The generalized Riemann hypothesis for Dirichlet L-functions

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Volume XV

Definite theta kernels and the three global exchanges

Abstract

We construct the definite class kernels and their periods. For each of the three theta exchanges we compare the local test factors, the relevant automorphic lines and the normalization of the global integral.

The guide in Volume I lists all twenty-seven volumes.

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77 Definite periods and the carrier integral

77.1 The periods on the complementary definite unitary group

We determine which functions on $U(W_j)$ are detected by the second factor of Corollary 73.25. Fix a node of the same interval and $1 \le j \le m$. Put

$$G = U(W_j), \qquad H = U(V^\circ), \qquad a = k + j - 1, \quad b = m - j, \quad N = \dim_E V^\circ = 5s - 9m - 1.$$

Thus $a + b = s$, $G(F_\infty) = \prod_\tau U(a, b)$ and $H(F_\infty) = \prod_\tau U(N)$. Write $[G] = G(F)\backslash G(\mathbb{A}_F)$, and likewise for H. Let J be the product of the source principal subgroups modulo π_v^2 at the old marks and the source lattice stabilizers elsewhere. Define J_H by the same rule on the target. Normalize each finite lattice stabilizer to have mass one. We use the bilinear lift $\Theta u(y) = \int_{[G]} u(g)\theta(g, y)\, dg$, and take Hermitian inner products linear in their second argument.

In the negative source columns order the coordinates by the numbering used for the minors in Section 73.10. At τ write $c_{h,\tau}$ for the number of negative columns of height h there. Only $h \le b$ can occur. Indeed, in the notation of Theorem 73.27, when $b < h < s$ one has $\sigma_{s-h} \ge H_0 - w_j$, where $H_0 = s^8$, and the other endpoint is at most $-w_j - (t-1)/2 < H_0 - w_j$. At heights $h \ge s$ the count is zero. After choosing an ordering of the target rows take the Fock polynomial $Q_\tau = \prod_h \Delta_h(Z_-)^{c_{h,\tau}}$, where Δ_h is the leading h-rowed square minor and Z_- consists of the N-by-b negative variables. It is nonzero at a matrix with an identity in these columns. Write v for the tensor of the corresponding Fock vectors, normalized to length one. Take the source and target circle characters of this pair to be respectively υ^{m-j} and $\upsilon^{m/2-j}$, with the rank normalization of the Weil action used above. They have trivial tame part. At each old mark take the indicator of $X \equiv z \bmod \pi_v^2$ on the integral N-by-s matrices, and elsewhere take the unit lattice vacuum. These are precisely the data of the complementary factor for both choices of twist.

Lemma 77.1 (The real projection selected by the negative columns)**.** *Put $\delta = j - m/2$. Write $g_\tau \in U(a, b)$ in blocks $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in positive and negative coordinates, and set $S_\tau = D - CA^{-1}B = (D^*)^{-1}$. The normalized oscillator coefficient on the source is*

$$K_\infty(g) = \langle v, \omega(g, 1)v \rangle = \prod_\tau (\det g_\tau)^\delta (\det S_\tau)^N \prod_{h=1}^{b} \Delta_h(S_\tau)^{c_{h,\tau}}.$$

An empty determinant is one. This continuous function is in $L^1(G(F_\infty)) \cap L^2(G(F_\infty))$. Put $d_\infty = \int |K_\infty(h)|^2\, dh > 0$. Then

$$P_\infty u(x) = d_\infty^{-1} \int_{G(F_\infty)} K_\infty(h) u(xh)\, dh$$

is an orthogonal projection on $L^2([G])$.

Proof. Here A, D are invertible by the defining unitary relations. Block inversion of g_τ, using $g_\tau^{-1} = \operatorname{diag}(1, -1) g_\tau^* \operatorname{diag}(1, -1)$, gives the formula for S_τ. For $k = \operatorname{diag}(k_+, k_-)$ the Fock action on the negative variables substitutes $Z_- \mapsto Z_- k_-$. Its scalar on the vacuum is $(\det k)^\delta (\det k_-)^N$. In fact the source acts on a Hom row by the inverse source matrix. The inverse determinant of its negative block in the Fock vacuum gives one $\det k_-$ per target row. The source action also includes the inverse power $m/2 - j$ of the determinant from the target circle character.

In a Cartan decomposition insert one real boost of size u_i in each paired positive and negative coordinate. In a single target row its lowering operator differentiates once in each coordinate

of the pair, so it kills a polynomial in negative letters. Its creation operator introduces a positive letter in every nonconstant term, orthogonal to any polynomial in negative letters. The middle, diagonal oscillator operator gives $(\cosh u_i)^{-1}$ and replaces the paired negative variable by $(\cosh u_i)^{-1}$ times itself, as in Lemma 142.12. After the two compact rotations the resulting substitution is $Z_- \mapsto Z_- S_\tau$. Indeed on the boost S_τ has diagonal entries $\operatorname{sech} u_i$ and ones, and under left and right compact rotations it acquires the corresponding negative blocks on the two sides. The scalar transforms in the same way as $(\det g_\tau)^{\delta}(\det S_\tau)^{N}$. For any matrix L on these variables the normalized coefficient of $Q_\tau(Z_- L)$ against $Q_\tau(Z_-)$ is $\prod_h \Delta_h(L)^{c_{h,\tau}}$. To see this, factor a matrix with nonzero leading minors as lower unitriangular, diagonal, and upper unitriangular. The upper unitriangular factor fixes the leading-column polynomial. The lower one fixes its dual functional, by unitary adjunction and the same upper triangular calculation. The diagonal entries give the stated product by telescoping. Polynomial continuation gives it for all L. This proves the formula.

All singular values of S_τ are at most one. Its determinant has modulus $\prod_i (\cosh u_i)^{-1}$, so the coefficient has modulus at most $\prod_i (\cosh u_i)^{-N}$ at this place. The Cartan density is bounded by a constant times $\exp(2(s-1)\sum_i u_i)$, by the same lemma. Here $N > 2s$ at sufficiently large nodes. Integrability follows, also when the group is compact. Its square is integrable and its value at the identity is one.

The vector is a joint highest weight harmonic. Indeed every mixed contraction differentiates in a positive letter and thus gives zero, while triangular changes of the ordered rows and columns give the two highest lines of the minors. In the harmonic decomposition for the compact dual member $U(N)$, its fixed target highest line generates an irreducible source representation. This is the compact unitary dual-pair decomposition of Davidson, §2, Theorem 2.7. Tensoring with the determinant characters preserves irreducibility. For $b = 0$ the vector is the vacuum and its source representation is the indicated character. The tensor over real places is irreducible and has a square-integrable coefficient. Schur orthogonality therefore gives $K_\infty * K_\infty = d_\infty K_\infty$. Also $K_\infty(h^{-1}) = \overline{K_\infty(h)}$. Integrating against the unitary right action proves the projection claim, with the sign of the kernel fixed by the bilinear lift. □

Lemma 77.2 (The finite part at the complementary rank)**.** *Let $f = [F : \mathbb{Q}]$. On $L^2([G])^J$ convolution with the finite oscillator coefficient is $c_{\mathcal{M}} U$, where*

$$c_{\mathcal{M}} = \prod_{v \in \mathcal{M}} \frac{q_v^{-2Ns}}{|\operatorname{GL}_s(\mathcal{O}_v/\pi_v^2)|}, \qquad U = U^*, \qquad \|U - I\| \ll f\, 2^{-s/3}.$$

The constant is absolute for all sufficiently large nodes. In particular U is positive and invertible there, with $\|U^{-1}\| \le 2$, and it commutes with P_∞.

Proof. The z-rows on the point coset are the identity modulo π_v^2. If both X and Xg are in its support, the two square blocks force $g \equiv 1 \bmod \pi_v^2$. Conversely this subgroup fixes the test. Thus its coefficient is its squared norm q_v^{-2Ns} on that subgroup and zero elsewhere. Integrating gives the corresponding factor in $c_{\mathcal{M}}$.

At every other place v let K_v be the lattice stabilizer and A_v the vacuum coefficient. It is one on K_v and bi-invariant. Write L_v for the integral source lattice on the quadratic algebra. The Heisenberg lattice calculation in Lemma 142.8 gives $|A_v(g)| \le [L_v : L_v \cap gL_v]^{-N/2}$: the Hom lattice here consists of N copies of the source dual. The same lemma's sublattice count bounds the number of cosets at an inert place with intersection index Q^i by $16Q^{2(s-1)i}$, where $Q = q_v^2$. At a split place the two indices in the chosen component are q_v^u, q_v^w and there are at most $16 q_v^{(s-1)(u+w)}$ cosets with these indices. Its inverse dual supplies the other component. These are all cases since E/F is unramified at finite places by Corollary 12.4.

Now $N/2-2(s-1)=(s-9m+3)/2\geq s/3$, using $s\geq x^5$ and $m\leq 2\sqrt{x}$. Hence the sum of absolute values over cosets other than K_v is bounded at inert places by $16\sum_{i\geq 1}Q^{-si/3}$, and at split places by $16\sum_{i\geq 1}(i+1)q_v^{-si/3}$. Both are at most $128q_v^{-s/3}$. In the space of K_v-fixed functions the K_v term of convolution is the identity. Each other coset contributes a translate with operator norm bounded by its absolute coefficient. The resulting bi-invariant operator is self-adjoint since $A_v(g^{-1})=\overline{A_v(g)}$. The operators at distinct places commute. Moreover $\sum_v q_v^{-s/3}\leq f\sum_{l=2}^{\infty}l^{-s/3}\ll f\,2^{-s/3}$. Taking the norm limit of the products bounds $\|U-I\|$ by $\exp(128\sum_v q_v^{-s/3})-1$, with the sum away from the marks. The degree $f\ll_{q_0,d}x^{12}$ on these intervals makes this less than $1/2$ eventually. The Neumann series and commutation of finite and real translations give the conclusions. □

Theorem 77.3 (The complementary Gram operator on the prescribed level)**.** *Let Θ use the data at the beginning of this subsection, with the unit vector v at infinity. There is a positive constant c_H, fixed by the Haar normalizations and the two spaces, such that*

$$\Theta^{*}\Theta=c_H\,c_{\mathcal{M}}\,d_{\infty}\,UP_{\infty}\quad\text{on } L^2([G])^{J}.$$

Here the lift on L^2 is the continuous extension of its integral on smooth rapidly decreasing automorphic functions of this level. In particular its squared norm on u, divided by $c_Hc_{\mathcal{M}}d_{\infty}$, is between $(1-\|U-I\|)\|P_{\infty}u\|^2$ and $(1+\|U-I\|)\|P_{\infty}u\|^2$.

Proof. Apply the convergent Siegel–Weil formula to the doubled source $W_j\oplus(-W_j)$ and the definite space of rank N. Its Siegel parabolic has Levi $\mathrm{GL}_s(E)$. Since the Siegel point $(N-s)/2$ is greater than $s/2$, the Eisenstein sum is absolutely convergent and the definite-group integral is its value, up to the positive Haar constant. See Yamana, Theorem 2 and the convergence range preceding it. Use the doubled tensor $\bar{\Phi}\otimes\Phi$ of the specified datum. Its theta integral on the doubled group, restricted to a pair g',g, is the integral over $[H]$ of $\overline{\theta(g',y)}\theta(g,y)$. This follows by using the direct sum polarization and summing over the two rational position vectors. The conjugate splitting on the first copy gives precisely the complex conjugate.

We verify the terms of the Eisenstein sum because the function to which the resulting identity is applied can be a product of automorphic functions. Each term corresponds to a rational maximal isotropic plane. Write a basis of it in blocks $\binom{K}{K'}$ with s columns. At one old split mark write X,X' for the two position matrices of the doubled tensor. The Siegel functional for this plane is supported where $XK-X'K'=0$, choosing the sign of the second coordinates accordingly. Indeed, in the polarization adapted to the plane it is evaluation at zero. In the split matrix polarization the partial Fourier transform giving this evaluation integrates over matrices annihilating the plane. Equivalently the functional is invariant under all multipliers $\psi_v(\mathrm{tr}(Z(XK-X'K')))$, whose characters separate points. In the translated point tests the matrices still have column rank s, since right translation at a split place acts by an invertible change of source coordinates. If $K'c=0$ then $XKc=0$, hence $Kc=0$ and consequently $c=0$. Thus K' is invertible at this place, and over E. Total isotropy says the plane is the graph of an isometry. Every rational isometry gives one plane, so the surviving sum is indexed by $G(F)$.

For the identity graph the partial Fourier transform at zero pairs the two data on $X=X'$. With self-dual additive measure this is their oscillator inner product. Equivariance takes the graph of $\gamma\in G(F)$ to the coefficient at $g'^{-1}\gamma g$, with trivial phase on the rational splitting. Thus the kernel is c_H times the sum of translates of the adelic coefficient $A(h)=\langle\Phi,\omega(h,1)\Phi\rangle$. Lemmas 77.1 and 77.2, including the absolute coset bounds in its proof, give $A\in L^1(G(\mathbb{A}_F))$. Unfolding against $\overline{u'(g')}u(g)$ therefore gives

$$\langle\Theta u',\Theta u\rangle=c_H\int_{G(\mathbb{A}_F)}A(h)\langle u',R(h)u\rangle\,dh,\qquad(R(h)u)(x)=u(xh).$$

Indeed the unfolded absolute integral is bounded by $\|A\|_1\|u'\|_2\|u\|_2$. For smooth rapidly decreasing inputs the original lift integrals also converge absolutely, since theta kernels have moderate growth uniformly on the compact target quotient. Factoring the last integral gives the stated identity on these inputs. Compactly supported smooth functions on the source quotient, averaged over J, form a dense subspace of the level space. The upper bound gives a bounded extension. The lower bound follows from self-adjointness of U and commutation with P_∞. □

Corollary 77.4 (What the complementary class evaluations span)**.** *Choose representatives y of $H(F)\backslash H(\mathbb{A}_{F,f})/J_H$. At each such class test against the polynomials in the span of the compact target translates of $\prod_\tau Q_\tau$. These finitely many classes, with a basis of that polynomial space, give linear period functionals whose common kernel is exactly* $\ker P_\infty$ *on the level space. Their Petersson dual vectors span $P_\infty L^2([G])^J$.*

In particular, if F_0 is a good source form and B is a first-factor theta function of Corollary 73.25 with the compatible splitting, the complementary periods of F_0B detect precisely $P_\infty(F_0B)$. When $j = m$ this is the ordinary compact-group integral of $F_0(xh)B(xh)\prod_\tau(\det h_\tau)^{m/2}$, using probability Haar measure at infinity.

Proof. The finite class set is finite by compactness of $[H]$. The finite test is right J_H-invariant and the span of the compact translates of v is finite dimensional. Their polynomial factors belong to the allowed negative-letter tests because rotating the N target rows expands each minor into a linear combination of minors on the same source columns by Cauchy–Binet. The determinant character on the target acts by scalars. Rational invariance and right translation at infinity express every value of Θu as a combination of these class evaluations. Conversely every evaluation in this span is a linear combination of values of the translated lift. Finite stabilizers merely impose relations between values. The resulting finite-dimensional space of target functions is closed in $L^2([H])$ and all its evaluation maps are continuous. The equivalence thus holds on the completed source. Theorem 77.3 identifies the common kernel. Taking orthogonal complements proves the spanning statement. Equivalently, the image of $\Theta^*\Theta$ is exactly the range of P_∞ since U is invertible on that range.

The good form is cuspidal and the first factor has moderate growth, so their product is rapidly decreasing. At the old marks it is fixed modulo π_v^2. At a new mark the character of the determinant of the marked rows cancels the source action on the first finite test, and the second factor is spherical. The product has full source level there and at every other unmarked place. It is therefore an admissible input. For $j = m$ the negative block is empty and Lemma 77.1 gives the displayed character integral. □

For the comparison of individual determinants, the remaining spanning assertion concerns the first factors. Their prescribed arithmetic family must give $P_\infty(F_0B) \neq 0$ for at least one member whenever $F_0 \neq 0$. The complex span of the complementary class evaluations by itself supplies no bound on the integral index of these product periods.

77.2 The carrier form and shortest vectors at its peaks

The first factors paired with P_∞ in Corollary 77.4 use a single form on $U(2m,m)$. We construct it on each of the fields already in use and locate the points where it and a good source form have large absolute value. The ranks here are smaller than that of the complementary group in the preceding subsection.

Lemma 77.5 (Counting self-dual neighbors of an unramified lattice)**.** *Let L be a self-dual Hermitian lattice of rank r over an unramified quadratic extension of a local field, with residue cardinality Q in the extension. For $i \ge 1$, the number of self-dual lattices L' for which $[L : L \cap L'] = Q^i$ is at most $8^i Q^{ri}$. Consequently at target rank $3r$, for the unramified unit vacuum on the fields of Corollary 12.4, the sum of absolute source coefficients over cosets outside the lattice stabilizer is $O(q^{-r/2})$ at every finite place of residue cardinality q in F, with an absolute constant for sufficiently large r.*

Proof. Fix the intersection $I = L \cap L'$ and the isomorphism type $A = L/I$ over the valuation ring O. Duality gives $I^{\#} = L + L'$. In $I^{\#}/I$ the modules A and L'/I are complementary. The perfect pairing with values in $\mathrm{Frac}(O)/O$ identifies L'/I with the conjugate dual of A, of the same elementary divisor type. If there is one complement, the others are graphs of maps from it to A. There are thus at most $|\operatorname{End}_O(A)|$ choices for L' for this I. On the other hand the number of possible I of this type is bounded by $Q^{ri}/|\operatorname{Aut}_O(A)|$, by counting surjections $L \to A$.

Write $A = \bigoplus_{h \ge 1} (O/\pi^h)^{e_h}$. The fraction of its endomorphisms that are invertible equals $\prod_{h : e_h > 0} \prod_{l=1}^{e_h} (1 - Q^{-l})$. Indeed reduction modulo π gives a block triangular matrix when one orders the block lengths increasingly. Its diagonal blocks are arbitrary and independent. By Nakayama’s lemma invertibility is equivalent to the invertibility of all those blocks. Since $\prod_{l \ge 1} (1 - 2^{-l}) > 1/4$ and at most i block lengths occur, each type costs at most $4^i Q^{ri}$. The types are partitions of i, of which there are at most 2^{i-1} by comparison with ordered compositions.

At an inert place the coset determines L', $Q = q^2$, and the vacuum bound of Lemma 142.8 is $Q^{-3ri/2}$. Thus the tail is bounded by $\sum_{i \ge 1} (8 Q^{-r/2})^i$. At a split place use the same lemma’s count of $16(i+1) q^{(r-1)i}$ for total index exponent i on the two components, and the coefficient bound $q^{-3ri/2}$. Summation proves the assertion, including at places of residue characteristic two. □

Theorem 77.6 (The prescribed full-level cuspidal carrier)**.** *Let $D^- = -I_m$ be the negative definite standard space, and $S = I_{2m} \oplus (-I_m)$. Take the theta lift of the constant function on $[U(D^-)]$ to $U(S)$, with circle characters $(\upsilon^m, 1)$, the full finite lattice vacua, and the archimedean polynomial*

$$\prod_{\tau} \det(Z_{\tau,-})^{2m}.$$

Here $Z_{\tau,-}$ uses the m negative target rows and all m source columns in the Fock coordinates for this pair. At every sufficiently large node this lift, denoted by ϕ, is a nonzero smooth cuspidal form fixed by the full finite lattice stabilizer. It is holomorphic of scalar weight $(0, \dots, 0; 3m, \dots, 3m)$ in the convention that positive coordinates come first.

Proof. At a real place let $u \in U(D^-)$. On the Fock vacuum it acts by $(\det u)^{2m}$. The $2m$ positive target rows have negative Hom sign and each contributes one determinant through the vacuum factor. The target circle character is trivial. The negative target rows have the opposite sign, so the indicated determinant polynomial acquires the character $(\det u)^{-2m}$. The test is nonzero and fixed by the compact source.

We verify first occurrence to control the constant terms. The lower members of the rational Witt tower are $I_m \oplus \mathbb{H}^l$, $0 \le l < m$, with $\mathbb{H}$ a hyperbolic plane. Over $\mathbb{R}$ their signatures are $(m+l, l)$. Use the splittings compatible with taking constant terms in this tower. The compact source vacuum character in the smaller oscillator is still $(\det u)^{2m}$. Indeed adjoining one hyperbolic plane tensors with the oscillator on its isotropic half in the Schrödinger polarization. On that space $U(D^-)$ acts by linear substitution with unit Jacobian and fixes the standard Gaussian. This Gaussian is the vacuum in its compatible Fock polarization. This fixes the source character on descent at each step.

Apart from that vacuum character, the polynomial algebra has $m+l$ copies of the standard source letters and l copies of the dual letters. The latter are the rows whose determinant had negative weight above. A summand, as a compact source representation, has the form $S_\lambda(\mathbb{C}^m) \otimes S_\mu(\mathbb{C}^m)^*$ with $\ell(\mu) \le l$. This is the polynomial Cauchy decomposition on each set of letters, obtained by taking simultaneous permutation invariants in the tensor powers of row and column spaces. For this summand to contain $\det^{-2m}$ one must have $\mu = \lambda + (2m, \dots, 2m)$, by Schur's lemma applied after moving S_μ across the Hom pairing. This requires m nonzero rows of μ, impossible for $l < m$. Averaging over the compact source on each homogeneous degree proves vanishing on the dense polynomial vectors and hence on the full oscillator. Every smaller global lift of the trivial source vanishes, already by averaging at one real place. The source is cuspidal because it is anisotropic. Rallis's tower property therefore gives zero for every proper parabolic constant term of the lift to S. We use the constant-term tower for the same splittings as in Gan, Qiu and Takeda, §11.2.

To verify nonvanishing for the chosen tensor, normalize its real vector to length one. Its real source coefficient is identically one. At finite places the unit vacua are fixed by the hyperspecial lattice stabilizers and have value one there in their source coefficients. All characters are unramified in the lattice splitting. By Lemma 77.5 the unmarked convolution on the constant source equals the identity up to an operator of norm $O(f 2^{-m/2})$. Indeed multiply the local operators and use $\sum_v q_v^{-m/2} \ll f 2^{-m/2}$, as in Lemma 77.2. Here $f \ll_{q_0,d} x^{12}$ and $m \ge \sqrt{x}$.

The doubled Siegel–Weil formula applies at rank $3m > 2m$. The target has Witt index m, so the difference $3m - m$ also exceeds the Siegel rank m. In particular both the theta integral and the Eisenstein sum converge absolutely by Yamana, Theorem 2. Every rational maximal isotropic plane of $D^- \oplus (-D^-)$ is a graph. Its intersection with either summand is isotropic in a definite space and hence zero. As in the proof of Theorem 77.3, unfolding these graphs expresses the squared norm of the lift as a positive Haar constant times the integral of the oscillator coefficient against the constant source pairing. The source quotient is compact, and the integral over the target of the diagonal squared theta kernel is the continuous convergent Eisenstein value on that quotient. Cauchy–Schwarz bounds the mixed integrals there. The real integral is the positive volume of the compact real source and the finite factor differs from one by $O(f 2^{-m/2})$. This proves strict positivity for large x.

On the target compact blocks k_+, k_- the vacuum contributes $(\det k_+)^{-m}$, from the negative Hom coordinates. The source circle character contributes $(\det(k_+ \oplus k_-))^m$, and the polynomial contributes $(\det k_-)^{2m}$. This gives the asserted weight. All noncompact lowering operators differentiate once in a positive and once in a negative target row, so they kill the polynomial. These are the holomorphic highest weight equations, and the theta functional intertwines the real group action. The specified finite vacua give full level. Smooth compact-finite cuspidal automorphic forms are rapidly decreasing and bounded on the quotient. □

For a signed space with matrix $J_0 = \operatorname{diag}(1, -1)$ in blocks, put $\Lambda_g = E^n \cap g_f \widehat{\mathcal{O}}_E^n$. Use the

Euclidean norm at the embeddings of the chosen CM type and set

$$\lambda(g)=\min\left(1,\ \min_{0\neq y\in\Lambda_g}\Big(\sum_{\tau}\|g_\tau^{-1}\tau(y)\|^2\Big)^{1/2}\right).$$

A peak of a nonzero bounded form L means a point g where $|L(g)|>\frac12\sup|L|$.

Lemma 77.7 (A shortest-vector bound from a holomorphic transvection)**.** *Let L be a bounded automorphic scalar coefficient, holomorphic of weight $u_{\tau,i}$ for an order placing positive coordinates before negative ones. This means that each translate at infinity extends holomorphically to $G(F_\infty)B$ with right character $\prod_{\tau,i}b_{\tau,ii}^{u_{\tau,i}}$ on the complex upper triangular group B. Suppose $|u_{\tau,i}|\le A$, $A\ge 2$, and every negative weight exceeds every positive weight at the same embedding by at least one. Let $\nu\in\mathcal{O}_E$ satisfy $\bar\nu=-\nu$ and $\tau(\nu)=it_\tau$ with $0<t_\tau\le T$, $T\ge 1$. Suppose the finite level contains the principal congruence subgroup modulo ν. Then any peak satisfies $\lambda(g)\ge(AT)^{-1/2}$.*

Proof. If a nonzero $y\in\Lambda_g$ violates the bound, put $v_\tau=g_\tau^{-1}\tau(y)$. Its Hermitian length y^*J_0y lies in $\mathcal{O}_F$ because the finite coordinates at g_f are integral. Its absolute value at every real place is less than one. The norm of a nonzero algebraic integer cannot have this property, so y is isotropic. In a definite space this already gives a contradiction.

Now $\gamma=1-\nu yy^*J_0$ is a rational isometry, and $g_f^{-1}\gamma g_f$ belongs to the finite principal level. Define

$$n_\tau(z)=1-it_\tau z\,v_\tau v_\tau^*J_0.$$

The rank-one nilpotent matrices give $n(z+1)=n(1)n(z)$. Automorphy and the finite level imply $L(gn(1)h)=L(gh)$ for every real h, where the right multiplication is in the archimedean component.

Write $Y=\Im z$ and $\epsilon_i=(J_0)_{ii}$. Set $a_l=\sum_{i\le l}\epsilon_i|v_{\tau,i}|^2$ and $B_l=1+2t_\tau Ya_l$, suppressing τ. The positive-first order and isotropy imply $0\le a_l\le\|v_\tau\|^2/2$, with equality on the right after the last positive coordinate. Also $a_0=a_n=0$. The leading Gram determinants of $n_\tau(z)^*J_0n_\tau(z)$ are the corresponding determinants of J_0 times B_l. Indeed the Gram matrix is $J_0+2t_\tau YJ_0v_\tau v_\tau^*J_0$, to which the rank-one determinant formula applies. Since $A\sum_\tau t_\tau\|v_\tau\|^2<1$, all B_l are positive on $Y>-1$. Indefinite Gram–Schmidt in this order now gives $n_\tau(z)=h_\tau b_\tau$ with h_τ unitary and b_τ upper triangular, $|b_{\tau,ll}|^2=B_l/B_{l-1}$. Therefore the holomorphic extension evaluated at $gn(z)$ defines a holomorphic function on this half-plane, periodic by the preceding translation and triangular equivariance.

Let $C=\sup|L|$. For $Y\ge 0$ the diagonal moduli are at least one on the positive block and at most one on the negative block. Subtracting the largest positive weight from every weight does not affect the modulus of the character since $|\det b_\tau|=1$. The weight gap bounds its modulus by the product of the negative diagonal moduli, namely $(1+t_\tau Y\|v_\tau\|^2)^{-1/2}$. Thus the periodic function tends to zero as $Y\to\infty$. For $-1<Y<0$ the two inequalities for the diagonal moduli reverse. The absolute weight bound gives a character modulus at most $(1-t_\tau\|v_\tau\|^2)^{-A}$. The product of these bounds over the real places is at most e^2, since $-\log(1-z)\le 2z$ for $0\le z\le 1/2$. The periodic function is consequently bounded by Ce^2. In the disk coordinate $\exp(2\pi i(z+i))$ it extends with value zero at the origin. The Schwarz lemma at $z=0$ gives $|L(g)|\le Ce^{2-2\pi}<C/2$, contradicting the peak. □

Theorem 77.8 (Simultaneous minima for the two forms)**.** *Let $M=p_0p_1$, and let $F_0\neq 0$ be any good form on $U(W_j)$ on its specified real line. Choose any peak g_1 of the carrier and any peak g of F_0. At every sufficiently distant node of the fixed-(q_0,d) family,*

$$\lambda_S(g_1)\ge(3m\sqrt{d})^{-1/2},\qquad \lambda_{W_j}(g)\ge(5b_xM\sqrt{d})^{-1/2},\qquad \lambda_S(g_1)\lambda_{W_j}(g)\ge b_x^{-12}.$$

The last bound also holds after adjoining the standard padding $(-I_j) \oplus I_{3m-j}$ to W_j and extending g by the identity.

Proof. Use the sign of $\sqrt{-d}$ whose imaginary part is positive at the chosen embeddings. The CM type is induced from a single embedding of $\mathbb{Q}(\sqrt{-d})$, as in the proof of Theorem 15.3. Theorem 77.6 supplies a bounded nonzero holomorphic form of full level. Apply Lemma 77.7 with $\nu = \sqrt{-d}$ and $A = 3m$.

For the good form apply the lemma to $\overline{F_0}$, which has the same peaks. We spell out the holomorphic convention. The ordered Harish–Chandra entries for W_j are the $k-1$ centered consecutive integers followed by $H_0 - w_1, \dots, H_0 - w_m$, with the largest $k+j-1$ entries positive in signature. Negate the entries for the conjugate representation and sort each compact block decreasingly, calling the result η. Every negative-block entry then exceeds every positive-block entry by at least $2H_0$ when both blocks occur. The separation is the adjacent gap between $H_0 - w_j$ and $H_0 - w_{j+1}$. The fixed line becomes the compact highest line in this reversed order. The compact-Cartan description of the holomorphic discrete series gives its weight $u = \eta + \rho_{\mathrm{n}} - \rho_{\mathrm{c}}$, with the half-sums of positive noncompact and compact roots in the chamber of η. In this chamber all positive noncompact roots point from the smaller positive-sign block entries to the larger negative-sign block entries. The lowest compact type is killed by the opposite noncompact root spaces. Its compact highest line is also killed by the compact upper root spaces. Together these are exactly the upper triangular root spaces for the positive-first order. To obtain the holomorphic extension in the lemma, extend to the open set $G(F_\infty)B$ by the triangular character. This is well-defined since $G(F_\infty) \cap B$ is the compact diagonal torus. The root equations and the diagonal weight are the Cauchy–Riemann equations for this extension. This also describes the compact case $j = m$ by the usual algebraic highest weight representation. For the holomorphic model of the noncompact groups see Davidson, §§3–4.

Each entry of $\rho_{\mathrm{n}} - \rho_{\mathrm{c}}$ has modulus at most s, since at most $s-1$ roots involving that coordinate enter the half-sums. Thus the gap is at least $2H_0 - 2s \geq 1$. The weight bound $w_j \leq 2b_x$ from Theorem 73.27 holds also for w_m, and $k, s, H_0 \ll x^{40}$, so $|u_i| \leq 5b_x$ everywhere for large x. The good form is cuspidal and bounded. Its finite level contains the subgroup principal modulo $M\sqrt{-d}$, since divisibility by M gives the prescribed congruence at every old mark. Apply the lemma with that element and $T = M\sqrt{d}$.

Peaks exist for both forms by the definition of supremum. Multiplying the two estimates gives $(15mb_xMd)^{-1/2}$. Since $M \leq b_x^{22}$ on these intervals and $15md \leq b_x$ eventually, this is at least b_x^{-12}. On the padded summands at the identity a nonzero integral coordinate has the sum of its squared archimedean moduli at least one, by the norm of an algebraic integer. The squared Euclidean lengths add in the direct sum, proving the last assertion. □

These estimates locate the points where the first-factor theta kernels must approximate the prescribed real coefficients. Such an approximation still has to use the finite residue data and level of the integral periods.

77.3 Approximating the real kernel at the prescribed finite level

Keep the carrier space S, the good form F_0, and the two peaks of Theorem 77.8. Put $T = W_j \oplus (-I_j) \oplus I_{3m-j}$, and let g_2 denote the extension of the peak of F_0 by the identity on the padding. We work at (g_1, g_2). Let $r = 3m,\ r' = s + 3m,\ f = [F : \mathbb{Q}]$, and set $n_* = 2ft^3$. Both ranks are at most $t = 5s$. The finite residue data will be functions on $\mathrm{Mat}_{r',r}(\mathcal{O}_E/\mathfrak{a}^2)$, where $\mathfrak{a}$ is the product of the designated old places of E. Thus $\mathfrak{a}\bar{\mathfrak{a}} = (M)$ with $M = p_0p_1$. At the other places we keep the lattice vacuum. Write b_x for the large weight of the node.

Lemma 77.9 (Separation of the rational Hom vectors)**.** *For a rational E-linear map $y : S \to T$ write $y(g_1, g_2) = g_2^{-1} y g_1$ for its adelic coordinates. Let α be the finite scale of the self-dual oscillator lattice, so the local vacuum lattice on Hom in these coordinates is $\alpha\,\mathrm{Mat}_{r',r}(\widehat{\mathcal{O}}_E)$. Suppose*

$$\alpha^{-1} y(g_1, g_2)_f \ \in\ \mathfrak{a}^2\,\mathrm{Mat}_{r',r}(\widehat{\mathcal{O}}_E).$$

At sufficiently large nodes the real Fock coordinates of these rational maps are b_x^6-separated in Euclidean distance. In particular every nonzero such vector has length at least b_x^6.

Proof. Let $K_0 = q_1 d$. The field E is contained in the K_0-th cyclotomic field, so $K_0\mathfrak{D}_E^{-1} \subset \mathcal{O}_E$, with $\mathfrak{D}_E$ the absolute different. Indeed the different of the full cyclotomic field is generated by $\Phi'_{K_0}(\zeta_{K_0})$, which divides K_0 by differentiation of $X^{K_0} - 1$. Transitivity of the different proves the assertion for the subfield. In particular $|D_E|^{1/[E:\mathbb{Q}]} \le K_0$. In the prescribed oscillator normalization the imaginary scalar is $-1/\sqrt{-d}$, and $q_1\alpha$ is integral. For the latter fact, the finite scale is one or the reciprocal of the local generator of the cyclotomic relative different, with the inverse placed in the specified component. Since d is a prime congruent to 7 mod 8 and $d \nmid q_1$, the discriminants of $\mathbb{Q}(\sqrt{-d})$ and $\mathbb{Q}(\mu_{q_1})$ are coprime. Their integer rings tensor to the integer ring of E, so the relative different over the quadratic field is generated by $\Phi'_{q_1}(\zeta_{q_1})$, a divisor of q_1.

Use the Euclidean realization of the source lattice Λ_{g_1} in Lemma 77.7, of real dimension $2fr$. Its Euclidean dual is obtained by applying $2J_0$ to the lattice in these coordinates corresponding to $\mathfrak{D}_E^{-1}\Lambda_{g_1}$. Indeed the absolute trace of its Hermitian pairing is twice the real Euclidean pairing after inserting J_0, and Λ_{g_1} is Hermitian self-dual at every finite place. Thus the shortest dual length is at least $2\lambda_S(g_1)/K_0$. The Euclidean transference inequality $\lambda_D(L)\lambda_1(L^\vee) \le D$ in dimension D gives $2fr$ independent source vectors of length at most $2frK_0/\lambda_S(g_1)$. See Banaszczyk, *New bounds in some transference theorems in the geometry of numbers*. These vectors span over $\mathbb{Q}$ and hence over E. A nonzero integral Hom map sends at least one of them to a nonzero vector of the target lattice. The sum-of-squares Frobenius norm of its real matrices is therefore at least $\lambda_S(g_1)\lambda_T(g_2)/(2frK_0)$.

Minkowski's convex body theorem on the fractional ideal $\mathfrak{a}^{-2}$ gives $c_0 \neq 0$ in this ideal with $\max_\tau |\tau(c_0)| \le 2\sqrt{K_0}/M$. In fact the real covolume is $2^{-f}\sqrt{|D_E|}\,M^{-2f}$. The product of the f disks of the indicated radius has greater than 2^{2f} times this volume. This choice works for any designated member of each old split pair. Multiplying y in the statement by q_1c_0 makes it an integral Hom map between the moved lattices. Its archimedean coordinates grow by a factor of at most $2q_1\sqrt{K_0}/M$. In Fock coordinates Euclidean matrix length is multiplied by $(2\pi/\sqrt{d})^{1/2}$, from the imaginary scalar above. The lower bound for a nonzero displacement is consequently

$$(2\pi/\sqrt{d})^{1/2}\frac{M\,\lambda_S(g_1)\lambda_T(g_2)}{4frq_1K_0^{3/2}}.$$

Here $M \ge b_x^{20}$, and the product of minima is at least b_x^{-12}. All factors in the denominator are polynomial in x of total order at most 77, with constant depending only on q_0, d. Indeed

$f \ll x^{12}$, $r \le t \ll x^5$, and $q_1 \le 2\varphi(q_1)^2 \le 8f^2$, the first inequality following prime power by prime power in Euler's formula. Since $b_x \ge x^{400}$, the bound exceeds b_x^6 for large x. Subtracting two admissible vectors preserves the finite congruence, giving the separation. □

Lemma 77.10 (Decay under a Heisenberg displacement)**.** *In real Fock coordinates let $\rho(z)$ be displacement by a real phase-space vector z, and $\omega(U)$ a unitary metaplectic operator covering a real linear symplectic map U. For a polynomial vector v of degree at most D, an integer $L \ge 1$, and $z \ne 0$,*

$$|\langle v, \rho(z)\omega(U)v\rangle| \;\le\; \|v\|^2 \left(\frac{C(1+\|U^{-1}\|)\sqrt{D+L+1}}{|z|}\right)^L$$

with an absolute constant in the Euclidean Fock normalization, independent of the number of variables.

Proof. Let X_e be the self-adjoint generator of displacement in the unit direction e. Choose e symplectically dual to $z/|z|$. The Euclidean metric and the standard symplectic form are compatible, so the commutator $[X_e, \rho(z)]$ is a scalar of modulus $|z|$ times $\rho(z)$, up to the fixed normalization of the central character. Covariance gives $\omega(U)^{-1} X_e \omega(U) = X_{U^{-1}e}$. The field operator X_w is a sum of one creation and one annihilation operator in the complex direction of w. On total degrees through d_0 their norms are bounded by $|w|\sqrt{d_0+1}$ times a fixed constant, by using an orthonormal basis with that direction first. A field operator raises the maximum degree by at most one. The L-fold commutator with X_e, paired between v and $\omega(U)v$, gives a binomial sum. In its term with l operators on the right, move them through $\omega(U)$ and bound the two norms by the preceding degree estimate. Summation gives $\|v\|^2 (C\sqrt{D+L+1})^L (1+\|U^{-1}\|)^L$. Division by $|z|^L$ proves the result. All differentiated vectors belong to Schwartz space. □

Theorem 77.11 (Interpolation by the prescribed residue functions)**.** *Let v be any archimedean Fock polynomial vector on $\mathrm{Hom}_E(S,T)$ of degree at most $b_x^{3/2}$. For the unitary splitting characters of this pair there exists one function Ψ on the residue matrices modulo $\mathfrak{a}^2$ such that, for all real $u \in U(S)$ and $u' \in U(T)$,*

$$\left|\theta_{\Psi,v}(g_1 u, g_2 u') - \langle v, \omega_\infty(u,u')v\rangle\right| \le b_x^{-20n_*} \|v\|^2 \left(\|u\|_{\max}\|u'\|_{\max}\right)^{4n_*}.$$

The maximum is the largest Euclidean operator norm over the embeddings. The residue function depends on the two peaks and v and works simultaneously for all their real translates. The threshold on the interval is independent of the peaks and j.

Proof. Write $\mathcal{D}$ for the additive group of residue matrices, and use all its additive characters χ. At the chosen split places let the finite position test be the lattice indicator times χ of the residue matrix, using probability mass for the position lattice. Use unit vacua at all other places. Call this test ϕ_χ. By linearity the function

$$\Psi = |\mathcal{D}|^{-1} \sum_{\chi \in \widehat{\mathcal{D}}} \overline{\theta_{\chi,v}(g_1,g_2)}\, \chi$$

gives a kernel equal to the average of the products $\overline{\theta_{\chi,v}(g_1,g_2)}\theta_{\chi,v}(g_1u, g_2u')$. The characters form the full dual even when the modulus has several prime factors.

We give the Poisson identity evaluating this average. For any two adelic Schwartz vectors v_1, v_2 in a rational position polarization X, the product of theta functionals $\overline{\theta(v_1)}\theta(v_2)$ equals $\sum_y \langle v_1, \rho(y) v_2\rangle$ over the full rational phase space. To prove it, group the two rational position

variables by their difference $a \in X(F)$. Poisson summation of $\overline{v_1(x)}v_2(x+a)$ in x introduces all dual rational frequencies. The half-pairing in the Weyl operator has trivial character on rational points. Use self-dual measures, for which the rational quotient has volume one. The matrix coefficients on phase space are Schwartz, being partial Fourier transforms after a linear change of the two position variables. Hence these sums converge absolutely. Local changes of polarization are unitary and preserve the inner products.

Apply this identity to the two translates in the average. Moving the common (g_1, g_2) through the Heisenberg operator replaces the rational vector y by $y(g_1, g_2)$. The common splitting phase cancels. In the split position model the finite inner product of a character test with its displacement by position a and frequency b' is zero unless the normalized a, b' are integral. Indeed the supports of the two lattice indicators must meet, and the frequency must integrate trivially over the lattice. On that support the coefficient is the character of the position residue of a, up to a fixed phase independent of χ. Reversing the displacement convention inverts all the residue characters and gives the same average. At unmarked places the unit vacuum has coefficient of modulus one on its self-dual phase-space lattice and zero outside, by commutation with that lattice. Orthogonality of $\widehat{\mathcal{D}}$ thus leaves exactly the finite condition of Lemma 77.9, with a phase of modulus one. In particular the $y = 0$ term is precisely $\langle v, \omega_\infty(u, u')v\rangle$.

For the other terms use $L = 4n_*$ in Lemma 77.10. The symplectic map on Hom has inverse norm at most $V = \|u\|_{\max}\|u'\|_{\max}$, because each unitary matrix preserving a sign form has inverse of the same Euclidean norm. Thus $V \geq 1$. In real dimension $d_0 = 2frr'$ the remaining displacements have separation $R = b_x^6$ and are at distance at least R from zero. Balls of radius $R/2$ about these points have disjoint interiors. The shell $2^h R \leq |z| < 2^{h+1}R$ contains at most $(1 + 2^{h+2})^{d_0}$ points by comparison of volumes. Since $d_0 \leq n_*$, summing over shells gives $\sum_{z\neq 0} |z|^{-L} \leq (6/R)^L$. Finally $n_* \ll_{q_0,d} x^{27}$ and $D \leq b_x^{3/2}$. For sufficiently large nodes $12C\sqrt{D + L + 1}/R \leq b_x^{-5}$. The asserted estimate follows on taking the L-th power. □

Corollary 77.12 (The determinant polynomial of the first factor)**.** *The interpolation theorem applies to the following actual polynomial with the splitting characters* $(\upsilon^m, \chi^0\upsilon^{m+j+(t+k-1)/2-2js^8})$. *Here* χ^0 *is the fixed character of weight* $-w_0$ *and trivial tame part. Put* $a = k + j - 1$, $p = s + 9m + 1$, $\Delta' = s^8 - (t + 1)/2$, *and let* $c^+_{h,\tau}$ *be the positive column counts of Theorem 73.27. For each* $1 \leq h < a$ *form a* $2m$*-rowed positive square minor* $D_{h,\tau}$ *using the* $2m$ *positive carrier columns. For the* i*-th row,* $0 \leq i < 2m$, *take positive row* $h + i$ *of* W_j *if* $h + i < a$, *and otherwise positive padding row* i, *numbering from zero. Form the* m*-rowed square* $D_{-,\tau}$ *on the negative carrier columns and all negative padded target rows, with the* j *padding rows first. The polynomial is*

$$\prod_{\tau} \left(D_{-,\tau}^{\,p+m+\Delta'+\sum_{1\leq h<a} c^+_{h,\tau}} \prod_{1\leq h<a} D_{h,\tau}^{c^+_{h,\tau}} \right).$$

In particular there is a linear combination of the elementary residue-character tests whose value at the two peaks is nonzero and differs from the squared Fock norm of this polynomial by at most $b_x^{-20n_*}$ *times its squared norm.*

Proof. Each square uses distinct rows, with at least $2m$ available positive padding rows. Each determinant is therefore a nonzero polynomial and their product is nonzero. At one embedding write $A_\tau = \sum_{h<a} c^+_{h,\tau}$. The degree there is $2mA_\tau + m(p + m + \Delta' + A_\tau)$. The telescoping count in Theorem 73.27 gives $A_\tau \leq w_j + k \ll b_x$, and the height-zero count vanishes. Also $p+m+\Delta' \ll b_x$. The total degree is $O(fmb_x) \leq b_x^{3/2}$ on the same intervals. Apply the theorem and take $u = u' = 1$. □

The coefficients in the displayed residue average are complex values of theta functions. The result supplies the approximation used to test complex spanning. The arithmetic comparison still requires pairing these tests with the carrier and controlling the prescribed integral first factors.

77.4 Integrating the theta kernel against the carrier at infinity

Fix the field, the two old marks, and the index j of Section 77.3. Let ϕ be the carrier, g_1 any peak of ϕ, and g_2 the padded peak of a nonzero good form. Write $G_S = \prod_\tau U(2m, m)$ and $G_T = \prod_\tau U(s + 2m, m)$ for the two real groups. Let v be the determinant polynomial of Corollary 77.12, regarded as a Fock vector. Define

$$c_v(h, u) = \langle v, \omega_\infty(h, u)v\rangle, \qquad c_v(1, 1) = \|v\|^2 > 0.$$

Here and below real translates have identity finite component. We shall integrate in h and compare the resulting value to an explicit positive mass.

At one place write $h = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in positive and negative coordinates. For integers $a_0 \le b_0$ put

$$H_{a_0,b_0}(h) = \frac{\det(h)^{b_0}}{\det(A)^{b_0-a_0}}.$$

The relation $A^*A - C^*C = I$ guarantees that the denominator is nonzero. On $K = U(2m) \times U(m)$ the character of this function on either side is $\det(k_+)^{a_0} \det(k_-)^{b_0}$. For a list of weight pairs u_0 indexed by the real places, write H_{u_0} for the product of these functions.

Use $p = s + 9m + 1$, $\Delta' = s^8 - (t + 1)/2$, and set

$$A_\tau = \sum_{1 \le h < a} c^+_{h,\tau}, \qquad \mu_\tau = A_\tau + \Delta', \qquad a = k + j - 1.$$

Define four weights, giving their positive and negative entries in that order,

$$\begin{aligned} u_\phi &= (0, 3m), & u_1 &= (-\Delta', 6m + 1),\\ u_{2,\tau} &= (\Delta', \mu_\tau + s + 2m), & u_{3,\tau} &= (0, \mu_\tau + p + 2m). \end{aligned}$$

The identity $u_\phi + u_1 + u_2 = u_3$ holds entry by entry.

Lemma 77.13 (The carrier variable of the real coefficient)**.** *For $h \in G_S$,*

$$c_v(h, 1) = \|v\|^2 H_{u_2}(h)\overline{H_{u_3}(h)}.$$

Proof. The target circle character acts on the source determinant with exponent $-e_\tau$, where $e_\tau = m - \mu_\tau$. To check the exponent, use the list σ_i in the proof of Theorem 73.27. Through index $a - 1$ its entries are at least $\sigma_{a-1} = s^8 - w_{j,\tau}$, which exceeds $(t - 1)/2 - w_{j,\tau}$. Its consecutive entries differ by at least one there. The sum of the positive column counts is therefore $\sigma_0 - \sigma_{a-1} - (a - 1) = w_{j,\tau} - s^8 + k/2 - a$. It follows that $\mu_\tau = w_{j,\tau} - j - (t + k - 1)/2$. Since $w_{j,\tau} = w_{0,\tau} + 2js^8$, the exponent of the circle character in Corollary 77.12, including the weight of χ^0, is $m - \mu_\tau$.

We compute the normalized coefficient at each real place by compact multiplication and on a real diagonal boost. There are $s + 2m$ positive and m negative target rows. On $k = (k_+, k_-)$ a positive target row contributes $\det k_-$ to the vacuum and a negative target row contributes $\det k_+$. Indeed the vacuum multiplier is the inverse determinant of the action on negative phase-space coordinates. The action on a Hom row is by the inverse source matrix, and exactly the source columns of opposite sign are negative coordinates. The splitting character contributes $\det(k)^{-e_\tau}$. A positive minor in v uses all positive source columns, and the negative minor uses all negative ones. In the variables of like signs the action on these determinants contributes $\det(k_+)^{-A_\tau} \det(k_-)^{-\kappa'_\tau}$, with $\kappa'_\tau = p + m + \mu_\tau$. Thus the two compact weights are

$$m - e_\tau - A_\tau = \Delta', \qquad s + 2m - e_\tau - \kappa'_\tau = s - p.$$

They are the weights of $H_{u_2}\overline{H_{u_3}}$. As the vector transforms by a unitary character, its coefficient has this character on both sides.

Now pair m positive and negative source coordinates in a Cartan boost with parameters $\xi_1, \dots, \xi_m$. In each target row the two oscillator variables for each pair have opposite phase-space signs. Every letter in the polynomial has positive phase-space sign. In the creation, diagonal, annihilation factorization of the boost, the annihilation terms kill it because they differentiate in an opposite-sign pair. Every nonconstant creation term is orthogonal to it because it contains a negative letter. The diagonal term supplies $(\cosh \xi_i)^{-1}$ for the vacuum in each row and rescales each paired coordinate by $(\cosh \xi_i)^{-1}$, as in Lemma 142.12. Each determinant uses every paired column of its own sign exactly once. The normalized coefficient on the boost is therefore

$$\prod_{i=1}^{m} (\cosh \xi_i)^{-(s+3m+A_\tau+\kappa'_\tau)}.$$

The determinant of the boost is one, and $\det A$ is $\prod_i \cosh \xi_i$. The displayed exponent is the sum of the two gaps $u_{2,-} - u_{2,+}$ and $u_{3,-} - u_{3,+}$. The Cartan decomposition gives equality on the whole group. Taking the tensor product over the real places proves the lemma. □

Lemma 77.14 (A comparison of two masses on the real carrier group)**.** *With the same Haar measure in both integrals, let $J = \int_{G_S} |H_{u_3}(h)|^2 \, dh$. Uniformly for large nodes on these intervals,*

$$0 < J < \infty, \qquad \int_{G_S} |H_{u_1}(h)| \, \|h\|_{\max}^{4n_*} \, dh \;\le\; b_x^{n_*} J.$$

Proof. Write $r = 3m$. The map $hK \mapsto Z = CA^{-1}$ identifies the symmetric space with the matrix ball $\mathcal{B} = \{Z \in M_{m,2m}(\mathbb{C}) : I - ZZ^* > 0\}$. Indeed Z gives the graph of a positive $2m$-plane. Orthonormal bases for this plane and its negative orthogonal complement recover h up to K. Put $\delta(Z) = \det(I - ZZ^*)$. The identity $A^*(I - Z^*Z)A = I$ gives $\delta = |\det A|^{-2}$. In particular $|H_{a_0,b_0}| = \delta^{(b_0-a_0)/2}$.

The quotient measure in this chart is $c\,\delta^{-r}\,dZ$ for a constant $c > 0$, where dZ is real Lebesgue measure. To verify it, under a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ the new coordinate is $Z' = (C + DZ)(A + BZ)^{-1}$, and its differential sends X to $(D - Z'B)X(A + BZ)^{-1}$. The block determinant formula gives $\det(D - Z'B) = \det(h)/\det(A + BZ)$. Hence the real Jacobian is $|\det(A + BZ)|^{-2r}$. Preservation of the Hermitian form gives $\delta(Z') = \delta(Z)/|\det(A + BZ)|^2$. These formulas give the asserted invariant density and the same constant for all integrals at this place.

The largest singular value of a boost of parameter $\xi \ge 0$ is $e^{\xi} \le 2\cosh \xi$. All factors $\cosh \xi_i$ are at least one. Thus $\|h\|_{\max} \le 2 \prod_\tau |\det A(h_\tau)|$. Since $s \gg x^5$, $n_* = 2ft^3 \ll x^{27}$, and $\Delta' = s^8 - (t+1)/2$, one has $(6m + 1 + \Delta' - 4n_*)/2 \ge r$ on this interval for all sufficiently large x. The weighted integral in the statement is bounded above by 2^{4n_*} times the product of the Lebesgue volumes of the balls, each with its measure constant. This follows because each remaining exponent of δ is nonnegative and $\delta \le 1$.

Put $\ell_\tau = \mu_\tau + p + 2m$ and $L = 1 + \max_\tau \ell_\tau$. We have $r \le \ell_\tau$ and $L \ll b_x$ by the column count and the bounds of Corollary 77.12. The local integrand for J against $c\,dZ$ is $\delta^{\ell_\tau - r}$, with values at most one. On the smaller ball $\eta\mathcal{B}$, where $\eta = (2mL)^{-1/2}$, it is at least $(1 - \eta^2)^{mL} \ge e^{-1}$. This uses $\log(1 - y) \ge -2y$ for $0 \le y \le 1/2$. As the complex dimension is $2m^2$, this smaller ball has $(2mL)^{-2m^2}$ times the volume of the full ball. This proves positivity and finiteness and bounds the ratio of the two integrals by

$$2^{4n_*} e^f (2mL)^{2m^2 f}.$$

Its logarithm divided by $n_* \log b_x$ tends to zero. In fact $\log(2mL) = O(\log b_x)$ and $2m^2 f/n_* = m^2/t^3$ tends to zero on the original interval. This proves the claimed uniform comparison. □

Theorem 77.15 (Nonvanishing after integration against the carrier)**.** *Choose Ψ from Theorem 77.11 for the two peaks and v. For $u \in G_T$ put*

$$B_0(u) = \int_{G_S} \phi(g_1 h) c_v(h,u) H_{u_1}(h)\, dh,$$
$$B_\Psi(u) = \int_{G_S} \phi(g_1 h) \theta_{\Psi,v}(g_1 h, g_2 u) H_{u_1}(h)\, dh.$$

Both integrals converge absolutely and B_0 is bounded. At every sufficiently large node they satisfy

$$B_0(1) = \|v\|^2 \phi(g_1) J \neq 0,$$
$$|B_\Psi(u) - B_0(u)| \le 2 b_x^{-19 n_*} \|u\|_{\max}^{4n_*} |B_0(1)|.$$

In particular $B_\Psi(1) \neq 0$ at the padded peak of every nonzero good form. At each such peak some elementary additive character of the prescribed residue group also has a nonzero integral of this kind.

Proof. The carrier satisfies $\sup|\phi| < 2|\phi(g_1)|$. Unitarity gives $|c_v(h,u)| \le \|v\|^2$ for every h, u. The preceding lemma gives absolute convergence and a bound independent of u for the first integral.

At one real place a holomorphic scalar group function of weight (a_0, b_0), divided by H_{a_0,b_0}, descends to a holomorphic function on $\mathcal{B}$. We recall the trivialization to fix the orientation. The weight on the compact blocks is scalar and extends on the complex block upper parabolic to $\det(P_{11})^{a_0} \det(P_{22})^{b_0}$. The upper triangular transformation law and the compact block law give this parabolic law by factoring each diagonal block into a unitary and an upper triangular matrix. For $Z = CA^{-1}$, the graph matrix $\left(\begin{smallmatrix} I & 0 \\ Z & I \end{smallmatrix}\right)$ is a holomorphic section. Its inverse times h is block upper with diagonal $A, D - ZB$. The character of this block upper factor is precisely $\det(h)^{b_0} / \det(A)^{b_0 - a_0}$. Evaluating the holomorphic extension on the section proves the assertion. The same argument works at all embeddings together.

Apply this to $h \mapsto \phi(g_1 h)$ of weight u_ϕ from Theorem 77.8, and write the quotient as $Q(Z)$. Since $H_{u_\phi} H_{u_1} H_{u_2} = H_{u_3}$, Lemma 77.13 identifies the integrand for $B_0(1)$ with $\|v\|^2 Q(Z) |H_{u_3}(h)|^2$. Both its mass measure and the product of the balls are invariant under simultaneous rotation $Z_\tau \mapsto e^{i\theta} Z_\tau$. The circle average of Q on each orbit is $Q(0) = \phi(g_1)$. Indeed the function of one variable $z \mapsto Q((z Z_\tau)_\tau)$ is holomorphic on a disk of radius greater than one. Absolute integrability and rotation invariance justify averaging under the integral. This gives the first identity.

For the second, Theorem 77.11 bounds the pointwise difference of kernels by

$$b_x^{-20 n_*} \|v\|^2 \|u\|_{\max}^{4n_*} \|h\|_{\max}^{4n_*}.$$

After multiplication by the carrier and H_{u_1}, Lemma 77.14 proves absolute integrability of the difference and bounds its integral in modulus by $2|\phi(g_1)| \|v\|^2 b_x^{-19 n_*} J \|u\|_{\max}^{4n_*}$. At $u = 1$ this error is strictly smaller than $|B_0(1)|$.

For clarity, each individual residue character χ also defines an absolutely convergent integral at $u = 1$. Apply the two-vector Poisson identity in the proof of Theorem 77.11 to the squared modulus of its kernel, with both translates equal to $(g_1 h, g_2)$. Each finite coefficient has modulus at most one and vanishes outside the moved integral phase-space lattice. Before the real translation by h, the corresponding rational vectors form a fixed lattice $\mathcal{L}$ in real Hom

coordinates. Indeed intersecting the local integral lattices gives a full fractional integer lattice in the rational Hom space, and the fixed real coordinates differ by an invertible linear map. For $z \neq 0$ in $\mathcal{L}$, the further displacement has length at least $|z|/\|h\|_{\max}$. Lemma 77.10 with $U = 1$ and $L' = 4n_*$ bounds the absolute sum of the nonzero terms by a fixed multiple of $\|h\|_{\max}^{L'} \sum_{z \neq 0} |z|^{-L'}$, which is finite since L' exceeds the real dimension. Taking square roots shows that the individual kernel is $O(\|h\|_{\max}^{2n_*})$, with a constant allowed to depend on these fixed data. The mass bound gives the required integrability. Expanding the finite sum defining Ψ under the integral now proves the last assertion. □

The integral here is over G_S with g_1 fixed. The arithmetic comparison uses periods on the adelic quotient of the carrier group. Relating these integrals requires the definite auxiliary theta kernels and their prescribed residue functions and row choices.

77.5 The adelic periods of the first factor

Write $r = 3m$, $N = 6m+1 = 2r+1$, and $V^\flat = I_N$. We use $U(V^\flat)$ to express the carrier integral of Theorem 77.15 in terms of the prescribed periods on $[U(S)]$. The small excess of N over $2r$ requires a more precise calculation at the finite places. In this subsection a positive lower bound is needed on each fixed field. The real approximation will retain its uniform constants.

Lemma 77.16 (Absolute convergence at rank $2r+1$)**.** *At an unmarked finite place of the present fields let q be the residue cardinality in F, K_v the stabilizer of the integral lattice in S, and A_v the normalized coefficient on $U(S)$ of the unit lattice vacuum for $V^\flat$. Give K_v mass one. Then*

$$\|A_v - \mathbf{1}_{K_v}\|_1 \ \ll_r \ \begin{cases} q^{-2}, & E_v/F_v \text{ inert}, \\ q^{-3/2}, & E_v/F_v \text{ split}. \end{cases}$$

Proof. These are all places by Corollary 12.4. The coefficient is one on K_v. At an inert place let $Q = q^2$ and let L, L' be the integral lattice and a translate. Fix $I = L \cap L'$ with $|L/I| = Q^i$ and put $A = L/I$. For a fixed module type of A write $\lambda_1, \dots, \lambda_b$ for its nonzero elementary divisor exponents, and $d_A = \sum_{u,v} \min(\lambda_u, \lambda_v)$. There are at most $Q^{ri}/|\operatorname{Aut}(A)|$ possible kernels I, by counting surjections out of L. The endomorphism ring has order Q^{d_A}, and

$$|\operatorname{Aut}(A)| \geq c_r Q^{d_A}, \qquad c_r = \left(\prod_{a=1}^{\infty} (1 - 2^{-a}) \right)^r > 0.$$

Indeed an endomorphism is invertible if and only if its reduction on $A/\pi A$ is invertible. In order of elementary divisor lengths the reduction is block triangular. The equal-length diagonal blocks vary independently over all square matrices, and the proportion invertible in a block of size b' is $\prod_{a=1}^{b'} (1 - Q^{-a})$.

In $I^\#/I$, A is isotropic and L'/I is an isotropic complement perfectly paired with it. Indeed $I^\# = L + L'$ by self-duality, and the intersection in the quotient is zero. If there is one such complement B, every other is the graph of a map $B \to A$. The graph is isotropic precisely when the associated pairing on B, with values in $E_v/\mathcal{O}_{E_v}$, is skew Hermitian. There are $Q^{d_A/2}$ such pairings. For a pair of different cyclic factors one off-diagonal entry ranges over $Q^{\min(\lambda_u, \lambda_v)}$ choices and determines the other. On a diagonal factor of length λ_u there are q^{λ_u} choices, since the trace onto $\mathcal{O}_{F_v}/\pi^{\lambda_u}$ is onto in the unramified extension. The uniformizer is taken in F_v. This count includes every L' with intersection I.

There are at most $(i+1)^r$ types, and $d_A \geq i$. The count of translates is consequently at most $c_r^{-1}(i+1)^r Q^{(r-1/2)i}$. Lemma 142.8 gives $|A_v| \leq Q^{-Ni/2}$. The sum for $i \geq 1$ is $O_r(q^{-2})$. At a split place the same lemma bounds the number of cosets of total index exponent i by $16(i+1)q^{(r-1)i}$ and the coefficient by $q^{-Ni/2}$. Summation gives $O(q^{-3/2})$. □

Lemma 77.17 (Positivity at each unmarked place)**.** *For every unitary representation R of $U(S)(F_v)$ and every K_v-fixed vector w,*

$$\int A_v(g) \langle w, R(g)w \rangle \, dg \ \geq \ \beta_v \|w\|^2, \qquad \beta_v = \frac{|\operatorname{GL}_r(\mathcal{O}_{E_v}/\pi)|}{|\operatorname{Mat}_r(\mathcal{O}_{E_v}/\pi)|} > 0.$$

At a split place the numerator and denominator use both components. The integrals are absolutely convergent.

Proof. Locally the target lattice is a sum of r hyperbolic planes and a line. In an unramified quadratic algebra one can split a pair of unit diagonal entries in I_N by choosing a unit of norm -1. The residue norm is onto and the norm lifts through successive principal-unit quotients

by the surjective residue trace. Pair the resulting primitive isotropic vector with a vector of pairing one. Adding an appropriate multiple of the first makes the second isotropic by trace surjectivity. The same operations hold in a split quadratic algebra.

Choose one isotropic line of each plane for the position space. The resulting part of the oscillator acts on r rows of $S^\vee$, hence on $\mathrm{Mat}_r(E_v)$, by $X \mapsto Xg$. Include the scalar splitting factors in the unitary action $T(g)$ of the remaining line. Indeed the source preserves the position polarization and the absolute norm of its determinant is one. In the coordinates of the source and target integral lattices, the unit vacuum is $\mathbf{1}_C \otimes \eta$ with $C = \mathrm{Mat}_r(\mathcal{O}_{E_v})$, C of mass one, and η of norm one. At an odd inert place the translation part of the self-dual Heisenberg lattice fixes the indicator of C, and its dual multipliers force support in C, which characterizes this vacuum line. The finite different scale can be absorbed into the position coordinates and their duals. At a split place this also follows from the split matrix model by Fourier transforming r of the $2r+1$ rows. A lattice indicator goes to the indicator of its dual with the corresponding unit normalization. This covers the dyadic places, which split since the quadratic field of discriminant $-d$ already splits at two and complex conjugation acts nontrivially on it. As the full vacuum is K_v-fixed, $T(k)\eta = \eta$ for $k \in K_v$.

Put $T_w = R \otimes T$ and $a = w \otimes \eta$. The integral in the statement is the integral over X, g of

$$\mathbf{1}_C(X)\mathbf{1}_C(Xg)\langle a, T_w(g)a\rangle.$$

Its absolute integral is at most $\|w\|^2 \int |A_v(g)|\, dg$, since the X-integral of the two indicators is the intersection volume multiplying the line coefficient in $A_v(g)$. The full-rank matrices have full additive measure, in both components at a split place. On them disintegrate the additive measure over the right $U(S)(F_v)$-orbits in $\mathrm{GL}_r(E_v)$. The quotient integration formula for Haar measure applies because additive measure there is Haar measure times $|\mathrm{Nm}_{E_v/F_v} \det X|_v^r$ up to a constant, and this factor is constant on each orbit. Both groups are unimodular.

For an orbit with representative Y, the integral in the two group variables is

$$\left\| \int_{U(S)(F_v)} \mathbf{1}_C(Yg)\, T_w(g)a\, dg \right\|^2 .$$

In fact expansion of the square and the substitution $(u, v) = (u, ug)$ give exactly the preceding pairing. On a fixed orbit the integrand of this vector integral has compact support. In fact $Yg \in C$ bounds the entries of g, and the unitary relation with the fixed unimodular sign matrix bounds the entries of g^{-1} as well. Thus every orbit contributes nonnegatively. For any orbit meeting $C^\times = \mathrm{GL}_r(\mathcal{O}_{E_v})$, choose $Y \in C^\times$. Then $Yg \in C$ precisely for $g \in K_v$, and these are exactly the points there with $Yg \in C^\times$. Its contribution is $\|w\|^2$. Since K_v has mass one the measure of these orbits is the additive mass of $C^\times$, which is β_v by reduction modulo the uniformizer. This proves the inequality. □

Theorem 77.18 (The definite auxiliary group reproduces the real integral)**.** *Let J_S be the principal subgroup modulo the squared old marks on $U(S)$, with the full lattice stabilizers elsewhere. On $U(V^\flat)$ use the analogous $J_\flat$. Use characters $(v^m, 1)$, the indicator δ_e of the standard coordinate inclusion e of r columns in N rows modulo the squared marks, and the unmarked unit vacua. At infinity take in every factor the Δ'-th power of the minor on the first $2m$ rows of this inclusion and all positive source columns. Call the product $Q^\flat$.*

For each finite class c of the definite target at level $J_\flat$, choose its representative y_c and its prescribed full-rank matrix of short rows. In every minor of $Q^\flat$ one may choose any $2m$ rows from that matrix applied to the variable matrix. Denote by $Q_{c,\rho}$ the products over all such selections, with a separate choice in every factor and embedding. For any $g_1 \in U(S)(\mathbb{A}_F)$ there

exist complex numbers $d_{c,\rho}$, independent of ψ, such that for every smooth rapidly decreasing J_S-fixed function ψ on $[U(S)]$,

$$\int_{G_S} \psi(g_1 h) H_{u_1}(h)\, dh = \sum_{c,\rho} d_{c,\rho} \int_{[U(S)]} \psi(x) \theta^{\flat}_{\delta_e, Q_{c,\rho}}(x, y_c)\, dx.$$

The representatives have identity real component.

Proof. The real coefficient of the vector $Q^{\flat}$ at $(h, 1)$ is $\|Q^{\flat}\|^2 H_{u_1}(h)$. Under the two compact blocks its weights are $(-\Delta', N)$ by the positive full-column determinant and the N positive target rows. On a boost its normalized coefficient is $\prod_i (\cosh \xi_i)^{-(N+\Delta')}$ per embedding, since the contractions kill the positive-letter polynomial and creation terms are orthogonal to it. These are precisely the compact characters and boost values of H_{u_1}, as in Lemma 77.13. This coefficient is integrable by Lemma 77.14.

At an old mark the point test has coefficient equal to its squared norm on the principal subgroup and zero elsewhere. The identity submatrix forces $g \equiv I$ modulo the squared mark if both X and Xg are in its support. To see this even before g is known to be integral, the selected submatrices of X and Xg are invertible over the integer ring and congruent to the identity. Their quotient is g. Right convolution therefore acts on the level space by a strictly positive scalar at that mark.

At the other places the operators $U_v = \int A_v(g) R(g)\, dg$ on the level space are self-adjoint. Here R is right translation and $A_v(g^{-1}) = \overline{A_v(g)}$. Bi-invariance preserves that space. Lemma 77.17 gives $U_v \geq \beta_v I$. For a fixed field the sum of $\|U_v - I\|$ is finite by Lemma 77.16. Indeed it is bounded by a constant depending on r times $[F : \mathbb{Q}] \sum_{n \geq 2} n^{-3/2}$. These operators commute at different places. Their product converges in operator norm to an invertible operator U. To see invertibility, put aside the finitely many factors with $\|U_v - I\| > 1/2$. Each has a bounded inverse by the positive lower bound. The other inverses satisfy $\|U_v^{-1} - I\| \leq 2\|U_v - I\|$, so their product converges in norm as well. The absolute local coset sums also give an integrable adelic coefficient.

Let $\Theta^{\flat}$ be the auxiliary theta lift using $Q^{\flat}$ and the indicated finite data. Apply the convergent doubled Siegel–Weil identity and unfolding in the proof of Theorem 77.3. Here $N = 2r+1 > 2r$, the target is definite, an old split point test has full column rank, and the real and finite coefficients together are in L^1. In particular the full-rank test eliminates every nongraph term even for a rapidly decreasing input that has a nonzero constant term. With a positive measure constant c_*, the identity reads

$$(\Theta^{\flat})^* \Theta^{\flat} = c_*\, U\, R_{\infty}, \qquad (R_{\infty}\psi)(y) = \int_{G_S} \psi(yh) H_{u_1}(h)\, dh.$$

For precision this follows first as a pairing on rapidly decreasing test functions. The L^1 convolution bounds the pairing, so the lift extends boundedly from the dense smooth compactly supported level space to its L^2 completion. Thus if $\Theta^{\flat}\psi = 0$ then $R_{\infty}\psi = 0$ in L^2, by invertibility of U. For the stated inputs this last convolution is continuous by boundedness of ψ and L^1-domination. The quotient Haar measure has full support, so it vanishes at g_1.

If the periods on the right of the theorem are all zero then $\Theta^{\flat}\psi$ vanishes everywhere. In fact the definite target quotient has finitely many finite classes at the given level. Its right translates at infinity rotate the rows of $Q^{\flat}$ by unitary matrices and multiply by a scalar character. Any rotated row is a complex linear combination of the prescribed full-rank rows at that class. Multilinearity of the determinant expresses each rotated minor in their $2m$-rowed minors. Taking products gives exactly the span of the $Q_{c,\rho}$. Finite level invariance and rational invariance give the asserted vanishing at every point. Consequently evaluation of $R_{\infty}\psi$ at g_1

is a linear functional vanishing on the common kernel of a finite list of period functionals. It descends to their joint image in a finite-dimensional coordinate space. Extending the functional to that space supplies the coefficients in the assertion. □

Corollary 77.19 (Detection of each good-form peak by an adelic first factor)**.** *Fix g_1, a peak of the carrier, and let g be any peak of a nonzero good form on $U(W_j)$ at its specified level. For adelic $z \in U(W_j)$ write $\widetilde{z} \in U(T)$ for its extension by the identity on the padding. Let $\mathcal{B}_i(z)$ be the following finite collection. For $i = (\Lambda, c, \rho)$, integrate $\phi(x)$ on $[U(S)]$ against the theta kernel for $T \oplus V^\flat$ at $(x, \widetilde{z} \oplus y_c)$, with characters $(\upsilon^m, \chi^0 \upsilon^{\,m+j+(t+k-1)/2-2js^8})$, polynomial $vQ_{c,\rho}$ on the disjoint row blocks, and elementary residue character indexed by the matrix Λ. All unmarked data are the unit vacua. There are complex a_i such that, with B_0 of Theorem 77.15, for every real $u \in U(W_j)(F_\infty)$,*

$$\left| \sum_i a_i \mathcal{B}_i(gu) - B_0(\widetilde{u}) \right| \le 2 b_x^{-19 n_*} \|u\|_{\max}^{4n_*} |B_0(1)|.$$

In particular some $\mathcal{B}_i(g) \neq 0$ at every sufficiently large node on the given intervals.

Proof. Use the residue function Ψ of Theorem 77.11 for $g_1, \widetilde{g}$. For fixed adelic z put $\psi_z(x) = \phi(x)\theta_{\Psi,v}(x, \widetilde{z})$. This is smooth and rapidly decreasing, since ϕ is cuspidal and the theta kernel and its derivatives have moderate growth for fixed $\widetilde{z}$. The principal subgroup preserves every marked matrix residue and the unmarked stabilizers fix the lattice vacua. Since the carrier is fixed by all these subgroups, ψ_z is J_S-invariant. Apply the preceding theorem to ψ_z.

In each of the resulting integrands the two theta kernels multiply to the kernel for $T \oplus V^\flat$ with residue function $(X, Y) \mapsto \Psi(X)\delta_e(Y)$. This follows by multiplying the two sums on the direct sum of their rational position spaces. The oscillator vectors tensor, and the character on the source is the product of the two source actions, with the auxiliary determinant twist trivial. On the two target blocks restriction of the determinant character υ^m gives the original characters. The tensor product of the unmarked vacua is the unit vacuum.

Expand this residue function in the additive characters of all the matrices. They are exactly the trace characters indexed by Λ. On every local integer quotient modulo the squared mark the defining additive character is primitive, so the pairing $(x, y) \mapsto \chi(xy)$ is nondegenerate because every nonzero principal ideal contains the last nonzero ideal, on which the character is nontrivial. The entrywise trace pairing and the finite product over marks retain this property. Finite Fourier inversion gives the required expansion with coefficients independent of z. All the sums are finite and the quotient integrals converge absolutely by rapid decrease. It follows that the real integral at z is $\sum_i a_i \mathcal{B}_i(z)$ simultaneously for all z. At $z = gu$ it is exactly $B_\Psi(\widetilde{u})$ of Theorem 77.15. Since $\|\widetilde{u}\|_{\max} = \|u\|_{\max}$, this gives the estimate and the nonvanishing at $u = 1$. □

These first factors have now been obtained on the original adelic quotient. Detecting the good form itself requires pairing its product with a first factor against the real projection of Corollary 77.4. The linear combination above is over $\mathbb{C}$.

77.6 Holomorphy of the adelic first factors

Let $\mathcal{B}_i$ be the first factors of Corollary 77.19. We compute their right derivatives on $G_\infty = U(W_j)(F_\infty)$ from their defining carrier integrals. Put $a = k + j - 1,\ b = m - j,\ s = a + b$, $p = s + 9m + 1,\ N_0 = 5s - 9m - 1$, and let $H = s^8$. The source of the theta kernel in that integral is S of signature $(2m, m)$. Its target $T \oplus V^\flat$ contains W_j and has exactly m negative rows, counting the j padded ones. This equality with the number of negative source columns is essential below.

At a real place use the following names for matrices of Fock variables, with target rows on the left and source columns above,

	$+$	$-$
$+$	X	V
$-$	U	Y

The Fock complex structure takes the original complex coordinate in a like-sign slot and its conjugate in an opposite-sign slot. Each of the four arrays consists of ordinary polynomial variables in this complex structure. Y is square of size m. The archimedean polynomial used in $\mathcal{B}_i$ depends only on X, Y and is divisible by $\det Y$ at every real place. Indeed its negative determinant has exponent $p + m + \mu_\tau \geq 1$ in Corollary 77.12, and the short-row factors of Theorem 77.18 use only X.

Lemma 77.20 (Transfer of a noncompact root derivative)**.** *For this pair, let P be a polynomial in X, Y divisible by $\det Y$ at a fixed place. Let E^W_{ij} be the complex root matrix with entry one in positive target row i and negative target column j, and $E^S_{\alpha\beta}$ the root with positive source row α and negative source column β. There are polynomial vectors $Q_{\alpha\beta}$ such that*

$$d\omega(E^W_{ij})P = \sum_{\alpha,\beta} d\omega(E^S_{\alpha\beta})Q_{\alpha\beta}.$$

The identity holds with the prescribed determinant twists.

Proof. For any polynomial L in the like-sign variables, the two root actions satisfy

$$d\omega(E^W_{ij})L = c_1 \sum_{\alpha \text{ positive}} X_{i\alpha}U_{j\alpha}L,$$
$$d\omega(E^S_{\alpha\beta})L = c_2 \sum_{l \text{ negative}} U_{l\alpha}Y_{l\beta}L.$$

The constants c_1, c_2 are nonzero and independent of the indices. To verify the actions, decompose a real boost in an opposite-sign plane into its two complex root operators. For one spectator column or row the infinitesimal boost in Fock coordinates is a nonzero multiple of creation in the pair and its negative adjoint annihilation. This follows by differentiating the two-mode boost in the proof of Lemma 142.12. Its off-diagonal operators are quadratic creation and annihilation, and its diagonal factor has derivative zero at $t = 0$. The two operators have opposite torus weights, so taking the root component keeps exactly one. On the target torus the weights of the X, Y letters are the corresponding row weights, and those of U, V are their negatives. For the root $e_i - e_j$, a positive spectator column therefore contributes multiplication by $X_{i\alpha}U_{j\alpha}$. A negative spectator column contributes differentiation in $V_{i\beta}$ and $Y_{j\beta}$, which kills L. On the source torus the like-sign letters have the negative of their column weights since Hom acts through the inverse source matrix. For $e_\alpha - e_\beta$ a negative spectator row contributes multiplication by $U_{l\alpha}Y_{l\beta}$. A positive spectator row differentiates in X and V and gives zero. The independent spectator actions add. Permutations within each sign block give

the same constants for its indices. Determinant characters have zero derivative on off-diagonal root matrices.

In the row order used for Y take

$$Q_{\alpha\beta} = (c_1/c_2)\, X_{i\alpha}(Y^{-1})_{\beta j} P.$$

The inverse here denotes the adjugate divided by $\det Y$, so this expression is polynomial and involves only X, Y. Applying the second action and summing contracts Y with its inverse on the negative target indices. The result is the first action applied to P, first on $\det Y \neq 0$ and then identically as a polynomial. □

Proposition 77.21 (The compact weights of the first factors)**.** *At each embedding number coordinates $0, \dots, s-1$ in their original order, the first a positive. In each sign block use decreasing original index as highest-weight order. Write $\sigma_0, \dots, \sigma_{s-1}$ for the decreasing Harish–Chandra list of Theorem 73.27 at this embedding. The first-factor vector is compact highest on W_j of weight*

$$v_i = \begin{cases} \sigma_0 - \sigma_i - i, & 0 \le i < a, \\ p + \mu_\tau, & a \le i < s. \end{cases}$$

In particular this is the right compact weight of each period $\mathcal{B}_i$, whenever the period is nonzero.

Proof. The splitting character on the target has weight m. The inverse determinant of its action on negative Hom coordinates contributes $-m$ on positive rows and $-2m$ on negative rows. These are respectively the numbers of opposite-sign source columns with a minus sign. After the splitting character the vacuum weights are therefore $0, -m$. In like-sign letters a compact target matrix k sends the variable row array to k^{t} times the array. This is the standard action on row-labelled generators, since the linear creation operator transforms by the target action on its basis vector. On the negative rows the polynomial uses all rows of Y in one determinant of power $p + m + \mu_\tau$. Its contribution on the negative block of W_j is that determinant character irrespective of the ordering of the padding.

For the positive block, a minor of height index $h < a$ has exponent $c^+_{h,\tau}$. The consecutive initial $k-1$ entries in the Harish–Chandra list make this exponent zero for $1 \le h < k-1$. For $k-1 \le h < a$ the number $a-h$ is at most $j \le m$. The $2m$ rows of the minor consequently include every row $h, \dots, a-1$ of W_j, and its other rows lie in the padding. A triangular matrix upper in the specified reversed order acts through its transpose on that suffix of rows, preserving it and multiplying its determinant by the suffix diagonal product. Hence the positive weight at i is $\sum_{1 \le h \le i} c^+_{h,\tau}$. Through $\sigma_{a-1} = H - w_{j,\tau}$ every entry exceeds $(t-1)/2 - w_{j,\tau}$. Thus at $1 \le h < a$ the column count is $\sigma_{h-1} - \sigma_h - 1$. Its sum gives the stated formula. All strict compact upper roots kill the vector, also when a sign block is empty. The auxiliary short-row factors use rows outside W_j. □

Theorem 77.22 (Holomorphic weight of the first-factor periods)**.** *Let u be the weight of the conjugate of a good form on W_j described in Theorem 77.8, and u° the weight of the complementary real coefficient K_∞ in Lemma 77.1. At any adelic point z the function $h \mapsto \mathcal{B}_i(zh)$ is holomorphic of weight $u - u^\circ$ for the order above, with the positive coordinates first. In particular $h \mapsto \mathcal{B}_i(zh) K_\infty(h)$ is holomorphic of the same weight as the conjugate good form.*

Proof. Differentiating the integral defining the first factor in an upper noncompact root direction of W_j replaces its polynomial by $d\omega(E^W)P$. By Lemma 77.20 this gives a sum of integrals of the form $\int_{[U(S)]} \phi(x) R_{E^S} \theta_Q(x, y)\, dx$. Right invariance of measure turns this into

$-\int_{[U(S)]}(R_{E^S}\phi)(x)\theta_Q(x,y)\,dx = 0$. Indeed Theorem 77.6 makes ϕ holomorphic with the positive coordinates first, so every such upper root kills it. For real derivatives the integration by parts follows by right translating under the integral and differentiating at the identity. The carrier and its derivatives decrease rapidly, and the theta kernel and its derivatives have moderate growth uniformly for the translates in a compact neighborhood. This justifies the differentiations, and complex linearity gives the root identity. Compact upper roots already kill the polynomial by Proposition 77.21.

We identify the resulting weight v. Write $\delta = j - m/2$. On a positive index $u_i^\circ = \delta$. On a negative index $i = s-1-l$, $0 \le l < b$, the complementary weight is

$$u_i^\circ = \delta + N_0 + \sum_{h=l+1}^{b} c_{h,\tau}^-.$$

This follows from the determinants on the ordered negative columns in Lemma 77.1. In its block formula the Schur complement $S_- = D - CA^{-1}B$ of the positive block transforms under right block upper matrices by $S_- \mapsto S_- b_{--}$. In particular its leading minors transform by the corresponding upper triangular characters. This also proves that K_∞ extends holomorphically with the stated weight.

The conjugate Harish–Chandra list $-\sigma_i$ decreases in reversed index order within each sign block. For the weight $u = -\sigma + \rho_{\mathrm{n}} - \rho_{\mathrm{c}}$, the half-sum of positive noncompact roots has entries $-b/2$ in positive coordinates and $a/2$ in negative ones. The compact half-sum at a positive i is $i - (a-1)/2$, and at the l-th negative coordinate it is $(b-1)/2 - l$. Therefore

$$u_i = \begin{cases} -\sigma_i - i + (a-b-1)/2, & i < a, \\ -\sigma_i + l + (a-b+1)/2, & i = s-1-l \ge a. \end{cases}$$

At a positive index the equality $u_i = v_i + u_i^\circ$ now follows from $\sigma_0 + \delta = (a-b-1)/2$. When $b > 0$, put $b' = (t-1)/2$ and suppress τ from w_j. For $1 \le h < b$ the negative count is $\sigma_{s-h-1} - \sigma_{s-h} - 1$, since both entries are at most $\sigma_a = -w_j - H$. At $h = b$ the upper endpoint is truncated to $-w_j - b'$, so that $c_{b,\tau}^- = H - b' - 1 = \Delta'$. Thus the sum in u_{s-1-l}° is $-w_j - b' - \sigma_i - b + l$. Using $\mu_\tau = w_j - j - (t+k-1)/2$ from Lemma 77.13, $p + N_0 = 6s$, $t = 5s$, and $k + m = s + 1$ gives $u_i = v_i + u_i^\circ$ on the negative block as well. For $b = 0$ the positive calculation covers the whole group.

We recall why the differential equations give holomorphy in the sense used here. For a fixed translate write its value as $F(h)$, let B be the ordered complex upper triangular group, and extend to the open set $G_\infty B$ by $F(hb) = F(h)\chi_v(b)$, using the diagonal character of weight v. The intersection of G_∞ and B is the compact diagonal torus, on which F has the stated character. The extension is smooth by local Hermitian Gram–Schmidt on the flags with positive first a-plane. It is holomorphic on each right B fiber. At a real group point its right differential on the complexified real Lie algebra agrees, on every upper triangular direction, with the derivative of this character times F. The strict roots and the diagonal weights give these equations. To check the Cauchy–Riemann condition, the derivative of the extension on such a direction is the sum of its holomorphic derivative there and its antiholomorphic derivative on the conjugate direction for the unitary real form. The holomorphic derivative already equals the character derivative by the fiber law, so the antiholomorphic derivative vanishes on those conjugate directions as well. The upper triangular algebra and its unitary conjugate, the lower triangular algebra, span the complex matrix algebra. Thus the antiholomorphic differential vanishes. The fiber transformation carries this condition to all of $G_\infty B$, proving the claim at every real place. □

77.7 Separation by products of theta periods

Fix the previous fields, levels, and $1 \le j \le m$. We prove that a nonzero good form F_0 on $U(W_j)$ has a nonzero projected product period using one of the carrier integrals $\mathcal{B}_i$ of Corollary 77.19. The peak estimates will be used for a circle of points on the same field.

Write $G_\infty = \prod_\tau U(a,b)$, where $a = k+j-1$, $b = m-j$, and $s = a+b$. Within the positive and negative blocks use the orders of Theorem 77.22, with the positive block first. At one place let $D_l(h)$ be the principal minor on the first l coordinates in this order. For an integral weight w decreasing inside each compact block put

$$M_w(h) = (\det h)^{w_s} \prod_{l=1}^{s-1} D_l(h)^{w_l - w_{l+1}},$$

numbering entries of the ordered list from 1 to s. The only proper minor that can have a negative exponent is $D_a = \det A$, where A is the positive square block of h. It is invertible by the unitary relations. This defines a continuous function on the group and a holomorphic function on $U(a,b)B$, where B is complex upper triangular. Its right character on B and its left character on the compact diagonal torus are the product of diagonal entries to the weights w_i. For a family of weights use the product of these functions at all embeddings, and write χ_w for the product character.

Lemma 77.23 (The circle average of a holomorphic group function)**.** *At all real places let $d(t)$ have successive diagonal entries $1, t, \dots, t^{s-1}$ in the indicated order, where $|t| = 1$. If L extends holomorphically with right upper triangular weight w, then*

$$\int_{|t|=1} \frac{L(d(t)h)}{\chi_w(d(t))}\, dt = L(1) M_w(h)$$

for normalized circle measure. In particular, whenever $L\overline{M_w}$ is integrable and $M_w \in L^2$,

$$\int_{G_\infty} L(h)\overline{M_w(h)}\, dh = L(1) \int_{G_\infty} |M_w(h)|^2\, dh.$$

Proof. First suppose all the leading minors of h_τ are nonzero. Factor each matrix as $n_\tau b_\tau$ with n_τ lower unitriangular and b_τ upper triangular. Ratios of consecutive leading minors are the diagonal entries of b_τ, so $\chi_w(b) = M_w(h)$. The lower triangular matrix $n(z) = d(z) n d(z)^{-1}$ has polynomial entries for $z \neq 0$ and extends with $n(0) = I$. It lies in the holomorphic flag domain for $|z| \le 1$. Indeed that domain consists of matrices whose first a columns span a positive plane, by Hermitian Gram–Schmidt in the given order. The first a-plane of n_τ equals that of h_τ, the graph of a contraction Z_τ of operator norm less than one. For $0 < |z| \le 1$ its new graph is $d_-(z) Z_\tau d_+(z)^{-1}$, of norm at most $|z| \|Z_\tau\|$ when both blocks occur. At zero the plane is the positive coordinate plane. For $b = 0$ the domain condition holds everywhere on the invertible matrices.

Thus $z \mapsto L(n(z))$ is holomorphic in a neighborhood of the closed disk. On its boundary the right character law gives $L(d(t)h)/\chi_w(d(t)) = M_w(h) L(n(t))$. The mean value formula proves the first identity. The nonvanishing condition on the minors is dense, since the individual minors are real analytic on the connected group and equal one at its identity. Continuity extends the identity to every h. Finally average $L(h)\overline{M_w(h)}$ under left translation by the same circle. The left character of M_w cancels the factor $\chi_w(d(t))$. Haar invariance and the assumed absolute integrability justify the exchange of the two integrals. □

Lemma 77.24 (The complementary mass relative to the good-form weight)**.** *Use the weights u, u° of Theorem 77.22 and put $J_u = \int_{G_\infty} |M_u(h)|^2\, dh$. Then $M_{u^\circ} = K_\infty$, $0 < J_u < \infty$, and,*

uniformly in j at large nodes on the original intervals,

$$\int_{G_\infty} |K_\infty(h)|\,\|h\|_{\max}^{4n_*}\,dh \le b_x^{n_*} J_u.$$

Proof. By Lemma 77.1 the complementary coefficient at each place is the product of a determinant character, a power of $\det S_-$, and leading minors of $S_- = D - CA^{-1}B = (D^*)^{-1}$. Block elimination gives $D_{a+l}(h) = \det A\, D_l(S_-)$ and $\det h = \det A \det S_-$. Substitution of the weights in Theorem 77.22 gives $K_\infty = M_{u^\circ}$. For $b > 0$, $\|S_-\| \le 1$, and the negative count at height b is $\Delta' = s^8 - (t+1)/2$. Therefore

$$|K_\infty(h)| \le \prod_\tau |\det A_\tau|^{-(N_0+\Delta')}, \qquad N_0 = 5s - 9m - 1.$$

For $b = 0$ it is a unitary character. For the good weight, every proper minor $D_l(h_\tau)$ has modulus at most $|\det A_\tau|$. For $l < a$ the product of the l largest singular values of A_τ bounds the minor, and all the other singular values are at least one. For $l > a$ use block elimination and contraction of S_-. Thus for $b > 0$, $|M_u(h_\tau)| \le |\det A_\tau|^{u_1 - u_s}$ in the ordered list at this place. All the nonboundary powers are nonnegative, and the boundary factor is exactly $\det A_\tau$. The gap $u_s - u_1 \ge 2s^8 - 2s$ was computed in the proof of Theorem 77.8. When $b = 0$, the group is compact and $|M_u| \le 1$.

Use on G_∞ the product Haar measure, with the compact factors in the matrix ball decomposition of mass one. Put $\mathcal{D} = \{Z \in M_{b,a}(\mathbb{C}) : \|Z\| < 1\}$. At a place, the measure in the chart $Z = CA^{-1}$ is $c_\tau \delta^{-s}\,dZ\,dk$, where $\delta = \det(I - ZZ^*) = |\det A|^{-2}$. The Jacobian calculation in Lemma 77.14 applies with these two block sizes. When $b = 0$ give the one-point ball volume one. The last bounds on M_u prove square integrability. Also $\|h\|_{\max} \le 2\prod_\tau |\det A_\tau|$, by the singular values of the Cartan boosts. Since $N_0 + \Delta' - 4n_* \ge 2s$ at every sufficiently large node, in both cases

$$\int |K_\infty(h)|\|h\|_{\max}^{4n_*}\,dh \le 2^{4n_*} \prod_\tau c_\tau \operatorname{vol}(\mathcal{D}).$$

For the lower bound put $\varepsilon = [1000 s^2(1 + b_x)]^{-1}$. Take $Z \in \varepsilon\mathcal{D}$ and $\|k - I\| \le \varepsilon$. The group representative of Z with positive Hermitian diagonal blocks has these blocks $(I - Z^*Z)^{-1/2}$ and $(I - ZZ^*)^{-1/2}$. Its lower left block is $Z(I - Z^*Z)^{-1/2}$ and its upper right block is the adjoint of the lower left. It consequently differs from the identity by at most 3ε in operator norm. After right multiplication by k the bound is 4ε. Every leading submatrix then has singular values in $[1 - 4\varepsilon, 1 + 4\varepsilon]$, giving $\big|\log|D_l|\big| \le 8s\varepsilon$. All weights of u have absolute value at most $5b_x$, by Theorem 77.8. Summing the absolute exponents shows that the local $|M_u|^2$ is at least e^{-1} on this set.

The indicated compact neighborhood has mass at least $(\varepsilon/3)^{2s^2}$. Indeed a maximal ε-separated subset of the compact group has at most $(1 + 2/\varepsilon)^{2s^2}$ elements, by packing operator-norm balls of radius $\varepsilon/2$ in the ambient complex matrices. Its translates of the given neighborhood cover the group. The smaller matrix ball has fraction ε^{2ab} of the full Euclidean volume and $\delta^{-s} \ge 1$. It follows that $J_u > 0$ and the mass ratio in the assertion is at most

$$2^{4n_*} e^f (3/\varepsilon)^{(2s^2 + 2ab)f}, \qquad f = [F : \mathbb{Q}].$$

Here $\log(3/\varepsilon) = O(\log b_x)$ and $f s^2/n_* = s^2/(2t^3) \to 0$. The logarithm of the last bound is at most $n_* \log b_x$ for all sufficiently large nodes, uniformly in the block sizes. □

Theorem 77.25 (Separation of good forms by the product periods)**.** *At every sufficiently large node of the fixed-(q_0, d) intervals, let F_0 be a good form on $U(W_j)$ at the prescribed level*

and on its specified real line. If

$$\int_{[U(W_j)]} F_0(x)\mathcal{B}_i(x)\theta^{\circ}_{\delta_e,Q}(x,y)\,dx = 0$$

for all first-factor indices i, all complementary finite class representatives y, and the complementary negative-column tests Q of Corollary 77.4, then $F_0 = 0$. In particular every nonzero good form has $P_\infty(F_0\mathcal{B}_i) \neq 0$ for some i.

Proof. Each $F_0\mathcal{B}_i$ is a smooth rapidly decreasing input at level J of Section 77.1. Indeed the carrier is rapidly decreasing and the kernel in $\mathcal{B}_i$ has moderate growth jointly in its two group variables. Integrating in the carrier variable leaves a function of moderate growth, also after real differentiation. The finite data are fixed by the prescribed principal subgroup at the old marks and the full stabilizers elsewhere, and the good form is cuspidal. The common-kernel assertion of Corollary 77.4 gives

$$\int_{G_\infty} F_0(xh)\mathcal{B}_i(xh)K_\infty(h)\,dh = 0 \qquad \text{for all } x, i.$$

The convolution is continuous because the input is bounded and $K_\infty \in L^1$. Thus vanishing in L^2 gives the pointwise statement.

Suppose $F_0 \neq 0$, write $F = \overline{F_0}$, and choose a peak z of F. The function $F(zh)$ is holomorphic of weight u by Theorem 77.8. Every $z_t = zd(t)$ is another peak since the compact torus acts on the right by a unitary character. Fix a carrier peak g_1. The bounded function B_0 of Theorem 77.15 depends on this carrier peak and the polynomial, and its definition uses no good-form peak. Hence the same function $D(h) = B_0(\widetilde{h})$, with $D(1) \neq 0$, occurs in Corollary 77.19 at every z_t. For each t choose a combination $L_t(h) = \sum_i a_i(t)\mathcal{B}_i(z_t h)$ satisfying

$$|L_t(h) - D(h)| \le \eta\|h\|_{\max}^{4n_*}, \qquad \eta = 2b_x^{-19n_*}|D(1)|.$$

Set $L = L_1$, where 1 is the identity circle element. With $W_t(h) = \overline{F(z_t h)/\chi_u(d(t))}$, the convolution identity gives $\int W_t(h)L_t(h)K_\infty(h)\,dh = 0$. Since $\sup|F| < 2|F(z)|$ and $|L - L_t| \le 2\eta\|h\|_{\max}^{4n_*}$, Lemma 77.24 gives, for each t,

$$\left|\int W_t(h)L(h)K_\infty(h)\,dh\right| \le 4\eta|F(z)|b_x^{n_*}J_u.$$

The left member inside the absolute value now uses the fixed L, and its integrand is jointly integrable in h, t. This follows from boundedness of W_t, D and the mass estimate. Thus the next averaging step needs no measurable choice of the coefficients $a_i(t)$. Lemma 77.23 gives the mean of $W_t(h)$ as $\overline{F(z)M_u(h)}$. By Theorem 77.22 the product $L(h)K_\infty(h)$ is holomorphic of weight u. Its pairing with $\overline{M_u}$ is integrable since $|M_u| \le 1$ by the proof of Lemma 77.24. The second assertion of Lemma 77.23 therefore evaluates that pairing as $L(1)J_u$. Averaging the displayed bound and cancelling $|F(z)|J_u > 0$ gives $|L(1)| \le 4\eta b_x^{n_*}$.

On the other hand $|D(1) - L(1)| \le \eta$. Dividing by $|D(1)| > 0$ now gives

$$1 \le 2(4b_x^{n_*} + 1)b_x^{-19n_*},$$

which fails for all sufficiently large nodes. This contradiction proves separation, including for $j = m$ when the real group is compact. □

Corollary 77.26 (A nonsingular matrix of complex periods)**.** *If the good space has dimension $d_j > 0$, then for any complex basis one can select d_j of these product periods whose matrix on that basis has nonzero determinant. The complementary polynomials can be taken to be products of minors with integral coefficients.*

Proof. The list of first factors and of finite classes is finite. By multilinearity, every compact translate of the indicated complementary polynomial is a combination of products of minors on the prescribed source columns and choices of target rows. These products have integral coefficients and there are finitely many of them. The theorem implies that the joint evaluation matrix has column rank d_j, so it has the required nonzero square minor. □

The determinant obtained here is a complex determinant for the periods involving the carrier integrals. An estimate for the arithmetic determinant also requires their comparison to the integral periods of the definite single lift, with the normalizing factors at the old and new marks retained under the twists.

77.8 Integral coordinates of the definite first-factor polynomial

The periods in Theorem 77.25 are defined with the carrier. We examine the polynomials used to compare them to the definite single lift. The size of a column in these polynomials is nearly the full rank. Taking complementary minors makes the relevant denominator depend only on the number of boxes missing from full columns.

For a p-by-p matrix X write $\Delta_{I,h}(X) = \det X_{I,[h]}$, where $[h]$ is its first h columns and I has h elements in increasing order. Fix c_h columns of each height $1 \le h < p$. As in Lemma 73.26, $C(c)$ is the integral span of products of these flag minors with the prescribed numbers, and $D(c) = \mathrm{Hom}_{\mathbb{Z}}(C(c), \mathbb{Z})$. Use the canonical map from the Weyl lattice to flag polynomials,

$$T_c\ell(X) = \ell_Y K_c(Y, X), \qquad K_c(Y, X) = \prod_h \det\Bigl(Y^{\mathsf{t}}_{:,[h]} X_{:,[h]}\Bigr)^{c_h}.$$

Cauchy–Binet expresses K_c in $C(c) \otimes C(c)$. Over the real or complex numbers it defines a positive Hermitian metric by taking complex conjugation in one argument. Products over embeddings use tensor products and distinct variables.

Lemma 77.27 (A denominator from the complementary diagram)**.** *Let $c_{\tau,h}$ be the diagrams at the embeddings, put $\widehat{c}_{\tau,p-h} = c_{\tau,h}$, and $d = \sum_{\tau,h}(p-h)c_{\tau,h}$. There is a positive integer $H_c \le \max(1,d)^{3d}$ depending only on these diagrams and p such that for any subring $A \subset \mathbb{C}$ and any $P \in C(c) \otimes A$, the unique complex vector $T_c^{-1}P$ belongs to $H_c^{-1}(D(c) \otimes A)$. The same statement holds with coefficients polynomial in auxiliary variables over A.*

Proof. First take an arbitrary diagram ν in one block. Let $m_\nu(X) = \prod_h \Delta_{[h],h}(X)^{\nu_h}$. The bilinear Fischer pairing is $[F, G] = F(\partial)G(0)$, with the differential operator formed from the same matrix entries. Monomials are orthogonal and have self-pairing the product of their factorials. If $G \in C(\nu)$ then

$$[m_\nu, G] = [m_\nu, m_\nu]G(I).$$

Indeed by diagonal row scaling only products of the same row multidegree as m_ν can contribute. Row one must occur in every factor. Once the rows strictly before k have been placed, row k must occur in every factor of height at least k. This inductively forces the principal product. Every other product also vanishes at I.

Row substitution by Y^{t} is adjoint for this pairing to substitution by Y, by the chain rule. Hence $[K_\nu(Y,\cdot), G] = [m_\nu, m_\nu]G(Y)$. For ℓ equal to evaluation at Y, the functional $[T_\nu\ell, \cdot\,]/[m_\nu, m_\nu]$ is exactly ℓ. Evaluations span the dual over $\mathbb{C}$, since a polynomial vanishing at all matrices is zero. Thus T_ν is invertible there and its inverse has this normalized pairing formula. In particular the positive integer $[m_\nu, m_\nu]$ clears the denominator on $C(\nu)$.

Let J_p reverse the p coordinates and $w = \sum_h c_h$. On invertible matrices put

$$(S_cF)(X) = \det(X)^w F(X^{-\mathsf{t}}J_p).$$

The complementary-minor identity sends each factor, after its one determinant multiplier, to a signed flag minor on the complementary rows and first $p-h$ columns. It follows that S_c maps $C(c)$ onto $C(\widehat{c})$. The inverse is integral by the same formula, whose double application is a sign. Furthermore applying S_c in both variables to K_c gives $K_{\widehat{c}}$. Indeed the new Gram matrix before the determinant multipliers is $J_p^{\mathsf{t}}(Y^{\mathsf{t}}X)^{-\mathsf{t}}J_p$; its leading h-minor is the leading complementary $(p-h)$-minor of $Y^{\mathsf{t}}X$ divided by $\det(Y^{\mathsf{t}}X)$. Thus $T_{\widehat{c}} = S_cT_cS_c^*$, with S_c^* the integral dual map. We can use the denominator of the complementary pairing.

For all embeddings take $H_c = \prod_\tau [m_{\widehat{c}_\tau}, m_{\widehat{c}_\tau}]$. The product polynomial in these independent variables has total degree d. The sum of the absolute values of its coefficients is at most

$\prod_{\tau,h}((p-h)!)^{c_{\tau,h}} \le d!$ by the determinant expansion. The self-pairing is at most $d!$ times the square of this sum. This proves the asserted bound, including $d=0$. The constructions are integral linear maps after multiplication by H_c, so extension of coefficients gives the last two assertions. □

Return to $p = s+9m+1$, $a = k+j-1$. Besides the a positive columns of W_j, use the added columns from

$$I_{3m-j} \oplus I_{6m+1} \oplus I_{3m}$$

in that order. In total these two sets have $p+2m$ columns. For each positive diagram column of height $h < a$ at τ put $2m$ constant rows above the variable matrix $(X_\tau \mid Z_\tau)$ of p rows. Numbering from zero, constant row l selects coordinate $h+l$ of W_j if $h+l<a$, and otherwise coordinate l of the first added summand. For a column of height a use instead the $2m$ selected rows of the short matrix $\tau(Y)$ of the definite class in I_{6m+1}, supported in the second added summand. There are $c^{+}_{h,\tau}$ columns of the former kind and Δ' of the latter. Indeed the positive count at a is $(H-w_j)-((t-1)/2-w_j)-1=\Delta'$, by the list in Theorem 73.27, and subsequent counts vanish. Here $H=s^{8}$. Denote the product of these square determinants at all embeddings by $P(X,Z;Y)$, using one fixed column ordering. Write c for the full list of these column multiplicities.

Proposition 77.28 (The coefficient vector in the added columns)**.** *As a polynomial in X, P belongs to $C(c)$ with coefficients polynomial in Z and the conjugate entries $\tau(Y)$. Its associated vector $\zeta = T_c^{-1}P$ is homogeneous of degree $d=\sum_{\tau,h}(p-h)c_{\tau,h}$ in Z. Put $q_*=\sum_{\tau,h}c_{\tau,h}$. At any complex specialization where the entries of Z and $\tau(Y)$ have modulus at most M^4, one has*

$$\|\zeta\|^2 \le M^{9(12m+1)q_*}, \qquad M=p_0p_1.$$

The metric is the one defined by K_c.

Proof. In a factor of height $h<a$ all the columns $h,\dots,a-1$ are selected by distinct coordinate rows, since $a-h\le j$ for every nonzero count. Subtract multiples of these rows to clear the corresponding entries in the bottom p rows. None of the other columns changes in this operation. For height a there are no columns to clear. The first h columns are now the first h columns of X_τ with $2m$ zero entries appended above each. The remaining matrix D of $p+2m-h$ columns has entries among $0,1$, the entries of Z_τ and $\tau(Y)$. Expanding on the first h columns gives a sum of the flag minors $\Delta_{I,h}(X_\tau)$. Their coefficients are signed maximal minors of D, taking all the top rows and the $p-h$ bottom rows outside I. Every contributing monomial has exactly $p-h$ entries of Z_τ, one from each remaining bottom row. This gives the integral polynomial statement, and taking products gives the degree. The inverse T_c^{-1} acts only on the X-coefficients.

For one factor the single-column Weyl metric is the usual exterior-power metric, as follows from Cauchy–Binet for K_c. The sum of the squared moduli of the indicated minors of D is at most $\det(D^{*}D)$, the sum over all row selections. The latter is at most the product of the squared column lengths. Each length squared is at most $(p+2m)M^{8}\le M^{9}$ for large nodes, and $p+2m-h\le 12m+1$ since $h\ge k-1=s-m$.

To pass to the product, tensor the exterior-power spaces of all its columns and embeddings. Evaluating the first h columns of X_τ gives their wedge in each factor. The inner product of the resulting tensors at two matrices is K_c with conjugation in the first argument. On their span this identifies the tensor metric with the metric defined by K_c. Representing a product of linear functionals amounts to restricting their tensor to this span, so its squared norm is bounded by the product of the individual squared norms. This gives the estimate. □

Theorem 77.29 (A common arithmetic denominator on the prescribed family)**.** *On the same fields and intervals, at each j a single positive integer H_c works for all definite classes, row choices, old residue matrices and tame twists. Every coordinate of $H_c\zeta(Z;Y)$ in the Weyl lattice is a polynomial of degree d in the added columns with coefficients in the ring generated over $\mathbb{Z}$ by the $\tau(Y)$. One has*

$$d \le (10m+1)q_*, \qquad \log H_c \ll_{q_0,d_{\mathrm{K}}} fmb_x \log b_x,$$

where d_{K} is the fixed positive quadratic discriminant parameter of the field family. The short-row matrices may all be chosen with entries of modulus $\le M^4$ in every embedding. At any specialization with the same bound on the added entries, the squared Weyl norm of $H_c\zeta$ is at most $\exp(O_{q_0,d_{\mathrm{K}}}(fmb_x \log b_x))$. These assertions about size apply to every conjugate for which the indicated specialization bounds hold. Multiplication by the prescribed added residue datum preserves both the integrality and this estimate.

Proof. Apply Lemma 77.27 over the coefficient ring of Proposition 77.28. In every nonzero diagram column $p-h \le 10m+1$. The total column count is $\sum_\tau \mu_\tau = O(fb_x)$ by Lemma 77.13 and the weight bound in Theorem 73.27. Thus $\log H_c \le 3d \log \max(1,d) = O(fmb_x \log b_x)$. Here $f \ll x^{12}$, $m \ll \sqrt{x}$, and $b_x \ge x^{400}$. All dimensions and multiplicities used for H_c are fixed before the choices and twists. The norm estimate follows from Proposition 77.28 and $\log M \le 22 \log b_x$.

To obtain the short rows, move the self-dual lattice of I_{6m+1} by the finite representative of its class. The Hermitian length of any nonzero lattice vector is a totally positive integer of F, since the finite lattice is integral. Its trace is at least f by the arithmetic–geometric mean inequality and the positive integral norm. Thus its Euclidean length over all embeddings is at least one. As in Lemma 77.9, the shortest Euclidean dual length is at least $2/(q_1 d_{\mathrm{K}})$, using the absolute different and Hermitian self-duality. The Euclidean transference inequality in that lemma supplies $2f(6m+1)$ independent lattice vectors of length at most $f(6m+1)q_1 d_{\mathrm{K}}$. They span over E. Choose an E-independent subset of $6m+1$ and use these vectors as the rows of Y. The bound is polynomial in x, since $q_1 \le 8f^2$, and is smaller than M^4 at large nodes. The Euclidean bound covers one embedding of each conjugate pair and thus every embedding in absolute value. The row construction can be used at all the twists. At a finite place the polynomial conclusion applies in particular to any integral specialization of its row and added entries in the indicated frames.

Finally let $R = \mathcal{O}_E/\mathfrak{a}^2$ be the residue ring of the old marks, and allow the modulus of the added matrix to include the new marked primes of the twist. Its first two added blocks are required to be the prescribed inclusion columns in that modulus. If this test holds, its residue value is

$$\sum_{\gamma \in \mathrm{GL}_{3m}(R)} \psi_R(\mathrm{tr}(\Lambda Z_{\mathrm{last}}\gamma))\,,$$

where Z_{last} is the last $3m$ columns reduced to R and ψ_R is the primitive additive character. It is zero if the inclusion test fails. Each summand is a root of unity. Thus the value is an algebraic integer and its modulus in every conjugate is at most $|R|^{9m^2} = M^{18fm^2}$. Multiplication adds $O(fm^2 \log M)$ to the logarithmic bound for the squared norm. This is absorbed in the stated estimate. □

This result controls the coefficients inserted in the definite theta kernels. To obtain the integral period comparison one must also control the arithmetic evaluations of those kernels and the scalar relating their class sums to the carrier periods.

77.9 Fourier coefficients at the twist primes and definite theta values

Fix a field and an index j from Theorem 77.29. Set $p = s + 9m + 1$, $t_j = 9m + 1 - j$ and $M = p_0 p_1$. Write $R = \mathcal{O}_E/\mathfrak{a}^2$ for the ring of the two old marks. For the ideal $\mathfrak{q}$ of an allowed tame twist put

$$S = \mathcal{O}_E/(\mathfrak{a}^2\mathfrak{q}) = R \times R_{\mathfrak{q}}, \qquad R_{\mathfrak{q}} = \mathcal{O}_E/\mathfrak{q}.$$

The equality is the Chinese remainder isomorphism. For the unit ideal the second additive group has one element. Put $G = \mathrm{GL}_{3m}(R)$. The following calculation keeps the orbit multiplicities in the trace characters. This matters even for the old matrix $\Lambda = 0$.

For a finite additive group A, let $\widehat{A}$ be its group of complex characters, and use the Fourier convention

$$f(x) = \sum_{\lambda \in \widehat{A}} \widehat{f}(\lambda)\lambda(x), \qquad \widehat{f}(\lambda) = |A|^{-1} \sum_{x \in A} f(x)\overline{\lambda(x)}.$$

The three positive definite tests have the following residue functions. The first is the unmarked constant vacuum. The second is the point indicator on $S^{p\times j}$ at the prescribed inclusion, multiplied in its argument by a unit of norm minus one in the comparison convention. Denote the resulting point by I_1. The third is a function of $(y, z) \in A \times B$, where $A = S^{p\times t_j}$ and $B = S^{p\times 3m}$. It is

$$\mathbf{1}_{y=I_2} \sum_{g \in G} \chi_g(z), \qquad \chi_g(z) = \psi_R(\mathrm{tr}(\Lambda \bar{z} g)) .$$

Here I_2 fixes the inclusion on the two constrained blocks of $3m - j$ and $6m + 1$ columns, $\bar{z}$ means reduction to R, and ψ_R is the prescribed additive character.

Proposition 77.30 (Exact Fourier coefficients of the three residue tests)**.** *The Fourier coefficients of the point test and the added test are, respectively,*

$$\frac{\overline{\lambda(I_1)}}{|S|^{pj}}, \qquad \frac{\overline{\alpha(I_2)}}{|S|^{pt_j}} \#\{g \in G : \chi_g = \beta\} \quad (\alpha \in \widehat{A},\ \beta \in \widehat{B}).$$

The sums of the absolute values of the three coefficient rows, at every complex embedding of their fields of definition, are $1, 1, |G|$*. The tensor product of the three rows has the same sum* $|G|$*. Its entries become algebraic integers after multiplication by* $|S|^{p(9m+1)}$*. At any prime away from* $N_{E/\mathbb{Q}}\mathfrak{q}$ *it suffices to multiply by the single old integer*

$$D_{\mathrm{old}} = M^{2fp(9m+1)}.$$

It works simultaneously for all j*, old matrices and twists at the given field.*

Proof. Fourier inversion for the indicator at I_1 gives the first formula. In the added test the Fourier sum in y gives the point coefficient at I_2, while orthogonality in z contributes one for each equality $\chi_g = \beta$. Thus all terms with the same frequency have the same phase, including when the stabilizer of the trace character is nontrivial. Their multiplicities sum to $|G|$. In a complex embedding each root of unity still has modulus one and all multiplicities remain nonnegative integers. This proves the norm statements, since the unmarked vacuum has a single coefficient equal to one. The tensor denominators have exponent $pj + pt_j = p(9m + 1)$. The chosen half of the primes above each old mark has total residue degree f, the old marks are unramified, and hence $|R| = M^{2f}$. The remaining factor in $|S|$ is $N_{E/\mathbb{Q}}\mathfrak{q}$, which is a unit at the stated primes. □

Proposition 77.31 (An integral retraction on the finite Fourier rows)**.** *For the same reduction of the inclusion points at the old marks, the passage to the twisted residue tests on the tensor*

product of the three finite coefficient spaces is a split injection over the cyclotomic integers with $N_{E/\mathbb{Q}}\mathfrak{q}$ inverted. At every complex embedding this injection preserves the sum of absolute values of the coefficients. It applies to arbitrary linear combinations of old rows with these fixed new inclusion points.

Proof. Separate the old and new matrix variables by the Chinese remainder isomorphism. The trace characters factor through the old variables. On the new constrained columns the residue function is a point indicator at the pair $I = (I_{1,\mathrm{new}}, I_{2,\mathrm{new}})$ in the additive group $A' = R_{\mathfrak{q}}^{p\times(9m+1)}$. On the other new columns it is constant. Thus on Fourier rows the map tensors with the row $v_\lambda = \overline{\lambda(I)}/|A'|$ on $\widehat{A'}$ and with the unit vector at the trivial frequency on the other new columns. These factors have norm one for the sum of absolute values. To retract, keep that trivial frequency and sum the entries over λ with multipliers $\lambda(I)$. This sends the tensor factors back to one. All coefficients of the two maps lie in the indicated ring. If the old unit of norm minus one is changed, the old point row is multiplied coordinatewise by the character values of the displacement. This diagonal operation and its inverse are integral and have norm one at every embedding. □

We next evaluate the three positive definite kernels in the marked CM normalization of Theorem 67.8. In the notation (r_i, s_i, d_i) for the source rank, target rank and polynomial degree they have parameters

$$(m, 3m, 0), \qquad (j, p, 0), \qquad (12m + 1 - j, p, d).$$

The real test on the third kernel is the Weyl-vector-valued polynomial $P_2 = H_c\zeta(Z; Y)$ of Theorem 77.29. The other polynomials P_0, P_1 are one. The vector-valued evaluation means to apply the same complex linear theta functional to each coefficient. Use real identity arguments and allow arbitrary finite pair arguments, including the block diagonal arguments occurring in a definite class sum. Denote the resulting kernels by $K_i(P_i)$. The three residue tests above specify their finite vectors in the unit lattice vacuum. Conjugating a positive kernel value is allowed by conjugating its normalization as well.

Theorem 77.32 (Conjugate bounds for the positive definite class kernels)**.** *At each field and j there are nonzero scalars a_i, depending on the marked oscillator identifications, the ranks and the polynomial degrees, such that $a_iK_i(P_i)$ are algebraic in the indicated scalar or Weyl coordinates. At every complex embedding ρ of their joint coefficient field,*

$$\prod_{i=0}^{2} \max(1, \|\rho(a_iK_i(P_i))\|) \le \exp\bigl(C_{q_0,d_{\mathrm{K}}}\, fmb_x \log b_x\bigr).$$

The same constants and scalars apply to all classes, old matrices and allowed twists on the original intervals. The quadratic CM field and marked elliptic differential are fixed as the field varies in the interval.

Proof. Write $D_i = fr_is_i$, $c = \pi/A$ in the marked differential. The products r_is_i are even since m is even and $p = s + 9m + 1$ is even. Theorem 67.8 identifies each kernel, multiplied by $a_i = C_i^{-1}c^{d_i/2}$, with $P_i(\partial)G_{\phi_i}(0)$. Here G_{ϕ_i} is the marked section of its finite Weil vector. The scalar C_i is fixed by the unit product vacuum in that rank. Changes of the finite pair arguments and of the characters in the pair action change the finite vector in this formula.

In the split position model a character of an integral residue box multiplies its unit vacuum by that character on the box. This multiplication is a frequency translation. Indeed the dual of the quotient of the position lattice by the residue sublattice is its annihilator quotient in the

frequency space. Any two lifts of the same frequency act identically on the box. At all places off this box the vector is the vacuum. Taking products gives the finite expansion in rational torsion translates of the standard vacuum, with the Fourier coefficients of Proposition 77.30. Rational representatives of the frequency cosets exist by additive approximation. Correcting the Weyl convention or changing such representatives can insert roots of unity. The sums of absolute values are therefore bounded at every embedding by $L_0 = L_1 = 1$ and $L_2 = |G|$.

First use finite pair representatives supported at completely split primes outside the level, residue modulus and discriminant, as in Theorem 65.11. The local calculation in Lemma 65.12 sends the unit vacuum to the normalized vacuum on the moved Hom lattice. It transports each of the frequency vectors into a torsion vector relative to that lattice by conjugation of the Heisenberg action. The determinant character factors of these pair actions are algebraic and have modulus one in every embedding, by the ray class argument of Theorem 65.13. Theorem 65.9 now gives algebraic Taylor coefficients and, for every embedding,

$$|G^{\rho}_{\phi_i}(v)|\, e^{-\pi\|v\|_2^2/(2A_\rho)} \le L_i\big[e\max(1, 2r_i s_i\,\mathrm{rd}(E)/\pi)\,\nu_\rho\big]^{D_i/2}.$$

In fact a torsion translate has the same metric supremum as its vacuum. Apply the triangle inequality to the translated sections after conjugation. This argument places no upper bound on the orders of their torsion points.

For arbitrary finite arguments choose open compact subgroups fixing the finite test under the two pair actions. Such subgroups exist by smoothness. At these levels express each argument as a rational point times a representative just used times a right level element. The right elements fix the test. Using rational automorphy transfers the rational points to the real polynomial. Their effect on the matrix variable at each embedding is a left and a right unitary substitution and on the vacuum line a circle character. The substitutions have algebraic entries in the CM field and their conjugates are still unitary, since field conjugation commutes with its complex involution. The circle character on a rational determinant is algebraic and all its conjugates have modulus one by the same determinant argument. For the third kernel a matrix argument in the unit polydisc is thereby sent to one with entries of modulus at most $\sqrt{D_i}$, by preservation of the Hilbert–Schmidt norm. This is at most M^4 on the interval by the bounds in Lemma 77.9. Thus for every ρ the conjugate of the transformed polynomial has supremum in Weyl norm on the unit polydisc at most $B_2 = \exp(O_{q_0,d_{\mathrm{K}}}(fmb_x \log b_x))$, by Theorem 77.29. For the constant inputs take $B_0 = B_1 = 1$. These arguments also prove the asserted algebraicity for arbitrary finite arguments.

For completeness let $P(v) = \sum_{|\alpha|=d_i} v_\alpha v^\alpha$ be one of the resulting conjugate polynomials. The Cauchy formula on the unit polytorus bounds $\|v_\alpha\|$ by B_i, including for the vector polynomial by integration in the finite-dimensional Hilbert space. On the same torus the last display bounds the section by its right side times $\exp(\pi D_i/(2A_\rho))$. The Cauchy formula for the section therefore gives

$$\|P(\partial)G^{\rho}_{\phi_i}(0)\| \le B_i(D_i+1)^{d_i} d_i!\, L_i\,[e\max(1, 2r_i s_i\,\mathrm{rd}(E)/\pi)\nu_\rho]^{D_i/2} e^{\pi D_i/(2A_\rho)}.$$

Indeed there are at most $(D_i+1)^{d_i}$ multi-indices of degree d_i, and $\prod_a \alpha_a! \le d_i!$. The case of degree zero gives the value of the section at zero.

Only finitely many marked differentials occur among the conjugates of the fixed elliptic data, so A_ρ^{-1} and ν_ρ are bounded with K and the marking fixed. Lemma 77.9 gives $\mathrm{rd}(E) \le q_1 d_{\mathrm{K}}$, and the interval has $f \ll x^{12}$, $s \ll x^5$, $m \ll \sqrt{x}$ and $b_x \ge x^{400}$. Thus $D_i \log(2r_i s_i\,\mathrm{rd}(E))$ and D_i are absorbed in the claimed logarithm. The degree satisfies $d \ll fmb_x$, and $\log(d+2)$ and $\log(D_i+1)$ are $O(\log b_x)$. Finally $L_2 = |G| \le M^{18fm^2}$ and $\log M \le 22 \log b_x$. Multiplying the three bounds proves the theorem. □

Proposition 77.31 concerns the finite coefficient rows before applying the theta functional. Theorem 77.32 controls the archimedean sizes after evaluation. An integral comparison of the projected periods still requires finite-place bounds for those evaluated derivatives and the identity relating the definite class sums to the projected carrier integrals.

77.10 An all-prime normalization of the definite class kernels

The comparison with the marked elliptic curve in Theorem 77.32 uses a rational orthonormal basis. A suitable choice makes the lattice index in that comparison small enough at the finite places. We make this choice once per field and pair of ranks, before specifying any twist or testing any prime.

Write $K = \mathbb{Q}(\sqrt{-d_{\mathrm{K}}})$ and $E = K(\zeta_{q_1})$. The prime divisors of q_1 form a fixed set on the original interval, since $q_1 = 3q_0l_0 2^e$ with l_0 a power of three. Put $u = \mathrm{rad}(q_1)$, $E_0 = K(\zeta_u)$, and choose the primitive roots compatibly. Set $h = q_1/u = bw^2$ with b squarefree and $w > 0$. There are finitely many possibilities for b, all divisors of u.

Lemma 77.33 (A bounded change of lattice to product coordinates)**.** *Let rr' be even, $D = frr'$, and give $E^{rr'}$ the K-Hermitian trace form $H(X,Y) = \mathrm{Tr}_{E/K}(\sum X_i\bar{Y}_i)$. There is a positive integer C_0 depending only on q_0, d_{K}, common to all ranks, with the following property. One can choose a K-orthonormal basis spanning a product period module B over $\mathcal{O}_K$, such that for $n = C_0wq_1$ and the standard oscillator period module L one has*

$$nB \subset L, \qquad nL \subset B, qquadnV,\ nV^* \text{ have algebraic integer entries.}$$

Here V is the matrix of the basis vectors in the oriented CM embeddings. In particular $I = [L : L \cap B]$ divides n^{2D} and $\log n = O_{q_0,d_{\mathrm{K}}}(\log(2x))$ at each node x.

Proof. The cyclotomic and quadratic discriminants are coprime, and $[E : E_0] = \varphi(q_1)/\varphi(u) = h$. Thus $T^h - \zeta_u$ is the minimal polynomial of ζ_{q_1} over E_0. Moreover $\mathcal{O}_E = \mathcal{O}_{E_0}[\zeta_{q_1}]$: the integer rings of the quadratic and the cyclotomic fields multiply to the integer ring of their compositum by coprimality of the discriminants, both at q_1 and at u. Consequently

$$\mathcal{O}_E = \bigoplus_{a=0}^{h-1} \zeta_{q_1}^a \mathcal{O}_{E_0}.$$

The roots of the minimal polynomial are all the h-th roots of ζ_u. The trace to E_0 of $\zeta_{q_1}^a\overline{\zeta_{q_1}^{a'}}$ is h when $a = a'$ and zero otherwise, by summing the powers of the h-th roots of unity. The summands in the display are therefore mutually orthogonal for the trace to K.

Group the rr' entries into pairs. On the fixed space E_0^2 choose for each possible b a K-orthonormal basis for $b\,\mathrm{Tr}_{E_0/K}(x_1\bar{y}_1 + x_2\bar{y}_2)$. For existence, diagonalize the form on one copy over K. Each positive rational diagonal entry a, including the factor b, occurs twice. As in Lemma 65.8, Meyer's theorem on the indefinite rational form $X^2 + d_{\mathrm{K}}Y^2 + Z^2 + d_{\mathrm{K}}T^2 - S^2/a$ gives $z_1, z_2 \in K$ with $|z_1|^2 + |z_2|^2 = 1/a$. The columns (z_1, z_2), $(-\bar{z}_2, \bar{z}_1)$ orthonormalize the pair. The finitely many resulting $\mathcal{O}_K$-spans B_b are full lattices. Choose one integer C_0 such that $C_0B_b \subset \mathcal{O}_{E_0}^2$ and $C_0\mathcal{O}_{E_0}^2 \subset B_b$ for every b. Commensurability of full integer lattices proves existence of such an integer.

In each pair and each cyclotomic summand multiply the chosen basis by $\zeta_{q_1}^a/w$. The trace identity and $h/w^2 = b$ prove orthonormality, and the direct sum of all these bases spans $E^{rr'}$. Their integral span satisfies

$$C_0wB \subset \mathcal{O}_E^{rr'}, \qquad C_0\mathcal{O}_E^{rr'} \subset wB \subset B.$$

The oscillator period module is obtained from $\mathcal{O}_E^{rr'}$ by the inverse half of the relative different at the specified split components. Locally its scalar is one or the inverse of a local generator of that different. Thus $\mathcal{O}_E^{rr'} \subset L$ and $q_1L \subset \mathcal{O}_E^{rr'}$ by the different calculation in Lemma 77.9. This proves both inclusions in the statement. The first inclusion of the last display also makes the

entries of C_0wV, and hence their complex conjugates, algebraic integers. Finally $L/(L\cap B)$ embeds in $(n^{-1}B)/B$, a group of order n^{2D}. The size estimate follows from $q_1 \ll_{q_0} x^{11}\log(2x)$ and $w^2 \le q_1$. □

For the three positive kernels in the preceding subsection set

$$e_0 = 0, \qquad e_1 = pj, \qquad e_2 = p(9m+1-j), \qquad T = 3d_{\mathrm{K}}M^2n^2.$$

Use the basis of the lemma in the definition of the marked scalars a_i of Theorem 77.32. Define

$$A_i = a_i\, M^{2fe_i}\, n^{D_i}\, T^{d_i}.$$

These scalars and H_c are chosen before the finite arguments, the subgroup fixing the residue function, the embedding of the coefficient field and the prime of reduction.

Theorem 77.34 (Integral normalized class kernels at the prescribed levels)**.** *Let ℓ be any rational prime with $\ell \nmid N_{E/\mathbb{Q}}\mathfrak{q}$, and fix a place $\lambda \mid \ell$ of the algebraic numbers. If the two finite arguments of the i-th positive definite kernel are integral at all places above ℓ, then for every homogeneous polynomial P of its specified degree with λ-integral coefficients,*

$$A_iK_i(P) \in \mathcal{O}_{\bar{\mathbb{Q}},\lambda}.$$

The conclusion applies coordinatewise to an integral lattice of polynomial-valued vectors. In particular it applies to $H_c\zeta$ whenever its row parameters are integral in the input frames at λ. For the actual polynomials and short rows in Theorem 77.32, the bound there on the product of conjugate sizes also holds with these A_i, with a constant depending only on q_0, d_{K}.

Proof. Fix i, λ. First arrange the finite pair arguments at split primes away from ℓ, the discriminant, the level and all the residue primes. This is possible in their double classes by Theorem 65.11. Choose each right level inside the integral group at ℓ, small enough to fix the test for the given finite pair actions. If the original arguments were integral above ℓ, the rational left factors relating them to these representatives are integral there with integral inverses. Automorphy transfers those factors to the real polynomial. The resulting substitutions have integral entries at λ and preserve integrality of its coefficients. Indeed the matrix entries and their inverses are integral at every place of E above ℓ, which also covers the permuted embeddings in the CM coordinates. The determinant characters on the rational factors are units there as well: their integral powers are products of conjugates of unit determinants and their inverses. The corresponding character factors of the finite split representatives are λ-units. In the ray class argument of Theorem 65.13 a positive power of a lift supported away from ℓ is principal times finite units. Its principal generator is a unit at all places above ℓ, giving the assertion by the same character formula.

We now express the section as a sum of torsion translates on the product of marked elliptic curves. For the first passage, from B to L, Lemma 65.8 sums the translates by $L/(B\cap L)$ with signs and factor $I^{-1/2}$. Lemma 77.33 implies $n^{D_i}/\sqrt{I}$ is an algebraic integer, since its square is the integer n^{2D_i}/I. For the next passage, from L to the moved lattice L', the normalized sum of Lemma 65.12 has denominator $[L' : L\cap L']^{1/2}$. The local modules agree above ℓ, so this index is prime to ℓ. Its normalization is a λ-unit. The remaining coefficients from the finite test are those of Proposition 77.30. For the individual leg they become λ-integral after multiplication by M^{2fe_i}, since the rest of their denominator is a power of $N_{E/\mathbb{Q}}\mathfrak{q}$. The signs, torsion composition phases and changes of rational frequency representatives contribute roots of unity. These facts give λ-integral coefficients for the whole sum after the first two integer factors in A_i/a_i.

Consider the order at ℓ of a translation point after these operations, in product coordinates modulo B. Write $L_\ell = L \otimes \mathbb{Z}_\ell$, and likewise for the other modules. The first set of coset representatives is in L_ℓ and the second is in $L'_\ell = L_\ell$. For the further finite frequency translate the class modulo L'_ℓ is killed by M^2. Indeed at a place over ℓ it is dual to a quotient of the integral position box by the old residue ideal or is trivial. This quotient is killed by the rational integer M^2, and the new residue ideal is a unit at ℓ. These assertions persist on taking a rational representative of the adelic coset. Composing the three translations adds their point coordinates. It follows that multiplying the local coordinate of the sum by nM^2 puts it in B_ℓ. Equivalently its finite order modulo B has ℓ-part dividing the ℓ-part of nM^2. This equivalence follows by tensoring the finite cyclic group generated by the point with $\mathbb{Z}_\ell$. It also holds as an upper bound for each of its elliptic product coordinates.

Use $v_\lambda(\ell) = 1$. For an elliptic translate of order N, Theorem 66.35 bounds its derivative of order k below in valuation by

$$-k\delta - \lfloor k/2 \rfloor (\kappa_\ell - 2\delta)_+, \qquad 0 \le \delta \le v_\ell(N).$$

Here $\kappa_\ell \le 1$, and it is zero away from $3d_{\mathrm{K}}$ by Theorem 67.12. For the coordinates just found this loss is at most $kv_\ell(3d_{\mathrm{K}}nM^2)$. This includes $k = 0$. A multi-derivative on an elliptic product distributes across the single-coordinate factors. Therefore multiplying by $(3d_{\mathrm{K}}nM^2)^{d_i}$ clears every derivative of total order d_i in product coordinates. The change from the embedding coordinate Z to product coordinates is $z = V^*Z$. Iterating the chain rule introduces products of d_i entries of V^*, all cleared by the further factor n^{d_i}. Hence T^{d_i} clears the undivided derivatives in the matrix variables. Contraction with the integral coefficients of P and addition preserve integrality. The marked identity of Theorem 67.8 now proves the claimed bound for the kernel. For a vector, apply this argument to its coordinates in an integral basis.

To retain the archimedean estimate, the extra logarithm for the three kernels together is

$$2fp(9m+1)\log M + \Big(\sum_i D_i\Big)\log n + d\log T.$$

The bounds of Lemma 77.33 and Theorems 77.29 and 77.32 make this $O_{q_0,d_{\mathrm{K}}}(fmb_x \log b_x)$. All the added factors are positive rational integers. Their sizes in every embedding are the same. □

The normalization now controls the three evaluated positive kernels at integral arguments as well as their complex conjugate sizes. For the projected period one must also track the output Weyl lattice at a general finite representative, the class integration weights and the equality with the carrier-defined factors.

77.11 The definite class sum in its Weyl lattice

We pass from integral arguments in Theorem 77.34 to arbitrary finite representatives, keeping track of the lattice in the target. Let $C(c)$, $D(c) = C(c)^{*}$ and T_c be the flag-minor module, its dual and the map of Lemma 77.27. All three include the tensor product over the oriented embeddings. For a matrix A in one block set

$$R_A f(X) = f(A^{\mathsf{t}} X), \qquad \rho_D(A) l(f) = l(R_{A^{\mathsf{t}}} f).$$

These formulas define representations since transposition reverses products. Substitution and Cauchy–Binet show that they preserve their respective integral lattices for any invertible integral matrix with integral inverse. The identity $K_c(Y, A^{\mathsf{t}} X) = K_c(AY, X)$ proves $R_A T_c = T_c \rho_D(A)$. We use R and ρ_D componentwise for tuples of matrices.

For a finite target argument $h \in U_p(\mathbb{A}_{F,f})$ and a place $\lambda \mid \ell$ of the algebraic numbers, put

$$\mathcal{D}_{h,\lambda} = \rho_D\big((h_{\lambda \circ \bar{\tau}})_{\tau}\big)\big(D(c) \otimes \mathcal{O}_{\mathbb{C}_\ell}\big).$$

Here τ is one of the chosen embeddings of E into the complex numbers, $\bar{\tau}(x) = \overline{\tau(x)}$, and $h_{\lambda \circ \bar{\tau}}$ is the E-matrix component at the corresponding finite place transported to $\mathbb{C}_\ell$. We choose a coefficient embedding inducing λ to read this formula. The conjugate embedding occurs because the output vector in the class sum is complex conjugated.

Let b_0 be a representative of a definite class of rank $6m+1$, and take the short matrix Y of Theorem 77.29 in that lattice. In particular the columns of Y^{t} belong to $b_0 \widehat{\mathcal{O}}_E^{6m+1}$. Let v, w be finite arguments of ranks $m, 3m$, and put $g = \operatorname{diag}(1_{3m-j}, b_0, w)$. Use the positive kernels, the residue tests and the scalars A_i from the last two subsections. Set

$$k_0(v,w) = A_0 K_0(1; v, w), \qquad k_1(h) = A_1 K_1(1; 1, h), \qquad V(g,h;Y) = A_2 K_2(H_c \zeta(\,\cdot\,; Y); g, h).$$

The character on the first source is the tame twist times the prescribed power of the weight-zero character, and the first target character is trivial. For the second pair the source character is trivial. On both members of the third pair the character has exponent

$$e_\tau = j + (t + k - 1)/2 - w_{j,\tau} = -\mu_\tau.$$

Here μ_τ is the number of determinant factors in its stacked polynomial, as computed in the proof of Lemma 77.13.

Lemma 77.35 (Rational changes of the short rows and target frame)**.** *Choose rational unitary matrices $\gamma_b, \gamma_w, \gamma_h$ of ranks $6m+1, 3m, p$. Put $\Gamma = \operatorname{diag}(1_{3m-j}, \gamma_b, \gamma_w)$ and $Y' = Y \gamma_b^{\mathsf{t}}$. Then*

$$V(\Gamma g, \gamma_h h; Y') = \rho_D\big((\tau(\gamma_h))_\tau\big) V(g, h; Y).$$

The scalar k_0 is invariant under rational left changes of both its finite arguments. The scalar k_1 is invariant under a rational left change of its finite target argument.

Proof. In a definite pair with source and target splitting exponents e_s, e_t, a rational left change (γ, γ') of the finite arguments can be removed by the global rational action. The remaining real element is the inverse pair. Write $A = \tau(\gamma'^{-1})$, $B = \tau(\gamma)$. Its action on creation vectors in the matrix space is $L \mapsto ALB$. On polynomial generators, taking the coefficient of the matrix unit E_{ij} consequently substitutes $Z \mapsto A^{\mathsf{t}} Z B^{\mathsf{t}}$: the image of the generator is the polynomial with coefficients $(AE_{ij}B)_{kl}$. Multiplication gives the substitution on every monomial. The compact splitting on the vacuum line has target determinant exponent e_s and inverse source determinant exponent e_t for the real pair. For the inverse pair the resulting factor is $(\det A)^{e_s} (\det B)^{e_t}$. One

can see the character on diagonal elements from the vacuum weight on each circle. Both it and the determinant formula are characters of the compact groups, so equality there gives equality everywhere by unitary diagonalization. Multiplying the factors over embeddings yields the change for the theta kernel. For k_0 both exponents are zero. For k_1 only the target varies, its source exponent is zero and the test polynomial is constant. This proves the two scalar assertions.

For the third pair take $B = \tau(\Gamma)$ and adjoin its identity block on the first $a = k + j - 1$ columns, writing $B' = \operatorname{diag}(1_a, B)$. For each determinant of the stacked polynomial $P_Y(X, Z)$ in Proposition 77.28, the constant rows for Y' are the old ones times $(B')^{\mathsf{t}}$. Indeed the coordinate selectors are on the fixed columns, and the rows from Y are on the γ_b block. Substituting $A^{\mathsf{t}}X$ and $A^{\mathsf{t}}ZB^{\mathsf{t}}$ in this new determinant therefore multiplies the original full square matrix on the left by $\operatorname{diag}(1_{2m}, A^{\mathsf{t}})$ and on the right by $(B')^{\mathsf{t}}$. Its determinant gains $\det A \det B$. In the product of the determinants this has exponent μ_τ at τ. The multiplier from the vacuum is its inverse, since both exponents are $-\mu_\tau$.

Now apply the kernel only in the Z variables and retain X as polynomial variables. The cancellation just proved says that applying $R_{(A)_\tau}$ to the new flag polynomial gives the old flag polynomial. Taking T_c^{-1} converts R into ρ_D, since $P_Y = T_c\zeta(\,\cdot\,; Y)$ as a polynomial in X. The constants H_cA_2 occur on both sides. Inverting $A = \tau(\gamma_h^{-1})$ gives the desired formula. □

Let $\mathcal{C}_t$ be the full-level definite classes of rank t and d_z the order of the rational stabilizer of the corresponding lattice. Use one finite representative for each class. Put $B_* = |\operatorname{GL}_{6fm}(\mathbb{F}_3)|$. Define the positive definite class expression, in the normalizations above, by

$$\mathcal{Y}(h) = B_*^2 \sum_{w \in \mathcal{C}_{3m}} \sum_{v \in \mathcal{C}_m} \frac{\overline{k_0(v,w)\,\overline{k_1(h)}\,V(g,h;Y)}}{d_w d_v}.$$

The weights are those of full-level class integration with the real compact group and the finite level each of volume one. The estimates below hold for every choice of representatives.

Theorem 77.36 (Integral class vectors and their conjugate sizes)**.** *For every permitted twist and every finite target h, the algebraic class vector $\mathcal{Y}(h)$ belongs to $\mathcal{D}_{h,\lambda}$ at every $\lambda \mid \ell$ with $\ell \nmid N_{E/\mathbb{Q}}\mathfrak{q}$. At every complex embedding,*

$$\|\rho(\mathcal{Y}(h))\|^2 \le \exp(C_{q_0,d_{\mathrm{K}}} f m b_x \log b_x).$$

These conclusions use the same choices of A_i, H_c, B_* for all twists and the same fields and intervals as Theorem 77.25.

Proof. We first move the finite arguments into the integral group above ℓ. Rational unitary matrices are dense in the product of the local unitary groups at these finitely many places of F. To see this directly, use the Cayley chart $X \mapsto (1 - X)(1 + X)^{-1}$ on the skew-Hermitian matrices. Its inverse has the same formula in the group variable where $1 + g$ is invertible. Every local open set meets that chart. One can multiply a point by central norm-one scalars arbitrarily close to one, avoiding the finitely many scalars that give eigenvalue -1. There are infinitely many such local scalars by the same one-dimensional Cayley formula. Ordinary weak approximation in the rational vector space of skew-Hermitian matrices now gives simultaneous approximation inside all the prescribed open sets. This works also above two.

Choose rational left multipliers moving v, w, b_0, h into the integral groups at these places. Use the same multiplier γ_w for both occurrences of w. For the added pair use the resulting Γ of Lemma 77.35. The transformed short rows are integral above ℓ since

$$(Y')^{\mathsf{t}} = (\gamma_b b_0)(b_0^{-1} Y^{\mathsf{t}})$$

is there a product of integral matrices. By Theorem 77.29, $H_c\zeta(\,\cdot\,;Y')$ has integral coefficients at each coefficient place above ℓ. Theorem 77.34 applies to all three kernels in their moved frames. In particular k_0 and k_1 themselves are integral there by their invariances.

To treat the conjugated vector at λ, use that theorem also at the place composed with the original complex conjugation. The vector $\overline{V(\Gamma g,\gamma_h h;Y')}$ is consequently integral in the standard Weyl basis at λ. Lemma 77.35 gives this vector as the action of $\rho_D((\bar\tau(\gamma_h))_\tau)$ on the original $\overline{V(g,h;Y)}$. Since $\gamma_h h$ is integral above ℓ with integral inverse, it preserves the standard lattice under each of the indicated substitutions. Applying the inverse action of γ_h therefore puts the original vector exactly in $\mathcal{D}_{h,\lambda}$. Multiplication by the two integral scalars preserves this membership.

Every stabilizer used in the sum acts faithfully on the underlying integer lattice of rank $2fm$ or $6fm$. It is finite since the trace form is positive definite. Extending by the identity if necessary gives a finite subgroup of $\mathrm{GL}_{6fm}(\mathbb{Z})$. Reduction modulo three is injective on each such subgroup. Indeed a nonidentity element of prime order in the kernel would have the form $1+3^aU$ with $a\ge 1$ and U nonzero modulo three. Its power at a prime $l\neq 3$ differs from one modulo 3^{a+1}. Its cube differs from one modulo 3^{a+2}, by expansion. A nontrivial finite kernel would contain an element of prime order, a contradiction. Hence d_v, d_w both divide B_*, and $B_*^2/(d_v d_w)$ is integral. The local assertion follows on summing.

For the archimedean bound use each of the individual bounds of Theorem 77.34, at ρ or at its composition with complex conjugation as required. The Weyl metric is defined by the real Gram kernel, so the norm is the one used in that theorem. The logarithm of B_* is at most $(6fm)^2\log 3$. The full-level genus count in the proof of Lemma 121.17 bounds the logarithm of the number of classes in either rank by $O((6fm)^4\log(2\cdot 6fmq_1d_{\mathrm{K}}))$. Its hypotheses hold since $E\subset\mathbb{Q}(\zeta_{q_1d_{\mathrm{K}}})$. With $f\ll x^{12}$, $m\ll\sqrt{x}$ and $b_x\ge x^{400}$, these two logarithmic costs are absorbed in $fmb_x\log b_x$. The triangle inequality proves the result, also for a vanishing sum. All coordinates are algebraic by construction. □

This establishes integrality in the moving output lattice for the positive definite double class expression. To apply the bound to the projected periods of Theorem 77.25, the change of sign in the source space and the successive global theta integrations must identify that expression with the prescribed first factor, including its scalar across the twists.

77.12 The change of sign in the definite class series

The middle kernel in the prescribed definite class series has source $-I_j$ and target I_p. We relate it to the positive kernel of Theorem 77.36. The scalar in this relation must be chosen before the level, the finite target argument and the twisting characters.

Fix the integral finite unit u_f with $u_f\bar{u}_f = -1$ used to frame the negative source, and write $u = (1, u_f)$ at the real and finite places. Such a unit exists on these fields. Indeed E/F is unramified at every finite place: away from d_{K}, inertia acts trivially on K, and at d_{K} it acts trivially on $\mathbb{Q}(\zeta_{q_1})$. Complex conjugation acts nontrivially on each of these two fields, so it belongs to none of these inertia groups. At a split place take a pair of units with product -1. At an unramified quadratic place the norm on residue units is surjective, and on successive quotients of the principal units it induces the surjective residue trace. Lifting successively and taking the limit gives a unit of norm -1, also above two. We use the reductions of the one chosen u_f at every modulus.

Use the pair splitting whose two external characters a, b multiply the action on the Hom space by $a(\det h)b(\det g)^{-1}$ at arguments (g, h). Write υ_1 for the restriction of the fixed unitary splitting character to norm-one ideles. Denote by $K^{+}_{a,b}(\Psi; h)$ and $K^{-}_{a,b}(\Psi; h)$ the kernels of (I_j, I_p) and $(-I_j, I_p)$ at polynomial one, real identity, source identity and finite target h. Here Ψ is the position function on the marked integral residue box, and the unit vacuum is taken at the other finite places.

Lemma 77.37 (One phase for reversing the source form)**.** *There is a number ϵ_j of absolute value one such that for every such datum and every h, a, b,*

$$\overline{K^{+}_{a,b}(\Psi; h)} = \epsilon_j \, K^{-}_{a^{-1}\upsilon_1^{j}, b^{-1}\upsilon_1^{p}}\big(x \mapsto \overline{\Psi(u_f^{-1}x)}; h\big).$$

The unit in this argument is reduced in the indicated residue ring. The character of the source determinant in the displayed negative kernel has no effect at the identity source argument.

Proof. First consider just the Heisenberg data on the two Hom spaces. In the negative coordinates let D be the identity on the real coordinates and multiplication by u_f^{-1} on the finite coordinates, mapping to the positive coordinates. At infinity the signs of the Hermitian form have been reversed. In the standard finite frames the identity $u_f^{-1}\overline{u_f^{-1}} = -1$ changes the finite Hermitian pairing by a minus sign. Consequently the Weyl multiplication factors in the two coordinates are complex conjugates. The Gaussian has the same magnitude in the two real models, and the standard finite box is preserved since u_f is a unit. There is an antiunitary map J taking the positive unit vacuum to the negative unit vacuum and taking a Weyl translate by v to the translate by $D^{-1}v$. To construct it on linear combinations of vacuum translates, conjugate the coefficients and send each translate to the one specified above. Their inner products are conjugated: the Weyl multiplication factor has just been checked, the real vacuum coefficient is a Gaussian, and the finite vacuum coefficient is the lattice indicator in the corrected Weyl convention. Any quadratic correction in the symmetric convention is conjugated by the same pairing calculation. Thus the map is isometric on the span and extends, since these translates span densely by irreducibility. At a split residue box the position action is $f(x) \mapsto \overline{f(u_f^{-1}x)}$, as is also seen on the Fourier expansion of Proposition 77.30. The real polynomial one maps to one.

The map D identifies the rational translation subgroups in the two framed spaces. In fact the coordinates of a rational map L with source frame x_S and target frame x_T are $x_T^{-1}Lx_S$. In the negative space the source frame is ux_S, so the same rational map has coordinates $ux_T^{-1}Lx_S$, which are taken back to the first coordinates by D. It follows that the complex-linear functional $f \mapsto \overline{\vartheta_{-}(Jf)}$ is invariant under the same rational Weyl translations as ϑ_{+}.

For clarity, such invariant distributions span one line. In a rational Schrödinger polarization, invariance under rational frequencies forces support in the rational position points. Indeed the annihilator of the rational frequencies in the adelic positions is exactly the rational positions, by adelic additive duality. Wherever a rational modulation differs from one, multiplication by its difference from one is locally invertible. At a single rational support point a distribution in the finite variables depends only on the value of the locally constant test in a neighborhood of that point. In the real variables a distribution supported at a point is a finite sum of derivatives of the point mass. Translate the point to zero. The Fourier transform of this real point distribution is a polynomial. Rational modulations in a rational basis of the real dual give it a full set of independent periods: for these finitely many modulations take the finite neighborhood small enough that all their finite characters are one. A polynomial with these periods is constant, by restricting successively to the corresponding lines. This rules out the derivatives of positive order. Invariance under rational position translations then makes all masses equal. The equality with a multiple of the rational summation distribution follows first on compact tests and then by continuity on Schwartz tests. For the two normalized functionals the proportionality factor has modulus one: their mean squares on the quotient of the Heisenberg translation space by its rational lattice both equal the squared test norm by Lemma 67.7. The isomorphism D preserves quotient Haar probability and J preserves the norm. We obtain $\overline{\vartheta_+(f)} = \epsilon_j \vartheta_-(Jf)$ with a single ϵ_j for all data.

It remains to compare the finite pair actions under J. The linear action G of the finite pair on its Hom space is the same in the two frames and commutes with scalar multiplication by u_f. The two Weil operators, after conjugating one by J, intertwine the same Heisenberg automorphism. Their ratio is a scalar by irreducibility. We compute it on the vacuum coefficient, retaining its phase. Write L_f for the standard finite lattice in normalized coordinates, and choose a finite matrix B with $BL_f = L_f \cap G^{-1}L_f$. In the unramified lattice splitting the determinant multiplier is

$$\Delta(G) = \chi_f(\det B), \qquad \chi_f(z) = \upsilon_f(z)|z|_f^{-1/2}.$$

Equivalently the unitary lift of the lattice change fixes the phase of the vacuum overlap to $\Delta(G)/|\Delta(G)|$. The squared magnitude of this overlap is $[L_f : L_f \cap G^{-1}L_f]^{-1}$, by orthogonality of the coset translates in the normalized lattice sum of Lemma 65.8. Since $|\Delta(G)|^2 = [L_f : L_f \cap G^{-1}L_f]$, the overlap itself is $1/\overline{\Delta(G)}$ in both finite frames.

Let $z = \det B$. The fractional ideals of $(\det G)z$ and $\bar{z}$ agree. At a finite place with Smith exponents a_k of G, the exponent of z is $\sum_k \max(0, -a_k)$, and adding $v(\det G)$ gives $\sum_k \max(0, a_k)$. The exponents at the conjugate place are the negatives of the original ones as a multiset, because $G^{-1} = G^*$ in the standard finite frame. The character χ_f is unramified and $\chi_f(\bar{z}) = \overline{\chi_f(z)}$. For the latter identity the positive norm factors agree, and the unitary splitting character is trivial on $z\bar{z}$, since its restriction to the base ideles is the quadratic character of the extension and $z\bar{z}$ is a norm. Also $|\det G|_f = 1$. Consequently

$$\frac{\overline{\Delta(G)}}{\Delta(G)} = \upsilon_f(\det G).$$

This is the ratio of the conjugated vacuum overlap to the negative one. At the arguments of this lemma the real operators are the identities, and G acts on the j columns by $L \mapsto hL$. Hence $\det G = (\det h)^j$. Conjugating the external unitary character at h replaces a by a^{-1}; the overlap ratio then supplies exactly $\upsilon_1(\det h)^j$. This proves the stated change of splitting. Combining it with the functional identity and the residue-box calculation proves the lemma. □

Theorem 77.38 (Arithmetic bounds for the prescribed definite class series)**.** *Fix the same field, j, old matrix, class with short rows and tame twist as in Theorem 77.36. Write ν for the*

third pair's character, and let $\Upsilon(h)$ *be the class series with its original signs. Explicitly it is the negative kernel of* $(-I_j, I_p)$ *at the inclusion indicator, characters* (υ_1^j, ν) *and finite arguments* $(1, h)$*, times*

$$\sum_{w\in\mathcal{C}_{3m}}\sum_{v\in\mathcal{C}_m}\frac{K_0(1;v,w)\,K_2(\zeta(\,\cdot\,;Y);g,h)}{d_v d_w}.$$

All terms here use the unscaled kernels and polynomial. There is one nonzero scalar S_j*, independent of the old matrix, short-row choice, twist and* h*, for which*

$$\overline{S_j\Upsilon(h)} = \mathcal{Y}(h).$$

In particular the left side belongs to $\mathcal{D}_{h,\lambda}$ *at every prime away from the new twist conductor, and its squared norm in every complex embedding has the bound in Theorem 77.36.*

Proof. In the positive middle kernel take $a = 1$, $b = \nu^{-1}\upsilon_1^p$ and $\Psi(x) = \mathbf{1}_{u_f x = I_{\mathrm{incl}}}$. Then $b^{-1}\upsilon_1^p = \nu$, and $\overline{\Psi(u_f^{-1}x)}$ is the prescribed negative inclusion indicator. The lemma makes the conjugate of the unscaled positive kernel ϵ_j times the negative one at every h. The first positive kernel is the summand defining the carrier coefficient at w. Expanding that coefficient gives precisely the inner sum over v in the statement. The third kernel is linear in its polynomial, hence inserting $H_c\zeta$ multiplies it by H_c. Substitute these identities into the definition of $\mathcal{Y}$, without changing either class weight. The claimed equality follows with

$$S_j = B_*^2 H_c A_0 \overline{A_1} A_2 \epsilon_j.$$

Each factor is nonzero and was fixed for the given field and j before the residue data or twist were chosen. The arithmetic conclusions follow by equality with the very same algebraic vector of Theorem 77.36. □

The series in this theorem is defined on the definite group of rank p. The first factors of Theorem 77.25 are carrier integrals on $U(W_j)$. Their comparison requires the intervening global theta integrals with their levels, polynomials and character projections.

77.13 The row stabilizer in the definite theta integral

Fix one field and one interval of Theorem 77.25, a lift index j, a choice i of old matrix Λ, short rows Y_0 and definite class b_0, and a prescribed twist ϑ. Write $s = \dim W_j$, $p = s + 9m + 1$ and $a = k + j - 1$ for the positive index of W_j. The rank p space

$$W' = W_j \oplus (-I_j) \oplus I_{3m-j} \oplus I_{6m+1}$$

has m negative directions. In its ordered coordinate set let e be the p-by-s matrix including the coordinates of W_j. Put $R = \mathcal{O}_E/\mathfrak{a}^2$ and $S = \mathcal{O}_E/(\mathfrak{a}^2\mathfrak{q})$ in the chosen split components. Here $\mathfrak{q}$ is the new part of the twist modulus. For residue matrices a bar in this subsection means the reduction $S \longrightarrow R$. We write conj for ordinary complex conjugation.

Let χ be the residue character of $\chi_M^j\vartheta$, extended by zero on nonunits. Let w be any matrix of rows over R for which $\Lambda = uw$ with u a $3m$-rowed matrix over R. We can always take $w = \Lambda$. The freedom to choose w allows the same calculation with the rows occurring in an integral projection. Define

$$F_\Lambda(y) = \sum_{\gamma \in \mathrm{GL}_{3m}(R)} \psi_R(\mathrm{tr}(\Lambda \bar{y} \gamma)), \qquad y \in \mathrm{Mat}_{p,3m}(S).$$

Thus $F_\Lambda(yB) = F_\Lambda(y)$ for $B \in \mathrm{GL}_{3m}(S)$, by reindexing γ. Write $G_S = \mathrm{GL}_p(S)$, $P = 1 - ee^{\mathsf{t}}$, and set

$$H = \{A \in G_S : AP = P,\ w\bar{A} = w\}.$$

This is a subgroup, since both conditions are stabilizer conditions. Let K be the full integral level of the definite unitary group $U(I_p)$, and K_H the inverse image of H in K. The reduction map from K onto G_S is surjective. At each chosen split place lift the residue matrix arbitrarily to an integral matrix. Its determinant is a unit, so the inverse transpose gives the integral action on the dual component. Use the identity at the unmarked places.

For a p-by-s matrix x over S, set

$$D_w(x) = \mathbf{1}_{w\bar{x} = w\bar{e}}\, \chi(\det(e^{\mathsf{t}}x)).$$

For $z \in \mathrm{Mat}_{p,p}(S)$ define

$$\Phi(y,z) = \begin{cases} \chi(\det z) F_\Lambda(z^{-1}y), & z \in G_S, \\ 0, & z \notin G_S, \end{cases}$$

$$\Phi_H(y,z) = D_w(ze)\, \mathbf{1}_{zP=P}\, F_\Lambda(y).$$

Write $\Pi(A)f(y,z) = \chi(\det A) f(A^{-1}y, A^{-1}z)$.

Lemma 77.39 (Division into cosets with coefficient one)**.** *One has* $\Phi_H(y,z) = \mathbf{1}_{z \in H}\Phi(y,z)$ *and*

$$\Phi = \sum_{AH \in G_S/H} \Pi(A)\Phi_H.$$

Each term is independent of the representative of its coset.

Proof. In coordinates with the e-columns first, $zP = P$ means that z has block form $\left(\begin{smallmatrix} D & 0 \\ B & 1 \end{smallmatrix}\right)$. Thus $\det z = \det D = \det(e^{\mathsf{t}}ze)$, and $w\bar{z} = w$ is equivalent to $w\overline{ze} = w\bar{e}$. A nonunit determinant contributes zero. If $z \in H$, then $\Lambda\bar{z}^{-1} = \Lambda$ because $\Lambda = uw$. This gives $F_\Lambda(z^{-1}y) = F_\Lambda(y)$ and proves the first equality. The determinant multiplier cancels under $\Pi(A)$ on Φ, so $\Pi(A)\Phi = \Phi$. Hence $\Pi(A)\Phi_H$ is Φ restricted to $z \in AH$. These supports partition the invertible matrices, and all terms vanish elsewhere. $\square$

For a level $J \subset K$, let $\mathcal{C}_J$ be a set of representatives of $U(I_p)(F)\backslash U(I_p)(\mathbb{A}_f)/J$. For $h \in \mathcal{C}_J$ put $d_h^J = |h^{-1}U(I_p)(F)h \cap J|$. The stabilizers are finite because the real group is compact and the rational group is discrete in the adelic group. These class sets are finite by the finiteness theorem for definite lattices, as in Theorem 77.36. Write $\mathcal{A}_J(f) = \sum_{h \in \mathcal{C}_J} f(h)/d_h^J$.

Let $\Theta_\Xi(g, h', h)$ be the theta kernel from $I_{3m} \oplus W'$ to I_p, with finite datum $\Xi(y, z)$ and real polynomial $P(X, Z; Y_0)$ of Proposition 77.28. The variables y, z are the columns from I_{3m}, W', respectively. The source point has entries $h', g, 1, 1, b_0$ on the five summands, with identity real part on the definite class variables. The target point has finite part h and identity real part. In the convention of Lemma 77.37, the two splitting characters are $(\alpha, \nu) = (\nu\upsilon_1^m, \nu)$. The compact restriction of α is χ. At a real embedding its exponent is $m - \mu_\tau$, where μ_τ counts the determinant factors of P at that embedding. Indeed ν has exponent $-\mu_\tau$ there, and $\nu\upsilon_1^m/(\chi_M^j\vartheta)$ has trivial tame part.

Proposition 77.40 (Refinement of the definite class integral)**.** *For every g, h' the following unnormalized class sums agree:*

$$\mathcal{A}_K(\Theta_\Phi(g, h', \cdot)) = \mathcal{A}_{K_H}(\Theta_{\Phi_H}(g, h', \cdot)).$$

Proof. At the marked split places the integral target change $k \in K$ acts on the position function by pullback by its inverse, times the character of its determinant. This follows by substituting in the position coordinates of a self-dual lattice box. The absolute determinant multiplier is one since k preserves the lattice. The marked splitting character gives $\chi(\det k_S)$, where k_S is the reduction of k. The unmarked unit vacuum is fixed. It follows that $\Theta_{\Pi(k_S)\Xi}(g, h', h) = \Theta_\Xi(g, h', hk)$.

Both kernels are left invariant under rational changes of the finite target point. To see this, first use rational automorphy on the full adelic target. The simultaneous change at infinity can be dropped, as the vector there is fixed. Indeed, for $v \in U(p)$ at one real place, substitution on the p variable rows multiplies each stacked determinant by $\det v$: the operation on its rows is $\mathrm{diag}(1_{2m}, v^{\mathsf{t}})$. In the Fock splitting the vacuum contributes $(\det v)^{-m}$, one inverse determinant per negative source column. Multiplication by the external character contributes $(\det v)^{m-\mu_\tau}$. The product with the μ_τ polynomial factors is one. This calculation holds at each real place. The lemma now gives right invariance at K for the first kernel, at K_H for the second, and expresses the first as the sum of right translates of the second over K/K_H.

Fix a full-level representative h and put $\Gamma = h^{-1}U(I_p)(F)h \cap K$. The refined classes above h correspond to the Γ-orbits on K/K_H by $kK_H \mapsto hk$. The summand is constant on each orbit by the two invariances. The contribution of an orbit, divided by $|\Gamma|$, is its common value divided by the order of $\Gamma \cap kK_Hk^{-1}$. Conjugation by k identifies this last group with the rational stabilizer at the refined class hk. Summing all the orbits gives the equality. In particular this refinement requires no division by $[K : K_H]$, even at primes dividing that index. □

Use the unscaled first definite kernel of Theorem 77.38 to set $C_\vartheta(h') = \sum_{v \in \mathcal{C}_m} K_0(1; v, h')/d_v$. Define the following function of the actual adelic variable $g \in U(W_j)(\mathbb{A})$:

$$L_{i,\vartheta}(g) = \sum_{h' \in \mathcal{C}_{3m}} \frac{C_\vartheta(h')}{d_{h'}} \mathcal{A}_K(\Theta_\Phi(g, h', \cdot)).$$

For $\xi \in D(c) \otimes \mathbb{C}$ write $\mathcal{L}(\xi)$ for $T_c\xi$ evaluated on the positive columns of the W_j variable in their flag order. All active heights are at most a, so the other columns of its formal square argument may be filled with zero. Write $\Theta_j[D, Q](g, h)$ for the theta kernel from W_j to I_p with characters (υ_1^{m-j}, ν), residue datum D and polynomial Q.

Theorem 77.41 (The arithmetic coefficient in the theta class integral)**.** *With the single scalar S_j and the Weyl vectors $\mathcal{Y}_\vartheta(h)$ of Theorem 77.38,*

$$S_j L_{i,\vartheta}(g) = \sum_{h \in \mathcal{C}_{K_H}} \frac{1}{d_h^{K_H}} \Theta_j\big[D_w,\, \mathcal{L}\big(\operatorname{conj} \mathcal{Y}_\vartheta(h)\big)\big]\,(g,h).$$

Conjugation uses the real structure of the integral Weyl basis. The equality also holds after taking any of the cuspidal product periods of Theorem 77.25, using the same character projection on both sides.

Proof. Refine the inner class sum by Proposition 77.40. Reorder the source summands as $W_j, -I_j, I^{\mathrm{add}}$, with $I^{\mathrm{add}} = I_{3m-j} \oplus I_{6m+1} \oplus I_{3m}$. On these blocks Φ_H is a product. Its first factor is D_w. On the j negative columns it requires the inclusion columns. On the first two added blocks it imposes the remaining inclusion columns, and on the last $3m$ columns its value is $F_\Lambda(y)$. These are precisely the negative and added tests used in Theorem 77.38. The added source point is $\operatorname{diag}(1, b_0, h')$.

In any integral basis (b_β) of $D(c)$, write $\zeta(Z;Y_0) = \sum_\beta b_\beta\, \zeta_\beta(Z;Y_0)$. The polynomial identity of Proposition 77.28 gives $P = \sum_\beta \mathcal{L}(b_\beta)\zeta_\beta$. The negative block has polynomial one. On an orthogonal direct sum the three theta kernels multiply for each of these pure tensors. In detail, take the direct sum of rational polarizations of the three Hom spaces. The summation set in the position model is the Cartesian product of the three rational position spaces. The product Schwartz test gives the product of their summands. Each rational Schwartz sum is absolutely convergent at a fixed adelic argument. The factorization persists after the pair action, since the oscillator action is the tensor product in this polarization, with the unit vacua and Gaussian transforms normalized by products. The three characters on the shared target multiply to $\upsilon_1^{m-j}\upsilon_1^{j}\nu = \alpha$, and the character ν on the block source determinant factors over the blocks. Thus the normalizations are the ones used in the three individual kernels.

Now interchange the two finite class sums. At a given h, the negative scalar kernel and the sum over h' of $C_\vartheta(h')/d_{h'}$ times the added kernel on ζ_β are exactly the β-coordinate of $\Upsilon_\vartheta(h)$ in Theorem 77.38. Linearity of Θ_j in the real-place polynomial gives the asserted sum with $\mathcal{L}(\Upsilon_\vartheta(h))$ before multiplication by S_j. That theorem says $S_j \Upsilon_\vartheta(h) = \operatorname{conj} \mathcal{Y}_\vartheta(h)$, which proves the formula including the location of the conjugation.

Finally, multiplication by the complementary test, translation, and integration against a fixed cusp form are linear in the factor just computed (or conjugate-linear if the conjugate pairing is used on that factor). The fixed theta tests have moderate growth and the cusp form is rapidly decreasing on Siegel sets, as in Theorem 77.25. The products are therefore integrable. Finite class sums and the finite character projection commute with the integral. In a conjugate-linear pairing S_j is conjugated on passing outside that pairing. □

The theorem inserts the previously bounded algebraic vectors into an adelic function on the prescribed $U(W_j)$. Identifying this function, up to the required scalar, with the separating carrier integral $\mathcal{B}_i$ still requires the global comparison on the other side of the theta exchange.

77.14 The matching of the residue data and the carrier integral

Keep the field, interval, i, j, ϑ, and the matrices of Section 77.13. Write $r = 3m$ and $n = p + r$. We identify the actual data on the two sides of the rank comparison. On the carrier side this includes the multiplicity with which a single elementary character occurs in the integral.

Lemma 77.42 (Complementary minors over the marked residue ring)**.** *Over $S = \mathcal{O}_E/(\mathfrak{a}^2\mathfrak{q})$, put*

$$\Psi\begin{pmatrix} A \\ B \end{pmatrix} = \chi(\det A) F_\Lambda(B), \qquad A \in \mathrm{Mat}_{r,r}(S),\ B \in \mathrm{Mat}_{p,r}(S).$$

Let the first r columns of $k \in \mathrm{GL}_n(S)$ be $\binom{A}{B}$, and the last p rows of k^{-1} be $(y\ z)$. Then

$$\Phi(y, z) = \chi(\det k)^{-1} \Psi\begin{pmatrix} A \\ B \end{pmatrix}.$$

Every point where Ψ is nonzero occurs as these columns of some k. Every point where Φ is nonzero occurs as these rows of the inverse of some k.

Proof. Set $D = \left(\begin{smallmatrix} 1 & 0 \\ y & z \end{smallmatrix}\right)$. The bottom rows of Dk are $(0\ 1)$ and its upper left block is A. Computing the determinant of Dk in two ways gives $\det z \det k = \det A$ over the given ring, including its old nonreduced factors. Thus A and z are simultaneously invertible. If they are singular in the sense of having nonunit determinant, both functions vanish. If they are invertible, $yA + zB = 0$ gives $z^{-1}y = -BA^{-1}$. The right invariance of F_Λ under $-A^{-1}$ and the determinant identity now give the formula. For the support assertions, complete the columns by $\left(\begin{smallmatrix} A & 0 \\ B & 1 \end{smallmatrix}\right)$ when A is invertible. Complete the rows by taking $k^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ y & z \end{smallmatrix}\right)$ when z is invertible. □

The real data associated to these complementary dimensions can be compared entry by entry. Let $S_0 = I_{2m} \oplus (-I_m)$ denote the carrier Hermitian space. At an embedding τ let X_τ^+ be the $(p + 2m)$-by-$2m$ matrix from the positive coordinates of S_0 to the positive coordinates of $I_r \oplus W'$. Let X_τ^- be its separate m-by-m negative-to-negative block. Give the rows in each block the fixed order used for the minors of Corollary 77.12. For each of the μ_τ determinant factors in the stacked polynomial of Proposition 77.28, let $C_{\tau l}$ be its $2m$-by-$(p + 2m)$ matrix of constant rows. The polynomial attached to these rows on the smaller source is

$$Q = \prod_\tau \left\{ \det(X_\tau^-)^{\,\mu_\tau + n - 2m} \prod_{l=1}^{\mu_\tau} \det(C_{\tau l} X_\tau^+) \right\}.$$

Proposition 77.43 (The real polynomial on the carrier space)**.** *The variables in Q belong entirely to S_0 and W'. On these spaces Q is the product $vQ_{b_0,\rho}$ of the padded polynomial and the short-row polynomial defining $\mathcal{B}_i$ in Corollary 77.19.*

Proof. All m negative rows lie in $W_j \oplus (-I_j)$. Their exponent is $\mu_\tau + p + m = \kappa'_\tau$, the negative exponent in Lemma 77.13. For a positive diagram column of height $h < a$, row t of $C_{\tau l}$, starting with $t = 0$, selects coordinate $h + t$ in W_j when $h + t < a$, and coordinate t in I_{3m-j} otherwise. Therefore $C_{\tau l} X_\tau^+$ lists exactly those rows of the variable matrix and its determinant is the corresponding padded minor, in its given order. Its multiplicity is the prescribed $c^+_{\tau,h}$. For height a the $2m$ rows of $C_{\tau l}$ are the selected rows of the short matrix at b_0, supported entirely on I_{6m+1}. Multiplication against X_τ^+ gives the $2m$-by-$2m$ short-row determinant of $Q_{b_0,\rho}$. There is one such factor for each prescribed selection, including any repeated selections. These exhaust the factors by the description preceding Proposition 77.28. None uses a coordinate in I_r. The fixed orders of the rows give equality with the stated product, including its sign. □

Let ϕ be the carrier of Theorem 77.6. For a finite residue function f on the p-by-r position box let $\mathcal{I}_i(f;g)$ be the integral of $\phi(x)$ times the kernel from S_0 to W' with datum f, real polynomial Q just computed and target point $\mathrm{diag}(g,1,1,b_0)$. Integrate on $U(S_0)(F)\backslash U(S_0)(\mathbb{A})$ with the quotient measure defining $\mathcal{B}_i$. Use the same unmarked vacua and characters (υ_1^m,β_0) as there. In particular the character β_0 applied to the source determinant is trivial on its finite integral units. For the finite box we allow either the old modulus or the old modulus enlarged by $\mathfrak{q}$.

Theorem 77.44 (The exact multiplicity in the carrier period)**.** *For every $g \in U(W_j)(\mathbb{A})$,*

$$\mathcal{I}_i(F_\Lambda;g) = N_R\mathcal{B}_i(g), \qquad N_R = |\,\mathrm{GL}_{3m}(R)|.$$

The integer N_R is the same for every new twist on the given field and interval. With $M = p_0p_1$ one has $\log N_R \le 18fm^2\log M$.

Proof. First consider $f_\Lambda(B) = \psi_R(\mathrm{tr}(\Lambda B))$ at the old modulus. Proposition 77.43 and the definition of the first factor give $\mathcal{I}_i(f_\Lambda;g) = \mathcal{B}_i(g)$. Inflating this test from R to S gives the identical finite Schwartz vector. On the old integral boxes the value is read on the same coset. At a new mark the pullback is constant in that component on its integral position box and zero outside. This is the unit vacuum in the split position model. Thus inflation has scalar one.

For $\gamma \in \mathrm{GL}_r(R)$ lift γ at the old split marks to an element k_γ of the full finite lattice stabilizer of $U(S_0)$. The dual component is forced by the Hermitian pairing, and away from the old marks take $k_\gamma = 1$. The split position action for a source change substitutes $B \mapsto B\gamma$. The lattice Jacobian has absolute value one, and the source determinant multiplier is one by the compact restriction of β_0. The integral with test $f_\Lambda(\cdot\,\gamma)$ therefore has as integrand the original integrand translated by $x \mapsto xk_\gamma$: the carrier is fixed at this full level by Theorem 77.6. Right invariance of the quotient measure gives the same integral. The carrier is cuspidal, hence rapidly decreasing on Siegel sets, and the theta function is of moderate growth for fixed target point. Thus each integral is absolutely convergent. Summing over all γ gives the formula by linearity, also when Λ has a nontrivial stabilizer.

At an old split component with residue field of size q, the reduction from $\mathrm{GL}_r(\mathcal{O}/\varpi^2)$ onto $\mathrm{GL}_r(\mathcal{O}/\varpi)$ has kernel $1+\varpi\,\mathrm{Mat}_r(\mathcal{O}/\varpi^2)$ of order q^{r^2}. The residue group has order $\prod_{h=0}^{r-1}(q^r-q^h)$, by successive choices of linearly independent columns. Multiplication over the old components gives N_R; in particular $N_R \le |R|^{r^2} = M^{2fr^2}$, as claimed. □

For the untwisted character, Theorem 77.25 now applies to the family of functions $\mathcal{I}_i(F_\Lambda;\cdot)$ itself: replacing every $\mathcal{B}_i$ in a nonzero projected product period by this function multiplies the period by the same positive integer N_R. In particular at least one of the matched residue and real tests above has a nonzero carrier product period for each nonzero good form. The comparison with $L_{i,\vartheta}$ requires an identity of global integrals; the pointwise relation of residue functions in Lemma 77.42 supplies its finite tests.

77.15 Constancy of the first equal-rank theta average

The first of the three changes of rank involves two copies of the definite space I_m. We give a direct argument for the dependence of its class integral on the source point. The argument works at the old marks and at the new twist conductor.

Write $G = U(I_m)$ over F, and $[G] = G(F)\backslash G(\mathbb{A})$. Take a unitary character δ of the adelic norm-one group, trivial on rational points and of weight zero at every real place. Its finite restriction to units is unramified outside the marked split places and is given there by a residue character χ_δ. The application takes $\delta = (\chi_M^j \vartheta)^{-1}$. Use the pair (I_m, I_m), splitting characters (δ, δ), and real polynomial one. At every unmarked place take the standard unit vacuum. At a marked split place v use the position test on $\mathrm{Mat}_m(F_v)$

$$f_v(X) = \begin{cases} \chi_{\delta,v}(\det(X \bmod \mathfrak{m}_v)), & X \in \mathrm{Mat}_m(O_v), \\ 0, & \text{otherwise}, \end{cases}$$

where characters of the finite ring are zero on nonunits. Let $\theta_\delta(g,h)$ be the resulting rational theta sum, and let $I_\delta(g) = \int_{[G]} \theta_\delta(g,h)\,dh$. With Haar mass one on $G(F_\infty)$ and on the full finite stabilizer, this integral is the definite target class sum. Indeed these groups fix the target vector and the orbit of a finite representative has volume the reciprocal of its finite rational stabilizer order. An overall change in the quotient measure multiplies all these integrals by the same constant.

Lemma 77.45 (The two spherical averages at a split place)**.** *Let v be any split finite place, let $K_v = \mathrm{GL}_m(O_v)$ and let $d \in \mathrm{GL}_m(F_v)$. Write ω_s, ω_t for the source and target actions, respectively, in the local position model. For a in a double coset use its normalized invariant measure. Then*

$$\int_{K_v d K_v} \omega_s(a) f_v \, da = \int_{K_v d^{-\mathsf{t}} K_v} \omega_t(a) f_v \, da.$$

At an unmarked split place $f_v = \mathbf{1}_{\mathrm{Mat}_m(O_v)}$.

Proof. Choose the same split component for the two copies of I_m. The position matrix transforms in the test by $X \mapsto h^{-1} X g$. Let $Tf(X) = f(X^{\mathsf{t}})$. We claim $T\omega_s(a)T^{-1} = \omega_t(a^{-\mathsf{t}})$. For the unitary pullbacks on positions, transposition conjugates $X \mapsto Xa$ to $X \mapsto a^{\mathsf{t}} X$, and the absolute Jacobians agree. This also gives agreement for the chosen Weil splitting. In fact the split unitary group of the whole Hom space is a general linear group on the m^2 positions. On this group unitary pullback with the square-root Jacobian implements its action on the Heisenberg translations. Any other splitting implementing that action differs by a scalar character, by irreducibility of the position representation and the multiplication law. A character has the same value on conjugate position changes. Finally, the external source multiplier is $\delta_v(\det a)^{-1}$ and the target multiplier at $a^{-\mathsf{t}}$ is $\delta_v(\det a^{-\mathsf{t}})$. They agree.

The local vector is fixed by both compact actions. On integral units the position Jacobian and the unramified Hom splitting are trivial. For the marked vector the substitution on the source supplies $\chi_{\delta,v}(\det k)$, and that on the target its inverse. The two respective external multipliers cancel them. The unmarked assertion follows by lattice invariance. More generally, every Schwartz vector b fixed by both compact actions satisfies

$$b(uXv') = \lambda(\det u)\lambda(\det v')b(X) \qquad (u, v' \in K_v),$$

where λ is the restriction of δ_v to units. The same calculation of the two compact actions gives this formula for any b. By Smith normal form, every matrix over F_v is uDv' with D diagonal and $u, v' \in K_v$, also for deficient rank: first clear a common denominator, diagonalize over

the discrete valuation ring and divide it out again. The factors in the displayed formula are unchanged on passing from uDv' to $(v')^{\mathsf{t}}Du^{\mathsf{t}}$. Thus $Tb = b$ on this entire space of compact-fixed vectors.

The average on the left of the lemma is fixed by the source compact group by invariance of double-coset measure and by the target compact group since the two actions commute. Transposition therefore fixes this average. Applying T inside it, using $Tf_v = f_v$, and changing a to $a^{-\mathsf{t}}$ proves the equality. This automorphism preserves K_v and carries one double-coset probability measure to the other. □

Theorem 77.46 (Constancy on the full definite source quotient)**.** *For every twist and tame character just specified, $I_\delta(g) = I_\delta(1)$ at every adelic source point. In particular the actual square class sum with character $(\chi_M^{j}\vartheta)^{-1}$ in the first exchange is a constant function.*

Proof. Both rational quotients are compact, since the group is anisotropic over F. Rational theta sums of Schwartz tests are continuous and bounded on their product. One can see continuity on compact sets of representatives by rapid Schwartz decay in a rational position lattice, with the finite support contained in a fixed fractional lattice on each such set. Consequently I_δ is continuous and square integrable. For an unmarked nonsplit finite place the lattice vacuum and the unramified characters make I_δ fixed by the full integral source stabilizer. At split places this invariance follows from the compact calculation in Lemma 77.45. At a real place the compact definite source and target both fix the Gaussian: the Hom signs are all positive, so the untwisted Fock vacuum has compact weight zero in the splitting of Theorem 77.38, and the external weights are zero.

Fix a split v and d. Apply the lemma before rational summation and target integration, with the other components held fixed. The local double-coset average of a smooth compact-fixed vector is a finite average of translates, so it commutes with both operations. On the target side each right translate gives the same integral by right invariance of quotient Haar measure. We obtain

$$\int_{K_v dK_v} R(a)I_\delta \, da = I_\delta, \qquad R(a)I_\delta(g) = I_\delta(ga).$$

Right translation on $L^2([G])$ is unitary. In particular

$$\int_{K_v dK_v} \|R(a)I_\delta - I_\delta\|_2^2 \, da = 2\|I_\delta\|_2^2 - 2\,\mathrm{Re}\left\langle \int_{K_v dK_v} R(a)I_\delta \, da, I_\delta \right\rangle = 0.$$

The integrand is continuous and the double-coset measure has full support. It follows that $R(a)I_\delta = I_\delta$ for each a in this coset. Continuity on the rational quotient upgrades equality in L^2 to pointwise equality. Varying d gives invariance under the full group at every split finite place.

Let J be the subgroup generated by these split local groups, all finite integral stabilizers, and the full real group. We finish by showing $G(\mathbb{A}) = G(F)J$. For $g \in G(\mathbb{A})$, the element $\mathrm{diag}(\det g, 1, \dots, 1)$ belongs to J. At a nonsplit finite place a norm-one element is a unit. At the split places there are only finitely many nonintegral components. Dividing on the right by this diagonal element puts g in $G^1(\mathbb{A})$ for $G^1 = \mathrm{SU}(I_m)$.

Choose one of the old split places v_0. For $m \geq 2$, G^1 is simply connected and absolutely almost simple: over an algebraic closure it is SL_m. At v_0 its local group $\mathrm{SL}_m(F_{v_0})$ is noncompact. The strong approximation theorem for simply connected groups over number fields therefore makes $G^1(F)$ dense in the finite adeles of G^1 away from v_0 (take the omitted places to be v_0 and all real places). The full hypothesis on the omitted places is precisely noncompactness of their product for this almost simple group. We use the number-field theorem of V. P. Platonov,

The problem of strong approximation and the Kneser–Tits conjecture for algebraic groups. The integral stabilizers away from v_0 form an open subgroup of this restricted product. Thus for any such g of determinant one there is $\gamma \in G^1(F)$ with $\gamma^{-1}g$ integral away from v_0 at the finite places. Its v_0 and real components already belong to J. This proves the factorization. In rank one the diagonal determinant argument alone gives it. Rational left invariance and right J-invariance now imply $I_\delta(g) = I_\delta(1)$. □

This proves the constant-function assertion before normalizing the first exchange. Its value is the scalar $I_\delta(1)$. Using its reciprocal in the chain requires nonvanishing and a formula for how its absolute value varies with the tame twist.

78 The first and second theta exchanges

78.1 The relative finite doubling factors of the first exchange

Let $m = 2d$ and retain the positive definite pair of Theorem 77.46. The variation with the twist at all split places can be computed before using a global inner product formula. Normalize multiplicative Haar measure on $\mathrm{GL}_m(k)$ by giving $K = \mathrm{GL}_m(O)$ mass one, and additive Haar measure on $\mathrm{Mat}_m(k)$ by giving the integral matrices mass one. Here $k = F_v$, q is its residue cardinality, and ϖ a uniformizer at a split finite place. Write

$$P_m(q) = \prod_{j=1}^{m} (1 - q^{-j}), \qquad H_m(q, s) = \prod_{j=1}^{m} (1 - q^{-2s-j}).$$

For $a \in \mathrm{GL}_m(k)$ with elementary divisor valuations $b_1, \dots, b_m$, set $\ell(a) = \sum_i |b_i|$. Let $C_{\delta,v}(a) = \int (\omega_s(a) f_v)(X) \overline{f_v(X)}\, dX$, where the subscript on ω_s means the source action. The complex parameter in the next formula occurs only in the scalar $q^{-s\ell(a)}$. Put

$$Z_{\delta,v}(s) = \int_{\mathrm{GL}_m(k)} C_{\delta,v}(a)\, q^{-s\ell(a)}\, d^{\times} a.$$

We record why this is the flat-section doubling integral for the constant source vector. At the theta point $s = 0$, evaluation of the doubled Schwartz tensor $f_v \otimes \overline{f_v}$ on the diagonal graph is the inner product of its two factors: in diagonal and difference coordinates set the difference to zero and take the zero Fourier coefficient in the diagonal coordinate. This is the calculation of Proposition 142.14. At a it gives the displayed $C_{\delta,v}(a)$, including the external source character. Moving the section from zero to s multiplies its value at each point by $|\det A/\det D|^s$, where A, D are the diagonal Levi blocks of the split Siegel parabolic in a parabolic-times-compact factorization.

Here that absolute value is $q^{-\ell(a)}$. Indeed choose an integral change of basis in the doubled split component carrying the diagonal plane to the first m coordinates. In these coordinates the bottom rows of the embedded point $\mathrm{diag}(a, 1)$ have the form $(1 - a, 1)$ up to integral invertible changes of basis, equivalently $(a, 1)$. For a matrix in parabolic-times-compact form the fractional ideal of maximal minors of its bottom rows is $(\det D)$: the last rows of the compact factor span a direct summand. For $(a, 1)$ the same ideal has valuation $\sum_i \min(0, b_i)$, by diagonalizing a with integral units. The determinant of the full point has valuation $\sum_i b_i$. Thus the valuation of $\det A/\det D$ is $\sum_i b_i - 2\sum_i \min(0, b_i) = \ell(a)$. This calculation also fixes which normalization of the complex variable is used below.

Theorem 78.1 (The marked and unmarked split integrals)**.** *Put $\nu = \delta^{-1}\upsilon_1^d$. At an unmarked split place write $u = \nu_v(\varpi)$, reading the norm-one torus on the chosen component. Then*

$$Z_{\delta,v}(s) = \frac{H_m(q, s)}{\displaystyle\prod_{j=1}^{m} (1 - uq^{d-j-s})(1 - u^{-1}q^{d-j-s})}.$$

This holds for $\mathrm{Re}\, s > d$ and by meromorphic continuation. At a marked split place with the prescribed determinant test one has $Z_{\delta,v}(s) = P_m(q)$ for all s.

Proof. At a mark the support of f_v is K, regarded as a subset of the matrix space. The support of a source translate is Ka^{-1}, disjoint from K unless $a \in K$. When $a \in K$ the full pair action fixes the test by Lemma 77.45. Its squared norm is the additive volume $P_m(q)$ of K, by reduction modulo ϖ and the count of invertible residue matrices. Since $\ell = 0$ on K,

the assertion follows at every complex parameter. This includes an old marked test written modulo ϖ^2 with a tame unit character.

At an unmarked place the source action on positions is $f(X) \mapsto \xi(\det a)|\det a|^d f(Xa)$, with ξ an unramified unitary character. To identify it in the lattice convention, take $a = \operatorname{diag}(\varpi, 1, \dots, 1)$. On the Hom points the action is $L \mapsto La^{-1}$. At the chosen component it has m elementary divisors ϖ^{-1} and at its conjugate m divisors ϖ, all others being units. In the vacuum overlap of Lemma 77.37, the intersection with the inverse image lattice thus has determinant ideal ϖ^m on the chosen component and a unit on the other. The overlap $1/\overline{\Delta}$ is $\upsilon_v^E(\varpi)^m q^{-m/2}$, where υ_v^E is the splitting character of the chosen E-component. On the two split components the restrictions of this character are inverse, since its restriction to the diagonal $k^\times$ is trivial. Therefore $\upsilon_v^E(\varpi)^m = \upsilon_{1,v}(\varpi)^d$. The external multiplier is $\delta_v(\varpi)^{-1}$. This gives $\xi(\varpi) = u$. For general a the unit vacuum has phase $u^{v(\det a)}$, and its overlap in absolute value is $q^{-d\ell(a)}$, by diagonalization and intersection of the two position lattices. Hence the desired integral is the integral of $u^{v(\det a)} q^{-(d+s)\ell(a)}$.

For an explicit evaluation put $t = d + s$ and $J(2s) = \int_{\mathrm{Mat}_m(O)} |\det X|^{2s}\, dX$. For fixed a one has

$$|\det a|^t \int_{\substack{X \text{ integral} \\ Xa \text{ integral}}} |\det X|^{2s}\, dX = J(2s) q^{-t\ell(a)}.$$

To verify this, reduce a to its elementary divisors. Column i of X must be divisible by $\varpi^{\max(0,-b_i)}$. Scaling this column contributes $q^{-(m+2s)\max(0,-b_i)}$, which proves the identity because $m + 2s = 2t$. Singular X have additive measure zero, as follows by successively choosing independent columns. Integrate the identity against $u^{v(\det a)} d^\times a$ and put $Y = Xa$. Using $dX = P_m(q)|\det X|^m d^\times X$ gives

$$J(2s) Z_{\delta,v}(s) = P_m(q) D(t,u) D(t,u^{-1}), \qquad D(t,u) = \int_{K'} |\det X|^t u^{v(\det X)}\, d^\times X,$$

where $K' = \mathrm{Mat}_m(O) \cap \mathrm{GL}_m(k)$. These changes of variables are absolutely convergent for $\operatorname{Re} s > d$. Indeed the following integral calculation with the real parts and $|u| = 1$ bounds both factors. For additive probability on integral matrices the generating function of $v(\det X)$ is $\prod_{j=1}^m (1 - q^{-j})/(1 - yq^{-j})$ for $|y| < q$. To see this, the first column is ϖ^h times a primitive column with probability $(1 - q^{-m}) q^{-mh}$. Carry the primitive column to the first basis vector by an integral row operation. Given this column, the other columns are still independent uniform vectors after the operation. Expanding the determinant reduces the problem to dimension $m - 1$, with a factor y^h. The geometric sum and induction give the formula. Consequently

$$D(t,u) = \prod_{j=1}^{m} (1 - uq^{m-j-t})^{-1}, \qquad J(2s) = P_m(q)/H_m(q,s).$$

Substitution proves the claimed rational function. □

Let δ_0, δ_1 be two of the prescribed weight-zero characters, with sets $\mathcal{S}_i$ of ramified split places (one chosen component per pair), and write $Q_i = \prod_{v \in \mathcal{S}_i} q_v$. The common old marks are included in both sets. Their tame characters, like the characters at the new marks, are nontrivial on residue units. Put $\nu_i = \delta_i^{-1} \upsilon_1^d$, and let $L_i(z)$ be the finite Hecke L-function over E of $a \mapsto \nu_i(a/\bar{a})$. Its ramified Euler factors are one. Define

$$\mathcal{R}(s) = \prod_{v \text{ split}} \frac{Z_{\delta_1,v}(s)}{Z_{\delta_0,v}(s)} \qquad (\operatorname{Re} s > d).$$

Theorem 78.2 (The conductor power in the relative doubling integral)**.** *The product $\mathcal{R}$ has a meromorphic continuation regular and nonzero at zero. Its absolute value there is*

$$|\mathcal{R}(0)| = \left(\frac{Q_1}{Q_0}\right)^{d^2} \prod_{z=1}^{d} \left|\frac{L_1(z)}{L_0(z)}\right|^2 .$$

Proof. At an unmarked split place the two factors in each pair in Theorem 78.1 are the inverse local Euler polynomials at the two conjugate places of E: the two base-change values on a uniformizer are u, u^{-1}. At a marked place the quotient by the same $H_m(q_v, s)$ is $P_m(q_v)/H_m(q_v, s)$. At a nonsplit place the two δ_i agree locally, since the norm-one group there is compact and the characters are unramified. Hence their Euler factors cancel. For $\operatorname{Re} s > d$ the absolutely convergent Euler products yield

$$\mathcal{R}(s) = \frac{\prod_{v\in\mathcal{S}_1} P_m(q_v)/H_m(q_v, s)}{\prod_{v\in\mathcal{S}_0} P_m(q_v)/H_m(q_v, s)} \prod_{j=1}^{2d} \frac{L_1(s+j-d)}{L_0(s+j-d)}.$$

The first fraction is regular with value one at zero.

The primitive Hecke functional equation gives the completed functions $\Lambda_i(z) = A_i^{z/2} G_\infty(z) L_i(z)$, where both have the same archimedean factor since the twists have weight zero. The finite conductor norm is Q_i^2: the base change is tamely ramified with exponent one on each of the two split components at a mark, and unramified elsewhere. Thus $A_i = |\operatorname{disc} E| Q_i^2$. At positive real integers z, $L_i(z) \neq 0$. For $z > 1$ this follows from the Euler product. At $z = 1$, let λ be the base-change character and omit its ramified factors in all three functions below. The Euler logarithms give $\zeta_E(\sigma)^3 |L(\sigma, \lambda)|^4 |L(\sigma, \lambda^2)| \geq 1$ for real $\sigma > 1$, by $3 + 4\cos t + \cos(2t) = 2(1 + \cos t)^2 \geq 0$. The angular type of v_1^d is nonzero, so λ and λ^2 both have nonzero angular type. Their Hecke functions are holomorphic at one while the omitted-factor zeta function has a simple pole. A zero of $L(1, \lambda)$ would force the product to tend to zero, which proves the assertion.

The integers $j - d$ occur in pairs $z, 1 - z$ for $1 \leq z \leq d$. For real arguments the functional equation relates $\Lambda_i(1 - z)$ to the complex conjugate of $\Lambda_i(z)$ by a factor of absolute value one. The continued ratios at $1 - z$ are consequently regular and nonzero, since their common archimedean factors cancel as meromorphic functions. The contribution of a pair to the absolute value of the product is

$$(A_1/A_0)^{z-1/2} \, |L_1(z)/L_0(z)|^2 .$$

Here the two conductors on the completed positive side give the power z, and removing the completion at z and $1 - z$ subtracts $1/2$. Summing $z - 1/2$ from 1 to d gives $d^2/2$. This proves the theorem. □

The theorem evaluates the relative product of the actual split local data at the equal-rank point. Passing from this continued product to the squared absolute values of the constants $I_{\delta_i}(1)$ requires the global doubled identity at that point.

78.2 Nonvanishing and normalization in the first theta exchange

Retain $m = 2d$, the definite group $G = U(I_m)$ and the weight-zero character δ. Let η be the quadratic Hecke character of F associated with E/F. For each prescribed δ, put $Q_\delta = \prod_{v\in\mathcal{S}_\delta} q_v$ and

$$\lambda_\delta(a) = (\delta^{-1}\upsilon_1^d)(a/\bar{a}), \qquad D_\delta = Q_\delta^{d^2}\prod_{z=1}^{d}|L_{E,f}(z,\lambda_\delta)|^2.$$

The subscript f denotes a product over finite places, with the local factor one at a place where the character is ramified.

Lemma 78.3 (The nonsplit and real factors of the square integral)**.** *At an inert finite place v the normalized unramified doubling integral for the square pair, with both full-level vectors of norm one and compact volume one, is*

$$Z_v(s) = \frac{\prod_{j=1}^{m} L_{E_w}(s+j-d,1)}{\prod_{j=1}^{m} L_{F_v}(2s+j,\eta_v^j)}.$$

At a real place the corresponding compact integral is a positive constant independent of s and δ.

Proof. There is no finite ramification of E/F by Lemma 77.37. At an inert place every norm-one element is integral, so δ_v is trivial. The finite space is the standard unimodular one. Each pair of basis lines of I_m splits a hyperbolic plane over E_w: a unit of norm -1 supplies a primitive isotropic vector, and the unramified residue trace lifts to a surjection of integer rings. In detail choose an integral vector pairing to one with the first. Subtract a multiple of the isotropic vector with trace equal to the norm of this second vector to make it isotropic as well. This argument also works in residue characteristic two. In the doubled lattice the diagonal has an integral isotropic complement, given by the vectors $(tx,(t-1)x)$ for $t\in\mathcal{O}_{E_w}$ with $t+\bar{t}=1$. Its basis with the diagonal has unit determinant. Thus the doubled spherical section is the section for the hyperspecial group, with value one at the identity by the unit-vacuum overlap. The Hom lattice and the additive pairing are self-dual in the prescribed finite frames.

We use the unramified doubling integral for a normalized spherical vector. In the unitary convention it gives the standard base-change Euler factor at $s+1/2$ divided by $\prod_{j=1}^{m} L_{F_v}(2s+j,\eta_v^j)$ in this equal, even-rank case. This is the spherical integral of J.-S. Li, Theorem 3.1, in the normalization of Gan, Qiu and Takeda, §11.6. Its hypotheses here are unramified groups, characters, hyperspecial vectors and the spherical section with value one. In general the denominator uses the restriction of the space-splitting character times η_v^{m-j} at the j-th argument. That restriction is $\eta_v^m = 1$ here, and $m-j$ has the parity of j.

For the constant representation the base-change Satake eigenvalues are $q_w^{(m+1)/2-j}$, $1\le j\le m$. Indeed the spherical constant occurs in the principal series with inducing parameter the inverse square root of the Borel modulus, up to Weyl conjugacy. On the diagonal the half-sum of the roots over E_w has entries $(m-1)/2,(m-3)/2,\dots,(1-m)/2$. Equivalently, on a split diagonal coordinate of U_m the two conjugate coordinates are inverses and the two opposite entries of this list contribute their difference to the relative modulus exponent; $q_w = q_v^2$. This gives the displayed Satake multiset after base change. The unramified space character on the common uniformizer of F_v, E_w equals $\eta_v(\varpi_v)^m = 1$. Hence the standard factor at $s+1/2$ has the m arguments $s+j-d$ in the lemma. At infinity the entire embedded definite group belongs to the doubled maximal compact group. The flat-section height is one there and the prescribed

Gaussian is fixed by the compact pair, by Theorem 77.46. Its squared norm integrated over this compact group is positive. □

Theorem 78.4 (The normalized first exchange on every prescribed twist)**.** *On each of the given fields there is a positive number $B_{F,m}$, independent of δ, such that on the entire adelic source*

$$|I_\delta(g)|^2 = B_{F,m} D_\delta.$$

In particular $\gamma_\delta = I_\delta(1)^{-1}$ is defined, $\gamma_\delta I_\delta(g) = 1$ for every g, and $|\gamma_\delta|^2 = B_{F,m}^{-1} D_\delta^{-1}$.

Proof. Use the same Haar choices on the two copies of G. The integral $J_\delta(h) = \int_{[G]} \theta_\delta(g,h)\,dg$ is constant by the proof of Theorem 77.46 with source and target interchanged. Indeed the local double-coset equality of Lemma 77.45 is reversible since $a \mapsto a^{-\mathsf{t}}$ is an involution. Full integral and real invariance hold on both sides. The same determinant and strong approximation argument therefore applies. Integrating both ways on the compact product shows $J_\delta(h) = I_\delta(1)$. Thus the squared norm of J_δ is $\mathrm{vol}([G])|I_\delta(1)|^2$.

Double the source as $I_m \oplus (-I_m)$, with its diagonal maximal isotropic subspace. In the doubled construction use the Schwartz tensor of the test and its conjugate. On the two factors the rational theta sum of this tensor restricts to $\theta_\delta(g_1,h)\overline{\theta_\delta(g_2,h)}$. This follows from the product polarization and from complex conjugation of the oscillator on the opposite copy, as in Proposition 142.14. The external target characters cancel, and the external source characters give $\delta(\det g_1)^{-1}\delta(\det g_2)$. Equivalently one applies these characters to the two source arguments after doubling with the underlying untwisted splitting. Keep the usual correction on the second source argument in the restriction of the doubled splitting. It is the character making the two actions there contragredient: on the diagonal the graph evaluation is the Hermitian inner product and is invariant under the simultaneous unitary action. With this convention the value at $(a,1)$ is precisely the oscillator coefficient used in Section 78.1. The diagonal modulus stays one throughout the flat-section family since a unitary determinant has norm one.

Let $\mathcal{Z}_\delta(s)$ be the integral on $[G]^2$ of the flat Siegel Eisenstein series with that doubled test and these source characters. Here the Siegel rank n is m, the dimension of the space integrated in the theta kernel is also m, and its rational Witt index is zero. The unitary Siegel–Weil formula at $m = n$ says that the Eisenstein series is holomorphic at $s = 0$ and its value is twice the theta integral in the measures of that formula. There is no regularization of the theta integral over our compact quotient. These are exactly the $m = n$ hypotheses of S. Yamana, *The Siegel–Weil formula for unitary groups*, Theorem 2. Changing the fixed Haar measures gives a positive common factor. Integration on the compact source quotients commutes with evaluation at zero: the meromorphic family of smooth Eisenstein series, holomorphic here, is locally uniform on compacta. Expanding the product of the two ordinary theta integrals then shows that $\mathcal{Z}_\delta(0) = c\,|I_\delta(1)|^2$ with $c > 0$ depending only on the field, spaces, and measures.

For $\mathrm{Re}\, s$ sufficiently large unfold the absolutely convergent Eisenstein series. Its terms are indexed by the rational maximal isotropic subspaces of the double. Each is the graph of a rational isometry of I_m: its intersection with either summand is zero by definiteness, so both projections are isomorphisms, and isotropy is exactly the isometry condition. There is one orbit under $G(F)^2$ and its stabilizer is the diagonal $G(F)$. Unfolding thus integrates the section on the open orbit, with one factor $\mathrm{vol}([G])$ from simultaneous translation. The other variable runs over $G(\mathbb{A})$; on that variable the section factors into the local doubling integrals. Absolute convergence permits the product in this half-plane. This is also the usual Eulerian unfolding in Gan, Qiu and Takeda, §§11.5–11.6. The section parameter has the split height computed in Section 78.1.

Combining Theorem 78.1 at all split places with Lemma 78.3 at the other places gives the meromorphic identity

$$\mathcal{Z}_\delta(s) = c' \frac{\prod_{j=1}^{m} L_{E,f}(s+j-d, \lambda_\delta)}{\prod_{j=1}^{m} L_{F,f}(2s+j, \eta^j)} \prod_{v \in \mathcal{S}_\delta} \frac{P_m(q_v)}{H_m(q_v, s)}.$$

Here $c' > 0$ includes the real compact integrals and measure normalizations. They are common to the twists. On an inert uniformizer the base-change character has value one as used in the lemma. The identity holds initially in the unfolding half-plane and extends by the Hecke functional equations and uniqueness of meromorphic continuation.

All factors on the right have finite nonzero values at zero. For the denominator, the $j = 1$ value is $L_{F,f}(1, \eta)$, which is positive by taking the ratio of residues in $\zeta_E(z) = \zeta_F(z) L_{F,f}(z, \eta)$. The other denominator arguments exceed one. The correction at each mark is one at zero. In the numerator, Theorem 78.2 gives nonvanishing at the positive integers $1, \dots, d$. At each complex place of E the angular character of λ_δ has exponent $2d$ or $-2d$, since υ_1 has weight one on the norm-one circle and base change is $a \mapsto a/\bar{a}$. Its gamma factor is $\Gamma_{\mathbb{C}}(z + d)$. At $z = 1 - d, \dots, 0$ these gamma factors are finite and nonzero. The primitive Hecke functional equation now gives finite nonzero numerator values at these arguments from the paired positive ones. It follows that $I_\delta(1) \neq 0$.

Take the ratio of the meromorphic identities for any two twists. All nonsplit and real factors cancel. The ratio at zero is the ratio of the squared absolute values of the two actual constants, hence is positive. Its absolute value is $D_{\delta_1}/D_{\delta_0}$ by Theorem 78.2. Fixing any prescribed reference twist gives $B_{F,m} = |I_{\delta_0}(1)|^2/D_{\delta_0} > 0$. This proves the assertions for the entire family on that field. □

Corollary 78.5 (The signed constant for insertion in the second exchange). *Let $c_0 = \chi_M^j \vartheta$, take $\delta = c_0^{-1}$, and write χ for the finite residue character of c_0. Denote by $I^-(g)$ the definite target class sum with source $-I_m$, positive target I_m, splitting characters $(c_0 \upsilon_1^m, c_0 \upsilon_1^m)$, marked test $\chi(\det X)$ and real polynomial one. There is a scalar γ^- such that*

$$\gamma^- I^-(g) = 1 \quad \text{for all } g, \qquad |\gamma^-|^2 = B_{F,m}^{-1}/D_{c_0^{-1}}.$$

Proof. Use the same finite unit u_f of norm -1 as in Lemma 77.37. Its antiunitary identification and the uniqueness of the rational summation functional give a single phase ϵ for the pair of ranks m, m. The finite calculation in that lemma applies with both group arguments moving. In fact the Hom change $L \mapsto hLg^{-1}$ has determinant $(\det h)^m (\det g)^{-m}$, so the ratio of the conjugated and reversed lattice phases is $\upsilon_1(\det h)^m \upsilon_1(\det g)^{-m}$. Together with complex conjugation of both external characters these are the two specified negative-pair characters. At the real places the compact groups fix both tests: on the reversed pair the m negative Hom columns give inverse target and direct source vacuum exponents, canceled by the respective external weights m. The old and new tests transform by $\chi^{-1}(\det(u_f^{-1} X)) = \chi(u_f)^{-m} \chi(\det X)$ on units, with both sides zero off the units. Here $\chi(u_f)$ is evaluated in the marked residue components. Consequently $\overline{I_\delta(g)} = \epsilon \chi(u_f)^{-m} I^-(g)$ for every g, since the class weights are real. Multiplying by $\overline{\gamma_\delta}$ gives the assertion with $\gamma^- = \overline{\gamma_\delta} \epsilon \chi(u_f)^{-m}$. The two additional factors have modulus one. □

This gives the signed constant and its full twist dependence at the first step. The second exchange passes from that constant to the carrier on $U(2m, m)$. The third compares the carrier period with the arithmetic class integral on $U(W_j)$.

78.3 The two cuspidal forms in the second exchange

Put $r = 3m$, $S_0 = I_{2m} \oplus (-I_m)$, $G = U(S_0)$, $H = U(I_r)$, and $\kappa = \upsilon_1^m$. Fix a prescribed $c = \chi_M^j \vartheta$, with finite residue character χ. Let $C(h)$ be the theta lift of the constant on $U(I_m)$ to H with the two external characters one, the full lattice vacua, and polynomial one. Normalize compact measures so that integration on the definite groups gives the previous stabilizer-weighted class sums. Both compact real group actions fix this definite Gaussian. The coefficient $C_\vartheta(h)$ in Theorem 77.41 is $c(\det h)C(h)$: its finite and real test vectors are the same and the external character c acts on the target determinant alone.

Let $\Theta_c(x, h)$ be the kernel for (S_0, I_r) with characters $(c\kappa, c)$, polynomial one, the marked square test $\chi(\det X)$ on integral matrices, and unmarked unit vacua. The unnormalized form in the second exchange is

$$A_c(x) = \int_{[H]} C(h)\Theta_c(x, h)\, dh.$$

Indeed moving the factor $c(\det h)$ from the coefficient into the kernel changes the first splitting character from κ to $c\kappa$. Let ϕ be the prescribed nonzero carrier of Theorem 77.6, a theta integral from $-I_m$ to S_0. We compare A_c and ϕ on this same group.

Proposition 78.6 (Level, weight and cuspidality of the class integral)**.** *The function A_c is a smooth cuspidal automorphic form on G, fixed by its full finite integral stabilizer. It is holomorphic with compact weight $(0, \dots, 0; r, \dots, r)$, in the order of Theorem 77.6.*

Proof. At a marked place the source change by an integral unit k multiplies the raw determinant test by $\chi(\det k)$; the splitting multiplier is its inverse. The corresponding target change acquires the inverse factor from the test and the direct factor from the splitting since κ is unramified. Unmarked finite vacua are fixed. At a real place each negative source column contributes one inverse target determinant on the vacuum, so the target compact action has weight $-m+m = 0$. For the source action, each target row contributes one copy of the determinant on the negative source block. The positive block has weight zero. This gives the two claimed source weights, since c has weight zero. In the polynomial Fock model the noncompact lowering operators are contractions, each summing a product of derivatives in a positive and a negative column over the target row index. They annihilate the polynomial one. These are the holomorphy equations for the positive-first order, as in Theorem 77.6. Rational invariance and smoothness follow from the theta construction and compactness of $[H]$.

Let P be a rational maximal parabolic of G, attached to a nonzero totally isotropic E-space $U \subset S_0$. We show that even the average over the center of its unipotent radical vanishes. Complete U to a rational Witt decomposition $U \oplus S_1 \oplus U'$, with U' dual to U. In the mixed position model the oscillator has position coordinates $y \in \mathrm{Hom}_E(U, I_r)$ and a remaining oscillator on S_1 and I_r. The rational theta sum accordingly sums these coordinates over their rational values. On the term of coordinate y, a central shear with parameter B multiplies by the additive character paired with the Gram matrix y^*y. Indeed the shear sends a dual isotropic coordinate to itself plus B times an isotropic coordinate. In position coordinates a shear of a momentum by a linear function of position multiplies by the additive character of its quadratic form. Here that form is a fixed nonzero trace multiple of $\mathrm{tr}(By^*y)$; the imaginary unit of the unitary splitting identifies the skew and Hermitian conventions. The shear fixes the remaining oscillator. The trace pairing between the two rational matrix spaces is nondegenerate, as one sees on diagonal entries and on each pair of off-diagonal entries. Integration over the adelic center modulo rational points therefore requires $y^*y = 0$, by additive character orthogonality. The rational theta sums may be integrated termwise on this compact unipotent quotient by

uniform Schwartz summability on compact sets. Definiteness of I_r at a real place forces every column of such a rational y to vanish.

The component at $y = 0$ is zero already at any one marked split place. In the split position coordinates before passing to the mixed model, the test is supported on invertible square matrices X. This persists after arbitrary local source and target translations, which compose X with invertible maps. Relative to the Witt decomposition, passing to the mixed model keeps the columns $X|_U$ on the chosen split component and Fourier transforms the columns on U'; the dual Fourier variables supply the conjugate components of y using the Hermitian pairing. In particular setting $y = 0$ sets $X|_U = 0$ before integration in the latter columns. An invertible X cannot vanish on a nonzero subspace. The other coordinates S_1 are unrestricted, so the entire mixed vector at zero is zero. This proves vanishing of the central average for every x, h, hence of the full parabolic constant term, also after integration against $C(h)$. The old split marks are present for every twist. Every proper parabolic is contained in a maximal one, so vanishing of their constant terms proves cuspidality. □

We next compare all the split spherical Hecke operators, including those at the conductor. At a split finite place with residue size q, identify the two rank r groups with $\mathrm{GL}_r(k)$ by their prescribed finite frames. For $0 \le b \le r$ let T_b denote right translation averaged with probability on the double coset of $\mathrm{diag}(\varpi 1_b, 1_{r-b})$ at full integral level. Write $\begin{bmatrix} N \\ h \end{bmatrix}_q$ for the number of h-dimensional subspaces of a residue vector space of dimension N, with value zero outside $0 \le h \le N$. Let ξ be the value of the unramified Weil splitting character at a uniformizer in the chosen E-component. Define

$$t_b = \xi^{-mb} q^{mb/2}, \qquad s_l = \xi^{-rl} q^{rl/2},$$
$$\lambda_b = \frac{t_b}{\begin{bmatrix} r \\ b \end{bmatrix}_q} \sum_{\substack{0 \le l \le \min(m,b) \\ b-l \le r-m}} \begin{bmatrix} r-m \\ b-l \end{bmatrix}_q q^{\,l(r-m-b+l)} \begin{bmatrix} m \\ l \end{bmatrix}_q s_l^{-1}.$$

Lemma 78.7 (A rectangular lattice count for the lower theta lift)**.** *At every split finite place $T_b C = \lambda_b C$ on H and $T_b \phi = \kappa_v(\varpi)^b \lambda_b \phi$ on G.*

Proof. Consider first the (r, m)-matrix test $f = \mathbf{1}_{\mathrm{Mat}_{r,m}(O)}$ with the two external characters one. A target element with b elementary divisors ϖ acts by $t_b f(h^{-1}X)$. A source element with l divisors ϖ^{-1} acts by $s_l f(Xg)$, with the other elementary divisors one in both statements. The positive absolute factors are the square-root position Jacobians. For the phases one may compute the overlap as in Theorem 78.1: a target change of valuation one gives m positive Hom exponents on the chosen component, hence m negative ones on the conjugate component. A source change of valuation minus one gives r positive Hom exponents on the chosen component. The conjugate uniformizer has splitting value ξ^{-1}, giving the two phases written above.

Both averaged tests vanish off the integral matrix lattice. For an integral X, let a be its residue rank. The target average divided by t_b is the fraction of codimension b residue subspaces containing its column space. It is $\begin{bmatrix} r-a \\ b \end{bmatrix}_q / \begin{bmatrix} r \\ b \end{bmatrix}_q$. The corresponding source average divided by s_l is the fraction of l-dimensional subspaces contained in $\ker \bar{X}$, namely $\begin{bmatrix} m-a \\ l \end{bmatrix}_q / \begin{bmatrix} m \\ l \end{bmatrix}_q$. Indeed gO^m is O^m enlarged by ϖ^{-1} times the chosen subspace. The two subspaces in these counts are uniform because the integral compact groups act transitively on the relevant Grassmannians.

Put $t = r - m$ and $n = m - a$. In a space $k_0^t \oplus k_0^n$ over the residue field k_0, count a b-plane by the dimension l of its projection to the second summand. Choose its $b-l$ dimensional intersection with the first summand and its l dimensional image in the second. For these choices the possible graphs are indexed by the linear maps from the image to the first summand modulo

the intersection, giving $q^{l(t-b+l)}$ possibilities. Thus

$$\begin{bmatrix} r-a \\ b \end{bmatrix}_q = \sum_l \begin{bmatrix} m-a \\ l \end{bmatrix}_q \begin{bmatrix} r-m \\ b-l \end{bmatrix}_q q^{l(r-m-b+l)}.$$

The support and this identity express the target average of f as the corresponding linear combination of the source averages, with coefficients whose sum is λ_b. Integrate over the definite source against the constant function. Each source translate has exactly the original integral by Haar invariance. This gives the identity for C. For ϕ the finite rectangular test and the two finite frames are identical: the negative definite source is framed using the norm-minus-one unit. The different real polynomial does not enter this local identity. The additional target multiplier is $\kappa(\det h)$, constant and equal to $\kappa_v(\varpi)^b$ on this double coset. The source integral is again against the constant. This proves the second identity. □

Theorem 78.8 (Agreement of the two split systems of Hecke equations)**.** *On the prescribed field and for every c, both A_c and the carrier ϕ satisfy*

$$T_b f = \kappa_v(\varpi)^b \lambda_b f$$

at every split finite place and every $0 \le b \le r$. This includes both sets of marked places and determines the action of the full split spherical Hecke algebra on these forms.

Proof. Use the square test defining Θ_c. At a split place it is fixed by both compact actions and is invariant under transposition. For the underlying Hom action transposition exchanges the source operator at a with the target operator at $a^{-\mathsf{t}}$, by the pullback argument of Lemma 77.45. The quotient of the external source and target factors now equals $\kappa(\det a)$, since they are $c(\det a)^{-1}$ and $(c\kappa)(\det a)^{-1}$, respectively. The compact restrictions of c and $c\kappa$ agree. As in that lemma, any vector fixed for both compact actions is fixed by transposition, by Smith normal form. Hence the source average for T_b equals $\kappa_v(\varpi)^b$ times the target average on the inverse-transpose coset. This holds equally for the determinant test at a mark and for the lattice indicator elsewhere.

Pass this equality through the theta sum and integrate over $[H]$ with C. Right translation of the target kernel by $a^{-\mathsf{t}}$ transfers to right translation of C by a^{t}, using the right invariance of the quotient measure. The transpose of a double coset has the same elementary divisors and the induced measure is its invariant probability. Thus the resulting right average on C is $T_b C$. Lemma 78.7 now gives the stated equation for A_c, and gives the same equation for ϕ directly. The spherical Hecke algebra of $\mathrm{GL}_r(k)$ is generated by these elementary double cosets and the inverse of the central coset $b = r$ (the elementary symmetric generators under Satake). The latter coset acts by an invertible translation, so its inverse action also agrees. □

The calculation locates A_c in the same cuspidal holomorphic weight space and split Hecke eigenspace as the nonzero carrier. The second exchange still requires showing that the two specified global vectors are proportional with the prescribed nonzero scalar and computing its norm.

78.4 The spherical parameters on the two groups of rank $3m$

Let $G = U(I_{2m} \oplus (-I_m))$ and $H = U(I_r)$, where $r = 3m$. Use the coefficient C, the carrier ϕ, the kernel Θ_c and the form A_c of Section 78.3. For a full-level function u on the compact quotient $[H]$, invariant at infinity, write

$$(\mathcal{T}_c u)(x) = \int_{[H]} u(h)\Theta_c(x,h)\,dh, \qquad B_c(h) = \int_{[G]} \phi(x)\overline{\Theta_c(x,h)}\,dx.$$

Thus $\mathcal{T}_c C = A_c$. We use inner products linear in the first argument. Then $B_c = \mathcal{T}_c^{*}\phi$ in the sense of these pairings. All the integrals exist: a cusp form is rapidly decreasing on Siegel sets and the theta kernel has moderate growth, uniformly for h in a compact set of representatives. The proof of Proposition 78.6 gives cuspidality and the same holomorphic compact weight for $\mathcal{T}_c u$. It only uses the compactness of $[H]$ and the marked support of the kernel. The function B_c is full-level and fixed by the compact real group of H.

For a positive integer b write

$$A_b(q) = \{q^{(b-1)/2-i} : 0 \le i < b\}$$

as a multiset. A spherical parameter at a split place is the Satake multiset for $\mathrm{GL}_r(k)$ in the chosen component, with the elementary positive coweights acting by the normalization of Lemma 78.7. At an inert place use the standard base-change multiset of Theorem 72.16.

Lemma 78.9 (The definite lower lift is nonzero)**.** *At every sufficiently large node of the prescribed family one has $C \neq 0$.*

Proof. Double the definite source I_m of the lift defining C. The target I_{3m} is anisotropic, the Siegel rank is m, and the Siegel point is $m > m/2$. Thus the convergent Siegel–Weil formula and the unfolding of Theorem 77.6 apply. Every rational maximal isotropic subspace of $I_m \oplus (-I_m)$ is a graph of an isometry. Consequently $\|C\|^2$ is a positive Haar factor times the integral of the source oscillator coefficient for the prescribed vacuum. The real source fixes the Gaussian and gives a positive compact volume. Each finite coefficient is one on the full stabilizer, of mass one. Off that stabilizer its integral in absolute value is $O(q_v^{-m/2})$ by Lemma 77.5. This lemma uses only the finite self-dual lattices and the two ranks $m, 3m$, which are the same for this positive pair. The sum of the errors is $O(f2^{-m/2}) = o(1)$ on the fields under consideration, as in Theorem 77.6. Absolute convergence bounds the difference of the finite product from one by $\exp(O(f2^{-m/2})) - 1$. Thus it is nonzero at sufficiently large nodes, proving the assertion. □

Theorem 78.10 (Full finite spherical systems of the carriers and adjoint)**.** *At every split v, put $q = q_v$, $\xi = \upsilon_w(\varpi_v)$ in the chosen component and $\kappa_v(\varpi_v) = \xi^{2m}$. Define*

$$P_{H,v} = \xi^{r-m} A_m(q) \ \sqcup\ \xi^{-m} A_{r-m}(q),$$
$$P_{G,v} = \kappa_v(\varpi_v) P_{H,v}.$$

At an inert place v with $w \mid v$, set

$$P_{H,v} = P_{G,v} = A_m(q_w) \sqcup A_{r-m}(q_w), \qquad q_w = q_v^2.$$

Every constituent of C, B_c at full level has parameter $P_{H,v}$, and every constituent of ϕ, A_c at full level has parameter $P_{G,v}$. Here a statement about constituents allows a form to vanish.

Proof. At a split place the probability double coset T_b of Section 78.3 has eigenvalue $q^{b(r-b)/2} e_b(P)/\begin{bmatrix} r \\ b \end{bmatrix}_q$ at multiset P. This is the minuscule coweight formula in the Satake

isomorphism: its unnormalized transform is the sum of the weight monomials with b distinct indices times $q^{b(r-b)/2}$, and its double coset consists of $\begin{bmatrix} r \\ b \end{bmatrix}_q$ right cosets. The elementary symmetric function of the segment of length t satisfies

$$e_l(A_t(q)) = q^{-l(t-l)/2} \begin{bmatrix} t \\ l \end{bmatrix}_q .$$

For instance, expand $\prod_{i=0}^{t-1}(1+zq^i)$ by degree, or induct using the last factor. In the union $P_{H,v}$ choose l entries in the first segment and $b-l$ in the second. Their phase is ξ^{-mb+rl}. After multiplying their symmetric function by $q^{b(r-b)/2}$, the exponent of q in this term is

$$\frac{b(r-b)-(b-l)(r-m-b+l)-l(m-l)}{2} = \frac{mb-rl}{2} + l(r-m-b+l).$$

The resulting coefficients are exactly those in λ_b of Lemma 78.7. Multiplication of every entry by $\kappa_v(\varpi_v)$ accounts for the factor $\kappa_v(\varpi_v)^b$ in the equations for ϕ, A_c.

For B_c, the transpose calculation of Theorem 78.8 before evaluating on C gives, on the whole full-level compact source space,

$$T_b^G \mathcal{T}_c u = \kappa_v(\varpi_v)^b \, \mathcal{T}_c T_b^H u.$$

Take adjoints and apply this to the eigenfunction ϕ. Spherical convolution is a commutative algebra closed under adjoints in the unitary right regular representation. Thus each operator is normal, and the eigenvalues of its adjoint on an eigenvector are the complex conjugates. It follows that $T_b^H B_c = \lambda_b B_c$. This argument includes every marked split place.

We detail the passage from global theta integrals to the local condition used at an inert place. The spaces of cuspidal automorphic forms with the given finite level and real holomorphic weight are finite dimensional, by finite cuspidal multiplicity at fixed level and infinitesimal character. On the compact group H the corresponding space is a finite class space. Decompose such forms in the cuspidal Hilbert sum. If a projection of a theta lift pairs nontrivially with an irreducible summand, there is a nonzero joint local oscillator pairing with its two constituents. Indeed fix pure tensors giving a nonzero global pairing and hold the data outside v fixed. The local form on the oscillator and the two test representations is equivariant, by right translation in the two quotient integrals. Its associated map from the smooth oscillator lands in the smooth dual of the tensor of the two local representations. These representations are admissible, so the smooth dual is the tensor of their contragredients; this follows on compact-fixed vectors by finite dimensionality. The nonzero image is an irreducible quotient isomorphic to that tensor. One may first project the input onto one of finitely many summands with nonzero contribution. Complex conjugation in an adjoint integral replaces the appropriate representations by their duals. At an inert place an unramified unitary-group parameter is invariant under conjugate inversion at the common uniformizer: its entries occur in reciprocal pairs, with a possible central entry one. Thus duals give the same multiset here.

All dyadic places of F split in E. In fact the imaginary quadratic subfield is split at two, and conjugation on E/F exchanges those two components. It cannot fix any place above two. Every remaining nonsplit finite place is unramified quadratic by Lemma 77.37. The even unitary spaces in question are quasi-split there, with hyperspecial stabilizers, by the hyperbolic lattice splitting in Lemma 78.3. Thus Theorem 72.16 applies at every inert place. Both source and target space characters on the uniformizer are one since the ranks are even. The finite restrictions of c, κ to the compact norm-one group there are trivial. Starting with the constant in rank m, whose multiset is $A_m(q_w)$, the positive lower lift to H and the negative lower lift to G both adjoin $A_{r-m}(q_w)$. This gives the systems for C and ϕ. For the square pair of ranks r, r

the same transfer theorem leaves the multiset unchanged. Apply the local pairing just proved to $\mathcal{T}_c C$ and to its adjoint on ϕ to obtain the two remaining assertions. □

At all finite places a spherical isomorphism class is determined by its Satake multiset here. At an inert place the relative torus parameter is determined up to signed permutation by its reciprocal pairs, which is the relative Weyl equivalence. Thus the theorem also fixes the finite spherical matrix coefficients with the unit-vector normalization, including at places above two.

78.5 Uniqueness of the two automorphic lines

Fix a sufficiently large node so that C and ϕ are both nonzero, as in Lemma 78.9 and Theorem 77.6. Let D stand for either $H = U(I_{3m})$ or $G = U(I_{2m} \oplus (-I_m))$. Write $\mathcal{E}_D$ for the space of smooth cuspidal automorphic forms on $[D]$ with full finite level and Hecke eigencharacter specified by $P_{D,v}$ in Theorem 78.10 at every finite v. On H require invariance at infinity. On G require the scalar compact weight $(0^{2m}; (3m)^m)$ and holomorphy at every real place. The compact group H has no proper rational parabolic, so its cuspidality condition is vacuous.

Lemma 78.11 (The matrix coefficient on a prescribed scalar line)**.** *For every nonzero $f \in \mathcal{E}_D$ the function*

$$M_f(a) = \frac{\langle R(a)f, f\rangle}{\langle f, f\rangle}, \qquad a \in D(\mathbb{A}),$$

is the same. It is a product of normalized local coefficients. At a real place the coefficient is determined by the compact character and the holomorphy equations. This real assertion also holds for a smooth unit vector satisfying the same equations in any unitary representation of the real group.

Proof. At a finite place insert the orthogonal projections onto the K_v-fixed vectors on both sides of $R(a_v)$ in the pairing. This compressed operator is convolution by probability on $K_v a_v K_v$. Its eigenvalue is determined by the specified spherical character. It commutes with translation at all other places. Removing the finite components one by one multiplies the archimedean coefficient by these eigenvalues. Only finitely many factors differ from one for any fixed adele.

At a real place of H the coefficient is one. For G use the complexified Lie algebra $\mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$, where the antiholomorphic derivatives $\mathfrak{p}^-$ annihilate the vector. Complex conjugation of the real form interchanges $\mathfrak{p}^+$ and $\mathfrak{p}^-$. In a unitary representation the adjoint of a complexified derivative Z is $-\bar{Z}$ on smooth vectors. Order monomials by the Poincaré–Birkhoff–Witt theorem with $\mathfrak{p}^-$ on the right and $\mathfrak{p}^+$ on the left. In a derivative of the coefficient at the identity any ordered monomial containing a right lowering factor annihilates the vector. Any remaining monomial with a left raising factor pairs to zero by the adjoint relation. Every term left over is evaluated by the differential of the one-dimensional compact character. Thus all derivatives of the normalized coefficient at the identity are determined.

Here these derivatives determine the function globally. Indeed choose real tangent directions in the noncompact part in orthonormal pairs X, JX for its invariant complex structure. For $Z = X + iJX$, $X^2 + (JX)^2 = (Z\bar{Z} + \bar{Z}Z)/2$. One of $Z, \bar{Z}$ is lowering. On the given vector the sum therefore reduces, up to sign and the factor $1/2$, to the commutator in $\mathfrak{k}_{\mathbb{C}}$. Each compact direction also acts by a scalar. The sum of the squares of this real basis, including a basis of the compact part, consequently acts on the vector by a fixed scalar. The coefficient solves the corresponding left-invariant elliptic differential equation with real analytic coefficients. Analytic elliptic regularity makes it real analytic. The real unitary group is connected by its Cartan decomposition and the connectedness of its maximal compact subgroup. Equality of all derivatives implies equality everywhere. On the product of real groups use the union of the bases. The derivatives at the identity factor by place on pure monomials, and the product of the local normalized coefficients therefore gives the same analytic function. For example one may use at each place the restriction to that factor of the coefficient of any one of the specified unit vectors. □

For $D = H$ let $L = U(I_m)$ and let $K(l, z)$ be the definite rectangular theta kernel defining C. For $D = G$ take $L = U(-I_m)$ and use the rectangular kernel with polynomial $\prod_\tau \det(Z_{\tau,-})^{2m}$

defining ϕ. In both cases put

$$Tu(z) = \int_{[L]} u(l) K(l,z)\, dl, \qquad T^* f(l) = \int_{[D]} f(z)\overline{K(l,z)}\, dz.$$

Write $f_0 = T1$. The integrals against cusp forms converge absolutely by rapid decrease and the moderate growth of the theta kernel. Bounds are uniform over the compact quotient $[L]$. These facts give the adjoint pairing even when the form Tu is only known to have moderate growth.

Theorem 78.12 (Uniqueness on the two groups of rank $3m$). *One has $\mathcal{E}_H = \mathbb{C}C$ and $\mathcal{E}_G = \mathbb{C}\phi$.*

Proof. We first prove that T^* is injective on $\mathcal{E}_D$. Let V be the space defining D, of dimension $r = 3m$, and double it as $V \oplus (-V)$ with diagonal maximal isotropic space. Use the tensor of the specified rectangular test with its conjugate, in the doubled polarization of Proposition 142.14. Its theta sum on the two factors at z_1, z_2 is $K(l,z_1)\overline{K(l,z_2)}$, with the usual restriction character on the second factor and the external determinant characters on the two arguments. Include these characters in the restriction throughout the flat Siegel family, as in Theorem 78.4. They are fixed independently of the choice of f. Simultaneous left translation of the two arguments by $D(\mathbb{A})$ leaves the corrected graph section invariant. On its diagonal Levi the determinant has norm one, and the unitary restriction character cancels there, equivalently by invariance of the Hermitian pairing.

Integrate the resulting Siegel Eisenstein series against $\overline{f(z_1)}f(z_2)$ on $[D]^2$ and denote the meromorphic pairing by $Z_f(s)$. The integrated theta space is definite of dimension m, and the Siegel rank is r. In the unitary normalization the point is $s_0 = (m-r)/2 = -m$. Yamana, Theorem 2 applies since $m \le r$. The ordinary theta integral suffices because L is anisotropic. The Schwartz section is holomorphic in the Eisenstein series at s_0, and the formula gives twice this theta integral in the measures of that theorem. Hence

$$Z_f(s_0) = b_L \|T^* f\|^2, \qquad b_L > 0.$$

The same constant works for all f. For a noncompact $[D]$ we use here the usual meromorphic continuation of Eisenstein series as smooth automorphic forms of locally uniform moderate growth after removal of poles. Pairing with rapidly decreasing cusp forms is continuous on such a family. More explicitly, on the two Siegel sets its bound by fixed powers of the heights on compact parameter sets is absorbed by choosing the two independent orders of cuspidal decay larger than those powers and the volume exponents. This permits evaluation inside the integral.

For $\operatorname{Re} s$ large, unfold the absolutely convergent series as in the doubling integral of Gan, Qiu and Takeda, §11.5. To see which terms remain, intersect a rational maximal isotropic subspace Y with the two summands and write X_1, X_2 for the intersections. Its projection to the i-th summand lies in $X_i^{\perp}$. Comparing dimensions for both projections gives $\dim X_1 = \dim X_2$ and equality of the projected spaces with $X_i^{\perp}$. The induced subspace modulo $X_1 \oplus X_2$ is the graph of an isometry between the two nonsingular quotients $X_i^{\perp}/X_i$. If $X_1 \neq 0$, the unipotent radical of the first parabolic stabilizing X_1 acts trivially on this quotient and preserves Y. It acts there with determinant one. The section is invariant under this radical at adelic points in the unfolding, so integration over its rational quotient gives the vanishing constant term of $\bar{f}$. If $X_1 = 0$, both intersections vanish and Y is a graph of an isometry of V. These graphs form one rational orbit with diagonal stabilizer. Writing $z_1 = za$, $z_2 = z$ in the unfolded integral gives the corrected section at $(a,1)$ paired with $\int_{[D]} \overline{f(za)} f(z)\, dz = \|f\|^2 \overline{M_f(a)}$. It follows from Lemma 78.11 that $Z_f(s)/\|f\|^2 = Z_{f_0}(s)/\|f_0\|^2$ first in this half-plane and then

meromorphically. At s_0 it follows that $\|T^*f\|^2/\|f\|^2$ has one common value for all nonzero $f \in \mathcal{E}_D$. For f_0 this value is positive, since $\langle 1, T^*f_0\rangle = \|f_0\|^2 > 0$. This proves injectivity.

It remains to determine the possible image on $[L]$. The lower compact real group fixes the rectangular vector, including the determinant polynomial for the negative source by Theorem 77.6. The finite integral groups fix it as well. At a split place use the probability Hecke operators for the positive coweight $(1, 0, \dots, 0)$. The pointwise rectangular count of Lemma 78.7 for $b = 1$, before the integration in l, gives

$$T_1^D T u = T(h_0 u + h_1 T_1^L u), \qquad h_1 \neq 0.$$

In fact the two summands in the count have $l = 0, 1$; the $l = 1$ counting coefficient before dividing by the nonzero normalization of its source average is $\begin{bmatrix} m \\ 1 \end{bmatrix}_q q^{r-m}$ times $t_1/\begin{bmatrix} r \\ 1 \end{bmatrix}_q$. The possible target character multiplies both terms by the same unit. The negative source coset in the kernel becomes the positive coset on u by the change of variables in the quotient integral. Applying this equation to $u = 1$ shows that T_1^D acts on f_0, and hence on all of $\mathcal{E}_D$, by $h_0 + h_1$. The Hecke operators on the full-level subspace are normal because the spherical algebra is commutative and stable under adjoints. Taking adjoint pairings against arbitrary full-level, real-invariant u therefore yields $(T_1^L)^* T^* f = T^* f$.

In a unitary representation, if a probability average of translates fixes a vector v, integration of $\|R(a)v - v\|^2$ gives zero. By continuity every element in the support fixes v. Thus the inverse fundamental double coset fixes T^*f, as do its inverses and $\mathrm{GL}_m(O_v)$. These generate $\mathrm{GL}_m(F_v)$ by the elementary-divisor decomposition and permutation of diagonal coordinates. This holds at every split finite place. The determinant and strong approximation argument of Theorem 77.46 now gives constancy on $[L]$. Indeed a norm-one determinant is integral at every inert place. After its diagonal removal, strong approximation for the simply connected special unitary group away from one split place and the real places reduces the remaining adele to integral components. The same argument works for the form with the opposite sign since it defines the same unitary group. The injective map T^* therefore has image in the one-dimensional space of constants. Each $\mathcal{E}_D$ contains the specified nonzero f_0, which proves the theorem. □

Corollary 78.13 (The scalar and its adjoint in the second exchange)**.** *Write $A_c = \mathcal{T}_c C$ and $B_c = \mathcal{T}_c^* \phi$ as in Section 78.4. For every prescribed twist there are unique scalars a_c, b_c with*

$$A_c = a_c \phi, \qquad B_c = b_c C, \qquad a_c \|\phi\|^2 = \overline{b_c}\|C\|^2, \qquad |a_c|^2 = \frac{\|B_c\|^2 \|C\|^2}{\|\phi\|^4}.$$

Proof. Their membership in the two spaces follows from Theorem 78.10 and the level, weight and cuspidality in Proposition 78.6. Apply Theorem 78.12, then use $\langle \mathcal{T}_c C, \phi\rangle = \langle C, \mathcal{T}_c^* \phi\rangle$ with the inner product linear in the first argument. □

78.6 Local norm factors for the adjoint of the second exchange

Use $m = 2d$, $r = 3m$, the carrier ϕ on G and the square kernel Θ_c of Section 78.3. To take the norm on the definite group H, integrate $\Theta_c(x,h)\overline{\phi(x)}$ first in x. The result is $\overline{B_c(h)}$. In the doubled norm the graph section at $(a,1)$ therefore multiplies the normalized coefficient $\overline{M_\phi(a)}$ on G. Lemma 78.11 gives its factorization by place. In particular at a split place the spherical parameter in this coefficient is $P_{G,v}^{-1}$, obtained by taking the inverse of each entry. Indeed complex conjugation in a unitary representation gives the contragredient, whose Satake character is inverted on the dual torus.

Here is a split local calculation for a general spherical input. Let k be a nonarchimedean local field with ring of integers O, uniformizer ϖ and residue size q. Give $K = \mathrm{GL}_r(O)$ multiplicative measure one and $M_r(O)$ additive measure one. For a split torus phase u of absolute value one write $u(a) = u^{v(\det a)}$. Use $P_r(q), H_r(q,s)$ from Section 78.1 with m replaced by r.

Lemma 78.14 (A split square integral with a spherical coefficient)**.** *Let $M(a)$ be the unit-vector spherical coefficient of an irreducible unitary spherical representation with parameter $\{\alpha_1,\dots,\alpha_r\}$ in the positive-coweight convention. If $\ell(a)$ is the sum of the absolute values of the elementary divisor valuations, then*

$$\int_{\mathrm{GL}_r(k)} M(a)u(a)q^{-(s+r/2)\ell(a)}\,d^\times a = \frac{H_r(q,s)}{\displaystyle\prod_{i=1}^{r}(1-u\alpha_i q^{-s-1/2})(1-u^{-1}\alpha_i^{-1}q^{-s-1/2})}.$$

The identity holds in a right half-plane and gives the meromorphic continuation of the integral.

Proof. Put $t = s + r/2$, let $K' = M_r(O)\cap \mathrm{GL}_r(k)$ and $J(2s) = \int_{M_r(O)} |\det X|^{2s}\,dX$. Diagonalizing a as in Theorem 78.1 gives

$$J(2s)q^{-t\ell(a)} = |\det a|^t \int_{X, Xa\in K'} |\det X|^{2s}\,dX, \qquad J(2s) = P_r(q)/H_r(q,s).$$

Singular matrices have additive measure zero. Multiply by $u(a)M(a)$, integrate, and set $Y = Xa$. The change $dX = P_r(q)|\det X|^r d^\times X$ leaves the product weight $|\det X|^t|\det Y|^t u(X)^{-1}u(Y)$ on $K'\times K'$, times $P_r(q)M(X^{-1}Y)$. The two integrated operators are bi-K-invariant. In the matrix coefficient of their composition one may therefore insert the projection onto the one-dimensional spherical line between them. This gives one factor from Y with (u,α_i) and another from X with (u^{-1},α_i^{-1}). The inverse parameter in the second factor follows from $M(X^{-1})$ being the coefficient of the dual spherical representation at X.

For clarity, the one-sided factor is

$$\int_{K'} |\det y|^t u(y)M(y)\,d^\times y = \prod_i (1-u\alpha_i q^{(r-1)/2-t})^{-1}.$$

One can evaluate each compact double coset on the normalized spherical principal series with this parameter. Its spherical vector in the induction model takes the value $\delta_B(a)^{1/2}\prod_i \alpha_i^{v(a_i)}$ at nak, where $a = \mathrm{diag}(a_i)$, B is the upper Borel and n is upper unitriangular. The spherical Hecke character on its K-line is the one of M, also when the principal series is reducible. Evaluating the convolution on this vector at the identity computes its eigenvalue. In Iwasawa integration the measure is $\delta_B(a)^{-1}dn\,d^\times a\,dk$, with integral unipotent, unit diagonal and K of mass one. Integrality of na requires $v(a_i)\ge 0$ and $n_{ij}a_j\in O$ for all $i<j$. The volume in n is $\prod_j |a_j|^{-(j-1)}$. Multiplication by $\delta_B(a)^{-1/2}$ turns this into $\prod_j |a_j|^{-(r-1)/2}$. Independent geometric sums over the nonnegative valuations give the formula. For $\mathrm{Re}\,t$ sufficiently large all these manipulations are absolutely convergent. For the change of variables this also follows

from $|M| \le 1$ and the one-sided integral for the trivial representation. Dividing the two-factor evaluation by $J(2s)/P_r(q)$ proves the lemma. □

Define two unitary Hecke characters of E by

$$\lambda_1(a) = (c\upsilon_1^d)(a/\bar{a}), \qquad \lambda_2(a) = (c^{-1}\upsilon_1^m)(a/\bar{a}).$$

Let η be the quadratic character of F associated with E/F, and $\mathcal{S}_c$ the marked split places of F. At a place v put

$$N_v(s) = \prod_{w|v} \left\{ \prod_{j=1}^{m} L_{E_w}(s+j-d, \lambda_{1,w}) \prod_{j=1}^{2m} L_{E_w}(s+j-m, \lambda_{2,w}) \right\}, \qquad D_v(s) = \prod_{j=1}^{r} L_{F_v}(2s+j, \eta_v^j).$$

The Euler factor of a ramified character is one.

Theorem 78.15 (Local factors for the adjoint square norm)**.** *For the flat doubled Schwartz section formed from the actual test of Θ_c and its conjugate, the graph integral against the normalized coefficient of $\bar{\phi}$ is $N_v(s)/D_v(s)$ at every unmarked finite place. At each marked v it is $P_r(q_v)$. At every real place it is independent of c, holomorphic in $\operatorname{Re} s > 0$, and has a finite strictly positive limit at $s = 0$.*

Proof. At a split unmarked place take the chosen component of the norm-one torus and write $\xi = \upsilon_w(\varpi)$. The source action has determinant multiplier c^{-1}. The lattice overlap for a source elementary divisor ϖ has phase ξ^r, one power for each target row. This is the calculation with the Hom lattice in Theorem 78.1; it uses only the number of rows and the chosen finite frames. Consequently at the theta point the unit-vacuum coefficient is $u^{v(\det a)} q^{-r\ell(a)/2}$, with $u = \xi^r c_v(\varpi)^{-1}$. The target multiplier is evaluated at the identity. Varying the section multiplies the graph value by $q^{-s\ell(a)}$, by the elementary-divisor calculation of Section 78.1. By Theorem 78.10, multiplication of the inverse spherical parameter by u gives the two segments

$$c_v(\varpi)^{-1}\xi^{-m} A_m(q), \qquad c_v(\varpi)^{-1}\xi^{2m} A_{2m}(q).$$

In Lemma 78.14 each segment is accompanied by its inverse. Since $\upsilon_{1,v}(\varpi) = \xi^2$, the first pair carries the two values of λ_1 above v, interchanged in order, and the second pair the two values of λ_2. In a segment of length $2e$ the arguments of the Euler factors are $s + j - e$, $1 \le j \le 2e$. At a split place $D_v(s)^{-1} = H_r(q, s)$. This proves the unmarked split assertion.

At a marked place the position test is supported on $\mathrm{GL}_r(O_v)$, and its absolute value there is one. A source translate has disjoint support unless $a \in K_v$. On K_v the character in the substitution is cancelled by the source determinant multiplier. Thus the overlap is its additive squared norm, $P_r(q_v)$. Both the flat height and the spherical coefficient equal one on K_v. This proves the asserted integral including at an old mark of modulus ϖ^2.

At every inert place the groups, level, section and constituent are unramified. The double has the integral diagonal polarization used in Lemma 78.3. The local norm-one group is compact, and the unramified space character of even rank has value one on the common uniformizer. Apply the spherical unitary doubling formula of J.-S. Li, Theorem 3.1, with the normalization and denominator checked in that lemma. Its numerator is the standard Euler factor at $s + 1/2$ for the contragredient. Both segments of $P_{G,v} = A_m(q_w) \sqcup A_{2m}(q_w)$ are self-inverse. The characters λ_1, λ_2 are unramified and equal one on $\varpi_w = \varpi_v$. This gives exactly the indicated $N_v(s)$, including the residue size $q_w = q_v^2$.

At a real place put $p = 2m$, $q' = m$ and use Cartan parameters $t_1 \ge \cdots \ge t_{q'} \ge 0$ of $U(p, q')$. Let $h(a)$ be the normalized source coefficient of the Gaussian of the square oscillator,

including its splitting character. Its compact character and holomorphy equations are those of ϕ by Proposition 78.6. By the real assertion of Lemma 78.11 the normalized coefficient of $\bar{\phi}$ is $\overline{h(a)}$. Consequently the integral at the theta point has integrand $|h(a)|^2$ times the positive squared norm of the Gaussian.

For a boost the two-mode oscillator factorization in Lemma 142.12 gives $|h(a(t))| = \prod_i (\cosh t_i)^{-r}$. In each row, the lowering factor kills the vacuum, the nonconstant terms from the raising factor are orthogonal to it, and the diagonal factor contributes $(\cosh t_i)^{-1}$ for the paired columns. The two compact factors only change its phase. The Cartan density is bounded by a rank-dependent constant times $\exp(2(r-1)\sum_i t_i)$, by the same lemma. The resulting square is integrable against this density since $r = p + q'$, with an exponential bound $O_r(\exp(-2\sum_i t_i))$. It is positive in a neighborhood of the identity.

To see continuity in the section parameter, take orthonormal coordinates on the real doubled space for its positive majorant. In the diagonal and orthogonal difference frame the lower r rows of the embedded element $(a, 1)$ have Gram determinant $\det((1+aa^\dagger)/2)$. This follows by writing them, up to a right unitary change of basis, as $2^{-1/2}(a, -1)$. For an upper-block parabolic times unitary factor, this determinant is $|\det D|^2$ of the lower Levi block. The upper block has the reciprocal determinant modulus. The flat modulus of the unitary Siegel section is the $\mathbb{C}/\mathbb{R}$ norm of the upper determinant, hence the height on the graph is $b(a) = \det((1 + aa^\dagger)/2)^{-1}$. It lies in $(0, 1]$, since $(1+y)/2 \geq \sqrt{y}$ for each positive eigenvalue and $|\det a| = 1$. In the alternative dual basis the inverse element gives the same determinant. Thus moving to s inserts $b(a)^s$. The integrable square dominates for $\operatorname{Re} s \geq 0$, giving continuity at zero and holomorphy in the open half-plane by Morera's theorem and Fubini. All real data are independent of the weight-zero twist. This proves the real assertion. □

78.7 The second global theta exchange and its normalization

For a prescribed twist $c = \chi_M^j \vartheta$ put $Q_c = \prod_{v \in \mathcal{S}_c} q_v$, with the old marks included. For a norm-one character α ramified at these marks, write

$$\mathcal{D}_e(\alpha) = Q_c^{e^2} \prod_{z=1}^{e} |L_{E,f}(z, \lambda_\alpha)|^2, \qquad \lambda_\alpha(a) = \alpha(a/\bar{a}).$$

We will use this with $\alpha = c\upsilon_1^d$ and $c^{-1}\upsilon_1^m$, so the marked primes are determined by α itself. The conductor power uses one residue cardinality for each conjugate pair over a split place of F.

Theorem 78.16 (The normalized carrier identity)**.** *At every sufficiently large node of the prescribed family there is a number $K_{12} > 0$, depending on the field, rank and measure choices, such that for every prescribed twist c there is a unique $\gamma_{12}(c)$ with*

$$\gamma_{12}(c) \int_{[H]} C(h)\Theta_c(x,h)\, dh = \phi(x) \quad (x \in G(\mathbb{A})), \qquad |\gamma_{12}(c)|^2 = \frac{K_{12}}{\mathcal{D}_d(c\upsilon_1^d)\mathcal{D}_m(c^{-1}\upsilon_1^m)}.$$

The same scalar works for every evaluation point and every subsequent period test at this node.

Proof. Double G with the diagonal Siegel parabolic and integrate on the definite space I_r. Form the doubled section with the square test of Θ_c and its conjugate, with the determinant restriction conventions of Theorem 78.4. In particular its corrected graph value at $(a, 1)$ at $s = 0$ is the source oscillator coefficient for this very test. The same choice on the diagonal makes the section invariant under simultaneous left translation there for every s. Let $\mathcal{Z}_c(s)$ be the integral of the restricted Eisenstein series against $\bar{\phi}(x_1)\phi(x_2)$ on $[G]^2$, divided by $\|\phi\|^2$. In the unitary Siegel–Weil formula the Siegel rank and the dimension of the integrated space both equal r. This space is anisotropic. Thus Yamana, Theorem 2 evaluates the holomorphic value of the series at $s = 0$ as twice the ordinary theta integral in its measures. The external target determinant characters on the two copies cancel. At that point the product of the two rational theta sums is $\Theta_c(x_1, h)\overline{\Theta_c(x_2, h)}$. The rapid decrease of ϕ absorbs the uniform moderate growth on the two Siegel sets, as in Theorem 78.12. Compactness in h and Fubini now give, in our fixed measures,

$$\mathcal{Z}_c(0) = b\, \|B_c\|^2 / \|\phi\|^2, \qquad b > 0.$$

Only the measures enter b.

Unfold in an absolute convergence half-plane. The non-graph orbits vanish by the proper parabolic constant terms of the cusp form, as proved in Theorem 78.12. In the graph orbit the quotient variable x supplies $\int_{[G]} \overline{\phi(xa)}\phi(x)\, dx$, leaving the coefficient $\overline{M_\phi(a)}$ after normalization. It factors by Lemma 78.11. The graph section is a restricted tensor product of the specified local sections, so absolute convergence gives the product of the local integrals. Equivalently one integrates over larger and larger sets of places, using compact volume one outside each set. The finitely many marked data enter with their actual squared additive norms. Theorem 78.15 therefore gives, up to a single positive Haar factor b',

$$\mathcal{Z}_c(s) = b' Z_\infty(s) \frac{\displaystyle\prod_{j=1}^{m} L_{E,f}(s + j - d, \lambda_1) \prod_{j=1}^{2m} L_{E,f}(s + j - m, \lambda_2)}{\displaystyle\prod_{j=1}^{3m} L_{F,f}(2s + j, \eta^j)} \prod_{v \in \mathcal{S}_c} \frac{P_r(q_v)}{H_r(q_v, s)}.$$

Here Z_∞ denotes the product of the real graph integrals. At a mark both numerator characters are ramified on each of the two components. The Euler numerator is one there and the reciprocal denominator is $H_r(q_v, s)$; the last product replaces it by the actual local answer.

All finite Hecke functions on the right have their classical meromorphic continuations. The real factor is holomorphic in $\operatorname{Re} s > 0$ by Theorem 78.15. Thus the identity holds meromorphically there. We can approach zero along positive real s. The real factor tends to its strictly positive value, and the correction at each mark tends to one. In the denominator the $j = 1$ value $L_{F,f}(1, \eta)$ is finite and positive by the quotient of the residues of ζ_E and ζ_F. The factors at the other integers are nonzero by absolute convergence.

For either numerator character put $e = d$ or $e = m$ respectively. Its angular exponents at the complex places of E have absolute value $2e$, because the finite-order twist has weight zero and base change is $a \mapsto a/\bar{a}$. Its finite Hecke function $L(z)$ is nonzero at $z = 1, \dots, e$. For $z > 1$ the Euler product suffices. At 1, the character and its square have nonzero angular type and hence holomorphic Hecke functions. Removing the ramified factors throughout, the Euler logarithms give $\zeta_E(\sigma)^3 |L(\sigma)|^4 |L(\sigma, \lambda^2)| \geq 1$ in the omitted-factor convention for $\sigma > 1$. This is the sum-of-squares trigonometric inequality in Theorem 78.2. A zero at one would outweigh the third power of the simple zeta pole. The archimedean gamma factors $\Gamma_{\mathbb{C}}(z + e)$ are finite and nonzero at $1 - e, \dots, 0$, since their smallest argument is one. The primitive Hecke functional equation now gives a finite nonzero value at each of these integers from the paired positive argument. The whole right side therefore has a nonzero limit at zero. Since $\mathcal{Z}_c$ is holomorphic there and its value is a positive factor times $\|B_c\|^2$, we have $B_c \neq 0$.

It remains to compare the constants across twists without introducing any dependence on x or on the period test. For either block the finite conductor ideal of the primitive character has norm Q_c^2, by nontrivial tame ramification in the two split components at every mark and unramified data elsewhere. Its completed conductor is $A_c^{\mathrm{cond}} = |\operatorname{disc} E| Q_c^2$. Compare to any reference twist c_* on the same field. For the two central-symmetric arguments $z, 1-z$, $1 \leq z \leq e$, taking absolute values in the two functional equations gives

$$\left| \frac{L_c(z) L_c(1-z)}{L_{c_*}(z) L_{c_*}(1-z)} \right| = \left(\frac{A_c^{\mathrm{cond}}}{A_{c_*}^{\mathrm{cond}}} \right)^{z-1/2} \left| \frac{L_c(z)}{L_{c_*}(z)} \right|^2 .$$

The gamma factors are common and cancel. Summing the exponents $z - 1/2$ gives $e^2/2$; the conductor power is consequently $(Q_c/Q_{c_*})^{e^2}$. Apply this calculation to both blocks of the displayed product. The other factors at zero are independent of c. It follows that

$$\frac{\|B_c\|^2}{\|B_{c_*}\|^2} = \frac{\mathcal{D}_d(c \upsilon_1^d) \mathcal{D}_m(c^{-1} \upsilon_1^m)}{\mathcal{D}_d(c_* \upsilon_1^d) \mathcal{D}_m(c_*^{-1} \upsilon_1^m)}.$$

By Corollary 78.13 the same ratio holds for $|a_c|^2/|a_{c_*}|^2$, and $a_c \neq 0$. Take $\gamma_{12}(c) = a_c^{-1}$, and define K_{12} by the claimed identity at c_*. This makes it positive and independent of the twist. The equality $A_c = a_c \phi$ holds in the cuspidal space and hence at every point by smoothness and full support of Haar measure. This proves the theorem. □

Corollary 78.17 (Removal of the first signed scalar)**.** *Let $\gamma^-(c)$ be the first signed scalar of Corollary 78.5. On the product in which that first class sum has already been multiplied by $\gamma^-(c)$, the remaining multiplier $\gamma_2(c) = \gamma_{12}(c)/\gamma^-(c)$ satisfies*

$$|\gamma_2(c)|^2 = K_{12} B_{F,m} / \mathcal{D}_m(c^{-1} \upsilon_1^m).$$

Proof. In that corollary $\delta = c^{-1}$, so $D_\delta = \mathcal{D}_d(c \upsilon_1^d)$. The ratio of the two squared norms cancels this factor. □

Thus the first two exchanges supply the carrier itself on the original group at the original level. The comparison between its period and the integral definite theta class sum in Theorem 77.41 concerns the third exchange on $I_r \oplus W'$.

79 The third theta exchange

79.1 The smaller-rank integral in the third exchange

Fix the field and interval of Theorem 77.25, a lift index j, and a choice $i = (\Lambda, b_0, \text{short rows})$ as in Section 77.13. Write

$$W = I_r \oplus W', \quad z_i(g) = \operatorname{diag}(g, 1, 1, b_0) \in U(W')(\mathbb{A}), \quad y_i(h', g) = \operatorname{diag}(h', z_i(g)).$$

Here S_0 has signature $(2m, m)$, $r = 3m$, $p = \dim W'$ and $n = p + r$. At a definite class point we take the identity real component. Retain $\kappa = \upsilon_1^m$, $c = \chi_M^j \vartheta$ and the character ν of the pair (W, I_p) in Section 77.13. That pair has characters (α, ν) with $\alpha = \nu\kappa$. The carrier kernel on (S_0, W') has characters (κ, β_0) where $\beta_0 = \nu\kappa/c$. Indeed removing the twist and old tame part c from the first-factor character ν gives its fixed untwisted part, and the carrier character is this part times κ. Thus $c\beta_0 = \alpha$ and $\alpha/\nu = \kappa$.

On (S_0, W) use the characters (κ, α). At the marked split places take the position datum $\Psi\binom{A}{B} = \chi(\det A) F_\Lambda(B)$ of Lemma 77.42. Elsewhere take the unit finite vacua. At the real places use the polynomial Q of Proposition 77.43, with its order of the rows. Denote this kernel by $\theta_{\Psi,Q}$. The smaller rank refers to the space S_0 integrated in

$$\mathcal{S}_{i,c}(g) = \sum_{h' \in \mathcal{C}_r} \frac{C_c(h')}{d_{h'}} \int_{[G]} \theta_{\Psi,Q}(x, y_i(h', g))\, dx, \qquad C_c(h') = c(\det h') C(h').$$

The weight $d_{h'}^{-1}$ is that of the definite class of the full stabilizer, as in Theorem 77.41.

Theorem 79.1 (The contracted smaller-rank integral)**.** *For every g and every prescribed twist one has*

$$\gamma_{12}(c) \mathcal{S}_{i,c}(g) = N_R\, \mathcal{B}_i(g), \qquad N_R = |\operatorname{GL}_r(R)|.$$

Each inner integral at the stated points is absolutely convergent. The formula holds for the entire family of i, g with the same $\gamma_{12}(c)$.

Proof. Write $K_{1,c}(x, h')$ for the kernel on (S_0, I_r) with characters (κ, c), determinant test and constant real polynomial, and $K_{2,i}(x, z_i(g))$ for the carrier kernel on (S_0, W') with characters (κ, β_0), datum F_Λ, and polynomial Q. On the displayed block diagonal one has exactly

$$\theta_{\Psi,Q}(x, y_i(h', g)) = K_{1,c}(x, h') K_{2,i}(x, z_i(g)).$$

Indeed the position spaces for the two target summands are a direct sum, and rational summation in the product polarization runs over all pairs of rational position vectors. The unmarked and marked Schwartz data are products. Proposition 77.43 puts all the polynomial variables in the second factor, leaving the Gaussian in the first. The two multipliers on the shared source multiply to $\alpha(\det x)^{-1}$. The two target multipliers are the restriction of κ on the full target determinant. The underlying oscillator on an orthogonal sum is the tensor product in these normalizations, as in the proof of Theorem 77.41. Absolute convergence of the rational Schwartz sums at each adelic point allows their multiplication.

Changing the first character of $K_{1,c}$ to $c\kappa$ multiplies it by $c(\det h')$ and gives exactly Θ_c of Section 78.3. For each fixed h', Proposition 78.6 gives cuspidality in x already for the kernel before class integration. Its holomorphic scalar weight, finite level and smoothness put it in the cuspidal space of the given weight, so it is rapidly decreasing on Siegel sets. The second

factor is of moderate growth in x for fixed $z_i(g)$. This proves absolute convergence. There are finitely many class terms. Passing the sum through the integral gives

$$\mathcal{S}_{i,c}(g) = \int_{[G]} \left(\sum_{h' \in \mathcal{C}_r} \frac{C(h')\Theta_c(x,h')}{d_{h'}} \right) K_{2,i}(x, z_i(g))\, dx.$$

Full level and compact real invariance in the combined summand identify the inner sum with $A_c(x)$. This uses the same quotient measure as in Theorem 78.16. Multiplication by $\gamma_{12}(c)$ therefore inserts $\phi(x)$. The resulting integral is precisely $\mathcal{I}_i(F_\Lambda; g)$, including the inflation at the new marks. By Theorem 77.44 it equals $N_R \mathcal{B}_i(g)$. This proves the identity. □

Corollary 79.2 (Separation by the smaller-rank integrals for every twist)**.** *Fix any nonzero good form covered by Theorem 77.25. Choose a complementary test, a character projection and a first-factor index i for which the carrier product period is nonzero. Replacing $\mathcal{B}_i$ by $\mathcal{S}_{i,c}$ gives a nonzero projected period for every prescribed twist c. Its squared absolute value is multiplied by*

$$\frac{N_R^2}{K_{12}}\, \mathcal{D}_d(c\upsilon_1^d)\mathcal{D}_m(c^{-1}\upsilon_1^m).$$

Proof. Hold fixed all the data other than the indicated first factor. The character projection and period functional in that factor are linear or conjugate-linear according to the pairing convention. Apply Theorem 79.1 and take squared absolute values. Theorem 78.16 evaluates the scalar, which is strictly positive. □

The larger-rank expression at these same arguments is $L_{i,\vartheta}(g)$ in Theorem 77.41. Its inner integral is on $U(I_p)$, has characters (α, ν), and uses the residue partner Φ and the stacked polynomial P. In particular the still needed comparison is between this expression and the specific $\mathcal{S}_{i,c}$ above. An identity $\delta_3(c)L_{i,\vartheta}(g) = \mathcal{S}_{i,c}(g)$ would put the multiplier $\gamma_{12}(c)\delta_3(c)/N_R$ in front of the arithmetic expression. The preceding proof determines the carrier and the integer N_R; it does not supply that remaining identity or the absolute value of $\delta_3(c)$.

79.2 Frame integrals at the split conductor of the third exchange

Retain $n = p + r$ and $\alpha = \nu\kappa$ from Section 79.1. At a split place write $k = F_v$, $O = O_v$, $K_b = \mathrm{GL}_b(O)$, and let ξ be the value of the Weil splitting character on the chosen uniformizer. Read norm-one characters on the chosen component. For $t \in k^\times$, $s \in \mathbb{C}$ put

$$\omega_s(t) = \alpha_v(t)^{-1}\xi^{nv(t)}|t|^{n/2+s}.$$

The parameter s varies a local determinant integral. At $s = 0$ its weight is that of the actual oscillator. We use Haar measures with K_b of mass one for all b.

Fix $k^n = k^r \oplus k^p$, put $E_r = \binom{1_r}{0}$, $F_p = (0 \;\; 1_p)$, and for $t \in \mathrm{GL}_n(k)$ set $Y_t = tE_r$, $X_t = F_p t^{-1}$. Thus Y_t is a basis of an r-plane and X_t a surjection with precisely this kernel. Define

$$J_p(\Phi; t, s) = \int_{\mathrm{GL}_p(k)} \omega_s(\det h)\Phi(hX_t)\, dh,$$
$$J_r(\Psi; t, s) = \int_{\mathrm{GL}_r(k)} \omega_s(\det u)\Psi(Y_t u)\, du.$$

Begin with locally constant functions compactly supported in the full-rank matrix loci, including zero extensions of the marked tests. These integrals then exist for every s. For example the map $h \mapsto hX_t$ parametrizes the surjections with the given kernel. It is a homeomorphism onto that fiber, which is closed in the full-rank locus. The inverse sends X to XT for any right inverse T of X_t. The same argument applies to $Y_t u$. Compact support in the full-rank locus therefore gives compact integration domains.

Theorem 79.3 (Equality of the marked sections in complementary frames)**.** *For the pair (Φ, Ψ) of Lemma 77.42, inflated at the marks to the integral boxes, one has*

$$J_p(\Phi; t, s) = \omega_s(\det t) J_r(\Psi; t, s)$$

at every marked split place and every $t \in \mathrm{GL}_n(k)$. At $s = 0$ this equality persists under every simultaneous translation by the actual split group $U(W)(F_v)$ on the two position models.

Proof. The additive trace character of the old finite ring factors over its local components. Moreover $\mathrm{GL}_r(R)$ is the product of the local groups, so its sum defining F_Λ is a product of independent local sums. On a new component the inflated F_Λ contributes the constant function on the integral box. Thus the marked tests in the assertion are the indicated local factors at old precision two and at new precision one.

Suppose first that $t \in K_n$. Then X_t is an integral surjection with integral right inverse, and Y_t is an integral injection with integral left inverse. The support of Φ requires its last square block to have unit determinant, so hX_t must be an integral surjection. This is equivalent to $h \in K_p$ as far as primitivity is concerned, since $X_t(O^n) = O^p$. Similarly $Y_t u$ must be an integral primitive frame, forcing $u \in K_r$. Indeed Y_t identifies O^r with the intersection of its column space with O^n, and a primitive frame in that space spans this intersection. Conversely multiplication by these unit groups preserves primitivity and the tests obey

$$\Phi(hX) = \chi_v(\det h)\Phi(X), \qquad \Psi(Yu) = \chi_v(\det u)\Psi(Y).$$

For the first identity the quotient $z^{-1}y$ is fixed by left multiplication of the two blocks. For the second, the determinant of the top block contributes $\chi_v(\det u)$ and the lower test is right invariant. The same equations hold at the zeros of the tests. On units ω_s is χ_v^{-1}. Thus the two integrals are respectively $\Phi(X_t)$ and $\Psi(Y_t)$, without a counting factor. Their relation is the determinant identity of Lemma 77.42 on the local residue ring.

For general t, the lattice in the column space given by its intersection with O^n is saturated, hence a direct summand over the discrete valuation ring. An integral basis of it extends to one of O^n. Consequently

$$t = k_0 \begin{pmatrix} a & b \\ 0 & d_0 \end{pmatrix}, \qquad k_0 \in K_n, \quad a \in \mathrm{GL}_r(k),\ d_0 \in \mathrm{GL}_p(k).$$

One has $Y_t = Y_{k_0} a$ and $X_t = d_0^{-1} X_{k_0}$. Changes of variables in the two Haar integrals multiply their values at k_0 by $\omega_s(\det d_0)$ and $\omega_s(\det a)^{-1}$, respectively. Their quotient relation is therefore the one asserted at t, including when both sides vanish.

Finally, at $s = 0$ write $\omega = \omega_0$. For $g \in U(W)(F_v) = \mathrm{GL}_n(k)$, the two translates have the forms

$$\Phi_g(X) = b(g)\Phi(Xg), \qquad \Psi_g(Y) = a(g)\Psi(g^{-1}Y).$$

If $z = \det g$, the position Jacobians and splitting characters give

$$b(g) = \nu_v(z)^{-1}\xi^{pv(z)}|z|^{p/2}, \qquad a(g) = \kappa_v(z)\xi^{-rv(z)}|z|^{-r/2}.$$

For the first substitution the p rows are acted on from the source, each giving the square-root Jacobian and one chosen splitting phase. In the second substitution the r columns are acted on from the target, with the inverse Jacobian and conjugate splitting phase. These are the source and target computations of Lemma 78.7. The external multipliers are the displayed characters, in their respective source and target conventions. Since $\alpha = \nu\kappa$, $b(g)/a(g) = \omega(z)$. Now $X_t g = X_{g^{-1}t}$ and $g^{-1}Y_t = Y_{g^{-1}t}$. Applying the identity there and inserting the two multipliers gives $J_p(\Phi_g; t, 0) = \omega(\det t) J_r(\Psi_g; t, 0)$. □

Proposition 79.4 (The kernel of integration along bases)**.** *On the test functions compactly supported in the full-rank locus, the kernel of $\Phi \mapsto J_p(\Phi; \cdot, s)$ is spanned by $\omega_s(\det h)\Phi(h\,\cdot) - \Phi$ for $h \in \mathrm{GL}_p(k)$. The kernel on the column side is spanned by $\omega_s(\det u)\Psi(\cdot\, u) - \Psi$ for $u \in \mathrm{GL}_r(k)$. In particular any linear evaluation on the row data invariant under the indicated weighted changes of basis has the same value for two data with the same frame integral. The corresponding assertion holds for column data.*

Proof. Invariance of the integral under the displayed operations follows by translation of Haar measure. For the converse, project the full-rank row space to the Grassmannian by sending a surjection to its kernel. On each coordinate minor chart this is a product with $\mathrm{GL}_p(k)$: use the frame with the chosen nonsingular p-column minor equal to the identity. Use a disjoint finite partition by compact open subsets inside such charts on the projection of the support. Such a partition exists by compactness and the compact open coordinate basis. A locally constant compactly supported function in a product chart is a finite sum of indicators of compact open rectangles, by compactness of its support. Refining the base partition writes it on each piece as one fixed fiber function $F(h)$.

If all frame integrals vanish, the ordinary integral of $\omega_s(\det h)F(h)$ on this fiber is zero. It is compactly supported and right invariant under a sufficiently small compact open subgroup J. Write it as a finite sum $\sum_l c_l \mathbf{1}_{h_l J}$. The cosets have equal measure, so $\sum_l c_l = 0$. The function is a linear combination of differences $\mathbf{1}_{h_l J} - \mathbf{1}_J$. Undoing the weight identifies these differences with differences under the stated weighted left change of basis. Extending each term by zero on the other base pieces proves the row claim. On the column side trivialize by a nonsingular row minor, use left invariance of the weighted fiber function under a compact open subgroup and take differences of its right translates. Both groups are unimodular. The statements about invariant evaluations follow by linearity. □

These assertions describe how a marked local functional can depend on its matrix test. Equality between the actual adelic functionals also depends on the spaces, real data and rational summation.

79.3 The split frame multiplier at the third critical block

The dimensions in Section 79.1 satisfy

$$e = (p - r)/2, \qquad n/2 = r + e, \qquad e = (k + 7m)/2.$$

Indeed $\dim W_j = k + m - 1$, $p = \dim W_j + 9m + 1$ and $r = 3m$; both k and m are even. In this subsection k in the dimension formula denotes the original source rank. At a split finite place write $q = q_v$ for the residue cardinality and $O = O_v$ for the local integer ring.

Lemma 79.5 (Relative multipliers of the two split lattice sections)**.** *At an unmarked split place use the two integral matrix indicators Φ^0, Ψ^0 of sizes p-by-n and n-by-r. Put $u_v = \alpha_v(\varpi)^{-1}\xi^n$ at that place. Then, in the notation of Section 79.2,*

$$J_p(\Phi^0; t, s) = D_{c,v}(s)\,\omega_s(\det t) J_r(\Psi^0; t, s), \qquad D_{c,v}(s) = \prod_{\ell=1}^{2e} (1 - u_v q^{-(s+\ell-e)})^{-1}.$$

This holds for all frames t in an absolute convergence half-plane, then as an identity of meromorphic functions. At a marked split place the analogous identity for the marked tests has $D_{c,v}(s) = 1$.

Proof. For $t \in K_n$ the matrix X_t has integral right inverse and Y_t integral left inverse. Thus hX_t is integral exactly when h is, and $Y_t u$ is integral exactly when u is. In dimension b the resulting one-sided integral on $\mathrm{GL}_b(F_v) \cap M_b(O)$ has weight $\omega_s(\det h)$. At an unmarked place its value is

$$\prod_{i=1}^{b} (1 - u_v q^{i-1-n/2-s})^{-1}.$$

This follows either by the elementary determinant generating function in Theorem 78.1, with additive and multiplicative Haar measures related by $dX = P_b(q)|\det X|^b\, d^\times X$, or by the one-sided integral in Lemma 78.14 evaluated at the constant representation. Initially take $\mathrm{Re}(n/2 + s) > p - 1$, so both integrals converge. Taking the ratio for $b = p$ and $b = r$ leaves just $i = r + 1, \dots, p$. As $n/2 = r + e$, the exponents in the remaining denominators are $-s - e, -s - e + 1, \dots, -s + e - 1$. This gives $D_{c,v}$ on integral frames, where $\omega_s(\det t) = 1$. For all frames apply the parabolic decomposition and change of variables of Theorem 79.3, initially where both integrals converge. That argument also gives their meromorphic extensions frame by frame. The last assertion, with the actual residue functions, is Theorem 79.3. □

Put $\sigma_c = \alpha^{-1} \upsilon_1^{r+e}$ on the adelic norm-one torus and $\lambda_c(a) = \sigma_c(a/\bar{a})$ on the ideles of E. This notation for σ_c is confined to this third block. Its finite phase at the chosen component of an unmarked split place is just u_v, since $\upsilon_{1,v}(\varpi) = \xi^2$. The conjugate component has the inverse value. The tame conductor at each old and new mark has exponent one on each of the two components. Let c_* be a reference twist with the same field and lift index. For $\mathrm{Re}\, s$ sufficiently large define

$$\mathcal{R}_3(s; c, c_*) = \prod_{v \text{ split}} \frac{D_{c,v}(s)\overline{D_{c,v}(\bar{s})}}{D_{c_*,v}(s)\overline{D_{c_*,v}(\bar{s})}}.$$

Taking the conjugate at $\bar{s}$ gives the second component as a holomorphic function of s. This is a relative product of frame multipliers.

Theorem 79.6 (The value of the relative split frame multiplier)**.** *The product $\mathcal{R}_3$ continues meromorphically and is regular and nonzero at zero. Its value there is positive and*

$$\mathcal{R}_3(0; c, c_*) = \frac{\mathcal{D}_e(\sigma_c)}{\mathcal{D}_e(\sigma_{c_*})}, \qquad \mathcal{D}_e(\sigma_c) = Q_c^{e^2} \prod_{z=1}^{e} |L_{E,f}(z, \lambda_c)|^2.$$

Proof. The unitary phase in the lemma gives

$$\mathcal{R}_3(s;c,c_*)=\prod_{\ell=1}^{2e}\frac{L_{E,f}(s+\ell-e,\lambda_c)}{L_{E,f}(s+\ell-e,\lambda_{c_*})}$$

in the initial half-plane. At an unmarked split pair its two Euler factors are the lemma and its conjugate. At a mark for one of the two twists its own two numerator-character Euler factors are one, as is its frame multiplier. This works also when the two sets of new marks differ. At an inert place the norm-one unit character is unramified and trivial, and a common uniformizer maps to one under $a\mapsto a/\bar{a}$. Thus the inert Euler factors coincide. There are no other finite places by Lemma 77.37. This proves the product identity and gives the continuation by the classical Hecke functional equations.

At a real embedding α has weight $m-\mu_\tau$ as in Section 77.13. The weight of σ_c is $a_\tau=\mu_\tau+2m+e$, independent of the twist. Hence its base-change character at the associated complex place has angular exponent of absolute value $2a_\tau$, with $a_\tau\geq e$. At the integers $1-e,\dots,0$ the common gamma factors $\Gamma_{\mathbb{C}}(z+a_\tau)$ are finite and nonzero. There are no positive-integer zeros: at $z>1$ use the Euler product, and at $z=1$ use the trigonometric inequality of Theorem 78.2; the character and its square have nonzero angular type. The functional equation then shows that the finite Hecke functions are nonzero throughout $1-e,\dots,e$.

For the sign in the claimed value, the finite Hecke function of each of these characters equals that of its inverse, by conjugation of ideals of E. Indeed conjugation sends λ_c to its inverse and preserves the absolute norm, and the ramified set consists of conjugate pairs. The common L-function is real on the real axis. Its root number in the completed functional equation could a priori contribute at the reflected arguments. Here it is $+1$. We give the local-to-global reason. The character $\lambda_c=\sigma_c\circ(1-\mathrm{conj})$ is trivial on the ideles of F. Use a fixed nonzero $\Delta\in E$ of relative trace zero and the standard additive character of $\mathbb{A}_E/E$ obtained by composing any nontrivial additive character of $\mathbb{A}_F/F$ with $z\mapsto\mathrm{Tr}_{E/F}(\Delta z)$. It is trivial on $\mathbb{A}_F$. At an inert finite place λ_c is trivial, whose local root number in unitary normalization is one for every nontrivial additive character. At a split place the two characters are inverse. The two additive characters are opposite under the canonical identifications with F_v, since the two components of Δ are opposites. The product of the two local epsilon factors at the central point is one by the local Tate functional equation applied twice, using opposite Fourier kernels and self-dual measures. This includes tame ramification. At a real place of F the associated character of $\mathbb{C}^\times$ and the additive character are independent of c, so their local root number is the same for c and c_*. In fact, for an exponent $2a$ the standard complex epsilon factor is $i^{|2a|}$ up to the Fourier sign convention, and scaling the real-trace additive character by a purely imaginary scalar contributes its angular character value $(\pm i)^{2a}$; their product is one. Consequently the global root numbers in particular have ratio one. For the asserted comparison only this last ratio is needed.

The completed conductor is $A_c=|\operatorname{disc}E|Q_c^2$. Apply the primitive functional equations to the pair $z,1-z$ for $1\leq z\leq e$. The root-number ratio is one, the real archimedean factors cancel, and at positive integers the dual value is the conjugate of the original one. The contribution of the pair to the ratio is

$$(A_c/A_{c_*})^{z-1/2}\left|\frac{L_{E,f}(z,\lambda_c)}{L_{E,f}(z,\lambda_{c_*})}\right|^2.$$

Multiplying these positive quantities gives the formula. The conductor exponent is e^2 for the product of the residue sizes on the F-side, by $\sum_{z=1}^{e}(z-1/2)=e^2/2$. □

The theorem evaluates the split frame contributions at the rank difference $p-r$. Identification of the relative norms of the two global theta periods with this product requires a comparison of their rational theta sums and their real polynomial data.

79.4 One standard residue test for the global comparison

We now use translations of the common group $U(W)$ in the two actual theta integrals. The translations can mix I_r and W'. We first justify the integral on the indefinite group at such points.

Lemma 79.7 (Rapid decrease from a split full-rank test on the source)**.** *Let S be a Hermitian space, T another Hermitian space, and use the theta kernel on $U(S)\times U(T)$ with Schwartz data. Suppose at one split finite place its position test on $\mathrm{Hom}(S_v, T_v)$ in the chosen component is compactly supported in the injective matrices. For a fixed target point the theta kernel is rapidly decreasing on Siegel sets of the source. The estimates are locally uniform in the target point and hold for Schwartz derivatives at infinity.*

Proof. If the source is anisotropic its quotient is compact. Otherwise choose a rational maximal isotropic flag in S and a dual flag, giving a rational Witt decomposition with isotropic basis $e_1, \dots, e_l$. In the mixed polarization the rational summation variables include the images $y_1, \dots, y_l \in T(E)$ of these basis vectors, written in the two split components at a split place. Denote remaining rational position coordinates by z. In the chosen split position model passage to this polarization keeps the columns of the matrix on the indicated isotropic subspace in the chosen component and Fourier transforms the dual columns, as in Proposition 78.6. In particular the mixed test vanishes at $y_1 = 0$: before the partial Fourier transform the retained column on e_1 would be zero. This remains true after arbitrary local source and target translations, since both act here by invertible matrix substitutions and scalar multipliers.

Apply adelic reduction theory to the restriction of scalars of $U(S)$, whose center is anisotropic. Its Siegel sets have the form $\Omega a K$, using a compact set Ω in the minimal parabolic, a compact K, and the positive split chamber with the simple roots bounded below. This is the adelic reduction theorem in Arthur, *An introduction to the trace formula*, Theorem 8.1. In Witt coordinates take a trivial at the finite places, with diagonal $a_1, \dots, a_l$ on the isotropic flag at each real embedding, and the inverse diagonal on the dual. These positive parameters are shared across embeddings for the rational split component. The ratios a_i/a_{i+1} and the last parameter a_l are bounded below. All a_i are therefore bounded below and are $O(1 + a_1)$, with constants allowed to depend on the fixed group and Siegel set. The conjugates $a^{-1}\Omega a$ lie in a compact set: inverse conjugation contracts positive root coordinates up to constants, and the Levi centralizes the split component. Fixed finite translates of these sets can be absorbed in the compact data.

Move a to the left in the pair action. The remaining Schwartz data range in a bounded archimedean Schwartz family, and their finite parts have common compact support in the mixed model. This follows from continuity of the oscillator on the Schwartz space and local constancy at the finite places. All retain the marked vanishing just proved. The rational coordinates that can contribute lie in a fixed product of fractional lattices in $(y_1, \dots, y_l, z)$. The action of a in the mixed model substitutes $a_i y_i$ for y_i, fixes z, and multiplies by determinant characters and the square root of the position Jacobian. It is the linear action on the isotropic position half and the identity on the remaining Witt summand. Its absolute multiplier is bounded by a fixed power of $1 + a_1$. Every occurring y_1 is nonzero, so its Euclidean norm over all embeddings is bounded below on its lattice. It follows that $1 + \|(a_1 y_1, \dots, a_l y_l, z)\|$ dominates both $1 + a_1$ and $1 + \|(y_1, \dots, y_l, z)\|$ up to fixed constants. Use an arbitrarily high power of Schwartz decay for the first bound and a power exceeding the lattice dimension for the second. Summing the latter lattice majorant gives $O_N((1 + a_1)^{-N})$ for every N, after absorbing the Jacobian. The height on these Siegel sets grows at most polynomially in $1 + a_1$. The same argument applies after differentiation and for target points in a compact set. This proves the lemma. □

At a marked place, for a locally constant function f of compact support on the p-by-r matrices, define the following functions in the open square-block charts:

$$\Psi_f\begin{pmatrix}A\\B\end{pmatrix}=\mathbf{1}_{A\in K_r}\chi_v(\det A)f(BA^{-1}),\qquad \Phi_f(y,z)=\mathbf{1}_{z\in K_p}\chi_v(\det z)f(-z^{-1}y).$$

A value outside the indicated condition is defined to be zero. Put $f^0=\mathbf{1}_{M_{p,r}(O)}$ and use the associated pair as the standard test at this mark. At all marks together the actual tests are obtained by taking f equal to F_Λ on the integral box and zero off it. Indeed F_Λ is invariant under right multiplication by units, including -1. At the new marks its factor is exactly f^0.

Let $V_{\Phi,P}(y)$ be the integral on $[U(I_p)]$ for the large pair at $y\in U(W)(\mathbb{A})$ and $V'_{\Psi,Q}(y)$ the integral on $[G]$ for the small pair at the same y, with the characters of Section 79.1. Write V^0_P, V'^0_Q for the integrals with the standard tests at every mark, retaining the same unmarked and real data. All have their ordinary meaning: the large group of integration is anisotropic, and the small test is supported on injections at each mark so Lemma 79.7 applies.

Theorem 79.8 (Common finite translates for both global integrals)**.** *For the prescribed F_Λ there are points g_C of $U(W)(\mathbb{A})$, supported at the old split places, and explicit numbers b_C, the same for the two sides, such that at every y*

$$V_{\Phi,P}(y)=\sum_C b_C V^0_P(yg_C),\qquad V'_{\Psi,Q}(y)=\sum_C b_C V'^0_Q(yg_C).$$

Here C runs over $M_{p,r}(R)$ at the old modulus and, with $N_R=|\,\mathrm{GL}_r(R)|$,

$$\sum_C |b_C|^2=N_R\,|\{\gamma\in\mathrm{GL}_r(R):\gamma\Lambda=\Lambda\}|.$$

The translates are independent of the new marks.

Proof. Work first at one mark. A ball in the graph coordinate has the form $B_0+\varpi^L M_{p,r}(O)$, with $L\ge 0$. Put

$$g=\begin{pmatrix}1_r & 0\\ B_0 & \varpi^L 1_p\end{pmatrix}\in U(W)(F_v),\qquad h=\varpi^L 1_p\in U(I_p)(F_v),$$

using the chosen split coordinates. Use $a(g),b(g)$ and $\omega=\omega_0$ of Theorem 79.3. On the small kernel the action on the standard test is $a(g)\Psi_{f^0}(g^{-1}Y)$. Its upper block is still A, and the lower block becomes $\varpi^{-L}(B-B_0A)$. Thus this vector is $a(g)\Psi_f$ for the indicator f of the ball. On the large kernel apply g at W and h at I_p. The matrix argument of the standard test is

$$h^{-1}(y_1,z)g=(\varpi^{-L}(y_1+zB_0),z).$$

The condition is $z\in K_p$ with $-z^{-1}y_1\in B_0+\varpi^L M_{p,r}(O)$, and the unit character is still $\chi_v(\det z)$. The two multipliers are $b(g)$ and $\omega(\det h)^{-1}$. The latter is the target determinant character α_v, the conjugate splitting phase for n columns and the inverse square-root Jacobian. Since $\det g=\det h$ and $b(g)/a(g)=\omega(\det g)$, their product is exactly $a(g)$. This proves the two equalities of local Schwartz vectors with the same scalar.

Use $L=2$ and any lift of the residue center C_v at each old place; at new places the standard box uses the identity. Extend the split matrices to adelic group elements, with the dual components forced by the unitary condition and identity elsewhere. Write g_C for the product of the g's. The old boxes partition the integral matrices, with F_Λ constant on each box. By linearity the coefficients are

$$b_C=F_\Lambda(C)\prod_{v\ \text{old}}a(g_{C,v})^{-1}.$$

In the larger integral the target translation by the product of the h's disappears by right invariance of Haar measure on $[U(I_p)]$. The smaller integral only has the translation by g_C in the external variable. Absolute convergence and finiteness of the sums justify the result on both quotients.

For the norm, $|a(g)| = |\det g|^{-r/2}$, so that the squared modulus of the product of $a(g)^{-1}$ is $|R|^{-pr}$. The primitive additive character of R gives a perfect matrix trace pairing: if one matrix entry is nonzero, multiplying that entry by a suitable element of R makes its additive character nontrivial, by primitivity. Using trace cyclicity, the Fourier frequencies in $F_\Lambda(C)$ are $\gamma\Lambda$, with their actual multiplicities. Orthogonality on the $|R|^{pr}$ matrices C yields

$$\sum_{C} |F_\Lambda(C)|^2 = |R|^{pr} \#\{(\gamma,\gamma') : \gamma\Lambda = \gamma'\Lambda\} = |R|^{pr} N_R \, |\operatorname{Stab}(\Lambda)|.$$

Multiplication by $|R|^{-pr}$ proves the formula. □

In particular an identity between the two standard global integrals at all y, with one scalar at the fixed twist, would pass to every prescribed residue choice with precisely that scalar. Restriction to $y_i(h', g)$ and summation against $C_c(h')/d_{h'}$ then give the two functions of Section 79.1.

79.5 The common real compact type in the third comparison

At a real embedding put $b = 2m$, $N = p + b$, $T = \mu_\tau$ and $L = T + n - 2m$. The positive block of W has dimension N and its negative block has dimension m. For arbitrary b-by-N complex matrices $\eta_1, \dots, \eta_T$, write the two real polynomial vectors of Section 77.14 in the form

$$P_\eta(Z) = \prod_{l=1}^{T} \det \begin{pmatrix} \eta_l \\ Z \end{pmatrix}, \qquad Q_\eta(Y, D) = (\det D)^L \prod_{l=1}^{T} \det(\eta_l Y).$$

Here Z is p-by-N, Y is N-by-b, and D is the m-by-m negative block on the smaller side. The orders of the source coordinates in the determinants are fixed as before. The constant polynomial has its usual Gaussian in either Fock space. For the norm computation take the unit Bargmann normalization in which a monomial of multi-index a has squared norm $\prod_j a_j!$.

Theorem 79.9 (The real identification and its norm on the polynomial type)**.** *The rule $P_\eta \mapsto Q_\eta$ extends uniquely to an isomorphism between the two spans of these vectors. With the characters on the actual two theta kernels it commutes with $U(N) \times U(m)$ in $U(W)(F_\tau)$. Each span is irreducible under $U(N)$. In the displayed normalization the map multiplies every squared norm by*

$$\frac{G_m(L)\, G_b(T)}{G_p(T)}, \qquad G_h(t) = \prod_{i=1}^{h} \frac{(t + i - 1)!}{(i - 1)!}.$$

In particular the factor is independent of the row selections and of the tame twist.

Proof. For a full-rank Z, exterior multiplication of its p rows identifies their wedge, using the fixed volume form, with a decomposable b-vector in $\bigwedge^b \mathbb{C}^N$. More explicitly, the map on a b-tuple of covectors η which sends it to $\det \binom{\eta}{Z}$ is the alternating functional obtained by contracting the ordered volume vector by these p rows. Its underlying b-plane is $\ker Z$: this follows in a basis in which the last p covectors span the rows of Z, and change of basis preserves the assertion. Choose as columns of Y a basis of this kernel scaled to have exactly that exterior product. Then $\det \binom{\eta}{Z} = \det(\eta Y)$ for every η at once. Conversely, every full-rank Y gives such a full-rank Z, by choosing and scaling a basis of the annihilator. Therefore a linear combination of the products on one side is zero precisely when its counterpart on the other is zero. One can test on full-rank matrices, which are dense; the independent factor $(\det D)^L$ can be tested at $D = 1$. This proves that the map of spans is well defined and bijective, including $T = 0$.

We detail the compact actions because determinant characters enter the asserted equivariance. For $k_+ \in U(N)$ on the larger side Z is substituted by $Z(k_+^{\mathsf{t}})^{-1}$. All rows and these columns have positive Hom sign, the positive source vacuum has weight zero, and the external source multiplier ν^{-1} contributes $(\det k_+)^T$. In each stacked determinant multiply all columns by k_+^{t}. This turns the transformed lower rows back to Z and the upper rows into $\eta_l k_+^{\mathsf{t}}$, with a factor $(\det k_+)^{-1}$. The T factors cancel the external multiplier. On the smaller side the variable substitution is $Y \mapsto k_+^{\mathsf{t}} Y$. Its target vacuum contributes $(\det k_+)^{-m}$ from the negative source columns, cancelled by the target action of $\kappa = \upsilon_1^m$. Thus both actions take η_l to $\eta_l k_+^{\mathsf{t}}$.

For $k_- \in U(m)$ in the negative block of W, the larger source action has vacuum exponent p and external exponent T; the variables of P_η are untouched. On the smaller target side the $b = 2m$ positive source columns each give one inverse determinant in the vacuum, and the external exponent is m. The negative-to-negative block has positive Hom sign and its determinant contributes $(\det k_-)^L$. Since $L - m = T + p$, both final actions have character $(\det k_-)^{T+p}$. These vacuum and substitution rules are the same Fock normalization used in Proposition 78.6, with signs assigned row by row. This proves equivariance for the full compact group.

To prove irreducibility it suffices on the smaller side to ignore the fixed factor in D. A polynomial in Y of degree bT which changes by $(\det u)^T$ under $Y \mapsto Yu$, $u \in \mathrm{GL}_b(\mathbb{C})$, lies in a single irreducible $U(N)$-space. Indeed the degree bT polynomial algebra on the two index sets has the Cauchy decomposition into $S_\lambda(\mathbb{C}^N) \otimes S_\lambda(\mathbb{C}^b)$, one summand per partition of bT with at most b rows, in the index action used here. To see the multiplicity, take the two tensor powers, apply Schur–Weyl duality on each set of indices, and then take invariants under simultaneous permutations. Two symmetric-group factors pair to the trivial representation once when their partitions agree and never otherwise. The determinant character selects only the rectangle $\lambda = (T, \dots, T)$ with b entries. The span of the products is a nonzero invariant subspace: the repeated frame $\eta_l = (1_b \;\; 0)$ gives the T-th power of a nonzero minor of Y. It is therefore the whole irreducible space. The proof includes $T = 0$.

The two invariant Hermitian norms on an irreducible compact type are proportional by Schur's lemma. Evaluate their ratio at the repeated frame. There P_η is the T-th power of the determinant of the last p columns of Z, and Q_η is the indicated power of $\det D$ times the T-th power of the determinant of the first b rows of Y. For an h-by-h matrix of independent standard complex Gaussians the $2t$-th absolute determinant moment is $G_h(t)$. In fact Gram–Schmidt makes the squared determinant a product of squared projected column lengths. Conditional on the preceding columns the next length is the sum of i independent exponential variables of mean one, with $i = h, h-1, \dots, 1$. Its conditional t-th moment is $(t+i-1)!/(i-1)!$. Successive conditioning proves the product formula. This Gaussian norm is exactly the stated monomial norm, by one-variable integration. Independence of the two square blocks in Q_η now gives the required ratio. □

At every real place choose the repeated standard frame in the fixed ordered positive coordinates of W, and take the tensor product over the places. Denote the resulting vectors by P_*, Q_*. Use the standard residue tests of Section 79.4 at all old and new marks. The pair of adelic functions so obtained is $V^0_{P_*}$ and $V'^0_{Q_*}$ on $[U(W)]$.

Corollary 79.10 (A single pair of global functions for all choices)**.** *Fix a field, a lift index and a twist. If a scalar $\delta_3(c)$ satisfies*

$$\delta_3(c) V^0_{P_*}(y) = V'^0_{Q_*}(y) \quad \text{for every } y \in U(W)(\mathbb{A}),$$

then for every prescribed i and g

$$\delta_3(c) L_{i,\vartheta}(g) = \mathcal{S}_{i,c}(g), \qquad \frac{\gamma_{12}(c)\delta_3(c)}{N_R} L_{i,\vartheta}(g) = \mathcal{B}_i(g).$$

Whenever the given node has a nonzero good form as in Theorem 77.25, the smaller standard function $V'^0_{Q_*}$ is nonzero. In that case a scalar fulfilling the first equality of functions on $[U(W)]$ is necessarily nonzero and unique.

Proof. The compact orbit of the repeated frame spans the irreducible space in Theorem 79.9 at each place. Since the map takes P_* to Q_* equivariantly, the same finite linear combination of compact translates takes the two vectors to any prescribed pair P_η, Q_η. Tensor the finite combinations over the real places. Translation of a real test on the shared $U(W)$ is right translation in y for both global functions. All integrals exist by Lemma 79.7 and compactness on the larger side. Thus the standard identity passes to the actual polynomials. Apply Theorem 79.8 at the finite places. Restrict to $y_i(h', g)$ and sum with $C_c(h')/d_{h'}$. The two resulting sides are precisely $L_{i,\vartheta}$ and $\mathcal{S}_{i,c}$ by their definitions. The second identity follows from Theorem 79.1.

If the smaller standard function vanished identically, its real and finite translates would all vanish. The span and translation formulas would force $\mathcal{S}_{i,c} = 0$ for every i, contradicting

Corollary 79.2 in the presence of a nonzero good form. This proves the nonvanishing assertion and the uniqueness statement. □

The remaining scalar problem here concerns the actual pair $V^{0}_{P_*}, V'^{0}_{Q_*}$ on the rational quotient. The compact norm ratio and the relative split factor in Theorem 79.6 have been computed on their specified local data.

79.6 The norm of the larger standard integral

On the common adelic quotient $[U(W)]$ of the third exchange write $F_c = V^0_{P_*}$ for the standard function of Corollary 79.10. Keep $r = 3m$, $n = p+r$, $e = (p-r)/2$ and $\lambda_c(a) = (\alpha^{-1}\upsilon_1^{n/2})(a/\bar{a})$. The determinant block in the marked matrix test has size p on this side. Measures in this subsection are fixed across all twists on the field at this lift index.

Theorem 79.11 (Global norm of the larger standard theta integral)**.** *The function F_c is square integrable and nonzero on $[U(W)]$. Up to a constant $B_p > 0$ independent of c, its squared norm is*

$$\|F_c\|^2 = B_p \frac{\prod_{j=1}^{p} L_{E,f}(j-e, \lambda_c)}{\prod_{j=1}^{p} L_{F,f}(r+j, \eta^j)} \prod_{v \in \mathcal{S}_c} \frac{P_p(q_v)}{H_p(q_v, r/2)}.$$

In particular this formula concerns the actual theta integral over the definite group with the standard marked tests.

Proof. Double the factor $U(I_p)$ in the product of the kernel and its conjugate, and integrate on $[U(W)]$. Using the second member of a unitary pair as the doubled member gives the same doubled theta construction: take the product of the rational polarizations for the pair and its conjugate. Their diagonal graph evaluation is the Hermitian pairing on the test, now for the I_p-action. The determinant correction on the opposite copy makes the two actions contragredient there, as in Theorem 78.4. On the first copy the external multiplier at the argument h is the actual target multiplier $\alpha(\det h)$. The two multipliers in the integrated variable cancel. Equivalently one can transpose the two members with the adjoint polarization and read the graph pairing back in the original coordinates.

Here the Siegel rank is p; the space integrated in the doubled theta integral has dimension n and Witt index $w \le m$ by its real signature. It satisfies $n - w \ge p + 2m > p$. The ordinary convergent theta integral therefore gives the holomorphic value of the Siegel Eisenstein series at $s_0 = (n-p)/2 = r/2$, with factor one in the measures of Yamana, Theorem 2. In particular the squared kernel, for a fixed point of $U(I_p)$, is integrable on $[U(W)]$. The integral of its square is bounded over a compact set of representatives of the definite quotient, by continuity of the doubled theta integral. Cauchy–Schwarz on $[U(I_p)]$, followed by integration in W, proves $F_c \in L^2$. The same bound permits Fubini in its norm on the three quotients.

Pair the restricted doubled Eisenstein family with the constant on the two definite quotients. By compactness evaluation at s_0 commutes with the pairing. The resulting value is a positive common measure factor times $\|F_c\|^2$. In the initial half-plane of absolute convergence every rational maximal isotropic subspace of the double of I_p is a graph, by definiteness. Thus the ordinary graph unfolding gives the product of the flat graph integrals for the constant representation of $U(I_p)$, up to the fixed diagonal volume.

At a marked place the row test is supported on $(y, z) \in M_{p,r}(O_v) \times K_p$ with value $\chi_v(\det z)$. A translate by h^{-1} in the row basis has no overlap unless $h \in K_p$, in which case the external character cancels the unit factor from the substitution. Its coefficient and hence its integral are $P_p(q_v)$, the additive measure of this support, for every section parameter. At an unmarked split place a target translate with elementary divisor valuations b_i gives the absolute lattice overlap $q^{-n\sum_i |b_i|/2}$, one copy for each of the n columns. The phase on the positive uniformizer is $\alpha_v(\varpi)\xi^{-n} = u_v^{-1}$. This follows by reversing the source and target lattice substitutions as in Theorem 79.3. Moving from the theta point s_0 to s inserts $q^{-(s-s_0)\sum_i |b_i|}$ by the split graph height in Section 78.1. As $n/2 - s_0 = p/2$, the split integral is precisely that of Theorem 78.1

in dimension p with phase u_v^{-1}. Its two inverse phases give both components of the Hecke character λ_c, independent of the ordering.

At an inert finite place all spaces and the section are unramified with hyperspecial data. The diagonal graph has the unimodular polarization of Lemma 78.3. The even-rank space character is unramified and trivial on the common uniformizer, since its restriction to $F_v^\times$ is $\eta_v^n = 1$. The unramified doubling formula in that lemma, in Siegel rank p, has standard numerator for the constant representation at $s + 1/2$ and denominator $\prod_{j=1}^{p} L_{F_v}(2s + j, \eta_v^j)$; both p, n are even. Its spherical section has value one at the identity, the unmarked vacuum norm. Thus the numerator arguments at this place are $s + j - p/2$ over E_w. The same arguments occur at the split places by the preceding computation.

At infinity $U(I_p)$ is compact, with graph height one. The polynomial P_* gains $(\det h)^T$ from its T full row determinants. Its target vacuum and multiplier have exponents $-m$ and $m - T$, respectively. They cancel the polynomial exponent. Hence the real coefficient is the positive squared norm of the real vector, constant on the integrated graph and independent of s, c. The finite Euler product in the unfolding region thus has numerator $\prod_{j=1}^{p} L_{E,f}(s + j - p/2, \lambda_c)$, the denominator above, and the correction $P_p(q_v)/H_p(q_v, s)$ at each mark. Equality of meromorphic continuations permits evaluation at s_0; individual local comparison factors at interior points are not evaluated as divergent integrals.

At this point the denominator arguments $r + j$ all exceed one. The numerator arguments are the integers $1 - e, \dots, r + e$. For $1 - e, \dots, e$ the gamma factors and the nonvanishing of these same Hecke values were verified in Theorem 79.6. At the remaining positive integers the Euler product is absolutely convergent and nonzero. Each marked correction is positive and finite. All changes of global measure and all real coefficients are positive and independent of the twist. The norm identity and the nonvanishing claim follow. □

79.7 The norm on the smaller side and the ratio of global norms

Write $F_c' = V_{Q_*}'^0$ on $[U(W)]$, with source $G = U(S_0)$ of rank r. Its integral in the source variable is absolutely convergent by Lemma 79.7. To compute its square norm we shall initially cut off the integration on $[G]$.

Lemma 79.12 (Exclusion of the boundary orbits in the smaller norm)**.** *In the doubled norm construction with source S_0, the section determined by the standard marked test vanishes on every non-graph rational orbit of the two source groups. Its theta-point coefficient on the graph is integrable in absolute value on $G(\mathbb{A})$.*

Proof. At a chosen marked split place use the two position coordinates $X_1, X_2 \in M_{n,r}(F_v)$ for the test and its conjugate. Their top r-by-r blocks are units, so both matrices are injective on their respective source spaces. In the split position model the double acts by invertible substitutions on the $2r$ columns, with the usual scalar determinant factor. The diagonal evaluation functional integrates the test on $X_1 = X_2$, as in Section 78.1. In intrinsic terms the variable $(X_1, -X_2)$, considered as a map on the doubled space, vanishes on the diagonal r-plane. Thus the section at a general doubled element integrates over maps vanishing on the transported r-plane. One sees this directly by applying the column substitution before diagonal evaluation; a change of Haar on the linear fiber only multiplies by a scalar. At a rational non-graph orbit this plane has nonzero intersection with a source summand, as proved in Theorem 78.12. This intersection remains nonzero on the chosen split component. Vanishing on it is impossible for the corresponding X_i. Translation by the two source groups preserves that obstruction. The flat continuation multiplies the fiber value by a height, so the same vanishing persists.

At a mark the graph overlap is compactly supported on K_r, by the unit top block. At unmarked finite places the vacuum overlap bound and the self-dual lattice count of Lemma 77.5 apply with source rank r and target dimension $n = p + r \geq 3r$ at our nodes. Indeed increasing the target dimension only increases the exponent of the intersection index in the denominator. Thus the absolute integral outside the full compact subgroup at an unmarked v is $O(q_v^{-r/2})$. The compact contribution is one, and the sum of the errors over the fixed number field is finite.

At a real place retain $b = 2m,\ T = \mu_\tau,\ L = T + n - 2m$ from Theorem 79.9. The maximal compact group of the source fixes Q_* including the vacuum and splitting factors. In fact on the positive source block the exponents from vacuum, external character and determinant polynomial are $m, T-m, -T$, and on the negative block they are $p+2m, T-m, -L$. Both sums are zero. Every monomial of the polynomial involves only letters of positive Hom sign. For each paired source-column boost the lowering operator therefore kills the vector and the nonconstant creation terms pair to zero against it, as in Lemma 142.12. The vacuum contributes one inverse hyperbolic cosine per target row. In addition every positive source column occurs to degree T in the positive determinant powers and every negative source column to degree L. The normalized coefficient on a Cartan element with parameters $t_i \geq 0$ is consequently

$$\prod_{i=1}^{m} (\cosh t_i)^{-(n+T+L)}.$$

The compact invariance just proved makes the coefficient positive on the entire real group. The Cartan density is bounded by a constant times $\exp(2(r-1)\sum_i t_i)$. Since $n > 2r$ on the prescribed fields, the real integral is finite and strictly positive. Together the local estimates prove the absolute integrability on $G(\mathbb{A})$. □

Theorem 79.13 (Global norm of the smaller standard theta integral)**.** *The function F_c' is square integrable and nonzero. There is a constant $B_r > 0$ independent of c on the given field*

and lift index such that

$$\|F_c'\|^2 = B_r \frac{\prod_{j=1}^{r} L_{E,f}(e+j,\lambda_c)}{\prod_{j=1}^{r} L_{F,f}(p+j,\eta^j)} \prod_{v\in\mathcal{S}_c} \frac{P_r(q_v)}{H_r(q_v,p/2)}.$$

Proof. Let u be smooth and compactly supported on $[G]$. Insert it in the integral defining F_c' and denote the result by $T_c u$. The doubled Siegel rank is now r. The space integrated in the Siegel–Weil formula has dimension n and Witt index at most m, so $n-m>r$. The ordinary convergent formula in Yamana, Theorem 2 applies at $s_1=(n-r)/2=p/2$. In fact $n>2r$, so the Eisenstein series is in the absolute convergence range already at the theta point. As in Theorem 79.11, the integrated square of the fixed-point kernel is finite and locally bounded in its source point by the convergent formula. Cauchy–Schwarz and the compact support of u justify Fubini for the norm of $T_c u$.

Unfold the doubled series against u and its conjugate at this point. On the compact support absolute locally uniform convergence permits termwise integration. Lemma 79.12 kills every non-graph term before integrating in a unipotent variable. On the graph the integrated diagonal variable gives the ordinary matrix coefficient of u, up to complex conjugation according to the ordering of the copies. In particular

$$\|T_c u\|^2 \le C\,\|u\|^2 \int_{G(\mathbb{A})} |K_c(a)|\,da,$$

where K_c is the actual theta-point graph coefficient and $C>0$ is a Haar constant independent of c. Here we used unitarity of right translation on the rational quotient. This proves boundedness on the dense compactly supported space, since the last integral is finite.

Choose smooth compact cutoffs $0\le u_j\le 1$ tending to one pointwise and in $L^2([G])$. Such cutoffs come from an exhaustion of this finite-volume quotient, or from compact exhaustions at the fixed finite level. Their images converge in L^2 by the bound. At each $y\in U(W)(\mathbb{A})$ they also converge to $F_c'(y)$ by the absolute convergence in Lemma 79.7. An almost everywhere convergent subsequence of the L^2 sequence identifies its limit with F_c'. Moreover the source matrix coefficients of u_j converge uniformly to the coefficient of the constant, namely $\mathrm{vol}([G])$. This follows by Cauchy–Schwarz and L^2 convergence on both sides of each unitary translation. Integrating against the L^1 graph coefficient therefore gives the ordinary product of local graph integrals for the squared norm of F_c', up to positive fixed Haar factors. The cutoff argument accounts for the noncompact source in this calculation.

At a mark the test is supported on $K_r\times M_{p,r}(O_v)$ in its top and bottom blocks. Two translates in the source have disjoint support unless the quotient belongs to K_r. There the test is fixed under the combined action, with norm $P_r(q_v)$. At an unmarked split place the source lattice coefficient has phase $u_v=\alpha_v(\varpi)^{-1}\xi^n$ at determinant valuation one, and modulus $q^{-n\ell(a)/2}$ in elementary divisor notation. Apply Theorem 78.1 in dimension r, with parameter $p/2$ since $n=r+p$. Its two Euler components are those of λ_c, and the arguments are $p/2+j-r/2=e+j$. At inert places the spherical unitary doubling computation in Lemma 78.3 applies in rank r for the even space character of rank n, trivial on the common uniformizer. With the constant input and section of value one its numerator has the same $e+j$ arguments, and its denominator has arguments $p+j$ and characters η^j. At the marks the correction displayed in the theorem gives exactly $P_r(q_v)$ after cancellation of the local reciprocal denominator. The real factors were proved positive, finite and independent of the twist in Lemma 79.12. This gives the formula. Its finite Hecke functions are taken at real arguments exceeding one, hence are nonzero. The norm is therefore nonzero. □

Corollary 79.14 (The third multiplier as a ratio of actual global norms)**.** *For a positive K_3 independent of c at the given node and lift index one has*

$$\frac{\|F_c'\|^2}{\|F_c\|^2} = \frac{K_3}{\mathcal{D}_e(\sigma_c)}.$$

In particular if the two functions are proportional, the squared modulus of their unique proportionality factor in Corollary 79.10 is the displayed value.

Proof. Divide the large norm by the small one. The arguments $e+1,\dots,e+r$ occur in both numerators and cancel. The remaining numerator values are exactly $1-e,\dots,e$. The denominators of the two global products are independent of c. At a marked place the relative correction is one since

$$P_p(q)H_r(q,p/2) = \prod_{i=1}^{p+r}(1-q^{-i}) = P_r(q)H_p(q,r/2).$$

Upon division by the corresponding ratio for a reference twist, the paired integer values give $\mathcal{D}_e(\sigma_c)/\mathcal{D}_e(\sigma_{c_*})$ by Theorem 79.6. Inverting and fixing the positive normalization at c_* gives K_3. □

Thus the test still needed on the common rational quotient is whether the two nonzero standard functions span the same line. Equivalently their mixed inner product must attain the Cauchy–Schwarz bound for the two norms just evaluated.

79.8 Holomorphy of the two standard functions on the common group

Use the compact type in Theorem 79.9 and the two nonzero square-integrable functions F_c, F_c' of Section 79.7. Fix a real embedding. Write i for a positive coordinate of W, β for a negative one, a for a positive coordinate of S_0, and d for a negative one. The holomorphic equations in the positive-first order use the root operators on $U(W)$ of compact torus weight $\epsilon_i - \epsilon_\beta$; equivalently these are the contractions on the oscillator with W as source and a positive definite target.

Theorem 79.15 (Holomorphy after integration on the indefinite source)**.** *Both standard functions are killed by all these operators at every real place. The assertion holds on their entire matched compact types, including the arbitrary row choices of Theorem 79.9.*

Proof. On the larger side a noncompact lowering operator is a sum of products of derivatives in a positive and a negative source column, one term per target row. The polynomial P_η is independent of every negative column. This proves the equation before integration.

On the smaller side retain the positive-sign Fock blocks $Y_{ia}, D_{\beta d}$ and denote the opposite-sign blocks by $Z_{id}, V_{\beta a}$. Under the compact torus of the target W, the four blocks Y, D, Z, V have weights $\epsilon_i, \epsilon_\beta, -\epsilon_i, -\epsilon_\beta$, respectively. Under the source torus the corresponding weights are $-\epsilon_a, -\epsilon_d, \epsilon_d, \epsilon_a$. Here the letter is dualized on the opposite-sign Hom coordinate. These rules follow by diagonal substitution on both indices; the vacuum weights are independent of the monomial.

We record the two infinitesimal actions used below. On a polynomial $H(Y, D)$ involving only these two blocks, the root operator of weight $\epsilon_i - \epsilon_\beta$ on W acts by a fixed nonzero scalar times

$$\sum_a Y_{ia} V_{\beta a} H.$$

On such a polynomial the root operator of weight $\epsilon_a - \epsilon_d$ on S_0 acts by a nonzero scalar times

$$\sum_{\beta'} V_{\beta' a} D_{\beta' d} H.$$

To verify the formulas, a matrix unit of either unitary Lie algebra switches the two indicated coordinates, and acts separately on each fixed index of the other space. The two modes in each switch have opposite Hom sign. The two off-diagonal complexified symplectic operators on these modes are multiplication by their product and differentiation by their mixed second derivative. This follows directly from the Heisenberg commutators $[\partial_x, x] = [\partial_y, y] = 1$: multiplication by xy sends each annihilation operator in its commutator to the other creation operator up to sign, and $\partial_x \partial_y$ performs the opposite switch. Their compact weights distinguish the two root directions. There is no scalar term for a nonzero torus weight. For the target root the other terms are derivatives in Z_{id} and $D_{\beta d}$, and for the source root they are derivatives in Y_{ia} and Z_{id}. They kill H. The creation coefficients are constant on each indicated sum by permutation of like-signed coordinates, and nonzero because the matrix unit acts nontrivially on each pair. Changes of root-vector normalizations can be absorbed in the scalar coefficients. The determinant splittings have zero differential on these off-diagonal operators.

Write $P(Y) = \prod_l \det(\eta_l Y)$ and $Q = (\det D)^L P(Y)$, with $L = T + n - 2m \geq 1$. For the fixed i, β, use the polynomials

$$H_{ad} = Y_{ia} \, (\operatorname{adj} D)_{d\beta} (\det D)^{L-1} P(Y).$$

They depend only on Y, D. The identity $D \operatorname{adj} D = (\det D) 1_m$ gives exactly

$$\sum_{a,d} \sum_{\beta'} V_{\beta' a} D_{\beta' d} H_{ad} = \sum_a Y_{ia} V_{\beta a} Q.$$

Consequently the unwanted target derivative of Q is a linear combination of source derivatives of the H_{ad}. This identity holds in the polynomial algebra even on the singular locus of D.

Pass to rational theta sums. A source derivative on the real test gives the corresponding right derivative in the variable integrated on $[G]$. Its integral is zero: right translation preserves quotient Haar measure, and the difference quotient can be integrated by the uniform rapid decay of Lemma 79.7 for compact families of source translations and Schwartz derivatives. The same lemma justifies taking the target derivative under the integral at every target point. Thus all the operators annihilate the smaller function as asserted. The argument works for each η and for their linear combinations. □

Corollary 79.16 (Real matrix coefficients of the two global functions)**.** *There is one function M_∞ on $U(W)(F_\infty)$, normalized by $M_\infty(1) = 1$, such that for any $f, h \in \{F_c, F_c'\}$ at the given twist,*

$$\langle R(a)f, h\rangle = M_\infty(a)\langle f, h\rangle \qquad (a \in U(W)(F_\infty)).$$

The normalization on the common repeated frame is as in Corollary 79.10.

Proof. The evaluations of the real polynomial vectors define compact equivariant maps from one irreducible compact type into L^2, using Theorem 79.9 to identify the two domains. Both maps are nonzero by Theorems 79.11 and 79.13. On this finite-dimensional type every pairing obtained from any two of these maps is a scalar multiple of one invariant Hermitian form, by Schur's lemma. This also allows zero for the mixed scalar.

For passage to the real group, the polynomial vectors used in the pairings are smooth for the Hilbert-space real action. On the larger side any derived real polynomial has square integrable image by the convergent Siegel–Weil argument with the compact source in Theorem 79.11. On the smaller side the estimate used for the cutoff in Theorem 79.13 applies also after a fixed archimedean differential operator. Such an operator gives a polynomial times the Gaussian. For any fixed finite degree the absolute Cartan coefficient on the integrated source is $O(\prod_i (\cosh t_i)^{-n})$. Indeed the lowering, diagonal and raising factorization of Lemma 142.12 first reduces degree a finite number of times, multiplies each monomial by the vacuum factor and powers of $\operatorname{sech} t_i$, and only finitely many subsequent raising terms can pair with the fixed degree on the other side. The remaining coefficients involve bounded powers of $\tanh t_i$, also uniformly under compact rotations. This bound is integrable since $n > 2r$. The marked rank support is unchanged by these real operators. Thus all distributional derivatives are in L^2, giving smooth vectors by the usual Sobolev criterion for a unitary Lie group action.

Order complexified real enveloping-algebra monomials with holomorphic raising operators on the left, compact operators in the middle and lowering operators on the right. As in Lemma 78.11, a lowering factor kills the vector and a left raising factor kills the pairing by unitary adjunction, using Theorem 79.15 for both entries. All derivatives at the identity reduce to compact derivatives, already determined up to the scalar of the invariant pairing. The matrix coefficients are real analytic. To check this here, take a compact-invariant orthonormal basis of the real tangent space in the sense of an invariant inner product, with the noncompact directions in pairs X, JX. On a holomorphic vector their sum of squares reduces to a sum of commutators in the compact algebra as in Lemma 78.11. This sum is compact-invariant and acts on the irreducible compact type by a scalar, as does the compact sum of squares. The coefficient therefore satisfies an analytic elliptic eigenfunction equation. The same argument applies to the product over real places. Equality of derivatives gives equality on the connected real group. Take M_∞ from the normalized self-pairing of F_c to conclude. □

The mixed scalar in this corollary is the global inner product at $a = 1$. Identifying the real coefficients fixes their dependence on real translations; the absolute value of that mixed scalar is still needed for proportionality.

79.9 The local representations and the standard adelic vector

Continue with the two functions F_c, F_c' on $[U(W)]$. At a split place, define two unitary characters of $F_v^\times$ by

$$\beta_B(z) = \nu_v(z)^{-1}\xi^{pv(z)}, \qquad \beta_S(z) = \kappa_v(z)\xi^{-rv(z)}.$$

Thus β_B/β_S is the unitary part u_0 of ω_0 in Section 79.2. We use the frame and the ordering of its two blocks fixed there.

Lemma 79.17 (The marked principal series and its specified vector)**.** *At a mark consider normalized induction to* $\mathrm{GL}_n(F_v)$ *from the upper block parabolic of type* (r,p) *of* $(\beta_B \circ \det_r) \boxtimes (\beta_S \circ \det_p)$*. It is irreducible. On either standard global integral, varying the local full-rank test induces a map from this same representation, with the same nonzero local vector for the standard pair.*

Proof. Use the model with translations on the left and parabolic covariance on the right. For $a_0(t) = \beta_S(\det t)|\det t|^{-r/2}$, the two functions of the frame are

$$a_0(t)^{-1}J_r(\Psi;t,0), \qquad a_0(t)^{-1}\omega_0(\det t)^{-1}J_p(\Phi;t,0).$$

Their left action is pure translation by Theorem 79.3. On right multiplication by a parabolic element with Levi (A,D), their common covariance is

$$\delta_P(A,D)^{-1/2}\beta_B(\det A)^{-1}\beta_S(\det D)^{-1}, \qquad \delta_P(A,D) = |\det A|^p|\det D|^{-r}.$$

Indeed the column integral transforms by $\omega_0(\det A)^{-1}$ and one must also divide by the value of a_0 on the parabolic. The row integral gives the same result by the row change of variables. On the standard pair the two sections coincide, with value one at the identity.

Both frame maps are onto the smooth induced space. On a Grassmannian coordinate chart trivialize the bases using an identity minor. A local section of compact open support is lifted by taking its scalar function of the plane times $\mathbf{1}_{K_b}(h)\omega_0(\det h)^{-1}$ in the basis coordinate h, where $b = p$ or r, incorporating the fixed displayed scalar for the chosen frame. Its fiber integral has the prescribed value; compactly many charts and a disjoint finite refinement suffice. By Proposition 79.4 the kernel of each map consists of sums of the weighted changes of basis minus the original vector. Each such difference pairs to zero in the global integral by right invariance of the Haar measure of the integrated group. The bigger integral is over a compact quotient; the smaller integral with these full-rank tests converges by Lemma 79.7. This proves factorization and equivariance of the global maps.

For irreducibility one may apply the segment criterion of Zelevinsky, *Induced representations of reductive p-adic groups II*, Theorem 4.2. In his segment subrepresentation convention each determinant character in rank b is the segment subrepresentation of the increasing sequence from $\beta|\cdot|^{-(b-1)/2}$ to $\beta|\cdot|^{(b-1)/2}$. The constant vector embeds in that normalized Borel induction after the determinant twist. The two segments here are on disjoint cuspidal lines since β_B/β_S is nontrivial on the residue units. They cannot be linked. The criterion gives irreducibility of their product. In particular, since the common standard vector has nonzero global image on both sides, each image of this local induced module is a copy of it. □

Theorem 79.18 (A common adelic representation with fixed local vectors)**.** *For a fixed twist every nonzero irreducible projection of* F_c *or* F_c' *in* $L^2([U(W)])$ *has the same abstract adelic representation* π*. In each such copy the projected vector lies on the same abstract tensor line. At the marks use the common vector in Lemma 79.17. At an unmarked finite place the vector is spherical. Its standard multiset is*

$$\beta_B(\varpi)A_r(q_v) \sqcup \beta_S(\varpi)A_p(q_v) \text{ at a split place}, \qquad A_r(q_w) \sqcup A_p(q_w) \text{ at an inert place}.$$

At the real places take the common holomorphic representation and repeated-frame vector determined in Theorem 79.15.

Proof. First the two vectors lie in the discrete spectrum. They are square integrable by the preceding norm formulas. Their compact types are finite dimensional and annihilated by the holomorphic lowering operators. They have a fixed infinitesimal character: the repeated frame spans an extremal line for the compact roots in the rectangular representation of Theorem 79.9. Choose the compact positive roots for which this is a highest line and adjoin the roots of the annihilating noncompact subspace. These form a positive system, obtained by putting all positive coordinates before all negative ones. The vector is a highest-weight vector for that system, so the center acts by the infinitesimal character of its highest weight. In the Langlands spectral decomposition the continuous part is a sum of unitary inductions integrated over imaginary split Levi parameters for proper parabolics. On each such parameter space the real Casimir eigenvalue is a nonconstant quadratic polynomial in the continuous parameter, by the infinitesimal-character formula for normalized induction. The norm-square term is definite on the imaginary split parameters. The locus of any specified eigenvalue has Lebesgue measure zero. The discrete data in the decomposition are countable. Thus no vector of this fixed infinitesimal character has a continuous part. This applies to the present unitary group with anisotropic center.

At infinity each nonzero irreducible projection with the specified vector has a fixed highest weight. The simple highest-weight module is the unique irreducible quotient of its Verma module, so the real constituent is determined. The highest-weight space in the simple module is a line. The repeated frame supplies that vector in both forms; equivalently the normalized real coefficients were identified in Corollary 79.16. The unitary globalization is unique since inner products on the cyclic module are determined by the norm of its highest line and unitary adjunction. This argument works on each real group separately.

At a marked place project the equivariant maps of Lemma 79.17 onto any nonzero automorphic summand. In the tensor decomposition of that summand, contraction against any smooth vector on the other places gives a local intertwiner, nonzero for at least one contraction. Irreducibility and Schur’s lemma show that the summand has the indicated local representation and that all such contractions of the standard vector lie on the same specified line. This also proves the tensor-line assertion there.

At an unmarked split place the smaller rectangular lift is from the integrated constant in rank r. The local Grassmannian computation of Lemma 78.7 works before integration and with arbitrary ranks. To recall its phases here, the positive elementary target coset of size h has phase $\beta_S(\varpi)^h$ and absolute multiplier $q^{rh/2}$; the source coset with l negative elementary divisors has phase $u_0(\varpi)^{-l}$ and absolute multiplier $q^{nl/2}$. The lattice count for residue rank a expands $\begin{bmatrix} n-a \\ h \end{bmatrix}_q$ using $\begin{bmatrix} r-a \\ l \end{bmatrix}_q$ with coefficient $\begin{bmatrix} n-r \\ h-l \end{bmatrix}_q q^{l(n-r-h+l)}$. Every source translate has the same integral against the constant. Dividing by the two averaging masses gives, by the elementary symmetric expansion in Theorem 78.10, the segments $\beta_S A_{n-r}$ and $\beta_S u_0 A_r$. Their phases are as asserted.

For the larger rectangular integral transpose the matrix and replace each group variable by its inverse transpose. The rank n target in these coordinates has phase β_B^{-1} and inverse square-root Jacobian on its p columns. The integrated rank p source has phase u_0 and direct Jacobian, since its former action had multiplier ω_0^{-1}. The same count now gives $\beta_B^{-1} A_{n-p}$ and $\beta_B^{-1} u_0 A_p$. Pullback by the inverse transpose inverts the multiset. Since the segments A_b are stable under inversion and $\beta_B/u_0 = \beta_S$, this agrees with the smaller integral. The proof is valid for arbitrary local unitary phases, even when some entries coincide.

At an inert place apply Theorem 72.16 to the spherical joint local quotient of the oscillator. We give the justification of the quotient in this application where one of the constants is on a noncompact group. With all other components fixed, vary the Schwartz test at this inert place. The integrated variable still acts trivially after integration by Haar invariance. The larger-integral map remains L^2-valued by the convergence argument of Theorem 79.11. For the smaller-integral map the other places include a split full-rank mark, so both rapid source convergence and the exclusion of boundary orbits survive. The local coefficient at the varying inert place for any fixed smooth test satisfies the same lattice overlap bound up to a test-dependent constant. Indeed in a self-dual lattice model a finite Schwartz vector is a finite linear combination of Heisenberg translates of the vacuum. An overlap of any two such translates is bounded by the same square-root intersection-index bound: the intersection of two translated lattices is empty or a coset of their intersection, and the extra phase is unitary. The proof of Lemma 79.12 therefore gives the needed local L^1 estimate and the cutoff argument of Theorem 79.13 applies. Projection and contraction in the other places now give a nonzero local map with trivial integrated member and the specified spherical output. Dual conventions do not change the reciprocal multiset here. Every inert place is odd and the spaces are unramified quasi-split with the given full stabilizers. Their dimensions n, r, p are even. Thus a lift from the constant in rank r adjoins A_p to A_r, and a lift from the constant in rank p adjoins A_r to A_p, with all uniformizer phases one.

At each unmarked place the multiset determines the spherical representation and its compact-fixed line has dimension one. A discrete irreducible automorphic representation is the restricted tensor product of its local constituents. The conditions just proved fix all these constituents. They also fix the one vector line in each factor: at a mark this comes from the common vector in the induced model, regardless of the dimension of its fixed space at a principal level. Consequently in any abstract copy all projections lie on the same tensor line. This proves the theorem, allowing at this stage several automorphic copies of the representation. □

79.10 Multiplicity one and the third theta comparison

Fix a sufficiently large node, a lift index j, and one of its prescribed twists $c = \chi_M^j \vartheta$. Write $U = U(W)$, $r = 3m$, $p = \dim W'$, and $n = p + r$. Both p and r are even and $p > r$. Theorem 79.18 identifies an irreducible adelic representation π of U and one line of vectors in its abstract tensor product. To compare functions on the automorphic quotient we must count copies of π there.

Proposition 79.19 (Automorphic multiplicity of the common representation)**.** *The representation π of Theorem 79.18 has multiplicity one in $L^2_{\mathrm{disc}}([U])$.*

Proof. Let μ be the fixed unitary Weil splitting character of $\mathbb{A}_E^\times/E^\times$ in dimension one, with $\mu|_{\mathbb{A}_F^\times} = \eta_{E/F}$. Its restriction to norm-one ideles is υ_1. Define two Hecke characters of E by

$$B(a) = \nu(a/\bar{a})^{-1}\mu(a)^p, \qquad S(a) = \kappa(a/\bar{a})\mu(a)^{-r}.$$

These characters are unitary. They are conjugate self-dual because $\mu(a\bar{a}) = \eta_{E/F}(N_{E/F}a) = 1$, and are trivial on $\mathbb{A}_F^\times$ because p, r are even. At a good split place, take the idele whose coordinates above that place are $(z, 1)$. Its quotient by its conjugate has coordinates (z, z^{-1}), so B, S on the chosen factor are the characters β_B, β_S of Theorem 79.18. At a good inert place the uniformizer can be taken in F_v, so both characters have value one on it.

For a unitary character λ, write $\lambda[b]$ for its tensor product with the b-dimensional irreducible representation of the Arthur $\mathrm{SL}_2(\mathbb{C})$. Its standard unramified parameter contributes the b twists with exponents $(b-1)/2, (b-3)/2, \dots, (1-b)/2$. The preceding computation and Theorem 79.18 identify the standard parameters of π at almost all places with those of $B[r] \boxplus S[p]$, including both places of E over a split place by conjugate duality.

Take $R = 2n$ and $M = 2R + 1$. Choose the quasi-split odd unitary group H over F of dimension M, given by R hyperbolic planes and a line, with the opposite Hermitian sign convention for the dual pair. Use a nontrivial adelic additive character and splitting characters $1, \mu$ for the spaces of dimensions n, M, respectively. Their restrictions to $\mathbb{A}_F^\times$ are $\eta_{E/F}^n$ and $\eta_{E/F}^M$, as required. Write σ for the restricted tensor product of the local small theta lifts of π_v to $H(F_v)$. Since $n < R$, Li's stable-range inequality gives

$$1 \;\le\; m_{\mathrm{disc,U}}(\pi) \;\le\; m_{\mathrm{disc,H}}(\sigma).$$

The first inequality follows also from the nonzero discrete vectors in Theorems 79.11, 79.13, and 79.18. The stable-range result applies to the discrete spectrum including its residual part. We use it in the form stated by Chen and Zou, Theorem 3.1, after J.-S. Li, Theorem A.

We determine the packet on H using its standard parameters. In the unramified theta correspondence from dimension n to dimension M, the input parameter is twisted by the inverse of the large splitting character times the small one, and a string of length $M - n$ with the small splitting character is adjoined. This is formula (4.2) of Chen and Zou. Thus the unramified data of σ are those of

$$\widetilde{\psi} = (B\mu^{-1})[r] \;\boxplus\; (S\mu^{-1})[p] \;\boxplus\; \mathbf{1}[M-n].$$

This is an elliptic global parameter for H, with ordinary base change. For the first two characters the restriction to the ideles of F is $\eta_{E/F}$. A conjugate self-dual character of this restriction has conjugate-symplectic sign. Tensoring it with an even-dimensional irreducible of SL_2, whose invariant form is alternating, gives conjugate-orthogonal sign. The trivial character and the

odd-dimensional last factor have positive sign as well. This is the required parity in odd dimension. The lengths are distinct since $M-n=3n+1>p>r$. Thus no simple summand is repeated.

The quasi-split discrete classification assigns one global parameter to the near-equivalence class of σ. To identify it, take standard base change at all good places of E. The isobaric cuspidal data for the assigned parameter agree there with the three strings above. Strong multiplicity one for isobaric representations of general linear groups identifies the multisets of cuspidal constituents with their real twists. These multisets determine the Arthur blocks. Indeed group the constituents by their unitary cuspidal part. The largest exponent in each group is $(b-1)/2$ for its longest block, with multiplicity the number of such blocks. Remove their full strings and iterate. Hence the assigned parameter is $\widetilde{\psi}$.

For this parameter we check the local multiplicities entering the global packet. At a split place the group is general linear and its packet has one member. At a nonsplit finite place the three Weil factors are unitary characters and the Deligne SL_2 acts trivially. Interchanging the two SL_2's therefore produces a tempered parameter. Its local packet has no repeated representation. The co-tempered packet is obtained by Aubert duality on those representations, with the component-group labels adjusted in the character identity. Aubert duality is an involution on irreducibles up to sign, so distinct underlying members stay distinct. This uses Atobe, Gan, Ichino, Kaletha, Mínguez and Shin, Theorem 5.4.1, with its tempered character identities on the quasi-split group. At each real place the multiplicity-one theorem for real unitary Arthur packets applies, including singular infinitesimal characters. See Mœglin and Renard, Theorem 1.3.

In the global formula each restricted tensor product of packet entries satisfying the global character condition contributes one summand. The local facts imply that the underlying representation σ can occur in at most one such tensor product, since its local representations are determined at every place. We have already identified its unique global parameter. This gives $m_{\mathrm{disc,H}}(\sigma)\le 1$, proving the assertion by Li's inequality. Here the global input is Mok's quasi-split multiplicity formula, Theorem 2.5.2. For its auxiliary intertwining and trace formula inputs we use Atobe and collaborators, Introduction and main theorems, and the twisted weighted fundamental lemma in the form proved in Halleck-Dubé's November 2026 preprint, §1.14. The auxiliary group H is used only for this upper bound. □

Theorem 79.20 (The three theta exchanges on one field and interval)**.** *There is a unique $\delta_3(c)\in\mathbb{C}^\times$ such that*

$$F_c'(y)=\delta_3(c)F_c(y)\qquad (y\in U(W)(\mathbb{A})).$$

Put $d=m/2$, $e=(p-r)/2$, $\sigma_c=\alpha^{-1}\upsilon_1^{n/2}$, and $b_{j,\vartheta}=\gamma_{12}(c)\delta_3(c)/N_R$. For every prescribed choice i and every $g\in U(W_j)(\mathbb{A})$ one has

$$b_{j,\vartheta}L_{i,\vartheta}(g)=\mathcal{B}_i(g),$$

with

$$|b_{j,\vartheta}|^2=\frac{K_{12}K_3}{N_R^2\,\mathcal{D}_d(c\upsilon_1^d)\,\mathcal{D}_m(c^{-1}\upsilon_1^m)\,\mathcal{D}_e(\sigma_c)}.$$

Here $K_{12},K_3>0$ and N_R are independent of the prescribed twist at the fixed node and lift index.

Proof. Both standard functions have nonzero norm. They are in the discrete spectrum and every nonzero irreducible projection is in a copy of π, on the specified tensor line by Theorem 79.18. The proposition shows that this copy occurs exactly once. Thus the two nonzero vectors span the same line in $L^2([U(W)])$. Their smooth representatives satisfy the equality

everywhere because a nonempty open set has positive quotient measure. The scalar is unique. Taking squared norms and using Corollary 79.14 gives $|\delta_3(c)|^2 = K_3/\mathcal{D}_e(\sigma_c)$. Corollary 79.10 extends the identity from the standard functions to every prescribed i and g. Theorem 78.16 gives the other two factors in the denominator. All fields, residue levels, interval endpoints, tests and measures used in the period identity are those fixed before the introduction of H. $\square$

80 References

[1] Arthur, *An introduction to the trace formula*, Theorem 8.1.

[2] Atobe, Gan, Ichino, Kaletha, Mínguez and Shin, Theorem 5.4.1.

[3] Banaszczyk, *New bounds in some transference theorems in the geometry of numbers*.

[4] Chen and Zou, Theorem 3.1, after J.-S. Li, Theorem A.

[5] Davidson, §2, Theorem 2.7.

[6] Gan, Qiu and Takeda, §§11.5–11.6.

[7] Halleck-Dubé's November 2026 preprint, §1.14.

[8] J.-S. Li, Theorem 3.1.

[9] Mœglin and Renard, Theorem 1.3.

[10] Mok's quasi-split multiplicity formula, Theorem 2.5.2.

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[12] V. P. Platonov, *The problem of strong approximation and the Kneser–Tits conjecture for algebraic groups*.

[13] Zelevinsky, *Induced representations of reductive p-adic groups II*, Theorem 4.2.