The generalized Riemann hypothesis for Dirichlet L-functions

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Volume XIX

Local orbital integrals and flag lattices

Abstract

We compute orbital integrals at finite places on Hermitian frames. At split auxiliary primes we compute the signed integral on each admissible core position and the elementary divisors for primitive terminal frequencies. We match the auxiliary compact row characters to the flag and determine the largest possible overlap of the original and transported auxiliary theta tests along the chain.

The guide in Volume I lists all twenty-seven volumes.

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98 Local measures on Hermitian frames

98.1 The symmetry of an orbital integral

Use the seeds of (92.15) and (92.16) at the level of Theorem 86.1, with $\beta = \varpi$. Put $F = F_+$. In unfolding the product of two periodizations, fix a rational double coset represented by $\gamma \in G(F)$. For a right finite coset $t_f K^0$ common to $B_f K^0$ and $\gamma B_f K^0$, put

$$\Gamma = B(F) \cap \gamma B(F) \gamma^{-1} \cap t_f K^0 t_f^{-1}, \qquad D = N \cap \gamma N \gamma^{-1}.$$

Intersection with a finite adelic subgroup in this notation means imposing the indicated condition on the finite component. Let W, W' be column-valued functions of types ρ, ρ' on the real group, with the transformation in (92.15). Write vw^* for the map $u \mapsto v\langle w, u\rangle$, with the inner product conjugate-linear in its first variable. The real map before evaluation on the finite column covectors is

$$J_{\gamma,t} = \int_{\Gamma \backslash G_\infty} W(g)\, W'(\gamma^{-1} g)^*\, dg. \tag{98.1}$$

The finite factors in (92.16) contract this tensor and give scalar phases and the volume of the finite coset. Indeed, after unfolding the first rational summation, the second splits into orbits of its left rational period group. The stabilizer of the ordered pair of flags is $B(F) \cap \gamma B(F) \gamma^{-1}$. On decomposing the finite variable further into right level cosets its stabilizer is precisely Γ.

Lemma 98.1 (The common unipotent character)**.** *The group Γ is a uniform lattice in D_∞. The integral (98.1) converges absolutely for smooth W, W' of compact support modulo B_∞. Whenever it converges absolutely, it vanishes unless*

$$\chi_{\varpi,\infty}(d) = \chi_{\varpi,\infty}(\gamma^{-1} d \gamma) \qquad (d \in D_\infty). \tag{98.2}$$

If the equality holds, its value is the covolume of Γ in D_∞ times the integral of the same tensor over $D_\infty \backslash G_\infty$, with compatible measures.

Proof. Write $t_f = bk$ in its first finite support. For every $\alpha \in B(F)$ stabilizing the finite coset, the projection to $H(F)$ stabilizes the column class of the H_f-component of b, by Theorem 86.1. That class has trivial rational stabilizer by Lemma 92.23. Hence $\alpha \in N(F)$. Apply the same reasoning to $\gamma^{-1} t_f K^0$ for the second flag. It follows that $\Gamma = D(F) \cap t_f K^0 t_f^{-1}$.

In either upper unipotent group, exponential and logarithm are mutually inverse finite polynomials and identify the group with its nilpotent Lie algebra in characteristic zero. On the intersection they therefore identify D with the intersection of the two Lie algebras, an F-vector space. Thus $D(F)$ is dense in D_f by additive approximation. The compactness of the adelic quotient of a unipotent group follows by induction along a central series with vector-group quotients, starting with the compact additive quotient $\mathbb{A}_F / F$. Compact sets of representatives lift across these quotients in polynomial exponential coordinates. Reducing its finite coordinate by density, as in Lemma 92.23, identifies the double quotient by $D_f \cap t_f K^0 t_f^{-1}$ with $\Gamma \backslash D_\infty$. This proves compactness.

For $\alpha \in \Gamma$, both finite unipotent characters are trivial. To see this for the first, $b^{-1} \alpha_f b$ belongs to $N_f \cap K^0$, and conjugation by B_f preserves the character. For the second use $\gamma^{-1} t_f$. The adelic character of a rational unipotent element is one, so the corresponding real characters are also one on Γ. This proves descent in (98.1). On the compact fibre $\Gamma \backslash D_\infty$, the integrand varies by the ratio of the two characters. If it is nontrivial, right translation by a point where it differs from one shows its integral is zero. Otherwise the integrand is constant along the fibre. The invariant quotient measures exist because both D_∞ and G_∞ are unimodular. Disintegration gives the two assertions. Equality of the real characters is also equivalent to

equality of their real differentials on the common Lie algebra, by applying the characters along each one-parameter exponential subgroup. In the frame coordinates these differentials are the real traces of the successive simple coordinates with the marked terminal vector. This gives the condition at `Formalism/SharpRestriction/CuspSpanNormBounds/BlockEnds.lean`, line 79.

For the asserted convergence, use as scalar envelopes the functions $\|W(g)\|$ and $\|W'(g)\|$ times the indicators of the respective finite supports before translation. Each descends to $B(F)\backslash G(\mathbb{A}_F)$ because the real transformations are unitary. Each has compact support on this quotient. Indeed, there are finitely many finite classes, and over a compact part of $B_\infty\backslash G_\infty$ their period fibres are compact by Lemma 92.23 and compactness of H_∞. The ordinary rational periodization of each envelope is bounded with compact support on the automorphic quotient. Locally one can lift each compact support to a compact set in the adelic group. Only finitely many rational elements can move this set to meet a given compact set, by discreteness and closedness of $G(F)$. Their two periodizations therefore have integrable product. Unfolding the nonnegative product by Tonelli bounds each individual double coset and each of its contributing finite cells, whose finite measure is positive. The norm of a rank-one tensor is the product of the two vector norms. This proves absolute convergence of (98.1). □

Theorem 98.2 (Restriction to the common compact group)**.** *Take C from Theorem 86.3 for the frames $y, \gamma y$. In any absolutely convergent cell, $J_{\gamma,t}$ intertwines the two restrictions to $U(C)_\infty$. If C has dimension k, this map vanishes for different words. For identical words it is a scalar times the unitary identification of the two irreducible actions, unique up to phase.*

If $\dim C = k-1$, let $\mathcal{M}$ be the set of highest weights of $U(C)_\infty$ occurring in both restrictions, and let d_μ be the dimension for such a weight. Then the rank is at most $\sum_{\mu\in\mathcal{M}} d_\mu$. If $\mathcal{M}$ is nonempty, write $d_* = \min_{\mu\in\mathcal{M}} d_\mu$ and put

$$A_{\gamma,t} = \int_{\Gamma\backslash G_\infty} \|W(g)\|\,\|W'(\gamma^{-1}g)\|\,dg.$$

The operator and Hilbert–Schmidt norms of $J_{\gamma,t}$ are at most $A_{\gamma,t}/d_*$ and $A_{\gamma,t}/\sqrt{d_*}$, respectively.

Proof. The case of unequal unipotent characters follows from Lemma 98.1. For equal characters use its integral over $D_\infty\backslash G_\infty$. A member $h \in U(C)_\infty$ fixes both frames, normalizes both unipotent radicals, and normalizes D_∞. Conjugation on D_∞ by this compact group has modulus one. Left translation by h hence preserves the quotient measure, whether or not it normalizes the discrete lattice. In either decomposition in the two period groups its unipotent part has trivial character. Indeed it fixes the u_i and the terminal vector, and so does its projected column action in the corresponding Witt splitting. Equivariance of W, W' now shows that $J_{\gamma,t}$ intertwines the two actions. Each identification of C with its image in a column space is complex linear and isometric at each real place, so these actions are unitarily equivalent to the usual inclusions in the two definite groups. For full dimension Schur's lemma gives the assertion. Two different words have different highest weights.

For dimension $k-1$, ordinary compact branching at one place gives precisely the weights μ for which $\lambda_i \ge \mu_i \ge \lambda_{i+1}$ for $1 \le i < k$, each with multiplicity one [1, Theorem 2.1]. Taking the product over the real places gives the same multiplicity statement there. An intertwiner vanishes between inequivalent irreducibles and has image in the sum over the common weights. This proves the rank bound.

More explicitly, average vw^* over the compact group with Haar probability. Between two copies of a common irreducible the average is a scalar times an isometric identification by Schur's lemma. Composing with its inverse and taking the trace bounds the modulus of that scalar by $\|v_\mu\|\|w_\mu\|/d_\mu$, where the subscripts denote orthogonal projections. Distinct blocks

have orthogonal domains and images. Taking the largest scalar bounds the operator norm by $\|v\|\|w\|/d_*$. For the Hilbert–Schmidt norm square the bound on each scalar, multiply it by d_μ, and sum. The result is at most $\|v\|^2\|w\|^2/d_*$. Averaging each pointwise tensor in the quotient integral leaves the integral unchanged. Fubini and the triangle inequality give both estimates, including the fibre covolume. □

For example, for the weights of Theorem 9.13, write $\lambda_i = u_i - (k+1-2i)/2$ at one place. The gaps in u are at least 3Δ. Interlacing gives, for $j - i \geq 2$,

$$\frac{\mu_i - \mu_j + j - i}{j - i} \;\geq\; \frac{u_{i+1} - u_j + 1}{j - i} \;\geq\; \frac{3\Delta + 1}{2}.$$

The adjacent factors are at least one. Weyl's dimension formula gives $d_* \geq ((3\Delta+1)/2)^{f\binom{k-2}{2}}$. Applying the norm estimates to a full period pairing still requires control of the scalar envelopes and their sum. The choice of a compact cutoff alone gives no uniform bound for that sum as it is removed.

98.2 The rank of an orbital integral between two different words

Use the fields and primes of Theorem 9.13 and its set $\mathcal{A}$ of words. Write $d(\rho_c) = \dim_{\mathbb{C}} V_{\rho_c}$ for the real-place representation dimension, to distinguish it from the mass-weighted column number d_c there. Use the level with terminal vector $z = \varpi e$ from Section 86.1. For any rational pair of frames and a fixed finite cell in the unfolded pairing, let $J_{c,c'}$ be the integral of the rank-one maps from $V_{\rho_{c'}}$ to V_{ρ_c} in (98.1), before evaluation on finite class covectors. The assertions below hold for compactly supported smooth seeds modulo the real period group and, more generally, whenever the individual tensor integral converges absolutely.

We first examine compact branching at one real place. Write $\lambda_i = u_i - (k+1-2i)/2$ for the highest weight of the word c. On restriction to $U(k-1)$ the summands are multiplicity-free. Their weights satisfy $\lambda_i \geq \mu_i \geq \lambda_{i+1}$. With $y_i = \mu_i + (k-2i)/2$, the choices of y_i fill the following product of intervals, with step size one. Explicitly,

$$I_i(c) = [u_{i+1} + 1/2,\, u_i - 1/2], \qquad 1 \leq i < k.$$

Successive intervals are separated by at least one, so each choice is a valid dominant weight. Give each summand the probability $\dim V_\mu / \dim V_\lambda$. These probabilities sum to one by the branching decomposition. By the Weyl dimension formula they are proportional, on this product, to $\prod_{i<j}(y_i - y_j)$. The normalization depends only on c and the place.

Lemma 98.3 (Conditional mass near either endpoint)**.** *There is an absolute $\delta > 0$ with the following property at every real place. For k sufficiently large and $k/64 \leq i \leq 63k/64$, fix the other $k-2$ coordinates at any allowed values. Each of the two end intervals of Δ consecutive points of $I_i(c)$ has conditional probability at least δ, uniformly in the place, word and Δ.*

Proof. For $2 \leq j \leq k-2$ the endpoints from (9.1) give

$$I_j(c) = [4\Delta(k-j-1) - \Delta c_{j+1},\; 4\Delta(k-j) - 1 - \Delta c_j].$$

It contains $w_j = \Delta(4 - c_j + c_{j+1})$ points, between 3Δ and 5Δ. Its real coordinates differ from $4\Delta(k-j)$ by a number in $[-5\Delta, 0]$. For real $y \in I_i(c)$, separate from the conditional density the two factors involving the immediate neighbors. The other factors form $R(y) = \prod_{|j-i|\geq 2} |y - y_j|$. For an interior j with $d = |j-i| \geq 2$, the difference $y - y_j$ differs from $4\Delta(j-i)$ by at most 5Δ. Taking reciprocals gives

$$\left| \frac{1}{y - y_j} - \frac{1}{4\Delta(j-i)} \right| \leq \frac{5}{4\Delta d(4d-5)}.$$

The total of these errors is $O(1/\Delta)$. The signed main sum on $2 \le j \le k-2$, omitting the two neighbors and i, is $(H_{k-2-i} - H_{i-2})/(4\Delta)$ with $H_n = \sum_{d=1}^{n} 1/d$. Both indices are bounded below by $k/128$ for large k, so their harmonic sums differ by an absolute constant at most. For $j = 1$, the constraint $y_1 \ge u_2 + 1/2$ puts it at distance $\gg k\Delta$ from $I_i(c)$. The other endpoint satisfies $y_{k-1} \le u_{k-1} - 1/2$ and has the same distance bound. These one-sided inequalities apply even at the high places. It follows that $|(\log R)'| \le C/\Delta$ on the interval for an absolute C. The ratio of its maximum to its minimum there is at most $\exp(5C)$.

Write L, U for the endpoints of $I_i(c)$. Of the lowest Δ points take those from $L + \lfloor \Delta/2 \rfloor$ to $L + \Delta - 1$. The neighbor below is at most $L - 1$. On these points its distance is at least $(\lfloor \Delta/2 \rfloor + 1)/w_i$ times its largest distance on the whole interval, as is seen by taking the closest allowed neighbor $L-1$. This ratio is at least $1/10$. The neighbor above is at least $U+1$, giving the ratio at least $(w_i - \Delta + 1)/w_i \ge 2/3$ for that factor. Reflection gives the same estimates on at least half the points of the top slice. In each case the chosen points constitute at least a tenth of the full interval. Comparing their weights with the maximum weight proves the lemma with $\delta = \exp(-5C)/1000$. □

Theorem 98.4 (Small relative rank for every mixed cell)**.** *There is an absolute $\kappa > 0$ such that, for all sufficiently large x on the common intervals and every $c \ne c'$ in $\mathcal{A}$,*

$$\operatorname{rank} J_{c,c'} \;\le\; e^{-\kappa f k} \min\{d(\rho_c), d(\rho_{c'})\}. \tag{98.3}$$

The bound holds for each fixed rational pair of frames and finite cell, with the same κ for every pair.

Proof. Theorems 86.3 and 98.2, give either full column rank for the common compact group, in which case a mixed map is zero, or a copy of $U(k-1)$ at each real place. In the latter case, fix one place and compare its two branching boxes. If $c_t = 0$ and $c'_t = 1$, the top Δ points of $I_t(c)$ have been excluded by the upper bound for $I_t(c')$. If $c_t = 1$ and $c'_t = 0$, the bottom Δ points of $I_{t-1}(c)$ have been excluded by the lower bound for $I_{t-1}(c')$. The definition of the boxes keeps the same lattice for the two words. Discarding mismatches whose assigned coordinate is outside $[k/64, 63k/64]$ costs at most $2k/64 + O(1)$ bits. At least $(k-2)/8$ bits differ by Theorem 9.13. Each coordinate can be assigned to at most two bits. For sufficiently large k this leaves at least $k/32$ different coordinates, each with an excluded end slice.

Under the dimension probability for c, the event of lying in the box of c' requires avoidance of all these slices. For each coordinate, even conditional on all the other coordinates, its avoidance probability is at most $1-\delta$ by Lemma 98.3. Multiply the indicators of the avoidance events and condition on all coordinates except one of those still present. Its indicator can be removed at cost at most $1-\delta$, since the other indicators are measurable under that conditioning. Iteration gives at most $(1-\delta)^{k/32}$. This bounds the fraction of dimension in summands also present for c'. Interchanging the two words proves the same bound relative to the dimension for c'.

The real column space is the tensor product over the f real places. Its dimension probabilities on tuples of summands are products, and the tuple is common precisely when it is common at each place. Thus $\kappa = -\log(1-\delta)/32$ works. The image of the intertwining integral lies in the sum of the common summands, and its kernel contains the entire orthogonal complement of those summands in the domain, by Theorem 98.2. This proves (98.3) in both dimensions. □

Corollary 98.5 (Simultaneous kernels for a collection of cells)**.** *For a fixed word c, take any $M \ge 1$ ordered frame pairs with finite cells at this level and identify their input column spaces with V_{ρ_c} by isometries. At each cell take every other selected word as an output. There is a subspace of V_{ρ_c} annihilated by all these maps whose codimension is at most*

$$M\,|\mathcal{A}| e^{-\kappa f k}\, d(\rho_c).$$

For $M \le \exp(C_0 f m^5 x^{\varepsilon})$ with fixed C_0, its relative codimension is at most $e^{-\kappa f k/2}$ for all sufficiently large x on these same intervals.

Proof. Intersect the indicated kernels. The codimension of an intersection is at most the sum of the individual codimensions by mapping to the direct sum of the corresponding quotient spaces. Apply (98.3) to each map. Finally $\log|\mathcal{A}| \le k\log 2$ and $f \to \infty$ by the field degree in Proposition 9.9. The lengths in the proof of Theorem 9.13 give $m^5 x^{\varepsilon} = o(k)$. The claimed exponential bound follows. □

These subspaces may depend on the cells. The corollary estimates how many dimensions can couple different words in a prescribed collection. Estimating the full period still requires the unrestricted sum over cells and the terms pairing a word with itself. The arithmetic normalization on the integral theta lattice is a further requirement even when the complex pairing has been bounded.

98.3 The finite measure of a nondegenerate Bessel orbit

Fix $\gamma \in \mathcal{R}$ as in Section 110.6. The rational label γ specifies the two decorated flags. The untruncated period also sums over finite right level cosets supporting both seeds. We compute the measure of this second sum. Let W be the nondegenerate span of the two frames, let $C = W^{\perp}$, and write $H_C = U(C)$. For $J = B \cap \gamma B \gamma^{-1}$, $D = N \cap \gamma N \gamma^{-1}$ and a finite place v of F, put

$$X_v = \{tK_v : t \in B_v K_v,\ \gamma^{-1} t \in B_v K_v\}, \qquad K_v = K_v^0, \qquad K^0 = \prod_{v<\infty} K_v.$$

Here and below a local algebra $E_v = E \otimes_F F_v$ may be a product of two fields. For $t_f K^0$ in the restricted product X_f of the X_v, set $\Gamma_t = J(F) \cap t_f K^0 t_f^{-1}$. The finite cells over γ are $J(F)\backslash X_f$.

Lemma 98.6 (The stabilizer of the two decorated flags)**.** *Let $h_1, \dots, h_l$ be the block lengths in Lemma 110.10. For U_h the upper unitriangular group of degree h, there are F-group isomorphisms*

$$D \simeq \prod_{\nu=1}^{l} \operatorname{Res}_{E/F} U_{h_\nu}, \qquad J = D \times H_C.$$

The number of independent E-coordinates of D is $a = \sum_\nu h_\nu(h_\nu - 1)/2$.

Proof. Choose rational representatives of the two flags in monomial normal form M. An element of J acts on them by upper unitriangular matrices U, V. It preserves C and $U^* M V = M$, so $V = M^{-1}(U^*)^{-1} M$. In positions $i < j$, an entry of U^{-1} can be nonzero only when $w(i) > w(j)$, since otherwise its conjugate would lie strictly below the diagonal of V. These inversions occur precisely within the individual reversed blocks. Thus U^{-1}, and hence U, is block upper unitriangular with no entries between the blocks. Conversely every such U gives an upper unitriangular V. These two matrices preserve the self pairings and their cross pairing. They therefore define an isometry on W. Extending by the identity on C gives a unipotent subgroup D', parametrized over F by the stated product. Any element of J is uniquely an element of D' times an element acting only on C. The two actions commute.

At a real place the image of D' in either column group is both unipotent and unitary for a positive definite form, so is trivial. This gives the same identity of algebraic maps over F, since the coordinate entries are polynomials and the real points of the displayed unipotent product are Zariski dense. The action of H_C in either column group is faithful by Theorem 86.3. Taking the kernels of the column actions gives $D = D'$. □

Define a finite set S_0 of places for the level and field. Include the places over $2TQ$, those ramified in E/F, and those where the chosen integral row lattice is not self-dual or K_v is smaller than its full stabilizer. Include any remaining places where the initial frame is not integral. These conditions hold at only finitely many places, since the level is a product of the prescribed local modifications of the integral lattice stabilizers. Enlarge S_0 to $S(\gamma)$ by adding places where $\mathcal{G}$ or a nonzero endpoint minor D_j is a nonunit in at least one factor.

Proposition 98.7 (The local orbits of finite level cosets)**.** *The set $J(F_v)\backslash X_v$ is finite at every finite place. Outside $S(\gamma)$ it consists of one orbit. The stabilizer there is conjugate in $J(F_v)$ to*

$$\prod_{\nu} U_{h_\nu}(\mathcal{O}_{E_v}) \times U(\Lambda_C)$$

for a self-dual lattice Λ_C in C_v.

Proof. First consider an arbitrary finite place. Write $tK_v = bK_v = \gamma b' K_v$ with $b, b' \in B_v$. The assignment $tK_v \mapsto b^{-1}\gamma b'$ induces a bijection

$$J(F_v)\backslash X_v \;\longrightarrow\; (B_v \cap K_v)\backslash\big((B_v\gamma B_v) \cap K_v\big)/(B_v \cap K_v).$$

Indeed the two choices of representatives produce the left and right compact factors on the right. Equal matrices from two pairs of representatives give an element of $J(F_v)$ carrying one tK_v to the other. Each matrix in the intersection gives such a pair.

The set $B_v\gamma B_v$ is closed in $G(F_v)$. To see this, take a convergent sequence in the orbit. The cross matrices converge. At each step of Gaussian elimination the entries before the prescribed pivot are zero and the pivot is the same fixed nonzero scalar for the whole sequence. All eliminations are continuous, since they divide only by that scalar. The limit has the same monomial normal form. Its concatenated Gram determinant is still $(-1)^d\mathcal{G} \neq 0$. The two pairs with this normal form are isometric on their spans. Hermitian Witt extension gives a local isometry of the whole space preserving both decorated flags in order. In the split case this extension amounts to extending a basis in the first factor and using the dual transformation in the second. Thus the limit belongs to the original double coset.

Each double coset of $B_v \cap K_v$ is open relative to that orbit. Here are details at a point g_0 of the orbit. For g in the orbit close to g_0, the elimination above chooses representatives for (Y, gY) continuously. Their concatenated Gram matrix is fixed. The map between the two normalized lists for g_0 and g is close to the identity on their spans. It admits a unitary extension tending to the identity. In fact, let P be orthogonal projection from the old complement to the new one. It tends to the identity by the projection formula using the inverse of the fixed Gram matrix. Thus P^*P on the old complement tends to 1. For sufficiently close g, the binomial series for $(P^*P)^{-1/2}$ converges, including at places over 2, since the n-th coefficient is $(-1)^n 4^{-n}\binom{2n}{n}$. The map $P(P^*P)^{-1/2}$ is the required isometry on the complement. The resulting extension b_g preserves the first decorated flag and carries g_0Y to the decoration of gY. Both b_g and $g_0^{-1}b_g^{-1}g$ lie in B_v and tend to the identity. They lie in $B_v \cap K_v$ for all sufficiently close g. This proves openness. The intersection of the closed orbit with K_v is compact. Its disjoint open double cosets give the finiteness assertion.

Now suppose $v \notin S(\gamma)$ and let $\Lambda_t = t\Lambda_v$. The decompositions $tK_v = bK_v = \gamma b' K_v$ supply integral representatives $bY, \gamma b' Y$ for both flags in Λ_t. Their cross matrix is integral. All its Gaussian pivots are units, because the endpoint ratios are h_ν-th powers of the initial block entries and the endpoints are units. The elimination can therefore be performed integrally, in both components at a split place. The resulting $2d$ vectors have a common, fixed Gram matrix with unit determinant. Their span over $\mathcal{O}_{E_v}$ is a self-dual lattice in W_v. Pairing with

this basis and inverting its Gram matrix gives the integral orthogonal decomposition of Λ_t into that lattice and a self-dual lattice in C_v.

For two choices of tK_v match the two normalized bases on W_v. This isometry preserves the two decorations. On C_v their self-dual lattices are isometric. For an unramified quadratic extension of odd residue characteristic, the reduction of a unimodular Hermitian form has a vector of nonzero norm by polarization. Lift the vector, split off its line, and continue. Each resulting unit coefficient is a norm of a unit. Indeed the residue-field norm is surjective and a principal unit in F_v has a square root by Hensel's lemma. This proves integral equivalence to the unit diagonal form. In the split case a lattice in one factor determines its dual in the other, and any two bases are related by an invertible linear map. The isometries on W_v and C_v together lie in $J(F_v)$ and carry one row lattice to the other. A row lattice determines its right K_v-coset at this level, proving transitivity. Nonemptiness follows from the assumed finite support of γ.

In the normalized integral bases, an element on W_v preserves the lattice precisely when U and V in Lemma 98.6 are integral. All block scalars are units, so integrality of U already suffices. Two normalized representatives for the same pair differ on W_v by an element of $D(F_v)$, which conjugates the stabilizer. On C_v the stabilizer is the full lattice unitary group, proving the stated description. □

Use the product of additive Haar measures in the entries of $D = \prod_\nu \operatorname{Res}_{E/F} U_{h_\nu}$, normalized by $\operatorname{vol}(\mathcal{O}_{E_v}) = 1$ at finite places and ordinary complex area at infinity. Write $c_E = 2^{-f}\sqrt{|\Delta_E|}$, where Δ_E is the absolute field discriminant. Choose product Haar measures on $G_f, H_C(\mathbb{A}_F)$, with the self-dual lattice stabilizers of volume one at the places outside $S(\gamma)$. Put

$$m_C = \frac{\operatorname{vol}(H_C(F)\backslash H_C(\mathbb{A}_F))}{\operatorname{vol}(H_C(F_\infty))}, \qquad L_{\gamma,v} = \sum_{tK_v \in J(F_v)\backslash X_v} \frac{\operatorname{vol}(K_v)}{\operatorname{vol}_{J_v}(J(F_v) \cap tK_v t^{-1})}.$$

Here $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ and the Haar measure on J_v is the product for $D_v, H_{C,v}$. Other fixed choices of measure can be used by retaining the corresponding local volume ratios.

Theorem 98.8 (Total finite cell measure of one rational orbit)**.** *With the actual level K^0 and its rational column stabilizers,*

$$\sum_{tK^0 \in J(F)\backslash X_f} \operatorname{vol}(K^0)\operatorname{vol}(\Gamma_t\backslash D_\infty) = c_E^a\, m_C \prod_{v<\infty} L_{\gamma,v}. \tag{98.4}$$

Every factor $L_{\gamma,v}$ outside $S(\gamma)$ equals one. All sums and the product in this formula are finite after removing these unit factors.

Proof. For a fixed $t_f K^0$ write $K_J = J_f \cap t_f K^0 t_f^{-1}$, $K_D = K_J \cap D_f$, and $K_C = \operatorname{pr}_{H_C}(K_J)$. These are compact open subgroups in the indicated groups. Lemma 98.1 gives $\Gamma_t = D(F) \cap K_D$ because a rational element stabilizing the finite coset projects trivially into both column groups at the marked level.

The rational points of D are dense in D_f by additive approximation in its matrix entries. Additive product measure is Haar since translation is triangular in the successive superdiagonals, with diagonal one. For the standard compact open of integral unitriangular matrices, the real lattice quotient has volume c_E^a. Indeed the integer lattice of E has Euclidean covolume c_E in the chosen embedding into $\mathbb{C}^f$, by the discriminant determinant formula. Reducing entries in successive superdiagonals by $\mathcal{O}_E$ gives one additive fundamental region per entry. Earlier superdiagonals stay fixed at each reduction. For any compact open K_D, intersect it with the

standard one. Density identifies the two finite group indices with the indices of their rational intersections. The comparison of covolumes gives

$$\operatorname{vol}(\Gamma_t\backslash D_\infty) = c_E^a/\operatorname{vol}(K_D).$$

The rational cells above one J_f-orbit are $J(F)\backslash J_f/K_J$. Projection to H_C identifies this set with $H_C(F)\backslash H_{C,f}/K_C$. To prove injectivity, lift a right K_C factor to K_J and a left rational column factor to $H_C(F) \subset J(F)$. Once the column components agree, density of $D(F)$ lets one align the remaining components modulo K_D. Surjectivity follows by inclusion of $H_{C,f}$. Each of the column class stabilizers is trivial. Indeed for a representative j_f and a rational column stabilizer, the possible unipotent components of its lifts in $j_f K_J j_f^{-1}$ form a nonempty open coset. Choose a rational component by density. The resulting rational element of J stabilizes $j_f t_f K^0$ and belongs to $D(F)$ by Lemma 98.1, so its column component was the identity.

The column quotient is compact at infinity and has finite class number, by finiteness of the genus of a positive definite Hermitian lattice and commensurability of compact open subgroups. Its class number at K_C is therefore $m_C/\operatorname{vol}(K_C)$. Indeed each class contributes the volume of $H_C(F_\infty) \times K_C$ to the adelic quotient by the just proved triviality of stabilizers. Across these representatives K_D changes only by conjugation in the unimodular group D_f. Finally $\operatorname{vol}_{J_f}(K_J) = \operatorname{vol}_{D_f}(K_D)\operatorname{vol}_{H_{C,f}}(K_C)$ by the product group formula, since each nonempty fibre is a coset of K_D. These identities give the contribution of one J_f-orbit. Its choices separate over finite places. Almost everywhere $\gamma_v \in K_v$, and Proposition 98.7 gives a unique local orbit with stabilizer of volume one. Thus the sum over adelic orbits factors into the asserted finite product. □

For the actual seeds (92.16), write B_ℓ and $B_{\ell'}$ for the maximal norms of the two covectors on the finite column class sets. Let $J_{\gamma,t}$ now denote the scalar contribution including its finite measure and both covectors. The common real quotient gives the explicit upper bound

$$\sum_{t\in J(F)\backslash X_f} |J_{\gamma,t}| \le B_\ell B_{\ell'}\, c_E^a m_C \Big(\prod_{v\in S(\gamma)} L_{\gamma,v}\Big) I_\infty(\gamma), \tag{98.5}$$

$$I_\infty(\gamma) = \int_{D_\infty\backslash G_\infty} \|W_1(g)\|\,\|W_2(\gamma^{-1}g)\|\,d\dot g.$$

Indeed the unipotent characters have absolute value one, the real product of norms is D_∞-invariant, and each contraction of the rank-one operator is bounded by this product times the two covector norms. Disintegration and (98.4) give the inequality. The untruncated inputs and their convergence before finite contraction are treated in Theorem 86.22. All quotient measures in (98.5) use the stated Haar on D_∞.

Theorem 98.9 (The primes of a nonunit local factor)**.** *Put $P_\gamma = \prod_{v\in S(\gamma)\setminus S_0} \mathrm{N}v$, where $\mathrm{N}v$ is the residue-field size. Then $P_\gamma \le w_\gamma = w(D(\gamma))$. For $0 \le \delta < 1/2$ and all sufficiently large Q in terms of δ,*

$$\sum_{\gamma\in\mathcal{R}} P_\gamma^\delta w_\gamma^{-1} \le \Big(\frac{128\log(2Q)}{Q^{4(1-\delta)}}\Big)^{fd}. \tag{98.6}$$

Proof. Every extra place divides the norm to F of some endpoint minor or divides $\mathcal{G}$ in F. All these quantities are nonzero algebraic integers. Thus

$$P_\gamma \le |N_{E/\mathbb{Q}}(\textstyle\prod_{j\in I} D_j)|\, N_{F/\mathbb{Q}}(\mathcal{G}).$$

The first factor is at most $w_\gamma^{1/2}$ by the embedding formula. At each real place $0 < \tau\mathcal{G} \le |\tau D_d|^2$, which bounds the second factor by $w_\gamma^{1/2}$. It suffices to sum $w_\gamma^{-1+\delta}$.

In the proof of Theorem 110.13 retain all the boxes, separations P_j, and multiplicities. For $b_1 = 2(1-\delta) > 1$ replace the one-coordinate density by $2b_1 r(1+r^2)^{-b_1-1}\,dr$, whose integral from u to infinity is $(1+u^2)^{-b_1}$. For $0 \le t < 2b_1 - 2$ the new moment is $M = \int_0^\infty 2b_1 r^{3+t}(1+r^2)^{-b_1-1}\,dr$. The same volume and binomial bounds give $(9M+2)^{fd}Q^{-(2+t)fd}$. Take $t = 2b_1 - 2 - \eta$ with $\eta = 1/\log(2Q)$, which is nonnegative for all sufficiently large Q. Integrating over $r \le 1$ and $r > 1$ gives $M \le 1 + 4/\eta$, since $b_1 \le 2$. The inequality $Q^\eta \le e$ gives (98.6). □

For example $\delta = 1/4$ allows a factor $P_\gamma^{1/4}$ with Q^{-3fd} up to the displayed logarithm. This estimate controls which primes can contribute. The sizes of the integrals $L_{\gamma,v}$ at those primes, the mass m_C, and the real integral in (98.5) also enter the mixed period.

Proposition 98.10 (Supported orbits outside the regular minor locus)**.** *For each period field, translate and level in Section 110.6 are infinitely many distinct relevant rational double cosets with $\gamma_f \in K^0$, $\mathcal{G} \neq 0$, and $D_j = 0$ for every $1 \le j < d$. Their common unipotent group is* $\mathrm{Res}_{E/F} U_d$.

Proof. Use the Witt basis in Section 121.16, with the last isotropic basis $v_1, \dots, v_s$ ordered dually to the first. Let $W_{\mathrm{mid}} = W_0 \perp Ee$. Choose a positive coordinate vector $w \in W_0$ of norm one in the signed row space. Take $a_0 \in E^\times$ with $\bar{a}_0 = (-1)^d a_0$, choosing $a_0 = 1$ for even d and any nonzero element of trace zero for odd d. Set $\alpha = \bar{a}_0/\bar{\varpi}$. Define $A : E^s \longrightarrow W_{\mathrm{mid}}$ by sending the first basis vector to $\alpha(w+e)$ and the others to zero. With $A^\dagger$ given by the Hermitian pairing of its columns with a middle vector, we have $A^\dagger A = 0$, since $w + e$ is isotropic. Let L be the s-square matrix with entries

$$L_{i,d+1-i} = (-1)^{i-1} a_0 \quad (2 \le i \le d-1)$$

and all other entries zero. The choice of $\bar{a}_0$ gives $L^* = -L$. In the ordered three summands define

$$g_n = \begin{pmatrix} 1 & 0 & 0 \\ nA & 1 & 0 \\ nL & -nA^\dagger & 1 \end{pmatrix} \qquad (n \in \mathbb{Q}).$$

The Witt matrix pairs the first and third summands by the identity. Direct multiplication, using $L + L^* = A^\dagger A = 0$, gives $g_n \in G(F)$ and $g_n g_m = g_{n+m}$. The induced map on finite adeles of $\mathbb{Q}$ is continuous since its entries are polynomials with fixed algebraic coefficients integral at almost all places. The preimage of K^0 is an open neighborhood of zero. It contains $N\widehat{\mathbb{Z}}$ for some $N \ge 1$. Take positive integral multiples n of N.

For these translates $Z = h(Y, g_n Y) = M_n - TE_{dd}$, where M_n is the full reversed block with i-th row entry $(-1)^{i-1} n a_0$. Indeed the middle antidiagonal entries come from nL. The top right entry is $n\bar{\alpha}\varpi = na_0$ and the bottom left is $-n\bar{\varpi}\alpha = -n\bar{a}_0$. The terminal vector keeps its middle component. Adding $T/(na_0)$ times the first row to the last row eliminates the extra diagonal entry. Thus $D_d = (na_0)^d$, all proper leading minors vanish, and $\mathcal{G} = D_d \overline{D_d}$ is totally positive. The distinct positive n give distinct final determinants.

Finally on this full reversed block any upper unitriangular U gives an upper $V = M_n^{-1}(U^*)^{-1}M_n$. Its simple entries in reverse order are the conjugates of those of U, since the ratio of consecutive antidiagonal entries is -1. Their real traces agree. Lemma 98.6 describes the entire common unipotent group, so this checks relevance on that group. □

The formal-degree orbital identity in `Development/AbelianVarietyTheory/Proofs/FlagMeasures/BesselOrbitalIntegralEqInvDegSmulStatement.lean`, line 7 takes its group element in the

regular set defined in `Development/AbelianVarietyTheory/Definitions/SemilocalFrames/MatrixOfBesselTwoSidedAction.lean`, line 77. That definition requires each proper leading minor and the Gram determinant to be nonzero. Proposition 98.10 exhibits supported, character-compatible data outside its domain. In the convention using the imaginary parts of the simple coordinates, one can realize constant antidiagonal entries directly. Choose a nonzero $a_0 \in E$ of trace zero, and set each displayed antidiagonal entry of L equal to a_0. Now $L^* = -L$ and the two extreme entries are both na_0. The simple entries of V are the negatives of the conjugates of those of U in reverse order. Their imaginary parts agree. All proper leading minors still vanish. The definition of the regular set uses only the minors and the Gram determinant.

Corollary 98.11 (Absolute integrability on the whole real group)**.** *Let W_1, W_2 be two nonzero smooth Bessel sections belonging to the prescribed real discrete series and finite under the right maximal compact group. For every translate in Proposition 98.10,*

$$\int_{G_\infty} \|W_1(g)\| \, \|W_2(\gamma^{-1} g)\| \, dg = +\infty.$$

Proof. Each section is real analytic. To verify this for scalar components, decompose the vector into its finitely many right compact types. On the Lie algebra of the real group use the invariant real trace form, negative on the compact summand $\mathfrak{k}$ and positive on the complementary summand $\mathfrak{p}$. In orthonormal bases for these signed forms the real group Casimir is $\Omega = \sum_{\mathfrak{p}} X_i^2 - \sum_{\mathfrak{k}} Y_i^2$. It acts by the infinitesimal character of the prescribed representation. The compact Casimir $\sum_{\mathfrak{k}} Y_i^2$ acts by a scalar on each compact type. Hence each scalar component in one type is an eigenfunction of the analytic differential operator $R(\Omega) + 2\sum_{\mathfrak{k}} R(Y_i)^2$. Its principal part is the sum of squares of a full invariant tangent frame and is elliptic. Analytic elliptic regularity gives the assertion (C. B. Morrey and L. Nirenberg, *On the analyticity of the solutions of linear elliptic systems of partial differential equations*, Ann. Scuola Norm. Sup. Pisa **11** (1957), 271–290).

The real group is connected. Indeed every factor has a Cartan decomposition with connected maximal compact group and connected exponential factor. A nonzero analytic scalar component therefore has open dense nonvanishing set. The two sections have positive product of norms on a nonempty open set even after translation. This product is invariant under left D_∞, since the two corresponding unipotent groups act through unitary characters. Thus it is positive on an open set of positive quotient measure. Here D_∞ is a noncompact unipotent group of positive dimension by Proposition 98.10, so its Haar measure is infinite. The quotient integral formula and Tonelli's theorem give the divergence. □

The integral in (98.5) is over $D_\infty \backslash G_\infty$. The rational lattice contributes the finite covolume in (98.4). The value of each signed projected matrix entry still depends on its sections and covectors.

98.4 The Bessel integral on a nonregular orbit

Fix a real place of a field in Section 86.7. Write $d = s+1$, $r = k+2d-1$, and let $B = N \rtimes U(k)$ be the stabilizer of the decorated frame $Y = (u_1, \dots, u_{d-1}, z)$. The negative index is d. We use the same Haar measures on G and B as in Theorem 106.20. For two words, write E_i for their finite-dimensional Hilbert spaces at this place and $\tau_i(nh) = \chi_i(n)\rho_i(h)$ for their unitary actions of B. The row representation σ is the same. Let $D = N \cap \gamma N \gamma^{-1}$, and write $d_0^\gamma = \gamma^{-1} d_0 \gamma$ for $d_0 \in D$. We assume that the two frames span a nondegenerate space with positive definite orthogonal complement and that their characters agree on D. These hypotheses hold for all the relevant orbits of nonzero Gram determinant in Section 110.6. The real versions of Lemmas 110.10 and 98.6 follow by the same matrix operations at this place.

Lemma 98.12 (Volume on a two-sided orbit)**.** *Give D any Haar measure, and give $\mathcal{X}_\gamma = (B \times B)/\{(d_0, d_0^\gamma) : d_0 \in D\}$ the corresponding right quotient measure. With $w(g) = \|g\|_{\mathrm{op}}$ in signed orthonormal coordinates, for $R_0 \geq 1$,*

$$\mathrm{vol}\{[b_1, b_2] \in \mathcal{X}_\gamma : w(b_1 \gamma b_2^{-1}) \leq R_0\} \ \ll_\gamma \ R_0^{8r^2(d^2+2)}.$$

The constant can depend on the actual real pair and the Haar measures.

Proof. Choose representatives Y_1, Y_2 of the two copies of the standard decorated frame so that $h(Y_1, \gamma Y_2) = M$ is monomial. These are fixed changes by upper unitriangular matrices. Put $x = b_1 \gamma b_2^{-1}$ and write $b_i Y_i = Y_i U_i$. Then

$$h(Y_1, xY_2) = (U_1^{-1})^* M U_2^{-1}.$$

All entries on the left are $O_\gamma(R_0)$. We first obtain a decomposition of this matrix into a lower unitriangular matrix, M, and an upper unitriangular matrix such that the two triangular factors and their inverses have norm $O_\gamma(R_0^{d^2})$.

Use the elimination in Lemma 110.10. If the pivot columns in successive rows are $w_1, \ldots, w_d$, order the columns temporarily in this order to calculate the residual matrix after each elimination. Its entry at remaining row i, column j, after t pivots is the determinant with ordered rows $1, \ldots, t, i$ and ordered columns $w_1, \ldots, w_t, j$, divided by the determinant with the first t rows and pivot columns. This follows by subtracting the linear combination of the first t rows that clears the selected entries, and expanding the bordered determinant in its last row. The denominator is the product of the first t pivots. Each pivot equals the fixed entry in M by uniqueness of the monomial form. Thus each eliminating multiplier is $O_\gamma(R_0^d)$. In the original column order only columns to the right of the current pivot require clearing. At one step each of the two eliminating matrices is the identity plus a matrix supported on one row or one column with square zero. Its inverse has the same bound. At most d such matrices are multiplied on either side, proving the assertion.

We can arrange that the U_i of an actual pair b_i realize this decomposition. Indeed two triangular decompositions with the same M differ by simultaneous right multiplication of the frame matrices U_i by A_1, A_2 with $A_1^* M A_2 = M$. The isometry on the two spans acts on $Y_1, \gamma Y_2$ by these matrices. Extend it by the identity on the orthogonal complement. The proof of Lemma 98.6 puts this element in D. Thus the change comes from right multiplication by (d_0, d_0^γ) in the required quotient.

For the resulting representative both $b_1 Y_1 = Y_1 U_1$ and $b_1 \gamma Y_2 = x Y_2 U_2$ are $O_\gamma(R_0^{d^2+1})$. Their original spans contain a fixed maximal negative subspace. A bound for the images of a basis of such a subspace bounds the operator norm of an isometry up to a fixed constant. To see this, first take the negative coordinate space, and write an isometry in positive and negative blocks as $\left(\begin{smallmatrix} A & B_0 \\ C & E \end{smallmatrix}\right)$. The relations $AA^* = 1 + B_0 B_0^*$ and $CC^* = EE^* - 1$ bound the other two blocks by the negative columns. A fixed isometry taking that coordinate space to the chosen negative subspace accounts for the change of basis. Hence $w(b_1) \ll_\gamma R_0^{d^2+1}$ and $w(b_2) = w(x^{-1} b_1 \gamma) \ll_\gamma R_0^{d^2+2}$.

Take a compact identity neighborhood of positive measure in D. Right multiplication above by this neighborhood multiplies the two bounds by at most a fixed constant. Along each fibre in the stated set the enlarged product of balls therefore has measure bounded below by the measure of this neighborhood. The quotient integration formula bounds its quotient volume by the product of the two Haar ball volumes divided by that positive number. Each ball of radius $S_0 \geq 1$ in B has measure $O(S_0^{4r^2})$ by Lemma 86.19. □

For column vectors e_i, $e_1e_2^*$ denotes the map $u \mapsto \langle u, e_2\rangle e_1$, with inner products linear in the first variable. Given two left equivariant sections put

$$K_\gamma(W_1, W_2) = \int_{D\backslash G} W_1(g) W_2(\gamma^{-1}g)^* \, d\dot{g}.$$

Equality of the two unipotent characters makes the integrand independent of its representative. For an operator-valued coefficient F of σ, put

$$\mathcal{O}_\gamma(F) = \int_{\mathcal{X}_\gamma} \tau_1(b_1)^{-1} F(b_1\gamma b_2^{-1}) \tau_2(b_2) \, d[b_1, b_2].$$

Here both τ_i act by matching scalars on the two images of D. These quotients also occur in `Formalism/SharpRestriction/CuspSpanNormBounds/RootMeasBig.lean`, line 27 and line 53 there, with the two integration variables inverted.

Theorem 98.13 (Schur orthogonality on the common unipotent quotient)**.** *Let $J_i v(g) = \sqrt{d_\sigma}\Lambda_i(\sigma(g)v)$ be the isometric sections of Theorem 106.20. Choose compact-finite u_i and $\xi_i \in E_i$ such that $\ell_i = \langle \Lambda_i(u_i), \xi_i\rangle \neq 0$. Such choices exist for each word. Set*

$$F(g) = \langle \sigma(g)u_2, u_1\rangle \, \xi_1\xi_2^*.$$

For compact-finite v_1, v_2, on every sufficiently large field with the spacings of Section 86.7, one has

$$K_\gamma(J_1v_1, J_2v_2) = \frac{\langle v_1, v_2\rangle}{\overline{\ell_1}\ell_2} \, \mathcal{O}_\gamma(F). \tag{98.7}$$

Both integrals are absolutely convergent, including for a single reversed block of length d.

Proof. For $a_i(g) = \langle \sigma(g)v_i, u_i\rangle$, start with

$$W_i(g) = \int_B a_i(bg)\tau_i(b)^{-1}\xi_i \, db.$$

Lemma 86.19 and the proof of Proposition 86.20 give absolute convergence. Testing against $z \in E_i^\vee$, the relative character in the proof of Theorem 106.20 gives $z(W_i(g)) = \overline{\ell_i}\, z(\Lambda_i(\sigma(g)v_i))$. Indeed the compact factor of that scalar coefficient pairs z with $\rho_i(h)^{-1}\xi_i$, and the unipotent factor is $\overline{\chi_i(n)}$. Nonvanishing of the functional and density of compact-finite smooth vectors give the choices of u_i, ξ_i.

There are two ways to integrate on the space with coordinates (b_1, b_2, g) and identification $(b_1d_0, b_2d_0^\gamma, g) \sim (b_1, b_2, d_0g)$. One first integrates in B^2 and then in $D\backslash G$. The other integrates in G and then in $\mathcal{X}_\gamma$. For clarity, first disintegrate B^2 by the right diagonal action in the first integral. In the fibre move d_0 to the left of g by the identification. The fibre integral followed by the $D\backslash G$ integral is precisely Haar integration on G. Tonelli gives this identity for nonnegative functions. The groups involved are unimodular, since N, D are unipotent and the compact groups preserve their Haar measures.

Apply the nonnegative identity to $|a_1(b_1g)a_2(b_2\gamma^{-1}g)|$. In the second integration order put $y = b_2\gamma^{-1}g$, $x = b_1\gamma b_2^{-1}$. Write $m_* = 8r^2(d^2+2)+1$. For the family at hand, $M - m_* > 4r^2$ for sufficiently large x in the interval parameter. Indeed $M = 8(f+1)(r+1)^3 + 8r^2 + 4$ and the ranks in Section 86.7 satisfy $r/d^2 \to \infty$. Here $2d = m_0 \asymp x^{1/2-\varepsilon/16}$ and $k \asymp x^{9/2+10\varepsilon}$ for the selected family. These give $r/d^2 \to \infty$. Since $w(x) \le w(xy)w(y)$, the product of the two coefficient majorants is bounded by

$$C\, w(x)^{-m_*} w(y)^{-(M-m_*)}.$$

The y-integral is finite by Lemma 86.19. Lemma 98.12 and summation over dyadic balls give finiteness of the remaining integral. The same ball estimate applied to F proves absolute convergence of its orbital integral. Thus Fubini applies coordinate by coordinate with the unitary column factors inserted.

Schur orthogonality now gives

$$\int_G a_1(xy)\overline{a_2(y)}\,dy = d_\sigma^{-1}\langle v_1, v_2\rangle\langle \sigma(x)u_2, u_1\rangle.$$

This follows by moving $\sigma(x)^{-1}$ onto u_1 in the first coefficient. Multiplication by the column factors and integration on $\mathcal{X}_\gamma$ evaluates $K_\gamma(W_1, W_2)$. Substituting $W_i = \overline{\ell_i}\, J_i v_i/\sqrt{d_\sigma}$ gives (98.7). □

Proposition 98.14 (A weighted bound with its field fixed)**.** *On the original level K^0, write $J_{\gamma,c}$ for the scalar contribution of a finite cell as in (98.5). Let $H_1(\gamma) = \prod_\tau \max(1, |\tau\zeta_1|)$, where ζ_1 is the scalar in the first reversed block. For $p, t \geq 0$ with*

$$h = 2pd(d+1) + t < L - 4fr^2$$

one has, for all relevant supported nondegenerate labels,

$$\sum_{\gamma,c} w_\gamma^p H_1(\gamma)^t |J_{\gamma,c}| < \infty, \qquad \sum_c |J_{\gamma,c}| \leq C_{x,p,t}\, Q^{-ft} w_\gamma^{-p}. \tag{98.8}$$

Here one constant works for every label and for unit input vectors in the prescribed finite-dimensional spaces at this x.

Proof. The first j frame vectors have exterior product $Y^{(j)}$ fixed by B. In signed coordinates the norm of the Hermitian pairing on exterior powers is one for the Euclidean exterior norm. Evaluate D_j as the pairing between $g^{-1}Y^{(j)}$ and $g^{-1}\gamma Y^{(j)}$. At every place,

$$|\tau D_j| \leq \|Y^{(j)}\|^2 \{\nu_\tau(g)\nu_\tau(\gamma^{-1}g)\}^j.$$

Indeed one can vary the representative on each side independently and $w(a) = w(a^{-1})$. With $C_Y = \prod_{j=1}^d (1 + \|Y^{(j)}\|^4)^2$ this bounds w_γ by $C_Y^f\{V(g)V(\gamma^{-1}g)\}^{2d(d+1)}$. The Euclidean squared norms here are 2^j for $j < d$ and $2^{d-1}T$ for $j = d$.

In a monomial matrix with first block length h_1, every entry with $i + j < h_1 + 1$ is zero. Consequently the entries on the first antidiagonal $i + j = h_1 + 1$ in that block stay fixed under both unitriangular changes. At least one such entry pairs two isotropic frame vectors, of Euclidean norm $\sqrt{2}$ each. Take $i = 1, j = h_1$ for $h_1 < d$, and $i = 2, j = d - 1$ for $h_1 = d$, using $d \geq 3$ on this family. The same two infima as above give $|\tau\zeta_1| \leq 2\nu_\tau(g)\nu_\tau(\gamma^{-1}g)$. Moreover $(\zeta_1/Q)^{h_1}$ is integral, by Proposition 110.11. When $h_1 = d$ the preceding leading minor vanishes. Integrality of ζ_1/Q follows because the ring of integers is integrally closed. Taking its nonzero absolute norm gives $H_1(\gamma) \geq Q^f$.

For each section use its positive seed $\mathbf{1}_{B_f K^0}\|W_i\|V^h$ on the adelic group. Its periodization over $B(F)\backslash G(F)$ is bounded because Lemma 86.21 gives $O_x(R_0^{4fr^2})$ indices in a height ball, while Proposition 86.20 gives decay $O_x(R_0^{h-L})$ on a dyadic shell. It is integrable by unfolding to the finite list of seed classes and using the weighted integral in that proposition. It therefore belongs to L^2. Multiply the two positive periodizations, integrate, and unfold by Tonelli as in Theorem 86.22. The weight inequalities bound the first sum in (98.8) by the product of the two L^2 norms times $2^{ft} C_Y^{fp} B_\ell B_{\ell'}$, with the finite covector norms of (98.5). Retaining one label and using $H_1 \geq Q^f$ gives the second inequality. At fixed x, expansion in orthonormal bases of the finitely many input spaces gives a common constant for the unit vectors. This proof does not bound its growth with x. □

98.5 Change of frame and the arithmetic cell measure

Keep the two sections in (98.7) and the Haar measure on G. At one real place allow terminal norms $T_1, T_2 > 0$ on the two sides. The application to (98.5) has $T_1 = T_2 = T$. Let $j_0 = 0 < j_1 < \cdots < j_l = d$ be the nonzero minor endpoints, $h_\nu = j_\nu - j_{\nu-1}$, and ζ_ν the first antidiagonal entry in block ν. Put

$$S_j = j(2d - j - 1), \qquad S = S_d = d(d-1),$$
$$e_\nu = 2d - j_{\nu-1} - j_\nu - 1, \qquad a = \sum_\nu h_\nu(h_\nu - 1)/2, \qquad H = \sum_\nu h_\nu(h_\nu^2 - 1)/6.$$

Write $\mathcal{G} = |D_d|^2 - T_1T_2|D_{d-1}|^2$. The local density P is the product of $|\zeta_\nu|^{2h_\nu(k-1+e_\nu)}$ over blocks, with the factor for a last block of length one replaced by $(|\zeta_l|^2 - T_1T_2)^{k-1}$. This is the positive density at the realized points of nonzero Gram determinant in `Formalism/SharpRestriction/CuspSpanNormBounds/RelevantZeroGramTerms.lean`, line 240.

Lemma 98.15 (The local density in endpoint minors)**.** *Every such orbit satisfies*

$$P = \mathcal{G}^{k-1} \prod_{\nu<l} |D_{j_\nu}|^{2(h_\nu + h_{\nu+1})} |D_d|^{2(h_l - 1)}. \tag{98.9}$$

For a relevant rational orbit supported at the prescribed finite level, let $P_\infty = \prod_\tau P_\tau$ in the arithmetic frames. Then $P_\infty^{1/2} \geq Q^{fd(k+d-2)}$.

Proof. For each block $|\zeta_\nu|^{h_\nu} = |D_{j_\nu}/D_{j_{\nu-1}}|$, where $D_0 = 1$. The factors involving $k-1$ multiply to $\mathcal{G}^{k-1}$. Indeed, if $h_l > 1$, then $D_{d-1} = 0$. If $h_l = 1$, then $\mathcal{G} = |D_{d-1}|^2(|\zeta_l|^2 - T_1T_2)$ and $e_l = 0$. For the remaining factors telescope the endpoint ratios using $e_\nu - e_{\nu+1} = h_\nu + h_{\nu+1}$ and $e_l = h_l - 1$. This proves (98.9).

By Theorem 86.4, $N_{F/\mathbb{Q}}\mathcal{G} \geq Q^{2df}$. For each proper endpoint the product of the f absolute values of D_j is at least Q^{jf}, since D_j/Q^j is a nonzero integer of E by Proposition 110.11. The same applies to D_d when $h_l > 1$, since its shifted congruence then has $D_{d-1} = 0$. Its exponent is zero in the other case. The sum of the endpoint exponents in the square root, weighted by their indices, is

$$\sum_{\nu<l} j_\nu(h_\nu + h_{\nu+1}) + d(h_l - 1) = \sum_\nu h_\nu e_\nu = \sum_{i=1}^{d} 2(d-i) = S.$$

The middle equality uses the fact that e_ν is the mean of $2(d-i)$ on the ν-th block. Combining the two norm bounds gives the assertion. □

Use the character $\exp(i\Re(c\sum_{i<d} U_{i,i+1}))$ in the frame Y, where $c \neq 0$. Give D the measure

$$dd_c = (|c|/(2\pi))^a \prod_{i<j \text{ in one block}} d^2U_{ij},$$

where d^2u is Euclidean area in $\mathbb{C}$. These are the upper entries of the first frame action on D in Lemma 98.6. Translation is triangular in order of increasing superdiagonal, with all diagonal Jacobian entries one. The formula therefore defines Haar measure. Passing back from the monomial frame by an upper unitriangular matrix conjugates each principal block by its own upper unitriangular principal block. This follows by block upper triangular multiplication and preserves the displayed area. Thus one may also read the within-block entries in the original frame, as in `Formalism/SharpRestriction/CuspSpanNormBounds/RootMeasBig.lean`, line 48. We denote the two choices of quotient kernel by K^{root} for dd_c and K^{area} for unweighted area.

Theorem 98.16 (Transport of the normalized orbital kernel)**.** *For $t,\kappa>0$ replace the terminal vectors by tz_i. Let A multiply u_i by $\lambda_i=t\kappa^{d-i}$ for $i<d$, multiply its isotropic dual by λ_i^{-1}, and fix the middle space. Put $\gamma'=A^{-1}\gamma A$. Write $j_B=|\det(\mathrm{Ad}_A|_{\mathfrak{b}})|$. The new coefficient of the character is $c'=\kappa c$. For the transported sections $W_i'(g)=j_B^{-1/2}W_i(Ag)$, with the two B-Haar measures the original ones, one has*

$$(P')^{1/2}K^{\mathrm{root}}_{\gamma'}(W_1',W_2')=t^{\,2(k-d)}\kappa^{-S-a}P^{1/2}K^{\mathrm{root}}_{\gamma}(W_1,W_2). \tag{98.10}$$

The transport preserves the $L^2(B\backslash G)$ norm of every section. The formula applies to every block decomposition.

Proof. Set $S_0=\mathrm{diag}(t\kappa^{d-i})_{i=1}^d$. If Y_i' is the new standard frame, then $AY_i'=Y_iS_0$. Thus the new cross matrix is $S_0^*ZS_0$. The action of $A^{-1}bA$ in the new frame is $S_0^{-1}US_0$ when b acts by U in the old frame. Each simple coordinate is divided by κ, giving the asserted c'. The column group action is preserved. These facts also apply after unitriangular changes to a monomial frame, by conjugating those changes by S_0. In particular

$$D_j'=t^{2j}\kappa^{S_j}D_j,\qquad \zeta_\nu'=t^2\kappa^{e_\nu}\zeta_\nu,\qquad \mathcal{G}'=t^{4d}\kappa^{2S}\mathcal{G}.$$

For the last equality $S_{d-1}=S_d$ and the product of the new terminal norms is $t^4T_1T_2$.

Let $E_*=\sum_{i=1}^d 4(d-i)^2$. The sum of squared deviations of the numbers $2(d-i)$ from their mean on a block of length h is $h(h^2-1)/3$, by the sums of the first h integers and their squares. Hence $\sum h_\nu e_\nu^2=E_*-2H$ and $\sum h_\nu e_\nu=S$. Substitution in the product defining P gives

$$P'/P=t^{4d(k+d-2)}\kappa^{2((k-1)S+E_*-2H)}. \tag{98.11}$$

A terminal singleton obeys the same scaling since $e_l=0$ and both terms of its difference acquire the factor t^4.

To compute j_B, take the three independent sets of coordinates of the unipotent Lie algebra in the Witt basis. In its isotropic upper triangular block $i<j<d$, the complex entry has factor λ_i/λ_j. Each of the $k+1$ complex middle entries in row $i<d$ has factor λ_i. In the skew-Hermitian block from the dual isotropic space to the isotropic space, a real diagonal coordinate has factor λ_i^2 and the complex off-diagonal coordinate $i<j$ has factor $\lambda_i\lambda_j$. These are all free coordinates by the equation $X^*J+JX=0$ for the Lie algebra. The compact column algebra is fixed. Taking real determinants yields

$$j_B=\prod_{i<d}\lambda_i^{2(k+2(d-i))}=t^{\,2(d-1)(k+d)}\kappa^{kS+E_*}.$$

In the disintegration of dg into $db\,d\dot g$, left multiplication by A preserves dg and carries each left fibre by $b\mapsto AbA^{-1}$. Consequently $\int_{B\backslash G}f(Ag)d\dot g=j_B\int f(g)d\dot g$. This proves the isometry assertion and gives the stated normalization for the transported sections.

The map $d_0\mapsto A^{-1}d_0A$ carries D onto D'. It multiplies the (i,j) coordinate within a block by κ^{i-j}. The sum of $j-i$ over these pairs is H, since on a block it is $\sum_{m=1}^{h-1}m(h-m)=h(h^2-1)/6$. The pullback of $dd'_{c'}$ is therefore $\kappa^{a-2H}dd_c$. Disintegrating group Haar measure on both sides of $g\mapsto Ag$ shows that the pushforward of the new quotient measure is κ^{2H-a} times the old one. Including both section scalars gives $K^{\mathrm{root}}_{\gamma'}(W_1',W_2')=j_B^{-1}\kappa^{2H-a}K^{\mathrm{root}}_{\gamma}(W_1,W_2)$. The integrals exist by Theorem 98.13, or by the change of variables from that theorem. Multiplying by the square root of (98.11) gives (98.10). □

Corollary 98.17 (The regular squared norm)**.** *Let $\Omega=\{q\in\mathbb{C}^d:\ q_1\cdots q_{d-1}\neq0,\ |q_d|^2>T_1T_2\}$. On this set let $\gamma(q)$ have diagonal cross matrix with entries $q_1,\dots,q_d$, and put*

$$\mathfrak{J}(U)=\int_U P(q)\|K_{\gamma(q)}(W_1,W_2)\|_{\mathrm{HS}}^2\,d^{2d}q\qquad (U\subset\Omega\ \textit{Borel}).$$

For $q_i' = t^2\kappa^{2(d-i)}q_i$ and the image U' of U,

$$\mathfrak{J}'(U') = t^{4k}\mathfrak{J}(U). \tag{98.12}$$

This holds on every truncation and on all of Ω as an identity of nonnegative extended integrals. Whenever the full norm is positive and finite, the pointwise quantity $P\|K_\gamma^{\mathrm{root}}\|_{\mathrm{HS}}^2/\mathfrak{J}(\Omega)$ is multiplied by $t^{-4d}\kappa^{-2(S+a)}$ at every realized nondegenerate orbit.

Proof. Diagonal cross data are realized by taking the first $d-1$ vectors of the second frame to be q_iv_i, where v_i are the dual isotropic vectors. For its last vector take $-q_dz_1/T_1 + \sqrt{|q_d|^2/T_1 - T_2}\,w_0$, with w_0 a positive unit middle vector perpendicular to z_1. Its norm is $-T_2$ and its pairing with z_1 is q_d. These choices lift locally smoothly to isometries taking the second standard frame to the displayed frame. On the hyperbolic planes interchange the isotropic basis and its dual with the reciprocal conjugate scalings. In the middle plane extend the negative unit vector to a signed orthonormal basis by projecting a positive vector onto its perpendicular and normalizing, and extend by the identity on the positive complement. A fixed change of the original middle basis can first put z_2 on its negative axis. This also gives measurable representatives. Any two representatives with these diagonal data are in the same B-double coset by Lemma 110.10 over $\mathbb{C}/\mathbb{R}$. For singleton blocks D is trivial. Changing the representative by $b_1\gamma b_2^{-1}$ multiplies the kernel on the two sides by $\tau_1(b_1)$ and $\tau_2(b_2)^*$, so its Hilbert–Schmidt norm is well defined. The integrands are measurable by integration of their measurable components over G and absolute convergence on every such orbit.

On Ω, equation (98.10) squared gives the factor $t^{4(k-d)}\kappa^{-2S}$. The real Jacobian of the map $q \mapsto q'$ is $t^{4d}\kappa^{2S}$. The change of variables for a nonnegative integral gives (98.12). Dividing the squared identity on an arbitrary realized orbit by the full regular norm, under the stated finiteness and positivity conditions, gives the last assertion. The regular density used here is the one in `Formalism/FormalDegreeBessel/StatementBF.lean`, line 146. □

Theorem 98.18 (The discriminant in the arithmetic quotient)**.** *Return to E/F and $T_1 = T_2 = T$. Set $R = |\Delta_E|^{1/(2f)}$. Choose any $A_0 > 0$ and apply Theorem 98.16 at all real places with $t = 1$ and $\kappa = A_0^{-1}$. Indicate transported data by primes. For the product quotient and the actual sections one has*

$$c_E^a P_\infty^{1/2} K_{\infty,\gamma}^{\mathrm{area}} = A_0^{-fS}(R/A_0)^{fa}(P_\infty')^{1/2}K_{\infty,\gamma'}^{\mathrm{root}}. \tag{98.13}$$

In particular $A_0 = R$ removes all dependence on the block stabilizer dimension from the scalar on the right. The finite cell contributions at the original level then satisfy

$$\sum_c |J_{\gamma,c}| \le B_\ell B_{\ell'}\, m_C\Big(\prod_{v\in S(\gamma)} L_{\gamma,v}\Big) R^{-fS}Q^{-fd(k+d-2)}\|(P_\infty')^{1/2}K_{\infty,\gamma'}^{\mathrm{root}}\|_{\mathrm{op}}. \tag{98.14}$$

Proof. In Section 121.16 the real character has $|c| = 4\pi$, since the trace from $\mathbb{C}$ to $\mathbb{R}$ is twice the real part. Thus the product of the root Haar measures is 2^{fa} times area and $K_\infty^{\mathrm{area}} = 2^{fa}K_\infty^{\mathrm{root}}$. The discriminant covolume in (98.4) is $c_E = 2^{-f}|\Delta_E|^{1/2} = (R/2)^f$. Multiply the inverses of (98.10) over the f places. The two area and lattice factors combine to $2^{fa}c_E^a = R^{fa}$, giving (98.13). The argument works first on tensors by Fubini, and then on the actual compact-finite vectors by their finite tensor expansions.

At a fixed finite cell the two covectors, including their finite unitary factors, are constant in the real variable. The real integrand between them descends to $D_\infty\backslash G_\infty$ before contraction by the relevance identity. Its integral over the rational unipotent quotient is the covolume in (98.4) times that contraction of $K_{\infty,\gamma}^{\mathrm{area}}$. Bound the latter in modulus by its operator norm times the two covector norms and sum the covolumes by Theorem 98.8. Now use (98.13) at $A_0 = R$ and Lemma 98.15. □

The two scales in the source can be compared explicitly. Its real block multiplier is R^{-e_ν}, whereas its parameter for the derivative of the additive character is $2\pi/(RQ)$. For the physical trace character the latter magnitude, after the conversion $s \mapsto -2is$, corresponds to $A_0 = RQ$. In (98.13) this choice gives the scalar $(RQ)^{-fS}Q^{-fa}$ and block multipliers $(RQ)^{-e_\nu}$. At $A_0 = R$ the block multipliers are R^{-e_ν} and the absolute coefficient is $4\pi/R$. Thus a use of the source's pointwise estimate at its joint data still requires a comparison at its specified character and block coordinates. See `Formalism/SharpRestriction/CuspSpanNormBounds/LocalDataRealPlace.lean`, line 16, `Formalism/SharpRestriction/CuspSpanNormBounds/LeadMinorK.lean`, line 168 and `Development/AbelianVarietyTheory/Definitions/GlobalTwisting/CcKPrimitive.lean`, line 109.

98.6 The contribution of the two auxiliary levels

The bound in Corollary 93.12 uses the largest norm of a column covector over all its finite classes. We now sum its squared norm over the refinements of a single class. This evaluates the effect of both auxiliary levels on the nondegenerate part of the original Gram matrix.

Retain $K = K^0$, the terminal vector ϖe, the two split primes g_1, g_2, and $Q = g_1 g_2$ from Section 93.4. Let K^* enlarge K to the full lattice stabilizer at $v \mid Q$, with $K^*_v = K_v$ elsewhere. Write $K^*_H = H_f \cap K^*$ and $K_H = H_f \cap K$ for the levels in the definite column group. Set

$$I_\sigma = [K^* : K], \qquad I_\pi = [K^*_H : K_H], \qquad c^*_\rho = \frac{\operatorname{vol}(K^*) v_N \operatorname{vol}(H_\infty)}{\operatorname{vol}(N_f \cap K^*)\, \dim \rho}.$$

Here all Haar measures are those of (92.17). The symbol c^*_ρ denotes this scalar even when a column class at the larger level has a nontrivial rational stabilizer. For a fixed relevant label of nonzero Gram determinant put $J = B \cap \gamma B \gamma^{-1}$ and $D = N \cap \gamma N \gamma^{-1}$. For either local level K' let

$$L_v(K') = \sum_{tK' \in J(F_v)\backslash((B_v K' \cap \gamma B_v K')/K')} \frac{\operatorname{vol}(K')}{\operatorname{vol}_{J_v}(J(F_v) \cap tK't^{-1})}.$$

Use the complement measures of Theorem 93.14.

Lemma 98.19 (The fraction of refinements supporting two flags)**.** *At $v \mid g$ for $g \in \{g_1, g_2\}$, a full-level coset supporting both flags has either zero or $i_{\pi,v} = |\mathrm{GL}_k(\mathbb{F}_g)/U_k(\mathbb{F}_g)|$ right K_v-cosets supporting both. Write $L_v^{=}$ for the part of $L_v(K^*_v)$ on which these refinements exist. Then*

$$L_v(K_v) = \frac{i_{\pi,v}}{i_{\sigma,v}} L_v^{=}, \qquad \frac{i_{\sigma,v}}{i_{\pi,v}} = g^k \prod_{j=k+1}^{r} (g^j - 1), \quad i_{\sigma,v} = [K^*_v : K_v]. \tag{98.15}$$

In particular $0 \le L_v^{=} \le L_v(K^*_v)$.

Proof. Use the first split factor and the Witt basis in Theorem 86.4. Put $s = d - 1$. The blocks have sizes $1, \dots, 1, k+1, 1, \dots, 1$, with s singletons on either side. Let $\bar N$ be the upper block unipotent radical over $\mathbb{F}_g$. On the middle block embed GL_k by $h \mapsto \mathrm{diag}(h, 1)$. The reductions of $B_v \cap K^*_v$ and K_v are respectively

$$\bar B = \bar N \rtimes \mathrm{GL}_k(\mathbb{F}_g), \qquad \bar K = \bar N \rtimes U_k(\mathbb{F}_g).$$

Indeed the singleton actions are one and the middle action of the period group fixes the terminal vector and its dual on the middle quotient. Its arbitrary invertible k-block and all the unipotent blocks lift entry by entry over the integer ring. The row level has exactly the additional upper unitriangular condition on that k-block. This follows as well from the zero entries in the

terminal column of the specified row reduction. The principal congruence subgroup is contained in K_v. It follows that $(B_v \cap K_v^*)K_v$ is the entire inverse image of the subgroup $\bar{B}$.

Choose $b_1, b_2 \in B_v$ and $h_1, h_2 \in K_v^*$ with $t = b_1 h_1$ and $\gamma^{-1} t = b_2 h_2$. A refinement represented by tu, $u \in K_v^*$, supports the first flag precisely when $\bar{u} \in \bar{h}_1^{-1}\bar{B}$, and the second precisely when $\bar{u} \in \bar{h}_2^{-1}\bar{B}$. These two left translates are equal or disjoint. When equal, their right $\bar{K}$-cosets are a translate of $\bar{B}/\bar{K}$, of cardinal $i_{\pi,v}$. The equality test is independent of the decompositions just chosen.

In the full-level orbit of tK_v^*, its stabilizer $J_t = J(F_v) \cap tK_v^* t^{-1}$ permutes the supporting refinements. For a J_t-orbit of length u_0, its stabilizer has volume $\mathrm{vol}(J_t)/u_0$. Its contribution at the finer level is therefore $u_0 \,\mathrm{vol}(K_v)/\,\mathrm{vol}(J_t)$. Summing proves the first equality. The second follows from $|\mathrm{GL}_n(\mathbb{F}_g)| = g^{n(n-1)/2}\prod_{j=1}^{n}(g^j - 1)$ and $|\bar{N}| = g^{(r(r-1)-k(k+1))/2}$. The local sums at both levels are finite by the compact-orbit argument in Proposition 98.7. □

At the other primes set $L_v^{=} = L_v(K_v)$. Define $\mathcal{M}_\gamma^{=} = \mathfrak{m}_{k-1}\prod_v L_v^{=}$ and $\mathcal{M}_\gamma^{*} = \mathfrak{m}_{k-1}\prod_v L_v(K_v^*)$. These are finite products after the factors one are removed. Lemma 98.19 and Theorem 93.14 give the exact equality $M_\gamma = (I_\pi/I_\sigma)\mathcal{M}_\gamma^{=}$. At the two split primes we can recover the other factor I_π in the estimate for unit columns as follows.

Proposition 98.20 (The column norms above one rational cell)**.** *Consider a rational cell at level K^* admitting a supported refinement at level K. Fix a representative t of such a refinement. Put*

$$\Gamma^* = J(F)\cap tK^*t^{-1}, \quad \Gamma_D = D(F)\cap tK^*t^{-1}, \quad a_t = [\Gamma^* : \Gamma_D], \qquad m_t^* = \frac{\mathrm{vol}(K^*)}{a_t}\,\mathrm{vol}(\Gamma_D\backslash D_\infty).$$

The two flags give column classes y_1, y_2 at level K_H^*. Choose finite adelic representatives for these classes and set

$$A_i = H(F) \cap y_i K_H^* y_i^{-1}, \qquad U_i(y_i)^2 = \sum_{y \mapsto y_i} \|\ell_{i,y}\|^2 \quad (i = 1, 2).$$

The sum runs over the column classes of the original level K_H mapping to y_i, and ℓ_i is an arbitrary tuple of covectors for the i-th word. For the original signed period summands above this cell,

$$\sum_{c \mapsto t} |J_{\gamma,c}| \le \|K_{\infty,\gamma}^{\mathrm{area}}\|_{\mathrm{op}}\,\frac{m_t^*}{I_\sigma}\sqrt{|A_1||A_2|}\,U_1(y_1)U_2(y_2). \tag{98.16}$$

Moreover the sum over all coarse rational cells admitting such a refinement has $\sum_t m_t^* = c_E^a \mathcal{M}_\gamma^{=}$.

Proof. Use $t = b_1 h_1$, $\gamma^{-1}t = b_2 h_2$ with $b_i \in B_f$ and $h_i \in K$. We may take $y_i = p_H(b_i)$ as the two representatives. Before dividing by Γ^*, the common refinements are parametrized by

$$\mathcal{C} = \prod_{v|Q} \mathrm{GL}_k(\mathbb{F}_g)/U_k(\mathbb{F}_g).$$

This follows from the proof of Lemma 98.19, since both $\bar{h}_i$ lie in $\bar{K}$. For a parameter $z \in \mathcal{C}$ lift it by an integral block $h(z)$ in H at the marked places, extended by one elsewhere. The lift specifies the finite row coset $th(z)K$. Its column class on side i is represented by y_i times a lift of $u_i z$, where u_i is the fixed tuple of k-blocks of the reductions of h_i. Indeed the reduction of $h_i h(z)$ has that middle block. Its unipotent part lifts to $N_v \cap K_v^*$, and all errors reducing to the identity lie in K_v. Thus each of the two maps to K_H^*/K_H runs through all I_π cosets once.

On passing to rational column classes the fibres are the A_i-orbits in these cosets, with the action conjugated by y_i. The groups A_i are finite because they preserve a positive definite

integral trace lattice. Each action is free, since at the original level the column stabilizers are trivial by Proposition 86.29 and Theorem 86.1. Consequently if $y_i(z)$ is the fine class on side i, then

$$\sum_{z\in\mathcal{C}} \|\ell_{i,y_i(z)}\|^2 = |A_i|\, U_i(y_i)^2.$$

The norms are independent of representatives because the archimedean column representation is unitary.

The projection of Γ^* into the first rational column group has kernel Γ_D, by Lemma 98.6, and its image lies in A_1. To see the latter inclusion conjugate by b_1 at the finite places and take the column component. This proves that a_t is finite. An element of Γ_D acts trivially on $\mathcal{C}$. In fact its conjugate by b_1^{-1} belongs to N_f, and at the marked places it is integral because $t = b_1 h_1$ with $h_1 \in K$. It reduces into $\bar{N}$, which is normal in $\bar{B}$ and fixes all of $\bar{B}/\bar{K}$. Conjugating also by h_1 does not change this property. At the other places the level has a single refinement. Conversely if an element of Γ^* fixes a fine coset, its rational column image fixes a fine column class in the sense of belonging to its compact stabilizer. That rational stabilizer is trivial. Thus Γ^*/Γ_D acts freely on $\mathcal{C}$ and every fine row stabilizer is Γ_D.

For each fine rational cell, the orbital integral contracts $K^{\mathrm{area}}_{\infty,\gamma}$ with its two column covectors and a scalar of modulus one. Its measure is $\mathrm{vol}(K)\,\mathrm{vol}(\Gamma_D\backslash D_\infty)$ by the disintegration giving (98.5). The product of the two covector norms is constant on each rational orbit of refinements. We may sum it over all $z \in \mathcal{C}$ and divide by a_t. Cauchy–Schwarz and the two marginal identities give (98.16), since $\mathrm{vol}(K) = \mathrm{vol}(K^*)/I_\sigma$.

For the assertion about m_t^*, count just the measures of these same cells. There are I_π/a_t above t. Their total is $m_t^* I_\pi/I_\sigma$. Summing and using (98.4) at the original level gives $(I_\pi/I_\sigma)\sum_t m_t^* = c_E^a M_\gamma$. Substitution of the formula following Lemma 98.19 proves the asserted identity, retaining possible torsion at the larger level. □

Theorem 98.21 (Auxiliary level indices in the normalized period)**.** *Let A_* be the largest order of a rational column stabilizer at level K_H^*. Suppose the two column tuples have unit norm for the original identity terms in (92.17) and the real row inputs are normalized as in Section 93.4. For each relevant supported rational label of nonzero Gram determinant,*

$$\sum_c |J_{\gamma,c}| \le \frac{A_*\,\mathcal{M}_\gamma^{=}}{\sqrt{c^*_{\rho_1} c^*_{\rho_2}}}\, R^{-fS} Q^{-f(dk+a)} (P_\infty^{\#})^{1/2} \|\widehat{K}_{\gamma\#}\|_{\mathrm{op}}, \tag{98.17}$$

$$\log A_* = O\big(fk\log^2(2fk)\big) = o(fk\log b).$$

This holds on the common cohomology intervals, with $S = d(d-1)$ and $a = \sum_\nu \binom{h_\nu}{2}$. One may use $\mathcal{M}_\gamma^*$ in place of $\mathcal{M}_\gamma^{=}$ on the right.

Proof. At $v \mid Q$ the integral unipotent blocks all reduce into $\bar{K}$. Hence $N_f \cap K^* = N_f \cap K$. Formula (92.17) gives $c_{\rho,K} = c^*_\rho/I_\sigma$. A unit tuple therefore has $\sum_y \|\ell_{i,y}\|^2 = I_\sigma/c^*_{\rho_i}$. This bounds each partial squared norm $U_i(y_i)^2$. The two square roots cancel the entire I_σ^{-1} in (98.16). Summing over t now bounds the period by

$$\frac{A_*\,\mathcal{M}_\gamma^{=}}{\sqrt{c^*_{\rho_1} c^*_{\rho_2}}}\, c_E^a \|K^{\mathrm{area}}_{\infty,\gamma}\|_{\mathrm{op}}.$$

Use (93.7) and $P_\infty^{1/2} \ge Q^{fd(k+d-2)}$ from Theorem 98.18. The resulting power of Q is $f(d(d-2)-a) - fd(k+d-2) = -f(dk+a)$, giving (98.17). The inequality $\mathcal{M}_\gamma^{=} \le \mathcal{M}_\gamma^*$ follows locally from Lemma 98.19.

Every coarse column stabilizer acts faithfully on the underlying rational space of dimension $n = 2fk$. Minkowski's theorem for finite subgroups of $\mathrm{GL}_n(\mathbb{Q})$ gives

$$v_p(|A_i|) \le \sum_{j\ge 0} \left\lfloor \frac{n}{p^j(p-1)} \right\rfloor \qquad \text{for every prime } p$$

[2, Lecture I, Theorem 1]. Only $p \le n+1$ occur and the sum is at most $np/(p-1)^2$. Summing its product with $\log p$ and bounding the sum by that over all integers $2 \le j \le n+1$ gives $O(n\log^2(2n))$, by the integral estimate for $\sum_{j\le n+1}(\log j)/j$. On the selected intervals f, k are bounded by fixed powers of x and $\log b = x^{\epsilon}$. This proves the claimed uniform estimate. □

For example, in the Cauchy–Schwarz estimate following Proposition 93.13 use (98.17) before summing over the rational labels. For any finite set $\mathcal{A}$ of such labels the resulting upper bound is

$$\frac{A_* R^{-fS} Q^{-fdk+2f}}{\sqrt{c^*_{\rho_1} c^*_{\rho_2}}} \left(\sum_{\gamma\in\mathcal{A}} \frac{(\mathcal{M}^{=}_{\gamma})^2}{w_\gamma Q^{2fa}} \right)^{1/2} \cdot \left(\sum_{\gamma\in\mathcal{A}} w(\beta_\gamma) P^{\#}_{\infty} \|\widehat{K}_{\gamma\#}\|^2_{\mathrm{HS}} \right)^{1/2}.$$

Indeed the comparison of the two weights in (93.6) and the integral injection there apply to the same original labels. Both sums are nonnegative so finite truncations may be increased. The remaining finite local integrals include those for the two full lattice stabilizers subject to the equality test in Lemma 98.19. Their dependence on the nonunit determinants, the archimedean sum, and the lower estimate for the specified integral pairing still enter the period comparison.

98.7 Local factors at divisors of the Gram determinant

The part $\mathcal{M}^{=}_{\gamma}$ in Theorem 98.21 contains local factors at the divisors of $\mathcal{G}$. We evaluate them at full lattice level when the preceding nonzero endpoint minors are units and the last block is a singleton. This includes every place outside the fixed set S_0 where the sole exceptional condition of Section 98.3 is divisibility of the Gram determinant.

Fix a finite place $v \notin S_0$. Write $F_0 = F_v$, $E_0 = E_v$, $O = \mathcal{O}_{F_0}$, $\mathcal{O} = \mathcal{O}_{E_0}$, π for a uniformizer of F_0, and $q = |O/\pi O|$. The residue characteristic is odd, T is a unit, and E_0 is either the split algebra or an unramified quadratic field. Give the full row lattice stabilizer volume one. Use the Haar measure of Theorem 93.14 on $H_C = U(C)$ and give $\mathcal{O}$ volume one in each common unipotent entry. Unit conditions in the split algebra refer to both factors.

Lemma 98.22 (Reduction to one integral vector)**.** *Let the last block have length one. Suppose every nonzero proper endpoint minor D_j, $j < d$, is a unit at v. Then the local mass $L_v(K^*_v)$ of Section 98.6 is the sum, over $H_C(F_0)$-orbits of self-dual lattices Λ in a rank k unramified Hermitian space V which contain a fixed vector w, of $\mathrm{vol}_{H_C}(\mathrm{Stab}(\Lambda))^{-1}$. Here $\mathrm{Stab}(\Lambda)$ denotes its stabilizer in $H_C(F_0)$. The vector has nonzero norm*

$$a_0 = \frac{\mathcal{G}}{T D_{d-1} \overline{D_{d-1}}}.$$

In particular $e = v(a_0) = v(\mathcal{G}) \ge 0$.

Proof. For a local coset supporting both flags, take representatives of both decorated frames in its row lattice. The changes to monomial cross matrix of Lemma 110.10 are integral here.

Every earlier block pivot is a unit since a power of it, up to a sign, is a ratio of unit endpoint minors. Elimination of the off-diagonal entries in the last row and column uses only those earlier pivots. The first $d-1$ vectors of the two frames together now span a unimodular lattice in the sum of their hyperbolic planes. Its Gram matrix has zero diagonal blocks and invertible off-diagonal blocks. The inverse matrix gives an integral orthogonal projection onto this lattice, so it splits off the whole row lattice.

Fix one rationally normalized pair of local frames. All pairs in the same decorations with the same monomial matrix differ by the block unitriangular action of $D(F_0)$, as in Lemma 98.6. Thus by this action the preceding hyperbolic lattice can be taken to be the lattice in the fixed normalized basis. The two remaining vectors z, z' are in its orthogonal complement. They have norm $-T$ and cross pairing $q_d = D_d/D_{d-1}$, since the last block is a singleton. The second vector has integral projection on the first because $q_d \in \mathcal{O}$ and T is a unit. Split off $\mathcal{O}z$ and let w be the perpendicular projection of z' onto the remaining space V. Its norm is $-T + q_d\bar{q}_d/T = a_0$, and $C = w^{\perp}$ inside V.

Conversely every self-dual lattice in V containing w gives an allowed row lattice by adjoining these fixed summands. Indeed the lattice in the remaining middle space contains both z, z', each of unit norm, so each individually splits off an orthogonal unimodular line. For either flag, use its first $d-1$ vectors with dual partners from the other flag obtained by inverting the unit cross matrix. The resulting hyperbolic bases, the chosen terminal vector, and a basis of its unimodular perpendicular give an integral isometry of the corresponding original marked lattice to the proposed lattice. In the unramified field case unimodular lattices of a given rank are integrally isometric. Over the residue field a nonsingular Hermitian form has a vector of nonzero norm, which lifts to split off a unit line. Induction and surjectivity of the norm on units give a basis of unit diagonal ones. The norm on units is surjective by the finite-field norm and Hensel lifting of principal units at odd residue characteristic. For the split algebra use a basis in one factor and the dual basis in the other. These isometries preserve the individual ordered decorations, which is precisely the two support conditions in X_v.

The subgroup $D(F_0)$ acts on the preceding hyperbolic summand and fixes the entire middle space because its final block has length one. The stabilizer of the chosen hyperbolic lattice in $D(F_0)$ consists of the integral block unitriangular matrices. Indeed the induced matrices on the other frame are their adjoint inverses conjugated by monomial blocks of units and are also integral. This subgroup has volume one. The other direct factor of the common stabilizer acts on V by $H_C(F_0)$, fixing w. This gives the asserted orbit masses, including for nonsingleton blocks earlier in the frame. □

Proposition 98.23 (The split local factor)**.** *Under the assumptions of Lemma 98.22, if* $E_0 = F_0 \times F_0$ *then*

$$L_v(K_v^*) = \sum_{i=0}^{e} q^{(k-1)i}.$$

Proof. A self-dual lattice in V consists of an O-lattice L in F_0^k and its dual. Write $w = (u, \eta)$, with u a vector and η a covector, so that $\eta(u) = a_0$. The two integrality conditions are $u \in L$ and $\eta(L) \subset O$. Put $C_+ = \ker \eta$. The stabilizer of the pair u, η acts as the full $\mathrm{GL}(C_+)$. It moves $L \cap C_+$ to a fixed lattice L_0 in C_+. Projection of L onto $F_0 u$ along C_+ has image $\pi^{-i}Ou$ for some $0 \le i \le e$, by the two conditions. Over this image L is generated by L_0 and a vector $\pi^{-i}u + y$. Exactly the classes $y \in \pi^{-i}L_0/L_0$ work, since $\pi^i y \in L_0$ is equivalent to $u \in L$. The compact group $\mathrm{GL}(L_0)$ has volume one and acts on this finite set. Each stabilizer has inverse volume equal to the length of its orbit. Summing therefore gives $q^{(k-1)i}$ at i, as claimed. □

Lemma 98.24 (The nonsplit masses at valuations zero and one)**.** *Let E_0/F_0 be the unramified quadratic field and let V admit a self-dual lattice. For a vector of unit norm the mass in Lemma 98.22 is one. For a vector u with $v(h(u,u)) = 1$ it is $q^{k-1}+1$.*

Proof. At unit norm the line splits from any containing self-dual lattice, and the perpendicular is self-dual. All such lattices form one orbit with stabilizer of volume one. For valuation one put $a = h(u,u)$. Every containing lattice has u primitive. In it choose a vector v_0 with $h(u,v_0) = 1$, by self-duality. There is a $b \in O$ such that $v_0 + bu$ is isotropic. Its norm is $h(v_0,v_0) + 2b + ab^2$, which has a simple root modulo π. The new pairing with u is a unit. Divide by it to get an isotropic v_1 with $h(u,v_1) = 1$. The plane on u, v_1 has unit Gram determinant, so the lattice splits as this plane and a unimodular lattice Λ_0 of rank $k-2$. All such marked lattices form one orbit by integral isometry of the remaining lattices. Such a lattice exists for any norm of valuation one. A unit diagonal plane contains a hyperbolic integral basis since -1 is a unit norm and 2 is invertible. In that basis $(1, a/2)$ has norm a. Matching this vector with the given one by Hermitian Witt extension gives a lattice in the required marked space.

Put $y = u - av_1$, of norm $-a$, and $L = \mathcal{O}y \perp \Lambda_0$. This is the intersection with $C = u^{\perp}$. On C use the scaled form $h' = \pi^{-1}h$. It gives the hyperspecial volume convention of Theorem 93.14. In fact the valuation of $\det h|_C$ is odd, since the determinant of V has even valuation and u has norm of valuation one. The two scalings therefore differ by a scalar of even valuation. This scalar is a norm, and scalar multiplication transports self-dual lattices while preserving their stabilizers. At unit norm the same argument applies with $h' = h$.

For the form h' one has $L^{\vee} = \mathcal{O}y \perp \pi\Lambda_0$. Put $P = \mathrm{Stab}_{U(C)}(L)$. The stabilizer of the original lattice in $U(C)$ is the subgroup satisfying $gy \equiv y \bmod aL$, since $v_1 = (u-y)/a$ and the original lattice is generated by L, v_1. Any $g \in P$ preserves $L^{\vee}$, so $gy \equiv \xi y \bmod \pi L$ for a scalar ξ whose norm is one in the residue field. This follows by evaluating its unit norm for h', since the $\pi\Lambda_0$ part has norm divisible by π. Every such scalar lifts to a norm-one unit acting on the y-line. Indeed lift the scalar arbitrarily, and correct its norm in $1 + \pi O$ by a principal unit. The index of the stabilizer in P is $q+1$.

It remains to determine $\mathrm{vol}(P)$. Write $t = (k-2)/2$, an integer since the column rank is even. Self-dual lattices M between $L^{\vee}$ and L correspond to maximal isotropic t-planes in the nonsingular $2t$-dimensional residue space $L/L^{\vee}$ with form $\pi h' \bmod \pi$. The condition $M^{\vee} = M$ means that the plane equals its perpendicular in this space. The group P is transitive on the choices, since the integral unitary group on Λ_0 induces its full residue unitary group. For a fixed M, the conjugate lattices L' containing M in its $\mathrm{Stab}(M)$-orbit correspond to the isotropic t-planes $\pi L'/\pi M$ in the $(2t+1)$-dimensional nonsingular space $M/\pi M$ with form $h' \bmod \pi$. Indeed $M \subset L' \subset \pi^{-1}M$ is determined by this plane, and $\pi h'(L', L') \subset \mathcal{O}$ gives isotropy. These planes all occur by transitivity. In both uses finite-field transitivity follows by extending an isotropic basis to a Witt basis. Reduction of the integral unitary group of a unimodular lattice is onto. If Y lifts a residue isometry, then $Y(Y^*Y)^{-1/2}$ is a unitary lift. The binomial series converges because $Y^*Y - 1$ vanishes modulo π and the denominators of its coefficients are powers of two.

For clarity, let $N_{n,t}$ count isotropic t-planes in a nonsingular residue Hermitian space of dimension n. The number of nonzero isotropic vectors is

$$z_n = (q^n - (-1)^n)(q^{n-1} - (-1)^{n-1}).$$

To count, sum a nontrivial additive character of the residue field over the norm form in a diagonal unit basis. The sum in one coordinate is $-q$, since a nonzero norm has $q+1$ preimages. Averaging over all additive characters gives $q^{2n-1} + (-1)^n(q-1)q^{n-1}$ vectors of norm zero, including

zero. Counting an isotropic line in an isotropic t-plane now gives $N_{n,t} = z_n N_{n-2,t-1}/(q^{2t}-1)$ for $t \geq 1$. Consequently

$$\frac{N_{2t,t}}{N_{2t+1,t}} = \frac{\prod_{j=1}^{t}(q^{2j-1}+1)}{\prod_{j=1}^{t}(q^{2j+1}+1)} = \frac{q+1}{q^{2t+1}+1}.$$

The ratio also holds at $t=0$. The two indices over $P \cap \mathrm{Stab}(M)$ calculated above show that this ratio is $\mathrm{vol}(P)$, because $\mathrm{vol}(\mathrm{Stab}(M)) = 1$. Dividing by $q+1$ gives the volume of the desired stabilizer. Its inverse is $q^{2t+1}+1 = q^{k-1}+1$. □

Theorem 98.25 (The factor at every terminal Gram valuation)**.** *For both splitting types under Lemma 98.22,*

$$L_v(K_v^*) = 1 + q^{k-1} + \cdots + q^{(k-1)e}. \tag{98.18}$$

Proof. Proposition 98.23 proves the split case. In the field case fix a reference self-dual lattice Λ of V and give its stabilizer in $U(V)$ volume one. On the surface $\Sigma_a = \{u : h(u,u) = a\}$ for $a \neq 0$, the group is transitive by Witt extension. The mass in the lemma for the vector u is the quotient measure of $\{g : u \in g\Lambda\}$ in $U(u^{\perp})\backslash U(V)$. Indeed each orbit of lattice cosets has measure equal to the inverse volume of its stabilizer. The map $g \mapsto g^{-1}u$ therefore gives this mass as $\mu_u(\Sigma_a \cap \Lambda)$ for the corresponding invariant measure on Σ_a.

There is a second invariant measure σ_a on the same surface. Give Λ additive volume one, and normalize additive measure on F_0 by $\mathrm{vol}(O) = 1$. For a compact open set $D' \subset V$, its surface measure on D' is

$$\lim_{n\to\infty} q^n \,\mathrm{vol}\{z \in D' : h(z,z) - a \in \pi^n O\}.$$

The derivative in direction y is $\mathrm{Tr}\, h(z,y)$, surjective onto F_0 at every point of Σ_a. By the local inverse function theorem the norm is one coordinate on a sufficiently small box around each such point, with constant absolute Jacobian there. Finitely many disjoint small boxes cover the compact intersection in D'. The compact remainder avoids the surface, hence makes zero contribution for large n. This proves the formula. The measure is positive and invariant since unitary linear maps preserve additive measure on V. Under multiplication by π one has $\sigma_{\pi^2 a}(\pi D') = q^{-2(k-1)}\sigma_a(D')$, by scaling the ambient $2k$-dimensional measure and the norm coordinate. In this formula intersection with the relevant surface is understood. By uniqueness of invariant quotient measure, write $\mu_u = c_u \sigma_a$ with $c_u > 0$. The stabilizers and their Haar measures for $u, \pi u$ coincide, so $\mu_{\pi u}$ is the image of μ_u under multiplication by π. It follows that $c_{\pi u} = q^{2(k-1)} c_u$.

For a unit target and for a target divisible by π, the primitive part of $\sigma_a(\Lambda)$ equals respectively

$$H_1 = 1 - q^{-k}, \qquad H_0 = 1 + (q-1)q^{-k} - q^{1-2k} = (1-q^{-k})(1+q^{1-k}).$$

Indeed for each nonzero residual vector in the specified fibre the trace derivative is onto the residue field. Each lift at level n therefore has q^{2k-1} lifts at level $n+1$. The limiting density is its residue count divided by q^{2k-1}. The zero-norm count is z_k in Lemma 98.24. All nonzero target norms occur equally often by multiplying by units, so each has $q^{2k-1} - q^{k-1}$ preimages. Here k is even.

Set $A = q^{k-1}$ and $\delta = e \bmod 2$ in $\{0,1\}$. The vector $u_0 = \pi^{-(e-\delta)/2} w$ has norm valuation δ. At valuations zero and one integral vectors are primitive. Lemma 98.24 therefore gives $c_{u_0} = 1/H_1$ if $\delta = 0$ and $c_{u_0} = (A+1)/H_0$ if $\delta = 1$. By the scaling rule $c_w = A^{e-\delta} c_{u_0}$. For the additive density at w, decompose its integral vectors according to their exact divisibility by π^j. The range is $0 \leq 2j \leq e$ and the contribution is $A^{-2j}H_0$ when $2j < e$, or $A^{-2j}H_1$ at

$2j = e$. For $e = 2h$, multiplication by c_w gives $1 + (1 + A^{-1}) \sum_{j=1}^{h} A^{2j}$. For $e = 2h + 1$ it gives $(A + 1) \sum_{j=0}^{h} A^{2j}$. Both expressions equal $\sum_{i=0}^{e} A^i$, proving the formula. □

Corollary 98.26 (The product at primes dividing only the Gram determinant)**.** *For any supported relevant label with $\mathcal{G} \neq 0$, let $\mathcal{U}_\gamma$ be the set of $v \notin S_0$ where every nonzero endpoint minor, including the final one, is a unit. Define the integral ideal $\mathfrak{b}_\gamma = \prod_{v \in \mathcal{U}_\gamma} \mathfrak{p}_v^{v(\mathcal{G})}$. For the marked-period local factors of Section 98.6,*

$$\prod_{v \in \mathcal{U}_\gamma} L_v^{=} = (\mathrm{N}\mathfrak{b}_\gamma)^{k-1} \sum_{\mathfrak{a} \mid \mathfrak{b}_\gamma} (\mathrm{N}\mathfrak{a})^{1-k}.$$

The divisor sum is between 1 and $1 + O(f 2^{-k})$, uniformly in the rational label on the chosen intervals.

Proof. At these unmarked places $L_v^{=} = L_v(K_v^*)$. When the last block has length greater than one, $D_{d-1} = 0$. Then $\mathcal{G} = D_d \overline{D_d}$ is a unit on this set, so the factor is one by Proposition 98.7. When the last block is a singleton, use Theorem 98.25. Factor $q^{(k-1)e}$ from each local polynomial and multiply over v, using unique factorization of ideals. For $k > 2$ the divisor sum is bounded by $\zeta_F(k-1) \le \zeta(k-1)^f$ by the Euler products, since above a rational prime p there are at most f places, each of norm at least p. Comparison with the integral beyond the first term gives $\zeta(k-1) - 1 = O(2^{-k})$ for large k. Since f is polynomial in x and $k \asymp x^{9/2+10\epsilon}$, $f 2^{-k} \to 0$. This proves the uniform conclusion. □

In particular the local product records the full valuations of the Gram ideal. Even a single factor with $e = 1$ contributes $1 + q^{k-1}$ to the finite mass. The analytic comparison must account for this factor together with the real density in (93.7). Primes of nonunit endpoint minors and the fixed finite levels can carry further factors.

98.8 The local mass of an ordered Hermitian frame

Let K be a finite extension of $\mathbb{Q}_p$, with integer ring O, uniformizer π, normalized valuation v, and residue field of order q. Let A be the unramified quadratic field over K or $K \times K$. Write $\mathcal{O}$ for its integer ring, $x \mapsto \bar{x}$ for its involution, and $\eta = -1$ or 1, respectively. Give the integer lattices in additive coordinates volume one. Let Λ be a self-dual Hermitian lattice of rank r, with form H. We use n columns, put $h = r - n$, and assume $n \ge 1$, h odd, and $m = h - n + 1 \ge 2$. In the application $n = 2d$ and $h = k - 1$. A tuple is *primitive* if its columns give a split injection $\mathcal{O}^n \to \Lambda$, or equivalently are independent after reduction in each residue field of A.

For a nonsingular Hermitian matrix S, let $\Sigma_S = \{X : X^* H X = S\}$. We use it when nonempty. On this space let μ_S be quotient measure for the ambient unitary group by the pointwise stabilizer of a frame. Give the stabilizer of Λ volume one. In the perpendicular space of dimension h, scale the form by the inverse of its determinant and give a self-dual lattice stabilizer volume one, as in Theorem 93.14. A change of basis multiplies this scaling by a norm, so it gives the same measure. Define

$$P_0 = \prod_{j=h+1}^{r} (1 - \eta^j q^{-j}), \qquad p(h, u) = \prod_{j=h-u+1}^{h} (1 - \eta^j q^{-j}).$$

Empty products are one. For a compact open test in the ambient matrix space, define σ_S on its intersection with Σ_S by the limit of the additive volumes of $X^* H X \equiv S \bmod \pi^j$, multiplied by $q^{j n^2}$. Here the target lattice consists of integral Hermitian matrices. The derivative $Y \mapsto$

$X^*HY + Y^*HX$ is onto over K at every point of Σ_S. Indeed X^*H is onto and the trace $A \to K$ is onto. The local inverse function theorem gives coordinates consisting of the Gram entries and complementary coordinates. On small compact open coordinate balls the absolute Jacobian is constant. A finite covering of the test near its compact intersection with the fibre proves that the limit exists and gives an invariant measure there.

Lemma 98.27 (Primitive frames)**.** *If S is integral and its reduction has radical dimension u, the σ_S-measure of its primitive frames is $P_0p(h,u)$. In the split algebra the two component matrices have the same rank.*

Proof. First count over the residue field and divide by q^{2rn-n^2}. In the split case choose an injection of the first n-space into the r-space. The number is $q^{rn}\prod_{j=h+1}^{r}(1-q^{-j})$. The second map, written as a map from the r-space onto an n-space, is prescribed on the first image. In the u-space quotient by its prescribed image, the remaining h-space must map onto. Of its q^{hn} unrestricted extensions the fraction doing so is $\prod_{j=h-u+1}^{h}(1-q^{-j})$, by counting independent rows. This proves the formula since $rn + hn = 2rn - n^2$.

In the field case diagonalize the residue form on the source into $n-u$ unit lines and a zero block of size u. A nonsingular residue Hermitian space has an orthonormal basis. Indeed, vanishing of all norms would imply vanishing of all pairings by polarization with arbitrary residue scalars and nondegeneracy of the trace pairing. Split off a vector of nonzero norm and use surjectivity of the residue norm on units. In dimension j the number of vectors of norm one is $q^{j-1}(q^j-(-1)^j)$, and the number of nonzero isotropic vectors is $z_j = (q^j - (-1)^j)(q^{j-1}-(-1)^{j-1})$. Both follow by additive-character orthogonality, since the sum of a nontrivial additive character of the norm in one coordinate is $-q$. Choose the unit vectors in successive perpendiculars. In the remaining space of dimension $h+u$, after choosing i independent isotropic vectors with pairings zero, the next one has $q^{2i}z_{h+u-2i}$ choices. It lifts a nonzero isotropic vector in the perpendicular quotient by their span. The unit-vector factors give normalized factors indexed from $h+u+1$ through r. The isotropic-vector factors give the adjacent pairs of indices $h+u-2i, h+u-2i-1$ for $0 \le i < u$. Their union is the interval from $h-u+1$ through r. The leading power of q is q^{2rn-n^2}, obtained by multiplying the leading powers $q^{2(r-i)-1}$ at steps $0 \le i < n$. This proves the residue count.

For a primitive residue tuple the map $Y \mapsto X^*HY$ is onto all n-square matrices, by a left inverse to the columns and self-duality. The map $B \mapsto B + B^*$ is onto Hermitian matrices, using the residue trace on the diagonal. Each solution therefore has exactly q^{2rn-n^2} lifts at each successive digit for the prescribed target. This argument also applies in residue characteristic two. Taking the defining limit proves the assertion. The residue counts correspond to `Formalism/LocalToGlobal/SiegelLatticeTheory/SiegelMassRatios/DensExact.lean`, line 22 and `Formalism/LocalToGlobal/SiegelLatticeTheory/SiegelMassRatios/DensExact.lean`, line 32. □

Proposition 98.28 (Quotient measure on the Gram fibre)**.** *For every represented nonsingular S,*

$$d\mu_S = P_0^{-1} q^{hv(\det S)}\, d\sigma_S.$$

Proof. Witt extension is transitive on Σ_S, so the two measures are proportional. For $P \in \mathrm{GL}_n(A)$ right multiplication by P transports the quotient measure unchanged, since it preserves the span and its pointwise stabilizer. On the additive matrix space its absolute Jacobian is $|N_{A/K}\det P|_K^r$. On the Hermitian target the map $B \mapsto P^*BP$ has Jacobian $|N_{A/K}\det P|_K^n$. For diagonal P multiply the diagonal and off-diagonal Jacobians. Integral invertible factors preserve the target lattice, so elementary divisors give the result for every P, separately in the

two factors in the split case. The fibre measure of a transported test is therefore multiplied by $|N_{A/K} \det P|_K^h$. This cancels the change of the factor $q^{hv(\det S)}$.

It remains to calibrate one target in each congruence class. The split algebra has a single nonsingular class. Over the unramified field there are two, represented by I_n and $\operatorname{diag}(I_{n-1}, \pi)$. To see this, diagonalize over the field. Every unit is a norm by residue norm surjectivity and successive lifts using the surjective residue trace. Reduce each valuation to zero or one. Two diagonal entries π, π make a hyperbolic plane, as do $1, 1$, since -1 is a norm. At I_n every integral frame is primitive and splits off a unimodular summand. Unimodular Hermitian lattices of the same rank are isometric, by unit-vector diagonalization and unit norms in the field case and dual bases in the split case. The compact orbit of integral frames thus has quotient volume one and additive fibre measure P_0.

At the other target every integral frame is again primitive. Otherwise its saturation would decrease the determinant valuation by at least two. Splitting off the $n-1$ unit columns leaves the choice of a vector of norm π in rank $h+1$. Lemma 98.24 gives this marked lattice orbit mass as q^h+1. The stabilizer of the unit columns has volume one in that intermediate group, by the same unimodular splitting. Here are the details extending the lemma to residue characteristic two. A primitive vector u_0 of norm π has an integral partner v_0 pairing to one. Choose $\lambda \in \mathcal{O}$ of unit trace. The norm of $v_0 + b\lambda u_0$, for $b \in O$, is

$$H(v_0, v_0) + b \operatorname{Tr}(\lambda) + \pi b^2 N_{A/K}(\lambda).$$

It has a simple root modulo π, which lifts. Rescale the new partner by a unit to restore pairing one. Existence follows in an integral hyperbolic plane by taking $(1, z)$ with $\operatorname{Tr}(z) = \pi$. Such a plane splits from every unimodular lattice of rank at least two by the unit norm of -1, self-duality and the surjective integral trace. A residue unitary isometry lifts a digit at a time using surjectivity of $B \mapsto B + B^*$ on residue Hermitian matrices. These facts supply the partner and reduction steps in Lemma 98.24. Its two finite isotropic-plane indices give the volume $(q+1)/(q^h+1)$ of the perpendicular lattice stabilizer used there. Fixing the marked generator has index $q+1$, giving the asserted orbit mass at every q. The perpendicular scaling by π^{-1} in that calculation differs from the determinant scaling by a norm since its determinant has odd valuation. By Lemma 98.27 the additive fibre measure at this target is $P_0(1+q^{-h})$, since h is odd. The ratio is q^h/P_0, as required. □

Theorem 98.29 (Integral frames in the stable range)**.** *For integral nonsingular S, let $M_0 = \mathcal{O}^n$ with form S, and let $M_0^\vee$ denote its Hermitian dual. Then*

$$q^{-hv(\det S)} \mu_S(\Sigma_S \cap \Lambda^n) = \sum_{\substack{M_0 \subset M' \subset M_0^\vee \\ S(M', M') \subset \mathcal{O}}} [M' : M_0]^{-h} p(h, u(M')).$$

Here $[M' : M_0]$ is cardinality of the quotient and $u(M')$ is the residue radical dimension in a basis of M'. For any measurable subset $\mathcal{D}$ of the integral frames containing all primitive frames and an absolute constant C,

$$e^{-Cq^{-m}} \le q^{-hv(\det S)} \mu_S(\mathcal{D}) \le e^{Cq^{-m}}. \tag{98.19}$$

Proof. For each integral tuple X, the lattice of $w \in A^n$ with $Xw \in \Lambda$ is the unique M' on which the induced tuple is primitive. The form is integral on M', hence $M' \subset M_0^\vee$. There are only finitely many such lattices. Write $M' = PM_0$, so $|N_{A/K} \det P|_K = [M' : M_0]$. The change $X \mapsto XP$ identifies this stratum with all primitive frames for P^*SP. The quotient measure is unchanged, and the determinant valuation factor is multiplied by $[M' : M_0]^{-h}$. The proposition and the lemma give the sum.

The term $M' = M_0$ gives the lower bound, since $|\log p(h,u)| \le C_1 q^{-m}$ by summing the geometric tail beginning at $h-n+1$. For the upper bound sum $[M' : M_0]^{-h}$ over all finite over-lattices. Over a discrete valuation ring of residue order q_0 the sum is $\prod_{i=0}^{n-1}(1-q_0^{i-h})^{-1}$. Indeed duality turns it into the sum over sublattices. A triangular normal form with diagonal $\pi^{b_1},\dots,\pi^{b_n}$ has $q_0^{\sum_{i=1}^{n}(n-i)b_i}$ possibilities, by reducing each entry to the right of its diagonal entry modulo that entry. Independent geometric sums give the product. Use $q_0 = q^2$ in the field case, and two such sums with $q_0 = q$ in the split case. Its logarithm is at most $C_2 q^{-m}$, since $m \ge 2$. This proves both estimates uniformly. An over-lattice expansion in the nonsplit case also occurs in the formula of Cho and Yamauchi given by Li and Zhang, Theorem 3.5.1. □

98.9 The finite orbital masses of the original period

Return to the fields, intervals and row levels of Section 98.6. Normalize the local row measure by $\mathrm{vol}(K_v^*) = 1$, using this same convention in the identity terms. The row lattice has diagonal entries ± 1 before its conjugation at T. After conjugation the two split factors there are mutually dual. Thus at every finite place K_v^* is the full stabilizer of a self-dual lattice Λ_v. Our E/F is unramified at all finite places, as in Section 93.4. Put

$$h = k-1, \quad n = 2d, \quad m = h-2d+1, \quad S_* = d(d-1).$$

Here $m \ge 2$ on the intervals under consideration for large x, since $k \gg d^2$. For the two frames Y_1, Y_2 take representatives in the monomial normal form of Lemma 110.10. Their two self pairings are $H_0 = -TE_{dd}$ and their cross pairing is M, with signed reversed blocks M_ν of length h_ν, parameter c_ν, and endpoint j_ν. Assume throughout that the rational label γ is relevant and supported and that $\mathcal{G} \neq 0$. Its $2d$ vectors span a nondegenerate space with perpendicular C.

At v put $A = E_v$, $\mathcal{O} = \mathcal{O}_{E_v}$, $q = q_v$ and $\mathcal{U} = U_d(A) \times U_d(A)$, where U_d is upper unitriangular. Normalize $\mathcal{U}$ by giving the integral subgroup volume one. Its subgroup D preserving M has the measure in Lemma 98.6, with unit integral entry volumes in its first frame. In $D\backslash\mathcal{U}$ put $Z = U^* M V$ at the class of (U,V). Let V_v be the quotient volume where Z is integral. Put $V_v^{=} = V_v$ away from $Q = g_1 g_2$. At $v \mid Q$ define $V_v^{=}$ by imposing both congruences

$$Z - H_0 \in \pi M_d(\mathcal{O}), \qquad 2H_0 - Z - Z^* \in \pi^2 M_d(\mathcal{O}). \tag{98.20}$$

Lemma 98.30 (Integral representatives of one decorated flag)**.** *A full row lattice admits a decorated flag precisely when some representative is integral and primitive. Such representatives form one right orbit under $U_d(\mathcal{O})$, acting freely.*

Proof. It suffices to use the reference lattice, by translation. Theorem 86.1 gives a primitive integral representative for the reference flag. Conversely take an integral primitive representative with the same self pairing. The first $d-1$ vectors are primitive isotropic columns. Unimodularity supplies integral dual partners with pairings the identity. Adding combinations of the isotropic vectors makes the partners mutually isotropic, since $B \mapsto B + B^*$ is onto the integral Hermitian matrices, including at two. Split off these hyperbolic planes. The terminal vector is orthogonal to the isotropic columns. Subtract its integral components along them, an allowed change of representative. Its image in the middle lattice is primitive by independence of the original columns after reduction.

At a nonsplit place its norm $-T$ is a unit, since T splits completely. Its line splits off. The remaining self-dual lattice is determined up to isometry by its rank, by the unit-vector argument in Proposition 98.28. At a split place the middle vector is a primitive vector and a primitive covector of fixed pairing $c = -T$. Make the vector the first basis vector and write the covector

as (c,t). If c is a unit, shears fixing the vector remove t. If c is a nonunit, t is primitive, so a basis change on the other coordinates makes it $(1,0,\dots,0)$. These integral normal forms show that the flag belongs to the reference lattice orbit, also at T. Finally a primitive integral matrix has an integral left inverse. Applying it to another integral representative shows that their change of basis is integral. Every integral unitriangular change is allowed, and independence of the columns gives freeness. □

Proposition 98.31 (Reduction of the finite mass to a matrix volume)**.** *Write $\delta_v = 1$ at $v \mid Q$, and 0 elsewhere. With the measures just chosen, the mass of compatible full-level cells satisfies*

$$L_v^{=} = q^{hv(\mathcal{G})+\delta_v(S_*-2dh)} V_v^{=} \exp(O(q^{-m})). \tag{98.21}$$

The implied constant is absolute. The same formula with $\delta_v = 0$ and V_v gives $L_v(K_v^*)$ at every place.

Proof. For g in the local row group set $(X_1, X_2) = (g^{-1}Y_1U, g^{-1}Y_2V)$. At full level integrate the indicator that both individual tuples are primitive and integral, first over $\mathcal{U}$. For each supported pair in $g\Lambda_v$ the integral is one, by the single coset of representatives for each flag in Lemma 98.30. For all other pairs it vanishes. Integration over $J\backslash G$ now gives $L_v(K_v^*)$, since each lattice orbit has measure $\mathrm{vol}(K_v^*)/\mathrm{vol}(J \cap gK_v^*g^{-1})$.

We may instead integrate over $D\backslash\mathcal{U}$ on the outside and over $U(C)\backslash G$ on the inside. Indeed $J = D \times U(C)$ by Lemma 98.6. If a change of two representatives is left multiplied by the image of $d_0 \in D$, replace g in the integrand by $d_0^{-1}g$. The quotient integration formula then absorbs the D-integral in the G-integral. All measures are Haar quotient measures of unimodular groups, and the integrands are nonnegative. For fixed U, V the inner integral is exactly μ_B on the fibre with Gram matrix

$$B = \begin{pmatrix} H_0 & Z \\ Z^* & H_0 \end{pmatrix}.$$

Its determinant is $(-1)^d\mathcal{G}$. Its test includes all jointly primitive integral frames and is contained in the set of all integral frames. Integrality of B is equivalent to integrality of Z. Integrating (98.19) over these U, V gives the full-level assertion.

At a marked place Lemma 98.19 tests whether the two reduced decorated flags agree in the full-level lattice. The stabilizer of the reduced decoration is precisely its $\bar{B}$ there. Indeed preserving the two split isotropic strings with fixed graded vectors gives the block unipotent radical, and fixing the terminal vector and covector on the middle quotient leaves GL_k. For equal decorations the fraction of pairs of integral representatives satisfying $X_1 \equiv X_2 \bmod \pi$ is q^{-S_*}. Once the first reduction is fixed, exactly one of the $|U_d(\mathcal{O}/\pi)| = q^{d(d-1)}$ changes of the second reduction matches it, by freeness and surjectivity of reduction. For different decorations that fraction is zero. Multiply the test of equal reduced representatives by q^{S_*} to obtain the indicator of compatibility in the integral above.

In its inner integral now use the ordered tuple $X_1, (X_2 - X_1)/\pi$. Its Gram matrix has off-diagonal block $(Z - H_0)/\pi$ and final diagonal block $(2H_0 - Z - Z^*)/\pi^2$. Its determinant valuation is $v(\mathcal{G}) - 2d$. The field change of basis preserves the quotient measure by Proposition 98.28. The new test is that both tuples be integral and the first primitive. Indeed the old second tuple then reduces to the old first and is primitive as well. The new test again lies between all jointly primitive frames and all integral frames. Theorem 98.29 applies exactly on (98.20). This proves the marked assertion. □

Theorem 98.32 (A uniform bound at all finite places)**.** *Set*

$$a_\nu = \binom{h_\nu}{2}, \quad a = \sum_\nu a_\nu, \quad b_\nu = 2h_\nu(d - j_\nu) + a_\nu, \quad W_\gamma = \prod_\nu |N_{E/\mathbb{Q}}c_\nu|^{-a_\nu}.$$

Let P_∞ be the product of real densities in Lemma 98.15 and $f=[F:\mathbb{Q}]$. Then

$$\mathcal{M}_\gamma^{=}=\mathfrak{m}_h|N_{F/\mathbb{Q}}\mathcal{G}|^h Q^{f(S_*-2dh)}\Big(\prod_{v<\infty}V_v^{=}\Big)\exp\big(O(f2^{-m})\big), \tag{98.22}$$

$$\frac{\mathcal{M}_\gamma^{=}}{\mathfrak{m}_h P_\infty}\le \exp\big(O(f2^{-m})\big)Q^{f(2a-S_*-2dh)}W_\gamma. \tag{98.23}$$

The constants are independent of the label and of the selected field. At the regular points, where every block has size one, $W_\gamma=1$.

Proof. By definition $\mathcal{M}_\gamma^{=}=\mathfrak{m}_h\prod_v L_v^{=}$. Each of g_1,g_2 has f split places of residue order the corresponding rational prime. The product of the Gram factors in (98.21) is the indicated ideal norm. Also $\sum_v q_v^{-m}\le f\sum_{j\ge2}j^{-m}\ll f2^{-m}$ for $m\ge2$. This gives (98.22).

We bound the remaining volume in explicit coordinates. Write $|c|_A=|N_{A/F_v}c|_v$. In $Z=LMV$, with $L=U^*$ lower unitriangular, choose the representative of $D\backslash\mathcal{U}$ having all diagonal blocks of L equal to the identity. This leaves every lower off-block entry and all upper entries of V free. These coordinates give additive product quotient measure. In fact write $L=L_0L_b$ with L_b its block diagonal part. Right multiplication by its blocks has determinant one on lower off-diagonal blocks. The accompanying change in V is a left translation, and involution and inversion have modulus one on the unipotent block coordinates of D.

Expose row and column block ν in order, keeping previous rows and columns fixed. The diagonal block of Z is a known matrix plus $M_\nu V_{\nu\nu}$. Each of its a_ν free entries is multiplied by $\pm c_\nu$ in a distinct position, so integrality gives volume at most $|c_\nu|_A^{-a_\nu}$. For $j>\nu$ the new row block has variable part $M_\nu V_{\nu j}$. For $i>\nu$ the new column block has variable part $L_{i\nu}M_\nu V_{\nu\nu}$. In each case the known summand uses only previously exposed coordinates. The factor $V_{\nu\nu}$ has determinant one. The volumes for all these rectangular blocks therefore contribute $|c_\nu|_A^{-2h_\nu(d-j_\nu)}$. At a marked place the first congruence of (98.20) prescribes a residue class in each of the same free output entries. Each such class has volume q^{-2} in the quadratic integer algebra. Dropping the second congruence gives the bounds

$$V_v^{=}\le q^{-2\delta_v(S_*-a)}\prod_\nu|c_\nu|_A^{-b_\nu},\qquad \sum_\nu b_\nu=S_*-a.$$

At almost all places all c_ν are units and integral free coordinates give volume one. The same bound is then one, so the product of the $V_v^{=}$ is finite after omitting its unit factors.

The finite product formula gives $\prod_{v<\infty}|c_\nu|_A^{-1}=|N_{E/\mathbb{Q}}c_\nu|$. By Lemma 98.15, with $e_\nu=2(d-j_\nu)+h_\nu-1$,

$$P_\infty=|N_{F/\mathbb{Q}}\mathcal{G}|^h\prod_\nu|N_{E/\mathbb{Q}}c_\nu|^{h_\nu e_\nu}.$$

Since $b_\nu-h_\nu e_\nu=-a_\nu$, division proves (98.23). This cancellation includes primes dividing several nonzero endpoint minors simultaneously. □

Corollary 98.33 (Finite measure in the squared real kernel)**.** *Write $R=|\Delta_E|^{1/(2f)}$, and use the original and transported kernels of Corollary 93.12. Then*

$$\mathcal{M}_\gamma^{=}\,c_E^{2a}\|K_{\infty,\gamma}^{\mathrm{area}}\|_{\mathrm{HS}}^2\le\mathfrak{m}_h\exp\big(O(f2^{-m})\big)R^{-2fS_*}Q^{f(d(d-3)-2dh)}\\ \times W_\gamma P_\infty^{\#}\|\widehat{K}_{\gamma^\#}\|_{\mathrm{HS}}^2. \tag{98.24}$$

For the integral relabeling of Lemma 93.10, put $B_\gamma^{\#}=|N_{F/\mathbb{Q}}\mathcal{G}^{\#}|\prod_\nu|N_{E/\mathbb{Q}}D_{j_\nu}^{\#}|$. Then

$$W_\gamma\le Q^{f(d-1)}(B_\gamma^{\#})^{(d-1)/2}.$$

Proof. Square (93.7) in Hilbert–Schmidt norm and multiply both sides by $\mathcal{M}_{\gamma}^{=}/P_{\infty}$. The exponent of Q after (98.23) is $f(2d(d-2)-S_{*}-2dh)=f(d(d-3)-2dh)$. For the weight let $A_{j}=|N_{E/\mathbb{Q}}D_{j}|$, $A_{0}=1$. Since $|N_{E/\mathbb{Q}}c_{\nu}|^{h_{\nu}}=A_{j_{\nu}}/A_{j_{\nu-1}}$, the exponent of each proper endpoint in W_{γ} is $(h_{\nu+1}-h_{\nu})/2$, and that of the final endpoint is $-(h_{l}-1)/2$. Every nonzero A_{j} is an integer at least one. Drop the negative exponents and bound the others by $(d-1)/2$. In (93.6) the reindexed endpoint norm is at least A_{j} for $j\geq 2$. At $j=1$ it is $Q^{-2f}A_{1}$. The reindexed endpoint and Gram norms are positive integers, proving the bound. This norm weight is defined in `Formalism/SharpRestriction/CuspSpanNormBounds/LeadMinorK.lean`, line 164. □

For unit column tuples the proof of Theorem 98.21 bounds the absolute period sum for a fixed label by $A_{*}\mathcal{M}_{\gamma}^{=}c_{E}^{a}\|K_{\infty,\gamma}^{\mathrm{area}}\|_{\mathrm{op}}/\sqrt{c_{\rho_{1}}^{*}c_{\rho_{2}}^{*}}$. Formula (98.23) supplies its finite mass at every place. Formula (98.24) controls one power of that mass times a squared kernel. Taking a square of the sum over cells introduces products from different cells. A bound on the sum and a lower bound for the designated integral pairing still enter the comparison in Volume I.

98.10 The finite measure above each rational label

In Corollary 110.17 the finite cells over one γ all have the same real unipotent stabilizer D_{∞}, but can have different rational lattices in that group. On the same fields E_{x} and at the level K^{0} these lattices and their multiplicities admit the following mass formula.

Theorem 98.34 (Finite cell measure on the period fields)**.** *Fix $\gamma\in\mathcal{R}$ of Theorem 110.16. Let W be the span of its two frames, $C=W^{\perp}$, $H_{C}=U(C)$, and $a=\sum_{\nu}\binom{h_{\nu}}{2}$ for its block lengths h_{ν}. Write $J=B\cap\gamma B\gamma^{-1}$, $D=N\cap\gamma N\gamma^{-1}$, and $K_{v}=K_{v}^{0}$. Set*

$$X_{v}=(B_{v}K_{v}\cap\gamma B_{v}K_{v})/K_{v},\qquad L_{\gamma,v}=\sum_{tK_{v}\in J(F_{v})\backslash X_{v}}\frac{\mathrm{vol}(K_{v})}{\mathrm{vol}_{J_{v}}(J(F_{v})\cap tK_{v}t^{-1})}.$$

All these local sums are finite. For X_{f} the restricted product of the X_{v} and $\Gamma_{t}=J(F)\cap tK^{0}t^{-1}$, the total finite measure with its real unipotent covolumes equals

$$\sum_{tK^{0}\in J(F)\backslash X_{f}}\mathrm{vol}(K^{0})\,\mathrm{vol}(\Gamma_{t}\backslash D_{\infty})=(2^{-f}\sqrt{|\Delta_{E_{x}}|})^{a}\,\frac{\mathrm{vol}(H_{C}(F)\backslash H_{C}(\mathbb{A}_{F}))}{\mathrm{vol}(H_{C}(F_{\infty}))}\prod_{v\in S(\gamma)}L_{\gamma,v}.\tag{98.25}$$

Use ordinary complex area and give the finite integer rings volume one in the independent entries of D. Away from the fixed bad places of the field and level and the divisors of the nonzero minors and $\mathcal{G}$, use volume one for the self-dual lattice stabilizers. This exceptional union is $S(\gamma)$.

Proof. The two nondegenerate frames determine their cross matrix up to unitriangular changes. In monomial form the common stabilizer on W has one upper unitriangular matrix over E_{x} in each reversed block. Its action fixes C pointwise. The remaining factor acts only on C, giving $J=D\times H_{C}$. At an ordinary place with unit minors and Gram determinant, integral Gaussian elimination produces a self-dual lattice on W. The orthogonal complement is a self-dual lattice in C_{v}, unique up to local isometry. There is one local $J(F_{v})$-orbit and its stabilizer has volume one. At a remaining place the compact level meets the closed double coset $B_{v}\gamma B_{v}$ in finitely many double cosets for the integral period group. Lemma 98.6 and Proposition 98.7 establish these assertions, including the local extension of isometries at places over 2.

For a finite representative let $K_{J}=J_{f}\cap tK^{0}t^{-1}$, $K_{D}=K_{J}\cap D_{f}$ and K_{C} be its projection to $H_{C,f}$. Density of $D(F)$ in D_{f} and the discriminant covolume of $\mathcal{O}_{E_{x}}$ give $\mathrm{vol}(\Gamma_{t}\backslash D_{\infty})=(2^{-f}\sqrt{|\Delta_{E_{x}}|})^{a}/\mathrm{vol}(K_{D})$. Above this adelic orbit the rational cells are the definite column

classes at level K_C. Their rational stabilizers are trivial by the two marked column level conditions, so their number is the definite quotient mass in (98.25) divided by $\mathrm{vol}(K_C)$. Since $\mathrm{vol}(K_J) = \mathrm{vol}(K_D)\,\mathrm{vol}(K_C)$, summing over the independent local orbits proves the identity. The covolume and class calculations in full are in Theorem 98.8. □

Corollary 98.35 (Allowable loss at the extra finite places)**.** *Let S_0 be the places exceptional for the field and level in Theorem 98.34, let $S'_\gamma = S(\gamma) \setminus S_0$, and let $P_\gamma = \prod_{v \in S'_\gamma} \mathrm{N}v$. For any fixed $0 \le \delta < 1/2$ and sufficiently large V,*

$$\sum_{\gamma \in \mathcal{R}} \frac{P_\gamma^{\delta}}{w_\gamma} \le \Big(\frac{128 \log(2Q)}{Q^{4(1-\delta)}} \Big)^{fd}.$$

For each fixed $B \ge 1$ and $\eta > 0$, throughout the same intervals for sufficiently large V,

$$\sum_{\gamma \in \mathcal{R}} \frac{B^{|S'_\gamma|}}{w_\gamma} \le \exp\!\big(-(64 - \eta) f d\, x^{\varepsilon}\big).$$

Proof. The product of the extra residue sizes divides the product of the absolute integer norms of the endpoint minors and $\mathcal{G}$. The former product is at most $w_\gamma^{1/2}$, and $0 < \tau\mathcal{G} \le |\tau D_d|^2$ gives the same bound for the latter norm. Thus $P_\gamma \le w_\gamma$. The integral of $2b_1 r (1 + r^2)^{-b_1 - 1}\, dr$ from u to infinity equals $(1 + u^2)^{-b_1}$. Taking $b_1 = 2(1 - \delta) > 1$ in the box counts of Theorem 110.16 bounds the sum of $w_\gamma^{-1+\delta}$ by the displayed expression, as computed in Theorem 98.9.

For the second conclusion choose $0 < \delta < 1/2$ with $64\delta < \eta$. Each place with $\mathrm{N}v \ge B^{1/\delta}$ contributes $B \le (\mathrm{N}v)^{\delta}$. The remaining places lie above rational primes at most $B^{1/\delta}$, with at most f places above each. Their factors together are at most $B^{f \lceil B^{1/\delta} \rceil}$. This has logarithm $O_{B,\delta}(f)$. Using $\log Q = (16 + o(1)) x^{\varepsilon}$ in the first estimate gives the assertion uniformly on the interval. □

For the integral itself, multiply the right side of (98.25) by the maximal norms of the two finite covectors and by

$$\int_{D_\infty \backslash G_\infty} \|W_1(g)\|\, \|W_2(\gamma^{-1} g)\|\, d\dot{g}.$$

This bounds the sum of the absolute cell contributions by contraction of the two real sections and integration along the common unipotent fibre. A uniform bound on this expression relative to w_γ^{-1} is still needed. It includes the values of the exceptional local factors and the mass of the definite complement. The integral lower bound for the designated theta pairing in Corollary 83.7 and the total order of the exceptional cohomology groups in (20.12) remain open.

Proposition 98.36 (A family to which the regular orbital identity does not apply)**.** *On every field and at every translate in Theorem 98.34 are infinitely many supported, relevant rational double cosets with $D_1 = \cdots = D_{d-1} = 0$ and $\mathcal{G} \neq 0$.*

Proof. In the signed Witt basis choose a positive coordinate vector w of norm one orthogonal to the negative terminal vector e. Their sum is isotropic. Use a shear of the first isotropic basis vector by a multiple of $w + e$, and its negative adjoint shear on the dual isotropic space. A skew-Hermitian antidiagonal matrix on the intervening indices fills out the remaining cross entries. Explicitly the resulting cross matrix is the sum of $-TE_{dd}$ and a single reversed block with successive nonzero entries $na_0, -na_0, \ldots$, where $\bar{a}_0 = (-1)^d a_0$. The three block matrices and the verification that they preserve the Hermitian form are given in Proposition 98.10. The shears form a rational one-parameter group. For every sufficiently divisible integer $n > 0$ its

finite component lies in K^0. Adding a multiple of the first row to the last removes the terminal diagonal entry. Then $D_d = (na_0)^d$ distinguishes the rational cosets and gives $\mathcal{G} = |D_d|^2$ at every real place. The alternating signs make the two sets of simple unipotent coordinates conjugate in reverse order, so their trace characters agree. □

For instance the source's formal-degree identity for a regular orbital integral requires every proper leading minor to be nonzero. The examples in Proposition 98.36 have a common unipotent group of positive dimension. For nonzero compact-finite discrete-series Bessel sections their product of norms has infinite integral on the whole real group. Indeed the two sections are analytic by elliptic regularity for the sum of the real and compact Casimir operators with the elliptic signs. On the connected group their product of norms is positive on an open set and is constant along the noncompact common unipotent fibres. Corollary 98.11 gives the full argument. The period formula uses the quotient by that unipotent group. An estimate on this quotient or a vanishing result for the particular projected entries is still needed.

98.11 The contribution of the common unipotent stabilizer

Keep the fields $E = E_x$, the translates, the prime $T = \varpi\bar{\varpi}$ and the level K^0 of Theorem 98.34. At a real place the two sections $W_i = J_i v_i$ for two words have the same discrete-series row σ. Their column actions τ_i are unitary. On a relevant orbit put $D = N \cap \gamma N \gamma^{-1}$ and fix Haar measure on D. The operator entering each finite contraction is

$$K_\gamma(W_1, W_2) = \int_{D\backslash G} W_1(g) W_2(\gamma^{-1} g)^* \, d\dot{g}.$$

Here $e_1 e_2^*$ maps u to $\langle u, e_2\rangle e_1$, with inner products linear in the first variable. Equality of the two characters on D makes the integrand well defined.

Theorem 98.37 (The orbital identity at the original level)**.** *For all sufficiently large x, the quotient kernel on every relevant orbit with $\mathcal{G} \neq 0$, including the one-block orbits in Proposition 98.36, equals*

$$\frac{\langle v_1, v_2\rangle}{\overline{\ell_1}\ell_2} \int_{(B\times B)/\Delta_\gamma D} \tau_1(b_1)^{-1} \xi_1 \xi_2^* \tau_2(b_2) \, \langle \sigma(b_1 \gamma b_2^{-1}) u_2, u_1\rangle \, d[b_1, b_2]. \tag{98.26}$$

Here $\Delta_\gamma d_0 = (d_0, \gamma^{-1} d_0 \gamma)$ acts on the right, and $\ell_i = \langle \Lambda_i(u_i), \xi_i\rangle \neq 0$ for a compact-finite choice of u_i, ξ_i in the normalized Bessel model. Both integrals converge absolutely.

Proof. Average the coefficient $\langle \sigma(bg) v_i, u_i\rangle$ against $\tau_i(b)^{-1}\xi_i$ on B. By the normalized relative character, this gives $\overline{\ell_i} W_i(g)/\sqrt{d_\sigma}$, where d_σ is the formal degree. Insert the two averages into the quotient integral and disintegrate $B \times B$ along the indicated diagonal group. Its fibre combines with $D\backslash G$ to give G. After $y = b_2 \gamma^{-1} g$, the two row coefficients have arguments $b_1 \gamma b_2^{-1} y$ and y. Their integral on G is $d_\sigma^{-1} \langle v_1, v_2\rangle \langle \sigma(b_1 \gamma b_2^{-1}) u_2, u_1\rangle$ by Schur orthogonality.

For the interchange, if $\|b_1 \gamma b_2^{-1}\|_{\mathrm{op}} \le U$, elimination in the cross matrix bounds representatives of both B-coordinates by $O_\gamma(U^{d^2+2})$ modulo the common unipotent group. On eliminating each pivot, the denominators are products of nonzero fixed pivot entries and the numerators are minors of order at most d. The two frames span all negative directions, so bounds on their coordinates bound the operator norms of the representatives. The quotient ball has measure $O_\gamma(U^{8r^2(d^2+2)})$. The row gaps of Theorem 110.2 give coefficient decay of greater order even after reserving $4r^2 + 1$ powers for integration in y. This proves absolute convergence in the expansion. The coordinate and measure computations are given in Lemma 98.12 and Theorem 98.13. □

Proposition 98.38 (A bound uniform over the labels of one field)**.** *For fixed x, put $L = 8(f+1)(r+1)^3$. For $p,t \ge 0$ with $2pd(d+1)+t < L - 4fr^2$, there is a finite $C_{x,p,t}$ such that at the original level*

$$\sum_c |J_{\gamma,c}| \le C_{x,p,t} Q^{-ft} w_\gamma^{-p}$$

for all relevant nondegenerate labels and unit inputs.

Proof. The determinant D_j pairs the first j frame vectors in the exterior power with the corresponding translated vectors. The product of the two quotient heights to power j bounds its absolute value up to the squared Euclidean norm of that wedge. In the first reversed block there is an antidiagonal entry pairing two isotropic frame vectors. Its modulus is bounded by twice the product of the two heights and equals the first scalar modulus. The first scalar divided by Q is a nonzero algebraic integer by the endpoint congruences. Thus the product over embeddings of the maxima of one and its unscaled modulus is at least Q^f. Weight the two positive seeds by the quotient heights to exponent $2pd(d+1)+t$, unfold their product, and use Cauchy–Schwarz. The height-ball count of exponent $4fr^2$ and the seed decay of exponent L put both periodizations in L^2. The bound follows with their two norms in the constant. The details, including the uniformity for unit inputs, are in Proposition 98.14. □

Theorem 98.39 (Arithmetic normalization of all nondegenerate cells)**.** *Write $R = |\Delta_E|^{1/(2f)}$, $S = d(d-1)$, and use the block stabilizer dimension a of Theorem 98.34. At a real embedding let P be the following positive density at the block endpoints $j_\nu = h_1 + \cdots + h_\nu$:*

$$P = \mathcal{G}^{k-1} \prod_{\nu<l} |D_{j_\nu}|^{2(h_\nu + h_{\nu+1})} |D_d|^{2(h_l-1)}.$$

Put $P_\infty = \prod_\tau P_\tau$. Scale the i-th isotropic vector and its dual by reciprocal factors $R^{-(d-i)}$ and R^{d-i}, keeping the terminal vector fixed. Use the conjugated orbit and the isometrically transported Bessel sections, denoted by primes. Then

$$c_E^a P_\infty^{1/2} K^{\mathrm{area}}_{\infty,\gamma} = R^{-fS} (P'_\infty)^{1/2} K^{\mathrm{root}}_{\infty,\gamma'}, \tag{98.27}$$
$$\sum_c |J_{\gamma,c}| \le B_\ell B_{\ell'} m_C \Big(\prod_{v \in S(\gamma)} L_{\gamma,v} \Big) R^{-fS} Q^{-fd(k+d-2)} \big\| (P'_\infty)^{1/2} K^{\mathrm{root}}_{\infty,\gamma'} \big\|_{\mathrm{op}}.$$

Area is Euclidean measure on the a complex entries of each real unipotent stabilizer. Root measure multiplies that area at each place by $(|c'|/(2\pi))^a$, where c' is the coefficient of the new simple-root character.

Proof. For a general scale $\kappa > 0$ in place of R^{-1}, the block scalar in positions $j_{\nu-1}+1, \dots, j_\nu$ acquires the factor κ^{e_ν}, with $e_\nu = 2d - j_{\nu-1} - j_\nu - 1$. Let $H = \sum_\nu h_\nu(h_\nu^2 - 1)/6$ and $E_* = \sum_{i=1}^d 4(d-i)^2$. The character coefficient acquires a factor κ. The root Haar pullback has factor κ^{a-2H}, since each upper entry within a block scales by κ^{i-j}. The two section normalizations together give κ^{-kS-E_*}, the inverse modulus of the frame change on B. Finally $P^{1/2}$ acquires $\kappa^{(k-1)S+E_*-2H}$ by the endpoint formula. Multiplying these factors, taking the inverse for the change in quotient measure, gives κ^{-S-a} for the scaled density times the scaled root kernel.

Originally $|c| = 4\pi$ since the additive character uses the complex trace. Hence $K^{\mathrm{area}}_\infty = 2^{fa} K^{\mathrm{root}}_\infty$. The discriminant covolume is $c_E = (R/2)^f$. Putting $\kappa = R^{-1}$ in the preceding computation gives (98.27). The proper endpoint congruences and the integral Gram determinant $\mathcal{G}/Q^{2d}$ imply $P_\infty^{1/2} \ge Q^{fd(k+d-2)}$. If the last block has length one, D_d occurs to exponent zero in the product of endpoint factors. Otherwise $D_{d-1} = 0$, so the full endpoint has the same homogeneous congruence as the others. Now contract the actual operator with the two finite

covectors, use their maximum norms $B_\ell, B_{\ell'}$, and sum the covolumes by Theorem 98.34. The general change of frame and the corresponding change of the regular squared norm are proved in Section 98.5. □

Proposition 98.38 gives a bound for all labels of each E_x. Corollary 110.17 requires a bound as x varies on the common intervals. For instance, taking $p = 1$ in the proposition would suffice for its hypothesis if $\log C_{x,1,t} - ft \log Q \le (64-\delta) f d\, x^{\varepsilon}$ for one fixed $\delta > 0$ at all sufficiently large x. Theorem 98.39 evaluates the discriminant and level factors entering this comparison. The growth of the remaining operator norms and finite factors, the lower bound for the designated integral theta pairing in Corollary 83.7, and the exceptional cohomology sum in (20.12) remain to be controlled on those intervals.

98.12 Summing the columns at the two auxiliary primes

Fix x on the interval of Theorem 20.13. At the marked places $v \mid Q = g_1 g_2$, enlarge $K = K^0$ to the full row lattice stabilizer and call the resulting finite level K^*. At the terminal prime T we use the prescribed level in both. Write $I_\sigma = [K^* : K]$, $I_\pi = [H_f \cap K^* : H_f \cap K]$. For a word with definite column representation ρ, let $c_{\rho,K}$ be its identity term constant, so the squared norm of a column tuple ℓ is $c_{\rho,K} \sum_y \|\ell_y\|^2$. Put

$$c^*_\rho = \frac{\operatorname{vol}(K^*)\, v_N \operatorname{vol}(H_\infty)}{\operatorname{vol}(N_f \cap K^*)\, \dim \rho}, \qquad v_N = \operatorname{vol}(N(F)\backslash N(\mathbb{A}_F)).$$

Lemma 98.40 (Two flags over the same residue field)**.** *At $v \mid g$, $g \in \{g_1, g_2\}$, each full-level row coset supporting both flags has either zero or $[\mathrm{GL}_k(\mathbb{F}_g) : U_k(\mathbb{F}_g)]$ supporting refinements. In the latter case each flag projects bijectively from the common refinements onto the local column cosets.*

Proof. Use the first split factor in the ordered Witt basis. Let $\bar N$ be the unipotent radical for the blocks of sizes $1, \dots, 1, k+1, 1, \dots, 1$. On the middle block the period subgroup acts by $\mathrm{diag}(h, 1)$. Its integral reduction is $\bar B = \bar N \rtimes \mathrm{GL}_k(\mathbb{F}_g)$. All these matrices lift by lifting their individual unipotent blocks and the invertible middle block. The row level contains the principal congruence kernel and reduces to $\bar K = \bar N \rtimes U_k(\mathbb{F}_g)$, by the zero entries in the terminal column of Proposition 94.6. For each of the two flags, the allowed residue refinements of a full row coset are the right $\bar K$-cosets in a single left translate of $\bar B$. The two translates agree if they meet. In that case the refinements are parametrized by $\bar B/\bar K$. Projection onto the middle k-block identifies them with $\mathrm{GL}_k(\mathbb{F}_g)/U_k(\mathbb{F}_g)$. The choice of a representative for either flag acts on this set by left translation, a bijection. □

Theorem 98.41 (A bound independent of the two level indices)**.** *For every relevant original label γ of nonzero Gram determinant and any two unit column tuples at K^0,*

$$\sum_c |J_{\gamma,c}| \le \frac{A_*\, \mathcal{M}^{=}_\gamma}{\sqrt{c^*_{\rho_1} c^*_{\rho_2}}}\, R^{-fS} Q^{-f(dk+a)} (P^{\#}_\infty)^{1/2} \|\widehat{K}_{\gamma\#}\|_{\mathrm{op}}.$$

Here the real inputs are normalized as in Corollary 93.19. The factor $\mathcal{M}^{=}_\gamma$ is $\mathfrak{m}_{k-1}$ times the product of the local sums of Theorem 98.34 at K^*, restricting at $v \mid Q$ to the row cosets for which the two translates in Lemma 98.40 agree. Moreover $\log A_* = O(fk \log^2(2fk)) = o(fk \log b)$, uniformly on the cohomology intervals.

Proof. Above a compatible rational coset tK^* there are I_π simultaneous refinements before dividing by rational symmetries. Choose t from a supporting fine refinement. On each side,

the I_π parameters project once onto the column level cosets by the lemma. Write A_i for the finite rational stabilizer of the underlying coarse column class y_i. It acts freely on that set, since the fine rational column stabilizers are trivial. Thus the sum of the squared norms of the covectors on side i over the common parameters is $|A_i| \sum_{y \mapsto y_i} \|\ell_{i,y}\|^2$.

Let $\Gamma_D = D(F) \cap tK^*t^{-1}$ and let a_t be its index in $J(F) \cap tK^*t^{-1}$. Its finiteness follows by projecting into A_1, with kernel Γ_D. At each marked place Γ_D acts through $\bar{N}$ on the refinements and fixes them all. The quotient of order a_t acts freely by triviality of the fine rational column stabilizers. Put $m_t^* = \operatorname{vol}(K^*) \operatorname{vol}(\Gamma_D \backslash D_\infty)/a_t$. The fine cells over t therefore have total measure $m_t^* I_\pi / I_\sigma$. Locally the same ratio is obtained by counting the supporting refinements in each full-level stabilizer orbit. Formula (98.25) now gives $\sum_t m_t^* = c_E^a \mathcal{M}_\gamma^=$, with $c_E = 2^{-f}\sqrt{|\Delta_E|}$.

To insert the actual covectors, sum the product of their norms over the common refinements and apply Cauchy–Schwarz to the two squared sums just calculated. The coefficient of this sum is m_t^*/I_σ times the operator norm of the real kernel. The unipotent levels agree because all integral unipotent blocks reduce into $\bar{N}$. Hence $c_{\rho,K} = c_\rho^*/I_\sigma$, and for a unit input its full sum of squared covector norms is I_σ / c_ρ^*. The two square roots in Cauchy–Schwarz cancel I_σ^{-1}. Bound $\sqrt{|A_1||A_2|}$ by the maximum A_* over the coarse column classes, sum over t, and apply the density and scaling identity of Corollary 93.19. This gives the asserted bound.

Each A_i acts faithfully on a rational vector space of dimension $n = 2fk$. Minkowski's finite subgroup theorem gives $v_p(|A_i|) \le \sum_{j \ge 0} \lfloor n/(p^j(p-1)) \rfloor$. Only $p \le n+1$ contribute, with valuation at most $np/(p-1)^2$. Summing their logarithmic contributions bounds $\log A_*$ by $O(n \log^2(2n))$. The ranks and degrees grow polynomially in x, whereas $\log b = x^\epsilon$. The local mass identity, the action of the rational stabilizers and the normalization appear in full in Section 98.6. □

Thus a column concentrated on one fine class also satisfies the estimate. A uniform upper bound for the remaining local masses and real orbital integrals, the lower comparison for the actual integral theta pairing in Corollary 83.7, and the total exceptional strict order in (20.12) remain necessary on these same fields and intervals.

98.13 The primes dividing only the Gram determinant

Fix an original relevant period term at K^0 with nonzero $\mathcal{G} = D_d \overline{D_d} - T^2 D_{d-1} \overline{D_{d-1}}$. The finite product in Theorem 98.41 has an explicit factor at the primes of $\mathcal{G}$ where all nonzero endpoint minors are units. The multiplicity with which a prime divides the determinant enters the answer.

Theorem 98.42 (Local masses at the terminal Gram determinant)**.** *Let $v \notin S_0$ be an unmarked place of odd residue characteristic and residue size q, with the full row lattice stabilizer of volume one and the complement measure of Proposition 93.20. Suppose the last block is a singleton and every nonzero proper endpoint minor is a unit at v, in both factors in the split case. With $e = v(\mathcal{G})$ the local mass in Theorem 98.41 is*

$$L_v^= = 1 + q^{k-1} + q^{2(k-1)} + \cdots + q^{e(k-1)}.$$

Here the sum has just the term 1 if $e = 0$.

Proof. In the integral pair of decorated frames, Gaussian elimination uses unit pivots in the first $d-1$ positions, in the order given by the reversed blocks. The corresponding vectors of both frames together split off a unimodular hyperbolic summand. The remaining two vectors have norm $-T$, a unit, and mutual pairing D_d/D_{d-1}. Split off the first vector and project the

second perpendicularly to it. This gives a vector w in rank k of norm $a = \mathcal{G}/(TD_{d-1}\overline{D_{d-1}})$. The local sum is over self-dual lattices containing w, modulo its unitary stabilizer, each counted with the inverse volume of its lattice stabilizer. The common unipotent subgroup acts only on the split-off hyperbolic summand, where its integral block entries have volume one. The integral extensions of both decorations and the orbit identification are given in Lemma 98.22.

At a split place, a self-dual lattice is given by a lattice L in a k-dimensional space and its dual. Write $w = (u, \eta)$, with $\eta(u) = a$. We need $u \in L$ and $\eta(L)$ integral. After fixing $L_0 = L \cap \ker \eta$ by a change of basis, the remaining generator is $\pi^{-i}u + y$, where $0 \le i \le e$ and y runs over $\pi^{-i}L_0/L_0$. Each orbit under the volume-one group $\mathrm{GL}(L_0)$ has size equal to the inverse volume of its stabilizer. There are $q^{(k-1)i}$ possibilities at i.

At an unramified field place the two initial masses, for norm valuations zero and one, are 1 and $A + 1$ with $A = q^{k-1}$. At zero the perpendicular lattice is self-dual. At one a primitive vector u extends to a unimodular plane with an isotropic partner v_1 pairing to one. In the perpendicular of u the intersection lattice is generated by $u - h(u, u)v_1$ and the unimodular rank $k-2$ summand. For the form scaled by π^{-1}, its lattice stabilizer has volume $(q+1)/(q^{k-1}+1)$. This is the ratio of the numbers of maximal isotropic $(k-2)/2$-planes in the residue Hermitian spaces of dimensions $k-2$ and $k-1$, by counting incident self-dual lattices. Fixing the distinguished generator modulo π has a further index $q+1$. Lemma 98.24 proves these assertions and computes both incidence counts.

For higher valuations, use additive volume on a self-dual lattice and form the density of the equation $h(z, z) = a$ by counting solutions modulo increasing powers of π. On primitive vectors the norm derivative is surjective modulo π. The density for a unit target is $H_1 = 1 - q^{-k}$, and for a nonunit target it is $H_0 = H_1(1 + A^{-1})$, by counting the residue norms in k coordinates. Multiplication of a vector by π scales additive surface measure by A^{-2} and transports the quotient measure with the same stabilizer. Decomposition according to the exact divisibility of z, with the two initial masses just computed, gives

$$\begin{cases} 1 + (1 + A^{-1}) \sum_{j=1}^{h} A^{2j}, & e = 2h, \\ (A + 1) \sum_{j=0}^{h} A^{2j}, & e = 2h + 1. \end{cases}$$

Both are $\sum_{i=0}^{e} A^i$. Theorem 98.25 gives the quotient measure comparison and the density calculation. □

Corollary 98.43 (The part of the Gram ideal away from the minors)**.** *Let $\mathcal{U}_\gamma$ be the places $v \notin S_0$ where all nonzero endpoint minors are units. Put $\mathfrak{b}_\gamma = \prod_{v \in \mathcal{U}_\gamma} \mathfrak{p}_v^{v(\mathcal{G})}$. For all the labels on the same cohomology intervals, uniformly as x grows,*

$$\prod_{v \in \mathcal{U}_\gamma} L_v^{=} = (\mathrm{N}\mathfrak{b}_\gamma)^{k-1} \sum_{\mathfrak{a} \mid \mathfrak{b}_\gamma} (\mathrm{N}\mathfrak{a})^{1-k} = (\mathrm{N}\mathfrak{b}_\gamma)^{k-1} (1 + O(f 2^{-k})).$$

Proof. If the last block has length greater than one, $D_{d-1} = 0$ and D_d is a unit on this set. Hence $\mathcal{G}$ is a unit and the ordinary factor is one by Theorem 98.34. Otherwise use Theorem 98.42. Factoring out the highest power in each local sum and multiplying gives the divisor sum of the ideal. Its excess over one is at most $\zeta_F(k-1) - 1 \le \zeta(k-1)^f - 1 = O(f 2^{-k})$ since $f 2^{-k} \to 0$ on these intervals. □

The resulting power of $\mathrm{N}\mathfrak{b}_\gamma$ enters the estimate for the original period. Its comparison with the archimedean orbital integrals and the remaining finite factors, the lower bound for the designated integral theta pairing, and the total cohomology order in (20.12) are open here.

98.14 The finite masses on the common field

Use the fields and intervals of (20.12). For a relevant, finitely supported pair of nonzero Gram determinant, let $h = k - 1$, $S = d(d-1)$ and $m = h - 2d + 1$. In its monomial cross matrix let c_ν denote the first antidiagonal entry of the block of length h_ν. Put

$$a_\nu = \binom{h_\nu}{2}, \qquad a = \sum_\nu a_\nu, \qquad W_\gamma = \prod_\nu |N_{E/\mathbb{Q}} c_\nu|^{-a_\nu}.$$

Give each full row lattice stabilizer K_v^* volume one. Keep the equality test on the reduced flags at both auxiliary primes in the mass $\mathcal{M}_\gamma^=$ of Theorem 98.41.

Theorem 98.44 (The finite mass relative to the real density)**.** *With P_∞ the product of the real point densities, uniformly in the field and the supported label,*

$$\frac{\mathcal{M}_\gamma^=}{\mathfrak{m}_{k-1} P_\infty} \le \exp\bigl(O(f 2^{-m})\bigr) Q^{f(2a - S - 2dh)} W_\gamma.$$

This includes every finite place of the original period.

Proof. Here E/F is unramified at every finite place, the row lattices are self-dual, and the individual d-frames are saturated in the reference lattice. At T the two components are the modified lattice and its dual. The ranks are $r = k + 2d - 1$, $n = 2d$, and $h = r - n$ is odd with $m \ge 2$ for large x. Thus the local frame theorem applies at every place. Its proof counts independent columns over the residue field, lifts them, and sums over the possible saturations of an integral tuple. For the full row orbit the resulting main factor is $q_v^{hv(\mathcal{G})}$, up to $\exp(O(q_v^{-m}))$ and the volume of integral cross matrices. At a marked place replace the second frame by its difference from the first divided by a uniformizer. This changes the Gram valuation by $-2d$. Counting equal representatives of the two reduced flags contributes q_v^S. Prescribing the residues in the $S - a$ free cross-matrix coordinates contributes at most $q_v^{-2(S-a)}$ in addition to the usual pivot Jacobians. The two marked rational primes split completely. After multiplication the Gram factors cancel those in P_∞. The pivot Jacobians leave exponent $-a_\nu$ on each norm of c_ν. The error follows from $\sum_{v<\infty} q_v^{-m} \ll f 2^{-m}$. The complete measure calculation is in Sections 98.8 and 98.9. □

Corollary 98.45 (The squared kernel for one rational pair)**.** *For the transported test and kernel of Corollary 93.19,*

$$\mathcal{M}_\gamma^= c_E^{2a} \|K_{\infty,\gamma}^{\mathrm{area}}\|_{\mathrm{HS}}^2 \le \mathfrak{m}_{k-1} \exp\bigl(O(f 2^{-m})\bigr) R^{-2fS} Q^{f(d(d-3)-2dh)} \times W_\gamma P_\infty^{\#} \|\widehat{K}_{\gamma\#}\|_{\mathrm{HS}}^2.$$

Proof. The squared change-of-variables identity contributes $R^{-2fS} Q^{2f(d(d-2)-a)} P_\infty^{\#}/P_\infty$. Apply Theorem 98.44 and collect the powers of Q. □

The corollary bounds one squared kernel with its finite orbital measure. The upper estimate for the period summed over rational cells, the designated lower estimate in Corollary 83.7, and the exceptional strict cohomology orders in (20.12) are still required.

99 Integral lattices at the marked primes

99.1 The two Hermitian column lattices

Use the inert place, lattices and notation of Proposition 99.5. Two self-dual column lattices are called neighbors if their intersection has one-dimensional residue quotient in each. Let A_v sum

the translations to all neighbors, with the column compact subgroup $K_H = U(\Lambda)$ of volume one.

Proposition 99.1 (Column position and multiplicity at an inert place)**.** *The two column lattices agree for the $P^{2h}(P-1)$ terms with $x = 0$. In every other term they are neighbors. There are $d = PM_k/(P^2-1)$ neighbors, forming one K_H-orbit, each with unsigned multiplicity $P^{2h}(P^2-1)$. On spherical columns the signed local factor is $a'I + u'A_v$, with*

$$|a'| \le P^{2h}(P-1), \qquad |u'| \le P^{2h}(P^2-1).$$

Proof. The first lattice $n(w,z)L$ projects to Λ in the column. For the second, project the intersection of $g^{-1}L$ with the perpendicular of the isotropic flag and terminal vector, modulo the flag, as in Proposition 99.11. For integral input (t, m, β), the middle and last coordinates of $g^{-1} = \gamma_c^{-1} n(w,z)$ are

$$m + X_0\beta/c, \qquad -ct - h_M(X_0, m) + (1 - Z_0)\beta.$$

The last must vanish. The other perpendicularity conditions put m in the integral sum of the remaining isotropic vectors and Λ, because X_0 belongs to that sum. Writing x_0 for its column component, the second lattice is

$$\Lambda' = \{y + x_0\beta/c : \ y \in \Lambda, \ \beta \in O, \quad h_M(x_0, y) \equiv (1 - Z_0)\beta \bmod c\}.$$

It is self-dual. Indeed $g \in KB(C)$, so the projected lattice is the image of Λ under the unitary column factor of g^{-1}. For $x = 0$ the formula gives exactly Λ. For $x \neq 0$ its intersection N with Λ is the inverse image of the hyperplane $x^\perp$. Integrality forces $c \mid \beta$, and adding an integral multiple of x_0 preserves this hyperplane since x is isotropic. Conversely every lift of a vector in that hyperplane occurs by taking $\beta = 0$. Modulo Λ the vectors of Λ' fill the fractional line generated by x_0/c, since the residue functional $h_M(x, \cdot)$ is onto. Both quotients by N have dimension one.

For any self-dual neighbor, its image after multiplication by π and reduction in $\Lambda/\pi\Lambda$ is an isotropic line. Indeed the pairings of its elements are integral before multiplication. Its intersection with Λ reduces into the perpendicular of this line and equals the inverse image by its index. There are $M_k/(P^2-1)$ such lines. Lift a generator of any line to a primitive isotropic $e \in \Lambda$. Starting with any lift, one corrects its norm successively modulo π^{j+1} by adding $\pi^j y$, $j \ge 1$. The derivative is the residue trace of $h_M(e, y)$, which is onto. Completeness gives the lift. Choose an integral dual vector f_1, and add a multiple of e to give f_1 norm zero by trace surjectivity.

In this hyperbolic plane N is generated by $e, \pi f_1$, with the full perpendicular lattice adjoined. Its dual $N^\#$ has plane generators $e/\pi, f_1$. Self-dual lattices between N and $N^\#$ correspond to isotropic lines in the plane $N^\#/N$ with pairing induced by $\pi h_M \bmod \pi$. This follows by taking the perpendicular of a line in that nondegenerate plane. Besides the line of f_1, which gives Λ, they are the lines of $e/\pi + tf_1$ with $t + \bar{t} = 0$ in κ. There are P. The integral shears $e \mapsto e + \pi z f_1$ with $z + \bar{z} = 0$ permute these lines transitively. Every residue skew scalar lifts to a skew integer, for c/π is a skew unit whose residue spans the trace-zero line also in characteristic two. Isotropic lines of the original residue lattice are conjugate under K_H by the preceding hyperbolic completion and the uniqueness of the unimodular complement of a given rank in Proposition 98.28. Thus all neighbors occur in the double coset that scales e, f_1 by π^{-1}, π, respectively.

Integral column isometries act on the successful unipotent cosets by conjugation and transport their column lattices, so the multiplicity per neighbor is the second count of Proposition 99.5 divided by d. Finally the average of a translation between two K_H-fixed vectors is

the same throughout a double coset. On the neighbor coset it is A_v/d, by decomposition into its d compact right cosets. Each unipotent term has a character of modulus one, or its compact average of modulus at most one, exactly as in Proposition 100.10. This gives the two signed coefficient bounds. □

99.2 The first-plane factor at every good prime power

Use the field and levels of Section 6.6. Let $F_v(c)$ be the signed finite orbital factor for γ_c between the original columns at a place of full lattice levels away from QT. Put $P = Nv$ and normalize by the identity orbital mass as in Theorem 99.10.

Proposition 99.2 (A uniform bound at arbitrary good valuation)**.** *For every $e = v(c) \geq 1$, at a split or inert place,*

$$\|F_v(c)\| \leq \left[4(k+1)^2\right]^e P^{(k+4s+1)e}.$$

The constants hold for all the prescribed words, all unit parts of c, and all residue characteristics.

Proof. Use the variables and truncated valuations t, u' of Lemma 99.6. For each successful unipotent coset its contribution is a column translation compressed between K_H-fixed vectors, with a character multiplier of modulus at most one. This follows by taking its two column coordinates and averaging over the right compact subgroup, as in Proposition 100.10. No cancellation between different cosets is needed here. Write $j = e - t - u'$ and $j_+ = \max(0, j)$. On a split column $2j_+ \leq d_- + d_+ \leq 2e$. On an inert column $j_+ \leq d \leq e$. Hence the bound per coset from Proposition 105.8 is at most

$$(k+1)^{2e} P^{-(k-2)j_+}.$$

At a split place integrality of X, Y, Z restricts the upper blocks to fractional entries of denominator dividing c. In each right integral coset choose the lifts of X, Y mod c first. Right multiplication by an integral unipotent adjusts the two blocks independently and changes Z by an integral scalar, since Y is integral. After they are fixed, only an integral central shift is possible, leaving at most P^e choices of Z mod c. The two middle blocks have length $n = r - 2 = k + 2s - 1$. Their column and dual column coordinates therefore allow at most $P^{k(2e-t-u')}$ choices with these depths. The remaining coordinates give at most $P^{2(n-k)e}$.

At an inert place fix X mod c in the quadratic integer ring. Then $Y = h_M(X, \cdot)$ is determined and $u' = t$. At this depth the column coordinates give at most $P^{2k(e-t)} = P^{k(2e-t-u')}$ choices, and the other coordinates at most $P^{2(n-k)e}$. Integral right translations permit a chosen lift of X, using the trace-one element as in Proposition 99.5. For that lift the admissible integral Z's satisfy the same difference equation as there, hence form at most one translate of O_{F_v}. The remaining integral central translations change Z by cO^-. Since c/π^e is a skew unit, this is precisely $\pi^e O_{F_v}$. There are at most P^e scalar choices.

Thus on either side the contribution in norm at fixed t, u' is bounded by

$$(k+1)^{2e} P^{(2(n-k)+1)e + k(e+j) - (k-2)j_+}.$$

For $j \geq 0$ the last two terms of the exponent equal $ke + 2j \leq (k+2)e$. For $j < 0$ they are bounded by ke. Now $2(n-k)+1 = 4s-1$. There are at most $(e+1)^2 \leq 4^e$ possible depths. Summing proves the bound, with the exact physical character still present. □

99.3 The first-plane mass at the three marked primes

Keep the physical level K^0 of Section 6.6, with $Q = g_1 g_2$ and terminal prime T. Write $F = E^+$, $f = [F : \mathbb{Q}]$, $n = r - 2$, $R = n + 1 = k + 2s$. For $0 \neq \xi \in \mathcal{O}_E^-$ let $\mathfrak{b}$ denote the ideal of F whose extension is $\xi\mathcal{O}_E$, as in Theorem 105.6. Factor it as $\mathfrak{b} = \mathfrak{g}\mathfrak{m}$, where $\mathfrak{m}$ is supported above QT and $(\mathfrak{g}, QT) = 1$. Set $c = Q\xi$. The level specified in Section 86.10 is the full lattice stabilizer away from g_1, g_2. The change of row lattice in Theorem 86.1 has support only over T. Thus every divisor of $\mathfrak{g}$ has the full levels used in Proposition 99.2.

Proposition 99.3 (A bound including every marked valuation)**.** *At $v \mid QT$, put $P = Nv$ and $a = v(\mathfrak{m})$. The unsigned first-plane mass relative to the diagonal is at most $P^{(2n+1)a}$. The signed operator at all finite places consequently has norm at most*

$$C_k^{\Omega(\mathfrak{g})} (N\mathfrak{g})^{R+2s+1} (N\mathfrak{m})^{2R-1}, \qquad C_k = 4(k+1)^2.$$

Here Ω counts prime factors with multiplicity. The bound holds at the original column level also when its reduction at the specified place is a proper subgroup.

Proof. All three primes split completely. Use the first split component and the two mutually dual middle lattices there. At T take a basis of the actual graph lattice in the middle and the dual basis on the other side. Together with the remaining Witt pairs these give integral middle coordinates of dimension n. The first pair u, v_1 is integral at every marked place by Proposition 111.15. In block order u, M, v_1, the first unipotent radical has unrestricted upper blocks (l, H, z), where l is a covector and H a vector. Its integral subgroup U_L consists of integral blocks in these bases. Theorem 99.10 gives the mass as the number of right U_L-cosets with $g = n(l, H, z)^{-1}\gamma_c \in K_v^0 B(F_v)$, at the graph and at the two unitriangular levels alike.

Put $\delta = 1$ above Q and $\delta = 0$ above T. Then $v(c) = a + \delta$. In the coordinates of Lemma 99.6, the middle part of gu is $-cH$, and the last row of g^{-1} is $(-c, -cl, 1 - cz)$. The first vector and first dual covector are fixed by the period subgroup. At T both images are integral. Above Q they agree with the reference ones modulo π because the reduction of the row level is upper unitriangular, fixing the first vector and the last covector. The integral change of first Witt partner adds a multiple of u and preserves these two tests, as in Theorem 99.10. It follows that

$$l \in \pi^{-a} O^{1\times n}, \qquad H \in \pi^{-a} O^{n\times 1}, \qquad z \in \pi^{-a} O.$$

For the graph these statements use the bases chosen above. There are at most P^{2na} choices of l, H modulo their integral lattices. Fix representatives of those choices by right multiplying by integral unipotents. The new z differs by an integral scalar plus the product of l with an integral column, so it is still in $\pi^{-a} O$. Once l, H are fixed exactly, right integral equivalence is translation of z by O, giving at most P^a choices. This proves the local count, including $a = 0$.

Each successful coset pairs two column arguments by unitary translation, possibly averaged over compact stabilizers. It has norm at most one, since translations are isometries, the character has modulus one, and the compact averages are probability averages. More explicitly, a finite coset class with stabilizer of relative index j has weight j in the integration of Theorem 99.10. Its pairing is the sum of j such terms, so that summing their norm bounds gives precisely the unsigned mass. Multiplying the marked bounds and Proposition 99.2 at the good divisors proves the displayed estimate. At the other places the minimal valuation has mass one. The product uses the unfolding of Theorem 105.6 and $2n + 1 = 2R - 1$. □

99.4 Integrating the covector in the terminal graph

Fix $v \mid T$. In the first split factor write $L = F_v$, $O = \mathcal{O}_L$, $\pi = T$ and $P = Nv$. With an integral lift of the primitive residue covector ℓ, the graph coordinates of Theorem 86.1 are (w, z'), where

$z' = (z - \ell w)/\pi$. In these coordinates the central part of the row lattice is $O^k \oplus O$. The row level K is the full stabilizer of the resulting row lattice. Put $C = \mathrm{GL}_k(O)$ and $y_0 = \ell/\pi$. In the central block consider the group

$$D = \left\{ \begin{pmatrix} a & 0 \\ d & 1 \end{pmatrix} : a \in \mathrm{GL}_k(L),\ d \in L^{1\times k} \right\}, \qquad p\left(\begin{smallmatrix} a & 0 \\ d & 1 \end{smallmatrix}\right) = a.$$

It acts on row covectors on the right by $yb = ya + d$. The original column group H is the stabilizer of y_0 in D. In fact preserving the physical last coordinate $z = \ell w + \pi z'$ requires $d = y_0 - y_0 a$. Let D_0 be the subgroup $a \in C,\ d \in O^{1\times k}$. Then $p(H \cap D_0) = C_\ell$, the prescribed subgroup fixing the residue covector. The orbit $S_0 = y_0 D_0 = \pi^{-1} O^{1\times k} \setminus O^{1\times k}$ is compact and open. Reduction modulo $O^{1\times k}$ gives all nonzero residue covectors by transitivity of C, and integral translations give each entire fiber.

Write N for the unipotent radical with the prescribed singleton blocks. It is normalized by D. The group $B' = N \rtimes D$ contains the period group $B = N \rtimes H$. Use the set of right U_L-cosets

$$\Omega' = \{n \in U(L) : n^{-1}\gamma_c \in KB'\}/U_L$$

in the first unipotent radical of Theorem 99.10. The two first endpoints still give its integrality tests, so it is finite by the denominator count in Proposition 99.3. Conjugation by D_0 preserves U_L and Ω', since it preserves the row lattice and commutes with γ_c.

Proposition 99.4 (The contribution from an orbit of the enlarged set)**.** *Fix $nU_L \in \Omega'$. Write $n^{-1}\gamma_c = k_0 b'$ with $k_0 \in K$, $b' \in B'$, and let $\beta \in D$ be the central block of b'. Put $A = p(\beta)^{-1}$. The sum of the actual signed column terms over the D_0-orbit of nU_L, divided by the size of this orbit, has norm at most $\Xi(A)$. A coset failing the original support condition contributes zero. The estimate uses the original finite additive character.*

Proof. The image of $B' \cap K$ on the central block is D_0. Indeed an integral invertible block triangular matrix in B' induces an integral automorphism there, and every element of D_0 embeds as an integral block. The remaining unipotent factor is then integral as well. It follows that $k_0 b' \in KB$ precisely when $\beta \in D_0 H$. Also $|\det A| = 1$, since n and γ_c have determinant one, $\det k_0$ is a unit, and $\det b' = \det p(\beta)$.

Push probability Haar measure on D_0 forward by $s \mapsto sns^{-1}U_L$. This gives uniform counting measure of total mass one on the finite orbit. The new central block is $s\beta s^{-1}$. Put $y = y_0 s$ and $y' = y\beta^{-1}$. The test $s\beta s^{-1} \in D_0 H$ is equivalent to $y' \in S_0$, since H fixes y_0 and S_0 is stable under D_0. In that case choose $s' \in D_0$ such that $y_0 s' = y'$. The identity

$$(s\beta s^{-1})^{-1} = (s\beta^{-1}(s')^{-1})(s's^{-1})$$

has first factor in H and second factor in D_0. Thus the true relative column translation in the unfolding of Theorem 111.8 can be taken to have matrix $p(s)Ap(s')^{-1}$. Changing s' changes this matrix on the right by C_ℓ. The unipotent terms and the two character factors contribute only their actual scalar phase. This follows by factoring off the central blocks in B', leaving N in the period subgroup, and using the right invariance of the row seed under K.

Let σ be a local column component. For a C_ℓ-fixed vector u define a vector-valued function $f_u(y_0 s) = \sigma(p(s))^{-1}u$ on S_0 and extend it by zero to $Y = L^{1\times k}$. Two possible s's differ on the left by $H \cap D_0$, so this is well defined. Use additive Haar measure dy with $\mathrm{vol}(S_0) = 1$. It is invariant under D_0, and its restriction to the orbit S_0 is the pushforward of Haar probability by $s \mapsto y_0 s$. In particular $\|f_u\|_2 = \|u\|$. Push the phase-weighted measure on D_0 forward by the same map. Its density $\phi(y)$ with respect to dy has modulus at most one, since the unweighted

pushforward on S_0 is dy. Use density zero elsewhere. The normalized orbital pairing of two original vectors is therefore

$$\int_Y \phi(y)\langle \sigma(A) f_v(y\beta^{-1}), f_u(y)\rangle \, dy.$$

Here the extension by zero includes the exact success test. A change to the adjoint convention conjugates the pairing and interchanges the two test vectors.

On $L^2(Y)$ the weighted composition $Sf(y) = \phi(y) f(y\beta^{-1})$ is a contraction, since the affine map has determinant of absolute value one. For $h \in C$, let $R_h f(y) = f(yh)$. Then f_u is fixed by $\sigma(h) \otimes R_h$, by using sh as the representative of yh. The displayed pairing is a coefficient of $\sigma(A) \otimes S$ between these fixed vectors. Proposition 101.4 supplies temperedness at this split place. Lemma 104.31 gives the asserted bound, with the adjoint of the scalar multiplier if the inner product is linear in its second argument. □

99.5 The first-plane flag at an inert place

Let $D/C = E_w/F_v$ be an unramified quadratic field extension at a finite place of full lattice level away from QT. Write O for the integers of D, π for a uniformizer of C, and κ/κ_0 for the residue extension, where $\#\kappa_0 = P$. Fix a skew c with $cO = \pi O$. In the first-plane decomposition $Du \oplus M \oplus Dv_1$ of Theorem 99.10, let $p_1, \dots, p_h$ be the remaining isotropic basis vectors and p_{h+1} the terminal vector, with $h = s - 1$. Its norm $-T$ is a unit. Complete these vectors by integral Witt partners and the reference column lattice $\Lambda \subset W_0$ of rank k. Write P_i for the residue span of $p_1, \dots, p_i$, with $P_0 = 0$.

Proposition 99.5 (The inert cosets at valuation one)**.** *Let M_k count the nonzero isotropic vectors of $\Lambda/\pi\Lambda$. In the unsigned local mass of Theorem 99.10 are $P^{2h}(P-1)$ cosets with zero residue column and $P^{2h}PM_k$ with nonzero residue column. The counts hold in every residue characteristic.*

Proof. Write the unipotent representative as $n(w, z)$ as in Theorem 99.10, set $X_0 = cw$, $Z_0 = cz$, and put $g = n(w, z)^{-1}\gamma_c$. Its first vector is

$$gu = (1 - \bar{Z}_0)u - X_0 + cv_1, \qquad Z_0 - \bar{Z}_0 = h_M(X_0, X_0)/c.$$

Here h_M is conjugate linear in its first variable. Thus integrality requires X_0, Z_0 integral and the residue X of X_0 isotropic. Conversely, for each such X fix a lift X_0. The displayed equation has integral solutions forming a translate of O_C. Indeed there exists $\lambda \in O$ of trace one, by surjectivity of residue trace followed by division by a unit. For an integral skew y, the difference $\lambda y - \overline{\lambda y}$ equals y.

Each right integral unipotent coset with residue X has a representative with this fixed X_0. To change w by an integral vector w', multiply on the right by $n(w', -\lambda h_M(w', w'))$. Its cross term in cz is integral since cw is integral. Once w is fixed, two solutions for Z_0 are equivalent precisely when their difference belongs to cO^-, where O^- is the lattice of skew integers. As c/π is a skew unit, $O^- = (c/\pi)O_C$ and $cO^- = \pi O_C$. There are therefore exactly P scalar classes for each X, before the other flag conditions.

Suppose i is the first index with $h_M(X, p_i) \neq 0$ in κ. Choose the lift of X with $h_M(X_0, p_j) = 0$ exactly for $j < i$, using the Witt coordinates. Then $gp_j = p_j$ there, while $gp_i = p_i - h_M(X_0, p_i)u/c$ has a pole of order one. If the flag admits an integral primitive representative, the earlier integral columns $gu, p_1, \dots, p_{i-1}$ are already independent modulo π. Their exterior product is unchanged by adding earlier columns to later ones. Clearing the pole by such additions would force the residue of u into their span. Indeed independence forces every

coefficient used in the additions to have valuation at least -1, after which one multiplies by c and reduces. The coefficient of gu in this residue relation must be nonzero. Projection to M therefore gives $X \in P_{i-1}$. Every member of P_{i-1} is orthogonal to p_i, a contradiction.

It follows that $X \in P_h \oplus (\Lambda/\pi\Lambda)$. Write x for its column component. It is isotropic. Choose X_0 in the corresponding integral direct sum, with zero column lift when $x = 0$. Now every $gp_i = p_i$. If $x \neq 0$, their reductions are independent together with gu for all P scalar choices. If $x = 0$, the norm of X_0 is zero and $Z_0 \in O_C$. Independence holds precisely for $Z_0 \not\equiv 1 \bmod \pi$, giving $P - 1$ choices. Dependence prevents primitivity by the exterior product test. In the independent case these integral columns have the required pairings because g is an isometry. Lemma 98.30 extends the primitive decoration to an integral isometry, which is precisely $g \in KB(C)$. There are P^{2h} choices in P_h, proving both counts. □

99.6 Two flag minors at arbitrary valuation

Fix a place outside QT with full row and column lattice levels. Let O be the integers in the first factor at a split place or those of E_w at an inert place. Use a uniformizer π of F_v in either case. Suppose the first-plane parameter c has valuation $e \geq 1$. In order u, M, v_1, write the three upper unipotent blocks as (l, H, z), a row, a column and a scalar. Put $Y = cl,\ X = cH,\ Z = cz$. In the inert case $l = -h_M(H, \cdot)$ and $Y = h_M(X, \cdot)$. For $g = n^{-1}\gamma_c$, multiplication gives

$$g^{-1} = \begin{pmatrix} 1 & Y/c & Z/c \\ 0 & I & X/c \\ -c & -Y & 1 - Z \end{pmatrix}, \qquad gu = (1 - Z + YX/c)u - X + cv_1.$$

If $g \in KB(F_v)$, its first vector and the first dual covector (the last row of g^{-1}) are integral. Hence X, Y, Z are integral. Write X_W, Y_W for the vector and covector components on the reference column Λ. For a vector or covector let v be the minimum valuation of its coordinates, with $v(0) = +\infty$. Put $t = \min(e, v(X_W))$, $u' = \min(e, v(Y_W))$ and $L_0 = \max(0, e - t - u')$.

Lemma 99.6 (Elementary divisors forced by the dual flag)**.** *Let Λ' be the column lattice of $g^{-1}L$ for a successful unipotent coset. Its elementary divisors relative to Λ have at most one negative and at most one positive exponent. Write these as $-d_-$ and d_+, taking zero for an absent exponent. Both $d_\pm$ lie between L_0 and e. At an inert place $u' = t$ and $d_- = d_+$.*

Proof. Let $\lambda_1, \dots, \lambda_s$ be the remaining dual flag covectors on M, the last being the terminal covector. At the present place its norm parameter is a unit. Use unit multiples and an adapted integral basis so that these are coordinate functionals. Write $q_1, \dots, q_s$ for their coordinate vectors, S for their span, and P_h for the span of the $h = s - 1$ remaining primal isotropic vectors. The middle lattice is $P_h \oplus \Lambda \oplus S$. Let $X_S = \sum_j q_j \lambda_j X$. The transformed list of dual rows starts with

$$\eta_0 = (-c, -Y, 1 - Z), \qquad \eta_j = (0, \lambda_j, \lambda_j X/c), \quad 1 \leq j \leq s.$$

Every minor of each initial list is integral. Indeed the rows can be made integral by adding earlier rows to later ones by Lemma 98.30, and such additions preserve the exterior product. This applies to the contragredient list at a split place and to the Hermitian dual list at an inert place.

For any column coordinate indexed by i, take the minor of $\eta_0, \dots, \eta_j$ on that coordinate, $q_1, \dots, q_{j-1}$, and the last coordinate v_1. Its value up to sign is $Y_{W,i}\lambda_j X/c$. On $q_1, \dots, q_s, v_1$ the determinant for the whole list is, up to sign, $D = 1 - Z + YX_S/c$. Thus

$$\pi^{u'} X_S/c \text{ is integral}, \qquad D \in O.$$

For the first conclusion when $u' = e$ use X_S integral.

An integral input (a, m, β) of g^{-1} has middle coordinate $m+X\beta/c$. For its column projection one requires zero last coordinate and zero S-coordinates of the middle vector, then projects modulo P_h. Write $m = m_P + w + m_S$. The conditions are

$$m_S = -X_S\beta/c \in S, \qquad Ym_P + Y_W w \equiv D\beta \pmod{cO}, \qquad w \in \Lambda.$$

Indeed the latter congruence is exactly the condition that the last coordinate vanish for an integral a. The projected vector is $w + X_W\beta/c$. Its image modulo Λ lies in a single cyclic module of length at most e, giving at most one negative exponent and $d_- \le e$. When $L_0 > 0$, choose $\beta = \pi^{u'}$, $m_P = 0$. Both conditions can be satisfied since $Y_W\Lambda = \pi^{u'}O$ and $D \in O$. The resulting vector has denominator of exact valuation $e - t - u'$, which proves $d_- \ge L_0$.

At an inert place Λ' is self-dual by the unitary column projection, so its exponents occur in opposite pairs. Also $u' = t$ by unimodularity of the column form. At a split place apply the same calculation to the dual component with endpoints in reverse order. The upper blocks of the reversed $n^{-\mathsf{t}}$ are $(-H^{\mathsf{t}}, -l^{\mathsf{t}}, lH - z)$, and the parameter of the reversed $\gamma_c^{-\mathsf{t}}$ is $-c$. Thus the two column depths interchange. The projected dual lattice is $(\Lambda')^\vee$, because $g^{-1}L$ is a $B(F_v)$-translate and its two Levi actions are contragredient. The negative exponents on that side are the negatives of the positive exponents on this side. The same bounds prove the assertion for d_+. □

99.7 The two unitriangular primes at every valuation

Take a place $v \mid Q = g_1 g_2$, with residue field κ of size P. Use the first split factor and put $O = \mathcal{O}_{F_v}$, $C = \mathrm{GL}_k(O)$. The column at this place is fixed by the inverse image in C of the upper unitriangular subgroup. Let $F_v(c)$ denote the signed first-plane factor, with the identity mass normalization of Theorem 99.10.

Proposition 99.7 (A local estimate at the prescribed unitriangular level)**.** *Take $c = Q\xi$ and $a = v(\xi) \ge 0$. The factor at $a = 0$ is the identity. For $a \ge 1$ one has*

$$\|F_v(c)\| \le [4(k+1)^4]^a P^{(k+4s+1)a+k}.$$

Proof. Let K be the original row level. By Lemma 98.19 it reduces to $\bar{N} \rtimes U_k(\kappa)$, where $\bar{N}$ is the upper unipotent radical for the blocks with central block of size $k+1$. Define K_1 to be the inverse image of $\bar{N}$ in the full row lattice stabilizer. Embed C in the column block. It normalizes K_1, the period subgroup $B = NH$, and the integral first radical U_L. It commutes with γ_c and preserves the Bessel character on N, since this character uses the flag and the fixed terminal vector. The integral first radical has only upper off-diagonal entries in the first row and last column, so $U_L \subset K_1$. The decomposition of the reduction, lifting the $U_k(\kappa)$ component in $H \cap K$, gives $KB = K_1 B$. Moreover the image of $B \cap K_1$ on the column is the principal congruence subgroup $C_1 = \ker(C \to \mathrm{GL}_k(\kappa))$. Indeed the middle block of an integral period element is $\mathrm{diag}(h, 1)$, and membership in K_1 requires $h \equiv 1$. The remaining factor lies in the integral unipotent on which the character is trivial.

The successful right U_L-cosets in Theorem 99.10 now form a finite C-set under conjugation. For a representative n choose $n^{-1}\gamma_c = k_1 b_n$ with $k_1 \in K_1$, $b_n \in B$. Write A_n for the column component of b_n^{-1}. In the common-column unfolding of Theorem 111.8 the first period at n translates the column by the identity, and the second, at $\gamma_c^{-1}n = b_n^{-1}k_1^{-1}$, translates it by A_n. The two unipotent factors contribute a scalar phase of modulus one. Using the adjoint convention exchanges A_n with its inverse and gives the same bounds below. The measure on cosets is counting measure by Theorem 99.10. A new decomposition changes A_n on the right

by C_1, which fixes each original column vector. A change of representative n by U_L can be absorbed into k_1, and its phase is trivial on U_L.

For $h \in C$ the conjugate decomposition gives hA_nh^{-1} as a representative for the translated coset. The average over each C-orbit of cosets is therefore a matrix coefficient of the average in Lemma 104.30, with a scalar multiplier of modulus at most one. More explicitly, push Haar probability forward by $h \mapsto hnh^{-1}U_L$. The image measure is uniform on this orbit. In each fiber the relative translation agrees on the test vectors with the chosen representative because the ambiguity lies in C_1. The actual phase can be used as the multiplier. The argument takes place in the whole local component before testing on the two original vectors. Proposition 101.4 applies to every such component.

Suppose $a \geq 1$. Put $e = a + 1$ and use $X = cH,\ Y = cl,\ Z = cz$ from Lemma 99.6. By the endpoint congruences in Proposition 99.3, all three are in πO in their respective spaces. Let t, u' be the valuations of $X_W/\pi, Y_W/\pi$ on the column and dual column, truncated at a. These depths are constant on each compact orbit, and also under a change of representatives modulo c. Forgetting the congruence level gives a successful coset for the full stabilizer of the same row lattice. The terminal norm is a unit at v, so the flag-minor proof of Lemma 99.6 applies there. The column lattice is A_nO^k, since k_1 stabilizes the row lattice and projecting b_n^{-1} to the column gives its Levi action. The two truncated depths for that lemma are $t + 1, u' + 1$. Its two possible nonzero elementary divisors satisfy

$$j_+ \leq d_\pm \leq a + 1, \qquad j = a - 1 - t - u', \quad j_+ = \max(0, j).$$

Lemma 104.30 bounds the norm of an orbit average by $k^{2(a+1)}P^{-(k-2)j_+}$, using the weaker exponent $k - 2$ for convenience.

Choose the lifts of $X/\pi, Y/\pi$ modulo π^a first, as in Proposition 99.3. At these depths their column coordinates allow at most $P^{k(2a-t-u')}$ choices. The other coordinates and the central scalar allow at most $P^{(2(n_0-k)+1)a}$ choices, where $n_0 = r - 2 = k + 2s - 1$. Multiplying each orbit average by its orbit size accounts for every coset exactly once. To bound their total use

$$k(2a - t - u') - (k - 2)j_+ = k(a + 1 + j) - (k - 2)j_+ \leq ka + k + 2a.$$

There are at most $(a + 1)^2 \leq 4^a$ depth pairs and $k^{2(a+1)} \leq k^{4a}$. This proves the assertion for positive a. At $a = 0$ the unique coset is represented by 1 and $\gamma_c \in K$. Both period arguments have the same column and trivial relative phase. □

99.8 Elementary divisors in the terminal graph basis

At a place $v \mid T$ use Ω', β and $A = p(\beta)^{-1}$ from Proposition 99.4. Put $e = v(c) \geq 1$ and $h = s - 1$. After taking out the first hyperbolic pair, write the remaining graph lattice as

$$M = P_h \oplus W \oplus Ot_0 \oplus S_h.$$

Here P_h consists of the h primal isotropic lines and S_h of their opposite lines. In the central graph coordinates $W = \{(w, 0) : w \in O^k\}$ and $t_0 = (0, 1)$. The quotient of the central block by its terminal line is identified with $\Lambda = O^k$. For the first unipotent blocks (l, H, z) put $X = cH,\ Y = cl,\ Z = cz$. Denote the W-coordinate of X by X_W and the restriction of Y to W by Y_W. Write $t = \min(e, v(X_W))$, $u' = \min(e, v(Y_W))$, taking the minimum over the coordinates and $v(0) = +\infty$.

Lemma 99.8 (A pair of opposite elementary divisors)**.** *For every $nU_L \in \Omega'$, the elementary divisors of A relative to Λ have the form $(-d, 0, \dots, 0, d)$ with*

$$\max(0, e - t - u') \leq d \leq e.$$

Proof. Write $g = n^{-1}\gamma_c = k_0 b'$. Both first endpoints are fixed by B', giving X, Y, Z integral as in Proposition 99.3. In the order (u_1, M, v_1) the matrix is

$$g^{-1} = \begin{pmatrix} 1 & Y/c & Z/c \\ 0 & I & X/c \\ -c & -Y & 1-Z \end{pmatrix}.$$

Let $\lambda_1, \dots, \lambda_h$ be the remaining isotropic dual covectors in their flag order, with coordinate sections q_j in S_h. Write X_S for the S_h-component of X. The first $j+1$ dual rows of g^{-1} in this ordering are

$$(-c, -Y, 1-Z), \qquad (0, \lambda_i, \lambda_i X/c) \quad (1 \le i \le j).$$

Their exterior product is integral. Indeed the upper block radical preserves each initial isotropic list by unitriangular changes, its central block D fixes the isotropic graded lines, and k_0^{-1} is integral. Taking the minor on a W-coordinate, $q_1, \dots, q_{j-1}$ and v_1 gives $\pm Y_{W,i}\lambda_j X/c$ at coordinate i. Taking the full minor on $q_1, \dots, q_h, v_1$ gives $\pm D_1$, where $D_1 = 1 - Z + YX_S/c$. Therefore $\pi^{u'} X_S/c$ is integral and $D_1 \in O$. If $u' = e$ the first assertion follows directly from integrality of X_S.

Project g^{-1} of the row lattice to the column as follows. Intersect with the rational subspace where the last coordinate and the S_h-coordinates vanish, then quotient by the span of u_1, P_h, t_0. At the reference lattice this gives Λ. The flag group B' preserves both subspaces and acts on the quotient by its p-component, since D fixes t_0 and N is trivial on the quotient. As $g^{-1} = b'^{-1}k_0^{-1}$, the output lattice is $A\Lambda$.

For an integral input (a_0, m, α), write $m = m_P + w + bt_0 + m_S$. The intersection conditions read

$$m_S = -X_S\alpha/c, \qquad Y(m_P + bt_0) + Y_W w \equiv D_1\alpha \pmod{cO}.$$

The congruence expresses existence of an integral a_0 making the last coordinate zero. The projected vector is $w + X_W\alpha/c$. Modulo Λ all projected vectors therefore lie in one cyclic module of length at most e. There is at most one negative exponent, of absolute value at most e. If $e - t - u' > 0$, take $\alpha = \pi^{u'}$ and $m_P = b = 0$. The first condition is integral, and $Y_W\Lambda = \pi^{u'}O$ permits the stated congruence. The projected vector has exact denominator length $e - t - u'$.

In the split dual lattice reverse the endpoints. The contragredient of the central block D preserves the annihilator of t_0, whose reference lattice has basis dual to W. On that annihilator it acts by $p^{-\mathsf{t}}$. Impose this extra vanishing along with the isotropic intersection conditions and project past the first primal lines on the reversed side. The projected dual lattice is $A^{-\mathsf{t}}\Lambda^\vee$. Indeed on the dual row the transformation of $g^{-1} = b'^{-1}k_0^{-1}$ is $g^{\mathsf{t}} = b'^{\mathsf{t}}k_0^{\mathsf{t}}$, and b'^{t} induces $A^{-\mathsf{t}}$ on the dual subquotient. For clarity, the dual transformation of g^{-1} is g^{t}. In reversed block order it has middle block I and a single column above the last diagonal entry, equal up to sign to Y^{t}/c. Its projection on the $W^\vee$-coordinates of an integral input is consequently an integral vector plus an integral multiple of this one projected column. The imposed vanishing conditions only restrict the allowable inputs. Thus the dual projection modulo $\Lambda^\vee$ is likewise contained in one cyclic module of length at most e. There is at most one positive exponent of A. Finally $|\det A| = 1$ by Proposition 99.4, so the two absolute values coincide. This proves the claim. □

99.9 The two split valuations and the common scalar

At a split finite place use the first component, with uniformizer π, residue cardinality P, and valuation v. Write c_+, c_- for the two components of $c = Q\xi$, $e_\pm = v(c_\pm)$, and $L = v(\lambda_+)$. Put

$\epsilon = 1$ above Q, zero elsewhere. Write $a_\pm = e_\pm - \epsilon \geq 0$, $h = s-1$, and $n = k+2s-1$. At T use the graph basis and the enlarged support KB' of Proposition 99.4. At other split places use the original support KB. For $g = n_1^{-1} w_{c,\lambda}$ in that support let A be the inverse column action of the period factor, or the inverse quotient action of its central block at T. Put $A_0 = \lambda_+ A$.

Lemma 99.9 (Elementary divisors after a central translation)**.** *Write the unipotent blocks of n_1 as (l, H, z). Set $X = c_+H$, $Y = c_-l$, $Z = c_-z$. On the column and dual column use the truncated depths $t = \min(a_+, v(X_W/\pi^\epsilon))$ and $u' = \min(a_-, v(Y_W/\pi^\epsilon))$. These are nonnegative on the indicated support. The only possible nonzero exponents of A_0 are $-d_-$, d_+, with*

$$0 \leq d_- \leq e_+, \quad 0 \leq d_+ \leq e_-, \qquad d_+ - d_- = e_- - e_+ - (2h+1)L.$$

If $S = t + u' + 2\epsilon$, then away from T one has $d_- + d_+ \geq e_+ + e_- - 2S + |L|$. At T the same bound holds for $L < 0$, and for $L \geq 0$ it holds with $-L$ in place of $|L|$.

Proof. In block order u, M, v_1 the inverse is

$$g^{-1} = \begin{pmatrix} 0 & 0 & 1/c_+ \\ 0 & \lambda_+^{-1} I & \lambda_+^{-1} X/c_+ \\ c_- & Y & Z \end{pmatrix}.$$

The middle part of gu is $-X$, and its first entry is $c_+(lH - z)$. Thus $X, Y \in \pi^\epsilon O$ in their blocks. Both Z and $c_+(lH - z)$ are integral and are $1 \bmod \pi$ above Q. These are the first primal and dual endpoint tests, which use the actual level above Q.

First use the primal projection of Lemma 99.8 by quotienting the terminal line. At other places use the integral splitting of the unit terminal line with its column. Write $q_1, \dots, q_h$ for the integral sections of the isotropic dual covectors η_i on M, and X_S for this part of X. The first dual row is (c_-, Y, Z). Each subsequent row in the isotropic list is $\lambda_+^{-1}(0, \eta_i, \eta_i X/c_+)$. Every initial wedge of these rows is integral, by the flag and the integral row factor. For the first $j+1$ rows, take columns $q_1, \dots, q_{j-1}, v_1$ and one additional column. Taking a W-coordinate or the initial u-coordinate gives, up to sign, $\lambda_+^{-j} a\, \eta_j X/c_+$, respectively for $a = Y_{W,i}$ or $a = c_-$. The whole list on $q_1, \dots, q_h, v_1$ gives, up to sign, $\lambda_+^{-h} D_1$, with $D_1 = Z - YX_S/c_+$. Set $u_c = \min(e_-, v(Y_W)) = u' + \epsilon$. Since the tested a's generate $\pi^{u_c} O$, the valuations of $\pi^{u_c}\eta_j X/c_+$ and D_1 are at least jL and hL, respectively.

Multiply the output column lattice by λ_+. It becomes $A_0\Lambda$. A projected vector has form $w + X_W\alpha/c_+$ for integral w, α, so its denominators give at most one negative exponent of length at most e_+. Take $\alpha = \pi^{u_c + h\max(0,-L)}$. The required opposite coordinates $-X_S\alpha/c_+$ are integral. With the two discarded primal components zero, the last equation is $Y_W w \equiv -D_1\alpha \bmod c_- O$. It is soluble because its right side is in $\pi^{u_c}O$. This also covers $u_c = e_-$. The middle vector now gives

$$d_- \geq e_+ - S - h\max(0, -L),$$

since $t + \epsilon$ is the column valuation of X_W truncated at e_+. A nonpositive lower bound needs no choice of vector.

Reverse the endpoints in the dual split component. Its two c's are interchanged, the central scalar is λ_+^{-1}, and the new vector and row are $-Y^{\mathsf{t}}, -X^{\mathsf{t}}$. This follows by taking the inverse transpose of the exchange and of the original upper unipotent. Away from T quotient the dual terminal line, again a direct integral summand. At T instead impose the additional equation of annihilating the primal terminal vector t_0, as in Lemma 99.8. The coordinate section for this last equation is $t_0^\vee$ in the graph dual basis, so the $W^\vee$ coordinates are precisely the duals of the preceding W-coordinates. The augmented dual list has one more row. Its initial wedges

are integral also for KB', since the group D fixes t_0 and the upper radical may add earlier isotropic vectors to it. The same minors and congruence argument, with $h' = h + 1$ at T and $h' = h$ elsewhere, give

$$d_+ \le e_-, \qquad d_+ \ge e_- - S - h' \max(0, L).$$

The dual output after scaling is $A_0^{-\mathsf{t}}\Lambda^\vee$. Its negative denominators are cyclic as in the primal projection, so A_0 has at most one positive exponent.

The exchange has determinant $-c_+/c_-$ on the first plane and λ_+^n on M. The row integral factor has unit determinant and the unipotents have determinant one. Hence $v(\det A_0) = e_- - e_+ - (n-k)L$, where $n - k = 2h + 1$. For $L < 0$, add this determinant valuation to twice the lower bound for d_-. For $L \ge 0$, subtract it from twice the lower bound for d_+. This gives the stated lower bounds for the sum. □

99.10 The support of the finite test on the first plane

Use $u, v, M, J, \gamma = \gamma_c$ of Proposition 111.15 at a finite place of F. Write O for its integers, π for a uniformizer, O_E for the local quadratic integer algebra, and $L = (O_E u + O_E v) \perp L_M$ for the lattice with its first hyperbolic plane split off. At T use the actual graph lattice in M. Put $K = K_v^0$, $J_K = J(F_v) \cap K$ and

$$X = (B(F_v)K \cap \gamma B(F_v)K)/K, \qquad m_v(\gamma) = \sum_{tK \in J(F_v)\backslash X} \frac{\mathrm{vol}(J_K)}{\mathrm{vol}(J(F_v) \cap tKt^{-1})}.$$

This quotient of volumes is independent of the normalization on $J(F_v)$.

Theorem 99.10 (A first-vector test for finite cells)**.** *Let U be the unipotent radical of the stabilizer of Eu and $U_L = U(F_v) \cap \mathrm{Stab}(L)$. With U_L of volume one, $m_v(\gamma)$ is the local unsigned finite mass divided by the identity mass, in quotient measures. It equals the number of right U_L-cosets in*

$$\Omega = \{n \in U(F_v) : n^{-1}\gamma \in KB(F_v)\}.$$

Write $a = 1$ above Q and $a = 0$ elsewhere. If $cO_E = \pi^a O_E$, there is one such coset. Both decorations then have the same finite column coordinate on every cell at this place.

Proof. In order u, M, v write the elements of U as

$$n(w, z) = \begin{pmatrix} 1 & -w^* & z \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix}, \qquad z + \bar{z} = -h(w, w), \qquad w^* = h(w, \cdot).$$

This follows by imposing preservation of the hyperbolic pairing and of the form on M. Inversion sends (w, z) to $(-w, \bar{z})$. Integrality is equivalent to $w \in L_M$, $z \in O_E$, since L_M is self-dual. The first-plane parabolic decomposition gives $B = J \ltimes U$. At full lattice level, a product belongs to K precisely when both factors do, by its diagonal block on M. This applies also to the graph at T. At a place above Q the integral elements of U have, on the first split factor, only upper off-diagonal entries in the first row and the last column. The prohibited entries of Proposition 94.6 are inside the middle block. Replacing the original opposite first vector by v adds a multiple of u integral at these odd places and preserves these conditions. Thus $U_L \subset K$ also there. The middle diagonal block again gives $B(F_v) \cap K = J_K \ltimes U_L$. Conjugation by $J(F_v)$ preserves Haar measure on $U(F_v)$. It acts unitarily on the E_v-space of w, with F_v-determinant of absolute value one, and fixes the center.

Every element of X has the form jnK, $j \in J(F_v)$, $n \in \Omega$, since j commutes with γ. If $jnK = n'K$, the middle diagonal block of $(n')^{-1}jn$ gives $j \in J_K$. The remaining condition is $jnj^{-1}U_L = n'U_L$. The stabilizer of nU_L for this J_K-action is precisely $J(F_v) \cap nKn^{-1}$. Each summand in m_v is consequently the size of its orbit of right U_L-cosets. Multiplying its unnormalized stabilizer reciprocal by $\operatorname{vol} K$ gives its local mass in $J(F_v)\backslash G(F_v)$. For the identity orbit that mass is $\operatorname{vol} K/\operatorname{vol}(B(F_v) \cap K)$. The decomposition of this last compact group and the normalization of U_L give the stated ratio.

Every $b \in B(F_v)$ fixes u. Every $k \in K$ sends u to an integral vector and, above Q, fixes it modulo π in both factors. Indeed upper unitriangular reduction fixes the first column and, in the dual action, the last covector. Hence membership in Ω implies

$$n(w,z)^{-1}\gamma u - u = c\bar{z}\,u - cw + cv \;\in\; \pi^a L.$$

This bounds both w, z in fractional lattices, so the coset count is finite. At minimal valuation it forces $n \in U_L$. Conversely $U_L \subset \Omega$, since $\gamma \in K$. Every representative is then jK with $\gamma^{-1}jK = jK$, proving the last claim. □

99.11 The two column lattices of a first-plane orbit

Work at the split place of Theorem 101.1 in its coordinates. Write $\Lambda = O^k$ for the definite column lattice on the first split factor. Two lattices are neighbors here if their intersection has index b in both. Denote by x, y the $W, W^\vee$ components of the residues X, Y of that theorem.

Proposition 99.11 (The relative positions of the two columns)**.** *In the finite coset count of Theorem 101.1 the first column lattice is Λ. The second is a neighbor exactly when $i = j = \infty$, $x \neq 0$, and $y \neq 0$. Every neighbor occurs $u_b = b^{2s-2}(b-1)^2$ times. All remaining terms give Λ, with multiplicity*

$$a_b = b^{2s-2}(b-1)\left[(2 + 2s(b-1))b^k - 1\right].$$

Proof. For a lattice of the form $b'L$, $b' \in B(F_v)$, its column lattice is the image in W_0 of its intersection with the perpendicular of the isotropic flag space and of the terminal vector, under projection modulo the flag space. Indeed the unipotent factor of b' acts trivially on the middle quotient, and the Levi factor acts on the column by its H-block. The same description works on the first split factor by imposing annihilation by the opposite flag covectors. For $n(l, H, z)L$ the column is consequently Λ. For $g^{-1}L = \gamma_c^{-1}n(l, H, z)L$ use the inverse matrix in Theorem 101.1. An integral input (t, m, β) maps to a vector with middle component $m + X_0\beta/c$ and last component $-ct - Y_0 m + (1 - Z_0)\beta$. Thus the latter vanishes for an integral t exactly when $Y_0 m \equiv (1 - Z_0)\beta \bmod c$. The middle component must also be annihilated by $\lambda_1, \dots, \lambda_s$, including the terminal covector. Project its remaining coordinates to W_0.

If $i = j = \infty$ these conditions put m in the direct sum of the first $s-1$ flag coordinates and O^k. The second column is precisely

$$\Lambda' = \{w + x_0\beta/c : \; w \in \Lambda,\; \beta \in O,\; y_0 w \equiv (1 - Z_0)\beta \bmod c\},$$

where x_0, y_0 are the component lifts. When $y = 0$ the successful cosets have $1 - Z_0$ a unit, forcing $\beta \in cO$. This gives $\Lambda' = \Lambda$. When $x = 0$ the same unit permits any w, again giving equality. For $x, y \neq 0$, the intersection with Λ is the inverse image of $\ker y$ modulo c. Indeed integrality requires $\beta \in cO$ and $yx = 0$; conversely one may take $\beta = 0$ on that hyperplane. The image modulo Λ is the full line of x_0/c, since $y \neq 0$ permits every β. This proves the neighbor condition.

If $i < \infty$, X_0 has only entries in P_{i-1} and $1 - Z_0$ is a unit. For any desired $w \in \Lambda$ take $m = w$ and choose β by the congruence. There are no fractional column coordinates. If $j < \infty$,

the equation for λ_j forces $\beta \in cO$. Here Y_0 is supported in D_{j-1}, so every integral column is obtained with $m = w,\ \beta = t = 0$.

There are $b^{2s-2}b(b^k-1)(b^{k-1}-1)$ cosets in the neighbor case, by the count of nonzero orthogonal pairs and the unrestricted Z. The integral group $\mathrm{GL}_k(O) \subset J_K$ acts on the cosets by conjugation and on the resulting column lattices by its usual action. It is transitive on neighbors by elementary divisors, whose exponents are $1, -1$ and $k-2$ zeros. There are $d_b = (b^k-1)/(b-1)$ choices for the intersection as an index-b sublattice of Λ. Each has $d_b - 1$ index-b superlattices besides Λ, all producing distinct neighbors with that intersection. Dividing the count by $d_b(d_b-1)$ gives u_b for $k \geq 2$, as in the original ranks. Subtracting the neighbor count from Theorem 101.1 gives a_b. □

99.12 The last additive coordinate at a split auxiliary prime

Fix a point x of an interval in Section 6.6, its field $F = E_x^+$, and an admissible tuple ξ. Let $v \in Q(\xi)$ be one of the places where the twist is nontrivial on units. The conditions on the tuple say that v splits in E_x, that its ideal norm is a prime p, and that $p \notin v^2$. The residue degree and the ramification index at p are therefore both one. In particular $F_v = \mathbb{Q}_p$ over $\mathbb{Q}_p$, and we use $O = \mathbb{Z}_p$ and uniformizer p. The excluded set contains 2. The character on units kills $1 + pO$, so its inverse gives a nontrivial character χ of $\mathbb{F}_p^\times$. These conditions are recorded in `Formalism/LocalToGlobal/Divisibility/Tuples.lean`, line 63–67,110–128.

Take $s = m - 2 \geq 1$, $a_j = k_0 + j$, $n_j = a_j + jt$, $r = a_s + 1$ and $d = n_{s+1}$, with the marked planes and ordered bases of Theorem 123.9. Here A_j is the marked a_j-plane of the residue core at stage j, and $U = A_{s+1}$. Fix a row $l \in O^{1\times d}$ with $\bar{l}|_U \neq 0$, and put $U_0 = \ker(\bar{l}|_U)$ and $C_s = O/p^sO$. For $\nu \in C_s^\times$ choose a lift $\widetilde{\nu}$ and use the frequency ball

$$\mathcal{B}_\nu = p^{-s-1}\widetilde{\nu}l + O^{1\times d}.$$

The Fourier convention is $\psi(-zv')$ for column v', with ψ of conductor O. The effective row z includes the scalar in the Fourier kernel. In the top test it is the last matrix row, multiplied by that scalar. In particular identifying a global Bessel datum specifies a scalar but does not specify the last matrix row. The two arguments in the top test appear in `Development/AbelianVariety Theory/Definitions/TwistedInnerProducts/StandardOrderExtendedBasisApplyLowBlock.lean`, line 796,919–920.

Write $z_0 = 0$, and for $1 \leq j \leq s$ let z_j be the marked core basis vector in the representative h_j of Theorem 123.9. Write $\mathcal{P}_l$ for the following set of integral core matrices $X_0, \dots, X_s$ and integral top matrices B. For $0 \leq j < s$ require $X_j z_j - z_{j+1} \in p^j O^{n_{j+1}}$. The a_j marked columns of X_j followed by $v_j = p^{-j}(X_j z_j - z_{j+1})$ must reduce to a basis of A_{j+1}. At stage s the marked columns of X_s must reduce to a basis of U_0. The marked columns of B must reduce to an isomorphism of its two marked r-planes. Unmarked columns are arbitrary integral columns.

For $j < s$ let u_j be the residue determinant of the indicated basis in its original orientation, including the fixed ordering sign. Let u_+ be the marked determinant of B. Appending a variable last column $v' \in U$ to the marked columns of X_s gives a determinant $L(v') = u_s\bar{l}(v')$ for a unique unit $u_s \in \mathbb{F}_p^\times$. Define

$$D = u_+ \prod_{j=0}^{s} u_j \in \mathbb{F}_p^\times, \qquad Y = p^{-1}lX_s z_s \bmod p^s \in C_s.$$

The second coordinate exists because $\bar{X}_s z_s \in U_0$. Use product additive Haar with integral lattices of volume one.

Theorem 99.12 (The signed integral on every core position)**.** *On the ball $\mathcal{B}_\nu$ the local path, with the compact weights and initial endpoint of Theorem 126.4, has support precisely $\mathcal{P}_l$ in its position variables. Each fiber of (D, Y) in this support has the same measure $\mathcal{V}$ as the position set in that theorem. On the fiber $(D, Y) = (b, y)$ the path has the constant value*

$$c_\chi\, \chi(\bar\nu)^{-1}\chi(b)\, \psi(p^{-s}\widetilde{\nu}\widetilde{y}), \qquad c_\chi = \eta_\chi\, p^{\,r-1-(s+1)d} G(\chi)^h, \qquad h = 1 + s(s+1)/2,$$

where $\widetilde{y}$ is any lift, $G(\chi) = \sum_{a\neq 0} \chi(a)\psi(-p^{-1}\widetilde{a})$ on $\mathbb{F}_p^\times$, and η_χ is the actual product of unramified splitting scalars in Theorem 123.9. As the positions vary, the full frequency support consists of the rows z for which $p^{s+1}z$ is integral and its reduction is nonzero on U.

For any integrable weight on the positions and this frequency ball, let $H_\nu(b, y)$ be its mean on the indicated fiber times the whole ball, with respect to Haar probability on that product. The weighted integral is

$$\mathcal{V} c_\chi\, \chi(\bar\nu)^{-1} \sum_{b\in\mathbb{F}_p^\times} \sum_{y\in C_s} H_\nu(b, y)\chi(b)\psi(p^{-s}\widetilde{\nu}\widetilde{y}). \tag{99.1}$$

It vanishes whenever $H_\nu(b, y)$ is invariant under adding $p^{s-1}C_s$ to y for each b.

Proof. Theorem 123.13 leaves a single chain of intermediate double cosets. In its representatives, integrality at an interior edge requires integrality of X_j and v_j. On the marked residue space the first $a_j + 1$ columns reduce to the indicated core columns, all with zero lower entries. They must span the whole marked target core, since the remaining $s - j$ columns must already span the flag quotient of that dimension. The conditions are sufficient by the triangular lower block calculation of Theorem 123.10. Each of its $s - j$ free diagonal residues contributes a factor $G(\chi)$; its marked core determinant contributes $\chi(u_j)$.

At the last edge use the variable change $v' = -X_s z_s + p^s v$. The core must again be integral. The integral in v is constant on residue classes except for its Fourier character. Translation by pO^d forces $p^{s+1}z$ to be integral. Integration on the marked plane then forces the residual frequency to annihilate the image of the first $r - 1$ marked columns. These columns must be independent and lie in U, by the determinant test. If the residual frequency vanishes on U, the last residue sum is a multiple of $\sum_a \chi(a) = 0$. On $\mathcal{B}_\nu$ the residual row is $\bar\nu\bar l$. Hence the image is exactly U_0. In the last determinant coordinate $L(v)$ the residue sum now has parameter $\bar\nu/u_s$. It contributes $\chi(u_s)\chi(\bar\nu)^{-1}G(\chi)$ by changing variables in the sum. This also proves sufficiency, since $|G(\chi)| = \sqrt{p}$ by Theorem 123.6. Every row satisfying the stated frequency condition is covered by taking $l = p^{s+1}z$ and $\nu = 1$.

The translation of v' has the remaining phase $\psi(zX_s z_s) = \psi(p^{-s}\widetilde{\nu}\widetilde{Y})$. Adding an integral row to the center has no effect. Changing a lift of ν by p^s times an integer can change the frequency ball, but gives the same phase on these positions. The top test supplies $\chi(u_+)$. The interior compact integrals cancel their reciprocal weights by Theorem 123.7. At the last orbit all contributing v' reduce to $-\bar X_s z_s$, a nonzero vector in U_0, since $s \geq 1$ and z_s is a marked basis vector. Thus the same primitive orbit and its entire stabilizer weight occur. Combining the Jacobians and splitting factors of Theorem 123.9 gives c_χ. The preceding calculation accounts for every factor whose value changes with the positions.

The units acting on the first marked column of B permute the values of D transitively and preserve Y and measure. Choose an integral column w with $lw = 1$, and an integral row e with $ez_s = 1$. For $u \in O$, translation $X_s \mapsto X_s + puwe$ fixes all the residue determinants and adds u to Y. It preserves measure and all the other positions. Hence all $(p - 1)p^s$ fibers have equal measure. The fiber $(1, 0)$ is the one counted in Theorem 126.4 for a marked coordinate row. For general l, counting ordered bases of U_0 gives the same number, since all hyperplanes of U have dimension $r - 1$. The lifting condition $Y = 0$ selects one of p^s equal fibers by the

displayed translation. The other factors in the count are independent of l. This proves the assertion about $\mathcal{V}$. The ball has measure one, so summing the integral over the fibers gives (99.1). The sum of the additive phase on $p^{s-1}C_s$ is zero, as ν is a unit and the character there has order p. □

For example a weight supported on a single b, constant in y, gives zero on every primitive frequency. The weight $\overline{\chi(b)\psi(p^{-s}\widetilde{\nu}\widetilde{y})}$ on all the fibers gives $(p-1)p^s\mathcal{V}c_\chi\chi(\bar{\nu})^{-1}$. Both calculations use the full position domain. For a discrete absolutely summable class expansion, sum its original weights in each fiber and omit the Haar factor $\mathcal{V}$ in (99.1). This follows by partitioning the original sum. At several places the same partition may be taken jointly, including when the class weights have global dependencies.

Theorem 99.13 (Elementary divisors at the residue characteristic)**.** *Use one common array of weights on $\mathbb{F}_p^\times \times C_s$ in (99.1). Take all primitive ν and any set of distinct nontrivial χ. Remove from each output its factor $\mathcal{V}\eta_\chi p^{r-1-(s+1)d}$. Work at any place above p of a cyclotomic splitting field, with $v_p(p)=1$. For $\chi=\omega^{-a}$ in this completion, where ω is the Teichmüller character and $1\le a\le p-2$, the nonzero elementary divisors contributed by χ have valuations*

$$\frac{ha}{p-1}+\frac{j+(p-1)\sum_{c=1}^{s-1}p^{c-1}\lfloor j/\varphi(p^c)\rfloor}{\varphi(p^s)},\qquad 0\le j<\varphi(p^s).$$

The full multiset is the union over the chosen χ.

Proof. Let $\zeta=\psi(p^{-s})$. It is a primitive p^s-th root. On the b-coordinate the character matrix splits integrally at p, by character orthogonality and the invertibility of $p-1$. Each retained character gives one copy of evaluation of $1,T,\dots,T^{p^s-1}$ at the primitive roots $\zeta^{u_0},\dots,\zeta^{u_{n-1}}$, where $n=\varphi(p^s)$ and $u_0<\cdots<u_{n-1}$ are the positive units below p^s. There is also the scalar $G(\chi)^h$ and the row unit $\chi(\bar{\nu})^{-1}$. Divide the powers T^j for $j\ge n$ by the monic Φ_{p^s} to remove redundant columns by integral column operations.

For $0\le j<n$ use the monic basis $P_j(T)=\prod_{i<j}(T-\zeta^{u_i})$, an integral change of basis with determinant one. Its evaluation matrix is lower triangular. If $i\ne k$, the valuation of $\zeta^{u_i}-\zeta^{u_k}$ is $p^{v_p(u_i-u_k)}/n$. Indeed the quotient of the roots has order $p^{s-v_p(u_i-u_k)}$, and $\Phi_{p^b}(1)=p$ gives valuation $1/\varphi(p^b)$ for the difference from one of a primitive p^b-th root. All its conjugate differences have the same valuation, since exponents prime to p change them by units.

For each $1\le c<s$ the list of the u_i modulo p^c cycles through all $\varphi(p^c)$ unit residues in increasing order in each successive block. Among the first j terms each residue occurs at least $\lfloor j/\varphi(p^c)\rfloor$ times, and the residue of u_j occurs exactly that often. Expanding $p^b=1+(p-1)\sum_{c=1}^{b}p^{c-1}$ in the sum of the j pairwise valuations shows that $v_p(P_j(\zeta^{u_j}))$ is the second summand in the formula. All entries below it have at least this valuation. After factoring the diagonal entry from each column the remaining lower triangular matrix is integral with diagonal one. Its inverse is integral. The diagonal valuations increase with j, which gives the claimed elementary divisors. Finally Lemma 92.8 gives $v_p(G(\chi))=a/(p-1)$. Changing the primitive additive character in that lemma multiplies the sum by a root of unity. This accounts for the first summand. □

99.13 Indices of the successive lattices in a split Bessel flag

Use the split coordinates and character of Lemma 86.33. In the chosen basis let Λ be the full coordinate lattice, with A-part O^a. For $d\ge 1$ let $K(d)$ be the group of automorphisms

congruent to one modulo π^d on Λ, and let $K(0)=\mathrm{GL}(\Lambda)$. Let c be characterized by

$$\{t\in k:\psi_i(tO)=1\}=\pi^cO,\qquad D=\max(0,d-c).$$

The simple characters ψ_i are all equal in this frame. For $x_i\in k^a$ define $M_0=O^a$, $M_i=M_{i-1}+Ox_i$, and let r_i be the least nonnegative integer with $\pi^{r_i}x_i\in M_{i-1}$. Put $r_0=0$.

Theorem 99.14 (A bound for every contributing tuple)**.** *Suppose the row F is $K(d)$-fixed. If the value $L(R(P)F)$, with $P=\prod_{i=1}^{s}X_i(x_i)$, is nonzero, then*

$$x_i\in\pi^{c-d-r_{i-1}}M_{i-1},\qquad r_i\le iD,\qquad [M_i:M_0]\le q^{Di(i+1)/2}.$$

Consequently the joint exchange integral is a finite sum of values of this same functional, with at most

$$q^{asd+aD\binom{s+1}{2}+D\binom{s+1}{3}}$$

terms. Each term carries the measure $\prod_i\mu_i(\pi^dO^a)$. The estimates hold with arbitrary fixed data at the other places, including right translates there.

Proof. Set $x_0=0$. One has $P^{-1}e_l=e_l-x_{l-1}$. For each i we can find $w\in\sum_{l\le i}Oe_l$ whose e_i-coefficient is $\pi^{r_{i-1}}$ and for which $P^{-1}w\in\Lambda$. For $i=1$ take e_1. For $i\ge2$, write $\pi^{r_{i-1}}x_{i-1}=m+\sum_{j<i-1}t_jx_j$, with $m\in O^a$, $t_j\in O$. Take $w=\pi^{r_{i-1}}e_i-\sum_{j<i-1}t_je_{j+1}$. Its displacement under P^{-1} is $-m$.

Let $y\in\pi^dM_{i-1}^{\vee}$, where $M^{\vee}$ consists of covectors integral on M. Define

$$b=y-\sum_{j\ge i}y(x_j)d_j^{*},\qquad n=1+wb.$$

The covector b annihilates the span of $e_1,\dots,e_i$. The matrix n belongs to $N(k)$ by the flag description. Conjugating it by P on the right gives

$$P^{-1}nP=1+(P^{-1}w)(bP),\qquad bP=y+\sum_{j<i}y(x_j)d_j^{*}.$$

All the entries of bP belong to π^dO, so this last matrix fixes F. This also holds for $d=0$, since the inverse is obtained by subtracting the same square-zero matrix. It follows from N-equivariance that a nonzero value at P forces $\chi(n)=1$.

The only simple coordinate of wb that can contribute is its (e_i,d_i) entry. The earlier d_j are annihilated by b and w has no later row entries. On the opposite half n fixes the columns of the simple character, by the same inverse-adjoint calculation as in Lemma 86.33. Thus $\psi_i(-\pi^{r_{i-1}}y(x_i))=1$ for every such y. The annihilator of $M_{i-1}^{\vee}$ is π^cM_{i-1}, as one sees in a basis of M_{i-1}. This proves the first inclusion. It implies $r_i\le r_{i-1}+D$. The quotient M_i/M_{i-1} is cyclic over O with annihilator $\pi^{r_i}O$, so its order is q^{r_i}. Multiplication of the successive indices proves the bound for M_i.

Each $X_i(\pi^dO^a)$ fixes F, so the joint integrand is constant on products of these cosets. The earlier choices modulo π^dO^a determine M_{i-1}. The allowed x_i lie in $\pi^{-iD}M_{i-1}$, which contains π^dO^a. The number of its cosets is at most $q^{a(d+iD)+D\binom{i}{2}}$. Multiplying over i gives the assertion and additivity of Haar measure gives the weight. □

For the integral on the original word u, take the product of these bounds over $v\in S_u$ with its own q_v,d_v,c_v. If the prescribed row is $R(g)F_0$, with F_0 fixed by $K(d_0)$, one may take $d=d_0+t_{\max}-t_{\min}$, where $t_{\max},t_{\min}$ are the largest and smallest elementary-divisor valuations of g on Λ. Indeed the two integral basis changes in a Smith decomposition preserve every $K(j)$, and conjugation by its diagonal part lowers any entry valuation by at most $t_{\max}-t_{\min}$. Checking the inverse gives the lattice-stabilizer assertion also when $d_0=0$. This evaluates the joint integral on the prescribed translate throughout the same interval.

99.14 The exchange integral for a lattice in the middle block

Retain the flag, the roots $X_i(x) = 1 + x d_i^*$ and the ordinary Bessel functional L of Lemma 86.33. At a terminal split place the annihilator of O under the simple additive character is O. Take $s \geq 1$. Write the full lattice on the chosen half as

$$\Lambda = \bigoplus_{i=1}^{s} O e_i \ \oplus M \ \oplus \bigoplus_{i=1}^{s} O f_i, \qquad M \subset A_k \oplus kz.$$

Here M is any full lattice in the indicated space. Put $J = M \cap A_k$, and define $d \in \mathbb{Z}$ by $\{b : w + bz \in M \text{ for some } w \in A_k\} = \pi^{-d} O$. Let F be fixed by the stabilizer of Λ, allowing a character of its determinant. All other local data are fixed.

Theorem 99.15 (The surviving cosets in the middle block)**.** *If $L(R(\prod_i X_i(x_i))F) \neq 0$, then*

$$x_1, \ldots, x_{s-1} \in J, \qquad z - x_s \in M.$$

For $d < 0$ the exchange integral is zero. For $d \geq 0$, choose $x_* \in A_k$ with $z - x_* \in M$. Its value is exactly

$$\left(\prod_{i<s} \mu_i(J) \right) \mu_s(\pi^d J) \sum_{x \in (x_* + J)/\pi^d J} L(R(X_s(x))F).$$

In this notation the sum uses one representative of each coset. There are q^{ad} such cosets, where $a = \dim_k A_k$.

Proof. Write $M^\vee$ for the covectors integral on M, and likewise for $J^\vee$. Every $y \in J^\vee$ extends to some $b_0 = y + \alpha z^* \in M^\vee$. In fact projection identifies M/J with the free rank-one module $\pi^{-d} O$. Lift a generator, prescribe an integral value there, then extend the resulting functional on M by k-linearity.

Proceed from $i = 1$ to s. Once the preceding x_j lie in J, their shears have determinant one and preserve Λ. They commute with all the other shears and may be removed from the argument of F. Write $P = \prod_{j=i}^{s} X_j(x_j)$, which fixes e_i, and take any $b_0 = y + \alpha z^* \in M^\vee$. Extend it by zero on the flags. Then

$$b = b_0 P^{-1} = y - \sum_{j=i}^{s-1} y(x_j) d_j^* + (\alpha - y(x_s)) z^*.$$

The element $n = 1 + e_i b$ is in $N(k)$. Each flag vector moves by an earlier one, the action on the middle quotient is trivial, and the inverse adjoint on the other half moves only f_i. Moreover $P^{-1} n P = 1 + e_i b_0$ has determinant one and preserves Λ, as does its inverse. Evaluating $L(R(nP)F)$ first by this stabilizer and then by N-equivariance shows that $\chi(n) = 1$ is necessary. The simple coordinate at i is the only contributing one. For $i < s$ its value is $-y(x_i)$. As y ranges through $J^\vee$, triviality of its character forces $x_i \in J$. Indeed, in a basis of J each coordinate is in the annihilator of O. For $i = s$ the simple coordinate is $\alpha - y(x_s) = b_0(z - x_s)$. Applying the same annihilator calculation to a basis of M gives $z - x_s \in M$.

This last condition has a solution exactly when $1 \in \pi^{-d} O$, and its solution set is then $x_* + J$. The first $s - 1$ integrations now give their respective volumes. For $t \in \pi^d J$ the final shear $X_s(t)$ adds $bt \in J$ to every $w + bz \in M$. It fixes the row, so the remaining integrand is constant on each indicated coset. Their number is $[J : \pi^d J] = q^{ad}$. □

For the transported terminal graph of Proposition 71.20, one has $z \in M$ and $d = 1$. The formula uses exactly one sum over $J/\pi J$ at each terminal place, with its original Haar weight. At all f such places this requires at most T^{af} ordinary Bessel evaluations. Some of these evaluations can vanish.

99.15 The exchange at the two auxiliary split primes

Fix a place over g_1 or g_2 of Theorem 7.11. Use the half marked for the first position in Section 86.11. Let k be this local field, O its valuation ring, π a uniformizer, and q its residue cardinality. Use the split flag of Lemma 86.33, with chosen half ordered as

$$e_1, \dots, e_s,\ A,\ z,\ f_s, \dots, f_1.$$

The auxiliary primes are odd and split completely, and ϖ in $z = \varpi e$ is a unit there. The two isotropic strings are integrally dual and the column lattice is unimodular at these places. Let Γ be the preimage of the signed symmetric row level of Proposition 94.6 and λ_Γ its character. In the half defining this compact type the residue group is upper unitriangular and the entries from the marked middle line to A are zero. Its character is the chosen nontrivial additive residue character on the sum of the adjacent entries other than those at the two specified junctions, together with the entry from z to e_s. This description is `Formalism/LocalToGlobal/Columns/GenericUnipotentCharacter.lean`, line 605. In the conjugate half take the inverse adjoint of these matrices.

Let $b_1, \dots, b_{2s+1}$ be the ordered string $e_1, \dots, e_s, z, f_s, \dots, f_1$, omitting A, and let $Z_j(t) = 1 + b_j t b_{j+1}^*$. In this section b_j^* is the chosen coordinate functional. Write $\psi_j(t) = \chi(Z_j(t))$ and c_j for the integer determined by $\{t : \psi_j(tO) = 1\} = \pi^{c_j} O$.

Lemma 99.16 (The residue row character on this flag)**.** *For $1 \le i \le s$, the subgroups $X_i(\pi O^a)$ and $Y_i(O^{a*})$ preserve the row and act trivially in its compact character. On $N \cap \Gamma$, the character λ_Γ is the product of its values on the $2s$ string entries: each $Z_j(O)$ belongs to Γ, and its character is nontrivial on $O/\pi O$. The global Bessel character on $N(k)$ is the product of the nontrivial additive characters ψ_j on those same entries.*

Proof. All $X_i(\pi O^a)$ reduce to the identity, in either half. In the half defining the compact type, Y_i adds an A-coordinate to e_i. This is an allowed upper entry. The only possible adjacent entry of that form is at the junction between e_s and the first column coordinate, which the character omits. In the conjugate half the inverse adjoint has an entry from f_i to A in the defining half. The intervening marked line z separates every column coordinate from f_i, so these entries are allowed and have trivial character as well.

On N the entire middle block is the identity. In the defining half all surviving terms of the compact character are therefore the successive entries of the string, including the extra entry from z to e_s. Each has a nonzero residue multiplier. On passing to the opposite half the string reverses and dualizes. On these entries inversion is negation: each joins consecutive graded blocks of the parabolic flag, including the middle block at its two ends. A product through an intermediate strictly upper block cannot contribute. The inverse adjoint therefore permutes the entries, multiplies each by a unit and introduces a sign. This proves the compact assertions in either convention. For the Bessel character, the chosen half gives the first s entries directly by its defining trace, as in Lemma 86.33. The inverse adjoint supplies the last s by the same calculation. The factors in the split trace, the Hermitian marked-line pairing and its chosen multiplier are nonzero, proving the last assertion. □

Theorem 99.17 (A common auxiliary translate for both positions)**.** *There is a diagonal $t \in G(k)$, acting identically on A, with the following properties. Its eigenvalues l_j on b_j satisfy $l_1 = 1$ and*

$$v(l_{j+1}) - v(l_j) = 1 - c_j, \qquad \psi_j((l_j/l_{j+1})w) = \lambda_\Gamma(Z_j(w)) \quad (w \in O).$$

On any row F of the specified compact type and any column, the complete exchange at this place on $R(t)F$ equals its ordinary period on $R(t)F$ multiplied by

$$C_v = \prod_{i=1}^{s} \mu_i(\pi l_{i+1}^{-1} O^a).$$

The character of the translated compact row agrees with χ on $N \cap t\Gamma t^{-1}$. The local compact type admits a smooth compactly supported test with nonzero ordinary local pairing for the matching column character.

Proof. The pairing $(a, w) \mapsto \psi_j(aw)$ identifies $\pi^{c_j-1}O/\pi^{c_j}O$ with the characters of $O/\pi O$. Indeed its left kernel is zero by the conductor definition, and the groups have the same order. Choose a representative a_j of the nontrivial character in the lemma. Set $l_{j+1} = l_j/a_j$. These prescriptions define the diagonal in the chosen split half and the inverse adjoint in the other. This torus normalizes N. For $n = tn_0t^{-1}$ with $n_0 \in N \cap \Gamma$, its j-th string entry is (l_j/l_{j+1}) times that of n_0. Multiplying the displayed character identities proves agreement on the full intersection.

The translated row is fixed by $Y_i(l_i O^{a*})$ and $X_i(\pi l_{i+1}^{-1} O^a)$, by the lemma and conjugation. Under ψ_i the annihilator of the first lattice is $\pi^{c_i} l_i^{-1} O^a$, equal to the second lattice by the valuation relation. The finite Fourier sum in Theorem 86.34 has just the zero coset. It thus removes this exchange for the volume stated. Apply this successively to the tail functionals, starting at $i = 1$. Determinant twists of the row do not affect these shears.

For the last assertion write $K_t = t\Gamma t^{-1}$ with transported character λ_t. Extend λ_t by zero from K_t to $G(k)$, and extend $\lambda_t|_{H\cap K_t}$ by zero from $H \cap K_t$ to $H(k)$. These are smooth tests of compact support and the row has the required right character. If the column is nonzero at h, the row at nh is supported exactly on $n \in N \cap K_t$. Character agreement cancels the unipotent character and the conjugate column cancels the value at h. The double integral is the product of the two positive compact subgroup volumes. □

In the absolute-trace normalization $\theta = -1/\sqrt{-d_0}$ up to sign, all the string multipliers are units at these split primes. Thus $c_j = 0$, $v(l_j) = j - 1$ and

$$C_v = q^{a\binom{s}{2}} \prod_i \mu_i(O^a).$$

The proof of the exchange identity itself uses just the two trivial shear actions. It also applies at these levels with a trivial row character and the indicated valuations; the positive local test in the theorem uses the stated signed character.

99.16 The auxiliary translate with least loss on the theta tests

Use an auxiliary half over g_1 or g_2 of Theorem 99.17 with its absolute-trace normalization. Write q for its residue size and π for a uniformizer. The ordered string of lines has length $d_0 = 2s + 1$. The rest of the first row is the column space A. Take t equal to the identity on A and with string eigenvalues $\pi^{-s}l_1, \dots, \pi^{-s}l_{d_0}$, where the l_i are those of Theorem 99.17. The resulting string valuations are $-s, -s+1, \dots, s$.

Proposition 99.18 (Exchange with the centered auxiliary translation)**.** *At this translate the character on the compact intersection still agrees with the Bessel character. The local scalar removing the first-position exchanges is*

$$q^{-as(s+1)/2} \prod_{i=1}^{s} \mu_i(O^a).$$

The determinant of t is a unit. Replacing the auxiliary translates in Theorem 71.25 by these elements at both auxiliary primes gives its global comparison with the displayed local scalars, the same C_T and the same D_T^s.

Proof. The characters on consecutive string entries depend only on ratios of consecutive eigenvalues. All ratios are unchanged by the common factor. In the proof of Theorem 99.17 the row is fixed by the two shear lattices with scales l_i and πl_{i+1}^{-1} respectively. Replacing the eigenvalues multiplies these two lattices by π^{-s} and π^s. They are still annihilators of each other, so the only surviving exchange coset is zero. Its measure at step i is the measure of $\pi^{s+1-i}O^a$ since $v(l_{i+1}) = i$. The product is as stated. The sum of the centered valuations is zero, giving the determinant claim. The local changes take place at disjoint factors, so the successive integrations of Theorem 71.25 apply. □

In the growing tower of ranks $n_i = r + it_{\mathrm{r}}$ use the matrix indicator on each auxiliary edge from Proposition 72.15. Its support is $Y_i + \pi^{c_i}\mathcal{M}_i$, where $Y_i = \binom{I_{n_i}}{0}$ and $\mathcal{M}_i = M_{n_{i+1},n_i}(O)$. Extend t at every vertex by the identity in the additional coordinates. Write g_i for these extensions, and put

$$L_i = \frac{d_0(d_0^2-1)}{6} + \left(\frac{n_i + n_{i+1}}{2} - d_0\right) s(s+1).$$

Theorem 99.19 (The optimal overlap on each auxiliary theta edge)**.** *The normalized absolute coefficient of the original and translated indicators at the auxiliary place is q^{-L_i}. On their common subdivision each indicator has exactly q^{L_i} cells. The values of the translated indicator on those cells are algebraic units.*

This coefficient is largest among simultaneous translations at the vertices which preserve all the centers Y_i and induce the prescribed character ratios on the original diagonal string. Here the initial translation preserves that string and acts diagonally on its lines, with any common shift of the valuations allowed. The extension on the complementary coordinates at each vertex may be arbitrary. One choice attains these bounds on every edge at once.

Proof. In the displayed extension the reference lattice splits into the string and the other coordinate lines at every vertex. The exponents on the Hom lattice are all target exponents minus source exponents, as in Theorem 71.27. Among string pairs half the sum of their absolute differences is $\sum_{k=1}^{d_0-1} k(d_0-k) = d_0(d_0^2-1)/6$. For each zero exponent in the source or target complement the other string contributes $\frac{1}{2}\sum_{k=-s}^{s}|k| = s(s+1)/2$. The remaining pairs contribute zero. This gives L_i. The sum of the signed differences is zero, so both intersection indices are q^{L_i}. The centers agree. Dividing the measure of the intersection by the geometric mean of the two support measures gives the coefficient. In the matrix Weil action the half density is one. Its splitting character at the pair is a root of unity. Indeed a smooth character of each general linear group factors through determinant by elementary matrices, and its restriction to the compact unit group has finite image. Thus the coefficient of each individual cell in the indicator is a unit.

For the upper bound consider any allowed sequence b_i. The identity $b_{i+1}Y_i = Y_i b_i$ identifies a stable copy of the same string at every vertex. Its lattice is a direct summand in each reference frame. We first record the effect of passing to a stable subspace. Let W be a lattice in a vector space with automorphism B preserving a subspace V_1. Write $W_1 = W \cap V_1$, W_2 for the quotient lattice, and B_1, B_2 for the induced maps. The intersection $J = W \cap BW$ meets V_1 in $W_1 \cap B_1 W_1$ and its image in the quotient is contained in $W_2 \cap B_2 W_2$. Multiplying the subspace and quotient indices shows that $[W : J][BW : J]$ is at least the product of the corresponding two index products on the subspace and quotient. This follows by applying the index formula to $J \subset W$ and then to $J \subset BW$.

Apply this observation to $Y \mapsto b_{i+1} Y b_i^{-1}$ on the matrix lattice. First use the subspace of maps with image in the string. On that subspace and on its quotient then use the maps zero on the source string. The four resulting lattices are Hom lattices between the source and target strings and their complements, since the reference string inclusions split integrally. Thus the negative base-q logarithm of the normalized overlap is at least the sum of the four graded costs. On the string the exponents are $m, m+1, \dots, m+2s$ for some $m \in \mathbb{Z}$ fixed by the first vertex. The string-to-string cost stays $d_0(d_0^2-1)/6$. For each Smith exponent v on a complementary quotient, the mixed cost includes $\frac{1}{2}\sum_{k=0}^{2s} |m+k-v|$. This is minimized at the median $v = m+s$, with value $s(s+1)/2$. This can be checked by comparing the sum at consecutive integers. The cost between the two complementary quotients is nonnegative. These bounds give L_i on every edge. The chosen centered diagonal has $m = -s$ and all complementary exponents zero, so all inequalities are attained. □

99.17 The integral cost of isolating a vector marking

At one unrefined ordinary initial class a take its actual rank k lattice M, the prescribed coefficient weights λ_τ, and the product of vector spaces Ω_0 of Section 73.2. Put $f = |S_T|$ and $Q = T^{fk}$. Enlarge the coefficient field J to contain a primitive T-th root ζ. Set $u = \zeta - 1$, and put

$$z_{\min} = \prod_{\tau} \tau(\bar{\pi})^{\lambda_{\tau,k}}, \qquad z_{\max} = \prod_{\tau} \tau(\bar{\pi})^{\lambda_{\tau,1}}.$$

Write D_a for the column Weyl lattice in the unrefined coefficient frame of M over $\mathcal{O}_J$. At a refinement ξ, denote the actual column lattice by $D_{a,\xi}$. It depends only on the tuple of lines spanned by the ξ_P. Let S_a° consist of the separately zero-sum tables with value at ξ in $D_{a,\xi}$. Independence of the refined initial coordinates at this class is proved in Section 119.7.

Theorem 99.20 (Evaluation and prescribed support in the transformed lattice)**.** *Fix $v = (v_P) \in \Omega_0$ with every $v_P \neq 0$. For each P choose $d_P \in k_P^\times$ different from one. Use the transform $\mathcal{F}_0$ of Theorem 73.2. Then*

$$\{(\mathcal{F}_0 H)(v) : H \in S_a^\circ\} = u^f z_{\min} D_a,$$

$$\left\{ y : \Big(\bigotimes_P (\delta_{v_P} - \delta_{d_P v_P}) \Big) y \in \mathcal{F}_0 S_a^\circ \right\} = Q\, u^{-f} z_{\max} D_a.$$

Both are identities of fractional lattices on every sufficiently distant period interval.

Proof. For one product of covector lines choose representatives c_P. The coefficient lattice $D_{a,c}$ is constant on that product of scalar orbits. The separate augmentation kernels there are generated by tensors of differences of point masses, as in Theorem 119.11. If $c_P(v_P) = 0$ at some place, each such generator transforms to zero at v. Otherwise each nontrivial difference gives there a factor which is a difference of distinct T-th roots of unity. Its ratio to u is an algebraic unit: $\zeta^j - 1$ is divisible by u and conversely u by $\zeta^j - 1$ when $T \nmid j$, by the geometric sum in each direction. Independence of the orbits shows that the evaluation lattice is $u^f \sum_c D_{a,c}$, where the sum is over tuples with $c_P(v_P) \neq 0$ at every P.

For the second assertion put $D(v) = \bigotimes_P (\delta_{v_P} - \delta_{d_P v_P})$. Fourier inversion gives the unique preimage of $D(v)y$ at ξ as

$$Q^{-1} \prod_P \left(\psi_P(-\xi_P(v_P)) - \psi_P(-d_P \xi_P(v_P)) \right) y.$$

It has separate sums zero since $D(v)y$ does. For a tuple of allowed lines its coefficient is $Q^{-1}u^f$ times a unit. For every other tuple its coefficient vanishes. The condition on y is therefore $y \in Q u^{-f} \bigcap_c D_{a,c}$, over the same allowed tuples.

We calculate the sum and intersection in the actual coefficient frames. Let $M(c)$ be the sublattice of M defined by the covector kernels at $P \mid \pi$. The lattice used for column coefficients is the ordinary linear dual of the Hermitian dual of $M(c)$, in each coefficient embedding. This follows from the definition of the column on the Hermitian dual of the marked kernel and the inverse transpose action on its coefficients, as in Proposition 104.34. Equivalently see `Formalism/LocalToGlobal/Columns/StdLatUnimodular.lean`, line 117, `Formalism/LocalToGlobal/Columns/CoefCol.lean`, line 89 and `Formalism/LocalToGlobal/Divisibility/CoefMod.lean`, line 326. At a split pair $P, \bar{P}$ the Hermitian form pairs the two local lattices perfectly. Taking the Hermitian dual exchanges the two places and takes the ordinary dual there. Taking the second, ordinary dual recovers the original sublattice from the opposite place. Thus the coefficient lattice at $\bar{P}$ is, in the identified unrefined frame, $W_{\lambda_\tau}(L_{c_P})$ of Lemma 120.7. At P it uses the entire lattice. This also proves that all refinements agree away from T.

At a fixed coefficient prime over T, distinct embeddings τ induce distinct primes of E: each completion $E_P = \mathbb{Q}_T$ has only one embedding over $\mathbb{Q}_T$ into an algebraic closure. The affected factors correspond to τ inducing $\bar{P} \mid \bar{\pi}$; each then depends on its own c_P. At such a factor apply Lemma 120.7 with $\mathcal{U} = \{v_P\}$. Its cutoff holds since all initial spreads are $O(xb + x^C)$ by (9.1) and $\log T \gg mx^{\varepsilon}$ on this family. Sums over the independently chosen c_P factor as tensors of sums. For intersections take ordinary lattice duals, tensor the sums of the duals and dualize again. The unrefined factors stay in their original frames. Normalizing the valuation by $v(T) = 1$, each $\tau(\bar{\pi})$ has valuation one in an affected factor and zero otherwise. Hence locally the sum is $z_{\min} D_a$ and the intersection is $z_{\max} D_a$. At every other prime both scalars are units and every lattice is D_a. These local identities prove the two global ones. □

Corollary 99.21 (The two weight extremes on the common fields)**.** *For the coded ordinary columns, the normalized logarithmic finite cost of evaluation at a single v relative to D_a is $o(b/x^A)$ for every fixed $A > 0$. If there is a high embedding, at some prime above the same T the valuation difference between the specified-support and evaluation lattices in the theorem, multiplied by $\log T$, is at least $(\lceil xb \rceil + \lceil xb/\exp(x^{\varepsilon/3}) \rceil) \log T$. Both assertions concern the same word and coefficient module.*

Proof. In rank k at the initial vertex the algebraic weight is obtained by subtracting $\rho_k(i) = (k+1)/2 - i$ from u_i. Thus $\lambda_k = -b_*$ at a high embedding and $\lambda_k = (k-2)/2$ elsewhere by (9.1), where $b_* = \lceil xb/\exp(x^{\varepsilon/3}) \rceil$. At the high embedding $\lambda_1 = \lceil xb \rceil$. The two extremes are independent of the word for a fixed translate. One has $v_T(u) = 1/(T-1)$, by evaluating the T-th cyclotomic polynomial at one and using the unit ratios above. The only finite cost for evaluation is therefore bounded in absolute value by $f(b_* + k + 1) \log T$, with supremum over all primes above T in the normalization $v(T) = 1$. Both f, k and $\log T$ are polynomially bounded in x. This proves the first estimate. Choose a prime dividing $\tau(\bar{\pi})$ for one high embedding. At this prime the difference in valuations is

$$fk - \frac{2f}{T-1} + \sum_{\tau' \colon v(\tau'(\bar{\pi}))=1} (\lambda_{\tau',1} - \lambda_{\tau',k}).$$

The first term after combining the two Fourier contributions is nonnegative, as are all spreads, and the chosen spread is $\lceil xb \rceil + b_*$. This proves the lower bound. □

99.18 Simultaneous interpolation of vector markings

Retain the actual coefficient lattice at the ordinary class a of Theorem 99.20. At each $P \mid \pi$ choose a set G_P of $s_P \geq 1$ distinct nonzero vectors in M/PM. Put $G = \prod_P G_P$, $s = \sum_P s_P$, and $D_a^G = \bigoplus_{v \in G} D_a$. Let E_G take the values of a vector table on G. Values outside G are free.

We use the same unnormalized Fourier transform, root $u = \zeta - 1$, and weight scalar $z_{\min}$ as in that theorem.

Theorem 99.22 (Integral interpolation on a product of residue sets)**.** *Let $h = \max_\tau(\lambda_{\tau,1} - \lambda_{\tau,k})$. Suppose $T > h + k$ and, at each P,*

$$s_P + 1 \le T - 1, \qquad T - \binom{s_P + 1}{2} > h.$$

Then the lattice of simultaneous evaluations satisfies

$$u^s z_{\min} D_a^G \;\subseteq\; E_G \mathcal{F}_0 S_a^\circ \;\subseteq\; u^f z_{\min} D_a^G.$$

For $s_P \le \lfloor \sqrt{T} \rfloor$ on every sufficiently distant common interval, these hypotheses hold for all selected words. In that range one integer $B_x = T^{1+f(b_*+k)}$ satisfies

$$B_x D_a^G \;\subseteq\; E_G \mathcal{F}_0 S_a^\circ \;\subseteq\; B_x^{-1} D_a^G, \qquad \log B_x = o(b/x^A) \quad (A > 0 \text{ fixed}).$$

The integer is common to all the stated grids and words with this terminal prime.

Proof. The upper inclusion is Theorem 99.20 at each coordinate. For the lower inclusion enumerate the vectors at one place as $v_1, \dots, v_{s_P}$. Use the list consisting of all v_i and all $v_i - v_j$ for $i < j$. It has at most $\binom{s_P+1}{2}$ members, each nonzero. A covector c_P nonzero on this list takes distinct, nonzero values t_i on the v_i. Fix such a covector and an index j, and write $z_i = \psi_P(t_i)$. The z_i are distinct nonidentity T-th roots because the residue field has prime order. Use the coefficients of the polynomial

$$p_j(Z) = Z(Z-1)\prod_{i \ne j}(Z - z_i) = \sum_{d=1}^{s_P+1} p_{j,d} Z^d$$

as the coefficients of point masses at dc_P on the covector side. All d are distinct nonzero residues. Their coefficients have sum zero since $p_j(1) = 0$. The transform of this array at v_i is $p_j(z_i)$. It vanishes for $i \ne j$. At v_j it is u^{s_P} times a unit of $\mathcal{O}_J$, by the unit calculation for differences of roots in Theorem 99.20.

Take the tensor product of these arrays for one chosen slot of G and multiply it by a vector in $D_{a,c}$. All its nonzero entries have the same product of covector lines, hence belong to their prescribed coefficient lattice. Its transform on G is supported on the chosen slot and takes every value in $u^s D_{a,c}$ there, by multiplication by the inverse of the unit. Vary the good covector independently at each place. At any prime over T the lattice computation in the proof of Theorem 99.20 applies with the lists just chosen. Lemma 120.7 gives $\sum_c D_{a,c} = z_{\min} D_a$ after localization. Elsewhere the same equality holds since the sets of choices are nonempty and the lattices coincide. Equality follows globally by localization. Summing these constructions on all the slots proves the first assertion.

On the given family h/T tends to zero uniformly, by (9.1) and Theorem 9.13. For $s_P \le \sqrt{T}$ the number of excluded hyperplanes is at most $(T + \sqrt{T})/2$, which gives the two cutoff inequalities. In the normalization $v(T) = 1$ one has $s\, v(u) \le f\sqrt{T}/(T-1) < 1$ eventually, since f is polynomial in x and $\log T \gg m x^\varepsilon$. At any prime above T, $|v(z_{\min})| \le f(b_* + k)$ by Corollary 99.21. Outside T all the scalars are units. Comparing valuations gives the stated two integer inclusions. Finally $b_* = \lceil xb/\exp(x^{\varepsilon/3}) \rceil$ while $k, f, \log T$ are polynomially bounded in x. The claimed rate for $\log B_x$ follows. □

100 Local factors for a reversed block

100.1 The split factors for two unrestricted local valuations

Use $a_{\pm}, L, \epsilon$ of Lemma 99.9 and put $m_0 = a_+ + a_-$, $C_k = 4(k+1)^4$. Write $F_v(c, \lambda)$ for the orbital operator on the original local columns, relative to the identity mass.

Proposition 100.1 (A uniform split bound for one moving line)**.** *At a split place on the support with $m_0 > 0$ one has*

$$\|F_v(c,\lambda)\| \le C_k^{m_0} P^{(k+4s)m_0/2+k\epsilon} \begin{cases} 1, & v \nmid T, \\ P^{m_0/2+s|L|}, & v \mid T. \end{cases}$$

When $m_0 = 0$ the norm is at most one. If the original support is empty, its operator is zero.

Proof. The first-plane decomposition $B = J \ltimes U$ and the commutation in Proposition 111.16 give the counting measure and relative translations of Theorem 99.10. Indeed the stabilizer ratio of a finite cell is the size of its orbit of right U_L-cosets under $J \cap K$, by the same diagonal block argument. At a full column level its coefficient is a translation compressed to $C = \mathrm{GL}_k(O)$ invariants. At $v \mid Q$, the support is $K_1 B$ and C acts on its cosets as in Proposition 99.7. The new isometry still commutes with C. The compact conjugation average therefore uses the same principal-congruence ambiguity and the same scalar character phases. At these places each orbit average is bounded by $\Xi(A_0)$ by Lemmas 104.30 and 99.9. Here translating by a central scalar preserves the spherical majorant, since the inducing character for Ξ is trivial on the center.

At T sum first over the enlarged compact orbits of Proposition 99.4, inserting a zero at every unsuccessful coset. The decomposition, the success test $y\beta^{-1} \in S_0$ and the density bound $|\phi(y)| \le 1$ there use only commutation with D_0, which still holds. The affine change $y \mapsto y\beta^{-1}$ now has linear part A of arbitrary determinant. The weighted composition has norm at most $|\det A|^{-1/2}$ by additive change of variables. Lemma 104.31, after division by that scalar, gives the local orbit majorant

$$P^{(e_- - e_+ - nL)/2}\, \Xi(A_0).$$

Indeed $v(\det A) = e_- - e_+ - nL$. The two exponents of A_0, and this determinant, are constant on each compact orbit. Assign its majorant to each member to sum the orbit sizes. The depths may vary among the members.

We count using X, Y, Z from Lemma 99.9. Right multiplication by an integral first unipotent chooses representatives of H, l modulo the integral vector and row lattices. At depths t, u' there are at most $P^{k(m_0-t-u')+(n-k)m_0}$ such choices. With H, l fixed, the two endpoint tests for z each prescribe a coset of a fractional integral lattice, of respective denominator lengths a_-, a_+. Their intersection contains at most $P^{\min(a_-,a_+)}$ classes modulo O. This continues to hold after the choice of lifts, because right integral translations preserve support. For $m_0 = 0$ this gives at most one coset and a single unitary translation has norm at most one.

For $m_0 > 0$ the spherical bound is $k^{d_-+d_+} P^{-(k-1)(d_-+d_+)/2}$. Use $e_+ + e_- - 2S = m_0 - 2(t + u') - 2\epsilon$ in Lemma 99.9. Apart from the L-terms, the contribution of the k coordinates and the negative spherical exponent is at most

$$(k+1)m_0/2 - (t+u') + (k-1)\epsilon$$

in the exponent of P. The other coordinates and the center cost at most $(n - k + 1/2)m_0$. Their sum is at most $(k+4s)m_0/2 + k\epsilon$. Away from T the remaining L-term is nonpositive.

At T combine it with the determinant contribution. For $L \geq 0$ the sum of the L-terms is $((k-1)-n)L/2 \leq 0$. For $L < 0$ it is $(n-k+1)|L|/2 = s|L|$. The remaining determinant term $(e_- - e_+)/2$ is at most $m_0/2$ there. Finally there are at most $(a_+ + 1)(a_- + 1) \leq 2^{m_0}$ depth pairs and $d_- + d_+ \leq m_0 + 2\epsilon \leq 3m_0$. These factors fit in $C_k^{m_0}$, proving the bound. □

100.2 Finite mass for a single reversed block

Proposition 100.2 (A bound from the two prefix determinants)**.** *Let w be the supported representative in Proposition 100.7 with cross block $C = cS_j$, and put $A_\xi = |N_{E/\mathbb{Q}}\xi|^{1/2}$. The product of its finite unsigned masses relative to the identity is at most*

$$(Q^f A_\xi)^{2rj}.$$

The same expression bounds its ordinary finite column operator in norm on the original level.

Proof. At a split place let $c_\pm$ be the two components, $e_\pm = v(c_\pm)$, and $P = Nv$. Work in the actual integral first-component basis, including the graph at T. Write the upper blocks of $n_0 \in U$ as l, H, Z and the row lattice as Λ. Put $C_+ = c_+ S_j$. The intersection of $g^{-1}\Lambda$ with the first j-plane for $g = n_0^{-1}w$ is $C_+^{-1}L_+$, where

$$L_+ = \{v \in O^j : Hv \in O^n,\ (lH - Z)v \in O^j\}.$$

Indeed applying $g^{-1} = w^{-1}n_0$ to an integral input with last coordinate v gives zero middle and last coordinates precisely when its middle coordinate is $-Hv$ and its first coordinate is $(lH - Z)v$. If $g \in KB$, its inverse lattice is a period translate of Λ. The first j-plane is stable under B, and B acts on it by upper unitriangular matrices. Its intersection lattice therefore has the same covolume as O^j. Consequently $[O^j : L_+] = P^{m_+}$, where $m_+ = je_+$. Right multiplication of n_0 by U_L preserves $g^{-1}\Lambda$, so this intersection and L_+ are well defined for a coset.

In the reversed dual component the three upper blocks are $-H^{\mathsf{t}}, -l^{\mathsf{t}}, (lH - Z)^{\mathsf{t}}$. Their rectangular column and their product minus the central block are $-l^{\mathsf{t}}$ and Z^{t}. The same intersection calculation gives a lattice $L_- \subset O^j$ of index P^{m_-}, $m_- = je_-$, on which these two matrices are integral. The sign permutation of the exchange is integral on both sides. All arguments apply to the original row level because it stabilizes the chosen row and dual lattices.

There are at most $(2P^{j-1})^m$ sublattices of index P^m in O^j. Each is reached by a chain of m steps of index P, and from any free rank-j lattice there are $(P^j - 1)/(P - 1) \leq 2P^{j-1}$ such steps. For fixed L_+, each row of H, modulo integral rows, belongs to $L_+^\vee/(O^j)^\vee$ of order P^{m_+}, by elementary divisors. There are n such rows. For fixed L_-, the n rows of l^{t} and the j rows of Z^{t} allow at most $P^{(n+j)m_-}$ residue choices. To use this count, choose the exact lifts of H, l first by right integral multiplication. The two kernel lattices persist under this operation. Thereafter Z varies only modulo the integral center in a given right coset. Thus the local mass is at most

$$(2P^{j-1})^{m_+ + m_-} P^{nm_+ + (n+j)m_-} \leq P^{rj(e_+ + e_-)},$$

using $2 \leq P$ and $n + 2j = r$.

At an inert place use the quadratic integer ring with residue cardinality $P' = P^2$, and put $e = v_w(c)$. The same lattice L_+ has index $(P')^{je}$. Now the other rectangular block is the negative adjoint of H for the middle form and $lH - Z = Z^*$, with star on the square center denoting conjugate transpose. Thus L_+ alone gives at most $(P')^{(n+j)je}$ row residue choices for H, Z^*. To lift an integral shift of H to U_L, choose one central entry from each off-diagonal pair and solve the diagonal trace equations by integral trace surjectivity in the unramified

quadratic extension. For a fixed rectangular lift, two admissible central blocks with the same residue modulo integral matrices differ by an integral skew matrix and are equivalent. Including the choices of L_+ gives at most $(P')^{rje} = P^{2rje}$.

The factors of $|N_{E/\mathbb{Q}}c|^{1/2} = Q^f A_\xi$ are $P^{(e_+ + e_-)/2}$ and P^e, respectively. Multiplication proves the mass estimate by Proposition 100.7. Each term in its coset count acts by a relative unitary column translation, with compact averaging if needed, and a scalar character of modulus one. This gives the same bound for the operator. □

100.3 The two prefix lattices at the marked primes

Proposition 100.3 (The finite mass with the row congruences imposed)**.** *Write $A_\xi = |N_{E/\mathbb{Q}}\xi|^{1/2}$ for the nonzero integer in $c = Q\xi$ from Proposition 100.7. The full finite unsigned mass relative to the identity is at most*

$$Q^{j^2 f} A_\xi^{2rj}.$$

The same bound holds for the finite operator norm on the given column.

Proof. Use the notation of Proposition 100.2 at a place over $Q = g_1 g_2$. Both primes split completely. Put $P = Nv$, $\pi O = \mathfrak{p}_v$, $a_\pm = e_\pm - 1$, and let Λ be the row lattice in the first component. For a supported $g = u_0^{-1} w = k_0 b_0$, with k_0 in the actual row level and $b_0 \in B$, its first-plane image satisfies

$$gP_j \cap \Lambda = k_0(P_j \cap \Lambda).$$

The reduction of the original row level preserves the first j-plane in this lattice. This follows from its upper block form in Lemma 98.19. Therefore both the middle and last projections of every vector in this intersection vanish modulo π. Its vectors have last projection $v \in L_+$ and middle projection $-Hv$, and every $v \in L_+$ occurs, by the explicit description in Proposition 100.2. Hence

$$L_+ \subset \pi O^j, \qquad HL_+ \subset \pi O^n.$$

In the reversed dual component the reduction also preserves the first j-plane. It is the annihilator of the initial subspace of codimension j in the original upper block flag. The identical argument gives $L_- \subset \pi O^j$ and $l^{\mathsf{t}} L_- \subset \pi O^n$.

Now $L'_\pm = \pi^{-1} L_\pm$ have indices $P^{ja_\pm}$. Each row of H is integral on L'_+ and each row of l^{t} on L'_-. The j central rows of Z^{t} are integral on $\pi L'_-$, of index $P^{j(a_- + 1)}$. Counting lattices and row residues as in Proposition 100.2 gives, with $m'_\pm = ja_\pm$, a local bound

$$(2P^{j-1})^{m'_+ + m'_-} P^{nm'_+ + (n+j)m'_- + j^2} \le P^{rj(a_+ + a_-) + j^2}.$$

Choosing fixed row lifts by right integral multiplication preserves support and the kernel lattices, as in that proof. At places away from Q use its unmarked bounds directly. Their factors and $P^{rj(a_+ + a_-)}$ above Q multiply to A_ξ^{2rj}. There are f places of F over each marked prime, giving the claimed prefactor. Taking norms of unitary translations, also after compression, proves the operator assertion. □

100.4 The finite operator for all blocks in a composition

Proposition 100.4 (Finite cosets over the internal unipotent)**.** *Put $\delta_k = 100(s+1+\log_2(2(k+1)))$. For any supported composition of Proposition 100.9 with $2 \le j \le s$, the finite signed operator has norm at most*

$$|N_{E/\mathbb{Q}} G_j|^{(k+\delta_k)/2} \prod_{p=1}^{j} |N_{E/\mathbb{Q}} G_p|^{u_p}.$$

Proof. At a finite place fix z in the prefix unipotent. To count the accompanying u_0 use $g = u_0^{-1} w_{C(z),\lambda}$. On the support write $g = k_0 b$ with k_0 in the original row level and b in the period subgroup, enlarged to B' when counting the affine orbits over T. On the j-plane let b_1 be the matrix of b^{-1}, which is upper unitriangular. The first j columns of $gb^{-1} = k_0$ are integral in the row basis. Their bottom block is $C(z)b_1 = z^* C b_1$, because the last row block of u_0^{-1} is $(0, 0, I_j)$. In the split case this argument applies in both the primal and reversed dual components.

Apply Lemma 107.5. All its minors are integral, so $v(G_p) \geq 0$ and its u_p entries in column p lie in $G_p^{-1} O$. At an inert place their coordinate volume costs at most $\prod_p (P^2)^{u_p v(G_p)}$, with $P = Nv$. At a split place conjugation interchanges the two components of z. Counting them both gives at most $\prod_p P^{u_p(v(G_{p,+}) + v(G_{p,-}))}$. These volume bounds also bound the numbers of possible right $\mathcal{V}_L$-cosets. Indeed if an element z admits a supported u_0, the same holds for each zv_0 with $v_0 \in \mathcal{V}_L$, by replacing u_0 with $\widehat{v_0}^{-1} u_0 \widehat{v_0}$. This changes the original unipotent by right integral multiplication. Each such coset has coordinate volume one. Multiplication of the local bounds gives the displayed product over p.

For the remaining estimate at fixed z, the prefix lengths $m_\pm$ of Proposition 100.13 are now $v(\det C(z)_\pm) = v(G_{j,\pm})$. To see that its proof applies, the intersection of the inverse lattice with the j-plane is still $C(z)_+^{-1} L_+$, and L_+ is still the kernel defined by $H, lH - Z$ in Proposition 100.2. The determinant of the triangular period block on that plane is one. The same statements in the reversed dual give $m_- = v(G_{j,-})$ and the kernel defined by $l^{\mathsf{t}}, Z^{\mathsf{t}}$. The middle projection in Lemma 100.8 still has the formula $m + Hv$ with $lm + Zv$ integral. It only uses that the outer exchange blocks are invertible. Thus both projected-length estimates and the determinant relation for the column hold with these lengths. The compact groups used in the split proof commute with $\widehat{z}$ and with w. At T their action on the middle block and its success test are the original affine ones. Hence the same compact-orbit estimate applies at each fixed z, allowing the actual scalar phases of its terms. The residue count for the rectangular and central coordinates used only the lengths of the two kernels. We obtain the split bound in Proposition 100.13 with $M = v(G_{j,+}) + v(G_{j,-})$.

At an inert place the adjoint kernel and the unimodular column calculation in Proposition 101.7 apply with $m = v(G_j)$. Indeed $lH - Z = Z^\dagger$ and the middle projection are unchanged, and $C(z)$ enters the index calculation only through its determinant. Taking the product of these bounds for the fixed-z fibers costs at most $|N_{E/\mathbb{Q}} G_j|^{(k+\delta_k)/2}$ by the final comparison in Proposition 101.7. This bound is independent of z. Multiplying by the number of possible internal cosets proves the claim. □

100.5 Combining the real and finite transverse factors

For a supported transverse label with $\mathcal{G} > 0$, use the arithmetic endpoints a_i of Proposition 126.5. Write $j_i = J_i,\ h_i = j_i - j_{i-1}$, and put

$$H_i = \prod_\tau \max(1, |\tau a_i|), \qquad A_- = \max\left(1,\ Q^{-df} \prod_\tau (|\tau D_d| - T|\tau D_s|)\right).$$

The terminal height H_t is calculated from $\Delta_+ = D_d + TD_s$ at every place. In particular $A_- \leq H_t$ by the triangle inequality. Let B_γ denote the sum of absolute normalized cell traces in Proposition 112.4.

Theorem 100.5 (A bound for a transverse label at all places)**.** *Take any retained fraction $\vartheta \geq 1/2$ as in that proposition. For arbitrary $\theta_i \geq 0$ of total weight $L = \sum_i j_i \theta_i \leq r^2/2$, one*

has

$$B_\gamma \le \frac{\exp(Cfkd + CfdL\log b)}{(k!)^{df}}\, Q^{-dfk} A_-^k \prod_{i=1}^{t} H_i^{-3-\theta_i}.$$

All original finite places, the terminal covector level and the restriction to the specified terminal character are included.

Proof. Set $G = |N_{F/\mathbb{Q}}\mathcal{G}|$ and $S = d(d-1)$. For the original block scalars c_i put $b_i = 2h_i(d - j_i) + \binom{h_i}{2}$. The finite formula and the free-entry volume bound in Theorem 98.32 give

$$\frac{\mathcal{M}_\gamma^{=}}{\mathfrak{m}_{k-1}} \le \exp(O(f2^{-(k-2d)}))\, G^{k-1} Q^{f(2a-S-2d(k-1))} \prod_i |N_{E/\mathbb{Q}} c_i|^{b_i}.$$

Here the Q-power includes the coincidence condition on the two reduced flags. In multiplying the finite and real bounds we keep this saving.

Put $B_i^+ = \prod_\tau |Q^{j_i}\tau a_i|$, with $B_0^+ = 1$. These are at least one since the $Q^{j_i}a_i$ are nonzero algebraic integers. At each block with $b_i > 0$ one has $|N_{E/\mathbb{Q}}c_i| = (B_i^+/B_{i-1}^+)^{2/h_i}$. Indeed this is the ratio of successive endpoint determinants. Only a final singleton can have its pivot changed by forming Δ_+, and then $b_t = 0$. Telescoping gives the exact finite pivot factor

$$\prod_{i<t} (B_i^+)^{h_i+3h_{i+1}}\, (B_t^+)^{h_t-1}.$$

The exponents are nonnegative and sum to at most $4d$. Each $B_i^+ \le Q^{j_i f} H_i$. Also, for every p with $u_p > 0$, the product of the absolute arithmetic pivots up to p, taken over all real places, is at least one. Inside block i it is the geometric interpolation of B_{i-1}^+ and B_i^+, because the pivots there have equal moduli. Consequently the inverse between-block factors in Proposition 126.5 cost at most $(2T)^{2f\sum_p p u_p}$. This argument also covers a nonintegral individual block scalar.

In the product of the remaining real powers the factors $G^{-\kappa}$ cancel all but G^s in the finite mass, since $k - 1 - \kappa = s$. The power of $\prod_\tau L_\tau$ is $2\kappa - E_0 = \kappa - 2ds - s^2$, between zero and k on our intervals. Hence that power contributes at most $Q^{df(2\kappa - E_0)} A_-^k$. To bound the residual G^s, expand $\mathcal{G} = |\Delta_+|^2 - 2T\,\mathrm{Re}(\Delta_+\overline{D_s})$. At a real place this is at most $(1 + 2T)\max(1, |\tau\Delta_+|)^2 \max(1, |\tau D_s|)$. Thus G^s costs at most $\exp(O(fd^2\log b))H_t^{2s}$ times H_{t-1}^s when $D_s \neq 0$. In that case $j_{t-1} = s$. When $D_s = 0$ the last factor can be omitted.

Combining these powers with the finite pivot factor leaves positive exponents v_i on the H_i whose sum is at most $4d + 3s$. Use $\eta_i = v_i + 3 + \theta_i$ in Proposition 126.5. Then $\sum_i j_i\eta_i \le O(d^2) + L \le r^2$. The cost of the fixed $O(d^2)$ in these tests fits in $\exp(Cfkd)$.

The terminal class count contributes $(T^k - 1)^{-f}$ by Proposition 112.4. After including it, the remaining exponent of T, divided by f, is

$$dE_0 + 2\sum_p p u_p - s(k+d) - k = O(d^3).$$

Here $(1 - T^{-k})^{-f} \le \exp(Cf)$. Since $\log T = O(d\log b)$ and $d^3 \log b = o(k)$, the remaining T-cost is at most $\exp(Cfkd)$. The powers of two have the same bound. Adding the finite pivot bounds, the Q-exponent divided by f is at most

$$-2d(k-1) + d(2\kappa - E_0) + O(d^2) \le -dk + O(d^2).$$

Its $O(d^2)$ term also fits the stated exponential. Finally Theorem 93.14 gives $\mathfrak{m}_{k-1}/\mathfrak{m}_k \le e^{Cfk}(k!)^{-f}$, since $|\Delta_F|, \zeta_F(k) \ge 1$. Together with the s factorials per place in the real integral this gives d factorials and proves the theorem. □

100.6 The stabilizer volume of a single terminal flag

Take the original fine row level K^0. For a supported transverse cell let Γ_D be its rational common unipotent stabilizer in the monomial frame coordinates of Proposition 112.4. Put $a_0 = \sum_i \binom{h_i}{2}$ and $c_E = 2^{-f}\sqrt{|\Delta_E|}$.

Proposition 100.6 (A uniform covolume for counting individual cells)**.** *If $\mathcal{G} > 0$ and $H_* < b^{Mfd}$, the original unipotent covolume at every fine cell satisfies*

$$\operatorname{vol}(\Gamma_D \backslash D_\infty) \le c_E^{a_0} \exp(C f d^6 \log b).$$

Proof. Choose primitive integral representatives of the two decorations in the row Λ, with cross matrix $Z = LM_0V$ as in Lemma 119.2. Put $u = a\bar{a}$. The changes from the monomial frames to these integral frames can be taken to be L^*, V. Indeed their cross matrix is Z. Any initially given pair of upper changes yielding the same Z differs from this choice by a simultaneous stabilizer of M_0. It has inverse-adjoint paired actions on the two frame spans and extends by the identity on their perpendicular. It belongs to the common unipotent group in Lemma 98.6. Inner changes of coordinates in this group preserve its Haar measure. A transport of the whole framed row simply transports the measure with it. Thus covolume can be computed in these frame coordinates.

In monomial coordinates an element of D acts on the first frame by a block diagonal matrix of upper unitriangular blocks A_i, on the second by the conjugate inverse transpose transported through M_0, and on C by the identity. Inside each block the scalar in M_0 cancels. For a nonzero integral element l, if $A_i \equiv 1 \bmod l$ entrywise and $l\mathcal{O}_E$ is conjugation invariant, then the second matrix is also the identity modulo l. Both L^*, V have entries in $u^{-1}\mathcal{O}_E$ and their inverses in $u^{-(d-1)}\mathcal{O}_E$. Conjugating the two actions into the integral bases therefore gives the identity modulo $lu^{-d}\mathcal{O}_E$.

Take $l = Q\mathcal{G}\,u^d$, which is a nonzero integer of F. Put $U_0 = \Lambda \cap Y_0$, $U_1 = \Lambda \cap Y_1$ and $U' = U_0 \oplus U_1$. In the integral frames its Gram determinant is $\pm\mathcal{G}$. Thus $\mathcal{G}(U')^\vee \subset U'$ by the adjugate identity. The projection of Λ onto $Y_0 + Y_1$ lies in $(U')^\vee$ by pairing against U'. For the elements just constructed, the difference from the identity maps U' into $Q\mathcal{G}U'$ and hence maps Λ into $QU' \subset Q\Lambda$. Its inverse has the same property. Such an element preserves the lattice at all places, including its modification above T, and is the identity on the residue row above Q. It therefore belongs to the original fine stabilizer.

The product of the upper groups with all off-diagonal entries in $l\mathcal{O}_E$ has covolume $(c_E|N_{E/\mathbb{Q}}l|)^{a_0}$ in D_∞. In entry coordinates Haar measure is product area. Successively quotienting by the entries on each upper diagonal gives the product of their additive lattice covolumes, since the cross terms in multiplication involve earlier diagonals. This subgroup is contained in Γ_D. The bound $\log G = O(fd^2 \log b)$ follows from the height calculation in Theorem 102.4. Lemma 119.2 gives $\log|N_{E/\mathbb{Q}}l| = O(fd^4 \log b)$, including $|N_{E/\mathbb{Q}}\mathcal{G}| = G^2$. Since $a_0 \le d^2$, the proposition follows. □

100.7 A reversed block of arbitrary length

Fix $2 \le j \le s$. In the first j signed planes choose integral Witt pairs as in Proposition 111.15. Write P_j, P_j^* for their primal and opposite subspaces and M_j for the perpendicular of these planes, of dimension $n = r - 2j$. Let S_j denote the signed permutation matrix whose antidiagonal reads $1, -1, \dots, (-1)^{j-1}$. Put $C = cS_j$. In block order P_j, M_j, P_j^* consider the isometry

$$w = \begin{pmatrix} 0 & 0 & (C^*)^{-1} \\ 0 & \lambda I & 0 \\ C & 0 & 0 \end{pmatrix}, \qquad \lambda\bar{\lambda} = 1,$$

where star on C is conjugate transpose. Use the one-word orbital operator O_w in identity normalization.

Proposition 100.7 (The common stabilizer and its unipotent quotient)**.** *This representative covers the supported relevant labels with a single reversed prefix block of length $j < s$ in Theorem 110.9. It also covers the labels with one reversed isotropic block in the full common column case of Proposition 109.40 at $j = s$. In either case $c = Q\xi$, $0 \neq \xi \in \mathcal{O}_E$, and $T\xi^j\lambda$ is integral. Put $J = B \cap wBw^{-1}$. Then $B = J \ltimes U$, where U is the unipotent radical for the whole j-plane P_j.*

Proof. The first j vectors pair across the two flags by C, and w multiplies the other vectors and the terminal vector by λ. The two cited classifications give the rational double coset from these data. The last prefix determinant divided by Q^j is a sign times $(c/Q)^j$. Its integrality implies integrality of c/Q by taking every prime valuation. For $j < s$, Proposition 94.20 gives integrality even of $\xi^j\lambda$. For $j = s$ the terminal congruence in Proposition 109.40 gives the assertion with T.

An element of J preserves P_j and P_j^*. It preserves their nondegenerate sum and its perpendicular, so it is block diagonal. Its primal block ranges over upper unitriangular j-square matrices, with the inverse adjoint on the opposite space. The signed reversal conjugates lower unitriangular matrices into upper unitriangular ones, so each such upper block preserves the second prefix as well. On M_j the two residual Bessel groups coincide because the second frame is scalar times the first. Thus J is the product of this prefix unitriangular group and the residual Bessel group. The block diagonal projection of B has image exactly J, with kernel U. The exchange normalizes J. It centralizes the middle group. On the prefix group it takes inverse adjoints and reverses the indices with unit signs. This preserves Haar measure at every place, as can also be seen by filtering upper entries by superdiagonal. Conjugation by J preserves Haar on U because the prefix determinants are one, the middle action is unitary and the center transforms by congruence on the prefix.

Let $U_L = U \cap \mathrm{Stab}(\Lambda_{\mathrm{row}})$ at a finite place. It is contained in the actual row level. Above Q its only nonidentity blocks go from a later block to an earlier one across the first j-plane or its opposite. These entries belong to the row unipotent radical of Lemma 98.19. At the graph and the other places the row level is the full stabilizer. Passing to the diagonal blocks gives $B \cap K = (J \cap K) \ltimes U_L$. Hence the ratio of finite unsigned mass to the identity mass is the count of right U_L-cosets satisfying $u_0^{-1}w \in KB$. Indeed every supported first representative is bu_0K with $b \in J$, and the stabilizer ratio counts the $J \cap K$-orbit of u_0U_L, exactly as in Theorem 99.10. Normalization of J by w is sufficient for this argument. □

100.8 The projected column lattice of a whole block

Use w from Proposition 100.7 at a finite place. In either split component use the local integer ring O, and at an inert place use the integer ring of the quadratic extension. Write $P_0 = \#(O/\pi)$ and use lengths over O. For a rectangular matrix M, write $t(M)$ for the length of the image of $z \mapsto Mz \bmod O^u$ on the standard integral domain, where u is the number of rows. This length is the sum of the absolute values of the negative elementary divisors and is unchanged by transpose. Let H, l, Z be as in Proposition 100.2.

In the middle space of dimension $n = k+2(s-j)+1$, write a vector in the order $m = (p, b, s_0)$, where b consists of the k column coordinates. The coordinates s_0 are set to zero on taking the column subquotient, and the coordinates p are then discarded. Put $q = \dim s_0$. In the first split component $q = h = s - j$, since one sets the residual opposite entries to zero and quotients by the residual primal vectors and the terminal line. Use the same description in the

reversed dual component away from T. At T use the integral graph basis and the enlarged group B' of Proposition 99.4. In the reversed dual component also impose annihilation of the primal terminal line, giving $q = h + 1$. These are the integral subquotients in Lemma 99.8. In particular the two column coordinate blocks are dual to one another. At an inert place use the decomposition with $q = h$ and the unimodular column form.

Lemma 100.8 (An index calculation for the projected lattice)**.** *Suppose $g = u_0^{-1} w \in KB$, or $g \in KB'$ for the enlarged counting set at T. Let A be the inverse column action of the corresponding period factor and put $A_0 = \lambda A$ in the chosen component. Set*

$$X = t(H_W), \qquad Y = t(l_W), \qquad L = v(\lambda), \qquad m_- = t([\, l \quad Z \,]),$$

where W denotes the k column coordinates. If d_- is the sum of the negative denominator lengths of $A_0 O^k$ relative to O^k, then

$$X + Y - m_- + qL \;\le\; d_- \;\le X.$$

There are at most j negative elementary divisors. In the split case $m_- = jv(c_-)$ in the first component. In the inert case $m_- = jv(c)$.

Proof. The inverse lattice is $g^{-1}\Lambda$, with Λ the row lattice in the chosen coordinates. Intersect it with the space where the last j entries are zero and quotient by the first j entries. Multiply the resulting middle lattice by λ. Inverting the exchange and applying it after u_0 as in Proposition 100.2 shows that this scaled middle lattice consists of

$$m + Hv, \quad (m, v) \in \mathcal{T}, \qquad \mathcal{T} = \{(m, v) \in O^n \oplus O^j : lm + Zv \in O^j\}.$$

Indeed the omitted first input is forced to be $-lm - Zv$ to make the last output zero. The first output is unrestricted. The volume of $\mathcal{T}$, with unit integral cubes, is $P_0^{-m_-}$. In the split case m_- is the length of O^j / L_- in Proposition 100.2, by transposing $[\, l \quad Z \,]$ and taking its kernel modulo the lattice. That proof uses only the two preserved prefix planes and unit determinants and therefore applies to B' as well. In the inert case the adjoint test is equivalent to integrality of $Hv, Z^\dagger v$ on the primal kernel, since the middle form is unimodular and $lH - Z = Z^\dagger$. This gives the asserted length there.

Map $\mathcal{T}$ onto the s_0-projection of the scaled middle lattice by $(m, v) \mapsto s_0 + H_S v$, with S denoting those rows. Its image has Haar volume P_0^{-qL}. In fact $g^{-1}\Lambda = b_0^{-1}\Lambda$ for an element b_0 of the period group in use. After taking the first middle subquotient, its action on the s_0-quotient is triangular with all diagonal entries one. At T the extra dual coordinate is a quotient on which the contragredient affine group acts by one. Multiplication by λ contributes the factor $|\lambda|^q = P_0^{-qL}$, using the absolute value normalized by P_0.

Let $V_0 \subset O^j$ consist of v for which the zero fiber of this map contains a point above v. In that fiber one must take $s_0 = -H_S v$. At a fixed admissible v, the allowed (p, b) form a coset of the kernel modulo O^j of the matrix $[\, l_p \quad l_W \,]$. Set $b_* = t([\, l_p \quad l_W \,]) \ge Y$. The fiber volume in (p, b, v)-coordinates is therefore $P_0^{-b_* - [O^j : V_0]_{\mathrm{len}}}$, where the bracket denotes the length of the quotient. The change of additive coordinates from s_0 to $s_0 + H_S v$ has determinant one. Each nonempty fiber of the map on $\mathcal{T}$ is a translate of its kernel, with this same volume. Fubini's theorem now gives

$$[O^j : V_0]_{\mathrm{len}} = m_- - qL - b_*.$$

The projected column lattice is $A_0 O^k$, by the two invariant subspaces defining the subquotient. Its image modulo O^k is exactly $H_W V_0 \bmod O^k$, because the other summand b is integral. The full image $H_W O^j \bmod O^k$ has length X and the loss on restriction to V_0 is at most $[O^j : V_0]_{\mathrm{len}}$. This proves both bounds. The image is generated by at most j elements since V_0 is a lattice in O^j. □

100.9 The unipotent quotient for a composition of the prefix

Suppose $j = j_1 + \cdots + j_t$, where every $j_i \geq 1$. Let $J_i = \sum_{a \leq i} j_a$. On $1 \leq p \leq j$ put $i(p) = \min\{i : p \leq J_i\}$ and define the products

$$G_0 = 1, \qquad G_p = \prod_{a=1}^{p} c_{i(a)}, \qquad u_p = j - J_{i(p)}.$$

At an endpoint G_{J_i} agrees up to sign with its leading cross determinant. These products are defined also at intermediate positions, where a leading cross minor can vanish. Use the ordinary single-word operators O_w. The level and the terminal frequency are those of Section 6.6.

Proposition 100.9 (The stabilizer and its quotient for several blocks)**.** *Every relevant supported pair with $0 < j = s - \dim(X_0 \cap X_1) < s$, and every transverse pair with full common column ($j = s$), has a representative of the block form in Proposition 100.7, with*

$$C = \bigoplus_{i=1}^{t} c_i S_{j_i}.$$

Here $a_i = G_{J_i}/Q^{J_i}$ are nonzero integers of E, $Ta_t\lambda$ is integral, and $\lambda\bar{\lambda} = 1$. Let $\mathcal{V}$ be the group of block upper unitriangular j-square matrices over E with identity diagonal blocks of sizes j_i. Write $\widehat{z} = \operatorname{diag}(z, I, z^{-*})$. If U is the radical of the whole j-plane, the unipotent quotient in $J\backslash B$ has unique coordinates $\widehat{z}\,u_0$ with $z \in \mathcal{V}, u_0 \in U$. Its measure is product Haar.

Proof. Theorem 110.9 and Proposition 109.40 classify the indicated pairs by the blocks, the cross coefficients and the common scalar on the remaining frame. The displayed isometry realizes these data on the fixed rational Witt pairs. The supported endpoint minors divided by the indicated powers of Q are integral by Proposition 110.11. The terminal assertion comes from Proposition 94.20 for $j < s$ and from Proposition 109.40 for $j = s$.

Set $J = B \cap wBw^{-1}$. Its elements stabilize both opposite j-planes and are therefore block diagonal in the decomposition by these planes and the middle. On the primal prefix a matrix in J is upper unitriangular. Its inverse adjoint, transported by the signed reversals, must also be upper unitriangular for the second prefix. The order of the blocks is the same in the two lists, whereas each list within a block is reversed. Consequently the permissible upper entries are exactly those inside a single diagonal block. Equivalently the two triangular conditions on the block indices impose both upper and lower triangularity, and impose just upper triangularity inside each diagonal block. Thus J is the product of these diagonal unitriangular groups and the residual middle Bessel group. Conjugation by w preserves J and its Haar measures, by inverse adjoint and reversal inside each block and the identity on the residual group.

Every upper unitriangular prefix factors uniquely into its block diagonal part and an element z of $\mathcal{V}$. Splitting off U from B now proves the quotient description. These coordinates have product measure since the action of the prefix unipotents on U has determinant one. All entries of $\mathcal{V}$ strictly above the block diagonal have independent additive Haar measure, as follows from triangular multiplication. At a finite place write $\mathcal{V}_L$ for their integral group and U_L for the integral radical. Both are in the actual row level, including above Q by the block description of Lemma 98.19. Taking diagonal blocks of integral elements gives the level decomposition used in Proposition 100.7. In particular the finite mass relative to the identity counts the supported right $(\widehat{\mathcal{V}}_L U_L)$-cosets of $\widehat{z}u_0$. For fixed representatives of $z\mathcal{V}_L$, count right U_L-cosets in the fiber. The compact column groups used for averaging fix z, since they act in the middle block.

In these coordinates

$$\widehat{z}^{-1} w = w_{C(z),\lambda}, \qquad C(z) = z^{*} C.$$

The same identity holds after the common Euclidean Witt normalization at infinity, which replaces C by $C/2$. Changing the rational opposite vectors is an upper central change of frame as in Lemma 100.12. It acts trivially on the section character, and conjugation has determinant one on the full upper unipotent. On the additional coordinates z the change from rational to unit primal vectors is uniform scaling of the entire prefix and leaves its matrix unchanged. Normalizing each of the $\sum_p u_p$ extra additive coordinates to adelic quotient volume one contributes the reciprocal of $2^{-f}\sqrt{|\Delta_E|}$. These factors cost at most $2^{f\sum_p u_p}$, since $|\Delta_E| \geq 1$. □

100.10 The signed factor at an added split prime

Let $c_j = jQ\sqrt{-q}$, with j in the set $\mathcal{T}$ of Theorem 111.6. At a place v over a prime $p \mid j$ we have $F_v = \mathbb{Q}_p$, the row and column split, and the prescribed levels are the full lattice stabilizers. Put $K_H = \mathrm{GL}_k(\mathbb{Z}_p)$. Normalize the finite factor by the identity orbital factor as in Theorem 111.5. The additive character of the ordinary seed is retained in both occurrences.

Proposition 100.10 (A bound for the signed extra factors)**.** *At $v \mid p \mid j$ the finite factor between K_H-invariant columns is of the form*

$$F_v(c_j) = a'_v I + u'_v A_v, \qquad |a'_v| \leq a_p, \qquad |u'_v| \leq u_p,$$

where a_p, u_p are the unsigned multiplicities of Proposition 99.11 with residue size p. In particular, writing $R = r - 1 = k + 2s$ and $H_x = 4(k + s + 1)^2$, its operator norm on the prescribed word is at most $H_x p^R$. If $m_f(c_0)$ denotes the unsigned finite mass for $c_0 = Q\sqrt{-q}$, the full finite operator for c_j has norm at most

$$m_f(c_0)\, H_x^{f\omega(j)} j^{Rf},$$

where $\omega(j)$ counts distinct rational prime factors and $\omega(1) = 0$.

Proof. The first split component of c_j has valuation one. Thus Theorem 101.1 and Proposition 99.11 apply to its actual unit part. In the unipotent fiber the quotient mass counts each successful integral right coset once. On any one such coset the two column arguments have relative position equal to the identity double coset or to the neighbor double coset. This follows by projecting the two integral lattices onto the middle column as in that proposition. The character multiplier has modulus one. Changing an integral representative leaves the correlation on spherical vectors unchanged because the column compact factors act trivially there and the unipotent lattice character is trivial on integral entries. These entries are integral for the physical unit ϖ at p. Equivalently one can average over choices of representatives, in which case the multiplier has modulus at most one.

For a unitary translation action R_H, let e_H denote the orthogonal average over K_H. The operators $e_H R_H(h) e_H$ agree for all h in one double coset by Haar invariance on its two sides. In the neighbor case this operator equals $A_v/[d_p(d_p - 1)]$ on spherical vectors, by summing the $d_p(d_p - 1)$ right cosets. Each distinct neighbor has unsigned multiplicity u_p, and the identity has total multiplicity a_p. Summing the possible phases now gives the two scalar inequalities without any condition on their signs. Spherical convolution at other places commutes with these operations.

The explicit counts are

$$a_p = p^{2s-2}(p-1)\{(2 + 2s(p-1))p^k - 1\}, \qquad u_p = p^{2s-2}(p-1)^2.$$

They give $a_p \leq (2 + 2s)p^{k+2s}$ and $u_p \leq p^{2s}$. Use the neighbor estimate of Proposition 101.2 and the triangle inequality to obtain $H_x p^R$.

At every place not dividing j, the unchanged valuations give the same unsigned upper bound as at c_0, by Theorems 99.10 and 111.4. Weight one there at a minimal valuation and the full unsigned mass at the places over q bound the operators on the entire finite column. More explicitly, the finite pairing disintegrates into the unipotent choices and a common integration over the column quotient, as in Theorem 111.5. Each tuple of local choices gives the commuting product of the corresponding local translations and phases. Averaging on spherical vectors at the new places therefore inserts exactly the local factors just bounded. There are f places of F of residue size p above each $p \mid j$, since p splits completely in E. Multiplying gives the global bound. Compression at the end can only decrease the operator norm. □

100.11 A local bound at every power of the terminal prime

Proposition 100.11 (The first-plane factor on the original terminal graph)**.** *At each $v \mid T$ with $e = v(c) = v(\xi) \geq 1$, the signed first-plane operator between the C_ℓ-fixed columns satisfies*

$$\|F_v(Q\xi)\| \leq [4(k+1)^2]^e (Nv)^{(k+4s)e}.$$

At $e = 0$ the factor is the identity. The bound is uniform in the unit part of ξ and the primitive residue covector specifying the graph.

Proof. Use the counting normalization of Theorem 99.10. The original summands belong to the finite set Ω' of Proposition 99.4. Insert zeros at all the other cosets in this set. The same proposition bounds the average on each orbit by $\Xi(A)$, with no compact index. The elementary divisors of A are constant on this orbit, since conjugating β by D_0 and changing the compact representative can multiply A only by elements of C on either side. By Lemmas 104.31 and 99.8, the bound for an orbit of size m_0 is at most $m_0 k^{2e} P^{-(k-1)d}$, where $P = Nv$. Equivalently assign the same weight $k^{2e} P^{-(k-1)d}$ to each member when summing this majorant. The depths t, u' in the last lemma may be read separately on each member.

For a fixed pair of depths there are at most

$$P^{k(2e-t-u')+(2(n-k)+1)e}$$

members, with $n = r - 2 = k + 2s - 1$. Indeed X, Y modulo c are a column and row over O/cO in the graph basis. Their k column coordinates on each side give the first exponent. Their other coordinates give $2(n-k)e$. After fixing representatives for those two blocks, the central variable allows at most P^e choices by the right integral multiplication used in Proposition 99.3. Both first endpoints are integral in the enlarged set, which suffices for this count.

Let $j = e - t - u'$. The remaining part of the exponent is

$$k(2e - t - u') - (k-1)d \leq k(e+j) - (k-1)\max(0, j) \leq (k+1)e.$$

Since $2(n-k)+1 = 4s-1$, this gives the stated power of P. The number of depth pairs is at most $(e+1)^2 \leq 4^e$. At $e = 0$, Theorem 99.10 gives the unique integral coset, and the representative 1 has $\gamma_c \in K$ and trivial relative character. The argument applies on each prescribed local constituent and hence on their automorphic sum. □

100.12 High powers of the physical block parameter

Lemma 100.12 (Two physical-frequency estimates for one block)**.** *For j, n as in Proposition 100.7, let $R = r - 1$, choose a real $1 \leq \ell \leq j$, and take $A = 3\ell R$ or $A = 6\ell R$. At a real place τ, the positive local orbital integral divided by its diagonal norm, with ordinary coordinate measure on the block radical, is at most*

$$\exp(CRj\ell s \log b)\, |\tau c|^{-jA}.$$

The field factor for the product in rational quotient measures is at most one.

Proof. First use unit terminal frequency. The upper central unipotent that replaces each integral opposite vector by its Euclidean orthogonal Witt partner fixes the isotropic flag. It is trivial on the Bessel character, as it has only entries from opposite to primal isotropic vectors. It commutes with the middle column group. The Euclidean Witt pairs can be used from the outset because their opposite vectors differ from the rational ones by multiples of the corresponding primal vectors. Changing the representatives by this upper central element in both flags transports their stabilizer by conjugation and preserves the quotient integrals. In the unit pairs the matrix in Theorem 107.4 is $C_0 = \tau C/2$.

On $J_\infty\backslash G_\infty$ use Cauchy–Schwarz with squared-integral weight $(l_j(g)/l_j(w^{-1}g))^A$ and its reciprocal, as in Lemma 107.11. The first squared section norm descends to $B_\infty\backslash G_\infty$. Its weight disintegrates over U by Proposition 100.7 and Theorem 107.4. The remaining diagonal moment has power $2A - 2(n + j)$ on $l_j(g)$. For the second squared norm, change variables by w. The exchange normalizes J with modulus one, and its inverse has cross block C_0^* of the same determinant modulus. Thus both squared-integral bounds agree. These positive disintegrations are justified by Tonelli. Since B fixes the exterior product of the first j unit isotropic vectors, $l_j(g) \le \nu(g)^j$, with ν the minimum norm of a coset representative. For these powers

$$0 < j(2A - 2(n + j)) \le 12j\ell R \le 12j^2 R \le 4r^2$$

for sufficiently large x, since $j \le s$ and $s^2 = o(r)$ by Theorem 6.29. The local moment in Theorem 105.3 costs at most $\exp(CRj\ell s \log b)$. For a nonintegral power use the next integer, since $\nu \ge 1$. The integrand defining $K_{n,j,A}$ decreases with A, and $A \ge 2n+3j$ on these intervals. At $\alpha = n + 3j/2$ the gamma quotients in Theorem 107.4 are at most one for these n. Thus $K_{n,j,A} \le \pi^{nj+j^2/2}$, also within this cost.

At terminal frequency ϖ_τ, the scaling that multiplies the primal flag vectors by ϖ_τ changes C_0 into C_0/T. Its modulus on U is $T^{j(n+j)}$. Indeed the rectangular coordinates receive one complex scalar of modulus $\sqrt{T}$ per entry in addition to a middle unitary matrix of determinant of modulus one, and the skew center receives scalar T in j^2 real coordinates. In the ratio of the J- and B-quotient densities this contributes $T^{-j(n+j)}$, by the change of variables in Proposition 105.14. The full physical factor contributed by the exchange is consequently

$$(2/|\tau c|)^{jA} T^{j(A-n-j)}.$$

Use $\log T = O(s \log b)$ from Theorem 93.17.

Finally choose rational compatible measures by giving U_L volume one at all finite places and $[U]$ adelic volume one. The rectangular additive lattice gives nj copies of complex covolume $2^{-f}\sqrt{|\Delta_E|}$, as in Proposition 105.19. Graph changes on a split pair have opposite determinant valuations and the total covolume is unchanged. The center contributes $j(j-1)/2$ further complex coordinates and j skew scalar coordinates of covolume $\sqrt{\Delta_F}$ each. The central extension has the product covolume by integration first on the center, since the rectangular projection of every integral radical is onto. Passing to Euclidean unit Witt pairs scales the rectangular entries by $\sqrt{2}$ and the central entries by 2, with real Jacobian 2^{nj+j^2} at each place. Thus the field factor multiplying the ordinary coordinate densities is the reciprocal of

$$(2^{-f}\sqrt{|\Delta_E|})^{nj+j(j-1)/2} (\sqrt{\Delta_F})^j 2^{(nj+j^2)f}.$$

This denominator is at least one. The normalization gives $[B]$ and $[J]$ the same volume, since $B = J \ltimes U$. The common compact real column contracts the two factors by Schur orthogonality as in Theorem 105.15. Its dimension factor cancels against the diagonal. On the other part of

J the two actions are scalar characters and coincide for relevant labels. This establishes the stated product normalization for the positive majorant. □

100.13 The split operator for a whole reversed block

At a split place let $P = Nv$, $m_\pm = jv(c_\pm)$, $M = m_+ + m_-$, $h = s - j$, $q_0 = n - k = 2h + 1$, and $L = v(\lambda_+)$. On the original local column let F_v be the signed orbital operator relative to the identity.

Proposition 100.13 (All split places, including the terminal graph)**.** *Uniformly for* $2 \le j \le s$,

$$\|F_v\| \le [2(k+1)]^M P^{(k/2+32(s+1))M}.$$

Proof. Use $A_0 = \lambda_+ A$ and X, Y of Lemma 100.8 in the first component. Write d_+, d_- for the sums of the positive exponents and of the absolute negative exponents of A_0, respectively. In the reversed dual, the rectangular blocks become $-H^{\mathsf{t}}, -l^{\mathsf{t}}$ in exchanged positions, the two prefix lengths interchange, and the scaled output lattice has matrix $A_0^{-\mathsf{t}}$. Applying that lemma in both components gives

$$X + Y - m_- + hL \le d_- \le X \le m_+,$$
$$X + Y - m_+ - h'L \le d_+ \le Y \le m_-,$$

where $h' = h + 1$ over T and $h' = h$ elsewhere. The upper bounds for X, Y follow from the kernel lattices L_+, L_- of Proposition 100.2. The determinant of $g = u_0^{-1} w$ has valuation $m_+ - m_- + nL$. All period graded pieces outside the column have determinant one. Scaling A on the column gives

$$d_+ - d_- = m_- - m_+ - q_0 L, \qquad q_0|L| \le 2M.$$

The second inequality uses $d_- + d_+ \le X + Y \le M$. Put $D = d_- + d_+$. For $L < 0$ add the determinant valuation to twice the lower bound for d_-. For $L \ge 0$ subtract it from twice the bound for d_+. Away from T this gives

$$D \ge 2(X + Y) - M + |L|.$$

At T it gives $D \ge 2(X + Y) - M - L$ for both signs.

We specify how the phases and original levels enter. The finite mass is the count of right U_L-cosets by Proposition 100.7. At a full column level their translations are compressed to compact invariants. Above Q write $KB = K_1 B$ as in Proposition 99.7. Conjugation by $C = \mathrm{GL}_k(O)$ stabilizes K_1, B, U_L and commutes with w, since the middle block of w is scalar. Two decompositions of the same matrix modulo K_1 change its column action by the principal congruence subgroup, which fixes each original vector. Thus averaging coset terms on each C-orbit averages the conjugates of A with phases of modulus at most one. Temperedness at these places is Proposition 101.4. The regular representation and compact averaging proof of Lemma 104.30 bounds this average by $\Xi(A)$ before using any exponents of A. This also bounds every full-level compression on a tempered constituent. The spherical coefficient is unchanged on multiplication by a central scalar, because its inducing character is trivial on the center. Hence $\Xi(A) = \Xi(A_0)$.

At T use instead the D_0-orbits of the enlarged cosets $u_0^{-1} w \in KB'$ in Proposition 99.4. This set is finite by the kernel count of Proposition 100.2, which still applies since B' preserves both prefix determinant tests. The scalar middle block again commutes with D_0, which preserves the integral radical. The original success test and the relative column translation are obtained

by factoring the central block β into the affine and stabilizer factors in that proposition. Its change-of-covector formula thus applies to every such orbit with the actual phases, extending vectors by zero outside the original shell. The composition by the affine map with linear part A has operator norm at most $|\det A|^{-1/2}$ by change of variables. The compact tensor estimate in Lemma 104.31, whose proof gives $\Xi(A)$ for every A, bounds the orbit average by

$$P^{(m_- - m_+ - nL)/2}\Xi(A_0).$$

The determinant and $\Xi(A_0)$ are constant on each averaging orbit. Its size can therefore be counted by giving this bound to each of its members.

For a fixed X, Y count the contributing cosets. The two prefix kernels have at most P^{jM} choices by Proposition 100.2. A rectangular k-by-j matrix with image length X modulo the lattice has at most $P^{(k+j)X}$ possible residues. Its kernel in O^j has index P^X and at most P^{jX} choices, and each row on that kernel allows at most P^X residues. Apply this to H_W and to l_W^{t}. The other rectangular rows cost at most $P^{q_0 M}$, and the central rows cost at most P^{jm_-}. As in the prefix count, choose fixed rectangular lifts by right integral multiplication before counting the center. Altogether the count is at most

$$P^{k(X+Y)+(3j+q_0)M}.$$

There are at most 2^M length pairs, since $(m_+ + 1)(m_- + 1) \le 2^M$.

At most j elementary divisors of each sign occur by Lemma 100.8. Put $v_0 = k - 4j + 3 > 0$. The spherical factor is at most $k^M P^{-v_0 D/2}$ by Lemma 102.2. Away from T its combination with $P^{k(X+Y)}$ has exponent at most $v_0 M/2 + (k - v_0)M \le (k/2 + 4j)M$. Above T the extra exponent from inserting the determinant and using the local lower bound for D is at most

$$M/2 + (v_0 - n)L/2.$$

For $L \ge 0$ the second term is nonpositive. For $L < 0$ it is at most $(q_0 + 4j)|L|/2 \le (1 + 4j)M$. These estimates and $q_0 = 2(s - j) + 1$ give the stated bound, including $M = 0$. An empty original support has zero operator. □

100.14 Summation over the rational flag parabolic

Let P stabilize the isotropic flag used to define N. We show that enlarging the summation from $B(F_+)$ to $P(F_+)$ preserves the full norm of every seed of Section 92.8. This evaluates an infinite family of rational cosets. Write $\mathbb{A} = \mathbb{A}_{F_+}$. Throughout, either of the two finite levels in that section can be used. In particular it is the original level at the auxiliary primes g_1, g_2, with unipotent reduction at both.

Lemma 100.14 (Character transport compatible with finite support)**.** *Let $\lambda \in P(F_+)$ and $\beta, \beta' \in \{1, \varpi\}$. Suppose that*

$$\chi_\beta(\lambda n \lambda^{-1}) = \chi_{\beta'}(n) \quad (n \in N(\mathbb{A})), \qquad \lambda_f \in B_f K B_f.$$

Then $\beta = \beta'$ and $\lambda \in B(F_+)$.

Proof. Use the Witt splitting from Section 121.16. On the one-dimensional flag quotients the Levi component acts by $a_1, \dots, a_s \in E^\times$. Its action on the paired quotients is by $\bar{a}_i^{-1}$, and its middle component is $M \in U(W)(F_+)$ for $W = W_0 \perp Ee$. This decomposition follows by reading the diagonal blocks of a flag-preserving isometry. Removing them leaves an element of $N(F_+)$, whose conjugation fixes every character of $N(\mathbb{A})$.

The first $s-1$ simple coordinates acquire factors a_i/a_{i+1}. Each can be tested on the elementary change $u_{i+1} \mapsto u_{i+1} + t u_i,\ v_i \mapsto v_i - \bar{t} v_{i+1}$. Varying t in one complex completion and using the nondegenerate real trace pairing forces $a_i/a_{i+1} = 1$. Write a for the common value, also when $s = 1$. To test the last simple block, take any linear form ξ on W at a complex place. Write $\xi(w) = h(z, w)$. There is a unipotent isometry given on the last Witt plane and W by

$$w \longmapsto w + \xi(w) u_s, \qquad v_s \longmapsto v_s - z - \tfrac{1}{2} h(z,z) u_s, \qquad u_s \longmapsto u_s.$$

The middle pairings are preserved, the cross terms with v_s cancel, and the last image has norm zero. This element has the specified last block and all preceding simple coordinates zero. Conjugating the block by the Levi replaces ξ by $a\,\xi \circ M^{-1}$. The character equality for all ξ therefore gives $\beta a M^{-1} e = \beta' e$. These are equalities of rational vectors, as can already be checked in a complex embedding. Since $h(e,e) = -1$ and M is unitary, we get

$$a \bar{a} = \frac{\beta' \bar{\beta}'}{\beta \bar{\beta}}. \tag{100.1}$$

In a finite decomposition $\lambda_f = b_1 k b_2$, the middle element k belongs to $K \cap P_f$. The two elements of B_f act trivially on every one-dimensional quotient of the isotropic flag. Thus the image of k on each such quotient is a. At every finite place the image of the compact subgroup $K_v \cap P(F_{+,v})$ in the multiplicative group of a field has valuation zero. This holds in both factors at split places. It follows that a is a global unit. The two different terminal frequencies have norms $1, T$, so (100.1) forces $\beta = \beta'$.

Now $a\bar{a} = 1$, and every conjugate of the algebraic integer a has modulus one. Thus a is a root of unity. Indeed the monic polynomials with roots the conjugates of a^j, counted over the embeddings of E, have integral coefficients bounded independently of j. Their roots comprise a finite set. At each auxiliary prime any element of the row level has unipotent reduction. Its characteristic polynomial in the standard lattice reduces to $(X-1)^r$. Applying this to the above k shows that a reduces to one at places above both g_1 and g_2. The order of a root of unity reducing to one in residue characteristic p is a power of p. Indeed its prime-to-p part z still reduces to one, and, if it had order $d > 1$, the equality $1 + z + \cdots + z^{d-1} = 0$ would reduce to $d = 0$ in the residue field. The two distinct auxiliary primes therefore give $a = 1$. We now have $Me = e$, so M acts on $W_0 = e^{\perp} \cap W$ by an element of $H(F_+)$. The Levi and unipotent components together put λ in $B(F_+)$. □

Theorem 100.15 (The full parabolic sum preserves the seed norm)**.** *For the seeds of* (92.16) *define the parabolic sum by*

$$(\mathcal{P}^P \Phi)(g) = \sum_{\gamma \in B(F_+) \backslash P(F_+)} \Phi(\gamma g) \quad \text{in } L^2 \text{ along the unipotent quotient.} \tag{100.2}$$

It defines an element of $L^2(P(F_+)\backslash G(\mathbb{A}))$. For any finite combination of the chosen types, frequencies and columns its squared norm there equals the squared norm of the input on $B(F_+)\backslash G(\mathbb{A})$. In particular its Gram operator on the refined zero-sum columns is precisely the scalar in (92.17)*. Distinct words and distinct terminal frequencies have orthogonal images.*

Proof. Fix g and representatives for the countable rational coset set. As a function of $n \in N(F_+)\backslash N(\mathbb{A})$, the term with index γ and frequency β is

$$\Phi(\gamma n g) = \chi_\beta(\gamma n \gamma^{-1}) \Phi(\gamma g).$$

Here the transformation law holds on the entire group, since left translation by N_f preserves the finite support. If two terms are both nonzero, their finite supports imply $(\gamma \gamma'^{-1})_f \in B_f K B_f$.

If their characters on $N(\mathbb{A})$ agree, conjugating the variable by γ' and using Lemma 100.14 gives equal frequencies and equal rational cosets. The distinct characters are orthogonal on the compact unipotent quotient. Right translation by an element on which their ratio differs from one proves this directly.

For a finite sum of inputs, first collect all types and columns at the same frequency into Φ_β. Put

$$D(g) = \sum_{\gamma \in B(F_+)\backslash P(F_+)} \sum_{\beta} |\Phi_\beta(\gamma g)|^2.$$

The nonnegative function D is invariant on the left by $P(F_+)$ and by $N(\mathbb{A})$. Unfolding by a fundamental region for the discrete group $P(F_+)$ and using Tonelli gives

$$\int_{P(F_+)\backslash G(\mathbb{A})} D(g)\, dg = \sum_{\beta} \int_{B(F_+)\backslash G(\mathbb{A})} |\Phi_\beta(g)|^2\, dg < \infty$$

by Theorem 92.24. Thus almost every unipotent fibre has square-summable coefficients. Orthogonality proves convergence of the series there in L^2, independently of the ordering. Its squared norm on that fibre with Haar probability is $D(g)$.

Here the base of the fibres is $P(F_+)N(\mathbb{A})\backslash G(\mathbb{A})$. The subgroup in this quotient is closed because it is the inverse image in $P(\mathbb{A})$ of the closed discrete rational Levi under the map to the adelic Levi. It is unimodular, since conjugation by the rational Levi on $N(\mathbb{A})$ has modulus one by the product formula for the determinant of the adjoint action on the unipotent Lie algebra. Haar disintegration therefore uses the invariant measure on $N(F_+)\backslash N(\mathbb{A})$ in each fibre, with the base measure scaled to allow Haar probability on that fibre. On a measurable section of the quotient the partial sums are measurable, and their fibrewise L^2 limits give a measurable L^2 function. Equivalently, the integrated squared tails tend to zero by dominated convergence with dominating function D on that section. Changing a representative translates the fibre or permutes the rational indices and conjugates the unipotent variable by a rational Levi element. These operations preserve the fibre measure and the sum. This defines (100.2) on the stated quotient.

Integrating its squared norm gives the displayed unfolded integral. The different Φ_β are orthogonal on the input side by Theorem 92.24, which also gives the mixed inner products of the types within one frequency. Polarization now proves all the Gram assertions. □

The intermediate sum $\mathcal{P}^P$ includes every rational element preserving the isotropic flag. To obtain automorphic row forms on $G(F_+)\backslash G(\mathbb{A})$ one must also sum over $P(F_+)\backslash G(F_+)$. The terms from different such cosets couple different flags, whose unipotent radicals need not agree. The orthogonality proof above uses their common radical at the parabolic stage. The estimates for pairs of different flags and the comparison on the integral theta lattice remain open here.

100.15 One normalization for the identity term on the common interval

Keep the shortened field E, the interval $[V, V^{3/2}]$, and the common prime T and word family of Theorem 9.13. Put $f = [E^+ : \mathbb{Q}]$, $s = m/2 - 1$, $r = 2s + k + 1$. We use the probability measure on the compact column quotient $[H]$. Here an identity term is the contribution to the Gram pairing of two Poincaré sums from equal rational cosets in the two summation indices. Its computation at the original and translated row levels is given in Section 92.8.

Theorem 100.16 (Common normalization in the probability norm)**.** *At each of the two row levels K the identity term is C_K times the probability inner product on the multiplicity space of each selected column type at its induced column level. The same positive C_K serves all the words*

and both terminal frequencies. At the translated level it also serves the zero-sum refinement. The mixed identity term between two different word-frequency pairs vanishes. Writing the translated level as K'_σ, one has

$$\frac{C_{K'_\sigma}}{C_{K_\sigma}} = a_{x,T}^2, \qquad a_{x,T} = T^{\,fs(r-s)/2}(T^k-1)^{f/2} \in \mathbb{Z}_{>0}. \tag{100.3}$$

Thus the positive square root needed to compare these two identity-term norms belongs to $\mathbb{Q}$ on the original field family.

Proof. Write M_K for the number of finite column classes and d_ρ for the dimension of a column representation ρ. Fix one unit vector w in it. In the class of h_i, the scalar function corresponding to the class covector ℓ_i has value $\ell_i(\rho(h_\infty)w)$. The classes have equal mass $1/M_K$ and trivial stabilizers. Averaging the projection onto $\mathbb{C}w$ over H_∞ gives the identity divided by d_ρ with probability Haar measure, by Schur's lemma and the trace. The squared norm of the column is therefore

$$(M_K d_\rho)^{-1} \sum_i \|\ell_i\|^2.$$

The same formula for inner products follows by polarization. Varying the covectors gives one copy of the full multiplicity space, since evaluation at a fixed nonzero vector on the irreducible type is injective on the space of intertwining maps and its dimension is $M_K d_\rho$.

With the Haar measures of Theorem 92.24, the coefficient multiplying the probability inner product is

$$C_K = M_K \, \frac{\mathrm{vol}_f(K)\, v_N \, \mathrm{vol}(H_\infty)}{\mathrm{vol}_f(N_f \cap K)}.$$

In particular the factor d_ρ has cancelled. The stated mixed terms vanish by that theorem, and restriction to the zero-sum subspace preserves the formula. Each unrefined class has $(T^k-1)^f$ refinements by Corollary 9.14. Proposition 92.25 gives the unipotent index $T^{fs(r-s)}$, and the row volumes agree. Taking the ratio gives (100.3). Finally E^+ contains $C_{l'} = \mathbb{Q}(\zeta_{2^{l'+2}})^+$ of degree $2^{l'}$. Our choice has $2^{l'} \ge x^{2+8\varepsilon}/\operatorname{ord}(\psi)$, so $l' \ge 1$ for all sufficiently large V. The degree f is even. Both exponents in $a_{x,T}$ are integers. □

Corollary 100.17 (Integral cost and normalized trace on the same fields)**.** *Let D_{aux} be the denominator of the auxiliary finite Whittaker pairing on its integral residue bases. For every fixed $A > 0$, uniformly at all the points of the selected intervals,*

$$\log(D_{\mathrm{aux}} a_{x,T}) = o(b/x^A), \qquad b = \exp(x^\varepsilon). \tag{100.4}$$

If $J^{\mathrm{id}}_{c,\beta}$ denotes the Gram operator of the identity term for word c, restricted to the refined zero-sum multiplicity space with the probability norm, then

$$\frac{\Gamma_c^2}{D\, C_{K'_\sigma}} \operatorname{tr}(J^{\mathrm{id}}_{c,\beta}) = 1 + O\Big(\exp(-x^{1/4})\Big) \qquad (\beta = 1, \varpi). \tag{100.5}$$

Here $D = \mathcal{V}(G) I_\sigma \prod_\tau W_r(\lambda_\tau)$ is the row quantity of Theorem 9.13.

Proof. Corollary 92.19 gives

$$D_{\mathrm{aux}} = \prod_{q \in \{g_1, g_2\}} \left(\prod_{j=2}^{k} (q^j - 1) \right)^f.$$

Its logarithm is at most $fk(k+1)\log(g_1 g_2)/2$. Equation (100.3) gives $\log a_{x,T} \le f(s(r-s)+k)\log T/2$. The ranks and f are bounded by powers of x, $g_i \le 2(kb)^8$, and $\log T$ is bounded

by a power of x. For the last bound apply Lemma 87.1. The lists of Theorem 9.13 satisfy its parameter bound, and all the chosen primes lie in its terminal range. Thus the sum of the two logarithms is bounded by $x^{C_{\psi,\varepsilon}}$ for large x. For any fixed A, C, $x^{C+A}e^{-x^{\varepsilon}}$ tends to zero, which proves the asserted uniform bound.

Theorem 100.16 gives $J^{\mathrm{id}}_{c,\beta} = C_{K'_\sigma}\mathrm{Id}$. The dimension of the space is R°_{c}. Substitution of its estimate in Corollary 9.14 proves (100.5). □

This computes the finite denominator and the change in the identity-term norm on the same fields, with their costs in (100.4). A vector may have a positive identity contribution and zero projected Poincaré sum, because the remaining rational cosets can cancel it. Controlling those terms and identifying the resulting value on the integral theta lattice are still required for the period estimate in this section.

101 Split and inert Hecke operators

101.1 Counting the two decorated flags at a split prime

We compute the next valuation in Theorem 99.10. Let v be a split place outside QT, with full row lattice stabilizer and residue field κ of size b. Use $k = \dim_E W_0$, $h = s - 1$, and $n = k + 2h + 1$, the dimension of M on the first split factor.

Theorem 101.1 (Exact local mass when the pairing has valuation one)**.** *If $cO_E = \pi O_E$, the unsigned finite mass relative to the diagonal is*

$$m_v(\gamma_c) = b^{2h}\left[b^{2k} + (b - 3 + 2s(b-1)^2)b^k + 1\right].$$

Proof. Use the same letter c for its first component, a uniformizer. The first split row group is GL_{n+2}. Write an element of U by its three upper blocks (l, H, z), a row of length n, a column of length n and a scalar. In this proof H denotes that column. Set $g = n(l, H, z)^{-1}\gamma_c$, using block order $1, n, 1$. Direct multiplication gives

$$g = \begin{pmatrix} 1 + c(lH - z) & -l & lH - z \\ -cH & 1 & -H \\ c & 0 & 1 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 1 & l & z \\ 0 & 1 & H \\ -c & -cl & 1 - cz \end{pmatrix}.$$

The first column of g and the last row of g^{-1} must be integral. Thus cH, cl, cz are integral and $(cl)(cH) \equiv 0 \bmod c$. Right multiplication by the integral group adds integral vectors to l, H, and adds $z_0 + lH_0$ to z when its last column entries are H_0, z_0. Fix representatives of residue scalars, using zero for zero. Every right integral coset satisfying these conditions therefore has a unique representative $H = X_0/c$, $l = Y_0/c$, $z = Z_0/c$, where X_0, Y_0, Z_0 are the chosen lifts, coordinate by coordinate, of $X \in \kappa^n$, $Y \in (\kappa^n)^{\vee}$, $Z \in \kappa$ with $YX = 0$. Reducing l, H first keeps cz integral. Once both lifts are fixed the remaining choice is z modulo the integers. This also proves the converse for the two integrality conditions.

On the middle space take coordinate vectors $p_1, \dots, p_s$ for the remaining $s - 1$ primal flag vectors followed by the terminal line. Write $\lambda_1, \dots, \lambda_s$ for the corresponding dual flag in $M^{\vee}$. The two sets of coordinates are disjoint except that p_s, λ_s pair to one. Multiplying the terminal vector or covector by the fixed units of norm product $-T$ does not affect these tests. Put $P_i = \langle p_1, \dots, p_i\rangle$ and $D_i = \langle \lambda_1, \dots, \lambda_i\rangle$ over κ, with $P_0 = D_0 = 0$. The condition $g \in KB$ is precisely that on each side the transformed list can be made integral and primitive by adding earlier entries to later ones. Indeed these are the two factors of the column changes of Lemma 98.30. After such changes the two lists still have their original pairings. The integral isometry in that lemma carries them to the reference decoration.

Let i be the first index with $Yp_i \neq 0$, and j the first with $\lambda_j X \neq 0$. Use ∞ when the corresponding list vanishes. Put $A = 1 + Y_0X_0/c - Z_0$, $D = 1 - Z_0$, and write $\bar{A}$ for the residue of A. The primal first vector is $(A, -X_0, c)$. The following vectors are $p_a - (Y_0p_a/c)u$. When $i = \infty$ they are all the original coordinate vectors. The list then passes unless $X \in P_s$ and $Z = 1$. In fact its reduction is independent exactly when $\bar{A} \neq 0$ or $X \notin P_s$; if $X \in P_s$ the coordinate lifts have $Y_0X_0 = 0$. If a prefix has dependent reductions, its exterior product is divisible by c and every allowed change preserves that product.

If $i < \infty$, the first $i-1$ vectors after the first are unchanged. For a successful list this prefix must already span a primitive lattice. Removing the pole at p_i requires u mod c in the span of the prefix reductions. To see this, a linear combination of primitive independent columns lying in $c^{-1}L$ has all coefficients in $c^{-1}O$, by an integral left inverse. Multiply by c and reduce. Consequently $X \in P_{i-1}$ and $\bar{A} \neq 0$. In this case $Y_0X_0 = 0$, so $Z \neq 1$. Conversely assume $X \in P_{i-1}$ and $Z \neq 1$. That prefix spans the vector $Au + c\,v$ together with the earlier middle coordinates. Here v is the last coordinate vector. After the prefix, all unwanted multiples of u/c can be canceled with multiples of $Au + cv$, leaving integral vectors that project to p_a in the middle coordinates. All prefix reductions are independent, as the first vector has a unit u-entry. This proves sufficiency.

On the dual side the first row is $(-c, -Y_0, D)$ and the subsequent rows are $\lambda_a + (\lambda_aX_0/c)v^\vee$, with $v^\vee$ the last coordinate covector. The same argument using these rows gives the following test. For $j = \infty$ it passes unless $Y \in D_s$ and $Z = 1$. For $j < \infty$ it passes exactly when $Y \in D_{j-1}$ and $Z \neq 1$.

When $i < \infty$ and the primal test passes, $X \in P_{i-1}$ implies $j = \infty$. For each i there are b^{i-1} choices of X, $(b-1)b^{n-i}$ choices of Y, and $b-1$ choices of Z. The case $j < \infty$ has the same count upon exchanging vectors and covectors. These disjoint cases total $2s(b-1)^2b^{n-1}$.

If $i = j = \infty$, the two middle residues belong to $P_h \oplus W$ and $D_h \oplus W^\vee$, respectively, where W is the k-dimensional residue column. The flag components give b^{2h} choices. Write x, y for the two remaining components. The condition $yx = 0$ has $b^k + (b^k - 1)b^{k-1}$ solutions by separating $x = 0$. Each allows b choices of Z, except that $Z = 1$ is excluded for the $2b^k - 1$ pairs with $x = 0$ or $y = 0$. Adding the previous two cases and using $n - 1 = k + 2h$ proves the formula. □

101.2 The good Hecke operators on the prescribed column

Let $H = U(W_0)$ be the definite rank-k column of Section 121.16. Use any original word of (9.1), its column level, and its real type or its dual. At an unmarked split place v put $P = Nv$, $K_v = \mathrm{GL}_k(\mathcal{O}_v)$ and give K_v volume one. Write D_v for the unnormalized sum over the index-P sublattices, and A_v for the sum over the neighbors whose intersection has index P in each lattice.

Proposition 101.2 (Spherical neighbor bounds on the word space)**.** *On the K_v-fixed automorphic column space of each prescribed real type,*

$$\|D_v\| \le kP^{k/2}, \qquad \|A_v\| \le (k^2 + 2)P^k.$$

The bounds hold for every unmarked split v on the common fields and intervals, with the displayed constants independent of the field and the word.

Proof. We check the base-change input on this column. The group is compact at all real places and has no proper rational parabolic, since its Hermitian form is positive definite at a real place. Its automorphic constituents are discrete, unitary and cuspidal. The field $F = E^+$ has even degree and at least two real places, since it contains $\mathbb{Q}(\sqrt{qd_0})$. The two imaginary quadratic

subfields of coprime discriminants ensure that E/F is unramified at every finite place, as proved in Theorem 94.9. At a nonsplit finite place -1 is a norm, so each pair of positive diagonal entries splits as a hyperbolic plane there. Thus H is quasi-split at all finite places and is the definite unitary group in Labesse’s base-change theorem.

At each real place the column Harish–Chandra entries are the u_i. Consecutive interior entries differ by $4\Delta+\Delta(c_{i+1}-c_i)\geq 3\Delta$. The first gap is at least $4k\Delta$ by the endpoint formula. At an ordinary place $u_{k-1}-u_k=4\Delta-\Delta c_{k-1}\geq 3\Delta$, and the high-place choice only increases this gap. Duality reverses and negates the list and preserves the gaps. Thus all gaps exceed one for sufficiently large x. The compact types are cohomological. For a definite unitary group the archimedean discrete-series and pseudo-coefficient hypotheses of Labesse’s Corollary 5.3 hold because a compact-group character projects onto its specified irreducible by ordinary Schur orthogonality. This is also the definite case stated in Theorem 2.2 and Remark (1) of the paper of Guerberoff on modularity lifting cited below.

Base change consequently gives an isobaric sum of discrete automorphic representations of general linear groups over E. Its local parameter at a split place agrees with that of the column there, and its holomorphic infinitesimal character at a complex place is the column list, up to the dual convention. The discrete-spectrum classification of Mœglin and Waldspurger expresses a noncuspidal discrete summand as a single string of length at least two over a unitary cuspidal representation. The twists in such a string differ successively by the idelic norm. At a complex place $|z|_{\mathbb{C}}=z\bar{z}$, so this would give two holomorphic entries differing by one. There is no such pair on this column. Every discrete summand is therefore unitary cuspidal. The unramified parameter is their multiset union, with each constituent unramified, since the direct sum has trivial inertia and zero monodromy. As in the proof of Proposition 71.17, the unramified bound of Luo, Rudnick and Sarnak, also applied to the dual, places every Satake modulus between $P^{-1/2}$ and $P^{1/2}$. Unitary characters have modulus one. The inputs here are Labesse, Corollary 5.3, the definite-unitary base-change statement, Mœglin and Waldspurger on the discrete spectrum and Luo, Rudnick and Sarnak on unramified parameters.

For precision, the eigenvalue of D_v is $P^{(k-1)/2}\sum_{i=1}^{k}\alpha_i$ for one of the two dual Satake conventions. In the upper triangular normal form for an index-P sublattice, a diagonal uniformizer in position i allows P^{k-i} choices of entries to its right, taken modulo that uniformizer. The normalized spherical vector in the unramified principal series has value $P^{-(k+1-2i)/2}\alpha_i$ on these representatives by the Borel modulus. This proves the eigenvalue formula, which passes to the spherical constituent. Thus its absolute value is at most $kP^{k/2}$. Summation on a compact arithmetic quotient decomposes into these unitary automorphic constituents, so the bound holds on the whole word space. There are $d_P=(P^k-1)/(P-1)\leq 2P^{k-1}$ index-P sublattices. Going down to one and then up either returns to the original lattice or passes through the unique intersection with a neighbor. Thus $A_v=D_v^{*}D_v-d_PI$, with the same identity in the opposite order under the dual convention. Taking norms gives the asserted estimate. □

101.3 The eigenvalue of the inert neighbor operator

Retain the hyperspecial column at the inert place of Proposition 99.1. Its rank in Section 6.6 is even, $k=2l$. Write $X_1,\dots,X_l$ for the normalized unramified inducing characters evaluated at π on the diagonal torus of $U(k)$ in a hyperbolic basis.

Proposition 101.3 (An inert spherical bound on the original column)**.** *The eigenvalue of the neighbor operator is*

$$P^{k-1}\left(\sum_{i=1}^{l}(X_i+X_i^{-1})+\frac{1-P^{-k}}{1+P^{-1}}\right).$$

On the column of each prescribed word one has $\|A_v\| \le (k+1)P^k$. The signed finite factor of a first plane of valuation one consequently has norm at most $B_x P^{r-1}$, where $B_x = 4(k+s+1)^2$.

Proof. Use the upper triangular Borel with unipotent radical U_+ and diagonal t_ν of exponents $(\nu_1, \dots, \nu_l, -\nu_l, \dots, -\nu_1)$. In the normalized Satake transform of a double-coset indicator ϕ the coefficient at ν is $\delta(t_\nu)^{1/2} \int_{U_+} \phi(t_\nu n)\, dn$. The integral compact subgroup has volume one. This transform is invariant under permutations and inversions of the torus variables and evaluates to the Hecke eigenvalue at the inducing characters. Haines and Rostami, equation (9.2.2) and Section 10.1.

Let $t = \mathrm{diag}(\pi^{-1}, 1, \dots, 1, \pi)$. Use $\phi = 1_{K_H t K_H}$ by Proposition 99.1. Every minor of an element of this double coset has valuation at least -1, by exterior powers of its two integral factors and of t. In an upper triangular matrix the principal minor formed from all negative diagonal exponents therefore gives $\sum_i |\nu_i| \le 1$. For t itself the condition $tn \in K_H t K_H$ is equivalent to n integral. Indeed the first row of tn bounds the first row of n. For $1 < i < k$, the minor in rows $1, i$ and columns $1, j$, $j \ge i$, equals $\pi^{-1} n_{ij}$, which bounds all remaining entries. Conversely integral n belongs to K_H. Conjugation by t dilates the $k-2$ entries over D of the first-line unipotent radical by π^{-1} and its one skew central coordinate by π^{-2}. It fixes the middle unipotent quotient. Thus $\delta(t)^{1/2} = P^{k-1}$. Weyl invariance gives this same coefficient for each of the $2l$ nonconstant monomials.

To determine the constant, evaluate on the trivial representation, whose eigenvalue is the number d of neighbors. Let Z_j count all isotropic vectors, including zero, in a nonsingular residue Hermitian space of rank j. After diagonalizing, the norm of one coordinate takes each nonzero value $P+1$ times and zero once. Hence $Z_0 = 1$ and $Z_{j+1} = Z_j + (P+1)(P^{2j} - Z_j)$. Induction gives $M_k = Z_k - 1 = (P^k - 1)(P^{k-1} + 1)$ for even k. By Proposition 99.1,

$$\frac{d}{P^{k-1}} = \sum_{i=1}^{l} P^{2i-1} + \sum_{j=0}^{l-1} P^{-2j}.$$

On the trivial representation the normalized parameters, up to the Weyl action, are P^{k+1-2i}. Indeed in the upper triangular Lie algebra the root spaces for $e_i - e_j$ and $e_i + e_j$, $i < j$, each have dimension two over C, being paired matrix entries over D. Those for $2e_i$ have one skew entry. The half sum with multiplicities therefore has coordinates $k + 1 - 2i$, giving the stated values for the inducing character $\delta^{-1/2}$. Subtracting the $2l$ monomials from d/P^{k-1} leaves $(1 - P^{-1}) \sum_{j=0}^{l-1} P^{-2j} = (1 - P^{-k})/(1 + P^{-1})$.

For the prescribed definite column the proof of Proposition 101.2 gives an isobaric sum of unitary cuspidal representations over E, using the real gaps of the very same word. Labesse's transfer agrees with local base change at every unramified place. At this inert place its parameters are X_i, X_i^{-1} for $1 \le i \le l$. The unramified bound used in Proposition 101.2 now gives $|X_i|, |X_i^{-1}| \le (Nw)^{1/2} = P$. Labesse, Corollary 5.3 and Mínguez, Theorem 4.1. The constant inside parentheses in the eigenvalue is between zero and one. The asserted bound follows on each unitary constituent and hence on their orthogonal sum. Finally Proposition 99.1 bounds the signed operator by $P^{2h}(P-1) + P^{2h}(P^2-1)(k+1)P^k$. Here $2h + 2 + k = r - 1$, giving the claimed common bound. □

101.4 Temperedness of the column at split places

Proposition 101.4 (The split local components of the prescribed column)**.** *Let $F = E^+$. Every automorphic constituent σ of the definite column $H = U(W_0)$ with one of the real types in Proposition 101.2, or its dual, has tempered local component at every place of F that splits in E. This includes ramified representations at g_1, g_2 and T.*

Proof. Fix such a constituent. The proof of Proposition 101.2 verifies the hypotheses of definite unitary base change for these fields. The transfer is an isobaric sum $\boxplus_i \Pi_i$. The individual discrete summands are conjugate self-dual, and the local identification holds at every split place. Each summand is unitary cuspidal. Indeed a residual string of length at least two would contribute two entries differing by one to the holomorphic infinitesimal character at infinity. All the gaps of the ordered column list exceed one. Labesse, Corollary 5.3 and Guerberoff, Theorem 2.2.

Write d_i for the rank of Π_i. At each complex place, each of its two infinitesimal-character lists is a sublist of the corresponding list for the transfer. The entries of each full list are distinct and in $(k-1)/2+\mathbb{Z}$, by the highest weight formula for the compact column and its conjugate. The rank d_i requires entries in $(d_i-1)/2+\mathbb{Z}$ for an algebraic infinitesimal character. We describe explicitly a unitary conjugate self-dual twist to correct the parity.

Take $K=\mathbb{Q}(\sqrt{-q})\subset E$, with its quadratic idele class character η on $\mathbb{Q}$. On the image of $\mathbb{C}^\times$ in the idele class group C_K, prescribe $z\mapsto z/|z|$, using one complex embedding. On the image of $C_\mathbb{Q}$ prescribe η. To see that the prescriptions join continuously, write rational idele classes using a positive archimedean scalar and finite units $\widehat{\mathbb{Z}}^\times$. The positive scalars are already in the first subgroup. If a rational finite unit idele represents a complex scalar in C_K, their quotient is a principal idele. Its finite components are rational and fixed by conjugation. The principal element thus belongs to $\mathbb{Q}^\times$. Since it is a finite unit everywhere, it is 1 or -1. The prescriptions agree on these elements because η_∞ is the sign character and η is trivial on principal ideles. The product subgroup is closed. Indeed its norm-one part is the image of the compact group $S^1\times\widehat{\mathbb{Z}}^\times$, and positive real scalars split off by the idele norm. This also proves continuity of the joined character. The character extension theorem for locally compact abelian groups extends it to a unitary character μ of C_K. For $x\in C_K$ we have $\mu(x)\mu(\bar{x})=\eta(N_{K/\mathbb{Q}}x)=1$ by quadratic reciprocity in idele class form.

Pull back μ by $N_{E/K}$. The resulting character ν is unitary and conjugate self-dual. At each complex place its two infinitesimal entries are $1/2,-1/2$, in one order. If $k-d_i$ is odd twist Π_i by $\nu\circ\det$, and otherwise use Π_i itself. The two lists now have the required parity. After sorting each and subtracting the rank-d_i half sum of positive roots, both yield integral dominant highest weights, since consecutive entries differ by at least one. This proves regular algebraicity of the twisted cuspidal representation. The cohomological characterization of regular algebraic cuspidal representations of general linear groups gives the archimedean cohomology required in Caraiani's theorem. The twist has preserved conjugate self-duality. For $d_i\geq 2$ that theorem gives temperedness at every finite place of E. For $d_i=1$ this follows directly from unitarity. Removing a unitary character twist preserves temperedness. Grobner and Raghuram, Theorem 6.3, Guerberoff, proof of Theorem 2.3 and Caraiani, Theorem 1.2 and Corollary 5.9.

At a split place the local Langlands parameter of σ_v in either general linear factor is the direct sum of those of the $\Pi_{i,w}$, up to the dual for the other factor. A finite direct sum of tempered parameters is tempered, since its restriction to the Weil group in the Weil-group-times-SL_2 convention is bounded. The tempered local correspondence for GL_k proves the assertion. □

101.5 The inert coefficient and the global finite factor

Proposition 101.5 (The finite operator for every rank-one label)**.** *Let $c=Q\xi$ and λ represent a supported label of Proposition 111.16. Put*

$$A_\xi=|N_{E/\mathbb{Q}}\xi|^{1/2},\qquad D_Q(\xi)=\prod_{\substack{v\mid Q\\ a_+(v)+a_-(v)>0}} Nv.$$

With $C_k = 4(k+1)^4$, the finite operator has norm at most

$$D_Q(\xi)^k\, A_\xi^{k+6s+1+2\log C_k/\log 2}.$$

Proof. At a nonsplit place E_w/F_v is unramified quadratic and both levels are full. The norm-one scalar λ is a unit. Put $e = v_w(c) = v_w(\xi)$, $P = Nv$. In the unitary upper radical $l = -h_M(H,\cdot)$ in the usual Hermitian notation. Use $X = cH$, $Y = \bar{c}l$, $Z = \bar{c}z$. The first endpoint conditions in the exchange matrix make X, Y, Z integral. Here $Y = -h_M(X,\cdot)$, so the two truncated column depths coincide, say t.

The column denominators have at most one negative exponent of length $d \le e$. Moreover $d \ge e - 2t$. For detail, take the isotropic dual rows of the matrix in Lemma 99.9 over $\mathcal{O}_{E_w}$ with $c_+ = c$, $c_- = \bar{c}$. Use the quotient by the terminal line, an integral orthogonal summand here. The minors on one column coordinate or the initial coordinate give $\pi^t X_S/c$ integral, since λ is a unit. The full minor makes $Z - YX_S/c$ integral. The input with last coordinate π^t admits an integral column w solving the last congruence modulo $\bar{c}$. It gives denominator $e-2t$ if this number is positive. The upper bound and cyclicity follow from the single fractional column X_W/c in the projection. Self-duality of the physical column under its unitary Levi gives positive exponent d as well. The unit λ preserves this column lattice.

The spherical compression at d has norm at most $(k+1)^d P^{-(k-2)d}$ by Proposition 105.8. The integral right unipotents permit fixed lifts for X modulo c, since the trace map on the unramified quadratic integer ring is onto, including in residual characteristic two. In fact an arbitrary integral shift of H can be realized by completing it with a central coordinate of the required trace. For fixed H, the possible z's differ by trace-zero elements in $\pi^{-e}\mathcal{O}_{E_w}$, taken modulo integral trace-zero elements. This gives at most P^e choices, since the trace-zero lattice has rank one over $\mathcal{O}_{F_v}$. Thus at depth t the count is at most $P^{2k(e-t)+(2(n-k)+1)e}$. Writing $j = e - 2t$, we have $2k(e-t) - (k-2)\max(0,j) \le (k+2)e$. Together with $n - k = 2s - 1$ this gives the local bound $C_k^{2e} P^{(k+4s+1)e}$, including the at most $e+1$ depths and the factor $(k+1)^e$. At valuation zero the integral exchange and unit central scalar give one coset.

At a split place, integrality of $\xi\lambda$ in both components gives $-a_+ \le L \le a_-$. Thus $|L| \le a_+ + a_- = m_0$. Proposition 100.1 is bounded by $C_k^{m_0} P^{(k+6s+1)m_0/2}$ with the extra P^k at active places above Q. The local factors of A_ξ are $P^{m_0/2}$ at split places and P^e at inert places. Since $P \ge 2$, the sum of all the exponents of C_k just used is at most $2\log A_\xi/\log 2$. The common-column unfolding of Proposition 111.16 multiplies the finite factors. This proves the assertion. □

101.6 Bounds on the finite operator for every prescribed word

Let $F_q^{\mathrm{col}} = \prod_{v|q} F_v$ be the operator of Theorem 111.5, and set

$$d_b = \frac{b^k - 1}{b-1}, \qquad L_b = b^{2s-2+k}(b-1)(1+2s(b-1)), \qquad U_b = a_b + u_b d_b (d_b - 1)\frac{3b-1}{b(b+1)}.$$

The q in the subscript denotes the rational field prime. The column forms are on $H = U(W_0)$ at the finite level of Theorem 86.1, with the given irreducible real word type. Use its probability inner product.

Theorem 101.6 (Positive lower and upper bounds after terminal projection)**.** *On each prescribed word and every one of its terminal character spaces or zero-sum spaces, the finite operator satisfies*

$$\left(\prod_{v|q} L_{b_v}\right) 1 \;\le\; F_q^{\mathrm{col}} \;\le\; \left(\prod_{v|q} U_{b_v}\right) 1, \qquad b_v = \mathrm{N}v.$$

The same inequalities hold as quadratic forms on the coordinates retained in Corollary 110.7. On the common intervals its operator norm, divided by the unsigned mass $m_f(\gamma_c)$, is at most $4^{\#\{v|q\}}q^{-f/2}$ for all sufficiently large x.

Proof. At one place let D_v sum the translations to the index-b sublattices of the column lattice. On the spherical level its adjoint sums over index-b superlattices, by inversion of the compact double coset in the unimodular local group. A two-step return has d_b paths to the starting lattice, one through each intermediate lattice. A path to a distinct endpoint has its intersection with the starting lattice as the unique intermediate lattice. Thus $D_v^*D_v = d_b\,1 + A_v$. Reversing the two steps gives the same identity, with the sum of the lattices as intermediate lattice. It follows that

$$F_v = (a_b - u_b d_b)1 + u_b D_v^* D_v, \qquad a_b - u_b d_b = L_b > 0.$$

This proves the lower bound without a spectral assumption.

There is no vector of this word type fixed by $\mathrm{SL}_k(F_v)$. Indeed the simply connected group $\mathrm{SU}(W_0)$ is split and noncompact at v. Strong approximation away from v makes its rational points dense in its adeles away from v, including the real coordinates. For a smooth local-invariant function on $[H]$, approximate gag^{-1} away from v, with $a \in \mathrm{SU}(W_0)(\mathbb{A}_F)$. At v the difference lies in the local special linear group, which is normal. Rational left invariance and local right invariance give equality at g and ga by continuity. Such a function is therefore trivial under all real special unitary factors. The original word has nonconstant highest weight there, as the consecutive entries in its cloud are separated by the gaps of Theorem 9.13. The same conclusion holds for Hilbert vectors by smoothing: the invariant subspace is stable under translations and the real compact type projector. This is the application of strong approximation used in Theorem 90.8.

The operator $A_v/[d_b(d_b-1)]$ is probability convolution on the double coset with exponents $1, -1$ and zeros. The local matrix-coefficient estimate in the proof of Lemma 83.1 gives norm at most $(3b-1)/(b(b+1))$ on this subspace. Its local argument uses only three coordinates, the size of the residue field and the absence of special-linear invariant vectors. It applies since $k \geq 3$ on the original intervals. For the general local method see H. Oh, Uniform pointwise bounds for matrix coefficients. This proves $F_v \leq U_b\,1$. All the F_v commute, are positive, and commute with the terminal translations, which act at other places. Multiplication gives the bounds, and then compression preserves the quadratic-form inequalities.

By Theorem 101.1, $m_v = a_b + u_b d_b(d_b-1)$ and $a_b/m_v = O(sb^{2-k})$ uniformly. For the original ranks sb^{3-k} tends to zero uniformly for $b \geq q$. Hence $U_b/m_v \leq 4/b$ eventually. The product of all b_v over q is $q^{f/2}$ by the ramification computation in Theorem 111.4. □

101.7 The inert operator and the product over the finite places

Proposition 101.7 (The finite product for an arbitrary single block)**.** *At an inert place with $P = Nv$ and $m = jv(c)$, the signed local operator has norm at most*

$$[2(k+1)]^m P^{(k+32(s+1))m}.$$

For the full product over the finite places one has

$$\|F(c,\lambda)\| \leq (Q^f A_\xi)^{kj+\Delta_j}, \qquad \Delta_j = 100j\big(s+1+\log_2(2(k+1))\big).$$

Proof. Use the unramified quadratic ring with residue cardinality P^2. Since the terminal prime splits, the terminal line at this place is an integral orthogonal summand. The column is unimodular. Hence the column part of l is the negative adjoint of H_W with an integral

unimodular factor, and $X = Y = t$ in Lemma 100.8. Also $\lambda\bar{\lambda} = 1$ gives $v(\lambda) = 0$. The scaled column action is unitary and its exponents are paired with their negatives. Multiplication by λ is an integral central action and does not change its hyperspecial double coset. The lemma bounds the common denominator length by $2t - m \le d_- \le t \le m$, with at most j nonzero pairs.

The primal kernel has at most $(P^2)^{jm}$ choices. The column rows at length t have at most $(P^2)^{(k+j)t}$ residues, by the row-kernel count in Proposition 100.13. The $q_0 = n - k$ remaining rectangular rows and the j central rows cost at most $(P^2)^{(q_0+j)m}$. Integral rectangular changes lift in the unramified radical, including in residual characteristic two, by the trace argument of Proposition 100.2. Thus the count at t is at most $P^{2kt+2(3j+q_0)m}$. The hyperspecial compression is at most $(k+1)^m P^{-(k-4j+2)d_-}$ by Lemma 102.2. Insert $d_- \ge 2t - m$; the exponent from this last power and P^{2kt} is at most $(k+8j)m$, since $t \le m$. There are at most 2^m choices of t. As $j \le s$ and $q_0 = 2(s-j)+1$, the claimed local bound follows.

Multiply this estimate and Proposition 100.13. The local factors of $|N_{E/\mathbb{Q}}\mathfrak{c}|^{1/2} = Q^f A_\xi$ are $P^{m/j}$ at an inert place and $P^{M/(2j)}$ at a split place. Replace $2(k+1)$ by $P^{\log_2(2(k+1))}$, valid also for $P = 2$. The exponents in the two local estimates are then bounded by the corresponding exponents of the asserted product. The original constituent spaces and their tensor products obey the same bounds, by decomposing the unitary automorphic column into its local representations and taking the operator norm on each constituent. □

102 Unramified and transverse factors

102.1 The unmarked root integrals

Use Proposition 121.4 at a finite place v. Write $E_v = E \otimes_F F_v$, with integer ring O, also when E_v is split. Let U_ℓ denote the upper unitriangular group over E_v, and put $\chi(u) = \psi_E(\sum_{j<\ell} u_{j,j+1})$. The character ψ_E uses the absolute trace to $\mathbb{Q}_p$, p below v, and the standard additive character. Put $X_* = (db_1, \dots, d^\ell b_\ell)$. Write a test in the first position as a function of the *scaled* columns, so the indicators in that proposition are applied after dividing column j by d^j. Keep the second position p in the unscaled row coordinates. For a local test Φ set

$$\mathcal{E}_v(g) = \int_{U_\ell} \overline{\chi(u)}\,[\omega(g)\Phi](X_* u, p_*)\,du, \qquad R_v = \{g \in U(S)(F_v) : gX_* = X_* u \text{ for some } u \in U_\ell\}.$$

This integral takes values in the Schwartz space of the residual oscillator. It has compact integration support because X_* has independent columns and admits a linear left inverse. Haar measure on U_ℓ is product additive measure in its upper entries, by triangularity of multiplication.

Proposition 102.1 (Support at an unmarked finite place)**.** *For v unmarked, take the test of Proposition 121.4 with the residual lattice vacuum. Let K_v be the stabilizer of $\Lambda_{S,v}$. Then $\mathcal{E}_v(g) \neq 0$ holds exactly on $R_v K_v$.*

Proof. Write P for the stabilizer of the flag spanned successively by $b_1, \dots, b_r$, and N_P for its unipotent radical. We record how the evaluation changes under this parabolic. In the first polarization the action of $g \in U(S)$ is substitution $x \mapsto g^{-1}x$ followed by a fixed invertible operator on the residual variable, including the splitting scalar. Indeed the oscillator splits across the orthogonal hyperbolic part of T and T_0; on the first part the substitution intertwines both translations and dual modulations. Uniqueness of the Heisenberg representation leaves only a scalar. In particular the residual operator is independent of the upper entries of u.

For $g \in P$, the induced action on the second position is the action on the quotient of S by the perpendicular of $\langle b_1, \dots, b_r \rangle$. Evaluation at p_* after this action equals an invertible operator

on the core applied to evaluation at $g^{-1}p_*$. To see this directly, displacements into $\langle b_1, \dots, b_r \rangle$ act by all modulations of the position p. Conjugation by g preserves these displacements, with the dual change of their characters. The kernel of evaluation at p_0 is spanned by differences of a modulation and its value at p_0 acting on tests. Indeed a locally constant test vanishing at p_0 has compact position support away from p_0; cover it by finitely many compact open pieces, on each of which a modulation separates that piece from p_0. Subdivision into disjoint pieces gives the stated span. Applying g and then g^{-1} gives the invertible map on the evaluation quotients. This is the Heisenberg construction in the local Schrödinger model. S. Kudla, *Notes on the local theta correspondence*, Chapter I.

Every element of R_v is in P; on the isotropic columns its matrix is upper unitriangular in the scaled frame. On the opposite rows the matrix is consequently lower unitriangular. It fixes p_*, since that position is zero or is supported on the last row. Substituting $u_0 u$ for u in the integral under a left element of R_v changes the integral by an invertible core operator and a scalar of absolute value one from χ (the splitting is already in the operator). This proves left invariance of its nonvanishing set. In particular $N_P \subset R_v$, since adding earlier isotropic columns to the last, possibly nonisotropic, column is allowed. Right invariance under K_v follows from the integral vacuum in Proposition 88.15.

We use $U(S)(F_v) = PK_v$. For completeness, at an unramified nonsplit place intersect any rational isotropic flag of the given dimensions with the unimodular lattice. These saturated submodules have a primitive adapted basis. Unimodularity supplies integral vectors pairing dually to its isotropic vectors. Subtract one half of their Gram matrix times the isotropic frame to make the dual vectors isotropic. Here the residue characteristic is odd. The hyperbolic pairs split off; the remaining unimodular complements of the same rank are isometric by the orthonormal-basis argument of Lemma 111.18. This gives transitivity of K_v on such flags. In the split case the two isotropic components amount to a flag in a free space and the annihilators of a dual flag. Together they are nested because the original flag is isotropic. Intersecting with the lattice and taking a single adapted basis proves the same transitivity in all residue characteristics. Inverting the orbit decomposition gives PK_v.

Modulo the left and right operations already treated it suffices to evaluate $g = tg_0$, where $tb_j = t_j b_j$ for $j \le r$, t acts by $\bar{t}_j^{-1}$ on the opposite columns, and $g_0 \in U(S_0)(F_v)$. The diagonal position conditions on $g^{-1}X_* u$ give $t_j^{-1} \in O$ for all $j \le r$. If $c = 2r + 1$ there is also the condition

$$g_0^{-1} b_\ell \in (I_\ell^{(c)})^{-1} \Lambda_{S_0,v}.$$

Indeed division of column j by d^j leaves its diagonal entry unchanged. For $k < j$ its b_k-component is $t_k^{-1} d^{k-j} u_{kj}$. The original lattice is the direct sum of the stated lines and the core. Thus the upper entries range independently over

$$u_{kj} \in t_k \, (dAB)^{j-k} O \quad (k < j),$$

where products denote localized fractional ideals. This uses $I_k^{(c)} / I_j^{(c)} = (AB)^{j-k}$. Proposition 122.7 gives $(d)AB = \mathfrak{D}_{E/\mathbb{Q}}^{-1}$ locally at every unmarked place. The largest fractional ideal on which ψ_E is trivial is the inverse different. This holds for the split product as well, by varying one component at a time. Orthogonality in the entry $u_{k,k+1}$ therefore forces $t_k \in O$ for every $k < \ell$, since its integral is zero unless the additive character is trivial on the entire ideal. All other entries contribute ordinary volumes. The row and core operators in this computation are independent of u.

For $c = 2r$ the last remaining entry is t_r. On the opposite rows $g^{-1}p_*$ has last row $\bar{t}_r p_{*,r}$. Its value on the middle column of T must belong to O, by the row-ideal computation of Proposition 121.4. The value of $p_{*,r}$ there is a unit. Thus $t_r \in O$ as well. For $c = 2r + 1$

the simple roots have already treated all $t_1, \dots, t_r$. Moreover $I_\ell^{(c)} \bar{I}_\ell^{(c)} = 1$, so the lattice in the display for $g_0^{-1} b_\ell$ is unimodular, contains b_ℓ, and has the original lattice stabilizer on S_0. The vector has norm -1; the integral norm-orbit argument of Lemma 111.18 places it in the compact orbit of b_ℓ. Hence g_0 is a product of an element fixing b_ℓ and a lattice stabilizer. For $c = 2r$ all of $U(S_0)$ already fixes the frame. All t_j and t_j^{-1} are now integral, so t itself is in K_v. This proves necessity in both cases.

At $g = 1$ the row and column tests take value one on the product just computed, and every simple-root character is trivial there. Integration gives its positive volume times the nonzero residual vector. The left and right invariances prove sufficiency. □

102.2 Spherical bounds with several nonzero elementary divisors

Lemma 102.2 (A spherical bound uniform in the number of moving directions)**.** *Let $P = Nv$ and suppose at most j positive and at most j negative elementary divisors occur in a column translate a, with $2 \le j \le s$. Denote their total absolute length by D. At a split place the spherical coefficient $\Xi(a)$ of Lemma 104.30 satisfies*

$$\Xi(a) \le k^D P^{-(k-4j+3)D/2}.$$

At an inert place write $D = 2d_-$. The norm of the hyperspecial compression on the given automorphic column is at most $(k+1)^{d_-} P^{-(k-4j+2)d_-}$.

Proof. Normalize the lattice stabilizer to have mass one and let ϕ_a be the indicator of its double coset. Order the diagonal exponents increasingly in the split group. As in Proposition 105.8, the normalized Satake transform of ϕ_a is bounded coefficient by coefficient by the product of the transforms of D translates of one coordinate of valuation 1 or -1. Indeed the convolution of two double-coset indicators is at least the indicator of the product double coset, by integrating over the first right coset at the product representative. The Satake integral preserves positivity. The convolution transforms multiply. At the trivial torus parameter defining Ξ, each one-coordinate transform evaluates to $kP^{(k-1)/2}$ by the split calculation of Proposition 101.2.

The degree of ϕ_a is at least the upper Borel modulus $\delta_B(a)$. The normalized Satake coefficient at a is at least $\delta_B(a)^{1/2}$ by integration over the integral unipotents. Its opposite Weyl coefficient is equal to it. Evaluation of this opposite monomial at the trivial representation multiplies it by $\delta_B(a)^{1/2}$, and all contributions are nonnegative. For ordered exponents $v_1 \le \cdots \le v_k$ the logarithm to base P of this modulus is $\sum_i (2i - k - 1) v_i$. Nonzero entries occur in the first or last j positions, so this sum is at least $(k - 2j + 1)D$. Dividing the numerator by the degree gives the first bound.

At an inert place the hyperspecial Cartan decomposition in Proposition 105.8 gives j or fewer nonnegative exponents $a_1 \ge a_2 \ge \cdots$ on hyperbolic pairs, with $d_- = \sum_i a_i$. The representative is the product of d_- conjugates of the one-pair translate. At the absolute values of the Satake parameters of our column, the transform of this translate is bounded by $(k+1)P^k$ by Proposition 101.3. The same positivity argument bounds the numerator by its d_--th power. The Borel modulus is

$$P^{2\sum_i (k-2i+1)a_i}.$$

For each pair of indices $i < i'$ the two upper root coordinates are quadratic-field additive lines with weights $a_i - a_{i'}$ and $a_i + a_{i'}$. The long root is one ground-field line with weight $2a_i$. Multiplying these factors gives the formula, also in residue characteristic two. The degree argument applies to this group as well. Since $a_i = 0$ for $i > j$, division by the degree proves the second bound. □

102.3 Making both terminal vectors isotropic

Let $d = s + 1$ and let Y, Y' be a relevant supported pair with $\mathcal{G} > 0$ as in Lemma 110.10. Absolute squares over E mean $x\bar{x}$. Adjoin to the Hermitian space V an orthogonal vector t of norm one, and let the original group act on $\widetilde{V} = V \perp Et$ by fixing t. Add ϖt to the final vector of each frame. An upper unitriangular change of either old frame makes exactly the same change of its extended frame. Indeed the last row of such a change is the last row of the identity. An element fixing t preserves the extended decorated flag exactly when it preserves the original one.

Proposition 102.3 (Opposite maximal isotropic spaces and the fixed vector)**.** *Both extended frames are totally isotropic. Their cross matrix is $Z+TE_{dd}$, of determinant $\Delta = D_d+TD_s \neq 0$. Their span S is nondegenerate. Its perpendicular is positive definite of rank k at every real place. If $t_\perp$ is the component of t in this perpendicular, then*

$$h(t_\perp, t_\perp) = \mathcal{G}/|\Delta|^2.$$

The extended cross matrix can be put in monomial form with the original reversed blocks. For a final block of length at least two the original pivots are unchanged. For a final singleton its pivot increases by T.

Proof. The added summand cancels the norm $-T$ of each terminal vector and leaves its pairings against the preceding vectors zero. It contributes T to the terminal cross entry, whence the assertion about Z and the determinant by cofactor expansion. The strict inequality $|\tau D_d| > T|\tau D_s|$ of Proposition 110.11 implies $\Delta \neq 0$. In the ordered basis of both extended frames the Gram matrix of S has zero diagonal blocks and off-diagonal blocks M and M^*, with $M = Z + TE_{dd}$. This matrix is nonsingular and represents d hyperbolic planes. The extended space has signature $(k + d, d)$, giving the assertion about the perpendicular.

The column of pairings of the two frames against t consists of two copies of $\bar{\varpi}\,\mathbf{e}_d$, where $\mathbf{e}_d$ is the last coordinate vector. The inverse Gram matrix has off-diagonal blocks M^{-*} and M^{-1}. Thus the norm of the projection of t to S is $2T\,\mathrm{Re}(D_s/\Delta)$, since $(M^{-1})_{dd} = D_s/\Delta$. Subtracting this quantity from one gives the displayed formula by expanding $\Delta\bar{\Delta}$.

Take the representatives giving the monomial form for the original pair. If its final block begins at $q < d$, column q has just one nonzero entry, in row d, say b. Subtract (T/b) times column q from column d of the extended cross matrix. This is a permitted change of the second frame and restores its original monomial matrix. In the singleton case the extended matrix is already monomial and its last pivot is nonzero by $\Delta \neq 0$. In particular $\Delta \in Q^d\mathcal{O}_E$ for a finitely supported pair by Proposition 110.11. □

At a real place take the original Euclidean unit Witt vectors p_i, q_i for $i \leq s$, with $u_i = \sqrt{2}\,p_i$, and set $p_d = (t + e)/\sqrt{2}$, $q_d = (t - e)/\sqrt{2}$. The middle block in this basis is the positive k-space. Write P_0, P_- for the column matrices of the p_i, q_i. Conjugate the relative isometry by the frequency change of Proposition 105.14, which multiplies p_i by $\tau\varpi$ for $i \leq s$ and fixes e, t. It sends the original first extended frame to $\sqrt{2}\,\tau\varpi\,P_0$. Consequently the monomial cross matrix C in the unit-frequency coordinates is the one in the proposition evaluated at τ and divided by $2T$ in every entry. Write δ_p for the product of the moduli of its first p row pivots. In particular $\delta_d = |\tau\Delta|/(2T)^d$.

102.4 A common rational subspace of the two columns

In a matching cell of Theorem 112.7, let Λ be its row lattice, $Y_i = X_i + Ez_i$ its two framed spaces and $P_i = Y_i^\perp/X_i$ their positive columns. Write L_i for the image of $\Lambda \cap Y_i^\perp$ in P_i, and

let $L_C = \Lambda \cap C$ in the common positive perpendicular $C = (Y_0 + Y_1)^{\perp}$. The quotient maps induce isometric injections $p_i : C \to P_i$. Indeed the Hermitian form on $Y_i^{\perp}$ descends modulo its radical X_i, and its restriction to C is positive definite. Here $\dim_E C = k - 1$.

Theorem 102.4 (The rank of the difference of the column maps)**.** *The comparison gives a rational isometry $\beta : P_0 \to P_1$ with $\beta L_0 = L_1$. Set $D = p_1 - \beta p_0$, $h = \operatorname{rank}_E D$, $K = L_C \cap \ker D$ and $L' = (DC) \cap L_1$. Then*

$$\delta(K)\delta(L') \prod_{v \mid Q} q_v^{2(h-\ell_v(c))_+} \le 2^{2fh}\delta(L_C), \qquad 2fh \log(Q/2) \le \log \frac{G}{Q^{2df} I} + 2S(c).$$

Here $y_+ = \max(y, 0)$, $G = |N_{F/\mathbb{Q}}\mathcal{G}|$, and I is the saturation index of Theorem 109.32. On the complementary range in Theorem 112.7 one has $h = O(d^2)$ uniformly. Thus p_1 and βp_0 agree on a rational subspace of dimension at least $k - 1 - O(d^2)$ in C. On the columns retained in Theorem 109.38 one can also insert the lower bounds $\delta(K) \ge Y_{k-1-h}$, $\delta(L') \ge Y_h$, where $Y_0 = 1$.

Proof. Put $\gamma_0 = 1, \gamma_1 = \gamma$. The rational maps γ_i^{-1} identify P_i with the fixed model column. Transport the comparison α of Proposition 119.1 between these two coordinates to obtain β. In the finite decompositions $\gamma_i^{-1} t = b_i h_i$, $h_i \in K$, the action of b_i on the model column is its column component y_i. Thus the column lattice on side i is y_i times the model column lattice in these coordinates. At T the model lattice in the first split factor is the inverse image of the kernel of the terminal covector and in the second factor it is the full dual parent lattice, by Lemma 109.10. Thus K^*_{line} preserves this lattice as well as all other local column lattices. The condition $y_1^{-1} \alpha y_0 \in K^*_{\mathrm{line}}$ now gives $\beta L_0 = L_1$. In particular $DL_C \subset L_1$.

At $v \mid Q$ both isotropic frames reduce to the same submodule U in the row. Reduction of the two middle quotients gives $U^{\perp}/U$. Indeed restriction of the self-dual row pairing onto the dual of either primitive isotropic module is onto, and its kernel stays exact on reduction, as in Theorem 109.15. The terminal vector has unit norm here. Its residue line in the middle quotient agrees for the two frames because the supporting frame changes reduce into $\bar{N} \rtimes U_k$ of Lemma 98.19. Orthogonal projection off this line is integral and commutes with reduction. Both column reductions therefore identify with its common perpendicular. The two projections of each integral vector $c_0 \in L_C$ are represented in this quotient by the same residue row vector.

To identify the action of β, write a row vector in the frame of t. In the model frame of the i-th flag its image has matrix h_i after removal of b_i. On the residual column the radical of the flag acts trivially, so this change has matrix $u_{i,v}$. Going from row coordinates on side zero to row coordinates on side one by β thus has matrix $u_{1,v}^{-1} \overline{(y_1^{-1}\alpha y_0)_v} u_{0,v} = A_{\alpha,v}$ in the first split factor. In the other factor the matrix is its inverse transpose in dual bases, by the perfect pairing of the two residual columns. In both factors the difference from the identity has rank $\ell_v(c)$. It follows that the reduction of $D : L_C \to L_1$ has rank at most $\ell_v(c)$ in each factor. This implication only uses the images of integral row vectors and applies even if $p_i(L_C)$ has smaller rank on reduction.

The lattice L' is saturated in L_1, so reduction of L' injects locally into reduction of L_1. At either of the two primes of E above $v \mid Q$, at least $(h - \ell_v(c))_+$ of the h elementary divisors of $DL_C \subset L'$ are divisible by the local uniformizer. The residue field has size q_v in each factor. Their indices multiply, giving

$$[L' : DL_C] \ge \prod_{v \mid Q} q_v^{2(h-\ell_v(c))_+}.$$

For the archimedean inequality give L_C/K the orthogonal quotient metric. For any positive Hermitian lattice M of rank a, use its Euclidean covolume in the f complex embeddings divided

by c_E^a, with $c_E = 2^{-f}\sqrt{|\Delta_E|}$. This agrees with $\delta(M) = [M^\vee : M]^{1/2}$ when M is integral. For a free lattice the Gram determinant at each embedding is the real volume factor and their product is the square root of its norm from E. Passage to a free sublattice of finite index gives the same identity for a projective lattice by multiplicativity of indices and duality. A basis of K as a $\mathbb{Z}$-lattice extended by lifts of a basis of L_C/K shows $\delta(L_C) = \delta(K)\delta(L_C/K)$ by orthogonal projection of the lifts. The normalizing powers of c_E also multiply. The map induced by D on the real orthogonal quotient has operator norm at most two at each place, since it is the difference of two isometries on C. Its volume expansion in real dimension $2h$ at each of the f places is at most 2^{2h}. Consequently

$$[L' : DL_C]\delta(L') = \delta(DL_C) \le 2^{2fh}\delta(L_C)/\delta(K).$$

This proves the first inequality, with rank-zero determinants equal to one. Both K and L' are integral, so their determinants are at least one. Use $\sum_{v|Q} \log q_v = f \log Q$, $(h - \ell_v)_+ \ge h - \ell_v$, and the identity $\delta(L_C) = G/(Q^{2df} I)$ of Theorem 109.32 to obtain the second inequality.

In the remaining height range each $H_i < b^{Mfd^2}$. Writing $\Delta_+ = D_d + TD_s = Q^d a_t$, the identity $\mathcal{G} = |\Delta_+|^2 - 2T\,\mathrm{Re}(\Delta_+\overline{D_s})$ gives

$$G \le (1 + 2T)^f Q^{(2d+s)f} H_t^2 \max(1, H_{t-1})$$

when $D_s \ne 0$, because then $D_s = Q^s a_{t-1}$. For $D_s = 0$ simply use $G \le Q^{2df} H_t^2$. Since $\log Q \asymp \log b$ and $\log T = O(d \log b)$, we obtain $\log G = O(fd^2 \log b)$. The bound on $S(c)$ in Theorem 112.7 and $I \ge 1$ now imply $h = O(d^2)$.

Finally $p_1 K \subset L_1$ is an isometric copy of K. Its saturation has the same rank and determinant at most $\delta(K)$, and L' is already saturated. A matching retained pair has both columns in the retained set, which is a union of scalar orbits. Applying the primitive determinant bounds on L_1 gives the last assertions. □

102.5 The weighted local estimate on the last row

Use precisely the period field, row and normalizations in Theorem 67.11. Choose the representatives and the common orthonormal basis of Lemma 65.14 for the finite set S of primes $p \ge 5$ inert in K, unramified in E, and prime to the level parameter Q of the finite CM test. Use the marked vacuum and rational torsion translates of order dividing Q, with coefficients integral above S, as in Theorem 65.13. The fixing-level representatives may be taken to move only at good completely split primes. With the common basis these are precisely finite sums of the class germs to which Theorem 66.24 applies. This also allows several such tests with a common Q and integral coefficients. Let A_n, A_s be common multiples of the respective stabilizer orders. View the right side of Theorem 67.11 as a bilinear form $\mathcal{B}_{\mathrm{row}}$ on the arrays of exterior tensors. The coordinate minor formula extends it to the ambient tensor spaces. Thus its matrix can be studied before it is paired with a particular incoming automorphic form.

Theorem 102.5 (Divisibility of the matrix for the prescribed last row)**.** *For $p \in S$, choose one half H_C in each even cycle of π_p on the CM embeddings, as in Theorem 66.24, using the same choice for all matrix entries. Denote by $d_M(u,h) = \max(0, \lceil (u - h(M-1))/M \rceil)$. Put*

$$g_p = \sum_{C \text{ even}} \sum_{\nu \in H_C} \sum_{\substack{j \ge 1 \\ j \text{ odd}}} \Big[\max\{ d_{p^j}(L_\nu, ns),\ n\, d_{p^j}(\lambda_{\nu,n}, s) \} - L_{\pi_p^j \nu} \Big]_+ .$$

Every matrix entry of $A_n A_s \mathcal{B}_{\mathrm{row}}$ has valuation at least g_p at every place above p, where $v_p(p) = 1$. In particular it takes the same bound on integral arrays in the actual Cartan lattices. The

product of the compact dimensions in the period normalization satisfies $\log \mathfrak{d} = O_{\psi,\varepsilon}(x^{C})$ *for a fixed* C.

Let k *and* B_ν *be the rank parameter and weight offsets of Lemma 104.6, and* $b = \exp(x^{\varepsilon})$. *The guaranteed local bounds on the same interval satisfy*

$$\sum_{p\in S} g_p \log p \;\ge\; (k-1)\sum_{p\in S}(\log p)\sum_{C\ \text{even}}\sum_{\nu\in H_C}\sum_{\substack{j\ge 1\\ j\ \text{odd}}} [B_\nu/p^{j} - B_{\pi_p^{j}\nu}]_+ - O_{\psi,\varepsilon}\Big(x^{C'} e^{x^{\varepsilon/3}}\Big)$$

for a fixed C'. *The error is* $o(b^{\eta})$ *for every fixed* $\eta > 0$. *The choice of half cycles may depend on* p.

Proof. Fix a pair of class representatives and a pair of coordinate basis tensors. Their polynomial is a product of minors. At embedding ν every monomial has total degree L_ν. Moreover exactly $\lambda_{\nu,n}$ factors have size n and each uses all n columns. All other factors have nonnegative degree in every column. Thus the degree in each individual column is at least $\lambda_{\nu,n}$, regardless of the smaller subsets chosen in that tensor.

For one monomial denote by $\alpha_{ab,\nu}$ its entry exponents. With $M = p^{j}$, the sum of the floors of the ns entry exponents at ν divided by M is at least $d_M(L_\nu, ns)$. Indeed the sum of the remainders is at most $ns(M-1)$. Applying the same argument in each column gives the second lower bound $n\, d_M(\lambda_{\nu,n}, s)$. The sum of the entry exponents at the opposite embedding is exactly $L_{\pi_p^{j}\nu}$. In (66.16), the sum of the individual positive parts is at least the positive part of the sum. This proves the stated contribution for every ν, j of this single monomial. The same halves work throughout. Summing these contributions and taking integral linear combinations of the ordinary derivatives proves the bound for each minor product. The integers A_n, A_s clear its class denominators. This proves the matrix assertion uniformly over all class pairs, including any signed combination with integral coefficients.

The dimension formula for each of the two Schur representations has factors $(\lambda_i - \lambda_j + j - i)/(j-i)$, after padding by zeros. Each factor is at most $1 + \lambda_1$. The ranks and the number of embeddings are bounded by powers of x, and $\log(1 + \max_\tau L_\tau) = O_{\psi,\varepsilon}(x^{\varepsilon} + \log(2x))$ from (104.2) and the prescribed offsets. This gives the asserted dimension bound.

In the same degree formula write $L_\nu = (k-1)B_\nu + J_x$. The other terms of (104.2) do not depend on ν. The v_i, the spacings, and the ranks in that formula are polynomially bounded in x, so $|J_x| + ns \le x^{C_1}$ on sufficiently distant intervals. Since $d_M(L_\nu, ns) \ge L_\nu/M - ns$, the contribution to g_p of a positive $B_\nu/M - B_{\pi_p^{j}\nu}$ is at least $(k-1)(B_\nu/M - B_{\pi_p^{j}\nu}) - x^{C_1}$. A nonpositive difference needs no error term.

For completeness the actual offsets are given following the weight lists in Proposition 9.9 and the construction of the terminal prime in Section 9.9. Write $X' = \exp(x^{\varepsilon/3})$ and let n_0 be the degree parameter in $w'' = b/(n_0 X')$ there. It is bounded by a fixed power of x. The three initial offsets are $\lceil xb/X' \rceil + \lfloor b/X' \rfloor$, $\lceil b \rceil$, and $\lceil b/(n_0 X') \rceil$. A parity correction can add one. In particular $B_\nu > 0$ and $R = \max B_\nu / \min B_\nu \le x^{C_2} X'$. Only powers $p^{j} \le R$ can have a positive difference. The number of available ν's is at most f. Each integer at most R is a power of at most one prime, hence $\sum_{p^{j}\le R} \log p \le R \log(2R)$. The accumulated error is at most $f x^{C_1} R \log(2R)$. This has the stated order and is $o(\exp(\eta x^{\varepsilon}))$ for each $\eta > 0$. □

For an algebraic incoming and outgoing array one can subtract their denominator costs explicitly. If their least coordinate valuations at a place are a_v, b_v, the normalized value of $A_n A_s \mathcal{B}_{\mathrm{row}}$ on them has valuation at least $g_p + a_v + b_v$, by bilinearity. The matrix estimate itself does not require either vector to survive the preceding lifts. Using its value in a product formula for the full period requires the arithmetic size of those actual vectors and a nonzero contraction with the signed terminal projection.

102.6 Congruences for paired frames

At a marked place v, the split unitary group of S is $\mathrm{GL}_s(F_v)$, $s = \dim_E S$. Write K for the stabilizer of the lattice, $K(a)$ for its principal congruence subgroup, $Q = Nv$, and π for a uniformizer. Use the unscaled columns $b_1, \dots, b_\ell$ of Proposition 121.4. Their Gram matrix is zero when $c = 2r$. When $c = 2r + 1$ its only nonzero entry is the last diagonal entry, equal to -1. Denote these ranks by $h = 0, 1$, respectively.

Lemma 102.6 (The marked orbit of an ordered frame)**.** *Let $b'_1, \dots, b'_\ell$ have the same Gram matrix as the given frame. Then*

$$b'_j - b_j \in \pi^a L_{j,v} \quad (1 \le j \le \ell) \quad \Longleftrightarrow \quad (b'_1, \dots, b'_\ell) \in K(a)(b_1, \dots, b_\ell).$$

With the K-orbit assigned invariant probability, the measure of this congruence class for $a \ge 1$ equals

$$\left[Q^{(a-1)(2s\ell-\ell^2)} \prod_{j=0}^{\ell-1} (Q^s - Q^j)\, Q^{h(s-\ell)} \prod_{j=0}^{\ell-h-1} (Q^{s-\ell} - Q^j) \right]^{-1}.$$

An empty product is one.

Proof. Identify the split halves of the self-dual lattice with a free module M over $\mathcal{O}_v$ and its dual. Locally choose generators of $I_j^{(c)}$ in each half. Multiply the w-part of column j by its first generator and the covector from its $\bar{w}$-part by the second. The resulting columns form a matrix V in M^ℓ, and the rows form a matrix Y with values in $\mathcal{O}_v^\ell$. The columns extend to a basis, and the rows to a dual basis, since the staircase lines are individual integral direct summands. Their product $YV = N$ is prescribed by the Gram matrix and the chosen generators. In the central case the two generators have unit product since the middle ideal has product one with its conjugate. Hence the reduction of N has rank h. The conditions on b'_j are exactly $V' \equiv V,\ Y' \equiv Y \bmod \pi^a$, with $Y'V' = N$. The split group sends a pair to (gV, Yg^{-1}).

Prescribe $g_0 V = V'$, extending g_0 by the identity on the other elements of a basis extending the columns of V. Then $g_0 \equiv 1 \bmod \pi^a$. Put $D = Y - Y' g_0$. It vanishes on V and is divisible by π^a. The rows of Y' have a unit maximal minor and therefore admit an integral right inverse J. Now $g = g_0 + JD$ satisfies $gV = V'$, $Y'g = Y$ and $g \equiv 1 \bmod \pi^a$. In particular it is invertible. Conversely any such congruence transformation preserves the indicated residue classes. This proves the orbit identity.

Over the residue field there are $\prod_{j=0}^{\ell-1}(Q^s - Q^j)$ independent ordered columns. After fixing them, choose a complement of dimension $s - \ell$. The row map on the columns is fixed by N. Its map on the complement must surject onto the quotient of the target by $\operatorname{im} N$, of dimension $\ell - h$. There are $\prod_{j=0}^{\ell-h-1}(Q^{s-\ell} - Q^j)$ such surjections, by choosing independent rows. Each has $Q^{h(s-\ell)}$ lifts to the target. All these pairs lie in one orbit. Indeed the prescribed isomorphism between the two column spaces intertwines their row maps. Extend it on the intersections with the row kernels to an isomorphism of the two full kernels. On the sum of the column space and the kernel this gives a well-defined isomorphism. Complete it using lifts on each side of a basis of the target modulo $\operatorname{im} N$.

In lifting a solution from depth n to $n + 1$, the linear condition on the corrections over the residue field is $\Delta Y\, V + Y\, \Delta V = D_0$, where D_0 is prescribed. The first term alone attains every ℓ-square matrix because V has independent columns. There are therefore $Q^{2s\ell-\ell^2}$ lifts. They are in one orbit over $\mathcal{O}_v/\pi^{n+1}$ above the residue-field orbit. Lifting the residue-field transformation and applying the preceding right-inverse construction over this ring maps either pair to the other. Every invertible matrix over the quotient lifts to K by lifting its entries. Thus the count is exactly the number of distinct reductions of the compact orbit. The orbit identity identifies

each fiber with a translate of the congruence orbit. Their invariant probabilities are equal, which proves the formula. □

102.7 The marked integrals on the original core chain

Use the finite level from Theorem 71.10. Write S_0 for its marked places, $w(v)$ for the chosen half over v, and $\mathfrak{m} = \prod_{v\in S_0} w(v)^{a_v}$ for the original ideal. At step $i < m$ put $u = n_i$ and $N = n_i + d$. In the map from the larger lattice to the smaller one use the $w(v)$-position space at a marked place. Choose bases of the two lattices there, put $M_v = \mathrm{Mat}_{u,N}(\mathcal{O}_v)$, and let $M_v^\vee$ be its full additive dual on the other half. The depth equations are those of Proposition 95.3. Normalize the additive measure by $\mathrm{vol}(M_v) = 1$ and group measure by $\mathrm{vol}(\mathrm{GL}_u(\mathcal{O}_v)) = 1$.

Theorem 102.7 (All marked factors at the prescribed depth)**.** *For every original field and every step, one integral finite test at the marked places gives a scalar local doubling pairing on the entire subspace fixed by $K_u(a_v) = 1 + \varpi_v^{a_v}\,\mathrm{Mat}_u(\mathcal{O}_v)$. It is fixed on the larger side by $K_N(a_v)$. Its scalar at $s = d/2$ is*

$$D_{i,v} = (Nv)^{-a_v(uN+u^2)} \prod_{j=1}^{u} \left(1 - (Nv)^{-j}\right)^{-1}.$$

There is a common integral residue class giving all the marked tests for a given pair of lattices. Over the entire chain one has

$$(N\mathfrak{m})^{-\sum_{i=0}^{m-1}(2n_i^2+dn_i)} \;\le\; \prod_{i=0}^{m-1}\prod_{v\in S_0} D_{i,v} \;\le\; 1.$$

Proof. Fix $v \in S_0$ and abbreviate $a = a_v,\ q = Nv$. Proposition 122.7 makes the two halves of the original lattice M_v and $M_v^\vee$. The congruence on the chosen half replaces its displacement lattice by $\varpi_v^a M_v$, leaving the frequency lattice $M_v^\vee$. Take $\phi_v = 1_C$, where $C = (I_u \;\; 0) + \varpi_v^a M_v$. Its values are integers. Translation by $\varpi_v^a M_v$ preserves C, and modulation by $M_v^\vee$ is trivial on its support. For a mixed displacement with components x, ℓ in these two lattices, the phase in the symmetric Weyl operator for $2(x,\ell)$, besides the pure translations and modulations, is the character evaluated at $\pm 2\langle x, \ell\rangle$. It is one by duality, including in residue characteristic two. This verifies both depth equations for the actual test.

The determinant splitting characters at the auxiliary place have conductor exponent one, and those at every other marked place have exponent two, the stipulated depth there. Since the auxiliary depth is positive, both characters are trivial on $1 + \varpi_v^a \mathcal{O}_v$ everywhere. In the split Schrödinger action the two groups act on the two sides of the rectangle by change of basis, multiplied by the determinant characters and real powers of absolute determinants. Both sorts of scalar are one on the indicated congruence groups. Consequently these subgroups fix ϕ_v. For the exponents see `Development/AbelianVarietyTheory/Definitions/Divisors/BasePlaceNotMemChiGridBadPlaces.lean`, line 70–82,125–139.

The first u columns of any element of C make an invertible square matrix congruent to I_u. If hC meets C for $h \in \mathrm{GL}_u(F_v)$, two such square matrices have quotient h, forcing $h \in K_u(a)$. Conversely each such h fixes ϕ_v. Its oscillator matrix coefficient on the smaller group is therefore $q^{-auN} 1_{K_u(a)}$, also for the conjugated oscillator. At the theta point the doubling section restricts to this coefficient. Integrating it against the source matrix coefficient of b, b' fixed by $K_u(a)$ gives $q^{-auN}\,\mathrm{vol}(K_u(a))\langle b, b'\rangle$, with the conjugate pairing if the conjugate convention is used. Reduction onto $\mathrm{GL}_u(\mathcal{O}_v/\varpi_v^a)$ is surjective since invertibility is detected modulo ϖ_v. Counting independent columns modulo ϖ_v and then the $q^{(a-1)u^2}$ lifts gives the displayed scalar. This

identifies the indicator of Proposition 72.15 inside the original depth space in the orientation of this core.

To realize the residues simultaneously, set $P = \Lambda_{\mathrm{big}}$, $Q = \Lambda_{\mathrm{small}}$. Finite projectivity of P gives $\mathrm{Hom}(P, Q) = P^{\vee} \otimes Q$. Reduction modulo $\mathfrak{m}$ identifies the quotient with the full module of maps of the two reduced lattices. By the Chinese remainder theorem this is the product of the local matrix modules at the $w(v)$. The chosen residues therefore lift to one element of $\mathrm{Hom}_{\mathcal{O}_E}(P, Q)$. Finally $\sum_{v \in S_0} a_v \log Nv = \log N\mathfrak{m}$ because each place is split. Dropping the inverse Euler factors, all at least one, gives the lower bound. The upper bound follows from $\mathrm{vol}(C) \le 1$ and $\mathrm{vol}(K_u(a)) \le 1$. □

102.8 The opposite line and its finite Fourier coefficients

Let C_0, C_1, C_2 be the first three even cores on a field of Theorem 71.3. Write Y for the bottom row space, $S = C_1$, and $s = \dim_E S$. The first two horizontal targets are $V_1 = S$ and $Z = V_2$. In the given frames

$$C_0 = Y \perp Ez, \qquad Z = C_2 \perp Ez', \qquad h(z, z) = 1, \quad h(z', z') = -1.$$

The isometry $V_1 = S$ preserves the two lattices since the added summand there has dimension zero. Write $\mathfrak{A}$ for the ideal of Proposition 122.7 and $\mathfrak{m}$ for the original marking. The lattice of Ez is $\mathcal{O}_E z$.

Proposition 102.8 (The exact transfer of the marked line test)**.** *The lattice on Ez' is $\mathfrak{l} z'$, where*

$$\mathfrak{l} = \frac{\mathfrak{A}\mathfrak{m}}{\bar{\mathfrak{A}}\bar{\mathfrak{m}}}.$$

The Hermitian adjoint, identifying the two underlying lines by $z \mapsto z'$ and accounting for the opposite sign of their forms, carries the whole original depth displacement lattice in $\mathrm{Hom}(S, Ez)$ *onto that in* $\mathrm{Hom}(Ez', S)$*. Put $D_L = (N\mathfrak{m})^s$. In the marked positions the corresponding Fourier transform of the line indicator in Theorem 71.3 is D_L^{-1} times a sum of exactly D_L distinct admissible coset indicators, with root-of-unity coefficients. This is in the normalization giving the original unshifted position lattice volume one at the marked places.*

Proof. For a one-dimensional added block, the grid assigns the ratio of the two modified oscillator ideals to its generator. Those ideals are $\mathfrak{A}\bar{\mathfrak{m}}^{-1}$ and $\bar{\mathfrak{A}}\mathfrak{m}^{-1}$, giving the displayed formula. This also proves $\mathfrak{l}\bar{\mathfrak{l}} = 1$, as required for the negative line lattice to be self-dual. The rank-one specializations of the defining formulas are in `Development/AbelianVarietyTheory/Definitions/Divisors/CmHostImaginaryGeneratorNeZero.lean`, line 132–134 and `Development/AbelianVarietyTheory/Definitions/Divisors/BasePlaceNotMemChiGridBadPlaces.lean`, line 80–88,178–189.

Use h_S and the positive scalar form on the coordinate line in defining $x^{\dagger}$. The old alternating trace form is the trace of $-\xi_0 h(x, y)$ with the ordinary Hermitian map pairing h, since the source is $-S$. The new source is $-Z$, whose restriction to Ez' is positive. The new map pairing on $x^{\dagger}, y^{\dagger}$ is $\overline{h(x, y)}$ by cyclicity of the matrix trace. Since $\bar{\xi}_0 = -\xi_0$, the two alternating trace forms agree. This is an F-rational symplectic map. It intertwines right action by g^{-1} with left action by g, since $(g^{-1})^{\dagger} = g$ on $U(S)$.

At unmarked places the displacement coefficient ideal on a Hom lattice is $\mathfrak{A}$. At the selected component of a marked place it is multiplied by $\mathfrak{m}$, because $\mathfrak{A}$ is a unit there. Thus the old line displacement lattice is the finite completion of $\mathfrak{A}\mathfrak{m}\,\mathrm{Hom}(\Lambda_S, \mathcal{O}_E)$. Perfectness of the Hermitian form on Λ_S carries its adjoint onto $\bar{\mathfrak{A}}\bar{\mathfrak{m}}\Lambda_S$. The new line displacement lattice is the completion of $\mathfrak{A}\mathfrak{m}\,\mathfrak{l}^{-1}\Lambda_S$. The formula for $\mathfrak{l}$ identifies them at every place.

At a marked v, set $a = a_v,\ q_v = Nv$. Write M_v for the integral old row lattice of positions and $M_v^\vee$ for its additive dual, a lattice of columns. The adjoint exchanges the two split components. Returning to the chosen position component therefore Fourier transforms the old function $1_{x_0+\varpi_v^a M_v}$, where x_0 is the indicated coordinate row. Its transform is

$$q_v^{-as}\,\psi_v(\pm\langle x_0, y\rangle)\,1_{\varpi_v^{-a}M_v^\vee}(y).$$

Indeed integrating a character on the translated lattice gives zero unless its restriction to the lattice is trivial, and in that case gives its value at the translate times the volume. The phase is constant modulo $M_v^\vee$, and there are q_v^{as} cosets in the displayed support. In new line coordinates this support is the full integral Hom lattice since $\wr$ has valuation a on the chosen half. Its depth sublattice is precisely $M_v^\vee$. Each of these cosets is therefore an allowed integral test. Multiplying the formulas over $v \in S_0$ proves the claim. At an unmarked split place the unshifted indicator transforms into its dual lattice indicator with the reference volume as scalar. At an inert place transport the polarization of the hyperbolic factor S by the rational adjoint. The unshifted vacuum is then its transported vacuum. These common reference factors are independent of the marked residues. □

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[2] J.-P. Serre, *Bounds for the orders of the finite subgroups of $G(k)$*, in *Group Representation Theory*, EPFL Press, 2007. Lecture I, Section 1.1, Theorem 1.