the calculation on a finite set of split positions. Trivialize the local position model so that the original box vector is the indicator of a lattice U, retaining its original scalar normalization in every subsequent vector. For $L = \varpi^j U \subseteq U$, the indicator of L is the box multiplied by

$$[U : L]^{-1} \sum_{\eta \in L^\perp/U^\perp} \psi_f(\langle X, \eta\rangle).$$

This is the finite-character orthogonality formula on U/L. Under L^0 the frequency scalar a on the dual member permutes this quotient. On the position member a translate by t goes to a translate by at. Any lattice $A_v U$ is a union of cosets of some such L, and multiplication by the scalar unit a permutes these cosets. At an old mark $a = 1$ on positions, so the specified individual coset modulo ϖU is fixed. Apply these expansions to all the finitely many factors of the one global box. They show that the corresponding unnormalized product of indicators is fixed up to the common ℓ_σ. The normalized change D_A adds $\prod_v |\det A_v|_v^{-1/2} = t_y \prod_v |g_v|_v^{n/2}$. The second factor is rational since n is even. Conjugating the first gives exactly the quotient in (84.7). The identity before moving the representatives gives the formula for Φ. □

Theorem 84.10 (The conjugates of the high-weight to flat norm ratio)**.** *Let $N_\nu = \|\phi_{\nu,1}\|_{\mathrm{cl}}^2$ and $N_\eta = \|\phi_{\eta,1}\|_{\mathrm{cl}}^2$ with the common rank and the scalings of Corollary 84.6. For every embedding of these algebraic numbers in $\mathbb{C}$,*

$$\left|\log|\sigma(N_\nu/N_\eta)| - \log\dim V_{\mathbf{d}} - \sum_\tau \log(d_\tau!) - h\log(M/\Omega_E)\right| \leq 1 + 4f\log\zeta(2). \tag{84.8}$$

Proof. Extend the embedding to an automorphism of $\mathbb{C}$. Composing on the left with complex conjugation preserves absolute values. We may thus suppose σ fixes K. Write $\sigma' = c\sigma c$. Choose the same split-supported representatives for source and target when computing the two norms and their conjugates, by Theorem 65.11. Avoid the old marks, all ramified places, and the support of the ideles u for both automorphisms. The original class weights are rational reciprocals of stabilizer orders.

In these representatives the finite part of the untwisted kernel is the test $D_A\Phi$ multiplied by $\mu_f(g)^{-1}\delta_n(y)$, where $\delta_n(y)$ is the intrinsic unitary target splitting factor. To verify the source factor, at a split pair the restrictions of the splitting character to the two members are inverse since its restriction to the diagonal is trivial. Its $n = 2z$ position factors on a source scalar give υ_1^z. The external factor is $(\mu\upsilon_1^z)^{-1}$, giving the asserted cancellation as in Theorem 82.1. The same identity on split units follows from the same character restriction. There is no external target factor. Put $r_\sigma(y) = \sigma(t_y\delta_n(y))/(t_y\delta_n(y))$. The product $t_y^2\delta_n(y)\overline{\delta_n(y)} = t_y^2$ is rational, so

$$r_\sigma(y)\overline{r_{\sigma'}(y)} = 1. \tag{84.9}$$

After the transport in Lemma 84.9, the extra operator D_u is absorbed by sending g to gu^{-1}: its position matrix multiplies yXg^{-1} by u. The change of variables is normalized in both expressions and the source character contributes one fixed unit-modulus factor, independent of g, y. This identity is valid also on the split support of u, where the other representatives are the identity. Alternatively it is the normalized central translation of Lemma 50.2. Right translation permutes the torus classes and preserves their counting measure since the source is abelian. We have therefore obtained the actual class sum for μ^σ at y.

We spell out the real and scalar factors. The original $a_n q_{\mathrm{id}}^{h/2}$ converts each unscaled coefficient to the homogeneous derivative of the marked section by Theorem 67.8, where $q_{\mathrm{id}} = \pi/A$. Theorem 84.8 takes this derivative to the corresponding derivative in the conjugate differential, with the oriented labels permuted. In that differential the same theta-functional calculation converts the derivative back to the unit oscillator coefficient times $C_{\sigma,n}^{-1} q_\sigma^{h/2}$. Indeed rational Weyl translations intertwine the two realizations and the product period line is one dimensional, fixing $C_{\sigma,n} \neq 0$ already in degree zero. Differentiation of $\exp(q_\sigma \bar{w}\cdot Z)$ gives exactly the indicated power for $P(\sqrt{q_\sigma}\bar{w})$. In the rational matrix coordinates $w = \delta_\sigma \Phi(X)$, the Gaussian scale $q_\sigma |\delta_\sigma|^2$ is constant by Lemma 84.7. Thus the unit polynomial changes from the original realization only by a degree-dependent scalar of modulus one and the permutation of the oriented indices. All components of the vector are covered by this calculation: a symmetric tensor is obtained by polarization of a pure power, and the symmetrizing projector averages permutations with rational coefficients.

It follows that the vector of conjugated high-weight class values at y equals the permuted vector of $F_{\nu^\sigma}(y)$ times

$$(\sigma b_{\mathbf{d}}) q_\sigma^{h/2} C_{\sigma,n}^{-1} \ell_\sigma \, \epsilon_{\sigma,\nu} r_\sigma(y), \qquad |\epsilon_{\sigma,\nu}| = 1.$$

For the flat vector the identical formula holds with $b = 1$ and $h = 0$, with the same $C_{\sigma,n}^{-1}\ell_\sigma$. Apply this description also to σ'. In a monomial basis the symmetric tensor metric has rational Gram matrix: it is induced on symmetrized tensors by the orthonormal tensor basis, with the pure highest tensor of norm one. Tensoring over τ gives the same metric after permuting the places. In the class pairing $\sigma(\bar{v}) = \overline{\sigma'(v)}$. Both vectors then have the same character μ^σ and the same labels. Equation (84.9) cancels the target-dependent factors at each y. The two products of the constants $C^{-1}\ell$ cancel upon division by the flat norm, and the two remaining phases have modulus one. Consequently

$$|\sigma(N_\nu/N_\eta)| = |\sigma b_{\mathbf{d}}| \, |\sigma' b_{\mathbf{d}}| \, (q_\sigma q_{\sigma'})^{h/2} \, \frac{\|F_{\nu^\sigma}\|_{\mathrm{cl}}^2}{\|F_{\eta^\sigma}\|_{\mathrm{cl}}^2}.$$

The norms on the right are positive by Theorem 82.1. In their quotient the rank and level factors cancel. The dimension and the factorials are unchanged by permutation, and the logarithm of the ratio of the partial Euler products is bounded in absolute value by $4f \log \zeta(2)$, as in Corollary 84.6. Insert (84.4) to obtain (84.8). □

84.5 A signed bound for the terminal determinant ratio

Fix a node x and the simultaneous family of twists of Corollaries 52.21 and 73.23. For nonzero algebraic U independent of the coefficient embedding, and a family $Y = (Y_{p,\iota})$ of nonzero algebraic numbers, write

$$\mathcal{H}_x(Y;U) = \sup_{\sigma} \log|\sigma U| + \sum_{p} \log p \sup_{\iota \mid p} \max(0, -v_\iota(UY_{p,\iota})).$$

A sum of nonnegative terms denotes the supremum over finite subsums. The first term uses the ordinary logarithm. For two such data, $\mathcal{H}_x(YY';UU') \le \mathcal{H}_x(Y;U)+\mathcal{H}_x(Y';U')$, by the triangle inequality for positive parts and the product law for absolute values. The same observation applies to any nonnegative integral power. A family of powers of the new conductor has $\mathcal{H}_x(\,\cdot\,;1) = 0$, since the conductor of the twist attached to p, ι is prime to p.

For the two positive integral weights κ_τ chosen in Section 82.8 put

$$J_x = \sum_{\tau} \big(\log(\kappa_\tau!) + \kappa_\tau \log(M/\Omega_E)\big),$$
$$\mathcal{A}_x = J_x - 2\sum_{p} S_p \log p. \tag{84.10}$$

This last sum is finite. More explicitly its subtracted term is

$$2\sum_{p} S_p \log p = \sum_{\tau} \sum_{\substack{1 \le r \le x b_x \\ (r,q_1)=1}} \Lambda(r)(1-\xi(r))\big(\kappa_\tau/r - \kappa_{[-r]\tau}\big)_+. \tag{84.11}$$

Here Λ is the von Mangoldt function, ξ is the quadratic character of K, and $[-r]$ multiplies the embedding label modulo q_1. In fact, for $r = p^a$ the coefficient $1-\xi(r)$ is two for an odd inert power, one for every ramified power and zero otherwise. These are exactly $2/e_0$ on the allowed exponents of Section 82.8. All primes dividing q_1 split in K. No term with $p^a > b_x$ contributes because each κ_τ is at most b_x and at least one.

Proposition 84.11 (A common multiplier for a main critical ratio)**.** *For each main critical pair, with its prescribed twist power $\beta = \beta_{p,\iota}$, there is a nonzero algebraic $U_{\nu,z}$, independent of p, ι, β, such that*

$$\mathcal{H}_x\big((R_{\nu,z}(\beta)^{-1});U_{\nu,z}\big) \le -\mathcal{A}_x + C_{q_0,d_{\mathrm{K}}} B_x^{\det}.$$

This holds at every node of every sufficiently distant shorter interval and for all main pairs there.

Proof. Use the high-weight normalization of Theorem 84.5 and any one flat comparison pair of rank $n = 2z$ in Theorem 84.10. Denote its tame character by ψ. With the constants of Section 82.4 take

$$U_{\nu,z} = J_n(\psi,1)\, H_{\nu,z}\, T_d \overline{T_d}.$$

All factors are nonzero and independent of the twist. Since $J_n(\psi,1) = B_n b_n^2 N_\eta$ and $H_{\nu,z} = c_*^2/(|C|^4 N_\nu)$, Theorem 84.10 gives

$$\sup_{\sigma} \log|\sigma U_{\nu,z}| \le -J_x + C_{q_0,d_{\mathrm{K}}}\big[x^{70}(1+\log b_x) + f(1+b_x/k)\log(2b_x)\big].$$

Here are the bounds on the factors. The representation dimension is at least one, $d_\tau \ge \kappa_\tau$ by Theorem 84.2, and $M/\Omega_E \ge 1$ eventually since K is fixed, $\mathrm{rd}(E) \ge 1$ and $M \ge b_x^{20}$. These facts allow the factorials and linear terms to be replaced in the required direction. The finite integer $|C| \ge 1$ only improves the upper bound. The bound for T_d and its conjugate is in

Corollary 84.3. The factor c_* satisfies the bound with x^{70} by Corollary 82.4 and Theorem 82.3. Finally $\log B_n \le (2fn)^2 \log 3$ and $\log b_n = fn \log(Mt_E)$, by definition. Here $fn \ll x^{17}$, $\log t_E \ll \log(2x)$ and $\log M \le 22 \log b_x$, as in the proof of the same theorem. All these costs, including the Euler product error, fit in the displayed bound. The bracket is $o(B_x^{\det})$ uniformly: divide by $fxb_x(\log b_x)^3/(mk)$ and use $b_x \ge x^{400}$, $m \ll \sqrt{x}$, $k \ll x^5$.

By Corollaries 84.3 and 84.6,

$$-v_\iota(U_{\nu,z}\rho_\beta^2 R_{\nu,z}(\beta)^{-1}) \le p_{p,\iota} + 2S_p.$$

The sum of the logarithmic cohomology bounds $\sum_p (\log p) \sup_\iota p_{p,\iota}$ is at most $B_x^{\det}$ by Corollary 52.21, using its common bound on all integral torus tests. For the four-row factor defining ρ_β, take ψ_2 with the original prescribed tame character of order Q as allowed in Section 82.4. Its critical integer is 2 and its absolute angular weight is 2. Its character is ramified at both old marks, and the new twist powers are ramified at the new marks. Thus the incomplete Euler ratio in Theorem 82.3 is the R_J of Theorem 13.13 for this very twist, with inverse character conventions giving the same positive absolute value. That theorem gives $\mathcal{H}_x((\rho_\beta^{-1}); 1) \le 4 \log 2 + 1$ using total tolerance at most one in the simultaneous choice. Multiplying by ρ_β^{-2}, summing the bounds, and using (84.10) proves the claim. □

Theorem 84.12 (The signed estimate for the terminal class determinant). *Let $\mathfrak{Z}_\vartheta$ be the actual terminal class Gram determinant of Corollary 81.21, and let D_x be its source dimension. On the same shorter interval, with the same fields, weights and simultaneous twists, there is a common nonzero algebraic U_x such that*

$$\mathcal{H}_x\Big((\mathfrak{Z}_1/\mathfrak{Z}_{\vartheta_{p,\iota}}); U_x\Big) \le -D_x mk\, \mathcal{A}_x + C_{q_0,d_{\mathrm{K}}} D_x mk B_x^{\det}. \tag{84.12}$$

In dimension zero both determinants and U_x equal one.

Proof. Assume $D_x \ge 1$ and fix a lift index j. In $\mathcal{L}_j(\beta)/\mathcal{L}_j(1)$ from Section 81.8 there are three products of absolute finite Hecke values, each squared, of lengths $m/2, m, (k+7m)/2$. The remaining factor is a power of the new conductor, a unit at the coefficient prime. On the third product take $7m/2 < z \le (k+7m)/2$. The $k/2$ resulting ratios are the main ratios of the preceding proposition. Indeed the third character is $\alpha^{-1} \upsilon_1^{3m+(k+7m)/2}$; the factor α changes by β and its tame part at the identity twist is χ_M^j. Here one fixes the original character extension first and multiplies by the prescribed twist, as in Section 77.13. Passing to the inverse character gives the negative-weight convention of Section 82.1. Its angular weights are the ones computed in Section 82.8.

There remain $m/2 + m + 7m/2 = 5m$ ratios. In the first two products the characters are $c\upsilon_1^{m/2}$ and $c^{-1}\upsilon_1^m$. Their absolute angular weights are $m/2, m$, since υ_1 has circle weight one. In the third the angular weight at every place is at least $(k+7m)/2$ by the main-pair window. Hence every remaining integer is positive and at most its absolute angular weight. The old tame parts in these three products are $\chi_M^{\pm j}$, nontrivial at each old pair since the prescribed order $Q > m$, and the twist powers are among those in Theorem 13.13. Old ramified places contribute Euler factor one both before and after the twist. At the new pairs the numerator is ramified and the denominator has its original local factors. Thus each is exactly one of that theorem's critical ratios. Applying the theorem with the simultaneous list of both signs costs at most $4 \log 2 + 1$ for each inverse ratio with multiplier one. Squaring the products now gives

$$\mathcal{H}_x((\mathcal{L}_j(1)/\mathcal{L}_j(\beta)); U_j) \le -k\mathcal{A}_x + C_{q_0,d_{\mathrm{K}}} k B_x^{\det} + 10m(4\log 2 + 1), \qquad U_j = \prod_{7m/2 < z \le (k+7m)/2} U_{\nu,z}^2.$$

In Corollary 81.21 use tolerance at most one. Its family y is the terminal ratio times $\prod_j(\mathcal{L}_j(\vartheta^j)/\mathcal{L}_j(1))^{D_x}$. Its multiplier is $H_x=\prod_j\Delta_j(1)$. The two estimates of that corollary imply $\mathcal{H}_x(y;H_x)\ll_{q_0,d_{\mathrm{K}}} D_xmkB_x^{\det}$: for the archimedean term use $fm^2b_x\log b_x/(mkB_x^{\det})=m^2/(x(\log b_x)^2)$, and for the finite term use $s=k+m-1\ll k$. Multiply y by the m inverse families just bounded, each to power D_x. With $U_x=H_x\prod_j U_j^{D_x}$, subadditivity gives (84.12), since $m\ll k$ and $B_x^{\det}\to\infty$. For $D_x=0$ the empty determinants give the assertion directly. □

The estimate here is one side of a comparison. A bound on the other side for the same terminal lattice is needed to turn (84.12) into the prime-sum inequality of Theorem 1.2.

84.6 A product formula for full and partial terminal frames

Use the terminal map from the common good source of dimension $d = D_x$, with the metric of Proposition 73.10. Write $A_\vartheta(Y)$ for the matrix with columns the terminal values of the lifts of a tuple Y, with rows indexed by a finite class and a tensor coordinate of its compact representation. The terminal rank is $s + mt$ and the class set depends on the twist.

Lemma 84.13 (Integral pairings on the actual terminal classes)**.** *Every stabilizer in the terminal class set is trivial. In tensor coordinates the class Gram matrix of a tuple Y is $A_\vartheta(Y)^* A_\vartheta(Y)$. The local pairing of a Weyl vector integral at ι with a Weyl vector integral at ιc, in the corresponding slots, is integral.*

Proof. At an original mark $v \mid p_0$ the terminal level imposes identity reduction, as specified immediately before Proposition 73.10. Here $p_0 \geq 3$ splits completely in E. A conjugate of the local level lies in a conjugate of $1 + p_0 M_{s+mt}(\mathbb{Z}_{p_0})$. This group is torsion free. In fact a nonidentity element has the form $1 + p_0^a B$, $a \geq 1$, with some unit entry of B. Taking a positive power prime to p_0 preserves the least entry valuation a of its difference from the identity. Taking the p_0-th power increases it to $a + 1$ by the binomial formula and oddness of p_0. No positive power can be the identity. The finite rational class stabilizer injects into this local matrix group, so it is trivial. In the class metric each class therefore has weight one.

For each column height i, the coordinate wedges of the standard orthonormal basis are an orthonormal basis of $\bigwedge^i$. Include the coefficient representation in the tensor of these exterior powers, with the required determinant twist if necessary. In the dual flag-minor construction of Lemma 73.26, taking products of flag minors gives a surjection over $\mathbb{Z}$ from the tensor of the column-coordinate modules onto their product span. Its dual embeds the Weyl lattice integrally in the dual tensor. The extremal vector has a single unit coordinate, since only the choice of the top minor in each column gives that extremal monomial. Thus the restricted tensor metric has the prescribed highest vector of norm one and equals the class coefficient metric by irreducibility. Determinant lines and their duals have the same unit normalization. Tensor the construction over all real places. At a finite place the two local frames in a Hermitian contraction are dual: a change by a unitary matrix on the rational space acts by the inverse transpose on the conjugate slot. Alternatively choose all class representatives with identity component above the coefficient prime by Theorem 65.11. At that prime the ordinary integral frames suffice. The coefficient pairing in the ambient tensor has identity matrix, so summing over the classes proves the two assertions. □

We specify the hypotheses on columns for the product formula. At any coefficient prime use the same twist $\vartheta = \vartheta_{p,\iota}$ in the two slots labelled by $\iota, \iota c$. The local evaluations may be made on a number field containing the coordinates when they are algebraic. More generally one can extend an embedding to a field isomorphism $\mathbb{C} \to \overline{\mathbb{Q}}_p$ and evaluate there: both algebraically closed fields have characteristic zero and transcendence degree the continuum over $\mathbb{Q}$, so an extension is obtained by matching transcendence bases. Indeed $\overline{\mathbb{Q}}_p$ has the cardinality of the continuum and the algebraic closure of a field generated over a countable field by an infinite set has cardinality the maximum of that cardinality and the countable cardinal. All hypotheses on local evaluations below are required at every such embedding. For the ordinary archimedean norm of a matrix or vector of values, σA means applying σ to the entries in the original tensor coordinates, with the same fixed representatives. For a rectangular complex matrix A with n columns denote its squared exterior norm by $V(A) = \det(A^* A)$, including $V(\varnothing) = 1$.

Theorem 84.14 (Covolume and index of a completed terminal frame)**.** *Let $(Y,Z)=(y_1,\dots,y_n,z_1,\dots,z_r)$ be any basis of the common good source, so $n+r=d>0$. Suppose that*

$$\sup_{\sigma}\log V(\sigma A_1(Y))\le c_0,\qquad \sup_{\sigma,i}\log\|\sigma A_1(z_i)\|^2\le b_0.$$

For each p,ι suppose integral square matrices on the n columns, separately at ι and ιc, take the twisted lifts of Y to integral vectors. Suppose their determinants have valuation at most $e_p\ge 0$, with $\sum_p e_p\log p\le\delta$. On each of the r additional columns let a common scalar $\gamma_\vartheta\neq 0$ make its twisted lift integral at both evaluations. Suppose $\gamma_\vartheta\overline{\gamma_\vartheta}$ is algebraic and that for some fixed nonzero algebraic V_0'

$$\mathcal{H}_x(((\gamma_\vartheta\overline{\gamma_\vartheta})^{-1});V_0')\le b_1.$$

For $r=0$ omit all assumptions on these additional columns. On the shorter intervals of Theorem 84.12,

$$c_0+2\delta+r(b_0+b_1)\ \ge\ dmk\,\mathcal{A}_x-C_{q_0,d_{\mathrm{K}}}dmkB_x^{\det}. \tag{84.13}$$

Proof. Set $G_\vartheta=V(A_\vartheta(Y,Z))>0$. Positivity follows from injectivity of the row map on the good source, proved in Proposition 73.21. A change of basis on the full source multiplies both determinants by the same squared modulus, so $G_1/G_\vartheta=\mathfrak{Z}_1/\mathfrak{Z}_\vartheta$. We first check algebraicity of these individual determinants under the hypotheses. With $\sigma'=c\sigma c$, σG_1 is the mixed exterior pairing of the columns of $\sigma' A_1$ with those of σA_1. The Cauchy–Schwarz inequality on the exterior power bounds it by $\sqrt{V(\sigma' A_1)V(\sigma A_1)}$. Each factor $V(\sigma A_1(Y,Z))$ is at most $V(\sigma A_1(Y))\prod_i\|\sigma A_1(z_i)\|^2$: project the successive additional columns orthogonally off the preceding ones in the Gram determinant formula. Hence

$$\sup_{\sigma}\log|\sigma G_1|\le c_0+rb_0.$$

A transcendental number has unbounded orbit under field automorphisms of $\mathbb{C}$, since on a transcendence basis containing it one may multiply it by any positive integer and extend to the algebraic closure. Thus G_1 is algebraic, and so is G_ϑ by the algebraic ratio of Theorem 84.12.

At a fixed p,ι, use the two clearing matrices on Y. Multiply each remaining column by γ_ϑ in both evaluations. Lemma 84.13 makes every entry of the mixed Gram matrix integral. Taking its determinant costs at most $2e_p$ from the two clearing matrices. More precisely,

$$-v_\iota((\gamma_\vartheta\overline{\gamma_\vartheta})^r G_\vartheta)\le 2e_p.$$

In the conjugate slot one uses the values of ιc, so scalar multiplication contributes exactly $\iota(\overline{\gamma_\vartheta})$. This calculation uses the same actual twist and class representatives in the two slots, also when their clearing matrices differ.

The constant family G_1 is the product of this last family, the r-th power of $(\gamma_\vartheta\overline{\gamma_\vartheta})^{-1}$, and the terminal ratio. Subadditivity from Section 84.5 bounds its $\mathcal{H}_x$ with multiplier $U=U_x(V_0')^r$ by

$$2\delta+rb_1-dmk\,\mathcal{A}_x+C_{q_0,d_{\mathrm{K}}}dmkB_x^{\det}.$$

For any nonzero algebraic α, the product formula applied to $U\alpha$ gives

$$0\le\sup_{\sigma}\log|\sigma\alpha|+\mathcal{H}_x((\alpha);U).$$

Indeed the average archimedean logarithm of $U\alpha$ is bounded by the two separate suprema and the negative finite valuations in the formula by the displayed positive parts, with the normalized local degree weights summing to one above each rational prime. Apply this inequality to G_1 and use its archimedean bound to obtain (84.13). □

In particular any full frame whose twisted terminal columns are integral in the two slots at all the prescribed places has $\sup_{\sigma} \log V(\sigma A_{1}(Y))$ at least the right side of (84.13). For a partial frame the error is explicitly proportional to the number of columns used to complete it. Applying this inequality in the zero criterion still requires independent columns with simultaneously small upper bounds for their untwisted norms and their local indices, in positive dimension on the same nodes.

85 References

[1] H. Oh, *Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants*, Duke Math. J. **113** (2002), 133–192. Theorem 1.1 and Corollary 1.5.