The generalized Riemann hypothesis for Dirichlet L-functions

27 November 2026 · Edition 930

Volume XIV

Unitary branching and first occurrence in the theta tower

Abstract

We follow a prescribed representation through the unitary theta correspondence. Branching, seesaw identities and terminal Hecke operators determine the rows used for the global lift, its first occurrence and the norms on the even core.

The guide in Volume I lists all twenty-seven volumes.

Contents

XIV Unitary branching and first occurrence in the theta tower 1
70 Unitary branching and the global seesaw . . . 3
70.1 Integral projectors for compact unitary branching . . . 3
70.2 Projected coefficients of Hermitian lattice theta series . . . 4
70.3 The global seesaw for a definite unitary period . . . 6
70.3.1 Finite-order determinant twists through a chain . . . 8
70.3.2 The number of central characters on the common interval . . . 10
70.3.3 Integral comparison across determinant classes . . . 11
70.3.4 Twists with the same output central character . . . 12
71 The first core and the prescribed row . . . 13
71.1 The cuspidal realization of the prescribed first core . . . 13
71.2 The unramified factors on the full even core chain . . . 14
71.3 One core form and line test for a whole bottom row . . . 16
71.4 Realizing the bottom projection in the first theta test . . . 17
71.5 The exact lowest-type projection on the first core test . . . 18
71.6 From a whole discrete summand to the specified vector . . . 19
71.7 The omitted coordinate of a tensor projector . . . 20
71.8 Cuspidality of the first core kernel . . . 22
71.9 Cuspidal lifting on the even core chain . . . 23
71.10 The first restriction on the original fields and intervals . . . 25
71.11 Every smooth bottom vector has a compact-finite core partner . . . 27
71.12 Selecting the unramified Hecke character of the first core . . . 27
71.13 Descent to the period of a prescribed bottom row . . . 28
71.14 Fixed forms in the initial seesaw period . . . 30
71.15 Selecting a prescribed real row vector . . . 31
71.16 First occurrence at the real places of the even core . . . 32
71.17 Cuspidal base change at the bottom of the two towers . . . 34
71.18 An everywhere unramified finite parity character . . . 35
71.19 The terminal exchange on the prescribed row . . . 36
71.20 Transporting the terminal graph to the column . . . 38
71.21 Following the refined level through the whole terminal flag . . . 39
71.22 The two Bessel positions at one pair of finite translates . . . 41

71.23 Transport of the projected theta chain at the terminal prime . . . 41
71.24 All the marked tests in the same theta chain . . . 43
72 Global theta lifting through the chain . . . 44
72.1 First occurrence on the real row . . . 44
72.2 A nonzero global period on the first theta pair . . . 46
72.3 The signed real theta chain to the compact group . . . 50
72.4 The finite standard L-values in the theta chain . . . 52
72.5 A common trace pairing and the finite theta tests . . . 56
72.6 Spherical parameters of a nonzero local lift . . . 59
72.7 A nonzero global theta chain with a choice of tests . . . 60
72.8 One iterated theta operator on the prescribed row . . . 63
72.9 The first global theta lift of the Bessel row . . . 67
72.10 The entire theta chain at the real places . . . 68
72.11 The successive standard L-values on the common fields . . . 68
72.12 The trace pairing at the finite places . . . 69
72.13 A nonzero iteration over the complex numbers . . . 70
72.14 One theta operator for all cuspidal inputs of a row . . . 71
73 Terminal Hecke operators and the first global lift . . . 72
73.1 Terminal covectors through the definite theta chain . . . 72
73.2 Fourier transformation of the definite matrix recurrence . . . 74
73.3 The terminal Hecke operator on the marked initial classes . . . 75
73.4 The auxiliary level factor in the single-lift determinant . . . 77
73.5 The common finite source of the auxiliary determinants . . . 80
73.6 The terminal flag and the remaining Gram determinant . . . 83
73.7 Split doubling factors along the row . . . 86
73.8 Cuspidality and the real pairings of the prescribed row . . . 88
73.9 Change of the row determinant under a tame twist . . . 92
73.10 Integral gluing at the old marked places of a single lift . . . 96
74 Horizontal lifts and first occurrence . . . 100
74.1 Real first occurrence on each horizontal row . . . 100
74.2 An explicit integral terminal coefficient for a restriction . . . 101
74.3 Cuspidal ascent along the full horizontal row . . . 102
74.4 The adjoint on consecutive line oscillators . . . 104
74.5 Nonvanishing at the second horizontal lift . . . 105
74.6 A nonzero period after the third horizontal lift . . . 107
74.7 Doubling values with both lengths of theta strings . . . 108
74.8 The relative norm on the full horizontal row . . . 110
74.9 Cancellation of the splittings through the second core . . . 111
75 Norms on the even core . . . 112
75.1 The spherical bound at the smaller core spacing . . . 112
75.2 Relative norms of the even core lift on its Hecke spaces . . . 114
75.3 Exponential accuracy of the relative core norms . . . 115
75.4 The second horizontal restriction for a lowest-type row . . . 116
75.5 The finite position test on the first norm shell . . . 117
75.6 Unfolding the Jacobi period on a full frame . . . 118
75.7 Integral coefficient families with controlled metrics . . . 120
76 References . . . 122

70 Unitary branching and the global seesaw

70.1 Integral projectors for compact unitary branching

Let E/E^+ be a CM extension. A Hermitian lattice in this volume is a projective $\mathcal{O}_E$-module of finite rank with a nondegenerate Hermitian form. It is *unimodular* if the form identifies it with its Hermitian dual, and *definite* if the induced forms at all complex embeddings are positive definite. Fix a coefficient field J containing the images of the embeddings of E to be used. For an embedding $\tau : E \hookrightarrow J$ and a lattice A, put $A_\tau = A \otimes_{\mathcal{O}_E,\tau} \mathcal{O}_J$.

We use the irreducible algebraic representation of GL_n of highest weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$. We specify an integral model $W_\lambda(A_\tau)$. For the partition $\nu_i = \lambda_i - \lambda_n$, take the image of an integral Young symmetrizer of shape ν on $A_\tau^{\otimes|\nu|}$ and tensor with $(\det A_\tau)^{\lambda_n}$, using the dual line for negative powers. The empty partition gives the trivial representation. Young symmetrizers in characteristic zero give the desired irreducible space after extension of scalars. The image construction commutes with localization, and its image is torsion free and hence projective over the Dedekind ring $\mathcal{O}_J$. In a local basis the operators e_{ab} of the matrix Lie algebra preserve this lattice. Indeed, their action on a tensor power is a sum of integral actions on the individual factors and commutes with permutations. The derivative on the determinant factor is λ_n times the trace. At a complex embedding we use a Hermitian norm invariant under the unitary group of A; the tensor construction supplies such a norm.

Theorem 70.1 (A common denominator for the branching projections)**.** *Suppose A is definite and unimodular of rank n. Let*

$$0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = A, \qquad \operatorname{rank} F_j = j$$

be a flag of unimodular Hermitian sublattices. Write $h = \lambda_1 - \lambda_n$. For restriction along the corresponding compact groups $U(F_1) \subset \cdots \subset U(A)$, every joint isotypic projection $P_{F,\boldsymbol{\mu}}$ for a complete branching pattern is defined over J. A single positive integer clears all these projections, for every flag of this kind:

$$B_\lambda = \prod_{j=1}^{n-1} \big((h+j-1)!\big)^j. \tag{70.1}$$

In the indicated integral model $B_\lambda P_{F,\boldsymbol{\mu}}$ preserves the lattice. For a fixed flag the nonzero projections are mutually orthogonal, sum to the identity and have operator norm one.

Proof. For the inclusion $s : F_j \hookrightarrow A$, the adjoint sends A into F_j by unimodularity of F_j, and $s^* s = 1$. Thus $A = sF_j \oplus \ker s^*$ integrally and orthogonally. The same argument for consecutive members splits the full flag into orthogonal projective lines. The group $U(F_j)$ acts as the identity on its orthogonal complement. Over J we obtain the corresponding block $\mathfrak{gl}_j \subset \mathfrak{gl}_n$. At each finite place a basis can be chosen for all the lines simultaneously.

For this block use the column Capelli polynomial

$$C_j(z) = \sum_{\pi \in \mathfrak{S}_j} \operatorname{sgn}(\pi) \prod_{b=1}^{j} \big(e_{\pi(b),b} + \mathbf{1}_{\pi(b)=b}(z+j-b)\big).$$

Multiply the factors in increasing order of b. The classical Capelli identity makes its coefficients central in $U(\mathfrak{gl}_j)$. On highest weight $\gamma = (\gamma_1, \ldots, \gamma_j)$ its scalar is $\prod_{b=1}^{j}(z + \gamma_b + j - b)$. This is the normalization of Molev's formulas (2.21)–(2.22) after a translation of the variable by $j-1$. Centrality also gives invariance under a change of block basis, since GL_j in characteristic

zero is connected and generated by its torus and root groups. In any integral block basis $C_j(k)$ preserves $W_\lambda(A_\tau)$ locally for $k \in \mathbb{Z}$, by the integral formula above. Basis independence identifies these local operators with a single rational operator. Preservation locally everywhere gives preservation of the global lattice. [1, Sec. 2]

By the branching law for GL_{j+1} restricted to GL_j, the next highest weight interlaces the previous one, each with multiplicity one. This follows, for example, by expanding the Schur character in its last variable: removing all entries equal to the largest letter in a semistandard tableau leaves a smaller tableau and a horizontal strip. Its row lengths are precisely the interlacing possibilities, and each filling of the smaller shape has one extension for each admissible strip. The determinant twist shifts all entries equally. Restricting the resulting character identity to $U(j)$ and using independence of irreducible characters gives the assertion there. Iteration places every entry between λ_n and λ_1.

Fix a desired weight μ at level j, and write

$$I_j = [\lambda_n, \lambda_1 + j - 1] \cap \mathbb{Z}, \qquad S_\mu = \{\mu_b + j - b : 1 \le b \le j\}.$$

The j members of S_μ are distinct and in decreasing order, so they determine μ. Consider

$$D_{j,\mu} = \prod_{s \in I_j \setminus S_\mu} C_j(-s), \qquad a_{j,\mu} = \prod_{s \in I_j \setminus S_\mu} \prod_{d \in S_\mu} (d - s). \tag{70.2}$$

The integer $a_{j,\mu}$ is nonzero, with value one for the empty product. On a summand of type μ, $D_{j,\mu}$ is multiplication by this integer. On any other type there is a shifted entry s outside S_μ. The corresponding factor kills that entire summand. Thus $P_{j,\mu} = D_{j,\mu}/a_{j,\mu}$ projects onto the full isotypic part, including any multiplicity. If $I_j = [u, v] \cap \mathbb{Z}$, the absolute value of the product over $s \notin S_\mu$ for a fixed d divides $(d-u)!(v-d)!$, which divides $(v-u)!$. Multiplying over d proves $|a_{j,\mu}| \mid ((h+j-1)!)^j$.

The centers of the nested blocks commute, since $U(\mathfrak{gl}_j)$ lies in $U(\mathfrak{gl}_k)$ for $j < k$. The product of the single-level projectors over the rows of a pattern is therefore its joint projector. For a unitary representation, inequivalent isotypic spaces are orthogonal. The single-level projectors are self-adjoint, and their commuting products are the orthogonal projections for the successive restrictions. The branching law exhausts the original space. Multiplying the individual denominators proves (70.1). □

Corollary 70.2 (The induced integral decomposition)**.** *For a fixed flag set $M = W_\lambda(A_\tau)$. For each pattern with projector P, write $V_P = P(M \otimes J)$ and $M_P = M \cap V_P$. Then*

$$M_P \subseteq P(M) \subseteq B_\lambda^{-1} M_P, \qquad B_\lambda M \subseteq \bigoplus_P M_P \subseteq M.$$

If several embeddings have designated weights λ_τ, every tensor product of such projections on $\bigotimes_\tau W_{\lambda_\tau}(A_\tau)$ has denominator dividing $B = \prod_\tau B_{\lambda_\tau}$.

Proof. An element in M_P is fixed by P. For $v \in M$, $B_\lambda P v$ is integral and belongs to V_P. Summing these integral elements over the pattern decomposition gives $B_\lambda v$, and distinct subspaces have direct sum. Tensoring the integral multiples of the projectors gives the last statement. □

70.2 Projected coefficients of Hermitian lattice theta series

For $1 \le j \le n$ let $\mathcal{C}^{(j)}$ be the genus of the standard positive definite Hermitian lattice $\mathcal{O}_E^j$, taken up to isometry. Its members are unimodular because unimodularity can be checked at finite places. The genus is finite by the double-coset finiteness theorem for definite unitary

groups, and $w(A) = |\operatorname{Aut}(A)|$ is finite since a positive definite trace form is preserved. For each $A \in \mathcal{C}^{(n)}$, let $\mathcal{F}(A)$ consist of the embedded flags of Theorem 70.1 whose F_j belongs to $\mathcal{C}^{(j)}$. This is a set of flags inside a chosen representative, without dividing by its automorphisms. [2]

On $\mathcal{C}^{(n)}$ take equivariant assignments $v_A \in M(A)$ and $h_A \in M(A)^\vee = \operatorname{Hom}_{\mathcal{O}_J}(M(A), \mathcal{O}_J)$, where $M(A) = \bigotimes_\tau W_{\lambda_\tau}(A_\tau)$. Equivariance means that the value transports under isometries, so on representatives these are stabilizer-invariant vectors and covectors. Let f be any $\mathcal{O}_J$-valued function on $\mathcal{C}^{(1)}$. For a pattern at each embedding use P_F for the resulting tensor projector in the given flag.

Theorem 70.3 (A projected theta coefficient on the common family)**.** *The period*

$$\mathcal{P}(f, v, h) = \sum_{A \in \mathcal{C}^{(n)}} \frac{1}{w(A)} \sum_{F \in \mathcal{F}(A)} f([F_1])\, h_A(P_F v_A) \tag{70.3}$$

is a mass-weighted sum of polynomially weighted Hermitian theta coefficients. With $d_E = [E : \mathbb{Q}]$ and $D_n = |\operatorname{GL}_{nd_E}(\mathbf{F}_3)|$ one has

$$D_n B\, \mathcal{P}(f, v, h) \in \mathcal{O}_J.$$

Summing first from the rank-one end or first from the rank-n end gives the same period, with the same denominator. On the fields $E = E_x$ of Theorem 19.6, take $n = \lfloor \sqrt{x} \rfloor + 1$ and at most one weight per embedding in a CM type. If H is the largest weight width $\lambda_{\tau,1} - \lambda_{\tau,n}$, then

$$\log(D_n B) \ll_\psi x^7 + x^4 (H + \sqrt{x} + 1) \log(2 + H + \sqrt{x}).$$

Proof. Choose representatives $A_1, \dots, A_n = A$ from the corresponding genera. For $j < n$ let $s_j : A_j \hookrightarrow A_{j+1}$ run over all isometries onto their images. There are finitely many choices because the images of a fixed $\mathbb{Z}$-basis have fixed lengths in positive definite trace lattices. Put $S_j = s_{n-1} \cdots s_j$. A tuple of maps determines the flag $F_j = S_j A_j$. Conversely, for a flag of the specified classes, choose isometries $A_j \to F_j$ independently for $j < n$, using the identity at $j = n$. These choices recover a unique tuple, and their number is $\prod_{j<n} w(A_j)$. The projector depends only on the images. Consequently (70.3) equals

$$\sum_{[A_1], \dots, [A_n]} \frac{f([A_1])}{\prod_{j=1}^n w(A_j)} \sum_{s_1, \dots, s_{n-1}} h_{A_n}(P_{F(s)} v_{A_n}). \tag{70.4}$$

All terms here are finite sums, giving the claimed equality of summation orders. For scalar trivial weights $B = 1$ and (70.4) is the chain of Theorem 7.3.

We justify the theta coefficient interpretation of the inner sum. Use fixed E-bases for the rational spaces attached to A_j. In the block for the image of S_j, a matrix unit acts on the ambient space by $S_j e_{ab} S_j^*$, where S_j^* is the adjoint for the fixed Hermitian forms. On the isometry locus this agrees with its extension by zero on the orthogonal complement because $S_j^* S_j = 1$. For a general tuple of linear maps s_j the same expression is polynomial in their entries and their complex conjugates, with constants supplied by the fixed forms. Applying the derived representation is a linear operation on this matrix. Formula (70.2), with its constant denominator, thus gives a polynomial insertion on the full product of the spaces of maps whose value on isometries is $h_{A_n}(P_{F(s)} v_{A_n})$. This description uses rational bases only to identify the polynomial.

In detail, sum this polynomial over $\prod_{j<n} \operatorname{Hom}_{\mathcal{O}_E}(A_j, A_{j+1})$, recording in independent variables the pullback Hermitian forms $s_j^* H_{A_{j+1}} s_j$. This is the polynomially weighted Hermitian theta series meant here. It can either be read coefficientwise or multiplied term by term by

Gaussian weights from positive definite trace norms on these spaces of maps. Each coefficient is finite, and the series with these Gaussian weights converges absolutely because a polynomial times a Gaussian is summable on a lattice. Extract the simultaneous coefficient at the tuple of forms $(H_{A_1}, \dots, H_{A_{n-1}})$. Its terms are exactly the isometries in (70.4). This identifies the coefficient independently of any automorphic period normalization.

Now use (70.3). The integer B makes each projected weight integral by Corollary 70.2. Reduction modulo three injects the finite group $\mathrm{Aut}(A)$ into $\mathrm{GL}_{nd_E}(\mathbf{F}_3)$; the prime-order binomial argument is given in Lemma 19.2. Hence $w(A) \mid D_n$, and the claimed denominator follows. The flag count introduces no further denominator.

Finally $d_E \le 4\,\mathrm{ord}(\psi)2^l \ll_\psi x^3$ for $0 < \varepsilon < 1/52$, and $n \le 2\sqrt{x}$. Thus $\log D_n \le (nd_E)^2 \log 3 \ll_\psi x^7$. At one embedding $\log B_\lambda \le n^2(H+n)\log(2+H+n)$ by (70.1). Sum over at most $d_E/2$ embeddings to obtain the estimate. □

Proposition 70.4 (A test with distinct flags)**.** *Take $A = \mathcal{O}_E^n$ with its standard form, $n \ge 3$, and the defining representation at one embedding. In each complete integral flag choose the pattern whose projection is onto F_1. If P_F denotes this projection, then*

$$\frac{1}{n!}\sum_{F \in \mathcal{F}(A)} P_F = \frac{I}{n}.$$

There nevertheless exist distinct flags with $\mathrm{tr}(P_F P_{F'}) = 1$.

Proof. A unimodular sublattice determines an orthogonal idempotent P on $\mathcal{O}_E^n$ by integral splitting. At every archimedean embedding each diagonal entry of P belongs to $[0,1]$. Such an entry is an algebraic integer of E^+. If it is nonzero, its absolute norm is a positive integer at most one. All conjugates must then equal one. Hence every diagonal entry is zero or one. Since $P = P^* = P^2$, the sum of the squared moduli in any column equals its diagonal entry. This forces all off-diagonal entries to vanish. The unimodular sublattices are therefore coordinate summands. They all lie in the prescribed genera. The $n!$ complete flags correspond to orderings of the coordinates, and each coordinate is first in $(n-1)!$ of them. This proves the average. The orderings $(1,2,3,\dots)$ and $(1,3,2,\dots)$ give different flags and both project onto the first line. Their projectors have product of trace one. The calculation is on the weight space. It also applies at an endpoint with a full marking modulo three, whose stabilizer is trivial. □

This example distinguishes two estimates required in applying a period comparison. Averaging over all flags here gives an exact scalar operator. Translates in the proposed prime count select particular maps. For two rank-one images in an n-dimensional target, a bound of order $1/n$ for their squared overlap would already fail for the two flags above. Whether a given arithmetic family of translates has small overlaps depends on its maps and on their distribution.

70.3 The global seesaw for a definite unitary period

Write $k = E^+$, let $\mathbb{A}$ be its adeles, and put $[G] = G(k)\backslash G(\mathbb{A})$. Fix a nontrivial additive character of $\mathbb{A}/k$. Let $V = V' \perp D$ be a definite Hermitian space, where D is a line, and let W be a nonzero skew-Hermitian space such that $U(W)$ is compact at all real places. For example, multiply a positive definite Hermitian form by a nonzero scalar of E negated by conjugation. Set $G' = U(V')$, $H = U(W)$, and view G' inside $U(V)$ by extending every isometry by the identity on D.

We recall the normalization that matters for the identity. The unitary dual pair $U(V) \times H$ acts on the symplectic space over k obtained from $V \otimes_E W$ with its alternating trace form. Use

a rational polarization compatible with the orthogonal sum, with position spaces $Y = Y' \oplus Y_D$. For each of the three pairs use the global Weil representation for the fixed additive character and an automorphic unitary splitting. Such splittings are supplied by the standard unitary theta construction with splitting characters. They agree with the rational splitting on rational points. If ϕ is an adelic Schwartz function on the position space, the kernel is

$$\Theta_V(g,h;\phi) = \sum_{y \in Y(k)} (\omega_V(g,h)\phi)(y).$$

Use corresponding notation for V' and D. Haar measures are fixed throughout; in a given integral the same measure is used on both sides. These constructions and the rational invariance of the theta functional are the usual global Weil representation and Poisson summation. [3]

Theorem 70.5 (Restriction of a theta lift against a period)**.** *There are unitary automorphic characters δ_1 of $G'(\mathbb{A})$ and δ_2 of $H(\mathbb{A})$, determined by the chosen splittings, with the following property. Take $\phi = \phi' \otimes \eta$ and continuous automorphic functions a on $[H]$, b on $[G']$. Define*

$$L_\phi(a)(g) = \int_{[H]} \Theta_V(g,h;\phi)a(h)\,dh.$$

The codimension-one period of this theta lift is given by

$$\begin{aligned} &\int_{[G']} L_\phi(a)(g' \oplus 1)\,\overline{b(g')}\,dg' \\ &= \int_{[H]} \delta_2(h)a(h)\Theta_D(1,h;\eta)\left(\int_{[G']} \delta_1(g')\Theta_{V'}(g',h;\phi')\overline{b(g')}\,dg'\right)dh. \end{aligned} \tag{70.5}$$

For compatible splittings $\delta_1 = \delta_2 = 1$. The equality remains valid after an archimedean isotypic projection on the restricted form, by applying the same projection to b on the right.

Proof. In the direct sum of symplectic spaces the Heisenberg representation in the chosen polarization is the tensor product on the two position spaces. Under the standard homomorphism of the two metaplectic covers into the cover of the sum, the oscillator action is the tensor product. This can also be seen on the Schrödinger generators: changes of coordinates and quadratic characters separate, and the Fourier transform of a product in the direct sum is the product of the two Fourier transforms. On $G'(\mathbb{A}) \times H(\mathbb{A})$ the two splittings to compare are thus lifts of the same symplectic action. One is the restriction of the splitting for V; the other is the product of the smaller lifts, evaluated at (g',h) and $(1,h)$. Their ratio lies in the central circle and is a character because both lifts are homomorphisms. The rational splittings agree under the direct sum, so the ratio is trivial on rational points. A character on a direct product is the product of its restrictions to the two factors. Call these δ_1, δ_2. We have thereby fixed the direction of the ratio by the equality

$$\Theta_V(g' \oplus 1, h; \phi' \otimes \eta) = \delta_1(g')\delta_2(h)\,\Theta_{V'}(g',h;\phi')\Theta_D(1,h;\eta).$$

Indeed the rational points in the chosen position space are $Y'(k) \times Y_D(k)$.

The quotients in (70.5) are compact: a definite unitary group is anisotropic over k, including its norm-one center. Equivalently, finite class number at a finite open compact level and compact archimedean groups give compact representatives. On compact subsets of the adelic group the oscillator translates of a Schwartz function give uniformly bounded Schwartz seminorms at infinity and a common compact open support bound at the finite places. For the latter assertion use smoothness at finite places and a finite covering of the compact set by cosets

fixing the test functions. Thus the rational-point series are dominated by the sum of a sufficiently rapid inverse power of the archimedean norm over a fixed lattice. These sums converge uniformly. Both automorphic weights are bounded on their compact quotients. Substitute the kernel identity in the left integral and apply Fubini to obtain (70.5). Finite sums of factorizable test functions follow by linearity.

Finally, on the compact archimedean group $G'(k\otimes\mathbb{R})$, the orthogonal projector onto an irreducible type with character χ_μ is right convolution by $\dim(\mu)\overline{\chi_\mu}$ for probability Haar measure, in the convention $Pb(g) = \dim(\mu)\int \overline{\chi_\mu(t)}b(gt)\,dt$. Schur orthogonality gives this formula and self-adjointness. Right invariance of quotient measure moves the projector across the pairing on $[G']$. Since restriction to G' commutes with right translation by G', the assertion also applies to the restriction of the lifted form. The integral formula moves the projector on the whole restricted form, with the splitting characters kept as in (70.5). □

Theorem 70.5 applies to every E_x on the intervals of Volume IV. Several orthogonal line summands can be treated by factoring the Schwartz function over all the summands and repeating the same proof. This gives the complex period identity for a fixed W. The integral lattice in Theorem 70.3 is specified by polynomial insertions on definite Hermitian embedding coefficients. To identify it with the lattice of periods of an *iterated* lift with changing W and prescribed Hecke characters, the map on test vectors at each finite place must carry the specified lattices with controlled indices. The seesaw integral gives equality of complex values and has no index term in its formula.

In the notation of Volume I, the target is its inequality (6.3), $\Delta_u \le x^{5\varepsilon}\mathcal{B}$ for every translate u. It requires a lower estimate for finite divisibility that represents the Frobenius-weighted count and an upper estimate from archimedean norms and covolumes on the same period lattice. For example, the associated off-diagonal Gram estimates concern distinct Hecke translates; Theorem 70.1 gives orthogonality within one flag, and Proposition 70.4 computes what happens for two distinct flags. To deduce (6.3) one still needs the actual translate pairings and the integral identifications just described. Their uniform comparison on $[V, V^{3/2}]$ remains unresolved here.

Source of the comparison. Items 7, 8, 12, 15 and 16 on page 4 of Alpöge's *Remark* describe the branching, theta lifts, common Bessel period and an estimate between twists. In the supplied source, `Formalism/SharpRestriction/GlobalTheoremInputs.lean`, lines 65–89, specifies the rationality, common isometry and simultaneous height requirements. The assertions proved in this volume use the mathematical constructions above. [4]

70.3.1 Finite-order determinant twists through a chain

Let $T = \mathrm{Res}^1_{E/k}\,\mathbf{G}_m$, the norm-one torus, and write $[T] = T(k)\backslash T(\mathbb{A})$. Take $G_i = U(V_i)$, $0 \le i \le m$, definite at every archimedean place. Write $n_i = \dim_E V_i$. Use Hermitian and skew-Hermitian forms in alternating order so that consecutive spaces give unitary dual pairs. Once splittings and Schwartz functions are chosen, let $\Theta_i(g_{i-1}, g_i)$ be the corresponding global theta kernel for $1 \le i \le m$. All groups and the kernels in this subsection are over the same E/k. The Haar measures on their compact quotients are fixed.

Let χ_i be finite-order automorphic characters of $T(\mathbb{A})$, $1 \le i < m$, and set $\xi_i = \chi_i \circ \det$ on $G_i(\mathbb{A})$. For a continuous automorphic function f on $[G_0]$, form the actual iterated lift with these multipliers inserted at the intermediate groups:

$$(A_{\boldsymbol{\chi}} f)(g_m) = \int_{\prod_{i=0}^{m-1}[G_i]} f(g_0) \prod_{i=1}^{m} \Theta_i(g_{i-1}, g_i) \prod_{i=1}^{m-1} \xi_i(g_i) \prod_{i=0}^{m-1} dg_i. \tag{70.6}$$

Continuity and absolute convergence follow as in Theorem 70.5, using the finite product of compact quotients. Computing the integral successively gives the individual theta lifts. Changing the splitting at an edge multiplies its kernel by a character of either group on that edge, or by their product, by the central-ratio argument of Theorem 70.5. Characters from neighboring edges multiply at their common group. Formula (70.6) records a finite-order determinant product remaining at an intermediate group. If the products are trivial, only the two endpoint multipliers remain.

Lemma 70.6 (Integral endpoint multipliers)**.** *Realize the integral algebraic modular forms on $[G_i]$ at a finite open compact level by their values in prescribed equivariant $\mathcal{O}_J$-lattices at finite-adelic representatives. If χ has finite order, is trivial on the determinant of the level, and J contains its values, multiplication by $\chi \circ \det$ is an integral automorphism and a Petersson isometry. If all intermediate products from a splitting change are trivial and its two endpoint multipliers are of this kind, their action preserves the finite content of a rational endpoint pairing, as well as its norm at every archimedean embedding.*

Proof. The character is trivial on rational points, and its archimedean restriction is trivial since each compact real unitary group is connected. It is constant on each finite level coset and its values are roots of unity, hence units at every finite place of J. Multiplication and division both preserve each value lattice. At every archimedean embedding the scalar has modulus one, and integration in the Petersson norm gives the isometry. For finitely many characters one may take the intersection of their kernels in the original finite level. With the stated cancellation, expansion of the iterated integral leaves just the endpoint multipliers, which can be absorbed into the two test functions. As these are bijections of the integral test lattices, the ideals generated by all paired values agree. The same bijections are unitary for the archimedean operator norms. □

Theorem 70.7 (Central characters of the images)**.** *Put $s_i = (-1)^i$. For each edge there is a unitary automorphic character λ_i of $T(\mathbb{A})$, determined by its splitting, such that the following holds. Suppose f has central character ν_0, with the convention $f(zg) = \nu_0(z)f(g)$. The image in (70.6) belongs to the central character $\nu_{\boldsymbol{\chi}}$ determined by*

$$\nu_{\boldsymbol{\chi}}(z^{s_m}) = \nu_0(z) \prod_{i=1}^{m} \lambda_i(z^{s_{i-1}}) \prod_{i=1}^{m-1} \chi_i(z)^{s_i n_i}. \tag{70.7}$$

In particular two choices with different products $\prod_{i=1}^{m-1} \chi_i^{s_i n_i}$ have orthogonal images on the entire ν_0-space for the Petersson inner product.

Proof. Use the convention that the symplectic action at an edge is the action on the tensor product of the two spaces over E. Then $(zI_{V_{i-1}}, z^{-1}I_{V_i})$ acts trivially on that symplectic space. Its lift by the splitting lies in the central circle of the metaplectic group and acts on the Weil representation by a scalar. These scalars define $\lambda_i(z)$. Multiplicativity of the splitting makes it a character; rational compatibility makes it trivial on $T(k)$. This proves, for the same Schwartz function,

$$\Theta_i(zg_{i-1}, z^{-1}g_i) = \lambda_i(z)\Theta_i(g_{i-1}, g_i).$$

Evaluate (70.6) at $z^{s_m}g_m$ and substitute $z^{s_i}g_i$ for each previous integration variable. Haar measures are preserved. At consecutive groups the exponents have opposite sign, so each kernel contributes exactly its indicated scalar. The determinant of the scalar action z^{s_i} on V_i is $z^{s_i n_i}$, giving the remaining factors. This is (70.7). For two different output characters choose z where they differ. Applying central translation by z to their Petersson pairing multiplies it by a unit complex number different from one while preserving its value, so the pairing is zero. □

70.3.2 The number of central characters on the common interval

For a positive rational integer M, write $T(\widehat{\mathcal{O}}_k)$ for the finite-adelic norm-one integral units and $T(\widehat{\mathcal{O}}_k; M)$ for the subgroup congruent to one modulo $M\widehat{\mathcal{O}}_E$. Define

$$C(M) = T(\mathbb{A})/\big(T(k)T(k\otimes_{\mathbb{Q}}\mathbb{R})T(\widehat{\mathcal{O}}_k; M)\big).$$

Every character of this group is a finite-order determinant multiplier after composition with determinant. Write $d_E = [E:\mathbb{Q}]$, $w_E = |\mu(E)|$, and assume $E \subseteq \mathbb{Q}(\zeta_N)$.

Proposition 70.8. *The group $C(M)$ is finite and satisfies $\log|C(M)| \ll d_E \log(2NM)$, with an absolute constant. For any odd rational prime $P \nmid N$,*

$$|C(P)| \;\geq\; (P-1)^{[k:\mathbb{Q}]}/w_E, \qquad w_E \leq 2d_E^2. \tag{70.8}$$

Proof. At level one, valuations identify $C(1)$ with the group of fractional ideals I of E satisfying $I\bar{I} = \mathcal{O}_E$, modulo the ideals (a) with $a\bar{a} = 1$. At split places a norm-one idele has opposite valuations in the two factors, and any desired opposite pair is realized. At the other finite places its valuation is zero. The kernel of this description consists of the integral norm-one units and the archimedean factors in the definition.

The image of the map to the ideal class group has order at most h_E. For a principal ideal in its kernel, $I = (a)$, its norm $a\bar{a}$ is a totally positive unit of k. Changing the generator changes this unit by the norm of a unit of E; equality modulo such norms gives the same class of norm-one principal ideals. Thus the kernel injects into $\mathcal{O}_k^{\times,+}/N_{E/k}\mathcal{O}_E^{\times}$. The norm group contains all squares of units of k, so the unit theorem bounds this index by $2^{[k:\mathbb{Q}]}$. Passing from $C(1)$ to $C(M)$ adds at most $|(\mathcal{O}_E/M\mathcal{O}_E)^{\times}| \leq M^{d_E}$ by reduction of integral norm-one units. Minkowski's ideal bound and the inequality $\zeta_E(2) \leq \zeta(2)^{d_E}$ give $\log h_E \ll d_E + \log D_E$, as in Lemma 19.3. The conductor–discriminant formula gives $D_E \leq N^{d_E}$, proving the upper bound.

For the lower bound consider the image of the integral norm-one ideles in $C(P)$. Its kernel modulo $T(\widehat{\mathcal{O}}_k; P)$ consists of reductions of global norm-one units. These units belong to $\mu(E)$ by Kronecker's theorem, since all their archimedean absolute values are one. As $P \nmid N$, every local extension at P is unramified or split. At a split place of k with residue field of size Q, reduction of the norm-one local units has order $Q-1$. At an inert place it has order $Q+1$, the kernel of the residue norm. To see surjectivity in this case, lift any residue root of unity of order dividing $Q+1$ to its Teichmüller root of the same prime-to-P order. Conjugation raises this root to the Q-th power, so its norm is one. Independent local choices give at least $\prod_{v\mid P}(P^{f_v}-1) \geq (P-1)^{[k:\mathbb{Q}]}$ residue classes, since $\sum_{v\mid P} f_v = [k:\mathbb{Q}]$. Divide by the bound w_E for the global kernel. Finally $\mathbb{Q}(\zeta_{w_E}) \subseteq E$, and $\varphi(m)^2 \geq m/2$ for every m, as one sees prime power by prime power. The only factor $\varphi(p^a)^2/p^a$ less than one occurs at 2^1, where it is $1/2$. Thus $w_E \leq 2d_E^2$. □

Corollary 70.9 (Pairings for explicit twists of the chain)**.** *On the common intervals take $E = E_x$, $n_i = i+1$, $m = \lfloor\sqrt{x}\rfloor \geq 3$. Fix an odd prime $P \geq 5$ prime to $2F$. For all sufficiently large intervals $P \nmid N = 2^{l+2}Fqd$. For each $\omega \in \widehat{C(P)}$, put $\chi_1 = \chi_2 = \omega$ and take all other interior characters trivial in (70.6). On any one starting central character the corresponding images in $L^2([G_m])$ are pairwise orthogonal for distinct ω. The number of twist choices is at least $\exp(c_P x^{2+8\varepsilon})$ for all large x, where $c_P > 0$ is independent of x, q, d.*

Proof. Definite spaces of these ranks exist over each E_x, using standard forms with a purely imaginary scalar in every skew-Hermitian space. The two auxiliary primes eventually exceed P. In (70.7), the product contributed by the multipliers is $\omega^{-2+3} = \omega$; hence all output

characters are distinct. Apply Theorem 70.7 and (70.8). We have $[k:\mathbb{Q}] \ge 2^{l-1} \ge x^{2+8\varepsilon}/2$ and $\log w_E = O_\psi(\log(2x))$. These estimates give the claimed number of choices. □

The conclusion concerns orthogonality, including the possibility that a lift vanishes. Projecting all these images to a *fixed* output central character retains at most one of them. In particular the family in Corollary 70.9 alone cannot prove that many projected maps of positive rank contribute in one designated Bessel component.

70.3.3 Integral comparison across determinant classes

There is also a simultaneous comparison for the finite Fourier transforms that occur when the internal products of characters survive. It specifies the possible denominator in passing between all twists and all determinant classes. Fix a level M and write D_i for the image of determinant from $[G_i]$ in $C(M)$, $1 \le i < m$. This is a finite group: it is the image of $G_i(\mathbb{A})$ as rational determinants map to one. Put $D = \prod_{i=1}^{m-1} D_i$ and enlarge J to contain the values of its characters. Pair the output (70.6) with an endpoint test function $\overline{h}$. For $a = (a_i) \in D$, let $P_a(f,h)$ be the untwisted endpoint integral restricted at intermediate group i to the fiber $\det g_i = a_i$. For $\xi \in \widehat{D}$ let P_ξ^{tw} be the full integral with multiplier $\xi((\det g_i)_i)$.

Proposition 70.10. *For fixed endpoint test functions,*

$$P_\xi^{\mathrm{tw}} = \sum_{a\in D} \xi(a) P_a, \qquad |D| P_a = \sum_{\xi\in\widehat{D}} \overline{\xi(a)} P_\xi^{\mathrm{tw}}, \qquad \sum_{a\in D} |P_a|^2 = |D|^{-1} \sum_{\xi\in\widehat{D}} |P_\xi^{\mathrm{tw}}|^2.$$

If the fiber values on chosen integral test lattices lie in J and generate a fractional ideal I_{fib}, their twisted values generate an ideal satisfying

$$|D| I_{\mathrm{fib}} \subseteq I_{\mathrm{tw}} \subseteq I_{\mathrm{fib}}.$$

In particular these ideals agree at every finite place away from $|D|$ and, for nonzero ideals, their local valuations differ by at most the valuation of $|D|$. One has $\log|D| \ll m d_E \log(2NM)$.

Proof. The determinant fibers are open and closed, so partitioning the absolutely convergent integral gives the first formula. For $a, b \in D$, $\sum_\xi \overline{\xi(a)}\xi(b)$ equals $|D|$ if $a = b$ and vanishes otherwise. Indeed translating the character sum by a character nontrivial on ba^{-1} multiplies it by a number different from one. Substitution gives the inverse and the stated sum of squares. Character values and their conjugates are algebraic units, so both integral linear combinations imply the ideal inclusions on all test values. The size bound follows from $D_i \subseteq C(M)$ and Proposition 70.8. □

For the lattice theta coefficients of Theorem 70.3, the rationality condition in Proposition 70.10 can be realized directly. Use the genera with finite level restricted to the determinant kernel modulo M at each intermediate group. On each level class the determinant in $C(M)$ is well-defined because rational points and the level have trivial image. Equivalently, decorate each lattice by a coset of its level inside its original stabilizer. In the finite embedding count of (70.4), fix such decorated representatives, count embeddings respecting their induced decorations whenever these are prescribed, and use their actual stabilizers. For definiteness one may allow all isometric embeddings between the underlying lattices while summing over every decoration. Changing an intermediate representative by a decorated isometry permutes the tuples. Each underlying flag with fixed intermediate decoration choices inside the endpoint has $\prod_{j<n} w(A_j)$ descriptions by decorated representatives, with $w(A_j)$ now the corresponding stabilizer orders. This follows by transporting each decoration through $S_j : A_j \to F_j$, just as

in the proof of Theorem 70.3. Restricting the finite sum to any prescribed determinant fibers therefore leaves every value in $(D_nB)^{-1}\mathcal{O}_J$. For the explicit integral model the two ideals of Proposition 70.10 exist, since they lie in that fractional lattice over a Noetherian ring. At each large x, taking $M = P$ and the family of Corollary 70.9 gives $\log|D| \ll_{\psi,P} x^{7/2}\log(2N)$.

For the periods prescribed in Section 6.6, one still needs the lower divisibility that records its prime count and the upper slope estimate on its prescribed common target. Exact orthogonality before projection in (70.7) and an identity of complex theta integrals do not identify the integral fibers of that target. In particular, the terms pairing distinct twists with the *same* output central character, the sizes of their diagonal blocks, and the index of the actual finite-place test lattices require estimates. Section 70.3.4 computes the combined Gram expressions in one such character.

70.3.4 Twists with the same output central character

Keep the ranks $n_i = i + 1$, $m \geq 3$, and the finite torus group $C = C(P)$ of Corollary 70.9. Every determinant map $G_i(\mathbb{A}) \to T(\mathbb{A})$ is onto. Indeed, a nondegenerate Hermitian space has an orthogonal line defined over E, by diagonalization over the base field. The same holds for a skew-Hermitian space after scaling its form. Any norm-one idele can act on that line and fix the orthogonal complement. We may therefore take $D = C^{m-1}$ in Proposition 70.10. Write

$$B = \{(z^{s_1 n_1}, \dots, z^{s_{m-1} n_{m-1}}) : z \in C\}, \qquad H = D/B.$$

The first two coordinates z^{-2}, z^3 have product z, so $B \simeq C$ and $h = |H| = |C|^{m-2}$.

Fix a finite-dimensional subspace V of continuous input forms with one central character and a finite-dimensional Hermitian target W. At the output apply any fixed bounded linear map into W, for example the Petersson projection to a prescribed subspace. Evaluation against a fixed continuous Bessel test on a compact subgroup is also allowed on continuous outputs, since it is bounded in the supremum norm there. Use the same output map for every twist. Fix $\xi_0 \in \widehat{D}$. The twists having the same output central character as ξ_0 are precisely $\xi_0\eta$, with $\eta \in \widehat{H}$ inflated to D, where the character means the one prescribed by Theorem 70.7, even for a zero lift. Indeed, a ratio preserves that character exactly when it is trivial on B. Denote the resulting operators $V \to W$ by A_η. For $a \in D$ let $L_a : V \to W$ be the untwisted operator restricted to determinant fiber a, followed by the same output map. For $y \in H$ define

$$F_y = \sum_{a \bmod B = y} \xi_0(a) L_a, \qquad T = \sum_{y \in H} F_y F_y^*, \qquad C_g = \sum_{y \in H} F_y^* F_{yg} \quad (g \in H).$$

These are finite sums of operators given by the actual compact integrals of (70.6). In the definition of F_y, the notation for the summation means all $a \in D$ whose image is y.

Theorem 70.11 (Second moments after a common projection)**.** *With adjoints for the chosen Hermitian inner products and I_V the identity,*

$$\frac{1}{h}\sum_{\eta \in \widehat{H}} \|A_\eta^* A_\eta - I_V\|_{\mathrm{HS}}^2 = \|C_1 - I_V\|_{\mathrm{HS}}^2 + \sum_{g \neq 1} \|C_g\|_{\mathrm{HS}}^2, \tag{70.9}$$

$$\sum_{\eta \in \widehat{H}} A_\eta A_\eta^* = hT. \tag{70.10}$$

In addition, suppose the entries of the F_y paired on prescribed integral test lattices generate a fractional ideal I_F in a coefficient field containing the character values. The corresponding ideal I_A for all twists in this one central character satisfies $hI_F \subseteq I_A \subseteq I_F$.

Proof. Splitting the integral by its determinant fibers gives

$$A_\eta = \sum_{y \in H} \eta(y) F_y, \qquad hF_y = \sum_{\eta} \overline{\eta(y)} A_\eta.$$

This identity is preserved by restriction to V and by the linear map into W. In $A_\eta^* A_\eta$, group pairs of indices (y, v) according to $g = y^{-1}v$. This yields $A_\eta^* A_\eta = \sum_g \eta(g) C_g$. Subtract I_V from the constant coefficient and apply orthogonality of characters entry by entry in an orthonormal basis of V. Summing the resulting squares gives (70.9). In the sum of the products $A_\eta A_\eta^*$, orthogonality retains only $y = v$, which proves (70.10). Finally both Fourier relations involve algebraic integers, since character values are roots of unity. Apply them to each lattice test pair and then take the ideals generated by all such tests. □

In particular let $d = \dim W > 0$, $b = \dim V > 0$, and index the maps in Proposition 6.9, by this common family, using one copy of V for each twist. Its first hypothesis is exactly that the right side of (70.9) is at most eb. Its second hypothesis is

$$d \operatorname{tr}(T^2) \le (1 + \delta)(\operatorname{tr} T)^2,$$

where δ is the nonnegative parameter of that hypothesis. For $T \neq 0$ this asks how evenly the eigenvalues of this particular positive operator fill the common target. Both assertions now concern the same projected maps and need estimates on their actual fiber values.

71 The first core and the prescribed row

71.1 The cuspidal realization of the prescribed first core

Let $[G] = G(F)\backslash G(\mathbb{A}_F)$, with its quotient measure, and keep the notation of Proposition 71.8. Right translation is $(R(a)F)(s) = F(sa)$.

Theorem 71.1 (The first restricted kernel belongs to one copy of the core)**.** *For the chosen row, lowest-type vector and point in Theorem 71.10, the kernel P_g there belongs to a single closed isometric copy of the prescribed real discrete series D in $L^2_{\mathrm{cusp}}([G])^{K_f}$. It is a smooth vector in that copy.*

Proof. The real coefficient used for this kernel is $\overline{\langle \pi(s)v', v' \rangle} = \langle D(s)b, b \rangle$ with $b = \overline{v'}$ in the conjugate Hilbert space. By the construction of Theorems 71.15 and 71.6, v' is obtained by translating a lowest-type tensor and then integrating its translates over the compact group of the hyperplane, against a compact matrix coefficient. All intervening translates, including a compact translate used to select a non-pure prescribed row, are real. Thus $b = \int D(a) b_0 \, d\nu(a)$ for a compact-finite vector b_0 and a finite complex measure of compact support. The vector b is smooth, since on this compact set every derivative of the translate of the smooth b_0 is uniformly bounded. The kernel is exactly $P_g = Y_b(b)$ with its original finite characteristic function.

On the Harish-Chandra module D_K define $Y(u) = Y_b(u)$. Proposition 71.8 puts its values in cuspidal L^2. Right translation by the real compact subgroup replaces u by its translate. Termwise real differentiation gives $R_X Y(u) = Y(dD(X)u)$. This is a Hilbert-space derivative as well. Both functions are smooth on the quotient and square integrable. Integration of the pointwise derivative along $\exp(tX)$ gives the same identity in L^2, since right translation is strongly continuous on that Hilbert space and the integral of the translated derivative is a Bochner integral. For example, testing against compactly supported smooth functions first proves the identity weakly, hence in L^2. Iteration gives all higher derivatives. These facts make Y a homomorphism of $(\mathfrak{g}, K_\infty)$-modules.

For a translate of b_0 the series satisfies the elementary identity $Y_b(D(a)b_0) = R(a)Y(b_0)$ term by term. Consequently

$$P_g = \int R(a)Y(b_0)\,d\nu(a).$$

This equality holds pointwise by the compact-uniform convergence in Proposition 71.8, and in L^2 by strong continuity and testing on compact sets. Theorem 71.10 gives a nonzero restricted integral of P_g. It is continuous, so it is nonzero in L^2, whose measure has full support. In particular $Y(b_0) \neq 0$.

The form $\langle Y(u), Y(t)\rangle_{[G]}$ on D_K is invariant, nonnegative, and nonzero. Its radical is an invariant submodule and D_K is irreducible. It is therefore positive definite. By admissibility any invariant Hermitian form gives an intertwiner to the compact-finite Hermitian dual. The original unitary form identifies that dual with D_K, using finite multiplicity on each compact isotypic space. Schur's lemma now makes the two forms positive scalar multiples. It follows that Y extends to a scalar multiple of an isometry from the whole Hilbert space of D.

This extension intertwines the real group. Indeed compact-finite vectors of an irreducible unitary Harish-Chandra module are analytic in its unitary globalization. For each real Lie algebra generator X, the norms of the successive derivatives of $Y(u)$ are those of the corresponding derivatives of u times the same scalar. Thus both one-parameter orbit maps are analytic and their Taylor coefficients correspond at the identity. They agree there on a neighborhood and hence on the entire real parameter line by real analyticity and translation. Density extends the equality to all vectors. The real unitary groups are connected and generated by their one-parameter subgroups. Hence the closed image is an isometric copy of D upon normalization. The compact integral above places P_g in that very image and identifies it with the image of the smooth vector b under the unnormalized extension. The finite level is preserved throughout. □

71.2 The unramified factors on the full even core chain

Keep the fields, intervals, a_l, g_*, m, n and core parameters of Theorem 71.10. The ranks of the even core chain are $n_i = n + id$, $0 \le i \le m$, with $d = (r + 6) - 1 = n + 4$. At the step from i to $i + 1$, number the step starting with $i = 0$ and use its unitary splitting character κ_i on both sides in the upward parameter convention. Its angular exponent at τ is $-2c_{m-i,\tau}$, where $c_{l,\tau} = \lfloor (a_{l-1} + a_l)/2 \rfloor$. These are the inverses of the characters indexed by $m - i$ in the original grid. They are trivial on $\mathbb{A}_F^\times$, as prescribed for the two even ranks, and unramified away from the marked finite places. The definitions of the characters, their order and their real exponents can be found in `Development/AbelianVarietyTheory/Proofs/TwistedOperators/TestGridCommonDataPropertiesHold.lean`, line 31–59. Use these given characters throughout. Let S contain the exceptional finite places in F, and write S_E for all places over S. If $w \notin S_E$, write $Q_w = Nw$ and evaluate an unramified character at a uniformizer of E_w.

Theorem 71.2 (Nonvanishing at every unramified doubling value of the core)**.** *Start with the bottom multiset $B_{0,w}$ of a unitary spherical Hecke character as in Proposition 71.12, in the upward convention, taking its conjugate when the conjugate of the form enters the kernel. Put*

$$A_w = \{Q_w^{(d-1)/2-l} : 0 \le l < d\}, \qquad B_{i+1,w} = B_{i,w} \sqcup \kappa_i(w)A_w,$$

$$z_* = (d+1)/2, \quad \rho_{hi} = \kappa_h\kappa_i^{-1}, \qquad C_i = \prod_{h<i}\prod_{a=1}^{d} L_E^{S_E}(a, \rho_{hi}).$$

For every $0 \le i < m$, the partial standard Euler product of $B_i \otimes \kappa_i^{-1}$, defined initially sufficiently far to the right, continues holomorphically to a neighborhood of z_*. Its value there is nonzero,

$C_i > 0$, *and*

$$e^{-2fn\log\zeta(3)} \le \left|\frac{L^S(z_*, B_i\otimes\kappa_i^{-1})}{C_i}\right| \le e^{2fn\log\zeta(3)}.$$

The constants hold on every original interval, including for the Hecke character detected by the first period. The transferred good parameters at any stage still distinguish all distinct systems of bottom good parameters.

Proof. The recurrence is the unitary unramified theta parameter formula with equal splitting characters. The old block is twisted by their ratio, which is one, and the new block is a character times the d-term segment. For the split and inert formulas see the unramified transfer calculation accompanying Theorem 72.16. The statement about these Euler products does not require a choice of ramified test vectors. Induction gives one copy of $B_{0,w}$ and exactly the segments $\kappa_h(w)A_w$ for $h < i$. On dividing the multiset by $\kappa_i(w)$ and multiplying its Euler factors on a common half-plane of absolute convergence one therefore obtains

$$L^S(z, B_i\otimes\kappa_i^{-1}) = L^S(z, B_0\otimes\kappa_i^{-1})\prod_{h<i}\prod_{l=0}^{d-1} L_E^{S_E}(z-(d-1)/2+l, \rho_{hi}).$$

At z_* the shifted arguments run once through $1,\dots,d$. For $h < i$, the arithmetic progression $a_l = a_{l-1} - g_*$ gives $c_{m-i,\tau} - c_{m-h,\tau} = (i-h)g_*$ because adding an integer commutes with taking a floor. Thus ρ_{hi} has nonzero angular exponent $2(i-h)g_*$ at each selected complex embedding. By Lemma 72.9 its partial Hecke function is holomorphic near these integers and is nonzero there, also at one. It takes positive real values at the integers because the ratio character is unitary and trivial on $\mathbb{A}_F^\times$, and the omitted set is conjugation stable. In particular all C_i are positive and depend only on the splitting data and omitted set.

By Lemma 75.1, at z_* each local logarithmic series for the initial factor is majorized, per root, by $\sum_{a\ge1} Q_w^{-3a}/a$. These series converge locally uniformly for $\Re z > (n+1)/2$, by the same estimate with exponent $\Re z - (n-1)/2 > 1$. In particular their exponential is holomorphic and nonzero on this half-plane. There are at most $2f$ primes of E above each rational prime p, all of norm at least p. The sum of the absolute logarithmic series at z_* is consequently at most $2fn\log\zeta(3)$, which proves both bounds. This continuation is for the full partial product with the omitted set held fixed. Finally each added segment is the same for all initial Hecke characters. Subtracting its multiplicity at every value of the multiset recovers the preceding multiset, even when roots coincide. Repetition recovers $B_{0,w}$ at each good prime. At a split place this multiset determines the normalized Satake orbit. At an inert place it is the union of pairs z_j, z_j^{-1}. It determines the tuple up to permutation and inversion of individual entries, precisely the relative Weyl orbit. At a self-inverse root the number of pairs is half its multiplicity. This proves the asserted separation. □

For the normalized unramified doubling quotient at $s = d/2$ one must also divide by $d_{n_i}^S(s)$. In the normalization following Theorem 72.16 it is the product for $1 \le j \le n_i$ of $L_F^S(d+j, \omega_{E/F}^{d+j})$ there. All Euler factors at these integers are positive. Each logarithmic absolute value is bounded by $f\log\zeta(d+j)$. The sum over $j \ge 1$ is $O(f2^{-d})$ by the absolutely convergent zeta series. Thus the product of the absolute normalized good-place quotients over all m steps, divided by $\prod_i C_i$, lies between e^{-Cmfn} and e^{Cmfn} with an absolute C. The remaining local integrals at the exceptional places and at infinity are needed to infer nonvanishing of a global lift with the prescribed test.

71.3 One core form and line test for a whole bottom row

Use the bottom hyperplane $Y \subset X = Y \perp L$ and the next larger core S. Thus $\dim Y = r,\ \dim X = u = r+1,\ \dim S = N = u+d$. Use their original lattices with $\Lambda_X = \Lambda_Y \oplus \mathcal{O}_E z$ on the line $L = Ez$. Fix the cuspidal realization T of one bottom row from Theorem 71.10. It has trivial character at its old finite level J_Y. Denote the core congruence level on S by J_S. Use the first seesaw integral $J(p,q)$ and the splitting ratios of Theorem 71.14.

Theorem 71.3 (A nonzero first projected seesaw integral at the original depth)**.** *For this entire row one can fix one terminal coefficient of Proposition 95.4, all descending tests from that terminal group down to S, and hence one cusp form A on S. One can also fix the finite tests on Y, L and the real line test q_∞. For every nonzero smooth row vector Tw some product Hermite test p_∞ on the Y-factor then gives $J(p,q) \neq 0$. The same period survives averaging the large test over $J_S \times J_Y$ in the original splitting. If ϕ_f is the large finite test, and e denotes that projector, then*

$$\|e\phi_f\|^2 = \|\phi_f\|^2/D, \qquad D\,e\phi_f \text{ has integral values}, \qquad 1 \le D \le (N\mathfrak{m})^{r^2}.$$

Here D is the order of the image of J_Y on the chosen marked halves of $\Lambda_Y/\mathfrak{m}\Lambda_Y$.

Proof. At each marked place use a basis of X ending in z. The rectangular indicator of Theorem 102.7 then has its first r rows centered on the first r coordinate row vectors in $\mathcal{O}_v^N$, and its last row centered on the u-th coordinate row vector. Every entry has the same displacement modulus $\varpi_v^{a_v}$. This is the product of the two indicated coset indicators. At an unmarked split place the integral rectangular lattice also decomposes by rows, with the same fractional ideal on the two summands. At an inert place use the integral hyperbolic polarization of the larger core S in Proposition 122.8. Its position variable consists of $N/2$ copies of X. The position lattice decomposes in each copy as the direct sum of the fractional lattices from Y and L. Thus in the direct-sum polarization the original finite test is $\phi_f = p_f \otimes \overline{q_f}$ with both factors integral. Conjugation changes the form on the line in the seesaw formula. All these bases may be chosen before applying Theorem 71.9, whose proof uses only full row rank of the specified residues.

For one nonzero lowest-type row vector, choose the terminal coefficient and a successful descent by Proposition 95.4. Keep all its tests above S, giving A. Its first restriction period is now the integral $I(\phi)$ of Theorem 71.14, by substituting the last descending kernel into $\mathcal{B}(S_{0,p_0}A, Tw)$. At infinity take the positive structures and position polarization respecting the orthogonal sum $Y \perp L$. Product Hermite functions on these real spaces are dense in Schwartz space. With the finite test already a product, continuity of the seesaw integral gives one real product $p_\infty^0 \otimes \overline{q_\infty}$ on which it is nonzero. These Hermite functions are polynomial Fock vectors for the compatible compact subgroups. Call the resulting form on X by descent F^0. It is a compact-finite cusp form by the reverse occurrence in Theorem 71.13 and is bounded.

The functional $w \mapsto \mathcal{B}(F^0, Tw)$ is continuous in the row Hilbert norm, since $[U(Y)]$ has finite measure and the restriction of F^0 is bounded. Its value at the chosen lowest-type vector is nonzero. Thus the orbit of any nonzero w in the irreducible row contains a translate having nonzero value. Changing variables transfers this real translation to the Y-action on the large test. In the direct-sum splitting only p_∞^0 changes, up to the nonzero scalar given by the splitting ratio on Y. All the other data stay fixed. For this fixed smooth Tw, the resulting functional on the Y-Schwartz variable is continuous. Indeed A and Tw decrease rapidly on their Siegel sets and the theta bound of Theorem 71.14 uses a Schwartz seminorm. The smooth-vector decay at fixed level follows from Miller and Schmid, *On the rapid decay of cuspidal automorphic forms*.

Expanding just the translated Y-test in product Hermite functions now gives the asserted nonzero $J(p,q)$.

It remains to compute the projector on this particular test. The full J_S already fixes ϕ_f, and it commutes with J_Y. At every unmarked place J_Y preserves it by Proposition 122.8. At a marked place $k \in J_Y$ operates on the u rows by $k \oplus 1$ or its inverse. The leading u-by-u square is invertible modulo the marking, so the translated support intersects the initial support exactly when k is the identity modulo $\varpi_v^{a_v}$. In that case its determinant character is trivial by the conductor bounds in Theorem 102.7. In general the modulus factor on the compact group is one and the remaining scalar is a root of unity. The translates in distinct residue classes thus have disjoint support and identical norm. The projector is their average with coefficient $1/D$, so its squared norm and its denominator satisfy the stated formulas. The image size is bounded by the number of all r-square matrices over the marked residue rings, namely $\prod_v (Nv)^{a_v r^2}$. Both input forms are invariant at their indicated levels. Changing integration variables therefore leaves the period unchanged under this averaging, as asserted. □

71.4 Realizing the bottom projection in the first theta test

Keep $v = \bigotimes_\tau v_\tau, H(v), T_v, P_v, T^\dagger$ of Theorem 74.2. Let $\phi = \bigotimes_\tau \phi_\tau$ denote the first real test in the downward kernel, with $\|\phi\|_2 = \prod_\tau \|\phi_\tau\|_2 > 0$. We take its real monomials using the positive structures of Lemma 106.10. Such structures may be fixed before choosing the monomials, since all monomials in them span the polynomial Fock model, as in Theorem 71.9. Let L_τ be their energies. Define

$$C(v,\phi) = \prod_\tau \sqrt{\zeta(2)}\, A(D, L_\tau) M_\tau(v_\tau).$$

For a terminal coefficient e, write $F_e(\Psi)$ for its descent using Ψ in place of the first real test. In this notation $F_e(\phi) = T^\dagger e$. Use linear Schwartz variables in the downward theta kernel and probability Haar measure on the terminal quotient.

Proposition 71.4 (Approximation of the projected restrictions)**.** *Fix a smooth bottom row u, and let $\mathcal{E}$ be any finite list of terminal forms in the prescribed compact type. There are finite Hermite polynomials $\Phi_j = \sum_\alpha c_{j,\alpha} h_\alpha$ in the first real test space such that*

$$\limsup_{j\to\infty} \|\Phi_j\|_2 \le \|\phi\|_2, \qquad \sum_\alpha |c_{j,\alpha}| \le C(v,\phi)\|\phi\|_2,$$
$$\lim_{j\to\infty} \mathcal{B}(F_e(\Phi_j), u) = \mathcal{B}(P_v T^\dagger e, u) \qquad (e \in \mathcal{E}).$$

The same first test Φ_j applies to every e in the list. Each h_α is an orthonormal product on Y and the line, and across the real places. All finite tests are the original ones.

Proof. Use an inner product linear in its second variable. On $G_\tau = U(C_0)_\tau$ put

$$q_\tau(g) = d_{\pi_\tau} \overline{\langle v_\tau, \pi_\tau(g) v_\tau \rangle}.$$

This belongs to L^1 by Lemma 106.10. Schur orthogonality for the discrete series gives $\pi_\tau(q_\tau) w = v_\tau \langle v_\tau, w \rangle$. The coefficient identity at two translated vectors gives $q_\tau * q_\tau = q_\tau$. Indeed, in the convolution integral the middle pair is integrated by the orthogonality formula

$$d_{\pi_\tau} \int \langle a, \pi_\tau(g) v_\tau \rangle \langle \pi_\tau(g) v_\tau, b \rangle \, dg = \langle a, b \rangle.$$

Also $q_\tau^* = q_\tau$. Thus in every unitary realization the integrated operator is an orthogonal projection of norm at most one. This applies to the actual real oscillator ω_τ.

Let K_τ be the maximal compact in the signed coordinates. Sum its central type projectors for the finitely many types in v_τ, and denote the resulting character measure by b_τ. In any unitary representation it gives an orthogonal projection. For instance the individual measures on Haar probability are $(\dim \gamma)\overline{\operatorname{tr} \gamma(k)}\, dk$ by compact Schur orthogonality. In the oscillator K_τ preserves the chosen positive structure. Its linear action on the positive creation space preserves total degree. Its splitting character is a scalar, so $\omega_\tau(b_\tau)$ commutes with $\mathcal{N}$.

Put $q_{\tau,j} = \mathbf{1}_{\{W(g)\le j\}} q_\tau$ for integers $j \ge 2$. The sets are compact in the matrix group and exhaust it. Define the Schwartz test

$$\Psi_j = \bigotimes_\tau \omega_\tau(b_\tau)\omega_\tau(q_{\tau,j})\phi_\tau.$$

An integral of translates over a compact set defines a Schwartz vector here, since the Weil action is continuous on Schwartz space and every Schwartz seminorm of these translates is bounded on the integration set. The L^1 convergence of $q_{\tau,j}$ to q_τ implies convergence in operator norm in any unitary representation. Both limiting factors are contractions. Hence $\limsup \|\Psi_j\|_2 \le \|\phi\|_2$. Because the compact projector contracts the weighted number-operator norm, the proof of Lemma 106.10 gives

$$\sum_\alpha |\langle h_\alpha, \Psi_j\rangle| \le \prod_\tau \sqrt{\zeta(2)}\, A(D, L_\tau)\|\phi_\tau\|_2 \int_{G_\tau} |q_\tau(g)| W(g)^{2s}\, dg = C(v,\phi)\|\phi\|_2.$$

Here the Hermite coefficients of a pure tensor multiply.

Right translation of $F_e(\phi)$ on the bottom core equals the descent with the corresponding bottom Weil translate of ϕ, by the definition of the theta kernel. This equality can be integrated against the compactly supported measures just used. The remaining descent is cuspidal and the theta kernel has polynomial growth uniformly on compact sets of Schwartz tests, as in Theorem 71.14. Thus $F_e(\Psi_j)$ is the product of the same compact and truncated convolution operators in the automorphic representation applied to $T^\dagger e$. The reverse real occurrence in Theorem 71.13 puts this vector in the bottom core isotypic space. The convolution tends in Hilbert norm to $P_v T^\dagger e$, since each b_τ fixes v_τ. All outputs belong to the same finite-dimensional subspace. Indeed the indicated real representation has finite cuspidal multiplicity at the fixed level, and its retained compact types have finite multiplicity in that representation. The restriction against u is defined on this subspace of smooth cusp forms and is continuous there by finite dimensionality.

Finally truncate the Hermite expansion of each Ψ_j. For a Schwartz vector, applying powers of $1+\mathcal{N}$ gives coefficient decay faster than any power of the degree. The Schwartz seminorms of a normalized Hermite function grow at most polynomially in that degree, by the raising and lowering relations and Euclidean Sobolev embedding. Since the number of indices of a degree is polynomially bounded at a fixed field, these expansions converge in Schwartz space. For each e the restriction integral as a function of the first test is continuous in that topology. In the seesaw expression both the form on C_1 and the row u have rapid decay, and a Schwartz seminorm bounds the theta kernel on their Siegel sets by Theorem 71.14. Choose the truncation so its error in all the indicated periods is at most $1/j$. Orthogonal coordinate truncation increases neither of the two coefficient norms. This gives the assertions for Φ_j. □

71.5 The exact lowest-type projection on the first core test

Take the original even core descent of Theorem 71.13. Write $T^\dagger e = F_e(\phi)$ for its bottom output, where $\phi = \bigotimes_\tau \phi_\tau$ is its first real test. The output is in the bottom discrete-series isotypic space by reverse first occurrence. At a real place write π_τ for that representation, K_τ for its maximal compact and κ_τ for its lowest compact type. Choose a pure unit vector $v_0 = \bigotimes_\tau v_\tau$ in these

types, as in Proposition 106.3. On the bottom isotypic space P_{v_0} projects to that line in every copy.

Lemma 71.5 (A single first test realizing the projected descent)**.** *There is a first real test $\phi^0 = \bigotimes_\tau \phi^0_\tau$ such that, for every terminal coefficient e,*

$$F_e(\phi^0) = P_{v_0} T^\dagger e, \qquad F_e(\omega(a)\phi^0) = R(a) P_{v_0} T^\dagger e \quad (a \in G_\infty).$$

Each ϕ^0_τ is a polynomial Fock vector of energy at most that of ϕ_τ and has norm at most $\|\phi_\tau\|_2$.

Proof. In the local compact group, with Haar probability and inner product linear in the second variable, use the measure

$$dQ_\tau(k) = (\dim \kappa_\tau)\overline{\langle v_\tau, \kappa_\tau(k) v_\tau\rangle}\, dk.$$

On an irreducible compact representation inequivalent to κ_τ, its integrated action is zero. On a copy of κ_τ it is the rank-one projection with image the line of v_τ. Both facts follow by integrating pairs of matrix entries in an orthonormal basis with first vector v_τ. The lowest compact type of a discrete series occurs with multiplicity one. This standard lowest-type theorem applies to $U(p,q)$ by the semisimple theorem on $SU(p,q)$ and its compact central character, or directly by the Blattner formula. In the compact case the whole representation is its unique compact type. Hecht and Schmid, *A proof of Blattner's conjecture*. Thus on every copy of the prescribed discrete series the measure acts by precisely the desired line projection.

Define $\phi^0_\tau = \omega_\tau(Q_\tau)\phi_\tau$. The oscillator restriction to K_τ is unitary, so the same compact orthogonality makes this operator an orthogonal projection there as well. The compact action preserves the number operator in the compatible signed coordinates of Lemma 106.10, giving the asserted energy and norm. Equivariance of the first descending kernel moves the right compact translate to its Schwartz test. The compact integrals may pass through the descent by the absolute convergence in Proposition 71.4. Their product acts on $T^\dagger e$ by P_{v_0} because every real constituent of this descent has the prescribed type. Finally the same equivariance for one group element a gives the second equality. All these operations are at infinity. □

71.6 From a whole discrete summand to the specified vector

At one real place let $H \subset G$ be the stabilizer of the unit coordinate z, in ranks $n-1, n$. Use W for the defining operator norms on both groups. Let $j : \sigma \to \pi|_H$ be an isometric intertwiner and $P = j^*$. For the two discrete series take the parameter bounds of Theorem 106.9 with the same n, B, T_0. For the present local argument $\mathcal{T}$ denotes the quantity in Theorem 71.15 at this single place.

Theorem 71.6 (A quantitative transfer to any lowest-type vector)**.** *Let v_0 be a lowest-type unit vector of π. Suppose $A \geq 1$ and there is a $g_0 \in G$ with*

$$W(g_0) \leq T_0^{An}, \qquad \|P\pi(g_0)v_0\| \geq T_0^{-An^3}.$$

For every unit vector w in the lowest compact type of σ, there is an $h \in H$ such that $v = \pi(hg_0)v_0$ satisfies

$$W(hg_0) \leq T_0^{C_A n^3}, \qquad |\langle v, jw\rangle| \geq T_0^{-C_A n^5},$$
$$d_\sigma \mathcal{T}(v) \leq T_0^{C_A n^5}.$$

Here C_A depends only on A. In the product of the real groups, if these hypotheses hold at every place, any prescribed line of the product lowest type satisfies the initial-period inequality

of Theorem 89.3 on every sufficiently distant original interval, after choosing the test vector there. In this assertion assume multiplicity one, the row hypotheses of that theorem and $\log T_0 = O(\log b)$ *uniformly.*

Proof. Put $a = P\pi(g_0)v_0$, $\eta = \|a\|$, and $R = W(g_0)$. The vector a is smooth under H, since the bounded map P intertwines the group and the original vector is smooth. For a word of length j in the matrix units of H, move its derivatives through $P\pi(g_0)$ by equivariance. One obtains a word on v_0 with each of its matrix arguments conjugated by g_0^{-1}, hence of Hilbert–Schmidt norm at most R^2. The word bound in the proof of Theorem 106.9 gives $J_{n-1}(a) \le (T_0R)^{2n}$. For w it gives $J_{n-1}(w) \le T_0^{2n}$. Proposition 106.11 therefore bounds the second moment of $\langle \sigma(h)a, w\rangle$ by $(T_0R)^{Cn}/d_\sigma$. Its unweighted squared integral is η^2/d_σ.

Take $R_H = \sqrt{2(T_0R)^{Cn}}/\eta$ for the constant in that moment bound. At least $\eta^2/(2d_\sigma)$ of the squared integral lies in $W(h) \le R_H$, since $W(h) \le \|h\|_{\mathrm{HS}}$. The volume estimate in Theorem 106.9 bounds d_σ times the volume of this compact ball by $T_0^{Cn^2}R_H^{4(n-1)^2}$. The continuous coefficient attains its maximum on the ball. At a maximizer its absolute value is at least $\eta/(\sqrt{2}T_0^{Cn^2}R_H^{2(n-1)^2})$. The starting bounds give $R_H \le T_0^{C_An^3}$, so $W(hg_0) \le R_HR$ and the claimed lower bound holds for $\langle \sigma(h)a, w\rangle = \langle v, jw\rangle$.

Write $g_1 = hg_0$. The first inequality of Theorem 106.9 on G yields

$$|F_v(t)| \le T_0^{Cn^3} W(g_1)^{2L} W(t)^{-L}, \qquad L = s + 2D.$$

Use the representative b_1 of tH constructed after Theorem 89.3. For $y \in H$, $W(b_1y) \ge |tz|$ and $W(b_1y) \ge W(y)/(2|tz|)$. Thus multiplication by $|tz|^s$ in the definition of $\mathcal{T}$ leaves the integrable majorant $2^DW(y)^{-D}$ for this last power. Theorem 106.9 on H bounds its integral times d_σ by $2^DT_0^{Cn^2}$. Since $L = O(n^2)$, this proves the asserted bound for $\mathcal{T}(v)$.

In the product take the tensor of the local vectors found for any fixed pure unit vector in the product lowest type S. Its overlap with that vector has absolute value at least $T_0^{-C_Afn^5}$, while $d_\sigma\mathcal{T}(v) \le T_0^{C_Afn^5}$ by product integration. To select any prescribed unit vector $w \in S$, let e be the unit vector in the direction of the orthogonal projection of Pv to S. Its overlap is at least the pure-vector bound. Over the compact group use the matrix unit $d_S\overline{\langle \sigma(k)e, w\rangle}$ in place of the diagonal one in Theorem 71.15. Its action on S is $y \mapsto \langle y, e\rangle w$, with zero action on inequivalent types. Its squared L^1 norm is at most d_S by orthogonality. The same calculation gives the identity action on w with scalar $\|\operatorname{pr}_S Pv\|^2/d_\sigma$, and the same tail cost. Since $\log d_S = O(fn^2 \log T_0)$, the ratio for this real test has logarithm at most $C_Afn^5 \log T_0$. The depth bound $\log N\mathfrak{m} \ge C_{\mathrm{dep}} f^2 n^6 \log b$ and $\log C_n < 6n + 9$ make it smaller than $(n + 2)\log N\mathfrak{m} - f\log C_n - \log 4$ for large x. The proof of Theorem 89.3 now gives the positive lower bound at the original level, with this scalar. For the precise ratio of that theorem one can also use a compact translate of v. Indeed the compact average of $|\langle \pi(k)v, jw\rangle|^2$ is $\|\operatorname{pr}_S Pv\|^2/d_S$, and $\mathcal{T}(\pi(k)v) = \mathcal{T}(v)$. The extra factor d_S obeys the same logarithmic bound. All changes of vectors take place at the real places. □

71.7 The omitted coordinate of a tensor projector

Let $G = U(p, q)$, $n = p + q$, and let H fix a signed coordinate vector z of Euclidean length one. Order the basis with z last. Let π, σ be discrete series of G, H with Harish-Chandra parameters λ, μ, expressed for their respective diagonal Cartans with the usual half-sum shift. Suppose $j : \sigma \to \pi|_H$ is an isometric embedding and put $P = j^*$. Use the regularity and size assumptions of Theorem 106.9. On the smooth tensor of π with the dual defining representation, write P_i for the projector to parameter $\lambda - e_i$, and $I_za = a \otimes e^z$. If y is a tensor, y_z denotes its last component.

Theorem 71.7 (A scalar for the projection to the stabilizer)**.** *For every smooth vector a of G,*

$$P((P_i I_z a)_z) = \gamma_i P a, \qquad \gamma_i = \frac{\prod_{k=1}^{n-1}(\mu_k - \lambda_i + 1/2)}{\prod_{j \neq i}(\lambda_j - \lambda_i)}.$$

In particular $|\gamma_i| \leq (2B+1)^{n-1}$ if every entry of both parameters has absolute value at most B.

Proof. Work first in the enveloping algebra of $\mathfrak{gl}_m$. For its matrix units put $L_{ip}(u) = u\delta_{ip} + E_{ip}$. Antisymmetrize in the rows, taking products in order from left to right, and put

$$C_m(u) = \sum_{\tau \in S_m} \operatorname{sgn}(\tau) L_{\tau(1),1}(u+m-1) \cdots L_{\tau(m),m}(u).$$

The row antisymmetrization with an arbitrary ordered list of columns is also alternating in the column indices, keeping the shifts at their positions. For adjacent columns p, q the two shifts are $t, t-1$. Pairing the two orders of rows i, l, the sum obtained on adding the original and exchanged column expressions consists of the two commutators $[E_{ip}, E_{lq}] + [E_{iq}, E_{lp}]$ and the linear terms $\delta_{ip}E_{lq} + \delta_{iq}E_{lp} - \delta_{lp}E_{iq} - \delta_{lq}E_{ip}$. The quadratic scalar terms cancel. The bracket rule $[E_{ip}, E_{lq}] = \delta_{pl}E_{iq} - \delta_{iq}E_{lp}$ cancels the remaining terms. Factors on either side of the two positions are identical in each paired expression. This proves the assertion and in particular gives zero for any repeated column.

In the commutator with E_{ab}, the Leibniz rule acts on the row indices by replacing row b by row a and on the columns by subtracting the expression with column a replaced by column b. This follows from the same bracket rule, which holds for the shifted entries in this expression because its two additional scalars cancel. For $a \neq b$ both determinants vanish by alternation. For $a = b$ they cancel. Thus $C_m(u)$ is central. For an upper triangular choice of positive roots apply it to a highest vector of weight Λ. In a nonidentity permutation take its last moved position l. Then $\tau(l) < l$, and all factors to its right are diagonal. The raising operator at l kills the highest vector. The identity permutation has eigenvalue $\prod_l (u + m - l + \Lambda_l)$. In terms of the shifted parameter $\nu_l = \Lambda_l + (m + 1 - 2l)/2$ this is $\prod_l (u + \nu_l + (m-1)/2)$. By the Harish-Chandra isomorphism the same symmetric polynomial gives the action at every infinitesimal character. In particular the formula applies to the discrete series here, including those without a highest vector.

In rank n take the first $n-1$ columns with the shifts in $C_n(u)$ and use column k for the last one. Expansion in that last position gives polynomials $c_l(u)$, independent of k, with

$$\sum_l c_l(u) L_{lk}(u) = \delta_{kn} C_n(u), \qquad c_n(u) = C_{n-1}(u+1).$$

The second identity uses the upper-left subalgebra. The first holds for $k < n$ by alternation. Right multiplication of this row identity by the inverse of $L(u) = uI + E$ in formal Laurent series gives $C_n(u)(L(u)^{-1})_{zz} = C_{n-1}(u+1)$. The order of multiplication here follows from the expansion in the last position.

On the smooth tensor in question $L(u) = uI - A$, where the split Casimir has entries $A_{ip} = -d\pi(E_{ip})$. Its resolution in Theorem 106.9 is $(uI - A)^{-1} = \sum_i P_i/(u - \xi_i)$, with $\xi_i = -\lambda_i - (n-1)/2$. Apply the last-coordinate identity to a, then apply P. This map intertwines the H-derivatives on smooth vectors, so the rank $n-1$ central polynomial acts with parameter μ. The result is

$$\sum_i \frac{P((P_i I_z a)_z)}{u - \xi_i} = \frac{\prod_{k=1}^{n-1}(u + \mu_k + n/2)}{\prod_{j=1}^{n}(u + \lambda_j + (n-1)/2)}\, Pa.$$

Multiplication by the denominator gives a polynomial identity. Taking the residue at each distinct pole proves the formula. Each denominator factor in γ_i has absolute value at least one and each numerator factor has absolute value at most $2B+1$. This proves the bound. $\square$

71.8 Cuspidality of the first core kernel

Use the adelic groups $H \subset G$, the finite level $K_f = K(\mathfrak{m})$ and the real parameter of Theorem 71.10. Fix the point g of that theorem and write $c = \iota(g)$. Let D be the unitary discrete series of G_∞ prescribed for the core. Thus D is the conjugate of the representation π used to construct the conjugate coefficient in Theorem 89.3. Write K_∞ for its standard maximal compact subgroup. In the next arguments a compact integral of translates of a vector means $\int D(a)v\, d\nu(a)$, where ν is a complex measure of finite total variation with compact support in G_∞.

Proposition 71.8 (The periodized core coefficient is a smooth cusp form)**.** *Let u, v each be a finite sum of compact integrals of translates of K_∞-finite vectors of D. With inner product linear in the first variable, put*

$$Y_v(u)(s) = \sum_{\gamma \in G(F)} \mathbf{1}_{K_f}(c_f^{-1}\gamma_f s_f)\langle D(c_\infty^{-1}\gamma_\infty s_\infty)u, v\rangle.$$

The series and all its right real derivatives converge absolutely and locally uniformly. The resulting function is bounded and cuspidal in $L^2(G(F)\backslash G(\mathbb{A}_F))$, fixed on the right by K_f.

Proof. Set $w(s) = \prod_\tau \|s_\tau\|_{\mathrm{op}}$, so that w is submultiplicative, takes values at least one and satisfies $w(s^{-1}) = w(s)$. Put $L = 2(4n^2+2)+2n+5$ as in Theorem 106.9. The gap verification of Theorem 71.10 allows Lemma 86.19 with this exponent on every factor. Thus a coefficient between any two compact-finite vectors is bounded by $Cw(s)^{-L}$. Indeed a product compact-finite vector is a finite sum of pure tensors of factor compact-finite vectors, by admissibility of the discrete series. The constant in this proof may depend on the fixed vectors and field. Right differentiation replaces the first vector by an enveloping-algebra derivative, which is still compact-finite. For a translated first vector, commute the derivative across the translate by the adjoint action. On a compact set the resulting derivatives of a fixed order lie in the span of finitely many derivatives with uniformly bounded coefficients. Left and right translation of the group argument over compact sets costs a bounded factor in the weight estimate by submultiplicativity. These observations justify the same exponent for both compact integrals in the statement and for all right derivatives, with constants depending on their order.

Only the finite double coset $G(F)c_fK_f$ contributes. At the representative $s_f = c_f$ the sum is over the arithmetic subgroup $\Gamma = G(F) \cap c_fK_fc_f^{-1}$, embedded in G_∞. It is discrete because rational points are discrete in the adeles and the finite subgroup is compact. More explicitly, a compact set at infinity times this finite subgroup contains only finitely many rational points. Choose a relatively compact identity neighborhood U with the translates γU disjoint. On such a translate $w(t)^{-L} \geq (\sup_U w)^{-L} w(\gamma)^{-L}$. Since $\int_{G_\infty} w(t)^{-L}\, dt < \infty$ by Theorem 106.9 and product integration, $\sum_\Gamma w(\gamma)^{-L} < \infty$. Therefore the absolute sum of every differentiated series is bounded by $Cw(s_\infty)^L$ on this component, uniformly on real compact sets. The other representatives in the double coset are obtained by left rational translation and right translation by K_f. This proves smoothness and uniform moderate growth. The center of the enveloping algebra acts through the infinitesimal character of D, by termwise differentiation.

Consider a proper rational parabolic of G/F with unipotent radical N, in this component. The group $\Gamma \cap N(F)$ is a lattice in N_∞ with compact quotient, by the lattice theorem for unipotent groups. Integrate the series on that quotient after replacing s_∞ by ns_∞. Absolute uniform convergence on a compact fundamental set permits summing over right cosets of $\Gamma \cap N(F)$ in Γ. The contribution of each coset is the integral over N_∞ of a translated matrix coefficient. For compact-finite vectors it vanishes by Harish-Chandra's local cusp-form theorem for the discrete series. Its precise assertion here is that a compact-finite coefficient belongs to

the cuspidal Harish-Chandra Schwartz space, so its integral along the radical of a proper real parabolic is zero after any left and right translations. A proper F-parabolic stays proper under real base change. For a product, the vanishing follows factorwise. The finite central covering by the compact real center and the special unitary group gives the same assertion for our unitary group since the unipotent radicals lift isomorphically. This local argument in a Poincaré series is also given by Khare, Larsen and Savin, Section 4.1, pp. 16–17.

The compact integrals in the statement retain this vanishing. For absolute interchange, the sum of integrals of the absolute translated seeds over the compact fundamental set is bounded by the first part of the proof, uniformly in the two compact translation parameters. Unfolding this positive majorant dominates each integral on N_∞. Finally $N(\mathbb{A}_{F,f}) = N(F)(N(\mathbb{A}_{F,f}) \cap c_f K_f c_f^{-1})$. Indeed the exponential coordinates of the unipotent group in characteristic zero reduce this density assertion to density of F in $\mathbb{A}_{F,f}$. Thus the adelic constant term on the component is a finite Haar factor times the classical constant term just calculated. Conjugating the parabolic gives the same assertion at every representative. On another double coset one uses its own finite representative and the same density in N to express the adelic constant term there by a classical one of the zero function. This proves cuspidality.

The rapid decrease theorem for smooth cuspidal automorphic forms applies to this smooth function of moderate growth with a fixed infinitesimal character. Reduction theory covers an arithmetic component by finitely many Siegel sets up to rational translates and compact factors. The norm-one central torus has compact adelic quotient. The function is therefore bounded, and finite covolume gives square integrability. See Miller and Schmid, Theorem A. □

71.9 Cuspidal lifting on the even core chain

Fix an original E/F and interval as in Theorem 71.10. Let G_i be the unitary group of the i-th core and $K_{i,f}$ its given congruence level. Write $\mathcal{A}_i$ for the $K_{i,f}$-fixed cuspidal forms, finite under a real maximal compact group $K_{i,\infty}$, of the real type P_i at all places. Use the actual lattices of the chain. Fix for the finite test $\phi_{i,f}$ the indicators from Proposition 122.8 and Theorem 102.7. Their tensor is a finite adelic Schwartz vector. A rational polarization may be used for the global theta kernel because the changes from the indicated local polarizations identify the usual unramified vacuum lines at almost every place. Normalize the intertwiners there on these lines. In the complex calculation below any of the finitely many remaining nonzero scalar normalizations gives the same nonvanishing assertion.

Write $[G_i] = G_i(F)\backslash G_i(\mathbb{A}_F)$. For a polynomial Fock test p at the real places, denote the downward theta kernel, written with the small group argument first, by $\theta_{i,p}(g,h)$. Use the linear map on cusp forms

$$T_{i,p}f(h) = \int_{[G_i]} \overline{\theta_{i,p}(g,h)}\, f(g)\, dg.$$

The good Hecke parameters below use the quotient convention of Theorem 71.2. Thus at each step the conjugate of the physical source is the input to the local correspondence.

Theorem 71.9 (A nonzero ascent from every bottom core vector)**.** *For every $0 \neq f \in \mathcal{A}_0$ there are polynomial tests $p_0, \dots, p_{m-1}$ for which $T_{j-1,p_{j-1}} \cdots T_{0,p_0} f$ is nonzero in $\mathcal{A}_j$ for every $1 \le j \le m$. The finite tests are the same for all these input vectors. For an input in one system of good Hecke eigenvalues, every nonzero output at step j has good parameters B_j of Theorem 71.2. One can take a single Fock monomial at each real place of each step.*

Proof. Suppose at step i that an irreducible cuspidal constituent has the specified real type and good parameters B_i arising from a bottom system. Write $u = n_i,\ N = u + d,\ s_0 = d/2$. Every smaller target in the rational Witt tower has zero lift. Indeed it removes at least one

hyperbolic plane at a real place, and a nonzero global lift would give a nonzero local pairing there after the other variables are fixed. This contradicts Theorem 71.16. By the constant-term tower property the lift to this target is cuspidal whenever nonzero. If R is the rational Witt index of the target, its smaller real signature bounds it by $\lfloor (m-i)/2 \rfloor$. Thus $N - R > u$, since $m < d$. This verifies the convergence condition for the doubled Siegel–Weil integral. The center is anisotropic over F, so the Petersson products are taken on the indicated quotients.

Use the mixed Rallis inner product formula for two tests and two cuspidal vectors in the constituent. The Petersson product of the lifts is the value at s_0 of the global doubling integral times a nonzero measure constant. For $N > 2u$ one uses the convergent Siegel–Weil formula. For $u < N \leq 2u$ the constant Laurent coefficient gives the value as soon as the unfolded product is holomorphic. This is the formula used in the proof of Theorem 72.19. Gan, Qiu and Takeda, Propositions 11.1(ii) and 11.2 and Section 11.5.

Unfold first far to the right. Cuspidality removes the non-open orbits, and the open orbit gives the integral of a global matrix coefficient times the restricted section. For factorized data this gives the product of local integrals. At places outside S_0 the value is the partial standard function divided by the doubling denominator, by Proposition 122.8. Theorem 71.2 proves that this quotient is holomorphic and nonzero near its required value (shrinking the neighborhood if necessary). At a marked place the flat section restricts to a compactly supported coefficient multiplied by an Iwasawa height to the power $s - s_0$. On that compact set the height and its inverse are bounded. The local integral is therefore entire and at s_0 is $D_{i,v}$ times the source inner product. At infinity the discrete series is tempered, so the flat section integrals are holomorphic for $\Re s > -1/2$, by absolute convergence locally uniformly there. Theorem 71.16 gives a nonzero pairing on the real polynomial tests. These factors give the desired continuation and its actual value.

We spell out why this argument applies to a prescribed vector. Let D be the tensor of the real Harish-Chandra modules of the source in quotient coordinates. Denote the product of real pairings by $Z_\infty(p, p'; v, w)$, extending it to finite sums of tensors. The subspace of $v \in D$ annihilated for all p, p', w is stable under the real maximal compact group and the complexified Lie algebra. For a compact translate the invariance of Haar measure moves its action on v onto the first oscillator test. Differentiating the same covariance identity moves a Lie algebra action onto that test. The identity is one of smooth invariant pairings, and the derived oscillator action preserves Fock polynomials. The real pairing is nonzero since tests nonzero in each local pairing can be tensored. By irreducibility of the exterior tensor product D the annihilated subspace is zero.

Choose an orthonormal basis e_j of the constituent’s finite fixed space. At the good places the spherical lines are one-dimensional. At the marked places the finite pairing is scalar on the entire fixed space by Theorem 102.7. By multilinearity the mixed global product on $f = \sum_j v_j \otimes e_j$ against the lift of $w \otimes e_j$ is consequently a nonzero fixed scalar times $Z_\infty(p, p'; v_j, w)$, with complex conjugates of the vectors in the quotient convention. If $v_j \neq 0$, some such product is nonzero. This gives a nonzero lift of f. For orthogonal invariant cuspidal copies, even copies of the same representation, the mixed product between their lifts vanishes. Indeed the open-orbit unfolding pairs a translate in one copy with a vector in the other, giving zero identically in the absolute convergence region and hence on continuation. At fixed level and real infinitesimal character the cuspidal multiplicity is finite. Resolving a nonzero vector into the orthogonal constituents and testing against a companion in one nonzero constituent therefore proves the same assertion for their sum.

The output is compact-finite at infinity and fixed by the required finite groups. Local Howe duality and the factorization of a nonzero cuspidal theta lift give its real type by Theorem 71.16.

At the good primes its parameters are those of Theorem 72.16 and Proposition 72.17. Equal splitting characters leave the old multiset in place and add $\kappa_i(w)A_w$. At a split place the larger space was written in dual coordinates. Pulling back by inverse transpose and then conjugating the unitary representation for the next quotient input returns its transferred multiset. At an inert place the multiset is already reciprocal. This proves the required recursion. Each bottom system satisfies the initial spherical bound of Lemma 75.1, so induction applies through all m steps. At any step expanding the successful polynomial into products of monomials leaves at least one product with nonzero output. Subsequent tests can be chosen starting with that output. □

71.10 The first restriction on the original fields and intervals

Take $E = B_J(\sqrt{-q}, \sqrt{-d_0})$ and the intervals of Theorem 20.11, with $F = E^+$, $f = [F : \mathbb{Q}]$ and $b = \exp(x^{\varepsilon})$. Let $r = k + m - 1$, $n = r + 1$, and let $Y \subset X$ be the bottom hyperplane and its first core, of respective dimensions r, n. Write $H = U(Y)$, $G = U(X)$, with inclusion ι. Keep the level $K(\mathfrak{m})$ on X, $J = \iota^{-1}K(\mathfrak{m})$ at the finite places, and the old row level $K_H(m_0)$, where $m_0 = g_1 g_2$. A row here means a nonzero realization T of the specified bottom real series σ in cuspidal L^2 at that old level with trivial level character. Normalize the realization to be isometric. For a unit vector w in its lowest compact type set $u_w = Tw$. Use the smooth representative and the Haar measures of Theorem 89.1.

Theorem 71.10 (A uniform lower bound for the first restricted period)**.** *On every sufficiently distant original interval, at each x, for each such row and each prescribed w, there is a real matrix coefficient for the prescribed core, in the convention of Theorem 89.3, such that at every g satisfying $|u_w(g)| > \|u_w\|_\infty/2$ one has*

$$|\mathcal{P}_g| \;\geq\; \frac{\mathrm{vol}(J)}{d_\sigma}\,\|u_w\|_\infty\,\exp\{-C f n^5 \log b\}.$$

Here $\mathcal{P}_g$ is the restriction integral of the original finite-level Poincaré kernel against u_w, as defined in Theorem 89.1. The constant is uniform in the field, point of the interval, row and vector for the fixed family.

Proof. We identify the two real parameters and check the hypotheses of the local estimates. In the notation of (9.1) the integers $v_i = 4\Delta(k - 1 - i)$ decrease from v_1 to $v_{k-1} = 0$. Put $s_* = r + 6$, $N_* = k + m(t + s_* + 1)$, where $t = t_{\mathrm{r}}$ is the integer of Theorem 6.29. The gap used in the first core of this grid is

$$g_* = \max\{\lceil N_*^3 + n^7 + n^3\sqrt{v_1 - v_{k-1}}\rceil,\; 2N_*^3 + 2s_* + 3,\; \lceil 48n^2\sqrt{m}\rceil\}.$$

The spread in this formula is the maximum over the real places. It has the value displayed since all the v_i are the same at each place. This is the gap with scale constant one in `Development/AbelianVarietyTheory/Definitions/Divisors/LeastSplitAuxiliaryPrimeThetaChainSpec.lean`, line 121–142. It is polynomially bounded in x and at least n^7.

At a real place τ write $b_j = B_\tau + 2(j-1)H_0$ and $a_l = v_1 + 1/2 + (m + 1 - l)g_*$. In decreasing order the core parameter and the dual of the row parameter are respectively

$$\lambda = (a_0, \dots, a_m, v_1 - 1/2, \dots, v_{k-1} - 1/2),$$
$$\mu = (v_1, \dots, v_{k-1}, -b_1, \dots, -b_m).$$

These formulas follow by taking the empty added blocks in the grid parameter lists in `Development/AbelianVarietyTheory/Definitions/Foundations/StrictInterlaceIffDescEnum.lean`,

line 774–814. All cloud entries have sign +. For the a_l the sign is positive at even l, since m is even. On the $-b_j$ it is positive at odd j. Thus the larger group has one more positive coordinate and the same number of negative coordinates. On merging the two lists, the first block consists of $a_0, \dots, a_m$, with alternating signs ending in +. It is followed by $v_1, v_1 - 1/2, \dots, v_{k-1}, v_{k-1} - 1/2$, alternating between lists, all with positive sign. The final block is $-b_1, \dots, -b_m$, beginning with + and alternating in sign. Adjacent entries from one list have opposite signs, while those from different lists have the same sign. The corank-one branching theorem therefore embeds the discrete series with parameter μ into the restriction of the one with parameter λ, with multiplicity one. See H. He, Definition 1 and Theorem 0.1. Taking the complex conjugate of this isometric embedding gives the embedding of the row into the dual of the core for the conjugate coefficient in Theorem 89.3. Simultaneous duality preserves every absolute gap and absolute size. The parameter convention can also be located in `Development/AbelianVarietyTheory/Proofs/TwistedOperators/BranchingHypothesesGridParametersLargeScale.lean`, line 80–98.

Within the core the successive gaps are drawn from $g_*, g_* + 1, 4\Delta$. Within the row they are drawn from $4\Delta, B_\tau, 2H_0$. In the spacing used in Theorem 110.2, $\Delta > 8(f+1)n^3$, $H_0 \ge r^{19} f^5 x^{\varepsilon}$, and the least B_τ exceeds every fixed power of x eventually. Since n tends to infinity and is polynomially bounded, all the gaps exceed $24n^2 + 2$ uniformly. The formulas for the weights and the polynomial bounds on g_*, k, m give a common bound $B \le \exp(Cx^{\varepsilon})$ for their absolute values. Consequently $T_0 = 10^4 n^4 (B+1)$ satisfies $\log T_0 = O(\log b)$.

The two finite invariances in Theorem 89.3 hold for the same row. Its old level is the principal congruence on the lattice of Y modulo m_0, by the definition in `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/OffQOfSizeRows.lean`, line 25–33. Both g_i split completely in E, as in Section 92.6. For each place of F over either marked prime the modulus selects one of the two primes of E to positive depth. Its conjugate selects the other. Thus $\mathfrak{m}\bar{\mathfrak{m}} \subset m_0 \mathcal{O}_E$. The self-dual lattice calculation of Theorem 111.1 identifies the level on X with the congruence subgroup for this product ideal. Intersecting with H, using the orthogonal lattice decomposition $\Lambda_X = \Lambda_Y \oplus \mathcal{O}_E z$ in finite coordinates, gives $J \subset K_H(m_0)$. The fixed-level compact-finite cusp form u_w is bounded by rapid decrease of cusp forms on Siegel sets. There are finitely many such sets at the fixed level, and the norm-one center has compact adelic quotient. See Miller and Schmid, Theorem A. Its supremum is positive by isometry. No uniform constant in this qualitative boundedness assertion is used.

At each real place Theorem 106.2 now gives the translate required by Theorem 71.6 with an absolute A. Take the product of the local witnesses, including the compact translate for the specified vector in the product lowest type. By that theorem the resulting vector v has

$$|\langle v, jw \rangle| \ge e^{-C f n^5 \log b}, \qquad d_\sigma \mathcal{T}(v) \le e^{C f n^5 \log b}.$$

This use of compact averaging also covers a unit vector in the product type that is a sum of pure tensors. The Weyl dimension formula gives $\log d_S = O(f n^2 \log T_0)$. Hence the logarithm of the ratio in Theorem 89.3 is at most $C f n^5 \log b$. At the original auxiliary depth, $\log N\mathfrak{m} > C_{\mathrm{dep}} f^2 n^6 \log b$, as calculated in Theorem 89.2, and $\log C_n < 6n + 9$. It follows uniformly for large x that the ratio is at most $(N\mathfrak{m})^{n+2}/(4C_n^f)$. The compact projection in Theorem 71.15 supplies the real coefficient. The last inequality of Theorem 89.3 gives the claimed lower bound after increasing C to absorb the square and the factor four. □

This period pairs the bottom row with the first core kernel. The ensuing successive lifts, the prescribed integral projection and its normalization, and the unrestricted strict cohomology sum in (20.12) still enter the comparison leading to (W).

71.11 Every smooth bottom vector has a compact-finite core partner

In the notation of Theorem 71.10 use $[H] = H(F)\backslash H(\mathbb{A}_F)$. The first restriction is the bilinear integral

$$\mathcal{B}(F_1, Tw) = \int_{[H]} F_1(\iota h)\,(Tw)(h)\,dh.$$

It converges absolutely when F_1 is bounded on $[G]$ and $Tw \in L^2([H])$, since $[H]$ has finite measure. The symbol T denotes the isometric realization of the nonzero old cuspidal row.

Theorem 71.11 (Nonvanishing of the first period on every smooth row vector)**.** *For each such T and x as in Theorem 71.10 is one cuspidal copy of the prescribed core at $K(\mathfrak{m})$ with the following property. For every nonzero smooth vector w of the bottom row, some K_∞-finite vector F_1 in that same copy satisfies $\mathcal{B}(F_1, Tw) \neq 0$.*

Proof. Choose one unit lowest-type vector w_0, then a point above half the supremum in Theorem 71.10. That theorem gives a nonzero pairing $\mathcal{B}(P_g, Tw_0)$, and Theorem 71.1 places P_g in a copy $Y(D)$ of the required core, fixed for the rest of the proof. By Proposition 71.8 the function P_g is continuous and bounded on $[G]$. Its restriction to $[H]$ belongs to L^2, so $e \mapsto \mathcal{B}(P_g, Te)$ is a continuous linear functional on the row Hilbert space. This functional is nonzero at w_0. For any nonzero smooth w, irreducibility implies that the linear span of its H_∞-orbit is dense. Hence $\mathcal{B}(P_g, T\sigma(a)w) \neq 0$ for some $a \in H_\infty$. Right translation invariance of quotient measure and equivariance of T give

$$\mathcal{B}\big(R(\iota(a^{-1}))P_g, Tw\big) \neq 0.$$

Write $F_2 = R(\iota(a^{-1}))P_g$. This smooth bounded function lies in the same closed core representation and has the same level.

Put $A(k) = \mathcal{B}(R(k)F_2, Tw)$ on K_∞. It is continuous by dominated convergence with majorant $\|F_2\|_\infty |Tw(h)|$, and $A(1) \neq 0$. By Peter–Weyl orthogonality and density, some matrix coefficient p of an irreducible compact-group representation has $\int_{K_\infty} p(k)A(k)\,dk \neq 0$. For example, if every such integral vanished, uniform approximation of $\overline{A}$ by their finite linear span would give $\int |A|^2 = 0$, contradicting continuity at the identity. The compact integral $F_1 = \int p(k)R(k)F_2\,dk$ belongs to the same closed copy. Its compact translates span a finite-dimensional space, since the left translates of p do. Its level is unchanged, and Fubini’s theorem identifies $\mathcal{B}(F_1, Tw)$ with the nonzero integral just obtained. The preimage of F_1 in D is compact-finite. □

The core partner here may depend on the bottom vector. For lowest-type vectors Theorem 71.10 gives a quantitative estimate for its explicitly constructed kernel. The compact integration in the last proof gives qualitative nonvanishing for general smooth vectors. To use this period in the full theta argument one must still place a partner in the span of the columns reached by the subsequent lifts. The corresponding two assertions are distinguished in `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/OffQOfSizeRows.lean`, line 119–128.

71.12 Selecting the unramified Hecke character of the first core

At the level $K_f = K(\mathfrak{m})$ of Theorem 71.10, let S contain the exceptional finite places of F. Away from S use the spherical Hecke algebras of the bottom core, with each hyperspecial group of volume one. Let M_D be the multiplicity space of the prescribed real series D in $L^2_{\mathrm{cusp}}([G])^{K_f}$. Give it the inner product determined by composition of Hilbert intertwiners and the unitary form of D.

Proposition 71.12 (A first core partner with one system of spherical eigenvalues)**.** *Let h_D be the number of distinct simultaneous Hecke characters on M_D. For the lowest-type vector w_0 and the kernel P_g of Theorem 71.10, one of its projections $P_{g,\eta}$ to a single Hecke character satisfies*

$$|\mathcal{B}(P_{g,\eta}, Tw_0)| \geq h_D^{-1} |\mathcal{B}(P_g, Tw_0)| > 0.$$

That character and one of its cuspidal copies of D can be fixed for the entire bottom row. Every nonzero smooth vector of the row has a compact-finite partner with nonzero first period in that copy.

Proof. Harish-Chandra finiteness for cuspidal forms with a fixed infinitesimal character, finite level and compact type gives finite dimension for M_D. Indeed evaluation at any fixed nonzero vector of the lowest compact type is injective on this space of intertwiners by cyclicity. Its image lies in the specified finite-dimensional space of forms. The finiteness theorem on the arithmetic components applies since the adelic center is compact modulo rational points, as in Theorem 88.1. The fixed-level D-isotypic space is therefore $D \otimes M_D$. The good Hecke operators act on its second factor, since they commute with the real group. They preserve the finite level and the cuspidal subspace. Each spherical algebra is commutative by Satake's theorem and closed under Hilbert adjoints, and different primes commute. These operators are simultaneous normal matrices on M_D.

Their joint eigenspaces are mutually orthogonal. The projection to an eigenspace is a polynomial in finitely many of these operators. Indeed for each other eigencharacter choose an operator separating the two, subtract the value at the other character and divide by the difference of the two values. The product of these finitely many factors is the projection. Its action on the original fixed-level function is thus a finite linear combination of right finite-adelic translates because each compact double coset is a finite union of sets aK_f on which the integrand in the convolution is constant. This realizes the projected functions by bounded smooth cusp forms, with the same level. Their sum is P_g pointwise by continuity and equality in L^2.

By Theorem 71.1, $P_g = Y(b)$ for a single intertwiner Y and a smooth $b \in D$. If Y_η is the projection of Y in M_D, the corresponding function is $P_{g,\eta} = Y_\eta(b)$. The positive pairing of Theorem 71.10 in absolute value, linearity and the triangle inequality give the asserted lower bound for at least one η. In particular $h_D \geq 1$ and $Y_\eta \neq 0$. The image of this intertwiner is a closed copy of D, by Schur's lemma. Apply the translation and compact averaging argument of Theorem 71.11 with the bounded continuous starting function $P_{g,\eta}$. Real translations preserve that same image and hence its finite-place Hecke character. This gives all the claimed partners. The numerical loss h_D is retained explicitly. □

71.13 Descent to the period of a prescribed bottom row

Use the forms $\mathcal{A}_i$, tests and kernels of Theorem 71.9. In the opposite direction set

$$S_{i,p} b(g) = \int_{[G_{i+1}]} \theta_{i,p}(g,h) b(h)\, dh.$$

At $i = m$ the group G_m is compact at every real place. Its rational unitary group is anisotropic and its automorphic quotient is compact. Denote by $\mathcal{B}(F, u)$ the first restriction period of Theorem 71.11, with the original old row level and the given level on the core.

Theorem 71.13 (Complex span of the descending core periods)**.** *On each of the same fields the space $\mathcal{A}_0$ is spanned over $\mathbb{C}$ by the forms $S_{0,p_0} \cdots S_{m-1,p_{m-1}} b$, $b \in \mathcal{A}_m$. All finite tests may be fixed as in Theorem 71.9. For each nonzero smooth vector u in a bottom row of Theorem 71.11, there is a single such descent whose first restriction period against u is nonzero. Each real test in it can be a monomial.*

Proof. We first compute the real first occurrence in the reverse direction, since cuspidality of the input alone does not give cuspidality of its theta lift. Work in the quotient coordinates of Theorem 71.16, with $t = m - i$, $A = k + i(d + 1)$, $y = \lfloor t/2 \rfloor$, $z = \lceil t/2 \rceil$. Interchange the factors, moving the scalar $\sqrt{-1}$ from the smaller form to the larger at the real place. This gives the same oscillator with the groups interchanged. Equivalently one may conjugate the physical downward pairing to use exactly the quotient coordinates of the upward pairing. The two characters both have half exponent $-c_t$. In the centered parameter of the larger representation there is now a complete plus string $(d-1)/2, (d-3)/2, \dots, -(d-1)/2$. Every other entry has absolute value greater than $(d+3)/2$. The string is maximal, so $k_{\lambda'} = d$ in the reverse criterion. After removing it, all A positive entries have plus sign, and among the negative entries z have plus sign and y have minus sign. Hence $r_{\lambda'} = A + y$ and $s_{\lambda'} = z$. This is exactly the signature of G_i. In the equal-offset case of the discrete-series criterion the target $(r_{\lambda'} + l, s_{\lambda'} + l)$ gives a nonzero lift exactly for $l \geq 0$. Removing a hyperbolic plane would give $l = -1$, and removing more makes l smaller still. Local Howe reciprocity identifies the lift at $l = 0$ with $\check{P}_i$, since that smaller representation paired with $\check{P}_{i+1}$ in Theorem 71.16. Returning to the physical downward kernel gives P_i. Ichino, Sections 2.2 and 4.3, Theorem 4.3(ii).

It follows that every smaller rational lift in this target tower vanishes by localization at infinity. The tower property gives cuspidality at G_i. The given finite test is invariant under $K_{i,f}$, and the real test is compact-finite, so $S_{i,p}\mathcal{A}_{i+1} \subseteq \mathcal{A}_i$. This holds for arbitrary constituents in the indicated spaces. For cusp forms the two operators are adjoint for the Petersson inner products. With the first variable linear one has

$$\langle S_{i,p} b, f \rangle_{[G_i]} = \langle b, T_{i,p} f \rangle_{[G_{i+1}]}.$$

Indeed both sides integrate $\theta_{i,p}(g,h) b(h) \overline{f(g)}$ on the product of the two quotients. On Siegel sets the theta kernel for a Schwartz test has polynomial growth in the two heights, and each compact-finite cusp form decreases faster than every power in its own height. This gives absolute integrability and permits the change of order, as in Theorem 71.14.

Let $R_0 \subseteq \mathcal{A}_0$ be the algebraic span in the statement. For any nonzero $f \in \mathcal{A}_0$, choose the ascending tests of Theorem 71.9 and take $b = T_{m-1,p_{m-1}} \cdots T_{0,p_0} f$. Repeated adjunction gives a member of R_0 paired with f by the strictly positive number $\|b\|^2$. Hence R_0 has zero orthogonal complement inside $\mathcal{A}_0$. To obtain algebraic equality, fix a $K_{0,\infty}$-type τ. The group $K_{0,\infty}$ preserves R_0 because right translation at the bottom can be absorbed by the first polynomial test. Its type projection also preserves R_0, since for each individual compact-finite form the projection takes place in the finite-dimensional span of its compact translates. The τ-space of $\mathcal{A}_0$ is finite-dimensional by finite cuspidal multiplicity at this level and admissibility at infinity. If its projection of R_0 were proper, a nonzero vector in its orthogonal complement inside that type would be orthogonal to all of R_0, a contradiction. Every form in $\mathcal{A}_0$ is a finite sum of its type components. This proves the asserted span.

Finally Theorem 71.11 provides a compact-finite $F \in \mathcal{A}_0$ with $\mathcal{B}(F, u) \neq 0$. Express F as a finite linear combination of the displayed descents. At least one term has nonzero period. Expanding its polynomial tests in monomials and using the same argument gives the last assertion. The existence statement allows its terminal form and real monomials to depend on the prescribed row vector. □

The positive norm in the proof is for an ascending complex theta lift. The period used in the slope comparison of Section 6.6 includes an integral lattice and a specified Fourier projection. A bound for that projected period still requires quantitative control of the terminal form and of the coefficients of descent in the integral model.

71.14 Fixed forms in the initial seesaw period

Use $X = Y \perp L$ and S from Proposition 95.3 on the given interval. Write $[S] = U(S)(F)\backslash U(S)(\mathbb{A}_F)$, and use the same convention for Y. For $B = X, Y, -L$, write $\Theta_B(g, h; f)$ for the theta kernel for S, B, with g acting on S. At finite places use the depth spaces of that proposition. At infinity use Schwartz functions in the compatible real position polarizations. For automorphic functions A, B_0 on $[S], [Y]$, respectively, set

$$I(f) = \int_{[Y]} \int_{[S]} A(g) B_0(h) \Theta_X(g, h \oplus 1; f)\, dg\, dh.$$

For the bottom of a descending column, A is its first core form above the bottom and B_0 is the initial row form, with conjugation on both if the conjugate lifting convention is used. The character and the field in this definition are fixed.

Theorem 71.14 (Transfer of nonvanishing with both forms fixed)**.** *Suppose A, B_0 are smooth and rapidly decreasing on Siegel sets. Let δ_S, δ_Y be the ratios of the splitting for X to the product splitting for Y and the conjugate of $-L$, restricted to the two groups. For all product tests put*

$$J(p, q) = \int_{[S]} \delta_S(g) A(g)\, \overline{\Theta_{-L}(g, 1; q)} \cdot \left(\int_{[Y]} \delta_Y(h) B_0(h) \Theta_Y(g, h; p)\, dh \right) dg.$$

Then $I(p \otimes \overline{q}) = J(p, q)$. If $I(f) \neq 0$ for any Schwartz test of the given finite depth, some such $J(p, q)$ is nonzero with the finite components of p, q individual integral basis functions of Proposition 95.3. Their real components may be chosen as individual Hermite functions in the two position spaces.

More precisely, let $A(gj) = \chi_S(j) A(g)$ and $B_0(hk) = \chi_Y(k) B_0(h)$ at their original compact open levels. On the large test take the orthogonal projector e_ζ to the character $\zeta(j, k) = (\chi_S(j)\chi_Y(k))^{-1}$, using the actual splitting on X. Then

$$I(e_\zeta f) = I(f), \qquad I(e_\zeta (p \otimes \overline{q})) = J(p, q).$$

In particular an integral of the original forms already known to be nonzero survives that joint projection.

Proof. For a product test the Weil action for the sum is the tensor of the actions on the summands up to the two splitting characters. In Schrödinger coordinates this follows from the tensor product of the two Heisenberg representations and the direct-sum action of the metaplectic group, as in Theorem 70.5. Reversing the alternating form conjugates the additive character, the Weyl phases and the theta sum. The ratio of splittings is a character on a direct product and hence the product of its two restrictions. Both are trivial on rational points. The position lattice of rational points is the product of the two rational position lattices. Factoring its sum therefore gives precisely the integrand defining J, with both displayed characters. In the Fourier–Jacobi period at this step the isotropic flag has length zero. Its unipotent group is trivial and its Haar probability integral is evaluation at the identity.

We give the convergence argument because the groups at the bottom can be noncompact at infinity. Take finite representatives and Siegel sets for the two groups by reduction theory. For a fixed finite Schwartz test the translates by the remaining compact finite groups take finitely many values. For each such value the rational positions in its support lie in a finite

union of translates of a full real lattice, by clearing finite denominators in a rational basis. The absolute theta sum at the identity is consequently bounded by a Schwartz seminorm on the real test. One may use a finite sum of L^2-norms of words in coordinate multiplications and coordinate derivatives. These control weighted supremum norms by Euclidean Sobolev embedding. Conjugation by the real oscillator action carries each such first-order operator to a linear combination of the same operators, with coefficients given by the symplectic matrix or its inverse. Unitarity now bounds the theta sum at $(g, h \oplus 1)$ by a fixed seminorm times a polynomial in the real matrix norms of g, h and their inverses. The entries of the symplectic action are polynomial in these matrices and their inverses. On the finitely many Siegel sets rapid decrease absorbs this polynomial. The quotient measures are finite, since the centers of the unitary groups are anisotropic. Equivalently one can integrate the resulting bound in Siegel coordinates. This proves absolute convergence, justifies interchanging the integrals, and proves continuity on every fixed finite test space.

Expand the finite variables in the integral product basis of Proposition 95.3. The finitely many coefficient functions at infinity are Schwartz, since each is evaluation in a finite coordinate. Expand each in product Hermite functions. The expansion converges in the Schwartz topology: applying powers of the positive harmonic oscillator gives arbitrarily rapid decay of the Hermite coefficients by its eigenvalue equation and Cauchy–Schwarz. The creation and annihilation formulas bound every fixed word in coordinates and derivatives on a normalized Hermite vector by a polynomial in its degree. Multiplicities grow polynomially for the fixed real dimension, so the rapidly decreasing coefficients give absolute convergence in every such seminorm. Completeness of the Hermite functions in L^2 identifies the limit. Real Hermite functions may also be used in the conjugate factor. Applying the continuous functional I proves the nonvanishing assertion.

Finally, right translating both integration variables shows directly that $I(\omega_X(j,k)f) = \zeta(j,k)I(f)$. Integrating with weight $\overline{\zeta(j,k)}$ and Haar probability gives the first projector identity. The product formula gives the second one. The two automorphic forms in the argument have been held fixed throughout. □

71.15 Selecting a prescribed real row vector

Write G_∞, H_∞ for the two real groups in the first restriction, and suppress the embedding of H_∞ in G_∞. Let K be the product of the standard maximal compact subgroups of H_∞. Its elements preserve the Euclidean norms and fix z. Give K probability Haar measure. Use inner products linear in the first argument. For a vector v in a unitary representation π of G_∞, set

$$F_v(t) = \overline{\langle \pi(t)v, v\rangle}, \qquad \mathcal{T}(v) = \sup_{t\in G_\infty} \left(\prod_{\tau} |t_\tau z|_\tau^{2n+5}\right) \int_{H_\infty} |F_v(th)|\, dh.$$

Let σ be an irreducible discrete series of H_∞, with formal degree d_σ in this measure. Assume its compact-finite coefficients are integrable. Lemma 86.19 gives this for the spaced row parameters by taking a power larger than $4(\dim_E Y)^2$ at each place.

Theorem 71.15 (A matrix coefficient acting on the prescribed line)**.** *Suppose σ occurs with multiplicity one in the Hilbert restriction of π, with isometric embedding j. Let w be a unit vector in a single multiplicity-one K-type S of σ, of dimension d_S. If $\mathcal{T}(v) < \infty$ and $\alpha = \langle v, jw\rangle \neq 0$, there is a vector v' in the same representation π such that*

$$\sigma(F_{v'}|_{H_\infty})w = \frac{|\alpha|^2}{d_\sigma} w, \qquad \mathcal{T}(v') \le d_S \mathcal{T}(v).$$

Here $\sigma(f) = \int f(h)\sigma(h)\, dh$ for $f \in L^1$. One can use every line in the lowest K-type, including a prescribed linear combination of weight vectors there.

Proof. Put $P = j^*$. First consider an arbitrary a with $F_a|_{H_\infty} \in L^1$. We claim that on σ

$$\sigma(F_a|_{H_\infty})e = d_\sigma^{-1}\langle e, Pa\rangle Pa.$$

For a unit compact-finite e of σ, put $p_e(h) = d_\sigma\overline{\langle\sigma(h)e, e\rangle}$. It is integrable. Formal degree orthogonality gives $p_e = p_e^* = p_e{*}p_e$ and $p_e{*}\delta_h{*}p_e = \langle\sigma(h)e, e\rangle p_e$, by substituting the two matrix coefficients in each convolution. Consequently $Q_e = \pi|_{H_\infty}(p_e)$ is an orthogonal projection fixing je. If $Q_e x = x \neq 0$, its matrix coefficient on H_∞ is $\|x\|^2\langle\sigma(h)e, e\rangle$ by the second convolution identity. The assignment $\sigma(h)e \mapsto \pi(h)x/\|x\|$ therefore preserves the Gram matrix of every finite set of orbit vectors and extends to an isometric intertwiner. Multiplicity one places x on the line $\mathbb{C}je$. Hence $\langle Q_e a, a\rangle = |\langle a, je\rangle|^2$. The diagonal coefficient of $\sigma(F_a|_{H_\infty})$ at e is the conjugate of $d_\sigma^{-1}\langle Q_e a, a\rangle$. It is real and equals $d_\sigma^{-1}|\langle e, Pa\rangle|^2$. Homogeneity, polarization on the compact-finite vectors and density give the operator identity, since the integral is bounded by the L^1 norm.

On K form the compact matrix unit $p_w^K(k) = d_S\overline{\langle\sigma(k)w, w\rangle}$ and take $v' = \pi(p_w^K)v$. By compact Schur orthogonality and multiplicity one of S, $Pv' = \sigma(p_w^K)Pv = \alpha w$. The double integral for $F_{v'}(th)$ uses the values of $F_v(k_1^{-1}thk_2)$ at $k_1, k_2 \in K$, with absolute weights $|p_w^K(k_1)p_w^K(k_2)|$. Right translation by k_2 preserves Haar measure on H_∞, and $|k_{1,\tau}^{-1}t_\tau z|_\tau = |t_\tau z|_\tau$. The triangle inequality and Tonelli therefore give $\mathcal{T}(v') \leq \|p_w^K\|_1^2\mathcal{T}(v)$. Schur orthogonality on K gives $\|p_w^K\|_2^2 = d_S$, so $\|p_w^K\|_1^2 \leq d_S$. In particular the restriction to H_∞ is integrable. Applying the operator identity to v' and w gives the assertion. The lowest compact type of a discrete series has multiplicity one at each factor and also in the product. □

For the corank-one real unitary pair the Hilbert multiplicity assumption has a direct parameter test. Order the two signed Harish-Chandra lists together, in coordinates where the omitted line is positive. For a pair of adjacent entries from one list the signs must differ, and for adjacent entries from different lists the signs must agree. For regular parameters with no common entry this is exactly the interlacing in He, Definition 1 and Theorem 0.1. His branching theorem gives a single isometric copy in precisely that case. The simultaneous change of sign of the Hermitian forms gives the same groups for an omitted negative line, with the parameters expressed in the new signature. For a product, one takes the tensor product of the embeddings. For uniqueness take a matrix entry of any intertwiner against vectors in all the other Hilbert spaces. The resulting operator is a scalar multiple of the factor embedding. Thus its image lies in that factor's embedded space. Applying this at every place puts the image in the tensor product copy, where Schur's lemma applies.

71.16 First occurrence at the real places of the even core

Fix a field and point of the interval in Theorem 71.10. Use its a_ℓ, v_j, g_* and the notation $d = n + 4$, $n_i = n + id$ from Theorem 71.2. Suppress a selected complex embedding τ. Write $c_\ell = \lfloor(a_{\ell-1} + a_\ell)/2\rfloor$. In the signed parameter P_i of the i-th core, the entry a_ℓ, $0 \leq \ell \leq m$, is in the plus block if $\ell \geq m - i$ or $m - i - \ell$ is even. All $v_j - 1/2$, $1 \leq j \leq k - 1$, are in the plus block. The other plus entries are

$$c_\ell + (d+1)/2 - h, \qquad m - i + 1 \leq \ell \leq m, \quad 1 \leq h \leq d.$$

Write $\check{P}_i$ for the parameter obtained by negating the entries in each block and putting the block back in decreasing order.

Theorem 71.16 (The real theta lift on the even core chain)**.** *At every real place and every $0 \leq i < m$, use the quotient convention for the upward kernel, with both half angular exponents*

equal to $-c_{m-i}$. The discrete series with parameter $\check{P}_i$ has first occurrence at the specified target of rank n_{i+1} in its Witt tower. Its local theta lift has parameter $\check{P}_{i+1}$ in dual target coordinates. The local doubling pairing at $s = d/2$ is nonzero on polynomial Fock data and compact-finite vectors.

Proof. Here is the choice of coordinates. In the downward kernel a map from the larger space to the smaller belongs to the tensor of the smaller space with the dual of the larger. The larger Hermitian form occurs with a minus sign. Complex conjugation of the kernel changes the sign of the symplectic pairing for the fixed additive character and inverts the splitting characters. The two minus signs cancel. At the chosen embedding the purely imaginary scalar $-1/(N\mathfrak{c}\sqrt{-d_0})$ is a positive multiple of i. Rescaling by a positive real square root gives the skew form iH_i on the smaller factor and the Hermitian dual form on the larger factor. The real additive character is $u \mapsto e^{-2\pi i u}$. A form with parameter P_i enters the quotient convention through its conjugate, of parameter $\check{P}_i$. Pullback from the dual target changes the signs of the weights within each compact block because a diagonal matrix acts there by its inverse. These choices explain the two dual parameters in the statement. The kernel and character conventions are specified in `Development/AbelianVarietyTheory/Definitions/Divisors/BasePlaceNotMemChiGridBadPlaces.lean`, line 401–449,535–546.

Put $t = m - i,\ c = c_t,\ A = k + i(d+1),\ y = \lfloor t/2 \rfloor,\ z = \lceil t/2 \rceil$. The centered entries of $\check{P}_i$ are $c - u$ for entries u of P_i. Precisely A are positive. They consist of the $i+1$ top entries from a_t through a_m, the $k-1$ cloud entries and the id entries of the old strings. All have plus sign. The other centered entries, in order outward from zero, correspond to $a_{t-1}, \dots, a_0$, whose signs start with minus and alternate. Thus y of them have plus sign and z have minus sign. The nearest top entries have distance at least $g_*/2 - 1$ from zero. Also $c_\ell = c_t - (\ell - t)g_*$, so every old string stays at distance at least $g_* - (d-1)/2$ from zero. The cloud lies below a_m. The gap in Theorem 71.10 makes every distance greater than $(d+3)/2$. It also separates distinct blocks, and entries within each block are distinct half-integers.

In Atobe's criterion the absence of the two central half-integers and equality of the dimension parities give $k_\lambda = 0$. Counting positive plus and negative minus entries gives $r_\lambda = A + z$, and the other two counts give $s_\lambda = y$. The target signature is $(A + z + d, y)$, since its minus block has $\lceil (t-1)/2 \rceil = y$ entries. In the large-side formula write it as $(r_\lambda + l + 2T, s_\lambda + l)$ with $l = 0,\ T = d/2$. The required inequalities are $l \geq k_\lambda$ and $\#C_\lambda^{\pm}(l + T) \leq l$. A member of either counting set would have absolute value at most $T - 1/2$, even before the deletions in the definition of the sets. Both are empty. Removing $h > 0$ hyperbolic planes gives $l = -h$, which fails the necessary inequality. This proves first occurrence also when $y = 0$, in which case the tower starts with the compact target. Ichino, Theorem 4.3(ii), stating Atobe's discrete-series criterion.

For the parameter of this lift use Ichino's Theorem 4.1(i). The positive centered entries keep their compact signs, and the negative centered entries exchange signs. The middle block has signature $(d, 0)$. Its Levi factor $U(d)$ and the remaining n_i one-dimensional Levi factors are compact. To see the infinitesimal character explicitly, let $N = n_i + d$. Put the middle block between the positive and negative entries. For an old centered entry β in position b, the inducing weight is $\beta - ((N+1)/2 - b) - c$. In each middle position it is $A - n_i/2 - c$. These are the inducing weights calculated in the proof of Theorem 72.6, with the two half exponents both equal to $-c$. Adding the ordered half sum of the roots restores $\beta - c$ at an old position and gives $-c + (d+1)/2 - h$ at middle position $A + h$. The resulting list is strictly decreasing, which verifies the good-range inequalities between the Levi blocks. Induction in stages from the compact Cartan therefore gives a discrete series. Exchanging the signs on indices $\ell < t$ replaces the old threshold t by $t - 1$, and the middle adds exactly the reflected string centered

at $-c_t$. This gives $\check{P}_{i+1}$. Finally the source is discrete series, the target has larger rank, and $d/2 > -1/2$. The real local doubling integral is absolutely convergent there, and its pairing on polynomial Fock tests and compact-finite vectors detects this nonzero lift. Ichino, Sections 4.2 and 7.2, Lemma 7.1 and Proposition 7.2. □

71.17 Cuspidal base change at the bottom of the two towers

The elementary spherical bound in Lemma 75.1 allows the trivial representation at a good place. For the global cuspidal spaces of Theorem 71.10, the widely spaced infinitesimal character imposes a stronger bound at all those places simultaneously.

Proposition 71.17 (The good parameters of the actual bottom cusp forms)**.** *Let π be an irreducible cuspidal constituent in the indicated bottom core or bottom row on any sufficiently distant common interval. Use the prescribed real type in each case, or its dual. At every finite place v where π and the unitary group are unramified, every member α of its standard base-change Satake multiset at $w \mid v$ satisfies*

$$(Nw)^{-1/2} \le |\alpha| \le (Nw)^{1/2}.$$

Proof. Write N for the rank. The field $F = E^+$ has at least two real places, since E contains the prescribed growing real cyclotomic layer. Both groups are unitary groups of Hermitian spaces over E/F. The finite places of E/F are split or unramified, and the given unimodular lattices give hyperspecial groups at the unmarked places. At a real place the unsigned infinitesimal characters in decreasing order are, up to duality,

$$(a_0, \dots, a_m, v_1 - 1/2, \dots, v_{k-1} - 1/2), \qquad (v_1, \dots, v_{k-1}, -b_1, \dots, -b_m).$$

Their entries have parity $(N-1)/2 \bmod \mathbb{Z}$. Every two distinct entries differ by more than one by the gap estimates of Theorem 71.10. Thus these are real integral strongly regular infinitesimal characters. Indeed after subtracting the ordered half sum of roots the successive gaps are still positive. The discrete series with these parameters are cohomological for the algebraic coefficient with the corresponding infinitesimal character, using its dual when pairing in cohomology.

We check the regularity hypothesis for Labesse's cohomological base change. An irreducible unitary representation with this infinitesimal character is cohomologically induced from a unitary character on a theta-stable Levi, by Salamanca-Riba's strongly regular theorem. The Levi of such a parabolic in $U(p,q)$ is a product of unitary blocks. Within a block of size b its character has constant differential, and the half sum from the nilradical is also constant on this block. The half sum of the Levi roots therefore contributes b consecutive entries of step one to the infinitesimal character. This follows directly from the list $((b+1)/2 - j)_{1 \le j \le b}$ on its Cartan. Consequently each block has size one. The Levi is a compact Cartan and the corresponding cohomological modules are discrete series in the packet of the given infinitesimal character. Labesse's condition $(*)$ follows. A discrete-series pseudo-coefficient has trace zero on the other tempered members and the prescribed sign on the chosen member. This is the sufficient condition preceding his Corollary 5.3. S. Salamanca-Riba, the strongly regular unitary dual and J.-P. Labesse, Section 5.1.

Apply that corollary. It supplies an isobaric sum of discrete automorphic representations of general linear groups over E. Its infinitesimal character is the base change of the displayed one. It agrees with the local base change at the unramified places and at split places. In particular this applies at each spherical place under consideration. By the discrete spectrum classification for general linear groups, a noncuspidal discrete summand has cuspidal support a

string of length $b \geq 2$ on one unitary cuspidal representation, with norm twists $(b-1)/2, (b-3)/2, \ldots, -(b-1)/2$. At a complex place the factor $|z|_{\mathbb{C}}^{a} = (z\bar{z})^{a}$ adds a to the holomorphic infinitesimal character as well as to its antiholomorphic counterpart. Even one holomorphic entry of the cuspidal factor then produces two entries differing by one in the holomorphic list of the isobaric sum. Stable base change of the real infinitesimal character has the displayed list on one factor and its dual on the other. Each excludes such a pair. Hence every discrete summand is unitary cuspidal. Labesse, Corollary 5.3 and C. Mœglin and J.-L. Waldspurger, the residual spectrum of GL_N.

At a good place the constituents of the isobaric sum have unramified local parameters. For example, their Weil–Deligne parameters form the direct sum for the isobaric sum, so trivial inertia and zero monodromy for that sum impose the same conditions on each summand. The Satake multiset is their multiset union. For a unitary cuspidal representation of GL_b over a number field, the bound of Luo, Rudnick and Sarnak at unramified places gives exponent $1/2 - 1/(b^2+1)$ for $b \geq 2$. A unitary character in rank one has modulus one. Apply the same upper bound to the dual to obtain the lower bound. This proves the assertion also in the conjugate quotient convention. W. Luo, Z. Rudnick and P. Sarnak, bounds for cuspidal Satake parameters. □

71.18 An everywhere unramified finite parity character

Proposition 71.18 (A single parity character for a whole interval)**.** *On every sufficiently distant interval of Theorem 71.10 is a unitary Hecke character on the fixed biquadratic subfield $M = \mathbb{Q}(\sqrt{-q}, \sqrt{-d_0})$ whose pullback by $N_{E/M}$ has the following properties for every field E on the interval. It is unramified at every finite place, has angular character $z/|z|$ at every embedding in the CM type above $K = \mathbb{Q}(\sqrt{-d_0})$, and restricts to $\omega_{E/F}$ on $\mathbb{A}_F^{\times}$. In particular one can use it as the parity character η in both theta directions. On each of these fields one can simultaneously choose all horizontal characters γ_i with half angular exponents $c_{m-i,\tau}$, trivial restriction to F, conductor exponent two at the four level and type primes, and exponent one at the single marked auxiliary place. They can be taken trivial on residue roots of unity at both type primes and unramified elsewhere.*

Proof. Put $F_0 = M^{+} = \mathbb{Q}(\sqrt{qd_0})$. The two imaginary quadratic fields have coprime fundamental discriminants $-q, -d_0$. Either one generates M over F_0, so M/F_0 is unramified at every finite place by unramified base change. Write ω_f for the finite part of its quadratic idele class character, viewed as a character of fractional ideals. Let Φ be the product of $z/|z|$ over the two complex embeddings of M extending the chosen embedding of K. On $F_{0,\infty}^{\times}$ it is the product of the real signs.

Let $\mathfrak{p}$ be the unique prime of F_0 above d_0. Its residue field is $\mathbb{F}_{d_0}$. The primes in Theorem 5.6 are both 3 mod 4 and $d_0 \equiv -1 \bmod q$. Quadratic reciprocity gives

$$\left(\frac{d_0}{q}\right) = -1, \qquad \left(\frac{q}{d_0}\right) = 1, \qquad \left(\frac{-q}{d_0}\right) = -1.$$

Thus $\mathfrak{p}$ is inert in $F_0(\sqrt{-q}) = M$. For $\delta = \sqrt{-d_0}$ we have $(\delta) = \mathfrak{p}\mathcal{O}_M$, since both ideals have square $d_0\mathcal{O}_M$. Also $\Phi(\delta) = -1 = \omega_f(\mathfrak{p})$.

We first prescribe the finite character on the subgroup of fractional ideals of M generated by principal ideals and ideals extended from F_0. On (α) prescribe $\Phi(\alpha)^{-1}$ and on $\mathfrak{a}\mathcal{O}_M$ prescribe $\omega_f(\mathfrak{a})$. To check all relations, suppose $(\alpha) = \mathfrak{a}\mathcal{O}_M$. Then $\alpha/\bar{\alpha}$ is a unit of absolute value one under every embedding, hence a root of unity by Kronecker's theorem. The only roots of unity in M are ± 1. Indeed its Galois group has exponent two. A primitive root of prime order p would force $p - 1 \mid 2$, and its three quadratic subfields exclude both $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$ since

$q, d_0 > 3$. It follows that $\alpha = t$ or $\alpha = t\delta$ with $t \in F_0^\times$. In the first case the required identity is the principal idele relation for ω_{M/F_0}. In the second it follows by multiplying that relation by the one for δ. Extension of ideals is injective, as is seen from valuations. This verifies well-definedness, including the relation for units.

The unit circle is divisible, so the resulting ideal character extends to the group of all fractional ideals. Multiplying this extension on the finite idele ideal by Φ at infinity gives a continuous unitary character trivial on $M^\times$. It is trivial on all finite local units and has the asserted restriction on $\mathbb{A}_{F_0}^\times$ by construction. Finally $M \cap F = F_0$ and $E = FM$, since M supplies an imaginary generator of the quadratic extension E/F. Thus $N_{E/M}$ on $\mathbb{A}_F^\times$ is N_{F/F_0}. Functoriality of quadratic reciprocity gives $\omega_{M/F_0} \circ N_{F/F_0} = \omega_{E/F}$. Local norms preserve units, and the infinite norm multiplies exactly the coordinates above each chosen embedding. This proves the assertion about η with a single choice on M.

For γ_i work on $A = \ker N_{E/F}$ and then pull back along $z \mapsto z/\bar{z}$. At infinity prescribe powers $c_{m-i,\tau}$ on the unit circles. At every place over the four marked rational primes use the chosen split component to identify $A(\mathcal{O}_v)$ with $\mathbb{Z}_p^\times$. Prescribe a nontrivial character on $(1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p)$, extended trivially across the residue roots of unity. At the single auxiliary place prescribe a nontrivial residue character. The auxiliary rational prime is distinct from the two type primes by its defining congruence, and from g_1, g_2 by its upper bound b and their lower bounds $(kb)^8$. At all remaining places prescribe the trivial character on units. The rational points of A integral everywhere finite are exactly $\mu(E)$, by Kronecker's theorem. At one place over g_1, reduction is injective on $\mu(E)$, since $1 + g_1\mathbb{Z}_{g_1}$ is torsion free. Multiply the unit character there by a residue character canceling the product of the prescriptions on $\mu(E)$. Such a character extends from the subgroup of the cyclic residue group, and leaves the nontrivial conductor-two part intact. The type primes are distinct from g_1. The prescriptions now descend to the image of $A(F_\infty)\prod_{v<\infty} A(\mathcal{O}_v)$ in $A(F)\backslash A(\mathbb{A}_F)$. This image is open. Extend its character by divisibility of the circle as above. Continuity on the open subgroup gives continuity of the extension. Pullback doubles the angular exponents and preserves the local conductors because at a split place it is the ratio of the two coordinates. The construction works independently for every i with the same marked places and levels. □

71.19 The terminal exchange on the prescribed row

Fix the field, word and terminal prime $T = \varpi\bar{\varpi}$ of Theorem 9.13. At $v \mid T$ use the split half where ϖ has valuation one, and put $k = F_{+,v}$, $O = \mathcal{O}_k$, $q = |O/\pi O|$. Use the terminal frame of Theorem 86.1, with $z = \varpi e$. In the simple character of Lemma 86.33 the annihilator of O is O. Indeed T splits completely and the standard finite character of $\mathbb{Q}_T$ has this conductor. In the absolute-trace normalization of Theorem 94.27 the multiplier is a unit at T. Write $m_i = \mu_i(O^a)$ for exchange Haar volumes. Assume $s \geq 1$. For $s = 0$ the period already has its symmetric value.

Theorem 71.19 (The terminal sum at the original lattice level)**.** *Suppose the row F is fixed by the original standard lattice stabilizer at v. Let L' be the symmetric period, possibly with the prescribed exchanges already performed at other places. Put $b(x) = L'(R(X_s(x))F)$. The period after all exchanges at v is*

$$\left(\prod_{i=1}^{s} m_i\right) q^{-a} \sum_{\bar{x} \in (O/\pi O)^a} b(x).$$

If ϕ has the prescribed covector level $P_\ell = \{h \in \mathrm{GL}_a(O) : \ell h \equiv \ell \bmod \pi\}$ for a primitive linear

form ℓ, the residue sum equals

$$b(0) + (q^{a-1} - 1)b(x_*) + q^{a-1} \sum_{t \in (O/\pi O)^\times} b(x_t).$$

Here $\bar{x}_*$ is any nonzero vector in $\ker \bar{\ell}$, and $\bar{\ell}(\bar{x}_t) = t$. The middle term is omitted for $a = 1$.

Proof. The original middle lattice is $O^a \oplus Oe$, which in the coordinates A, z is $O^a \oplus \pi^{-1}Oz$. Thus $Y_i(O^{a*})$ fixes F for every i. For $i < s$ the group $X_i(O^a)$ also fixes it. At $i = s$ the group of integral $X_s(x)$ has $x \in \pi O^a$, since it sends z to $z + x$. For $i < s$, apply Theorem 86.34 to $L_{>i}$ with $U = O^a = V^\perp$. Removing each of these integrations leaves the factor m_i and the original F. The last exchange has $V^\perp = O^a$, $U = \pi O^a$ and $\mu_s(U) = m_s q^{-a}$. This proves the first formula.

For $h_0 \in P_\ell$, conjugation by h_0 sends $X_s(x)$ to $X_s(h_0 x)$, and $R(h_0)F = F$. In L', moving h_0 onto the column amounts to right translation of ϕ by h_0^{-1}, by invariance of the quotient measure on $[H]$. This fixes ϕ. Thus b is constant on the orbits of the stabilizer of $\bar{\ell}$. Every matrix in the residue stabilizer lifts to P_ℓ. In coordinates $\bar{\ell}(w, t) = t$ the action is $(w, t) \mapsto (Bw + tv, t)$ with $B \in \mathrm{GL}_{a-1}(O/\pi O)$ and v arbitrary. This gives precisely the orbits and sizes in the second formula, including the case $a = 1$. □

Applying the first formula at every terminal place contributes at most $T^{a[F_+:\mathbb{Q}]}$ summands on this lattice level, since each residue field has T elements. The other marked places can keep their original exchanges. All formulas are linear in the row, hence apply after any prescribed finite linear combination for which the indicated invariances hold.

Proposition 71.20 (The congruence lattice at the transported level)**.** *At the terminal level K^0 of Theorem 86.1 put $J = \{w \in O^a : \ell(w) \in \pi O\}$. For a K^0-fixed row, one may take $V = O^{a*}$ in every step of Theorem 86.34. On that row one may take $U = J$ for $i < s$ and $U = \pi J$ for $i = s$. The respective numbers of terms in the single-step transforms are q and q^{a+1}. The joint sum on the transported row satisfies Theorem 99.14 with $d = 2,\ c = 0$.*

Proof. In coordinates (w, b) for $w + bz$, the middle lattice is given by $w \in O^a$ and $b - \ell(w)/\varpi \in O$. It meets A in J. Its projections to A and the b-line are O^a and $\pi^{-1}O$, respectively. The last equality uses primitivity of ℓ. Adding $y(w)e_i$ consequently preserves the full lattice for $y \in O^{a*}$. For $i < s$, adding x to $d_i = e_{i+1}$ preserves it exactly when $x \in J$. For the last root the displacement is bx. This belongs to J for every possible b exactly when $x \in \pi J$. All are shears, so checking the negative displacement also verifies preservation by the inverse. As $[O^a : J] = q$, the two indices follow. These single-step indices concern evaluations on F. In a nested sum the later arguments can be translates of F.

Finally the graph lattice is $C\Lambda$ for the coordinate lattice Λ in A, z and the flag, with $C(w, b) = (w, b + \ell(w)/\varpi)$. Both C and C^{-1} have entries in $\pi^{-1}O$. For $g \in K(2)$ the entries of $C^{-1}(g - 1)C$ and the corresponding entries for g^{-1} are integral. Hence $K(2) \subset K_v^0$, which gives the joint bound. For a further specified translate its depth is given by the elementary-divisor calculation following Theorem 99.14. □

The formulas retain the values of the actual automorphic row at the listed translates. Bounding their signed sum after the specified cuspidal projection is still required for the comparison in Section 6.6.

71.20 Transporting the terminal graph to the column

Use the terminal prime and the flag of Section 71.19. Write $H(k) = \mathrm{GL}(A_k)$, with its given embedding fixing the flag and the marked line. In the middle coordinates $w + te$ let Λ_* have standard middle lattice $O^a \oplus Oe$. For a nonzero residue covector $\bar{\ell}$, take an integral lift ℓ. Give Λ_ℓ middle lattice

$$\{w + te : w \in O^a,\ t \in O,\ t \equiv \ell(w) \pmod{\pi}\}.$$

Both lattices have the same integral flags. Put $J_\ell = \{w \in O^a : \ell(w) \in \pi O\}$. Choose $h_\ell \in H(k)$ taking O^a to J_ℓ, and write a_ℓ for a linear transformation taking Λ_* to Λ_ℓ. One such choice on the middle block is $(w, t) \mapsto (w, \ell(w) + \pi t)$.

A unitary character κ of the compact determinant extends to $k^\times$ by assigning the value one to π. Work with $\widetilde{R}(g) = \kappa(\det g)^{-1} R(g)$ on the row and its restriction to $H(k)$ on the column. Thus rows of the given compact type are fixed in this action. Put $\mathcal{J}_\phi(F) = \int L_\phi(R(\prod_i X_i(x_i))F) \prod_i d\mu_i(x_i)$. Exchanges at other marked places may already be included in L_ϕ. Theorem 86.34 gives absolute convergence, and $\mathcal{J}$ is conjugate linear in ϕ.

Theorem 71.21 (Exact transport at the first Bessel position)**.** *Let F_* be fixed under $\widetilde{R}$ by the stabilizer of Λ_*, and let $s \geq 1$. Then*

$$\mathcal{J}_\phi(\widetilde{R}(a_\ell)F_*) = q^{-s}\mathcal{J}_{\widetilde{R}(h_\ell^{-1})\phi}(F_*).$$

Every row of the prescribed type at Λ_ℓ has the displayed form. The right side is unchanged on replacing ϕ by $\widetilde{R}(v)\phi$ for any $v \in H(k)$ preserving J_ℓ. This includes all scalars in $O^\times$.

Proof. If $u \in O^a$, then $X_s(\varpi u)$, with $z = \varpi e$, takes (w, t) to $(w', t) = (w + tu, t)$. Its image of the graph satisfies $(1 + \ell(u))t \equiv \ell(w') \pmod{\pi}$. The converse follows by setting $w = w' - tu$. Primitivity gives u with $1 + \ell(u) \equiv 0$. For that u the image lattice is $J_\ell \oplus Oe$ with the original flags, namely $h_\ell \Lambda_*$. Thus $h_\ell^{-1} X_s(\varpi u) a_\ell$ stabilizes Λ_*. Translation in the final exchange variable leaves the Haar integral unchanged, since the X_i commute. The left side therefore equals $\mathcal{J}_\phi(\widetilde{R}(h_\ell)F_*)$.

For $h \in H(k)$, moving h to the left of the exchange factors changes x_i to $h^{-1}x_i$. This gives $|\det h|^s$. In $L_\phi(R(h)F)$ put $h' = h_0 h$ in the integral over $[H]$. Right invariance of its measure moves $R(h^{-1})$ onto the column. Unitarity of κ and conjugate linearity there give exactly

$$\mathcal{J}_\phi(\widetilde{R}(h)F) = |\det h|^s \mathcal{J}_{\widetilde{R}(h^{-1})\phi}(F).$$

Now $|\det h_\ell| = q^{-1}$. Given a graph row F, take $F_* = \widetilde{R}(a_\ell^{-1})F$. Finally, for $vJ_\ell = J_\ell$, the element $h_\ell^{-1} v h_\ell$ belongs to $\mathrm{GL}_a(O)$, fixes F_* in the twisted action and has absolute determinant one. Applying the last formula with this element proves the column invariance. □

Proposition 71.22 (The integral cost of averaging the column)**.** *Suppose ϕ is fixed in the twisted action by the covector stabilizer P_ℓ of Theorem 71.19. In the transport formula its translated column can be replaced by the mean of $q^a - 1$ translates, fixed under $\mathrm{GL}_a(O)$. For a column fixed by the line stabilizer the corresponding number is $(q^a - 1)/(q - 1)$. At the f terminal places the total transport factor on the evaluated periods is T^{-sf}.*

Proof. In a basis with $\bar{\ell}$ the last coordinate, take $h_\ell = \mathrm{diag}(1, \dots, 1, \pi)$. Conjugating P_ℓ by h_ℓ^{-1} gives the inverse image in $\mathrm{GL}_a(O)$ of the stabilizer of the last basis vector. Indeed the last row of an element of P_ℓ reduces to $(0, \dots, 0, 1)$. Conjugation divides the off-diagonal part of this row by π and multiplies the entries above the last diagonal entry in the last column by π. The inverse calculation gives all elements with the specified last column modulo π. The residue

group is transitive on nonzero vectors. With the line in place of the covector, the last diagonal entry may reduce to any nonzero element and the orbit consists of lines. These two orbit sizes give the asserted finite averages. The spherical row pairs equally with every translate in the mean, by the transformation formula for $h \in \mathrm{GL}_a(O)$. Thus averaging creates no denominator in the evaluated equality. The independent terminal actions commute and $q = T$ at every one of the f places, giving the stated product. □

If one keeps the average as a vector in a lattice of integral class functions stable under the indicated translations and character values, the integer $H_x = T^{sf}(T^a-1)^f$ clears all the rational coefficients of this operation. An n-row minor uses at most H_x^n by multilinearity. Lemma 87.1 bounds $s, a, f, \log T$ by fixed powers of x, so $\log H_x \le f(s+a)\log T = o(\exp(x^{\varepsilon})/x^B)$ for every fixed $B > 0$. This estimate concerns the explicitly specified transport and the averaging of its tests. Comparing the integral values of the resulting global period with a theta column is a further step.

71.21 Following the refined level through the whole terminal flag

We retain the chosen split half and simple character of Section 71.19, and the column operator D of Theorem 86.37. The graph at the symmetric endpoint is the prescribed Λ_{ℓ_0}. We specify one translate at the first position that compares the two endpoint pairings on the separately zero-sum columns. Write $R_G^{\sim}$ for the compatible twisted action on rows. A lattice on the chosen half also specifies its dual on the opposite half.

Use the following coordinates on the string of length $s+1$:

$$b_i = \pi^{-1}e_i \quad (1 \le i \le s), \qquad b_{s+1} = \pi^{-1}z = e.$$

For $1 \le i \le s+1$, give the j-th string line lattice $\pi O b_j$ if $j < i$ and $O b_j$ if $j > i$. On the sum of the column space and $k b_i$ use the graph

$$w + t b_i, \qquad w \in O^a, \quad t \in O, \quad t \equiv \ell_0(w) \pmod{\pi}.$$

All dual flag lines $O f_j$ on the chosen half stay fixed. A common row fixed at the standard lattice in the twisted action defines a row F_i at this lattice by translation. Any two matrices taking the standard lattice to it give the same row. Write F_i^0 for the row with zero in place of ℓ_0. Thus F_{s+1} is an arbitrary row of the given graph type. The row F_1 is $R_G^{\sim}(g)F_{s+1}$, where g first interchanges e and e_1 and then multiplies each e_i by π^{-1}, fixing $A_k, e, f_1, \dots, f_s$ in this second operation.

Denote the Bessel pairing with integration in X_j for $j \ge i$ by $L_{i,\phi}$. In particular $L_{s+1,\phi}$ has no exchange at this place. Other places have the same data and the same exchange variables on the two sides of each identity.

Lemma 71.23 (One step with the original Haar measure)**.** *For each $i \le s$ and every P-fixed column ϕ with zero scalar sum at this place,*

$$L_{i,\phi}(F_i) = \mu_i(\pi O^a)\, q^{s-i} L_{i+1,D\phi}(F_{i+1}).$$

Proof. Choose an integral lift of ℓ_0. Then $F_i = R_G^{\sim}(Y_i(\pi^{-1}\ell_0))F_i^0$. The row F_i^0 is fixed by $Y_i(O^{a*})$, since its i-th line is $\pi O b_i = O e_i$. It is fixed by $X_i(\pi O^a)$, since the next line has lattice $O b_{i+1} = \pi^{-1} O d_i$. These determinant-one shears give the two lattices in Theorem 86.34. That theorem, at $y = \pi^{-1}\ell_0$, gives the finite sum

$$\mu_i(\pi O^a) \sum_{v \in V} \overline{\psi_0(\ell_0(v))} L_{i+1,\phi}(R_G^{\sim}(X_i(\widetilde{v}))F_i^0),$$

where $\widetilde{v} \in O^a$ lifts v. The zero term pairs with a K-fixed row. Moving a scalar unit to the column in the remaining exchanges has Jacobian one, so this term vanishes by the zero scalar sum of ϕ.

For $v \neq 0$, take $h_v = k_v h$ as in Theorem 86.37. We claim

$$R_G^{\sim}(X_i(\widetilde{v}))F_i^0 = R_G^{\sim}(h_v)F_{i+1}.$$

Indeed the left lattice in $A_k \oplus kb_{i+1}$ is generated by O^a and $b_{i+1} + \pi^{-1}\widetilde{v}$. Its projection to A_k is $A' = O^a + O\pi^{-1}\widetilde{v} = h_v O^a$. The b_{i+1}-coordinate modulo π is the unique residue functional on A' vanishing on O^a and taking value one on $\pi^{-1}\widetilde{v}$. This functional is $\ell_0 h_v^{-1} \bmod \pi$, since $k_v v_0 \equiv \widetilde{v}$. The allowed coefficient of b_{i+1} for each given $w \in A'$ is exactly that residue class. This describes the lattice on the right, and its other summands are identical. Twisted compact invariance proves the claim. Moving h_v onto the column as in Theorem 71.21 gives the factor $|\det h_v|^{s-i} = q^{s-i}$.

It remains to identify the resulting column in this pairing. The row F_{i+1} has P-invariance, so $L_{i+1,\xi}(F_{i+1})$ is unchanged on averaging ξ over P. With Haar probability dk on K, the column sum after this averaging is

$$(q^a - 1)\rho(h^{-1}) \int_K \psi_0(\ell_0(kv_0))\rho(k^{-1})\phi \, dk.$$

To check it, partition K into the cosets $k_v P'$. Conjugation by h^{-1} transports the normalized average of P' to that of P. The coefficient here is conjugated in the evaluated pairing, which is conjugate linear on columns. Now replace k by t^{-1}, which preserves Haar probability, and partition by $\eta = \ell_0 t^{-1}$. The integrand is constant on each $t_\eta P$ since ϕ is P-fixed. Each coset has measure $1/(q^a - 1)$. This gives exactly $D\phi$, proving the identity. □

Theorem 71.24 (Comparison at every terminal place on the original refined columns)**.** *Use all f places above the same T of Theorem 9.13. Let A be any row of the prescribed graph type at these places. Write $g_T = \prod_{v|T} g_v$ for the displayed translate, $D_T = \prod_{v|T} D_v$ on the column, and set*

$$C_T = T^{fs(s-1)/2} \prod_{v|T} \prod_{i=1}^{s} \mu_{i,v}(\pi_v O_v^a).$$

Let L^T have no exchange above T and let J^T have all the first-position exchanges there. On the refined columns with zero scalar sum separately at each terminal place one has

$$J_\phi^T(R_G^{\sim}(g_T)A) = C_T L^T_{D_T^s \phi}(A), \qquad \|D_T^s \phi\| = T^{afs/2}\|\phi\|.$$

This holds with common arbitrary data at all other places. On these column spaces the inverse of D_T^s is a finite integral combination of translations divided by T^{afs}, with coefficients extended by the finite character values.

Proof. The lemma can be iterated because D preserves the level and the zero scalar sum. The powers of q multiply to $q^{s(s-1)/2}$. Exchanges and translations at distinct places commute and are absolutely integrable by Theorem 86.34. Applying the one-place identity at each place proves the comparison. The norm identity and the formula for an integral multiple of the inverse follow by multiplying those in Theorem 86.37. □

After removal of the positive Haar factor $C_T T^{afs/2}$, the two terminal pairings therefore have identical row Gram forms on these columns. This conclusion covers a specified translation at T, with the original symmetric endpoint. Theorem 71.25 below compares the remaining exchanges at g_1, g_2 on a common auxiliary translate. The value of a symmetric projected period on the signed theta column also remains to be estimated.

71.22 The two Bessel positions at one pair of finite translates

At a point of an interval of Theorem 22.11 keep the original word, degree, flag, signed auxiliary row levels, terminal graph rows and definite refined columns. At the marked halves over g_1, g_2 take the product t_G of the diagonals in Theorem 99.17. At T take the translate g_T of Theorem 71.24. Write $\mathcal{B}$ for the ordinary symmetric Bessel pairing and $\mathcal{J}$ for the full first-position pairing, which integrates the exchanges at all these halves. At T use the twisted action on rows and columns, and write $R(t_G)$ for ordinary translation at the auxiliary primes.

Theorem 71.25 (Comparison of all the marked exchanges)**.** *On columns having zero scalar sum separately at each place over T, for every row A of the stated types,*

$$\mathcal{J}_\phi(R(t_G)\widetilde{R}(g_T)A) = C_T \prod_{v \mid g_1 g_2} C_v\, \mathcal{B}_{D_T^s \phi}(R(t_G)A).$$

Here v runs over the marked half at each base place. The scalar C_T uses the original Haar measures of Theorem 71.24, and each C_v is the volume in Theorem 99.17. After dividing by the additional factor $T^{afs/2}$, the column operator D_T^s is unitary. Thus for the two specified translates the row Gram forms on these full refined column spaces differ by the square of $C_T T^{afs/2} \prod_v C_v$.

Proof. Translations at the auxiliary places preserve the terminal graph type since the local groups commute. First apply Theorem 71.24 with every auxiliary exchange still present. The column $D_T^s \phi$ has its prescribed levels at the auxiliary places, which commute with all its summands. At each of those places apply Theorem 99.17 to remove all remaining exchanges. The product contains exactly the places in the definition of the first position and no exchange is left on the right. Absolute integrability in Theorem 86.34 justifies these changes of integration order. For the Gram assertion take an orthonormal basis of the separately zero-sum columns. The normalized D_T^s maps it to another such basis, so summing products of the two pairings over this basis gives the claimed scalar square. □

Both translates in this theorem are fixed before varying the row or column. The source comparison in `Formalism/SharpRestriction/Divisibility/ZetaProductPairs.lean`, line 199 uses two translates belonging to further theta and period data. For the pair above the equality of Bessel positions is exact. Its common symmetric Gram form, and the identification and integral size of its value on the prescribed signed theta image, still enter Section 6.6. The compact local witness at an auxiliary place gives no lower bound for that global value.

71.23 Transport of the projected theta chain at the terminal prime

Fix the fields, word, real vectors and measures in Section 6.6. On a sequence of theta edges let $R(u)$ denote right translation of automorphic forms. A Schwartz kernel on an edge is denoted by Φ. At any vertex one may apply a specified linear projector P, including the projector to a compact type with its prescribed character. In the following identities the splitting of the Weil representation on each edge is fixed.

Proposition 71.26 (Translation of the actual projected image)**.** *Choose finite adelic elements g_i at the vertices. Replace each test on an edge with ends i, j by $\omega(g_i, g_j)\Phi$, each projector at i by $R(g_i)P_iR(g_i)^{-1}$ and the input at its vertex by its right translate. Then the output is the right translate at the last vertex of the original projected theta form. This holds in either direction along the chain and preserves all Hilbert inner products between outputs, when the same translates are used for each input.*

Proof. For an edge directed from i to j use the theta series $\Theta(h,k;\Phi)$. Its defining sum gives $\Theta(h,k;\omega(g_i,g_j)\Phi) = \Theta(hg_i,kg_j;\Phi)$. The input factor at h is now the original factor at hg_i, also when the convention conjugates this factor. Change the variable from h to hg_i in the quotient integral defining the lift. Right invariance gives the original output at kg_j. This argument applies to the convergent cuspidal theta integrals used here. It also applies after reversing the two groups, with the same variable change on the integrated group. The relation $(R(g_j)P_jR(g_j)^{-1})R(g_j) = R(g_j)P_j$ permits iteration. Right translation preserves the automorphic inner product by the same change of variable. Compact-level character projectors are transported with the conjugate group and the transported Haar measure. Any scalar in an additional determinant twist of the right action is retained separately on the resulting translated form. □

At a terminal place take $g_0 = g$ of Theorem 71.24 and $g_i = \operatorname{diag}(g, I_{n_i-r})$ in compatible local frames for the increasing ranks $n_i = r + it_{\mathrm{r}}$. Write $\mathcal{M}_i = M_{n_{i+1},n_i}(O)$, and denote the additive lattice $g_{i+1}\mathcal{M}_i g_i^{-1}$ by $\mathcal{M}_i'$. Put $\mathcal{D}_i = \mathcal{M}_i \cap \mathcal{M}_i'$.

Theorem 71.27 (An integral common subdivision on every theta edge)**.** *Take the graph frame at the first vertex, with $h = s+1$, and set*

$$E_i = (s+2)(n_i+n_{i+1})/2 - (h^2+1), \qquad A_i = s(n_{i+1}-n_i)/2.$$

For any integer c, on subdividing by translates of $\pi^c\mathcal{D}_i$, one original matrix cell of lattice $\pi^c\mathcal{M}_i$ gives $q^{E_i+A_i}$ cells, and one transported cell gives $q^{E_i-A_i}$. These counts are exact. For the standard inclusion $Y_i = \binom{I_{n_i}}{0}$, the normalized absolute matrix coefficient of the indicator of $Y_i + \pi^c\mathcal{M}_i$ under $\omega(g_i,g_{i+1})$ is q^{-E_i}.

Use this subdivision at every terminal place in the projected chain of Proposition 71.26. For the fine first test of Theorem 93.5 and the other matrix indicators of Proposition 72.15, the expansion of the transported Schwartz tensor on these cells has coefficient ideal generated, at one terminal place, by

$$q^{s(n_m-r)/2}\prod_{i=0}^{m-1}\eta_i(g_i,g_{i+1}).$$

Here η_i is the splitting character relative to the matrix model. The formula concerns the tensor of tests before the integrations. The logarithm of the number of subdivisions per original tensor cell over all the terminal places is bounded by a fixed power of x on the intervals of Section 6.6, independently of the chosen congruence depths. The same holds per transported tensor cell.

Proof. In Smith frames let t_l and u_j be the exponents of g_i and g_{i+1}. The relative exponents of $\mathcal{M}_i'$ are $u_j - t_l$, one per matrix entry. Changing the two frames by units preserves the reference matrix lattice. By Proposition 83.24 each list contains h copies of -1, one copy of 1, and zeros in the other coordinates. In the square block of size n_i, half the sum of the absolute differences is $(n_i-h-1)(h+1)+2h = n_i(h+1)-(h^2+1)$. Each of the additional $n_{i+1}-n_i$ rows contributes $(h+1)/2$ to that half sum. The result is E_i. The sum of the signed differences is $s(n_{i+1}-n_i) = 2A_i$. The positive differences therefore sum to $E_i + A_i$ and the absolute values of the negative ones to $E_i - A_i$. These sums give the two indices of $\mathcal{D}_i$. Multiplying all lattices by π^c preserves their indices. The intersection is their largest common translation lattice, so each subdivision is the partition by all cosets it contains.

The center stays fixed since $g_{i+1}Y_i = Y_ig_i$. The absolute inner product of the two unit vectors supported on these cells is the measure of their intersection divided by the geometric mean of their measures. The indices just computed give q^{-E_i}. For the fine support at the first edge, choose c at least one, the depth, and the conductor depth of the compact character.

Then the original first test is constant on the ordinary cells because its invertible top block stays in its coset, with the same character, and the lower block stays divisible by π. At each other edge one can take the original indicator depth. The transported support is a union of transported cells with values multiplied by

$$\eta_i(g_i, g_{i+1})|\det g_i|^{n_{i+1}/2}|\det g_{i+1}|^{-n_i/2} = \eta_i(g_i, g_{i+1})q^{A_i}.$$

On the common subdivision evaluation on one cell is one coefficient. Every nonzero original value is a root of unity and each support is nonempty. This proves the ideal formula on multiplying the edge factors, since $\sum A_i = s(n_m - r)/2$. A theta integral or a period with the other inputs fixed can now be evaluated as the finite sum of its values on these cell indicators, with complex conjugation of coefficients on a conjugate-linear input. This is an equality for each actual form and each transported projector.

Finally the ranks n_i, the length m, the number f of terminal places, and $\log T$ are each bounded by a fixed power of x by Theorem 9.13 and the formula for n_i. Hence both $f\sum_i(E_i \pm A_i)\log T$ obey the claimed bound. The fine first edge uses the whole coset matrix of Theorem 83.25 when it is paired back against its original test space. □

These formulas follow a given projected automorphic form under a specified change of tests and projectors. The independent signed column in `Formalism/SharpRestriction/Divisibility/ZetaProductPairs.lean`, line 107 also prescribes the value at the original endpoint. To obtain its comparison in Section 6.6, one must evaluate the resulting signed sum of cell periods with that endpoint projector. The cell count bounds the number of terms. Their phases and values on the automorphic forms determine the cancellation in the sum.

71.24 All the marked tests in the same theta chain

Use the centered diagonal at every auxiliary half, the translate of Theorem 71.24 at the terminal halves, and the same fields and interval as in Section 6.6. Take all the test functions at the marked places of the common theta operator in Section 72.8. Keep its first-edge terminal depth and allow all residue cosets at that depth. Each later terminal edge and each auxiliary edge has its specified single matrix indicator. Write M_v for the normalized matrix of Theorem 83.25 at a terminal place. Every factor uses its original local norm to normalize inner products.

Corollary 71.28 (The coefficient ideal and rank on the marked tensor of tests)**.** *Up to the product of the scalar splitting characters on the singleton edges, the mixed Gram matrix of this tensor and its translate is*

$$\left(\prod_{v\mid g_1g_2} q_v^{-\sum_{i=0}^{m-1} L_i}\right) T^{-f\sum_{i=1}^{m-1} E_i} \bigotimes_{v\mid T} M_v.$$

Here L_i is the auxiliary cost of Theorem 99.19, E_i is the terminal cost of Theorem 71.27, and one marked half is used per base place. In the coefficient field with these scalars adjoined, the ideal of its entries is the product of the displayed scalar ideals, the splitting character ideals of the singleton edges, and the ideals (m_v) in Theorem 83.25. If b terminal places have positive first-edge depth, its rank fraction is at most

$$\left(\frac{\begin{bmatrix} s+1 \\ s \end{bmatrix}_T}{\begin{bmatrix} r \\ s \end{bmatrix}_T}\right)^b.$$

These formulas use the same pair of local translations as the comparison of Bessel positions in Proposition 99.18.

Proof. Choose the extension by identity on the partner coordinates at each successive vertex, independently at every marked place. For a singleton indicator its center stays fixed. Its normalized matrix coefficient is its splitting character times the positive overlap computed in the respective theorem. In the Bessel comparison the symmetric endpoint also carries the auxiliary translate. The matrix here measures the change from the original Schwartz tensor to the translate at the first position. Its auxiliary factors also measure the common auxiliary change at the symmetric endpoint. The fine first-edge computation at T retains all residue cosets in M_v. An inner product of simple tensors is the product of their inner products, which proves the identity entry by entry. Each local matrix has a nonzero entry, and all nonzero entries in each M_v are equal. Taking one such entry in each factor gives a generator of the asserted ideal. Ranks multiply under tensor products of finite-dimensional linear maps, giving the bound at the positive depths. The auxiliary splitting characters are units by Theorem 99.19. □

The rank here refers to Schwartz tests before integrating a theta kernel. Proposition 71.26 transports the full automorphic projector. An assessment of the period on the fixed original projector still needs the signed global sum of the subdivided tests, even when all the local matrix entries above are known. This is the endpoint requirement of Section 6.6.

72 Global theta lifting through the chain

72.1 First occurrence on the real row

Fix a field E/F, $F = E^{+}$, and a translate on the power intervals of Section 6.6. Write $t = t_{\mathrm{r}}$, $m = 2s + 2$, $r = k + 2s + 1$, and $w = r + t$. Write $G = U(Y_0)$ and $H_1 = U(Y_1)$. Their signatures at every real place are $(k+s, s+1)$ and $(w-s, s)$ by Proposition 92.1 and Theorem 6.29. We work with the forward theta lift having the row representation as a quotient of the local oscillator. For global functions this convention pairs the theta kernel with the complex conjugate of the row cusp form.

At a chosen complex embedding above each real place we use Atobe's Hermitian source, skew-Hermitian target and signature convention. The local unitary splitting character attached to the target has angular exponent $-2e_{1,\tau}$, so it sends z to $(z/|z|)^{-2e_{1,\tau}}$. Here an infinity exponent of a denotes $(z/\bar{z})^{a}$, interpreted as angular exponent $2a$. The first target has even dimension. The source character has odd angular exponent, and the choice $-2e_{1,\tau}-1$ gives the first pair of the archive. The half exponents are given at `Formalism/LocalRingCardinality/DataTheta.lean`, line 610 and in the alternating prescription on line 727 there. The scalar multiplying the target Hermitian form to make it skew-Hermitian may be chosen with the indicated imaginary sign at every selected complex embedding. Indeed, multiply a fixed nonzero trace-zero element of E by an element of $F^{\times}$ with prescribed real signs, using weak approximation. Positive real changes of scale give isometric real forms. More explicitly, the real additive character in Ichino's coordinates is $y \mapsto e^{-2\pi i y}$, the infinite component of the usual adelic trace character with this sign. Ichino identifies this normalization with the one in Atobe's criterion in Section 4.3 of his paper.

Theorem 72.1 (The first real lift in the specified tower)**.** *Let σ_τ have the signed row of Volume I, equation (87.4), as specified below. At every real place its theta lift to $U(w-s,s)$ is nonzero. Its lift to $U(w-s-h,s-h)$ is zero for each $1 \le h \le s$. These assertions hold simultaneously for every word.*

Proof. Drop τ from the notation. In decreasing order the row consists of $v_1, \dots, v_{k-1}$, all marked $+$, followed by $-b_1, \dots, -b_m$ marked $+, -, +, -, \dots, +, -$. Here $v_{k-1} = 0$, all $v_i \ge 0$,

$$b_j = B + 2(j-1)H_0, \qquad e_1 = B + H_0, \qquad B > 0, \qquad H_0 > (t+3)/2.$$

The signed entries specify the two compact blocks of the Harish-Chandra parameter λ.

Atobe's criterion first subtracts half the target angular exponent from each entry. Thus the entries here are $v_i + e_1$ and $(3-2j)H_0$. They are integers of absolute value at least H_0. In the plus block there are k positive and s negative entries. In the minus block there are zero positive and $s+1$ negative entries. In particular there is no zero in either block and no centered string of positive length. Definition 1.6 of Atobe therefore gives

$$k_\lambda = -1, \qquad r_\lambda = k + s + 1, \qquad s_\lambda = s.$$

The second count adds the positive plus entries to the negative minus entries. The third adds the other two counts.

For $k_\lambda = -1$ the nonvanishing criterion at signature $(r_\lambda + 2a + l + 1, s_\lambda + l)$, $a \geq 0$, requires $l \geq 0$ and at most l surviving entries of each sign ϵ with $0 \leq \epsilon\alpha - 1 < a + l$, where α is the shifted entry. The deletion procedure in the criterion only removes entries. Choose $a = (t-1)/2$, an integer because t is odd. For $l = 0$ even the original list contains no entry in the indicated range. Indeed $\epsilon\alpha \leq -H_0$ or $\epsilon\alpha \geq H_0 > a + 1$. The resulting signature is exactly $(w - s, s)$. Removing h hyperbolic planes gives the same a and $l = -h$, which violates the necessary lower bound on l. The parameters used here are common to the words at this place. We have used the discrete-series case of Atobe, Definition 1.6 and Theorem 1.7, which is unconditional. □

At a place v let ω_v be the oscillator restricted to G_v, with invariant inner product linear in the first variable. For a unitary representation π_v of G_v and smooth vectors u, u', write

$$Z_v(\phi, \phi'; u, u') = \int_{G_v} \langle \omega_v(g)\phi, \phi' \rangle \, \overline{\langle \pi_v(g)u, u' \rangle} \, dg.$$

Write $Z_v(\phi; u)$ for the diagonal value. We specify convergence for the data at which we use this integral.

Proposition 72.2 (Positive integrals on the chosen row vectors)**.** *At each real place, for every unit vector u in the lowest compact type of σ_τ, there exists a polynomial Fock test ϕ for which $Z_\tau(\phi; u) > 0$. At a terminal split place take the test $\phi_{1,j}$ of Theorem 93.5, writing $\phi_{1,0} = \phi_*$, $L_0 = K$, and $I_0 = 1$. For every local unitary row one has*

$$Z_v(\phi_{1,j}; u) = \|\phi_{1,j}\|^2 \operatorname{vol}(L_j) \|e_{L_j,\kappa} u\|^2.$$

In particular this is positive on every nonzero (L_j, κ)-vector. On an original (K, κ)-vector it is exactly $I_j^{-2} Z_v(\phi_*; u)$.

Proof. At infinity the rank inequality $w \geq r$ and Theorem 72.1 meet the hypotheses of the local inner product theorem for discrete series. Ichino, Proposition 7.2 gives a nonzero integral between two tests and two compact-finite vectors of σ_τ. All the row coefficients here belong to L^1 by Lemma 86.19. Multiplying the conjugate coefficients by the formal degree d_σ gives integrable matrix units. Polarization expresses a mixed coefficient as a linear combination of diagonal ones. Consequently $\omega_\tau(p_z) \neq 0$ for some unit compact-finite z, where $p_z(g) = d_\sigma \overline{\langle \sigma_\tau(g)z, z \rangle}$. The projection and orbit-vector argument in Theorem 86.22 embeds a Hilbert copy of σ_τ into the oscillator. On that copy $\omega_\tau(p_u)$ acts as the projection onto the line of u. This bounded orthogonal projection on the full oscillator is therefore nonzero. Fock polynomials are dense in its Hilbert space. In fact some coordinate monomial ϕ has $\omega_\tau(p_u)\phi \neq 0$, and

$$Z_\tau(\phi; u) = d_\sigma^{-1} \|\omega_\tau(p_u)\phi\|^2 > 0.$$

At $v \mid T$ the oscillator coefficient equals $\|\phi_{1,j}\|^2 \kappa(g) \mathbf{1}_{L_j}(g)$ by Theorem 93.5, including the same calculation with the entire support at $j = 0$. Integration gives the orthogonal character projector. The norm of the test squared and the volume of the subgroup each decrease by I_j from their values at $j = 0$. □

72.2 A nonzero global period on the first theta pair

Keep the splitting at infinity from Section 72.1. Fix compatible global unitary splitting characters. Their restrictions to the ideles of F are the required powers of the quadratic character of E/F. For completeness the real specifications above admit such characters. In the norm-one idele class group of E, take the images of the norm-one class group of F and of $\prod_{\tau\mid\infty} S^1$. Both images are compact. Their intersection consists of real sign tuples. Indeed, if their representatives differ by a principal idele $a \in E^\times$, its finite components lie in the local copies of F. This forces $a = \bar{a}$. The specified characters agree on the intersection by parity. They thus give a continuous character of the product of the two compact subgroups. Character extension for closed subgroups of locally compact abelian groups extends it to the norm-one class group. Extend trivially along the positive idele norm direction chosen in F. This argument establishes existence. The statements below apply to any choice with these real specifications, including preassigned compatible finite characters.

Write $[G] = G(F)\backslash G(\mathbb{A}_F)$ and use analogous notation for other groups over F. For a Schwartz test ϕ and a cuspidal row f_0, set

$$\Theta_\phi f_0(h) = \int_{[G]} \theta_\phi(g,h)\overline{f_0(g)}\, dg.$$

At every terminal place $v \mid T$ prescribe the test ϕ_{1,j_v} of Proposition 72.2, where $j_v = 0$ is allowed. At other places we shall choose the tests. We call a vector *of the prescribed terminal type* if L_{j_v} acts on it through κ_v for every $v \mid T$.

Lemma 72.3 (Finite places away from the terminal prime)**.** *For the ranks of Theorem 6.29, and all sufficiently large x, the following hold. At each nonterminal finite place and any sufficiently small principal subgroup J of the row group there is a Schwartz test whose row coefficient is a positive constant times $\mathbf{1}_J$. At almost every finite place let Q_v be the residue cardinality of F_v and give the hyperspecial row subgroup volume one. For the unit lattice test ϕ_v^0 and any unit spherical row vector u,*

$$|Z_v(\phi_v^0; u) - 1| \le \sum_{h\ge 1} Q_v^{-hD}, \qquad D = \lfloor (w-2)/2 \rfloor - 4r^2 \ge 3.$$

These unramified integrals are real and their product outside any fixed sufficiently large finite set is positive.

Proof. The ranks satisfy $w = 8[F:\mathbb{Q}]r + 4 + \lceil x^{377} \rceil n_W$ and $r = k+m-1$ with $k = O(x^{9/2+10\varepsilon})$, $m = O(\sqrt{x})$ in the range of ε of Volume I. In particular $w > 8r^2 + 10$ for large x. At a nonsplit finite place a Hermitian space has anisotropic dimension at most two. One can also see the bound by taking the underlying quadratic form over F_v. A Hermitian anisotropic subspace of dimension d gives an anisotropic quadratic form of dimension $2d$, and a nonarchimedean local field of characteristic zero has quadratic u-invariant four. This is also the dimension bound in the local classification of Hermitian forms by Jacobowitz. Witt decomposition gives at least r hyperbolic pairs in the target, including at dyadic places. In a mixed Schrödinger model each such pair supplies one row-space variable. In coordinates for r pairs they form a square matrix X, acted on by change of row-space frame. This follows by polarizing the hyperbolic pairs into their two isotropic halves. The isometry g acts on the position half by its linear action, with the square root of the Jacobian and a smooth splitting character. The orthogonal remainder supplies a separate oscillator factor.

Use $\mathbf{1}_{1+\mathfrak{p}_E^a M_r(\mathcal{O}_E)}(X)$ and a fixed nonzero smooth test on the remainder. Over a split place use the square top block of the matrix model (93.1) over F_v instead, and a smooth test on

its other rows. At both types of places the overlap of the two square-block supports implies $g \in J$, the subgroup congruent to the identity at depth a in the chosen frame. Indeed both X and its transform must lie in the same matrix subgroup. In dual coordinates this gives the same assertion after taking the dual frame. For large a, J fixes the remainder test and the splitting character is trivial on J. The Jacobian there is one. Hence the coefficient is precisely the squared norm of the test on J, and vanishes outside it.

Outside finitely many places the extensions and the data are unramified, the additive character has integral conductor, the lattices are self-dual, and all the hyperspecial groups fix the unit lattice test. At an unramified nonsplit place of odd residue characteristic one can split off at least $\lfloor (w-2)/2 \rfloor$ integral hyperbolic pairs from the unimodular lattice. To see this, an isotropic vector of the reduced nondegenerate Hermitian space lifts to a primitive isotropic vector by Hensel's lemma, since a nonzero residual vector has nonzero linear derivative in some direction. Such residual vectors exist in dimension at least two by surjectivity of the finite-field norm. Pair the lift with a vector of pairing one and subtract half its norm times the isotropic vector to give a hyperbolic pair. Its unimodular complement permits iteration. The lattice test in this polarization is a product of normalized lattice indicators and a unit vector for the remainder.

For a right coset gK_v put $\Lambda' = g\Lambda$ in the row space and let h be least with $\varpi^h \Lambda \subseteq \Lambda' \subseteq \varpi^{-h}\Lambda$. Take one component at a split place. At these good places K_v is the full stabilizer, so the lattice determines the coset. The quotient modulo $\varpi^h\Lambda$ is generated by the images of r basis elements of Λ'. There are at most $Q_v^{4hr^2}$ ordered choices since the coefficient residue degree is at most two. In the nonsplit case the two lattice covolumes are equal. The coefficient of one row variable then has absolute value $[\Lambda : \Lambda \cap \Lambda']^{-1} \le Q_v^{-h}$. Indeed the elementary-divisor exponents sum to zero and the largest absolute exponent is h. At a split place the coefficient in absolute value for each of the w row variables is

$$\big([\Lambda : \Lambda \cap \Lambda'][\Lambda' : \Lambda \cap \Lambda']\big)^{-1/2} \le Q_v^{-h/2}.$$

This follows by dividing the intersection volume by the geometric mean of the two volumes. The residual oscillator coefficient in the nonsplit case and the unit row coefficient in both cases have absolute value at most one. The integral over K_v is one. Summing over the shells $h \ge 1$ gives the bound, also for the integral of the absolute value off K_v. Inversion makes each diagonal integral real, and the displayed error is at most $1/7$. Finally $\sum_v Q_v^{-3} < \infty$ for a fixed number field, by comparison with $[F : \mathbb{Q}] \sum_{n \ge 2} n^{-3}$. The product of the positive factors therefore converges to a positive number. □

Theorem 72.4 (Detection of a cuspidal row on the first pair)**.** *Fix a unit pure tensor $u_\infty = \bigotimes_\tau u_\tau$ of lowest compact types of the original signed row. Let $A = S_{\boldsymbol{j}}(u_\infty \otimes \ell)$, with an arbitrary finite class vector ℓ at the chosen levels. Whenever $A \neq 0$, there is a factorizable complex Schwartz test ϕ, with the prescribed terminal factors, such that $\Theta_\phi A$ is a nonzero cusp form on $[H_1]$. Equivalently for this existence assertion, there is a cusp form D_1 on $[H_1]$ with*

$$\langle \Theta_\phi A, D_1 \rangle_{[H_1]} \neq 0.$$

The assertion includes $\boldsymbol{j} = 0$ whenever the original sum A is nonzero.

Proof. By Theorem 86.22 and its finite-level restriction in Section 86.8, the square-integrable vector A is fixed by the real matrix unit p_{u_∞}. It generates under the real group a copy of the discrete series $\sigma = \bigotimes_\tau \sigma_\tau$. At a fixed finite level the adelic quotient is a finite union of arithmetic real quotients. Restrict the real representation to each such component by the corresponding projection. Its nonzero image is again a Hilbert copy of σ, as one sees from the matrix-unit

argument. Its maximal compact-finite vectors are smooth automorphic eigenfunctions (the Casimir together with the compact Laplacian gives elliptic regularity). The discrete series is tempered. Wallach's constant-term theorem places its full square-integrable multiplicity on each arithmetic quotient in the cuspidal spectrum. This proves adelic cuspidality. Indeed the finite adelic unipotent group is the product of its rational points and any given compact open subgroup by strong approximation for unipotent groups. Thus the adelic constant term at a finite class representative is a positive multiple of the arithmetic real constant term for that representative. Wallach, Theorem 4.3.

Theorem 72.1 forces all theta lifts of these cuspidal vectors to smaller spaces in the same rational Witt tower to vanish. A smaller space is obtained by removing hyperbolic planes at every place. A nonzero global lift would give a nonzero equivariant pairing at a real place by fixing the other tensors and evaluating the lift. In particular the local oscillator there would have the row as a quotient. The tower property of the global theta correspondence expresses the constant term of the lift of a cusp form along a target isotropic parabolic in terms of the lifts to its smaller Witt space. Consequently every lift under consideration to H_1 is cuspidal. Gan, Qiu and Takeda, introduction and Section 11.

We give the norm argument since the vectors at T are fixed. The global Witt index d_1 of Y_1 is at most s, as seen at a real place. We have $w - d_1 > 2r$ and $w > 2r + 2$. Hence the ordinary doubled Siegel–Weil theta integral is in Weil's convergent range, and its Siegel Eisenstein series at $(w - r)/2$ converges absolutely. For cusp forms f_1, f_2 the convergent inner product formula at a common test gives, with a fixed $c > 0$,

$$\langle \Theta_\phi f_1, \Theta_\phi f_2 \rangle_{[H_1]} = c \int_{G(\mathbb{A}_F)} \langle \omega(g)\phi, \phi \rangle \, \overline{\langle R(g) f_1, f_2 \rangle_{[G]}} \, dg. \tag{72.1}$$

Here R denotes right translation. To recall the unfolding, the product of a theta kernel with the conjugate of a second one is the restriction of the doubled kernel. Under the polarization by the diagonal and antidiagonal, evaluation of the transformed tensor at zero gives the oscillator inner product. The determinant twist on the second factor cancels the corresponding twist in the doubling embedding. The convergent Siegel–Weil formula identifies the target integral with the Siegel Eisenstein series, up to the Haar normalization. Its rational terms are indexed by maximal isotropic planes in $Y_0 \oplus (-Y_0)$. An open-orbit plane is a graph of an isometry, with diagonal stabilizer. If a plane meets the first summand in $D \neq 0$, its projection to that summand lies in $D^\perp$. The unipotent radical stabilizing D acts trivially on $D^\perp/D$, preserves the plane and acts there with determinant one. Its integral on the first cusp form is zero. The same argument applies to the second summand, and all remaining planes are graphs. Unfolding the graph orbit gives exactly the integral in (72.1). Weil convergence and rapid decrease of cusp forms justify the kernel integrals. Absolute convergence of the open-orbit integral for the tests below also follows directly from Lemma 72.3, the compact support of the bad finite coefficients and Lemma 86.19 at infinity. These are the classical convergent doubling formula and its local evaluation at zero. Gan, Qiu and Takeda, Sections 1 and 3 and Ichino, Sections 6.3 and 7.2.

Project A onto an irreducible Hilbert summand Π of the cuspidal spectrum on which it has nonzero projection A'. Orthogonal projection commutes with R. The projected vector has the same terminal type and is still fixed by p_{u_∞}. At infinity it therefore factors as u_∞ times its finite part in the restricted tensor product realization of Π. At almost all finite places the finite part has the spherical line. At the remaining finite places it lies in a tensor product of finite-dimensional spaces of vectors fixed by compact open subgroups, by smoothness and admissibility. These facts follow also by applying the finite-level averaging projectors to the restricted tensor product. They allow the finite part to be a sum of pure tensors.

Choose the real tests by Proposition 72.2 at u_τ, and the good finite lattice tests by Lemma 72.3. At every remaining nonterminal finite place choose the small subgroup in that lemma to fix the entire finite-dimensional space of components of A'. The integrated local form is then a positive scalar times its Hilbert inner product on those components. The same holds on the terminal type by Proposition 72.2 and polarization. At the good places it holds on the one-dimensional spherical line, with positive scalar whose product converges. At the real places only the line u_τ occurs in A', and its scalar is positive. Expanding the finite part into pure tensors and using the absolutely convergent product of the local integrals, we obtain

$$\|\Theta_\phi A'\|^2 = c\|A'\|^2 \prod_v c_v > 0,$$

where c_v is the value of the local form on a unit vector in the indicated subspace. This argument permits arbitrary unitary finite components of Π. Finally $\langle R(g)(A - A'), A'\rangle = 0$ for every g. The mixed identity (72.1) yields $\langle \Theta_\phi A, \Theta_\phi A'\rangle = \|\Theta_\phi A'\|^2 > 0$. Taking $D_1 = \Theta_\phi A'$ gives the asserted period. □

Corollary 72.5 (One test away from the terminal prime)**.** *Fix the field, translate, splitting, pure real row u_∞ and levels away from T. There is a single choice of all theta factors away from T and a number $C_x > 0$ such that at every terminal depth $\boldsymbol{j}$ and for every cuspidal row A at these levels fixed by p_{u_∞} and of the prescribed terminal type,*

$$e^{-\delta_x} C_x I^{-2} \|A\|^2 \;\le\; \|\Theta_{\phi_{\boldsymbol{j}}} A\|^2 \;\le\; e^{\delta_x} C_x I^{-2} \|A\|^2, \qquad \delta_x = 4[F:\mathbb{Q}]\, 2^{-D}, \quad I = \prod_{v\mid T} I_{j_v}. \tag{72.2}$$

The estimate holds simultaneously for the Bessel columns of all words with these data. The chosen real factors can be Fock monomials. At finitely many nonterminal finite places the factors can be characteristic functions in the mixed coordinates of Lemma 72.3.

Proof. Choose a finite set of finite places containing the terminal places, all ramification and all deviations from the hyperspecial levels and lattice tests. At each bad nonterminal place the kernel of the specified level character contains an open neighborhood of the identity. Take J of Lemma 72.3 in this kernel, using any fixed lattice indicator for its residual test and shrinking J to fix that test. The two local factors in the coefficient calculation are then independent of the cuspidal representation, the column, and the terminal depth. For a finite list of level characters one can take the intersection of their kernels. Proposition 72.2 supplies the same monomial at each real place for the whole fixed real row.

In the product formula for an irreducible projection, let C_x be c times the product of the real scalars and the nonterminal bad finite scalars, times $\prod_{v\mid T} \|\phi_{*,v}\|^2 \operatorname{vol}(K_v)$. Each scalar is positive. The terminal contribution for a unit vector of the fine type differs from this contribution at depth zero by precisely I^{-2}, by the norm and volume calculation in Proposition 72.2. Each good factor z_v satisfies

$$|\log z_v| \le 2Q_v^{-D}, \qquad \sum_v Q_v^{-D} \le [F:\mathbb{Q}] \left(2^{-D} + \int_2^\infty y^{-D}\, dy\right) \le 2[F:\mathbb{Q}]\, 2^{-D}.$$

The first bound follows from the error at most $1/7$ and the geometric series in Lemma 72.3. Each rational prime has at most $[F:\mathbb{Q}]$ places above it. These estimates place the good product in $[e^{-\delta_x}, e^{\delta_x}]$ independently of the irreducible summand. Distinct orthogonal summands have zero mixed integral in (72.1). At any fixed depth the space of cusp forms of the given levels, compact type and infinitesimal character is finite-dimensional by the finiteness theorem for automorphic forms. Adding the squared norms in these summands proves (72.2) for A. □

For a nonzero Bessel column at depth $\boldsymbol{j}$, Corollary 72.5 now gives a nonzero global first lift with common tests away from T. At the original level it detects precisely the columns surviving periodization. The factor C_x contains the archimedean matrix coefficients of the chosen monomials and the volumes of the small subgroups. The estimate gives no arithmetic height or valuation for these factors. The period in Section 6.6, also uses the remaining steps of the specified theta chain.

72.3 The signed real theta chain to the compact group

Fix a real place and suppress its subscript on the weights. Write $t = t_{\mathrm{r}}$, $d = (t-1)/2$, $n_j = r+jt$, $s_j = \lfloor (m-j)/2 \rfloor$, and

$$h_j = -\tfrac{1}{2}\mathbf{1}_{j \text{ odd}}, \qquad G_j = U(n_j - s_j, s_j) \quad (0 \le j \le m).$$

The target G_m is compact. On the ordered pair G_j, G_{j+1} use the real additive character and the signature convention of Section 72.1. In the normalization of Ichino, the half angular exponents of the characters attached to target and source are, respectively,

$$a_T = -e_{j+1} + h_j, \qquad a_S = -e_{j+1} + h_{j+1}.$$

The parity conditions hold since the ranks differ by the odd integer t. These exponents occur in `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/IsStepWeil.lean`, line 25.

Define a list P_j of signed numbers as follows, sorting within each compact block after forming the list.

1. Put $v_1, \dots, v_{k-1}$ in the plus block.
2. Include $-b_i$ for $1 \le i \le m$. The entries with $i \le j+1$ have sign plus. If $i > j+1$, give the entry sign plus when $i-j-1$ is even and sign minus otherwise.
3. For each $1 \le i \le j$ put the entire string $-e_i + d, -e_i + d - 1, \dots, -e_i - d$ in the plus block.
4. Add h_j to every entry.

This is the signed parameter at `Formalism/LocalRingCardinality/DataTheta.lean`, line 538. It has n_j entries. All entries before the final shift are integers. Each string centered at $-e_i$ is strictly below $-b_i$ and, for $i < m$, strictly above $-b_{i+1}$, since $e_i - b_i = b_{i+1} - e_i = H_0 > d$ whenever the indicated indices exist. The strings and the other entries are therefore pairwise disjoint, and the final shift gives the parity $(n_j - 1)/2 \bmod \mathbb{Z}$. Exactly s_j entries have sign minus. The Harish-Chandra classification gives a discrete series σ_j with parameter P_j on G_j, including the usual irreducible finite-dimensional representation when $s_j = 0$. The parameter at $j = 0$ is the signed row in Theorem 72.1.

Theorem 72.6 (Every real lift and its first occurrence)**.** *For $0 \le j < m$ the real theta lift of σ_j to G_{j+1} is σ_{j+1}. Removing any positive number of hyperbolic planes from this target gives zero lift in the same Witt tower. Thus the forward real lifts of the original row give P_m at the compact endpoint.*

Proof. Subtract a_T from P_j. The entries arising from v_i become $v_i + e_{j+1} > 0$. Those from $-b_i$ become

$$e_{j+1} - b_i = (2j + 3 - 2i)H_0.$$

This is positive exactly for $i \le j+1$. Each old string becomes $2(j+1-i)H_0+d, \dots, 2(j+1-i)H_0-d$, hence is positive in its entirety. All the positive entries belong to the plus block, and their number is $A = k + j(t+1)$. Among the negative entries the initial sign is minus and the signs alternate. There are $y = s_{j+1}$ negative plus entries and $z = s_j$ negative minus entries. In particular $n_j = A + y + z$. All centered entries are integers of absolute value at least H_0.

In Atobe's notation this gives $k_\lambda = -1$, $r_\lambda = A + z$ and $s_\lambda = y$. As in the proof of Theorem 72.1, the count for each sign at d is zero because $H_0 > d+1$. With $l = 0$ the target is $(A + z + 2d + 1, y) = (n_{j+1} - s_{j+1}, s_{j+1})$. Removing any $h \ge 1$ hyperbolic planes changes l to $-h$. The necessary condition $l \ge 0$ then fails. This proves nonvanishing and the first occurrence for this row at every step. Atobe, Theorem 1.7.

To identify the nonzero lift, use the discrete-series correspondence of Ichino, Theorem 4.1(i) and the compact-Levi calculation in Section 5. We give the calculation because the signs of the negative entries change in this correspondence. In the parabolic defining the cohomologically induced target, positive entries stay in their original blocks. Negative entries exchange their blocks. The new middle block has signature

$$((n_{j+1} - s_{j+1}) - (A + z),\ s_{j+1} - y) = (t, 0).$$

All other Levi blocks are single compact lines because the old entries are distinct. Thus the Levi is compact. Order the old centered entries decreasingly and place the t slots of the middle block between the positive and negative entries. Put $N = n_{j+1}$. For an old centered entry β at combined position i, Ichino's inducing weight, inclusive of its scalar twist, is

$$\beta - \big((N+1)/2 - i\big) + a_S.$$

For a positive entry this follows by counting the larger positive entries ahead of it. For a negative one there are $A + t$ earlier slots before the negative list and then all larger negative entries, in either block. Equivalently there are $N - i$ smaller negative entries. Substitution into the two negative-entry formulas in Theorem 4.1(i) gives the displayed weight. For every middle slot the inducing weight is $A - n_j/2 + a_S$.

Refine the parabolic to the compact Cartan by putting the middle slots in decreasing order. For a compact Levi the cohomological induction of its character, in the regular good range, is the discrete series with parameter equal to the inducing weight plus the half sum of the positive roots of this ordering. Indeed one may first induce from the compact Cartan inside the compact Levi, giving its character by the compact highest-weight formula, and then use induction in stages. For the complex root system of GL_N the i-th entry of that half sum is $(N+1)/2 - i$. Adding it back gives $\beta + a_S$ at each old entry and, at the middle positions $A + i$ for $1 \le i \le t$, gives

$$A - n_j/2 + a_S + (N+1)/2 - (A+i) = a_S + (t+1)/2 - i.$$

The resulting entries strictly decrease in the combined ordering. This verifies regularity and the good-range inequalities just used, including across both ends of the middle block. The old entries have now undergone the full translation $a_S - a_T = h_{j+1} - h_j$. Their negative entries before translation were exactly those coming from $-b_i$ with $i \ge j+2$. Flipping those signs turns the old alternating tail into the prescribed tail for $j+1$. All old strings retain their plus signs. The middle slots give precisely the new string centered at $-e_{j+1} + h_{j+1}$, with plus signs. We have obtained P_{j+1}, proving the assertion. □

Corollary 72.7 (Real test vectors through the compact endpoint)**.** *Fix a nonzero compact-finite vector at the first row and a nonzero linear functional on the final compact representation.*

At this real place there are polynomial Fock tests, one at each step, whose successive local theta maps give a nonzero value for that functional. The tests can be chosen to be monomials in fixed Fock coordinates if only nonvanishing of the final value is prescribed. The product over all the real places is nonzero for pure initial vectors and terminal functionals of this kind.

Proof. Let V_j be the compact-finite module of σ_j. By local Howe duality and Theorem 72.6 the oscillator has the irreducible quotient $\sigma_j \boxtimes \sigma_{j+1}$ in the smooth local correspondence. On polynomial Fock vectors its image is the algebraic tensor $V_j \otimes V_{j+1}$. Indeed the polynomial vectors (in the Schrödinger model, Hermite polynomials times the Gaussian) are dense in the smooth Schwartz space and are stable under the complexified Lie algebras and compact groups of the pair. Their image is nonzero, consists of compact-finite vectors, and is an invariant submodule of an irreducible Harish-Chandra module. It must be the entire module.

Given a nonzero input $u_j \in V_j$ and a desired $u_{j+1} \in V_{j+1}$, choose a polynomial mapping to $(u_j/\|u_j\|^2) \otimes u_{j+1}$. Contracting the first factor against u_j in the invariant Hermitian pairing gives u_{j+1}. Choose all intermediate vectors nonzero and choose the final one with nonzero value under the given functional. This constructs the successive tests. Expand each polynomial into its finitely many monomials. Successive contraction is additive in each test, with complex conjugation on the scalar in alternating positions. If every tuple of monomials gave value zero, the polynomial tuple would give zero as well. Hence one monomial tuple suffices. Multiplying nonzero scalars over the finite set of real places proves the last assertion. □

Here P_j labels the forward quotient of the oscillator. The contragredient model negates its Harish-Chandra entries, as in `Formalism/LocalToGlobal/IndefiniteReal/ScalarOnPartsFull.lean`, line 15 and line 34 there. Complex conjugation of the entire intertwiner passes to the dual unitary representations and the conjugate oscillator. Individual monomial tests in the corollary have coefficient one in the chosen Fock model. Their images under the quotient maps depend on the scalars normalizing those maps. The integral comparison along the global chain requires those scalars and the prescribed finite data.

72.4 The finite standard L-values in the theta chain

Keep the fields and spacings of Section 6.6. Write $F = E^+$, $f = [F : \mathbb{Q}]$, $t = t_{\mathrm{r}}$, $d = (t-1)/2$, $z_0 = d+1$ and $n_j = r + jt$. Write S for a finite set of finite places of F, and S_E for all the places of E above S. An L-function with superscript S or S_E omits the indicated Euler factors and all gamma factors. We study the scalar at z_0 that occurs before the ramified local integrals in the doubling formula.

The splitting characters can be written

$$D_j = \eta^{-\mathbf{1}_{j\ \mathrm{odd}}}, \qquad \chi_{S,i} = \mu_i D_i, \qquad \chi_{T,i} = \mu_i D_{i-1}.$$

Here all the characters are unitary Hecke characters of E. The μ_i are trivial on $\mathbb{A}_F^\times$ and at a selected complex embedding τ have character $(z/|z|)^{-2e_{i,\tau}}$. The exponent for η is 1. We take S large enough to contain the finite ramification of the data. Put $\rho_{ik} = \mu_i \mu_k^{-1}$ for $i < k$. The exponent of ρ_{ik} at τ is $4(k-i)H_0$. In particular the common translation of all the $e_{i,\tau}$, which can be very large, disappears. The splitting convention is recorded in `Formalism/LocalToGlobal/IndefiniteReal/HasInfExpF.lean`, line 17.

Lemma 72.8 (A bound for the initial spherical parameters)**.** *Let $B_{0,w}$ be the standard unramified base-change parameters of a unitary spherical representation of the row group at a good finite place v, for $w \mid v$. Then*

$$\#B_{0,w} = r, \qquad |\alpha| \le (Nw)^{c_r} \quad (\alpha \in B_{0,w}), \qquad c_r = 10(r+1)^3.$$

Proof. Write q for the residue cardinality of F_v, let K be the hyperspecial subgroup, and give K volume one. At a split place take $G = \mathrm{GL}_r(F_v)$. At an inert place the unimodular odd-dimensional Hermitian space has the usual unramified group with relative Weyl group consisting of signed permutations on $(r-1)/2$ coordinates. In either cocharacter lattice put $\lambda_i = (1, \dots, 1, 0, \dots, 0)$ with i ones. At an inert place the diagonal in G also contains the reversed negatives of these coordinates. Write M_λ for the sum of the distinct Laurent monomials in the Weyl orbit of λ.

We use the normalized Satake isomorphism. The transform of the indicator of $K\lambda_i(\varpi)K$ is Weyl invariant, has nonnegative coefficients, is supported on dominant weights $\nu \le \lambda_i$ and their orbits, and has coefficient $q^{\langle \varrho, \lambda_i \rangle} \ge 1$ on M_{λ_i}. Here 2ϱ sums the positive relative roots with their multiplicities. These are the triangularity and leading coefficient in Satake's transform. Nonnegativity also follows directly by taking the unipotent constant term of an indicator and multiplying by the square root of the modulus. The sum of the coefficients is its spherical eigenvalue in the normalized principal series induced from the trivial unitary character. Unitarity bounds that eigenvalue, and the absolute value of the eigenvalue in the given representation, by the volume of the double coset. Satake, *Theory of spherical functions on reductive algebraic groups over p-adic fields*, Sections 5–7.

The volume is at most $V = q^{4r^2}$. Indeed, the right cosets give distinct lattices between $\varpi\Lambda$ and $\varpi^{-1}\Lambda$. Their images modulo $\varpi\Lambda$ are determined by r generators in that quotient, each with at most q^{4r} possibilities. At a split place one component suffices. In the split chamber, dominance bounds the largest coordinate by one and the smallest by zero from below, and fixes the sum i. At an inert place the coordinates are decreasing nonnegative integers, the largest is at most one and the sum at most i. Thus the only possible dominant weights are λ_i in the split case and $\lambda_0, \dots, \lambda_i$ in the inert case. At the Satake point of our representation, triangularity consequently gives $|M_{\lambda_i}| \le (2V)^i$. To see the induction, subtract the lower terms from the eigenvalue, use the bound V on the sum of their nonnegative coefficients, and divide by a leading coefficient at least one. Start with $M_0 = 1$.

For the split standard representation the elementary symmetric functions of the parameters are exactly these orbit sums. For the inert one the parameters are $z_1, \dots, z_a, 1, z_a^{-1}, \dots, z_1^{-1}$, where $a = (r-1)/2$. Indeed, a diagonal dual Frobenius $s\vartheta$ restricts over E_w to $s\vartheta(s)$, where ϑ reverses and inverts the diagonal. The z_i are the values on individual relative coordinates. Each elementary symmetric function is now a nonnegative integral combination of the same orbit sums, with coefficient sum at most 2^r, by expanding the subsets of this list. The elementary root bound for a monic polynomial gives $|\alpha| \le 1 + 2^r(2V)^r \le q^{c_r}$. At the other component of a split place apply the same argument to the contragredient, which is again unitary. Since $q \le Nw$, this suffices. □

Lemma 72.9 (Hecke characters at the endpoint of convergence)**.** *Let K be a CM field of degree $2f$. Let ρ be a unitary Hecke character with pure angular exponents $\nu_\tau \in \mathbb{Z}$, at least one nonzero. Let U contain its finite ramification and put*

$$M(\rho, U) = 1 + \log(2|D_K|N\mathfrak{f}_\rho) + f\log(3 + \max_\tau |\nu_\tau|) + \#U.$$

The value $L_K^U(1, \rho)$ is nonzero and

$$|\log|L_K^U(1,\rho)|| \ll M(\rho, U), \qquad \sum_{a=2}^{\infty} |\log|L_K^U(a,\rho)|| \ll f.$$

Both implied constants are absolute. If K/E_0 is CM quadratic, ρ is trivial on $\mathbb{A}_{E_0}^\times$ and U is stable under conjugation, all these values are positive real numbers.

Proof. Neither ρ nor ρ^2 is an imaginary power of the idelic norm, by the nonzero angular exponent. Hecke's continuation and functional equation give entire L-functions with complex factors $\Gamma_{\mathbb{C}}(s+|\nu_\tau|/2)$ and conductor $|D_K|N\mathfrak{f}_\rho$. The completed functions are bounded on closed vertical strips away from their poles by the global zeta integral, and the incomplete functions have finite order on strips by Stirling's formula. The trivial character gives the Dedekind zeta function with a simple pole at 1. Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, Chapters II and IV.

Here are uniform constants sufficient near 1. On $\Re s = 2$ the Euler product is bounded by $\zeta(2)^{2f}$. On $\Re s = -1$, the functional equation costs $(|D_K|N\mathfrak{f}_\rho)^{3/2}$ and a factor at most $C(3+|\nu_\tau|+|\Im s|)^3$ at each complex place. Indeed the absolute gamma quotient there reduces by conjugation and three applications of $\Gamma(u+1) = u\Gamma(u)$ to the absolute value of a cubic times an absolute constant. This expression extends across poles of the denominator. Apply the strip maximum principle of Phragmén–Lindelöf after dividing by $(s+3)^{3f}$. For ζ_K first multiply by $(s-1)/(s+3)$. The same bounds apply to ρ^2 since its conductor divides $\mathfrak{f}_\rho$ and its angular exponents double. Removing the local factors in U costs at most $2^{\#U}$ on $|s-1| \le 1/2$. Consequently on that disk

$$|(s-1)\zeta_K^U(s)|, \quad |L_K^U(s,\rho)|, \quad |L_K^U(s,\rho^2)| \ \le A = \exp(C_0 M(\rho,U))$$

for an absolute C_0 with $A \ge 2$. Cauchy's inequality on the smaller disk of radius $1/4$ gives $|(L_K^U)'(s,\rho)| \le 4A$.

For real $\sigma > 1$, take Euler logarithms and apply $3 + 4\cos u + \cos(2u) = 2(1+\cos u)^2 \ge 0$ to each prime power. This gives

$$\zeta_K^U(\sigma)^3 \, |L_K^U(\sigma,\rho)|^4 \, |L_K^U(\sigma,\rho^2)| \ge 1.$$

Hence $|L_K^U(1+h,\rho)| \ge h^{3/4}/A$ for $0 < h \le 1/4$. Choose $h = (16A^2)^{-4}$. Moving back to 1 costs at most $4Ah \le h^{3/4}/(4A)$, giving the claimed lower bound after taking logarithms. For $a \ge 2$, absolute Euler logarithms are bounded by $2f \log \zeta(a)$, whose sum is $O(f)$ by the geometric series over a.

For positivity, conjugation interchanges a split pair of Euler factors with their complex conjugates since $\rho(z)\rho(\bar{z}) = \rho(Nz) = 1$. At a nonsplit place a real value $+1$ or -1 on an unramified uniformizer follows from the same identity and conjugation invariance of its valuation. Each unramified local contribution at real $\sigma > 1$ is positive. Pass to the nonzero limit at one. □

Theorem 72.10 (The successive partial standard values)**.** *For each $w \notin S_E$ set $Q_w = Nw$, $\mathcal{A}_w = \{Q_w^a : -d \le a \le d\}$ as a multiset, and evaluate an unramified character at a uniformizer of w. Starting with any initial spherical row as in Lemma 72.8, form the successive parameter lists*

$$B_{i,w} = (\chi_{S,i}/\chi_{T,i})(w) B_{i-1,w} \ \sqcup \ \chi_{S,i}(w)\mathcal{A}_w.$$

For $0 \le j < m$ and $k = j+1$, the partial standard product of $B_j \otimes \chi_{T,k}^{-1}$ continues holomorphically to a neighborhood of z_0. Its value is

$$L_j^S(z_0) = L^S(z_0, B_0 \otimes \mu_k^{-1}) \prod_{i=1}^{j} \prod_{a=1}^{t} L_E^{S_E}(a, \rho_{ik}) \neq 0. \tag{72.3}$$

Suppose $U_0 = \max_{i<k} \log(2|D_E|N\mathfrak{f}_{\rho_{ik}})$, with $\log(2|D_E|)$ used for an empty set, and put $M_* = 1 + U_0 + f\log(3+4mH_0) + \#S_E$. For large x,

$$\sum_{j=0}^{m-1} |\log|L_j^S(z_0)|| \ll m^2 M_* + mfr\, 2^{-(z_0 - c_r)}.$$

After dividing each value by its segment product on the right of (72.3)*, its logarithmic absolute value is* $O(fr\,2^{-(z_0-c_r)})$*, uniformly in the initial row and in the splitting characters.*

Proof. For $\Re z$ large, the standard product of a multiset B is $\prod_{w\notin S_E}\prod_{\alpha\in B_w}(1-\alpha Q_w^{-z})^{-1}$. Induction on j gives

$$B_{j,w}=D_j(w)\Big(B_{0,w}\sqcup\coprod_{i=1}^{j}\mu_i(w)\mathcal{A}_w\Big).$$

Indeed the old terms gain D_j/D_{j-1}, and the new segment has scalar $D_j\mu_j$. Division by $\chi_{T,k}$ cancels D_j. Disjoint union multiplies Euler products, and a monomial Q_w^a shifts the Hecke argument from z to $z-a$. This gives (72.3) by continuation, since z_0-a runs exactly once through $1,\dots,t$.

For the initial factor put $u=\Re z-c_r$. For $u\ge 2$ the sum of the absolute logarithms of its local factors is at most $2fr\log\zeta(u)\ll fr\,2^{-u}$, by Lemma 72.8. Comparison uses at most $2f$ primes over each rational prime and $Nw\ge p$. The choice $t\ge x^{377}$ and the row rank of Lemma 72.3 give $z_0-c_r>3$ eventually. The initial product is thus holomorphic and nonzero there. All shifted factors are entire, and Lemma 72.9 applies at their positive arguments. Summing that lemma over the $m(m-1)/2$ pairs proves the stated bounds. The recursion and location of the special value can also be read in `Development/AbelianVarietyTheory/Definitions/LatticeFrames/ChainParSucc.lean`, line 121 and line 128 there. □

Corollary 72.11 (Characters of controlled conductor on the common fields)**.** *The* μ_i *with the required real exponents can be chosen unramified away from the existing split auxiliary primes, with conductor exponent at most one there. With this choice the logarithm of each ratio conductor is at most* $2f\log(g_1g_2T)$*. Multiplying all the* μ_i *by the same finite-order character preserves all segment factors for a fixed omitted set. For omitted primes above* $2Ng_1g_2T$*, where* $E\subset\mathbb{Q}(\zeta_N)$*, the sum of logarithmic absolute values in Theorem 72.10 is bounded by a polynomial in* x *on the common intervals.*

Proof. Let $A=\ker(N_{E/F}:\mathrm{Res}_{E/F}\mathbb{G}_m\to\mathbb{G}_m)$. In $A(\mathbb{A})$ prescribe the character of $A(F_\infty)$ with weights $-e_{i,\tau}$. The elements of $A(F)$ integral at all finite places form the roots of unity in E. They are units of absolute value one at all embeddings, so Kronecker's theorem applies. At any chosen $v\mid g_1$, reduction on these roots of unity is injective. Indeed E splits completely at the odd prime g_1, and $1+g_1\mathbb{Z}_{g_1}$ is torsion free. Choose a character of the cyclic residue-unit group at v canceling the prescribed character on these roots. Such a character exists by extending from the cyclic subgroup. One may first prescribe other residue characters at split auxiliary places and cancel their product in the same way. Take the trivial character on all remaining finite unit groups. The resulting character on $A(F_\infty)\prod_{v<\infty}A(\mathcal{O}_v)$ kills its intersection with $A(F)$. It descends to an open subgroup of $A(F)\backslash A(\mathbb{A})$ and extends to a unitary character there. For example, the circle is divisible, so an abelian character extends algebraically, and continuity on the open subgroup gives continuity throughout. Pull back under $z\mapsto z/\bar z$. This doubles the angular exponents, kills $\mathbb{A}_F^\times$, and gives the asserted finite conductors. The same conductor bound applies to each quotient.

For completeness one can choose η, whose conductor cancels from the formula, on the same fields. On $\mathbb{Q}(\sqrt{-d_0})$ take the angular character $z/|z|$ and the quadratic residue character on the units at $\sqrt{-d_0}$, trivial on the other finite unit groups. The roots of unity are $\{\pm1\}$ and $d_0\equiv 3 \bmod 4$, so the product on them is one. Extension from the subgroup of units and the infinite component, as above, gives a unitary Hecke character. Its restriction to the rational ideles is the quadratic character of this imaginary quadratic field. They agree at infinity and on all finite units, and $\mathbb{Q}$ has class number one. Composing with the idele norm from E gives η

with the desired exponents on the selected CM type. On $\mathbb{A}_F^\times$ it restricts to $\omega_{E/F}$ by quadratic base change. Its possible finite ramification lies over d_0, which divides N.

Finally $\#S_E \ll f(1+\log(2Ng_1g_2T))$. The conductor–discriminant formula gives $\log|D_E| \le 2f\log N$. The field degrees, m, H_0 and the logarithms of all the indicated rational primes and of N are polynomially bounded in the construction of Volume I. Substitution proves the assertion. □

The theorem concerns the standard parameters propagated by the displayed recursion. At a step where a spherical theta lift has these parameters, it computes its partial scalar. The nonzero local integrals for a prescribed Schwartz vector and the integral normalization of a global theta lift are additional requirements. They occur separately in `Development/AbelianVarietyTheory/Definitions/LatticeFrames/ChainParSucc.lean`, line 197 and line 215 there, which impose separate conditions on the local integrals and the resulting global lift.

72.5 A common trace pairing and the finite theta tests

Consider the actual fields $F = E^+$ and $E = B_J(\sqrt{-q}, \sqrt{-d_0})$ of Theorem 20.11. Here $d_0 \equiv 7 \bmod 8$, $C_J = \mathbb{Q}(\zeta_{2^{J+2}})^+ \subset F$, and $J \ge 2$ on sufficiently distant intervals. Choose the CM embeddings that are positive on $\sqrt{-d_0}/i$. In this section $\delta_0 = \sqrt{-d_0}$, $\mathfrak{d}_F$ denotes the absolute different of F, and $\omega = \omega_{E/F}$ denotes the quadratic character. At finite places E/F is unramified, including the split case, by Theorem 94.9. In particular ω is a character of the narrow ideal class group of F. All dyadic places split since 2 splits in $\mathbb{Q}(\sqrt{-d_0})$.

Lemma 72.12 (A narrow norm class)**.** *There is an integral ideal $\mathfrak{a}_0$ of F with $\mathfrak{a}_0\mathcal{O}_E = (\delta_0)$ and $\omega(\mathfrak{d}_F\mathfrak{a}_0) = 1$. An integral ideal $\mathfrak{b}$ of E and a totally positive $\alpha \in F^\times$ can be chosen with*

$$(\alpha)\,\mathfrak{d}_F\mathfrak{a}_0 = N_{E/F}\mathfrak{b}, \qquad \log N_{E/\mathbb{Q}}\mathfrak{b} \le \tfrac{1}{2}\log|D_E| + f\log(4/\pi). \tag{72.4}$$

Proof. The valuations of δ_0 agree at conjugate primes and the relative ramification index is one everywhere. They therefore define an integral ideal downstairs, supported above d_0. All its valuations at primes above d_0 are equal since $E/\mathbb{Q}$ is abelian and every conjugate of δ_0 is $\pm\delta_0$. The primes of F over d_0 also have the same value under ω. There are evenly many of them: $d_0 \equiv 7 \bmod 8$ splits in $\mathbb{Q}(\sqrt{2}) \subset F$, and the two primes there have the same number of prolongations by Galois symmetry. Hence $\omega(\mathfrak{a}_0) = 1$.

The global root number of ω is one, by the factorization of the completed Dedekind zeta function of E as the completed zeta function of F times the completed L-function of ω. Both zeta functional equations have sign one. For the global trace additive character, the local root number of an unramified quadratic character at v is $\omega_v(\varpi_v)^{v(\mathfrak{d}_F)}$. Indeed the local constant for unit conductor additive character is one, and scaling that character by a uniformizer multiplies the root number by its character value. Reversing the conductor convention inverts this sign and gives the same number. At each real place the character is the sign character, with root number i or $-i$ according to the single global sign convention. The number of real places is divisible by four since $C_J \subset F$ with $J \ge 2$. Multiplying the local constants gives $\omega(\mathfrak{d}_F) = 1$. These local constants follow from the unramified and real local functional equations in Tate, *Fourier analysis in number fields and Hecke's zeta-functions*, Chapters II and IV.

Let H^+ be the narrow Hilbert class field of F. The unramified-at-finite-primes abelian extension E/F is contained in H^+. By Artin reciprocity and Chebotarev there is a prime in the narrow class of $\mathfrak{d}_F\mathfrak{a}_0$. Its Frobenius in E/F is trivial by the first two paragraphs, so either prime of E above it has ideal norm that prime. Replace one such prime upstairs by an integral

ideal $\mathfrak{b}$ in its ordinary ideal class satisfying Minkowski's bound in degree $2f$:

$$N_{E/\mathbb{Q}}\mathfrak{b} \;\le\; (4/\pi)^f \,\frac{(2f)!}{(2f)^{2f}}\,\sqrt{|D_E|}.$$

The norm of a principal ideal from E has a totally positive generator in F, so this replacement preserves the narrow class of the norm. Equality of narrow classes now gives α and the asserted size estimate. □

Theorem 72.13 (Simultaneous self-dual lattices at the inert places)**.** *Put $\delta = \alpha\delta_0$ using Lemma 72.12. Use this one trace-zero scalar to make the Hermitian tensor pairing alternating at every step, with the global additive character obtained by trace from $\mathbb{Q}$. At each inert finite place v, let $L_{j,v}$ be the tensor of the two standard unimodular local Hermitian lattices at step $j \to j+1$. There are homothetic lattices $L_{j,v} \subseteq L'_{j,v}$ which are self-dual for the additive pairing and are preserved by both of the original local compact stabilizers. They are equal to $L_{j,v}$ away from finitely many v, and*

$$\sum_{j=0}^{m-1} \sum_{v \text{ inert}} \log[L'_{j,v} : L_{j,v}] \;\le\; \Big(\sum_{j=0}^{m-1} n_j n_{j+1}\Big) \log N_{E/\mathbb{Q}}\mathfrak{b}. \tag{72.5}$$

The right side is polynomially bounded in x throughout the intervals of Section 72.4. All the real signatures and the oriented real additive pairings of Section 72.3 can be used at the same time.

Proof. Fix v with prime w above it, and put $a = v_w(\mathfrak{b})$. Equation (72.4) gives $v_w(\delta) + v(\mathfrak{d}_F) = 2a$ since the residue degree of E_w/F_v is two. Write $\Lambda = L_{j,v}$. The pairing has the form

$$(x, y) \longmapsto \psi_{\mathbb{Q},p}\big(\mathrm{Tr}_{F_v/\mathbb{Q}_p}\,\mathrm{Tr}_{E_w/F_v}(\delta\, h(x, y))\big),$$

up to a sign or a factor $1/2$, both units here. The Hermitian tensor form h is unimodular. The annihilator of $\mathcal{O}_{E_w}$ for the absolute trace character is its inverse different, namely $\mathfrak{d}_F^{-1}\mathcal{O}_{E_w}$, because E_w/F_v is unramified. In a lattice basis inversion of the unimodular Gram matrix thus gives $\Lambda^{\#} = \varpi_v^{-2a}\Lambda$. Taking $L'_{j,v} = \varpi_v^{-a}\Lambda$ makes it self-dual. Both compact stabilizers preserve every such homothety. The tensor rank over E_w is $n_j n_{j+1}$, and $Nw = (Nv)^2$. Its index is $(Nw)^{a n_j n_{j+1}}$. Multiplication over inert primes and then the steps gives (72.5), because retaining all primes in the norm of the integral ideal $\mathfrak{b}$ only increases the bound. Polynomial growth follows from $\log|D_E| \le 2f \log N$ and the rank bounds. At each chosen embedding α is positive. The new trace-zero scalar consequently has the prescribed imaginary sign there. Any change of magnitude is absorbed by a positive real rescaling in the complex vector space. This verifies the assertion about the real pairs. □

At an inert place the self-dual lattice also determines a compact vector in the lattice model. To see the local normalization explicitly, take the Heisenberg group of the symplectic tensor space and induce the character of the subgroup above $L'_{j,v}$ that is trivial on the lattice and is the given additive character on the center. Integrality of the pairing and odd residue characteristic make this a character with the usual half-pairing in the group law. Self-duality gives the irreducible Heisenberg model. Every isometry preserving the lattice acts on this model by pullback, and fixes the function supported on the subgroup and equal there to its inducing character. This describes the vector for the lattice splitting over the compact stabilizer. Its use with a globally specified Weil splitting requires the agreement of the two compact characters.

Proposition 72.14 (Removing the finite inert ramification of the splitting characters)**.** *The characters μ_i, η of Section 72.4 can be chosen with the stated archimedean types and restrictions to $\mathbb{A}_F^\times$, unramified at every nonsplit finite place. In fact η can have finite ramification confined to the existing split prime g_1 with conductor exponent at most one.*

Proof. Use the μ_i from Corollary 72.11. Start with the η constructed at the end of its proof. Its finite unit characters have conductor exponent at most one at places above d_0: local norms carry principal units into principal units, and the original character at the quadratic prime is a residue character. At all other finite places it is unramified. At a place $v \mid d_0$ its product on $\mathcal{O}_{E,v}^\times$ is trivial on the diagonal $\mathcal{O}_{F,v}^\times$, since $\omega_{E/F}$ is unramified. Over the residue fields the quotient by these scalar units identifies via $z \mapsto z/\bar{z}$ with the group of norm-one elements. For an inert place this follows by counting in the cyclic group of the quadratic residue field. At a split place it is the ratio of the two coordinates. There is therefore a character on the local norm-one residue group whose pullback cancels the finite unit character. Prescribe these characters on the norm-one torus at all places over d_0 and the trivial character at infinity. Use one place above g_1 to cancel their product on the global roots of unity, exactly as in Corollary 72.11. Extend and pull back along $z \mapsto z/\bar{z}$. Multiplying η by this Hecke character preserves its infinite type and restriction to F, removes the ramification above d_0, and gives the claimed support and conductor. □

Proposition 72.15 (A common depth for the split local tests)**.** *At a split finite place put $q_v = Nv$ and let $K_n(a) = 1 + \varpi_v^a M_n(\mathcal{O}_v)$, $a \geq 1$. For any consecutive ranks $n \leq N'$, use the matrix Schrödinger model there and suppose both determinant splitting characters are trivial on $1 + \varpi_v^a \mathcal{O}_v$. There is an integral indicator fixed by $K_{N'}(a)$ such that its diagonal local doubling integral at the theta point, on every unit $K_n(a)$-fixed source vector, equals*

$$q_v^{-a(N'n+n^2)} \prod_{\ell=1}^{n} (1 - q_v^{-\ell})^{-1}. \tag{72.6}$$

Here the integral matrix lattice has additive volume one and $\mathrm{GL}_n(\mathcal{O}_v)$ has multiplicative volume one.

Proof. For $X \in M_{N',n}(F_v)$ take the indicator of

$$X \equiv \begin{pmatrix} 1_n \\ 0 \end{pmatrix} \pmod{\varpi_v^a M_{N',n}(\mathcal{O}_v)}.$$

Both congruence groups preserve it in the split model (93.1). The determinant modulus on each is one, and the character there is trivial by hypothesis. For the inner product with a source translate to be nonzero, the two top square blocks must lie in $K_n(a)$; their quotient then gives $g \in K_n(a)$. Conversely inside $K_n(a)$ the two indicators coincide. The oscillator coefficient is thus $q_v^{-aN'n} \mathbf{1}_{K_n(a)}$. The source matrix coefficient is one on that subgroup. Finally reduction modulo ϖ_v^a has image of cardinality $q_v^{an^2} \prod_{\ell=1}^{n} (1 - q_v^{-\ell})$. Integration gives (72.6). Since the output test is fixed by $K_{N'}(a)$, this construction can use the same depth at the following step. It applies at dyadic split places as well. □

The lattice index in (72.5) and the local integral in (72.6) measure different parts of a global comparison. The first bounds the finite cost of making the trace lattice self-dual for a permissible global scalar. The second evaluates a particular compactly supported zeta integral. The action of global projection on the designated arithmetic tensors and their periods requires a separate estimate.

72.6 Spherical parameters of a nonzero local lift

Let E_v/F_v be unramified quadratic with odd residue characteristic, $Q = \#\kappa_{E_v}$, and take the two unramified unitary groups of ranks n, N. The groups have their unimodular lattices and are quasi-split. Write χ_T, χ_S for the splitting characters attached to the spaces of dimensions N, n, respectively. At an inert place their respective values on the common uniformizer are $(-1)^N, (-1)^n$. This follows by restriction to $F_v^\times$. For a spherical representation π let $B(\pi)$ be its standard base-change Satake multiset. It has a reciprocal pair for each coordinate of the relative split torus and, in odd dimension, one additional entry 1. For $b \geq 1$ put $A_b = \{Q^{(b-1)/2-i} : 0 \leq i < b\}$ and put $A_0 = \varnothing$.

Theorem 72.16 (Spherical transfer at an inert place)**.** *Suppose that a spherical π in rank n and a spherical σ in rank N occur together as an irreducible quotient of the local oscillator. If $N \geq n$, then*

$$\chi_S(\varpi)^{-1} B(\sigma) = \chi_T(\varpi)^{-1} B(\pi) \ \sqcup \ A_{N-n}. \tag{72.7}$$

This assertion allows non-tempered spherical representations.

Proof. A spherical representation embeds in a normalized principal series with unramified torus character in its Satake orbit. Indeed it is a subquotient of an unramified principal series, has unramified cuspidal support, and the subrepresentation theorem embeds it in a principal series with that support. Composing the assumed quotient with such embeddings on both sides and using normalized Frobenius reciprocity gives a joint torus-character quotient of the normalized Jacquet module of the oscillator for the two minimal parabolics. We prove the corresponding multiset identity for any such unramified character quotient. We work in the free abelian group on $\mathbb{C}^\times$, extending A_b by $A_{-b} = -A_b$ for $b > 0$. Interchanging the two groups then negates both sides of the desired difference formula. The unramified splitting values and their inverses agree at this place.

When both ranks are at most one there are no reciprocal pairs. The middle entry in odd rank is 1. With ranks $(0, 1)$, the splitting value for the zero-dimensional space is 1; with ranks $(1, 1)$ both splitting values are -1. These facts verify the assertion in all base cases. Now take $N \geq n$ with $N \geq 2$ and take first the Jacquet module on the N-side for the parabolic of an isotropic line. Write u for the unramified character on its $\mathrm{GL}_1(E_v)$ factor, evaluated at ϖ. The two possible strata of Kudla's filtration have ranks zero and one. Normalized Jacquet functors are exact and transitive. A nonzero map from a finite filtration to a character restricts nontrivially to some successive quotient, by taking the last filtration subspace with nonzero image. We spell out both possibilities. Atobe and Gan, Lemma 5.1, with $k = 1$ and $\epsilon_0 = 0$.

In rank zero the line carries $\chi_S| \cdot |_{E_v}^{(n-N+1)/2}$, and the remaining oscillator has ranks $N-2, n$. Set $b = N - n$. After removing $\chi_S(\varpi)$ the two entries supplied by the line are $Q^{(b-1)/2}, Q^{-(b-1)/2}$. In the free abelian group one has

$$A_b - A_{b-2} = [Q^{(b-1)/2}] + [Q^{-(b-1)/2}] \qquad (b \geq 0).$$

This identity also covers $b = 0, 1$ by the signed definition. The inductive formula for the remaining ranks, interchanged if necessary, now proves the assertion in rank zero.

In rank one the other group also has an isotropic line. The quotient is normalized induction from its line parabolic. Its inducing representation is the oscillator in ranks $N-2, n-2$ tensored with the functions of compact support on the isomorphisms between the two conjugate lines. For the two general linear factors this last space is a copy of $C_c^\infty(E_v^\times)$, with left and right translation and the two splitting characters. The element (ϖ, ϖ) acts there by the scalar $\chi_S(\varpi)\chi_T(\varpi)$. The conjugate-linear isomorphisms cause no change because $\varpi \in F_v$. Taking

the further minimal Jacquet module on the n-side uses the geometric lemma for parabolic induction. Its finite Bruhat filtration places the line character in one Weyl coordinate or its conjugate inverse. The other coordinates belong to the minimal Jacquet module of the smaller group. This description follows by taking the unipotent coinvariants on the Bruhat cells of the line-parabolic flag variety. Normalized induction and Jacquet functors cancel the modulus on each cell. It applies to the smooth compactly supported function representation on the line as well.

If v' is the value of the coordinate so placed (after this possible inversion), then $uv' = \chi_S(\varpi)\chi_T(\varpi)$. Hence the two normalized reciprocal pairs on opposite sides coincide as multisets. Fixing any line function on which the character map is nonzero for some remaining tensor gives a nonzero character map for the two smaller minimal Jacquet modules. The rank difference is still b, so induction proves the formula in rank one also. The parity of each group stays the same whenever two dimensions are removed, and the splitting values in every invocation of induction are consequently the stated ones. This completes the proof. □

Proposition 72.17 (The split pair and its two components)**.** *At a split place with unramified data the same formula holds, with $Q = Nw$ on each component $w \mid v$, in the unitary tensor convention. Thus at either type of good place a nonzero spherical theta pair has exactly the parameter update of Theorem 72.10.*

Proof. Write w^c for the other component. The position $V_w \otimes W_w$ of the symplectic tensor can be viewed as $\mathrm{Hom}(W_{w^c}, V_w)$, since the Hermitian form identifies W_{w^c} with the dual of W_w. In the matrix splitting the right group is therefore the source in its w^c coordinates, and the left group is the target in its w coordinates. The normalized matrix action has the square roots of the two determinant Jacobians. The unitary splitting twists the two actions in their w coordinates by $\chi_{T,w} \circ \det$ and $\chi_{S,w} \circ \det$. Equivalently, the change from the trivial split characters is through the isomorphism $E_v^\times / F_v^\times \to E_v^1,\ z \mapsto z/\bar{z}$. The determinant with coordinates (a, a^{-1}) lifts to $(a, 1)$. This gives precisely the indicated component character. The same split unitary convention and its reduction to the matrix correspondence are used in Chen and Zou, *Arthur's multiplicity formula for even orthogonal and unitary groups*, Remark 6.4(2).

Remove both twists. A spherical source with list B in its w coordinates then has list $(B/\chi_{T,w}(\varpi))^{-1}$ in its w^c coordinates, where inversion is termwise. For the normalized type II matrix oscillator the larger representation has cuspidal support given by the contragredient of the smaller one together with the support of the trivial representation of GL_{N-n}. Mínguez, *Correspondance de Howe explicite*, Theorem 1(2). The second list is A_{N-n} by normalized induction from the diagonal. The contragredient reverses the inversion from w^c. Restoring the target twist proves the formula at w. At w^c both lists and both component characters invert, while A_{N-n} is invariant under inversion. □

72.7 A nonzero global theta chain with a choice of tests

There is another use of the narrow-class calculation of Lemma 72.12. For an existence assertion over $\mathbb{C}$ one can arrange that the additive character and the trace-zero scalar are separately unramified at every inert place. This choice may introduce split places at which the Schwartz function is allowed to vary.

Lemma 72.18 (Simultaneous unramified data off split places)**.** *On each field of Section 72.5 is a global additive character and a trace-zero scalar of the signs of Section 72.3 such that at every inert finite place the additive character has conductor $\mathcal{O}_v$ and the scalar is a unit. The choices in this statement use at most two additional prime ideals of F as representatives of narrow ideal classes, both split in E.*

Proof. The proof of Lemma 72.12 shows separately that $\omega(\mathfrak{d}_F) = \omega(\mathfrak{a}_0) = 1$. Choose distinct prime representatives $\mathfrak{p}, \mathfrak{q}$ of these two narrow classes outside the existing finite levels, by Chebotarev in the narrow Hilbert class field. Both split in E. There exist totally positive β, γ with

$$(\beta)\mathfrak{d}_F = \mathfrak{p}, \qquad (\gamma)\mathfrak{a}_0 = \mathfrak{q}.$$

Use the negative trace character evaluated at βx and take the skew scalar $\gamma\sqrt{-d_0}$ with the sign specified for the pair. The annihilator of $\mathcal{O}_v$ for the additive character is $\beta^{-1}\mathfrak{d}_{F,v}^{-1}$, which is $\mathcal{O}_v$ off $\mathfrak{p}$. The principal ideal of the skew scalar is $\mathfrak{q}\mathcal{O}_E$. Positivity of β, γ retains the real orientation. Positive real changes of magnitude are absorbed by an equivariant scalar isometry of the local tensor spaces. Together with Proposition 72.14 this gives unramified data at every inert finite place. The size estimate for the original trace character is the separate ideal bound (72.5). □

Theorem 72.19 (Cuspidal lifts to the compact endpoint)**.** *On a field and interval of Theorem 20.11, let Π_0 be an irreducible cuspidal automorphic representation of the row group with the real parameters P_0 and a hyperspecial fixed vector at every inert finite place. Use the splitting characters of Proposition 72.14 and the global data of Lemma 72.18. With Schwartz functions free at a finite set of split places and at infinity, the successive global theta lifts through $U(Y_m)$ are all nonzero and cuspidal. Their real parameters are P_j, and their standard parameters at the good finite places are precisely B_j of Theorem 72.10. In particular this applies to the cuspidal constituents of a nonzero row S_j at the levels of Section 72.2. It also gives a nonzero complex iteration on any nonzero compact-finite vector in such an irreducible constituent fixed by the good hyperspecial groups.*

Proof. In the quotient convention for theta lifting, the kernel is integrated against the complex conjugate of the input cusp form. A local nonzero pairing then gives the source representation and its forward lift as a quotient of the oscillator. Choose a finite set S of split places including the two primes of Lemma 72.18, the dyadic places, the auxiliary levels and any split places where Π_0 is ramified. Outside S, all spaces have unimodular lattices, the finite characters are unramified and the additive character has integral conductor. The standard unramified pair therefore has its lattice vacuum, fixed by both hyperspecial subgroups. In a self-dual lattice model this is the vector supported on the lattice. The unramified Weil splitting on the hyperspecial groups is the lattice splitting, equivalently the usual spherical data in the local doubling integral. Fix these vacua outside S.

Suppose Π_j has been obtained with the indicated real components and good finite parameters. Write $n = n_j$, $N = n + t$, $s_0 = t/2$. Theorem 72.6 excludes a nonzero local lift to any smaller member of the target Witt tower at every real place. A nonzero global theta integral to a smaller member would give such a local pairing by fixing all other test vectors. Thus all the smaller global lifts vanish. The constant-term tower property then makes the lift to Y_{j+1} cuspidal if it is nonzero. Moreover the rational Witt index R of Y_{j+1} is at most s_{j+1}, the smaller real signature, so $N - R \geq n + t - s_{j+1} > n$. This is Weil's convergent range for the doubled unitary pair at every step.

We apply the convergent-case Rallis inner product formula. For a pure cusp vector and a pure Schwartz function the squared Petersson norm is a nonzero Haar constant times the value at s_0 of the global doubling integral for the associated Siegel–Weil section. For $N > 2n$ this is the convergent formula used in Section 72.2. For $n < N \leq 2n$ it is Gan, Qiu and Takeda, Propositions 11.1(ii) and 11.2. The latter formula uses meromorphic continuation from large $\Re s$. We verify below that the unfolded product is holomorphic and nonzero at the required point, which also identifies its constant Laurent coefficient there.

At a good finite place take the spherical input and the fixed vacuum with norms one. The partial product of the local integrals is

$$\frac{L^S(s+1/2, B_j \otimes \chi_{T,j+1}^{-1})}{d_n^S(s)}.$$

Here $d_n^S(s) = \prod_{i=1}^n L_F^S(2s+i, \omega_{E/F}^{t+i})$ is the unramified doubling denominator. This evaluation is the unramified doubling calculation of J.-S. Li, *Non-vanishing theorems for the cohomology of certain arithmetic quotients*, Theorem 3.1. The unitary normalization is displayed in Ichino, Section 6.3, and the meromorphic integral is treated in Gan, Qiu and Takeda, Section 11.5. In the norm pairing the source coefficient is conjugated. One may equivalently use the dual list twisted by χ_T. Conjugation on E/F exchanges this with the list displayed above. Indeed $B_{j,w^c} = B_{j,w}^{-1}$ at split places, the inert list is reciprocal, and $\chi_T(z)\chi_T(\bar{z}) = 1$ because its restriction to $\mathbb{A}_F^\times$ is trivial on norms. The two partial products agree since all places above S are omitted. The calculation initially holds for $\Re s$ large. Theorem 72.10 continues the numerator to s_0 with nonzero value. The denominator has an absolutely convergent Euler logarithm near s_0, since $t+1 > 1$. Its logarithmic absolute value there at s_0 is $O(f\,2^{-t})$ by summing $f \log \zeta(t+i)$ over $1 \le i \le n$.

At $v \in S$ take a unit smooth vector fixed by $K_n(a_v)$, with $a_v \ge 1$ also large enough for the splitting characters. Use Proposition 72.15. The resulting local integral is strictly positive. It is holomorphic in a neighborhood of s_0. Indeed the restriction of a flat extension of the Siegel–Weil section at $(g, 1)$ equals the section at s_0 times its positive Iwasawa height to the power $s - s_0$. The local oscillator coefficient has compact support there. The multiplying height is bounded above and below on that compact set, so integration gives an entire function for this local test. Each place in S is split, where the matrix argument works for every rank $N \ge n$.

At a real place the input is the discrete series of Theorem 72.6, including a compact representation when the source has compact real points. Its local theta lift to the specified target is nonzero. For such a discrete series and $N \ge n$, the local Rallis integral is a nonzero sesquilinear pairing, and the flat section integrals converge in an open neighborhood of $s_0 > 0$. Ichino, Lemma 7.1 and Proposition 7.2. By polarization in each of the two vectors, some diagonal choice of a Schwartz vector and a source vector has nonzero integral. One may take compact-finite vectors by density and continuity, since the convergent local zeta integral is a continuous functional on smooth vectors. For a compact source the same holomorphy follows immediately by compact integration.

For these factorizable data the Eulerian unfolding of the doubling integral holds first in its half-plane of absolute convergence. Cuspidality annuls the non-open orbits there. The finite product of the integrals at S and infinity, times the partial product already computed, is its meromorphic continuation. Every factor is holomorphic and nonzero at s_0. The norm formula therefore gives a nonzero cusp form. The nonzero cuspidal theta lift of an irreducible cuspidal representation is irreducible and is the restricted tensor product of the local small lifts. In this setting this is the cuspidal global Howe correspondence as recalled in Ichino, Section 6.3. Our nonzero vector is fixed by the good target hyperspecial subgroups. By Theorem 72.6 its archimedean type is P_{j+1}, and Theorem 72.16 and Proposition 72.17 give exactly the required parameter recursion at every good finite place. This closes the induction from Π_0. The hypotheses on a surviving row constituent follow from Section 72.2 and its hyperspecial levels at the inert places.

Finally fix a nonzero compact-finite vector in an obtained Π_j, spherical outside S. The common kernel of all lifts using the fixed vacua outside S and variable tests inside S and at infinity is stable under the finite groups in S, the real maximal compact groups and the complexified real Lie algebras. This follows by translating or differentiating the Schwartz

function, using cuspidal decay in the theta integral. On the subspace spherical outside S the resulting smooth, compact-finite module is irreducible: it is the tensor of its irreducible local factors at the indicated places with one-dimensional spherical lines elsewhere. The common kernel is proper by the nonzero norm just proved and hence is zero. Some test therefore gives a nonzero output on the fixed vector. Its output is compact-finite for compact-finite tests and again spherical outside S. Repetition proves the assertion about a nonzero composition. □

The source distinguishes the standard L-value from the nonzero upward integral in `Deve lopment/AbelianVarietyTheory/Definitions/LatticeFrames/ChainParSucc.lean`, line 215. Theorem 72.19 obtains an actual nonzero complex iteration with its stated choices. The estimate for the arithmetic periods calls for particular rational tensors, a prescribed projection and quantitative norms on the integral lattices. It must also handle each required word at the original finite level.

72.8 One iterated theta operator on the prescribed row

Fix a field, a translate, a signed real row and the splitting data of Theorem 72.19. Fix one unit compact-finite vector $u_{0,\tau}$ at each real place, and set $u_0 = \bigotimes_\tau u_{0,\tau}$. Choose the finite exceptional set S there to include all places $v \mid T$. Let $\mathbf{j} = (j_v)_{v|T}$ be the terminal depths of the initial Bessel periodization in Sections 86.8–93.2, with the first-edge functions ϕ_{1,j_v} and the indices I_{j_v} of Section 93.2. Put $I = \prod_{v|T} I_{j_v}$. The character κ_v on the terminal row compact subgroup is fixed. On each $v \in S$ choose a depth $a_v \geq 1$ on which all determinant characters at that place are trivial. At $v \mid T$ also require $a_v \geq j_v$ and triviality of the top-block character κ_v/ζ_v on $K_r(a_v)$, in the matrix frame of Section 93.2. Here ζ_v is the scalar splitting character in that frame. At a nonterminal place one may increase a_v so that its principal subgroup is contained in the prescribed source level.

Let $\mathcal{H}$ be the space of cusp vectors on the row group with real factor in the specified line, spherical outside S, of the prescribed fine type at $v \mid T$, and fixed by $K_r(a_v)$ at the other places of S. The real line is taken in each copy of the fixed irreducible archimedean representation in the cuspidal spectrum. Define

$$\mathcal{J}_v = \begin{cases} \{1, \dots, m-1\} & v \mid T, \\ \{0, \dots, m-1\} & v \in S,\ v \nmid T, \end{cases} \qquad P(\mathbf{a}) = \prod_{v \in S} (Nv)^{-a_v \sum_{j \in \mathcal{J}_v} n_j (n_{j+1} + n_j)}.$$

The missing first edge in $\mathcal{J}_v$ at T has its own normalization $I_{j_v}^{-2}$.

Lemma 72.20 (Real monomials on prescribed successive lines)**.** *Suppose that at least one cuspidal constituent of the specified row is present. There are monomial Fock tests $\phi_{j,\tau}$ and unit compact-finite vectors $u_{j+1,\tau}$ at all real places and steps such that the following holds. Contraction of $\phi_{j,\tau}$ against $u_{j,\tau}$ in the local theta correspondence spans the line of $u_{j+1,\tau}$. The diagonal real Rallis integral for these data is nonzero. Both assertions use only the fixed representations and real tests.*

Proof. Consider a single step and place, with source vector u. Denote the local Rallis pairing by $Z(\phi, u; \phi', u')$, in the convention in which its diagonal gives the local factor of the theta norm. Ichino's nonvanishing criterion, applied as in Theorem 72.19, makes this four-vector pairing nonzero. If $Z(\phi, u; \phi, u) = 0$ for every ϕ, then polarization in the test would give zero for $Z(\phi, u; \phi', u)$ for all ϕ, ϕ'. Left and right invariance of Haar measure in the local integral allow a translate of u in either slot to be moved inversely to that slot's Schwartz test. Consequently all pairings would vanish on the two spans of translates of u, and then on their smooth closures. The compact-finite module generated by a nonzero u is irreducible. One may

also apply Lie derivatives in this argument, by continuity of the convergent local integral. This would contradict the nonzero pairing.

For this fixed u the diagonal in ϕ has a constant sign after multiplication by a fixed nonzero scalar. To see this, use one reference cuspidal constituent from Theorem 72.19 at this step. Fix its other local data with nonzero diagonal factors, including an arbitrary nonzero diagonal choice at the other real places. Inserting the variable test at the present place expresses a global squared norm as a fixed nonzero scalar times $Z(\phi,u;\phi,u)$. Thus the resulting local sesquilinear form is positive semidefinite. If it vanished on each monomial diagonal, Cauchy–Schwarz would kill every pair of monomials. Polynomial Fock vectors are dense in the smooth oscillator, so the whole form would vanish, a contradiction. Choose a monomial with nonzero diagonal. The same global norm computation shows that its contraction in the local small lift is nonzero. Local Howe multiplicity one makes the line of that contraction independent of the choice of a nonzero quotient map. Choose a unit vector on it and repeat. The reference constituent has every successive lift by Theorem 72.19, so the argument applies at every step. □

Theorem 72.21 (Norm of the full complex iteration)**.** *There are common choices of the real monomials, independent of* $\mathbf{j},\mathbf{a}$*, and a positive constant* C_x *depending on the fixed field, row, set* S*, splittings, real lines and measure normalizations. Use normal lattice vacua outside* S*, the fine test* ϕ_{1,j_v} *on the first edge at each* $v\mid T$*, and on every other edge at each exceptional place the matrix indicator of Proposition 72.15 at depth* a_v*. Write* $\mathcal{T}_{\mathbf{a},\mathbf{j}}$ *for their composed global theta operator. For every* $A\in\mathcal{H}$*,*

$$e^{-\Delta_x}C_xI^{-2}P(\mathbf{a})\|A\|^2\ \le\ \|\mathcal{T}_{\mathbf{a},\mathbf{j}}A\|^2\ \le\ e^{\Delta_x}C_xI^{-2}P(\mathbf{a})\|A\|^2,\qquad \Delta_x=Cmfr\,2^{-(z_0-c_r)}.\tag{72.8}$$

Here $c_r=10(r+1)^3$*,* $z_0=(t+1)/2$*, and* C *is absolute. In particular the same operator detects every nonzero vector in the fixed row space. After the displayed level factors have been extracted* C_x *is independent of both depth vectors and of the finite cuspidal constituent.*

Proof. If there is no cuspidal representation of the specified real row and good spherical type at any of the levels, the assertion is vacuous. Otherwise fix a reference representation of this type. Choose the real data by Lemma 72.20. At a terminal place the support and compact character of the first test are preserved by the target principal group $K_{n_1}(a_v)$. Indeed the change of the integral position matrix under left multiplication is zero modulo $\varpi_v^{a_v}$; its invertible top block thus stays in the same coset modulo $\varpi_v^{j_v}$ and in the same top-block character. The lower block stays divisible by ϖ_v, and the target splitting character is trivial. Away from these first terminal edges, the congruence test of Proposition 72.15 preserves the target depth a_v for the next lift. All good places preserve their spherical lines.

First take the input in one irreducible cuspidal summand. The archimedean line at the next step is the one constructed in the lemma for every finite part of this summand. Indeed the global theta map is equivariant in each pair of local variables and factors locally through the unique irreducible theta pairing. For a pure tensor this fixes the target archimedean line, and finite sums give the same conclusion for an arbitrary finite part. With unit vectors on the lines, the real factors of the Rallis norm formula at each step are therefore fixed scalars.

At $v\mid T$ on the first edge the oscillator coefficient is $\|\phi_{1,j_v}\|^2\kappa_v\mathbf{1}_{L_{j_v}}$, where $L_{j_v}=(B_v\cap K_v)K_r(j_v)$ for positive depth and $L_0=K_v$. Integration against the source coefficient projects onto the prescribed character there. On that space the local factor is $\|\phi_{*,v}\|^2\,\mathrm{vol}(K_v)/I_{j_v}^2$. One index comes from the test volume and one from the group volume, as in Section 93.2. On the other exceptional edges the same calculation with a principal subgroup acts as a scalar on its fixed vectors. By (72.6) their product is $P(\mathbf{a})$ times a positive constant independent of all the depths. These projection identities hold for arbitrary tensors of fixed local type. They can be checked first for pure tensors, then by taking the corresponding mixed pairings.

At step $j \to j{+}1$, write $k = j{+}1$. In the unramified part of the continued norm formula from Theorem 72.19 the denominator is fixed, as are all factors $L_E^{S_E}(a, \mu_i \mu_k^{-1})$ of (72.3). The only factor varying with the constituent is $L^S(z_0, B_0 \otimes \mu_k^{-1})$, whose logarithmic absolute value has size $O(fr\, 2^{-(z_0 - c_r)})$. This bound uses the initial spherical parameters throughout. In particular its exponent does not contain the growing rank n_j. The Haar constant in the global norm formula is fixed for each pair of groups. Expressing it using the input Petersson norm removes any scalar used to identify an automorphic copy with its abstract unitary tensor product. Take the product of the absolute values of all the common factors at all steps, including the real factors and the terminal depth-zero constants, to define $C_x > 0$. Taking absolute values is legitimate since each complete identity is a squared norm. The logarithmic error is additive over the m applications, giving (72.8) in one summand.

For two orthogonal invariant cuspidal summands the lifts are orthogonal, also for independently chosen tests. In the half-plane of absolute convergence the mixed doubling integral unfolds against the matrix coefficient $\langle R(g)A_1, A_2 \rangle = 0$. Its continuation is identically zero. The convergent-range Siegel–Weil formula used in Theorem 72.19 consequently makes the mixed theta inner product zero. This argument also applies to equivalent representations in orthogonal multiplicity spaces. Varying both tests gives orthogonality of the invariant lifted spans, so it persists through the iteration. At any fixed level the real type and infinitesimal character in question occur in a finite-dimensional space of cusp vectors of that level and compact type. Summing the squared norms proves (72.8) on $\mathcal{H}$. □

Corollary 72.22 (The full iteration on the Bessel columns)**.** *Restrict the Bessel periodization $S_{\mathbf{j}}$ to the unit input line u_0 and its finite column domain, with depths as above. Write $B = \mathcal{T}_{\mathbf{a},\mathbf{j}} S_{\mathbf{j}}$ on that domain. Its kernel equals $\ker S_{\mathbf{j}}$, including at the original depth. For the exponent $\eta = L - 4fr^2 > 0$ of Section 86.8 and its field-dependent constant C'_x,*

$$\left\| \frac{I^3}{C_x P(\mathbf{a})} B^* B - 1 \right\|_{\mathrm{op}} \le (e^{\Delta_x} - 1) + e^{\Delta_x} C'_x T^{-\eta \sum_{v \mid T} j_v}.$$

At sufficiently large total terminal depth for each fixed large x, this gives a nonzero full complex iteration on every nonzero finite column.

Proof. The hypotheses of the theorem on the nonterminal source groups hold after choosing a_v inside their given levels. Apply (72.8) to $S_{\mathbf{j}} \ell$. Positivity gives the kernel assertion. The estimate $\| I S_{\mathbf{j}}^* S_{\mathbf{j}} - 1 \|_{\mathrm{op}} \le C'_x T^{-\eta \sum_v j_v}$ from Sections 86.8–93.2 continues to hold on the fixed unit real line by compression. Use its upper and lower quadratic-form bounds in the preceding inequality. If its lower bound becomes negative, positivity of $B^* B$ still gives the stated absolute error. Taking the supremum over unit columns gives the formula. At sufficient depth the Bessel bound alone gives injectivity, and the kernel identity transfers it to the iteration. □

Proposition 72.23 (Cost of the two auxiliary split primes)**.** *The prime representatives in Lemma 72.18 can be chosen above distinct rational primes outside $2N g_1 g_2 T$ so that, on the intervals of Section 6.6,*

$$\log N\mathfrak{p} + \log N\mathfrak{q} \ll_{\psi,\epsilon} x^c$$

for a constant $c = c(\psi, \epsilon)$. For the prescribed row with its original exceptional places and these two primes, put $A = \max(1, \max_{v \in S} a_v)$. Then

$$0 \le -\log P(\mathbf{a}) + 2 \log I \ll_{\psi,\epsilon} A x^{c'}$$

for a constant c' depending only on the fixed family. In particular, for depths bounded by a fixed power of x, this explicit logarithmic loss is $o(b/x^B)$ for every fixed $B > 0$, where $b = \exp(x^\epsilon)$.

Proof. Let H^+ denote the narrow Hilbert class field of F. Write D_F for the absolute discriminant and $f = [F : \mathbb{Q}]$. We first find a prime representing any given narrow class while excluding an arbitrary finite set U of rational primes. For $p \in U$ take k_p least positive with $\varphi(p^{k_p}) > f$, and put $M = \prod_{p\in U} p^{k_p}$. If $k_p > 1$ then $p^{k_p-1}(1-1/p) \le f$, hence $p^{k_p} \le 2fp$. This bound holds also for $k_p = 1$. Consequently

$$\log M \le \sum_{p\in U} \log p + |U| \log(2f).$$

In the abelian extension $L = H^+F(\zeta_M)$ of F, every place above $p \in U$ ramifies. In fact its completion contains $\mathbb{Q}_p(\zeta_{p^{k_p}})$, whose absolute ramification index $\varphi(p^{k_p})$ exceeds the absolute ramification index of any completion of F at p.

The norm of the finite conductor of every character of $\mathrm{Gal}(L/F)$ is at most M^f. To see this, embed this Galois group by restriction in the product of $\mathrm{Gal}(H^+/F)$ and $\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})$. A character of a subgroup of a finite abelian group extends to the whole group. Adjoining one generator at a time requires only taking a root of a complex number of absolute value one. The first component is unramified at finite places. By functoriality of Artin reciprocity the second component on F is the pullback of a Dirichlet character by the norm. For $v \mid p$ and $y \in \mathcal{O}_v$ the norm of $1 + p^{k_p}y$ is 1 modulo p^{k_p}, as follows from the determinant of multiplication on the free $\mathbb{Z}_p$-module $\mathcal{O}_v$. Away from M this character is unramified. Its finite conductor thus divides $M\mathcal{O}_F$.

Lift the chosen narrow class to an element of $\mathrm{Gal}(L/F)$ via the surjective restriction to H^+, identifying classes with their Frobenius elements. The unconditional least-prime theorem of Thorner and Zaman, Theorem 1.1 gives an unramified prime v of this Frobenius with rational prime norm. Indeed, for an abelian extension one takes the full Galois group for the abelian subgroup in their theorem. The fixed field is then F, and the quantity called $\mathcal{Q}$ is the maximum finite conductor norm just bounded. Taking logarithms gives

$$\log Nv \ll 1 + \log|D_F| + f \log M + f \log(2f)$$

with an absolute implied constant. Such a prime lies outside U, and its narrow class is the required one. If the class lies in the kernel of $\omega_{E/F}$, then it splits in E, since $E \subset H^+$.

Start with the primes dividing $2Ng_1g_2T$ and the class of $\mathfrak{d}_F$. Adjoin the rational prime below the resulting $\mathfrak{p}$ to U and repeat for the class of $\mathfrak{a}_0$. Both classes have trivial quadratic character by Lemma 72.18. The bounds on the common intervals give $\log|D_F| \le f \log N$ and bound $f, \log(2Ng_1g_2T)$ by powers of x. Also $|U| \le \sum_{p\in U} \log p/\log 2$. Inserting the first prime's logarithmic bound into the second proves the assertion for both representatives.

For the stated choice of S, the sum of $\log Nv$ over its original places is at most $f \log(2Ng_1g_2T)$, by the residue degree formula. The two extra terms were just bounded. Each sum of rank factors defining P is at most $2mn_m^2$. At a terminal place, $I_{j_v} \le |\mathrm{GL}_r(\mathcal{O}_v/\varpi_v^{j_v})| \le (Nv)^{j_v r^2}$ for $j_v > 0$, while $I_0 = 1$. Apply $j_v \le a_v$ and the polynomial bounds on the ranks and on m. This proves the loss estimate. Exponential growth of b gives the last assertion. □

The finite tests here are explicit integral-valued indicators or indicators times finite character values, and each real test is a Fock monomial in the chosen coordinates. The Petersson constant C_x includes the normalization of the real theta pairings. The Gram estimate controls the norm in the full complex target. Passage to the prescribed arithmetic projection requires a bound for the image in its lattice, including the finite valuations and the absolute archimedean normalization. The depth needed for injectivity from the estimate in Corollary 72.22 also depends on C'_x.

72.9 The first global theta lift of the Bessel row

Keep the choices of Theorem 110.2. For the signed row (87.4) fix a pure unit vector u_∞ in its lowest compact type. At a terminal depth $j_v \ge 0$ use the residue test of Theorem 83.12, with the coarse test at depth zero. Put $I = \prod_{v|T} I_{j_v}$. The following period concerns the first forward lift $U(Y_0) \to U(Y_1)$ of the indefinite chain.

Theorem 72.24 (The first theta period at the prescribed terminal tests)**.** *Fix x on a common interval of Theorem 20.11, a translate and a compatible unitary splitting with the prescribed real exponents. A common choice of tests away from T and a positive real number C_x give*

$$e^{-\delta_x} C_x I^{-2} \|A\|^2 \;\le\; \|\Theta_{\boldsymbol{j}} A\|^2 \;\le\; e^{\delta_x} C_x I^{-2} \|A\|^2, \quad \delta_x = 4[F:\mathbb{Q}]\, 2^{-\lfloor (w-2)/2 \rfloor + 4r^2},$$

where $F = E^+$, $w = r + t_{\mathrm{r}}$, and $A = S_{\boldsymbol{j}}(u_\infty \otimes \ell)$ for any finite class column ℓ. The same tests and C_x work for every depth and every word with the fixed row, splitting and levels away from T. Every nonzero such lift is cuspidal. For sufficiently large total depth at each fixed x, the inequality gives a positive first period for every nonzero ℓ.

Proof. At a real place the first target character has half angular exponent $-e_1$. Centering (87.4) gives k positive and s negative entries in the plus block, and $s+1$ negative entries in the minus block. All their absolute values are at least $H_0 > (t_{\mathrm{r}} + 3)/2$. Theorem 72.1, applies the real theta nonvanishing theorem to these four counts. The first target in this Witt tower is the one of signature $(w - s, s)$. The lower lifts vanish, and the constant-term formula puts the global lift of a cusp form into the cuspidal spectrum. The square-integrable row A of Theorem 110.2 is cuspidal by the tempered multiplicity theorem, as proved in Theorem 72.4.

In the convergent doubled inner product formula, the local quadratic form on each terminal type has scalar $\|\phi_{*,v}\|^2 \operatorname{vol}(K_v) I_{j_v}^{-2}$. At infinity one can choose a Fock monomial with positive scalar on the line of u_τ. At each bad nonterminal place a matrix neighborhood of an integral frame gives a positive scalar on the entire specified level. The logarithm of the product of all other local factors has absolute value at most δ_x, uniformly in the cuspidal constituent. Lemma 72.3 and Corollary 72.5, prove these local estimates and sum the positive forms over the spectrum. This gives the asserted norm inequality. Lastly Theorem 83.12 bounds $\|I S_{\boldsymbol{j}}^* S_{\boldsymbol{j}} - 1\|_{\mathrm{op}}$ by $C_x' T^{-\eta \sum_v j_v}$, with $\eta > 0$. For fixed x this is less than one at sufficient depth. For a nonzero input take the Petersson pairing of its lift with itself to obtain the nonzero period. □

Corollary 72.25 (The scale of the first lifted norm)**.** *Put $d_{\boldsymbol{j}} = C_x' T^{-\eta \sum_v j_v}$. When $d_{\boldsymbol{j}} < 1$, each nonzero ℓ satisfies*

$$e^{-\delta_x}(1 - d_{\boldsymbol{j}}) \;\le\; \frac{I^3 \|\Theta_{\boldsymbol{j}} S_{\boldsymbol{j}}(u_\infty \otimes \ell)\|^2}{C_x \|\ell\|^2} \;\le\; e^{\delta_x}(1 + d_{\boldsymbol{j}}).$$

At the original level a common first theta test detects any column with $S_0(u_\infty \otimes \ell) \ne 0$.

Proof. The Gram bound in the preceding proof puts $I\|A\|^2$ between $(1 - d_{\boldsymbol{j}})\|\ell\|^2$ and $(1 + d_{\boldsymbol{j}})\|\ell\|^2$. Multiply the two inequalities of Theorem 72.24 by I^3 and substitute. Its lower bound at $I = 1$ proves the last assertion without a depth estimate at zero. □

The choices away from T give a complex first lift. The number C_x includes real integrals of the selected monomials and finitely many volumes. We still need a surviving ramified column at the original level for every required word, and a comparison with the *specified*, arithmetically normalized periods along the entire chain. The first-lift norm inequality provides neither their valuations nor the remaining strict cohomology orders in (19.15).

72.10 The entire theta chain at the real places

The first-period bound of Theorem 72.24 uses the prescribed real row and an actual global first lift. We can also determine all its successive *real* theta lifts. For $0 \le j \le m$ put

$$n_j = r + jt_{\mathrm{r}}, \quad s_j = \lfloor (m-j)/2 \rfloor, \quad h_j = -\tfrac{1}{2}\mathbf{1}_{j \text{ odd}}, \quad d = (t_{\mathrm{r}} - 1)/2.$$

At a real place form a signed list as follows. Take the v_i with sign plus. Include $-b_i$ with sign plus for $i \le j+1$ and alternating signs starting with minus for $i = j+2, \dots, m$. For each $1 \le i \le j$ add $-e_i + d, -e_i + d - 1, \dots, -e_i - d$, all with sign plus. Finally translate all entries by h_j. Call the result P_j.

Theorem 72.26 (The archimedean representations through the compact endpoint)**.** *On the same fields, intervals and weights as Theorem 20.11, the real row with parameter (87.4) lifts at each successive step to the discrete series with parameter P_j on $U(n_j - s_j, s_j)$. At each such step every smaller target in the same local Witt tower gives zero. At $j = m$ one obtains the irreducible compact representation whose highest weight is $\big(\lambda_i - (n_m+1)/2 + i\big)_{1 \le i \le n_m}$, where $\lambda_1 > \cdots > \lambda_{n_m}$ lists P_m. At every real place, monomial Fock tests exist along the entire chain with nonzero composed local pairing between any prescribed nonzero initial compact-finite vector and any nonzero final covector.*

Proof. For the step $j \to j+1$ its two half angular characters are $-e_{j+1} + h_j$ on the target and $-e_{j+1} + h_{j+1}$ on the source. Subtract the former from P_j. All $k + j(t_{\mathrm{r}} + 1)$ positive centered entries are in the plus block. The negative entries consist of s_{j+1} plus entries and s_j minus entries, all of absolute value at least H_0. Theorem 72.6, substitutes these counts into Atobe's first-occurrence criterion. The parabolic in Ichino's formula for that lift has a new compact block of size t_{r}. The old negative entries exchange their blocks, and the new plus string is centered at $-e_{j+1} + h_{j+1}$. Adding the half sum of the positive roots to the inducing weight gives exactly P_{j+1}. This proves the induction from P_0, and at the compact endpoint subtracting that half sum gives the stated highest weight. Corollary 72.7, constructs the local tests by successively contracting the irreducible oscillator quotients. Expanding the resulting polynomials proves the monomial assertion. □

For the global period these computations determine all the real representations. They leave a finite-place and arithmetic question at each subsequent lift. Already on the second pair, the source rank is $r + t_{\mathrm{r}}$ and the target rank is $r + 2t_{\mathrm{r}} < 2(r + t_{\mathrm{r}})$. Thus the large-rank bound used for the absolute convergence of the first doubled Eisenstein series in Theorem 72.4, does not extend by the same argument. A nonzero value for a product of local monomials also leaves undetermined its normalization against the specified global integral lattices.

72.11 The successive standard L-values on the common fields

Use the ranks and weights of Theorem 72.26. Put $t = t_{\mathrm{r}}$, $d = (t-1)/2$, $z_0 = d + 1$, and $f = [E^+ : \mathbb{Q}]$. At a finite prime w of E, away from the omitted set S_E, write $Q_w = Nw$. Let μ_i be the unitary splitting characters of half angular exponent $-e_{i,\tau}$, trivial on the ideles of E^+. Write $D_j = \eta^{-\mathbf{1}_{j \text{ odd}}}$. At step i the source and target splitting characters are $\mu_i D_i$ and $\mu_i D_{i-1}$, respectively.

Theorem 72.27 (The partial standard factor through the last step)**.** *Start with the standard spherical parameters B_0 of a unitary constituent of the initial row outside S_E. At each step i, multiply the existing parameters by D_i/D_{i-1} and append the segment with terms $(\mu_i D_i)(w) Q_w^a$*

for $-d \le a \le d$. Denote the result by B_i. For $0 \le j < m$, $k = j+1$, the partial standard L-function of $B_j \otimes (\mu_k D_j)^{-1}$ has the following nonzero value at z_0:

$$L^S(z_0, B_0 \otimes \mu_k^{-1}) \prod_{i=1}^{j} \prod_{a=1}^{t} L_E^{S_E}(a, \mu_i \mu_k^{-1}).$$

The characters can be chosen on the same E with finite ramification only at existing auxiliary or conductor primes. For this choice, and S_E above the primes of $2Ng_1g_2T$, the sum over $j < m$ of the absolute logarithms of the moduli of these values is bounded by a polynomial in x throughout the common intervals. Here $E \subset \mathbb{Q}(\zeta_N)$.

Proof. In the recursion all the entries at step j have a common factor D_j. Dividing it out and shifting the variable in each Euler product gives the displayed formula at first far to the right. The initial spherical roots satisfy $|\alpha| \le Q_w^{10(r+1)^3}$ by Lemma 72.8. Hence the initial factor has an absolutely convergent Euler logarithm of size $O(fr\,2^{-(z_0-10(r+1)^3)})$ at the required point. Each ratio character $\rho = \mu_i \mu_k^{-1}$ has angular exponent $4(k-i)H_0$ at every selected embedding. Hecke continuation applies both to ρ and to ρ^2. For the integer $a = 1$, Lemma 72.9, combines the functional equation with positivity of $3 + 4\cos u + \cos(2u)$ to give

$$\left|\log |L_E^{S_E}(1, \rho)|\right| \ll 1 + \log(2|D_E| N\mathfrak{f}_\rho) + f \log(3 + 4mH_0) + \#S_E.$$

The Euler logarithms of all the higher integer factors for one pair contribute $O(f)$ in total. There are fewer than $m^2/2$ pairs. Corollary 72.11, constructs the characters with $N\mathfrak{f}_\rho \le (g_1g_2T)^{2f}$. Together with $\log |D_E| \le 2f \log N$ and the common-family parameter bounds this proves the assertion. □

The partial standard value is the scalar denoted by the nonvanishing condition at `Develo pment/AbelianVarietyTheory/Definitions/LatticeFrames/ChainParSucc.lean`, line 128. The global theta integral on the designated input is a separate quantity in line 215 of that file. Establishing its arithmetic comparison still requires a surviving original-level column for each word and control of the projected rational periods. The remaining strict cohomology lengths are those in (19.15).

72.12 The trace pairing at the finite places

Write $F_+ = E^+$ in this paragraph and retain the fields and intervals of Theorem 20.11. The integral tensor of two Hermitian lattices enters the oscillator through its alternating trace form. Even a unimodular Hermitian lattice can have a nontrivial duality index for the trace down to $\mathbb{Q}$. This index can be treated simultaneously at all inert places.

Theorem 72.28 (The index for a single global trace pairing)**.** *There exists a trace-zero scalar $\delta \in E^\times$ of the required sign at every selected CM embedding, usable at all m steps. At any inert prime v the integral tensor lattice $L_{j,v}$ at step $j \to j+1$ has a self-dual homothety $L'_{j,v} \supseteq L_{j,v}$ for this trace pairing. The original compact stabilizers on both sides preserve this homothety, and*

$$\sum_{j=0}^{m-1} \sum_{v \text{ inert}} \log[L'_{j,v} : L_{j,v}] \;\le\; \Big(\tfrac{1}{2}\log|D_E| + [F_+ : \mathbb{Q}]\log(4/\pi)\Big) \sum_{j=0}^{m-1} n_j n_{j+1}.$$

The splitting characters can also be chosen unramified at every inert prime. At any finite collection of split primes, for source vectors fixed at a principal congruence depth on which the splitting characters are trivial, one can use the same depth at each subsequent step. For a

split prime of residue cardinality q_v and depth $a \geq 1$, the resulting local integral on a fixed unit vector from rank n to rank $n+t$ is

$$q_v^{-a((n+t)n+n^2)} \prod_{\ell=1}^{n} (1 - q_v^{-\ell})^{-1}.$$

The measures here give volume one to the full matrix lattice and to the integral general linear group, respectively.

Proof. Let $\delta_0 = \sqrt{-d_0}$ and $\mathfrak{a}_0 \mathcal{O}_E = (\delta_0)$. Lemma 72.12, proves that $\mathfrak{d}_{F_+}\mathfrak{a}_0$ is a narrow ideal norm class from E. For the different the obstruction is the finite product of local root numbers of ω_{E/F_+}, which is one since $4 \mid [F_+ : \mathbb{Q}]$. For $\mathfrak{a}_0$ it is one because d_0 splits in $\mathbb{Q}(\sqrt{2})$. Choose an integral ideal $\mathfrak{b}$ in a preimage class by Minkowski's theorem and write $(\alpha)\mathfrak{d}_{F_+}\mathfrak{a}_0 = N_{E/F_+}\mathfrak{b}$ with α totally positive. With $\delta = \alpha\delta_0$, at $w \mid v$ inert the dual of $L_{j,v}$ is $\varpi_v^{-2v_w(\mathfrak{b})} L_{j,v}$. Half this exponent gives the claimed homothety. Its index is $(Nw)^{v_w(\mathfrak{b})n_j n_{j+1}}$. Multiplication and Minkowski's bound give the displayed inequality, as in Theorem 72.13.

At the finite places above d_0, the original η-character on residue units factors through $z/\bar{z}$. Proposition 72.14, cancels it with a single character of the norm-one torus, using the existing split prime g_1 to enforce compatibility on roots of unity. For the final assertion take the indicator of matrices congruent to the first n coordinate columns modulo ϖ_v^a. Its overlap with a source translate is supported on the depth-a subgroup, on which that overlap is exactly the additive volume of its support. This gives the integral by the finite general linear group order and fixes the same target depth. The calculation is in Proposition 72.15. □

The theorem permits a choice of the global trace scalar and uses chosen matrix tests at split places. Its polynomial bound applies to the local trace-lattice indices. Identification with the specified projected global periods, including their arithmetic divisibility, remains necessary in Section 6.6.

72.13 A nonzero iteration over the complex numbers

There is a qualitative global consequence of the preceding calculations. It applies even at the steps where the doubled Eisenstein series is outside its half-plane of absolute convergence. The choices of finite tests and the scope of the conclusion are part of the statement.

Theorem 72.29 (The global theta chain of a surviving row constituent)**.** *On the fields and intervals of Theorem 20.11, let Π_0 be an irreducible cuspidal constituent of a nonzero row S_j with the levels of Theorem 72.24. There are a global additive character, a choice of skew pairing of the required real signs, and splitting characters with the prescribed real exponents, for which the forward theta chain reaches the compact endpoint with nonzero cuspidal lifts at every step. One can fix the unramified vacua outside a finite set of split finite places (and vary the archimedean Schwartz functions). The finite set may include two additional split prime ideals. The archimedean representations are those of Theorem 72.26, and the good finite parameters are those of Theorem 72.27. On any nonzero compact-finite input in the constituent fixed at the good places, a choice of successive tests gives a nonzero complex cusp form at the endpoint.*

Proof. Lemma 72.18, chooses split prime representatives for the separate narrow ideal classes of the different of E^+ and the ideal underlying $(\sqrt{-d_0})$. Rescaling the additive character and skew scalar by totally positive elements makes both unramified at every inert place. The splitting characters of Proposition 72.14, are also unramified there. At an inert place, taking the Jacquet module of the oscillator along an isotropic line gives two terms. One removes the

outer two parameters of the new segment. The other matches a reciprocal pair on the two sides. This gives the parameter update for every spherical constituent, even after the first step when temperedness is unavailable. At a split place the same update follows from the normalized matrix correspondence and the two dual components of the Hermitian space. The calculations are Theorem 72.16 and Proposition 72.17.

For the step from rank n_j to $n_j + t$, the global Witt index of the target is at most its smaller real signature $s_{j+1} < t$. The doubled theta integral is therefore in Weil's convergent range. Every smaller global lift vanishes by its real first occurrence, so the tower property gives cuspidality. The Rallis norm formula uses the continued open-orbit integral at $s = t/2$. Its unramified numerator is the nonzero partial standard value of Theorem 72.27. Its denominator has an absolutely convergent Euler product there. All the exceptional finite places are split, where Theorem 72.28 gives nonzero integrals on fixed vectors. For a real discrete series its nonzero local lift gives a nonzero local integral by Ichino's criterion. Thus the global norm is nonzero and the cuspidal lift is irreducible, with the local components just computed. This permits induction through all m steps. In an irreducible constituent the common kernel over the allowed tests is invariant and proper, hence zero. Theorem 72.19, gives the continuation and kernel arguments in detail. □

The new conclusion is existence for a choice of Schwartz functions on a surviving irreducible constituent. The prime count in (W) requires the projected values for the designated rational theta tensors with their prescribed integral normalization. In addition, the argument requires a surviving ramified column for every word at the original level and the strict cohomology estimate in (19.15).

72.14 One theta operator for all cuspidal inputs of a row

Fix a field and translate as in Theorem 20.11, a signed real row, and one unit vector in its lowest compact type at each real place. Use the splittings and separately unramified global data of Theorems 72.28 and 72.29. Let S consist of the exceptional split places, including those above T and the two auxiliary places. Denote the terminal Bessel depths by $j_v \geq 0$. Write $I = \prod_{v|T}[K_v : L_{j_v}]$ with the groups of Theorem 83.12. At each $v \in S$ take a principal depth $a_v \geq 1$ inside the kernels of the compact determinant characters. At $v \mid T$ require also $a_v \geq j_v$ and triviality of the top-block character on $K_r(a_v)$. At each other exceptional place take this subgroup inside the source level. The rank at step j is $n_j = r + jt$, where $t = t_{\mathrm{r}}$.

Theorem 72.30 (A common comparison through the last lift)**.** *Let $\mathcal{H}$ be the cuspidal space on this real line with the specified terminal character at L_{j_v}, principal fixed vectors at the other exceptional places, and spherical vectors elsewhere. There is a single iterated theta operator $\mathcal{T}$ on this space using monomial Fock tests at infinity, the prescribed terminal tests on the first edge, and explicit congruence tests at every other exceptional finite edge. Put*

$$\mathcal{J}_v = \begin{cases} \{1, \dots, m-1\} & v \mid T, \\ \{0, \dots, m-1\} & v \in S,\ v \nmid T, \end{cases} \qquad P = \prod_{v \in S} (Nv)^{-a_v \sum_{j \in \mathcal{J}_v} n_j (n_j + n_{j+1})}.$$

For a constant $C_x > 0$ independent of both depth vectors and of the finite cuspidal constituent, every $A \in \mathcal{H}$ satisfies

$$e^{-\Delta_x} C_x I^{-2} P \|A\|^2 \leq \|\mathcal{T} A\|^2 \leq e^{\Delta_x} C_x I^{-2} P \|A\|^2, \qquad \Delta_x = C m f r\, 2^{-((t+1)/2 - 10(r+1)^3)}.$$

Here C is absolute. For the original row the two auxiliary primes can be chosen so that $-\log P + 2 \log I \ll_{\psi,\varepsilon} x^c \max_{v \in S} a_v$ on these same fields and intervals.

Proof. Choose the real tests successively. For the unit source vector at a real place the local Rallis form is nonzero by cyclicity and the discrete-series nonvanishing theorem. One reference global constituent from Theorem 72.29 realizes a fixed nonzero multiple of its diagonal as a squared norm. Cauchy–Schwarz and the density of Fock polynomials then give a monomial with nonzero diagonal. The unique irreducible local theta pairing determines the next unit line. This choice uses only the real representations. When the specified cusp space is zero the estimate is vacuous.

At the first terminal edge the local integral is a scalar multiple of the projector onto the prescribed character. Its scalar is the depth-zero scalar times $[K_v : L_{j_v}]^{-2}$. The top-block support and character survive left multiplication by $K_{n_1}(a_v)$. On the other exceptional edges the integral of Theorem 72.28 supplies the displayed power of Nv and preserves the target principal depth. In the unramified formula of Theorem 72.27 all factors are common except the one from the original spherical row. Its logarithmic absolute value is $O(fr2^{-((t+1)/2-10(r+1)^3)})$ at every edge. Multiply the m norm identities. Orthogonal invariant cuspidal summands have orthogonal lifts because their mixed doubling integral unfolds to a zero matrix coefficient and hence has identically zero continuation. This permits summing the norm inequalities over the whole row.

For the prime bound take the two narrow ideal classes of the different and of the ideal below $(\sqrt{-d_0})$. To avoid previous rational primes, adjoin to the narrow Hilbert class field the roots of unity of orders p^{k_p} with $\varphi(p^{k_p}) > [E^+ : \mathbb{Q}]$ at those primes. Every one of them ramifies, while finite conductor norms of all the abelian characters are at most $(\prod p^{k_p})^{[E^+:\mathbb{Q}]}$. The unconditional least-prime theorem of Thorner and Zaman now bounds the logarithmic norm of each representative by a power of x. Both classes split in E. Counting matrix entries in the finite congruence subgroups gives the asserted depth cost. The full proofs of the local pairing, the simultaneous iteration and the least-prime estimate are in Section 72.8. □

Corollary 72.31 (Kernel on the finite Bessel columns)**.** *On the finite column space at the fixed real line let $S_{\mathbf{j}}$ denote the Bessel periodization and put $B = \mathcal{T} S_{\mathbf{j}}$. Then $\ker B = \ker S_{\mathbf{j}}$, including at the original depth. With the exponent $\eta > 0$ and field constant C_x' of the terminal-depth estimate, one has*

$$\left\| \frac{I^3}{C_x P} B^* B - 1 \right\|_{\mathrm{op}} \le e^{\Delta_x} - 1 + e^{\Delta_x} C_x' T^{-\eta \sum_{v \mid T} j_v}.$$

Proof. The lower bound of the theorem gives the kernel equality. Insert the upper and lower quadratic-form bounds for $I S_{\mathbf{j}}^* S_{\mathbf{j}}$ from Theorem 110.3 and the simultaneous-depth calculation of Section 93.2. Taking the supremum over unit columns gives the displayed inequality. □

Thus a column which survives at the original level can be detected through the compact endpoint by the common tests. The prime count requires a surviving ramified column for every word there, with a nonzero pairing against the specified arithmetic projection and a bound for its integral size. The remaining strict lengths in (19.15) also enter the estimate for (W).

73 Terminal Hecke operators and the first global lift

73.1 Terminal covectors through the definite theta chain

Take the base auxiliary tuple and the definite column used in Theorem 127.1. Keep its chosen period field E_x, its word, the marks at g_1, g_2, and $S_T = \{P : P \mid \pi\}$ for the same terminal generator π. We compute the finite sum on this column before comparing it with the analytic Bessel functional. At a vertex retain the lattice M_i, its frames and levels away from T, its

finite type index and all its coefficient data. A terminal refinement is a tuple $f_i = (f_{i,P})_{P\in S_T}$ of nonzero linear forms on $V_{i,P} = M_i/PM_i$. The unrestricted coefficient vector space at the vertex is common to these refinements. Its integral lattice can depend on the kernels of the $f_{i,P}$.

An edge term uses a rational matrix $x_i : M_i \otimes E \to M_{i+1} \otimes E$. Its terminal factor is one precisely when x_i is integral at the indicated P's and

$$f_{i,P} = f_{i+1,P}\,\overline{x}_{i,P} \qquad (P \in S_T),$$

and zero otherwise. Here $\overline{x}_{i,P} : V_{i,P} \to V_{i+1,P}$ is reduction. This follows directly by applying the two covectors to every element of the local lattice. In the finite theta sum the matrix ranges over the quotient of $c_A^{-1}\mathcal{H}_i$ by $c_B\mathcal{H}_i$, where $\mathcal{H}_i$ is the module of rational homomorphisms integral between the local lattices at the indicated finite places. The denominator c_A is prime to T, and $T \mid c_B$. Thus reduction of the matrix at S_T is defined on each coset. This quotient is finite: localizing at the finite set of places gives projective modules over a semilocal Dedekind ring, and reducing the homomorphism module modulo c_Ac_B gives a product of finite residue modules. This description also follows by choosing bases over each of the finitely many local rings and applying the Chinese remainder theorem.

The other factors in an edge term are its coefficient-valued theta distribution on that matrix coset and its local tests at the other marked places. They use M_i, M_{i+1} and the prescribed frames there. A lift of a frame at $w \mid g_1g_2$ is specified by its lattice and its reduction at w, so the same lift can be used for every T-refinement. In the final two coupled edges the additional vector coset is taken from the intermediate lattice with bounds only at the other marked places. Its tests on the two edges use this vector coset and the same frames. In particular these factors have no dependence on the intermediate terminal covectors. The edge sums and the three terminal indicators are given in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 636, lines 1115–1139 and 1162–1170 there.

Fix unrefined initial class representatives a. Fix a small test F at the endpoint, possibly after its finite type projection. Write γ for a choice of the initial and intermediate unrefined classes, the final small class b, the finite type indices, the matrix cosets at each edge, and any coupling coset used by the two adjoining edges. Write $W_{\gamma,F}$ for the following linear functional on the initial coefficient space. Compose the coefficient maps of these matrix cosets in their original order and apply the terminal lowering L_z. Contract with $F(b)$, and multiply by the product of the signed local tests outside T, including the weights of the chosen finite linear combinations. Divide by $|\Gamma_b|\prod_{i<m}|\Gamma_{a_i}|$, where m is the number of edges, Γ_b is the small class stabilizer and the earlier stabilizers are at the unrefined levels. All changes of frame use the same class representatives in the two incident maps. This prescription retains the coefficient maps of the actual cosets, including their dependence on the matrix residues at T.

Theorem 73.1 (Summing the intermediate terminal refinements)**.** *Let $e_N^\vee$ read the normal coordinate at the lifted final class $i(b)$. For a path starting at a, put*

$$\eta_{\gamma,P} = (\bar{\pi} \bmod P)\,\overline{e_N^\vee}\,\overline{x}_{m-1,P}\cdots\overline{x}_{0,P}.$$

Discard a path if any $\eta_{\gamma,P} = 0$. The signed period on an initial column vector ϕ is exactly

$$\mathcal{P}(\phi, F) = \sum_a \sum_{\gamma \text{ starting at } a} W_{\gamma,F}(\phi(a, \eta_\gamma)),$$

using the remaining paths and the stated finite class measure. In particular every path whose composite maps into the final marked hyperplane at even one of the places contributes zero. If the terminal covectors of the lifted class are scaled by $c \in A_T$ in this finite recurrence, the same formula reads the initial column at $c\eta_\gamma$.

Proof. Above an unrefined representative a_i the refined classes are the orbits of the finite group Γ_{a_i} on the tuples of nonzero covectors. The refined stabilizer is the stabilizer of the tuple in this action. In a contracted class sum a change by Γ_{a_i} transports the preceding and following matrix variables, and their coefficient transports cancel on the shared value. At the initial vertex the same statement uses the defining equivariance of ϕ. Reindexing the adjacent coset sums therefore gives a function constant on each orbit. Orbit–stabilizer replaces the sum over refined orbits with their inverse stabilizer orders by $|\Gamma_{a_i}|^{-1}$ times the sum over all tuples above this representative. One may make this replacement at every vertex, or take the product of the finite stabilizers and make it once on all the vertices. At the final small vertex the covector is already prescribed by the normal marking.

Now fix the remaining matrix choices and coupling data. Only the displayed edge indicators and ϕ depend on the intermediate covectors. At each P start with the final covector $(\bar{\pi} \bmod P)\overline{e_N^\vee}$ and pull it back through the matrices in reverse order. There is exactly one possible value at every vertex. If any pullback is zero, all the pullbacks back to the initial vertex are zero there. Conversely, nonzero initial pullback implies that all these values are admissible. The product of all the terminal indicators sums to one for that initial refinement, and to zero for every other one. Interchanging the finite sums gives the formula with precisely the unrefined masses already obtained. Multiplying the final covector by c_P multiplies every one of its pullbacks by c_P and changes none of the other factors. This gives the last assertion. □

73.2 Fourier transformation of the definite matrix recurrence

Keep the matrices, signed coefficient maps and class measures of Theorem 73.1. At a lattice M_i of rank n_i put $\Omega_i = \prod_{P|\pi} M_i/PM_i$ and $Q_i = |\Omega_i| = T^{fn_i}$, with $f = |S_T|$. Use $\Omega_i^\vee = \prod_{P|\pi}(M_i/PM_i)^\vee$. Choose the prescribed nontrivial residue additive character ψ_P at each place and write $\psi(\xi(v)) = \prod_P \psi_P(\xi_P(v_P))$. Define the unnormalized transform and its opposite by

$$(\mathcal{F}_i H)(v) = \sum_{\xi \in \Omega_i^\vee} \psi(\xi(v)) H(\xi), \qquad (\mathcal{F}_i^- G)(\xi) = \sum_{v \in \Omega_i} \psi(-\xi(v)) G(v).$$

A separately zero-sum table has sum zero on every scalar orbit in each coordinate. Extend it to the full product by zero wherever a coordinate vanishes. We use this convention on both sides.

For a residue matrix tuple $C : \Omega_i \to \Omega_{i+1}$ and the coefficient map W of that matrix term, the covector edge acts by $(K_C H)(\xi') = W H(\xi' C)$. On vector tables use

$$(K_C^{\rightarrow} G)(w) = \sum_{v:\, Cv = w} W G(v).$$

Denote by K the definite recurrence up to its large terminal table, before evaluating at the lift of a small test class. Use $K^{\rightarrow}$ for the same sum with the vector operations on its matrix terms. This keeps the order of the coefficient maps, the unrefined class weights, the common vector coset in the two coupled edges and all local tests outside T.

Theorem 73.2 (Exact vector recurrence with the original theta coefficients)**.** *On the separately zero-sum tables one has*

$$\mathcal{F}_m K = \frac{Q_m}{Q_0} K^{\rightarrow} \mathcal{F}_0, \qquad K = Q_0^{-1} \mathcal{F}_m^- K^{\rightarrow} \mathcal{F}_0.$$

In the class Hilbert norms $Q_i^{-1/2}\mathcal{F}_i$ is unitary between the two tables. In particular, when the large terminal value is read at a lifted small class b, its value before lowering is the sum of $(K^{\rightarrow}\mathcal{F}_0 H)(b, v)$ over the terminal vector tuples with coefficients $Q_0^{-1}\psi(-f_b(v))$, where $f_{b,P} =$

$(\bar{\pi} \bmod P)\overline{e_N^\vee}$. *The only Fourier inversion denominator in this evaluation is* $Q_0 = T^{fk}$, *whose logarithm is polynomially bounded on the period intervals.*

Proof. For each fixed vector, summing the characters over the full dual group gives its order at the zero vector and zero otherwise. For a nonzero vector this follows by translating the summation by a covector with nontrivial character value. This proves both $\mathcal{F}_i^- \mathcal{F}_i = Q_i I$ and the norm identity. At a zero vector coordinate the transform of an extended input vanishes, because the character there is constant on each scalar orbit. At a nonzero coordinate, summing the output over scalar multiples sums the input over the corresponding scalar action, by a change of variables. The result is zero. The same argument with the inverse shows that the two zero-sum spaces correspond. These transforms commute with changes of frame when the vector and covector frames are dual. At a fixed unrefined class the refined mass norm is the sum of all point norms divided by the unrefined stabilizer order, by orbit–stabilizer. This proves unitarity also on the equivariant class tables.

Insert $H = Q_i^{-1} \mathcal{F}_i^- G$ into K_C and apply $\mathcal{F}_{i+1}$. For each v, w the inner character sum over ξ' is that of $\psi(\xi'(w - Cv))$. It equals Q_{i+1} exactly when $w = Cv$. This proves the first identity on one edge, without a rank condition on C. The covector formula already gives zero if a pulled-back coordinate vanishes. Thus it uses exactly the terminal indicators of the original edge. The vector formula preserves the zero-sum space by Fourier conjugacy. In a product of edges the factors Q_{i+1}/Q_i telescope. Apply this term by term with the coupling data fixed, then sum the signed maps and class weights as in Theorem 73.1. Inverting the last transform gives the second formula and evaluation at f_b. Finally f, k and $\log T$ have polynomial bounds by Theorem 9.13. □

73.3 The terminal Hecke operator on the marked initial classes

Fix an ordinary word on the fields of Theorem 9.13. Write S for its full separately zero-sum integral initial table, summed over the ordinary classes. Its rank is d. At a chosen coefficient place $w \mid T$, let A_w be the set of $P \mid \pi$ induced at w by a high embedding τ. Write $s = |A_w|$, and use $\tau(P)$ for this embedding. The initial weights have $\lambda_{\tau,k} = -b_*$ at each such place. The operation required on this branch of the integral construction is H_w^m, where

$$H_w = T^{sb_*} \prod_{P \in A_w} U_P.$$

Here U_P sums over upward neighbors with unit coefficient. We give its local definition below. The scalar, the selected places and the power m are specified in `Formalism/LocalToGlobal/Columns/ScaledRamifiedLattice.lean`, line 219, and the use of the resulting image at a prime over T is in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1336. The initial lattice is the Weyl lattice of class functions with zero scalar sums at every marked prime, as in `Formalism/LocalToGlobal/Columns/ColForms.lean`, line 108 and `Formalism/LocalToGlobal/Columns/ScaledRamifiedLattice.lean`, line 159.

Theorem 73.3 (Incidence, integrality and determinant of the terminal Hecke move)**.** *The operator* H_w *maps* $S_w = S \otimes \mathcal{O}_{J_w}$ *into itself and is invertible on its fraction space. In the valuation* $v_w(T) = 1$ *it satisfies*

$$\frac{1}{d}\, v_w \det H_w = sb_* + \frac{s(k-1)}{2} + \frac{1}{k} \sum_{P \in A_w} |\lambda_{\tau(P)}|.$$

Here $|\lambda| = \sum_i \lambda_i$. *At each complex embedding all its singular values in the ordinary class Hilbert space are* $T^{sb_* + s(k-1)/2}$. *For* $s = 0$ *the map is the identity.*

Proof. At a split pair $P, \bar{P}$, write M for the rank k position lattice at P and $M^\vee$ for its partner under the perfect Hermitian pairing. Start with a nonzero marked covector c on M/TM. The neighbors read by U_P are indexed by $x \in M/TM$ with $c(x) = 1$. The position lattice of such a neighbor is $L_x = M + \mathbb{Z}_T T^{-1}\tilde{x}$ with $\tilde{x}$ any lift, and its partner is $L_x^\vee$. Its covector f_x vanishes on M and takes value one on $T^{-1}\tilde{x}$. Indeed $f_x(y) = c(Ty \bmod TM)$ on L_x. Away from this pair the lattices and markings carry over by the identity. This is the marked neighbor definition in `Formalism/LocalToGlobal/Columns/StdLatUnimodular.lean`, line 139, lines 145–163 there. Intersecting the prescribed local lattices in the rational space produces a global lattice with those localizations, by the Chinese remainder theorem after clearing a common denominator. The perfect duals at conjugate places give self-duality. A basis at the position place and its dual at the other place identify the two local Hermitian lattices with the standard split pair, so the new lattice is in the same genus. The residue vector x can also be lifted inside the global lattice by reduction modulo P.

Conversely, given the upward lattice and its covector, taking the kernel at P recovers M. The relation $f_x(T^{-1}\tilde{x}) = 1$ recovers x uniquely in M/TM: any two elements of L_x with covector value one differ by M, and multiplication by T gives the same residue. Thus the source markings above the common kernel are the nonzero vectors x, and the target markings are the nonzero covectors c on that kernel lattice. Scaling the original f_x scales x inversely. In particular the scalar augmentation is the same condition after this reindexing. These operations are equivariant for integral frame changes. The initial unrefined stabilizers are trivial by Section 119.7. Consequently summation on classes at this level in these coordinates uses the matrix $I(c, x) = \mathbf{1}_{c(x)=1}$ with coefficient identity and unit mass. Alternatively one can sum on all markings and divide by the common unrefined stabilizer, as in Theorem 73.1. At different split pairs all modifications are independent. Performing them simultaneously reindexes the composite as the tensor product of the incidence matrices, with all other data fixed.

The incidence matrix preserves the two scalar augmentation spaces. Indeed summing $I(c, x)$ over the nonzero scalar multiples of c gives $\mathbf{1}_{c(x)\neq 0}$, which is constant as x ranges over any scalar line. For two vectors x, y, the number of covectors taking value one on both is T^{k-1} when $x = y$, zero for unequal proportional vectors, and T^{k-2} for independent vectors. On an augmented table the independent terms vanish by summing over each of the other lines. Hence $I^* I = T^{k-1}$ there. The transpose gives the same argument on covectors. This proves invertibility and the claimed archimedean norm, including the product of the places and the positive scalar. Transport by rational unitary elements in the class representatives is isometric. Finite level character factors have absolute value one.

We compute the coefficient frames in this reindexing. Since T splits completely, there is at most one coefficient embedding inducing P at w. The embeddings form a CM type. If $\tau(P)$ induces P, its conjugate induces $\bar{P}$ and is outside that type. No other embedding then induces either member of the pair. At $\tau(P)$, the description by the ordinary dual of the Hermitian dual in Theorem 99.20 uses the lattice from the opposite place. The marking restricts the position place alone. Thus the target coefficient frame at c is $W_{\lambda_{\tau(P)}}(M^\vee)$, and the source frame at x is $W_{\lambda_{\tau(P)}}(L_x^\vee)$. In the dual position lattice one has

$$L_x^\vee = \{z \in M^\vee : z(\tilde{x}) = 0 \bmod T\}.$$

All other coefficient factors are unchanged for this move. Each selected source has precisely one coordinate multiplied by T in an adapted integral frame. Its least weight is $\lambda_{\tau(P),k} = -b_*$, by Lemma 120.7. Hence the scalar T^{sb_*} puts all entries into the target frame. It follows that $H_w S_w \subseteq S_w$.

Use the target coefficient frame also as a reference for the source at every vector marking. Relative to this reference, the unscaled source exponents are distributed as sums of the single-

coordinate weights at the selected embeddings, with their Weyl multiplicities. Indeed the source coefficient lattice is constant on each tuple of scalar lines. Each line contributes its integral basis of augmentation differences, exactly as in Theorem 120.8. Other marked places may first be grouped by their scalar lines. Their coefficient factors and their augmentation bases occur on both sides. By symmetry of the diagonal coordinates the mean weight at P is $|\lambda_{\tau(P)}|/k$, as in Corollary 120.9. On the reference lattices identify vector and covector indices by dual bases. The resulting single-place incidence endomorphism is symmetric and its square on the augmented array is $T^{k-1}I$. Its determinant therefore has valuation $(k-1)/2$ per dimension. Products add these means, as does the coefficient scalar. Integral coordinate changes in the class representatives preserve all of the actual source and target lattices. This gives the stated local determinant after summation over the classes. □

73.4 The auxiliary level factor in the single-lift determinant

On a shorter period field put $s = r_x,\ t = 5s,\ n = s + t$. For a single lift with source W_j, use the block test of Lemma 142.7 and its fixed archimedean vector. The good source of this lift is the marked space at the original principal levels. Its dimension d_j is independent of the tame twist. Normalize the source integral using the same Haar measure before and after twisting. On the definite target use the Hermitian class sum with reciprocal stabilizer weights, as in Theorem 52.24. A level character acts by a root of unity and hence disappears from the squared norm.

For residue cardinality q and a finite twist set $\mathcal{Q}$ put

$$F_{s,t}(q) = \prod_{a=t+1}^{t+s} (1 - q^{-a}), \qquad C_{\mathcal{Q}} = \prod_{w \in \mathcal{Q}} q_w^{st} F_{s,t}(q_w).$$

In products over marked or twist places we take one member of each split pair in E; it gives one place of E^+. In particular $C_{\varnothing} = 1$.

Lemma 73.4 (Conversion from maximal-compact measure to the target class sum)**.** *At an original level place the local factor in the class-sum norm is $F_{s,t}(q)/|\mathrm{GL}_s(\mathbb{F}_q)|$. At a new twist place it is $q^{st}F_{s,t}(q)$. These factors include the local source convolution on its level-fixed vectors and the change to the target class measure.*

Proof. Initially give each maximal compact group and the integral matrix lattice volume one. Write $g_s = |\mathrm{GL}_s(\mathbb{F}_q)|$. At an original level the marked matrix is one specified inclusion. By Lemma 142.7 its source coefficient is q^{-ns} times the characteristic function of the principal subgroup, which has volume $1/g_s$. At a twist place the top s-by-s block is invertible modulo the maximal ideal. Its nonzero values are character values, so its squared norm is g_s/q^{s^2}, the fraction of invertible blocks. The source determinant character cancels the change of this test under the maximal compact, giving that same scalar on its fixed vectors.

A compact level of index b in the target maximal compact has mass $1/b$ for the initial measure. Its class contributes $1/(b\,|\,\mathrm{Stab}\,|)$ under this measure; hence the class-sum convention multiplies the norm squared by b. At an original place the target level fixes an ordered s-frame. Its index is the number of full rank n-by-s residue matrices, namely

$$I_{n,s}(q) = \prod_{k=0}^{s-1} (q^n - q^k) = q^{ns} F_{s,t}(q).$$

Here each next column is chosen outside the span of the preceding ones. At a twist place the target level preserves the subspace of that frame. Its index is $I_{n,s}(q)/g_s$, because every such

subspace has g_s ordered bases. Passing to the parahoric with its prescribed unitary character gives the same measure on pairings of two forms with that character. The two products of source masses and target indices are therefore $q^{-ns} I_{n,s}(q)/g_s$ and $(g_s/q^{s^2}) I_{n,s}(q)/g_s$, as asserted. □

Theorem 73.5 (Exact conductor factor in the auxiliary Gram determinants)**.** *For a finite twist β let $\Theta_{j,\beta}$ be the single lift in the above normalization. Write $\mathcal{V}_j(\beta) = \det(\Theta_{j,\beta}^{*}\Theta_{j,\beta})$ on the good source, and $\mathcal{R}_{j,\beta}^{g}$ for the orthogonal compression to that source of the unmarked finite operator in Theorem 142.9. For the simultaneous twists of Corollary 52.21, at every sufficiently large node and every j one has*

$$\frac{\mathcal{V}_j(\vartheta^j)}{\mathcal{V}_j(1)} = C_{\mathcal{Q}}^{d_j} \frac{\det \mathcal{R}_{j,\vartheta^j}^{g}}{\det \mathcal{R}_{j,1}^{g}}.$$

All these determinants are positive, with empty determinant one. For the product over j, after division by $C_{\mathcal{Q}}^{\sum_j d_j}$, the absolute logarithm is at most

$$\left(\sum_j d_j\right) \log \frac{1+\varepsilon_s}{1-\varepsilon_s}, \qquad \varepsilon_s = 512[E_x^{+} : \mathbb{Q}]\, 2^{-s}.$$

Proof. The unfolded identity for two source forms in the proof of Theorem 142.15 gives a sesquilinear form equal to the pairing with the unmarked finite operator times the product of the real integrals, the positive Haar constant and the two sorts of level factors. To compare it with the algebraic-form class pairing, realize the fixed vector-valued form by its coefficient along the chosen highest line of the compact real target. For a unit vector u in its irreducible compact type, averaging the rank-one operators $\rho(k)u\,(\rho(k)u)^{*}$ over probability Haar gives the identity divided by the dimension of the type. Indeed the average commutes with the group and has trace one. Thus passage between that scalar coefficient norm and the vector class norm introduces a positive scalar depending only on the real type and the normalization of its vector. The finite measure change is exactly the one in the lemma. Equivalently one can apply the unfolded diagonal formula and polarize on the good source, since both forms are Hermitian.

The target and source for this j, their real polynomials, the Haar measures initially giving the maximal compacts mass one, and the scaling of the theta kernel are fixed as β varies. The real factors are positive and independent of finite twists by Theorem 142.13. The original level product is fixed as well. Consequently, for a positive constant A_j independent of β, the operator on the good source is

$$\Theta_{j,\beta}^{*}\Theta_{j,\beta} = A_j C_{\mathcal{Q}(\beta)} \mathcal{R}_{j,\beta}^{g}.$$

This formula uses the unscaled source integral. Multiplying that integral by any further scalar would multiply the operator by the squared modulus of that scalar. The estimates of Theorem 142.9 apply before and after twisting. The compressed operator is self-adjoint with all eigenvalues in $[1-\varepsilon_s, 1+\varepsilon_s]$, by its Rayleigh quotients. At the nodes in question $\varepsilon_s < 1/2$. This gives positivity on every good source, including those on W_j with indefinite real signature. Taking determinants and dividing by the untwisted identity cancels $A_j^{d_j}$. Every power ϑ^j, $1 \le j \le m_x$, ramifies at each chosen twist place, so $\mathcal{Q}(\vartheta^j) = \mathcal{Q}$. In each ratio of positive determinants, the logarithm lies between plus and minus $d_j \log((1+\varepsilon_s)/(1-\varepsilon_s))$. Adding proves the assertion for the product. □

Proposition 73.6 (Valuation of the common scalar at the coefficient prime)**.** *Suppose q is the norm of a new split place for a twist at the coefficient prime ℓ; in particular $\ell \nmid q$. For*

$q \equiv 1 \bmod \ell$ *one has*

$$v_\ell(q^{st} F_{s,t}(q)) = s\, v_\ell(q-1) + v_\ell\Big(\frac{(s+t)!}{t!}\Big) + \mathbf{1}_{\ell=2}\left(\Big\lfloor \frac{s+t}{2} \Big\rfloor - \Big\lfloor \frac{t}{2} \Big\rfloor\right)(v_2(q+1)-1).$$

If the tame order at w is $\ell^{h_w} > 1$, then $v_\ell(C_{\mathcal{Q}}) \geq s \sum_{w \in \mathcal{Q}} h_w$ for a twist of ℓ-power order. In particular a small valuation of $1 - \vartheta(\tau_w)$ at every new generator is compatible with a large valuation of the common scalar.

Proof. The factor q^{st} and all powers of q in the denominators are ℓ-adic units. We must sum $v_\ell(q^a - 1)$ for $t < a \leq s+t$. If ℓ is odd and $q \equiv 1 \bmod \ell$, raising q to a power prime to ℓ preserves $v_\ell(q-1)$ by the geometric series formula. Raising a number $1+u$, with $v_\ell(u) \geq 1$, to the ℓ-th power increases that valuation by one: in the binomial expansion the term ℓu has strictly smaller valuation than all the other terms of positive degree. Thus $v_\ell(q^a - 1) = v_\ell(q-1) + v_\ell(a)$. At $\ell = 2$, q is odd. An odd exponent preserves both the valuation of $q-1$ in the difference and the valuation of $q+1$ in the sum, by factoring and reducing the odd quotients modulo two. The first squaring gives $v_2(q^2-1) = v_2(q-1) + v_2(q+1)$; after a squaring the new base is $1 \bmod 8$, so each further squaring adds exactly one. The correction to $v_2(q-1) + v_2(a)$ is therefore $v_2(q+1) - 1$ precisely for even a. Summing proves the identity.

The tame character at the chosen split place factors through $\mathbb{F}_q^\times$, hence $\ell^{h_w} \mid q - 1$. Each of the s differences $q^a - 1$ is divisible by $q-1$, giving the stated lower bound after summation over w. On the other hand $v_\ell(1 - \vartheta(\tau_w)) = 1/(\ell^{h_w - 1}(\ell - 1))$ by Lemma 13.12. For a nonempty auxiliary set the latter number can be made arbitrarily small in the twisting construction, and doing so forces the lower bound for $v_\ell(C_{\mathcal{Q}})$ to grow. This accounts for the power of $C_{\mathcal{Q}}$ in the determinant identity before a comparison across different lifts. □

The determinant in this subsection concerns one prescribed lift. Applying it to the composite chain requires an identity comparing the composite Gram operator with the product of the single-lift operators, including the level scalar and the source multiplicities.

73.5 The common finite source of the auxiliary determinants

Fix a node of the shorter family of Corollary 12.4. Write $F = E_x^+$, $E = E_x$, $m = m_x$, and $s = r_x = k_x + m_x - 1$. Both k_x and m_x are even. Consider the rank s spaces

$$Y_0, \quad W_1, \dots, W_m.$$

For each V in this list let H_V be its marked automorphic space at the level of Theorem 73.5. At infinity we take the coefficient on the specified line of the discrete series. At finite places take the principal congruence subgroup at the chosen places above p_0, p_1 and the lattice stabilizer elsewhere. Write J for this compact open subgroup and e for the two commuting central projections of Section 130.1, taken together as a product.

The finite frames identify all these finite adelic groups, together with J and e. To see the identification explicitly, write B_V for the rational Hermitian matrix of a space and $u_{V,v}$ for an orthonormal frame of its specified local lattice. Then $u_{V,v}^* B_V u_{V,v} = 1$, and $u_{V',v} u_{V,v}^{-1}$ carries the lattice of V isometrically to that of V'. Conjugation by this isometry identifies their unitary groups. Such frames exist for the diagonal sign lattices and for the unimodular row lattice (Lemma 131.16). At an inert finite place E/F is unramified. A nondegenerate Hermitian space over the quadratic residue field has a vector of nonzero norm. Indeed, if all diagonal values vanished, variation of a scalar in the norm of $au + v$ would, by nondegeneracy of the residue trace pairing, force each pairing of two vectors to vanish. Lift such a vector, use surjectivity of the norm on local units to give it norm one, and split off its orthogonal complement. Surjectivity follows from the norm on the residue fields and the trace on each successive principal-unit quotient. This proves the frame assertion by induction, also in residue characteristic two. At a split place choose a basis on the prescribed member of the pair and its dual on the other member. In particular the chosen reduction coordinates can be used at the marked places. In these coordinates J and the finite residue characters defining e are identical for every V.

Lemma 73.7 (Two split supercuspidal components)**.** *Every irreducible automorphic constituent of H_V has a supercuspidal component at a place above each of p_0, p_1. It belongs to the cuspidal spectrum.*

Proof. The product of finite character projections defining e selects tuples with a primitive member above each rational mark. Decompose the J-fixed vectors of a constituent into isotypic spaces for the finite residue groups and take a tuple on which e acts as the identity. The two primitive members occur at distinct split places because $p_0 \neq p_1$. At either place, of residue cardinality q, the primitive label is one Frobenius orbit of length s, with partition consisting of one box. In Green's classification this is a cuspidal representation of $\mathrm{GL}_s(\mathbb{F}_q)$, so it is absent from all finite inductions from proper parabolics.

Let $K = \mathrm{GL}_s(\mathcal{O}_{F_v})$ and K^1 be its reduction kernel. If the local representation were a subquotient of a proper parabolic induction, its K^1-fixed vectors would be a subquotient of the K^1-fixed vectors of that induction. Averaging over K^1 gives exactness over $\mathbb{C}$. For a parabolic $P = MN$, the Iwasawa decomposition $G = PK$ identifies the latter fixed vectors with functions on K/K^1 induced from $P(\mathbb{F}_q)$ with coefficient space the $(M \cap K^1)$-invariants of the inducing representation. Indeed, restriction to K determines the function, its value is fixed by $M \cap K^1$, and the induction modulus is one on $P \cap K$. Thus the finite induction cannot contain the cuspidal label. This proves local supercuspidality at both places. The noncuspidal terms in the automorphic spectral decomposition are obtained from proper parabolic induction, including subquotients at poles of Eisenstein series. A proper global parabolic remains proper at either split place. The supercuspidal local factor therefore excludes these terms. □

Theorem 73.8 (The finite Hecke modules on the row sources)**.** *For $V = Y_0, W_1, \dots, W_m$, the spaces H_V are unitarily equivalent modules for the finite convolution algebra $e\mathcal{H}(G_f, J)e$. Here G_f is the common finite adelic group, $\mathcal{H}(G_f, J)$ consists of compactly supported J-bi-invariant functions, and e acts with its idempotent normalization. In particular*

$$\dim H_{W_j} = \dim H_{Y_0} =: D.$$

Proof. First take an automorphic constituent π on one of the spaces. Lemma 73.7 supplies two split places where the local group is the general linear group and the representation is supercuspidal. The rank s is odd. Theorem 2.2.2 of Harris and Labesse thus gives a cuspidal weak base change Π on $\mathrm{GL}_s(\mathbb{A}_E)$. Its unramified local parameters at almost all places of E are the ordinary base changes of those of π. [5]

We identify the global parameter to which the multiplicity formula applies. By Corollary 3.3.2 of Kaletha, Mínguez, Shin and White the unramified Hecke data of a discrete constituent, with the ordinary base-change embedding, are those of a global parameter for the quasi-split unitary group. Its image on the general linear group is an isobaric sum of cuspidal representations and their real determinant twists. The unramified data just computed agree with those of Π outside a finite set. Strong multiplicity one in the isobaric form identifies that entire sum with Π. In particular there is one cuspidal summand with multiplicity one and the auxiliary SL_2 is trivial. Indeed, a block of length a contributes a twists with successive exponents $(a-1)/2, \dots, (1-a)/2$, and the cuspidal support of Π consists of its single rank s summand. This is a simple generic global parameter. The correct unitary-group parity is included in the support statement of Corollary 3.3.2. [6, §§1.3.2, 3.3]

Each group here is a pure inner form of the quasi-split unitary group. Over an algebraic closure an isometry from a fixed reference Hermitian space to V gives the inner isomorphism on groups, and its Galois coboundary takes values in the reference unitary group itself. For a simple generic parameter on pure inner forms the global multiplicity formula is unconditional. In this case the component group governing the multiplicity is trivial. The commutant of the single summand is scalar, and conjugate duality fixes among these scalars just $1, -1$. Dividing by the Galois-fixed center removes both. Equivalently the local packet characters multiply to one on this center by the global product formula. Hence every restricted tensor product of local packet members for the given inner form occurs once. [6, §1.7, Theorem 1.7.1 and the paragraph following it]

At every finite place use the isometry of framed Hermitian spaces above and keep the same local representation. At a real place the unsigned Harish–Chandra parameters of the row sources agree. In fact the first $k_x - 1$ entries run from $k_x/2 - 1$ down to $1 - k_x/2$; the last m entries are the half-gap minus the successive lift weights. These decrease by the positive gap, with the first below $1 - k_x/2$ by the choice of the gap and weights. Thus all entries are distinct. The prescription for a row assigns the entries to positive and negative coordinates of its Hermitian space. For $U(p, q)$ a regular discrete parameter has precisely the discrete series obtained by assigning its distinct entries to the p positive and q negative coordinates, modulo permutation inside each group of coordinates. This is the compact-Cartan description of the real discrete-series packet. In particular the prescribed member on the second space belongs to this same local packet. Using the conjugate coefficient convention for all rows dualizes the entire list and gives the same comparison. Replacing the real components by these chosen members therefore gives an automorphic constituent on the second space, with multiplicity one. The reverse replacement gives the inverse correspondence.

The coefficient projection on the fixed real line has rank one within each such copy by Schur orthogonality, as in Lemma 142.11. A finite representation π_f contributes exactly $e\pi_f^J$. Its

nearly everywhere finite data determine its global parameter by the same isobaric uniqueness, so counting its contribution cannot count two distinct global parameters. We have obtained the same finite direct sum of the spaces $e\pi_f^J$ on each row. The finite convolution action on a summand is the ordinary compressed action in π_f. The invariant Hilbert structures of equivalent irreducible unitary representations agree up to a positive scalar under an intertwiner, and rescaling gives a unitary identification. Taking the direct sum proves the assertion. □

Corollary 73.9 (The exponent in the product of auxiliary determinants)**.** *Use the characters and the set $\mathcal{Q}$ of Theorem 73.5. On H_{Y_0} write $R^0_{j,\beta}$ for the compression of the normalized finite convolution for (j,β), transported in the finite frames. At every sufficiently large node,*

$$\prod_{j=1}^{m} \frac{\mathcal{V}_j(\vartheta^j)}{\mathcal{V}_j(1)} = C_{\mathcal{Q}}^{mD} \frac{\det(R^0_{1,\vartheta} \cdots R^0_{m,\vartheta^m})}{\det(R^0_{1,1} \cdots R^0_{m,1})}.$$

The absolute logarithm of the determinant quotient on the right is at most $mD \log((1+\varepsilon_s)/(1-\varepsilon_s))$, where $\varepsilon_s = 512[F:\mathbb{Q}]\,2^{-s}$.

Proof. In the normalized finite convolution of Theorem 142.9, finite partial sums of double cosets, averaged on each side by J and compressed by e, belong to $e\mathcal{H}(G_f, J)e$. The absolute tail bound of Lemma 142.8 gives norm convergence. Therefore the equivalence of Theorem 73.8 intertwines the limits as well. Apply Theorem 73.5 for each j, use $d_j = D$ in its scalar, and multiply. Determinants are multiplicative for endomorphisms of one finite-dimensional space even when the operators do not commute. The interval for the eigenvalues of each factor gives the asserted logarithmic bound; with the empty determinant convention the identity also holds when $D = 0$. □

This identifies the power of $C_{\mathcal{Q}}$ contributed by the auxiliary lifts. To cancel it in the comparison of periods one must compute the Gram determinant of the composite map from Y_0 to Y_m in its own lattice normalization.

73.6 The terminal flag and the remaining Gram determinant

Keep the short-family node and the notation of Theorem 73.8. Set $n_i = s + it$, $0 \le i \le m$. Let $\mathcal{M}$ be the set of old marked places of F, $\mathcal{Q}$ the set supporting the tame twist, and q_w the residue cardinality. Each of these places splits in E; use the prescribed member of each split pair. Write

$$g_b(q) = |\mathrm{GL}_b(\mathbb{F}_q)|, \qquad N_i(\vartheta) = \prod_{w \in \mathcal{M} \cup \mathcal{Q}} g_{n_i}(q_w),$$
$$M(\vartheta) = \prod_{w \in \mathcal{M} \cup \mathcal{Q}} q_w^{s(s+t)}, \qquad Q_* = \prod_{w \in \mathcal{Q}} q_w.$$

Here N_i is the index of the principal subgroup in the maximal compact at row i, taken over these places. $M(\vartheta)$ counts matrices with $s + t$ rows and s columns over the same product of residue fields.

The point test on the edge from Y_{i-1} to Y_i specifies the standard inclusion on reduction. At a new twist place it is convenient to write ψ_i for the tame character on the determinants of that edge. Use the split matrix action with target determinant character ψ_i and source determinant character ψ_i^{-1}. If the integral changes of coordinates on target and source have reductions h_i, h_{i-1} satisfying $h_i \iota_i = \iota_i h_{i-1}$, they preserve this point test up to $\psi_i(\det h_i / \det h_{i-1})$. Indeed, the matrix variable is changed by left and right multiplication. The displayed condition fixes its prescribed residue, the lattice is preserved, and both determinant absolute values are one. Only the indicated splitting characters remain. Thus the action on a terminal coefficient can be followed back to the initial coefficient by changing the variable on each source.

At $w \in \mathcal{Q}$ prescribe the parabolic $P_m(q_w)$ preserving the flag of dimensions $n_0, \dots, n_{m-1}$ inside $\mathbb{F}_{q_w}^{n_m}$. If h_m preserves the flag, take h_i to be its successive restrictions. Arbitrary invertible lifts of those restrictions to the compact groups suffice, since the test only sees their reduction. The determinant quotient at step i is the determinant on the i-th successive quotient of dimension t. The initial compact change fixes the initial form at this new place. The resulting terminal character is therefore the product of the ψ_i of these m quotient determinants. At $w \in \mathcal{M}$ take the identity reduction at the terminal level; all restrictions then have identity reduction and the initial principal level fixes the form. Denote by $J_{\mathrm{end}}(\vartheta)$ this specified compact level, with the full lattice stabilizer at the remaining finite places, and by I_ϑ its index in the product of the maximal compacts. This description specifies a level even for a vector that happens to have additional symmetry.

Proposition 73.10 (The metric at the terminal compact level)**.** *Give the terminal group at infinity and its finite maximal compacts volume one. Let d_∞ be the dimension of the fixed irreducible compact representation at infinity. If z, z' are terminal vector-valued forms of this weight and the specified unitary level character, and u, u' are their scalar coefficients on the same unit highest-weight vector, then*

$$\langle z, z' \rangle_{\mathrm{cl}} = d_\infty I_\vartheta \, \langle u, u' \rangle_{\mathrm{Pet}}.$$

Proof. Over a finite double class of representative c, the quotient measure divides the compact measure by the order of Γ_c, its rational stabilizer in the real compact group times $cJ_{\mathrm{end}}c^{-1}$. The stabilizer is finite since the rational group is discrete and this product is compact. The finite volume factor is $1/I_\vartheta$. The terminal character has modulus one, so integration over the finite level introduces no additional factor in the paired coefficients. Averaging the rank-one projector of the unit vector over the real compact group gives d_∞^{-1} times the identity by Schur orthogonality. Thus the scalar Petersson pairing above that class is $\langle z(c), z'(c) \rangle / (d_\infty I_\vartheta |\Gamma_c|)$.

Summing over the classes proves the formula with the reciprocal stabilizer weights of the class-sum pairing. □

Theorem 73.11 (The normalized terminal index)**.** *Put $u_m = mst + \binom{m}{2}t^2$, and set*

$$L_\vartheta = I_\vartheta \frac{N_0(\vartheta)N_1(\vartheta)^m}{N_m(\vartheta)M(\vartheta)^m}.$$

For $C_{\mathcal{Q}}$ as in Theorem 73.5,

$$\frac{L_\vartheta}{L_1} = C_{\mathcal{Q}}^m\, Q_*^{-u_m}.$$

In particular at the coefficient prime ℓ of the simultaneous twists the valuation of this quotient is exactly $m\, v_\ell(C_{\mathcal{Q}})$.

Proof. The successive diagonal blocks of $P_m(q)$ have sizes $s, t, \dots, t$. Its diagonal choices number $g_s(q)g_t(q)^m$. The entries strictly above these blocks can be chosen freely. There are mst such entries in the first block row and $\binom{m}{2}t^2$ in the rest. Consequently

$$[\mathrm{GL}_{s+mt}(\mathbb{F}_q) : P_m(q)] = \frac{g_{s+mt}(q)}{g_s(q)g_t(q)^m q^{u_m}}.$$

Reduction of the split integral group is onto, since a lift of a matrix invertible on reduction has determinant a unit. At each old place the terminal index is $g_{s+mt}(q)$, the same for every twist. The finite level is a direct product, so all its old factors cancel from L_ϑ/L_1.

At one new place the factor remaining in this quotient is

$$\frac{g_{s+mt}}{g_s g_t^m q^{u_m}} \, \frac{g_s g_{s+t}^m}{g_{s+mt} q^{ms(s+t)}} = q^{-u_m} \left(\frac{g_{s+t}}{g_t q^{s(s+t)}} \right)^m.$$

Counting ordered bases gives $g_b(q) = q^{b^2} \prod_{a=1}^{b}(1 - q^{-a})$. The expression in parentheses is therefore $q^{st} \prod_{a=t+1}^{t+s}(1 - q^{-a})$, exactly the local factor of $C_{\mathcal{Q}}$. Multiply over the new places. At a place of tame ℓ-power order q_w is prime to ℓ, so $v_\ell(Q_*) = 0$. This gives the last assertion. □

Corollary 73.12 (Reduction to the Gram form with fixed Haar measures)**.** *Take the scalar row integrals in the fixed Haar measures, using on the i-th edge the prescribed principal-level weight $N_{i-1}(\vartheta)$. Write U_ϑ for this weighted composite, before conversion of its terminal coefficient to a vector-valued form. On H_{Y_0} write*

$$K_\vartheta = \frac{N_m(\vartheta)M(\vartheta)^m}{N_0(\vartheta)N_1(\vartheta)^m} U_\vartheta^* U_\vartheta,$$
$$S_\vartheta = \prod_{i=1}^{m} \det R^0_{i,\vartheta^i}, \qquad A_\vartheta = \prod_{i=1}^{m} \mathcal{V}_i(\vartheta^i).$$

In the first line the adjoint uses the terminal scalar Petersson pairing. Whenever the row integrals give square-integrable terminal coefficients on this source, let Z_ϑ be their Gram determinant in the vector-valued terminal class sum. Then

$$\frac{Z_\vartheta}{A_\vartheta} = (d_\infty L_1)^D \frac{S_1}{A_1}\, Q_*^{-Du_m} \frac{\det K_\vartheta}{S_\vartheta}.$$

The identity permits a zero composite determinant.

Proof. Proposition 73.10 identifies the operator of the terminal class-sum pairing with $d_\infty I_\vartheta U_\vartheta^* U_\vartheta = d_\infty L_\vartheta K_\vartheta$. Thus $Z_\vartheta = (d_\infty L_\vartheta)^D \det K_\vartheta$. By Corollary 73.9, $A_\vartheta/A_1 = C_{\mathcal{Q}}^{mD} S_\vartheta/S_1$. The factors $A_1, A_\vartheta, S_1, S_\vartheta$ are positive by Theorem 73.5. Substitute the terminal index formula and divide by A_ϑ. □

The finite flag calculation cancels the entire explicit factor $C_{\mathcal{Q}}^{mD}$ in this ratio. The remaining expression contains the actual operator K_ϑ; an estimate or identity for its determinant relative to S_ϑ still requires control of the intervening theta integrals.

73.7 Split doubling factors along the row

At a place w of $\mathcal{M} \cup \mathcal{Q}$ put $q = q_w$, and write $P_b = \prod_{a=1}^{b}(1 - q^{-a})$, with $P_0 = 1$. For a row edge of ranks $n, n + t$ set

$$H_{n,t}(q) = \frac{P_{n+t}}{P_t} = \prod_{a=1}^{n}(1 - q^{-(t+a)}).$$

This is the inverse split spherical doubling denominator at the theta point. In the normalization of Theorem 72.16 its factors at parameter z are $1 - q^{-(2z+a)}$ for $1 \le a \le n$, since the quadratic character of E/F is trivial at a split place. Here $z = t/2$. We use $H_{n,t}$ only as an explicit rational scalar. The local integral for a ramified test is computed directly below.

Lemma 73.13 (The principal-level weight in the doubled integral)**.** *Use the specified inclusion-point datum on the row edge of ranks $n, n + t$. On the principal-level fixed space the split local doubled pairing at the theta point, including the two copies of the local edge weight $g_n(q)$, is the original pairing times*

$$B_{n,t}(q) = g_n(q)q^{-n(n+t)} = q^{-nt}P_n.$$

This holds at every stage, also when the source carries a nontrivial character on the larger flag subgroup.

Proof. In the split matrix model the test is the indicator of $X \equiv \left(\begin{smallmatrix}1_n\\0\end{smallmatrix}\right)$ in the integral $(n + t)$-by-n matrices. If X and Xg both lie in its support, their top square blocks belong to $1 + \varpi M_n(\mathcal{O}_{F_w}) =: J_n$. The quotient of those blocks is g, so $g \in J_n$. Conversely J_n preserves that residue class and its determinants are in the kernel of the tame character. Its action leaves the indicator fixed. The squared norm of the indicator is the additive volume $q^{-n(n+t)}$ of one residue class. Thus the matrix coefficient of the test is $q^{-n(n+t)}\mathbf{1}_{J_n}$.

The local doubling section at the theta point restricts on the open orbit to this oscillator coefficient, as in Proposition 72.15. On J_n the matrix coefficient of two level-fixed source vectors is precisely their inner product. With maximal compact volume one, $\mathrm{vol}(J_n) = 1/g_n(q)$. The scalar for the unweighted edge is consequently $q^{-n(n+t)}/g_n(q)$. Multiplication of each lift by its local edge weight $g_n(q)$ multiplies its polarized Gram form by $g_n(q)^2$. The output point test is fixed by the principal subgroup on the next group, by the same reduction argument. Starting with the initial fixed vectors therefore verifies the level condition at each successive stage. □

Theorem 73.14 (Cancellation along the complete split flag)**.** *Let $n_i = s + it$, $0 \le i \le m$. Denote by I_w the local factor of the specified terminal level index I_ϑ. The product of the m local row factors in Lemma 73.13 obeys*

$$\frac{I_w \prod_{i=0}^{m-1} B_{n_i,t}(q)}{\prod_{i=0}^{m-1} H_{n_i,t}(q)} = \begin{cases} 1, & w \in \mathcal{Q}, \\ |P_m(q)|, & w \in \mathcal{M}, \end{cases}$$

where $P_m(q)$ in vertical bars denotes the flag subgroup of Theorem 73.11.

Proof. Each quotient in the product before multiplication by I_w is $q^{-n_i t}P_{n_i}P_t/P_{n_{i+1}}$. The middle products cancel, giving

$$\prod_{i=0}^{m-1} \frac{B_{n_i,t}}{H_{n_i,t}} = q^{-u_m}\frac{P_s P_t^m}{P_{s+mt}}, \qquad u_m = \sum_{i=0}^{m-1} n_i t = mst + \binom{m}{2}t^2.$$

At a new place $I_w = g_{s+mt}/(g_s g_t^m q^{u_m})$. Substitution of $g_a = q^{a^2} P_a$ gives exactly the inverse of the last display, since $(s+mt)^2 - s^2 - mt^2 = 2u_m$. At an old place the prescribed level is principal, so $I_w = g_{s+mt}$. This increases the preceding result by $|P_m(q)|$. □

Corollary 73.15 (Normalized comparison with all single lifts at the marks)**.** *Use the unscaled auxiliary source integrals and the terminal class measures of Theorem 73.5. At each marked place divide the product of the weighted row factors with its terminal index by the product of the m auxiliary local factors. Then divide by the ratio $\prod_{i=0}^{m-1} H_{n_i,t}(q)/H_{s,t}(q)^m$ of inverse spherical doubling denominators. The resulting scalar is*

$$q^{-mst} \quad (w \in \mathcal{Q}), \qquad |P_m(q)|\, g_s(q)^m \quad (w \in \mathcal{M}).$$

At each node the old factor is identical before and after twisting, and the product of the new normalized factors has coefficient-prime valuation zero, including after taking the D-th power.

Proof. Lemma 73.4 gives $q^{st} H_{s,t}(q)$ for one auxiliary local factor at a new place and $H_{s,t}(q)/g_s(q)$ at an old one. The same set of new places occurs for each of the m characters. Divide the formula of Theorem 73.14 by the m-th power of the corresponding normalized auxiliary factor. The new factors multiply to Q_*^{-mst}, and $\ell \nmid Q_*$ for the simultaneous twists at ℓ. □

This calculation evaluates the marked local data in the doubled pairings. To apply it to $\det K_\vartheta$ in Corollary 73.12 one still needs the global inner product identities for the successive prescribed theta vectors, including their archimedean factors.

73.8 Cuspidality and the real pairings of the prescribed row

Fix one node of the shorter intervals of Corollary 12.4, with its original marked level and any one of the twists of Corollary 52.21. Use the row $Y_0, \dots, Y_m$ and the point tests and measures of Corollary 73.12. Write $s = k+m-1$, $t = 5s$, $n_j = s+jt$, $q_j = \lfloor (m-j)/2 \rfloor$, and $b = (t-1)/2$. Both k and m are even. The source at step $j+1$ has real signature $(n_j - q_j, q_j)$. Write $H = s^8$ for the half-gap. At a chosen complex embedding above a real place the weights of the successive characters are $-w_a$, where $w_a = \kappa + (2a+1)H$ and $\kappa > 0$ is the weight at that embedding. The same field, point of the interval and embedding are used throughout the calculation.

Lemma 73.16 (First occurrence for the bilinear row kernel)**.** *Let $h_j = -\frac{1}{2}\mathbf{1}_{j \text{ odd}}$. Form the real parameter P_j by adding h_j to each of the following entries.*

1. *The $k-1$ consecutive integers from $k/2-1$ down to $1-k/2$ have plus sign.*
2. *The entries $H - w_a$, $1 \le a \le m$, have plus sign through $a = j+1$. The subsequent signs alternate, starting with minus.*
3. *For $1 \le a \le j$, the t entries $-w_a + b, \dots, -w_a - b$ all have plus sign.*

Within each of the two compact blocks put the entries in decreasing order. For a physical input of parameter P_j, the first possible nonzero real lift in the target Witt tower is at Y_{j+1}. The local correspondence there gives parameter P_{j+1} on the physical target. This holds for all sufficiently distant nodes, for every $0 \le j < m$ and every specified twist.

Proof. The base weights are the centered $k-1$ consecutive integers and the m shifted weights $H - w_a$. After j insertions the underlying list has acquired the j displayed strings. On the a-th shifted weight put $e = a-1$. The row sign is $(-1)^{e+\min(e,j)}$, giving exactly the signs in the statement. The number of minus signs is q_j. Before the parity shift all entries are integers. Each inserted string lies between $H - w_a$ and $H - w_{a+1}$ when both are present, since the distance from its center to either neighbor is $H > b$. The first shifted weight is below the last consecutive base weight. Thus the parameter is regular and its parity is $(n_j - 1)/2 \bmod \mathbb{Z}$.

We check how this signed list enters the local correspondence. On the edge put $n = n_j$, $N = n + t$, and $\nu = \mu_{j+1}^0 \vartheta^{j+1}$. On the norm-one torus the rank-d splitting datum is $\nu\upsilon^{\lfloor d/2 \rfloor}$, where υ in this expression is restricted to that torus. Its character on the quadratic field is $\chi_d = \nu^{\mathrm{BC}}\upsilon^{-(d \bmod 2)}$. Indeed base change on characters of the torus is pullback by $z \mapsto z/\bar{z}$. Conjugate self-duality takes the restricted υ to υ^2, and the space character also has the factor υ^{-d}. The finite-order twist is trivial on the connected real norm-one torus. At the chosen embedding υ has angular exponent one. The half angular exponents of χ_N, χ_n are therefore

$$a_T = -w_{j+1} + h_j, \qquad a_S = -w_{j+1} + h_{j+1}.$$

The theta integral multiplies the input form by the kernel. Thus the source quotient of the oscillator is the contragredient of the physical input. In matrix coordinates the action is $X \mapsto hXg^{-1}$. Write its tensor space as $Y_{j+1} \otimes Y_j^\vee$. The source acts on the second factor by ${}^t g^{-1}$, which inverts the compact torus coordinates. Transporting the contragredient to this dual space therefore restores P_j, with the compact coordinates sorted in decreasing order. The target torus has its original coordinates.

One can check the additive-character orientation and both twists on the Fock vacuum. Put $(p, q) = (n - q_j, q_j)$ and $(r, u) = (N - q_{j+1}, q_{j+1})$. In a target row of sign ϵ the determinant shift is $\lfloor n/2 \rfloor - w_{j+1}$. Each of the $(n - \epsilon(p-q))/2$ oppositely signed columns contributes -1 from normal ordering. The vacuum weight there is $a_S + \epsilon(p-q)/2$. In a physical source column

of sign ϵ' the derivative of right inverse multiplication gives the opposite weight. Its constant before normal ordering is $N - \lfloor N/2 \rfloor + w_{j+1}$, and one subtracts the number $(N + \epsilon'(r-u))/2$ of target rows of the same sign. In dual source coordinates the result is $a_T + \epsilon'(r-u)/2$. A Fock variable has weight $+1$ on both corresponding tensor coordinates for equal signs and -1 on both for opposite signs. These are precisely the compact action and vacuum characters for Ichino's negative real additive character. This checks the polarisation on the tensor space. For a fixed symplectic action two unitary splittings can differ only by a character. On these connected unitary groups such a character is determined on the compact tori. See Ichino, §4.1 and Lemma 7.8.

Subtract a_T from P_j. The consecutive base entries are positive and greater than H, since $w_{j+1} \geq 3H$ and $H > k$. The a-th shifted entry becomes $(2j + 3 - 2a)H$. An old string is centered at $2(j + 1 - a)H$ and has radius b. Hence all centered entries have absolute value at least $H > b + 1$. There are $A = k + j(t + 1)$ positive entries, all of plus sign. The negative entries have alternating signs. Among them $y = q_{j+1}$ have plus sign and $z = q_j$ have minus sign, so $A + y + z = n$.

Apply Atobe's first-occurrence criterion in the form of Ichino, Theorem 4.3(i). The centered list is integral and misses zero, so its alternating central string has parameter $k_\lambda = -1$. The two signature counts are $r_\lambda = A + z$ and $s_\lambda = y$. Membership of either exceptional counting set at b requires a centered entry with absolute value between 1 and b. Thus both sets are empty. The required signature is $(r_\lambda + 2b + 1, s_\lambda) = (N - q_{j+1}, q_{j+1})$, with the common displacement $l = 0$ in the criterion. Deleting h hyperbolic planes preserves the rank parity and the splitting character and replaces l by $-h$. For every $1 \leq h \leq q_{j+1}$ this violates $l \geq 0$.

Finally, in the increasing-rank correspondence of Ichino's Theorem 4.1(i), old positive centered entries keep their signs and old negative entries change sign. The middle Levi factor has signature $(t, 0)$. Every other factor is a compact line. For an old centered entry ξ in combined position i the inducing weight is $\xi + a_S - ((N+1)/2 - i)$. The middle weight is $A - n/2 + a_S$. Adding the compact-Cartan half sum of roots, whose i-th entry is $(N+1)/2 - i$, restores $\xi + a_S$ and gives $a_S + (t+1)/2 - h$ in middle position $A + h$, $1 \leq h \leq t$. The strict gaps on both sides of this middle string and the strict order of the old entries give the regular good range. These computations are also the compact-Levi calculation of Theorem 72.6. The changed signs belong exactly to the shifted entries with $a \geq j + 2$. We have obtained the stated list for $j + 1$. □

Theorem 73.17 (Cuspidality of the prescribed composite)**.** *On H_{Y_0} of Theorem 73.8 all the successive bilinear row integrals exist. Every intermediate output is cuspidal, with real types P_j wherever it is nonzero. The weighted terminal coefficient defining U_ϑ belongs to the Petersson space of $U(Y_m)$. The doubled theta integrals on these edges are in Weil's convergent range.*

Proof. Lemma 73.7 gives cuspidality at Y_0. For a smooth compact-finite cusp form, the rapid-decay theorem on Siegel sets gives decay faster than every power of their height. A Schwartz theta kernel has at most polynomial growth in that height, locally uniformly in the other argument. The center of each unitary group here is the anisotropic norm-one torus. Consequently the defining integral converges absolutely for a fixed target argument. The same bounds apply to derivatives of the Schwartz test, giving a smooth lift. Multiplication by the finite principal-level index is harmless for this assertion.

Consider one irreducible cuspidal input and any smaller rational member in the target tower. Removing a rational hyperbolic plane removes one positive and one negative direction at every real place. A nonzero value of a smaller global lift would give a nonzero invariant pairing of the local oscillator with the physical input at such a place. Indeed one may translate the test

to evaluate at the target identity, take pure data by linearity, and hold every other local factor fixed. Right invariance of the source integral gives this pairing. On compact-finite vectors it yields a map to the contragredient Harish-Chandra module of the source. A nonzero map is onto, since this module is irreducible. The maximal source-isotypic quotient of the oscillator would then be nonzero and would have a nonzero irreducible quotient on the smaller target by local Howe duality. Lemma 73.16 rules this out.

Rallis's tower property now gives vanishing of all proper unipotent constant terms of the lift at the specified target. The hypothesis of that theorem is the vanishing of the lower global lifts of the whole source representation, which was just verified for arbitrary Schwartz tests. In particular the conclusion applies to our prescribed point test and highest polynomial. See Gan, Qiu and Takeda, §11.2. For a cuspidal target constituent with nonzero projection, the projected theta map gives a quotient of the local oscillator with that constituent as target and the contragredient physical input as source. This follows by pairing the lift with a target cusp form and fixing all other places as above. Local Howe duality and Lemma 73.16 give its real parameters P_{j+1}. The fixed target test is compact-finite, and its finite part is fixed by an open compact subgroup. At fixed infinitesimal character, compact types and finite level the cuspidal space is finite dimensional by automorphic finiteness. Thus the argument applies to the sum of the resulting constituents at the next step. Induction proves cuspidality all the way through Y_m, including the possibility of a zero output.

For the last assertion let R be the rational Witt index of Y_{j+1}. Scalar extension of a rational isotropic space to a real place gives $R \le q_{j+1} \le m/2 < t$. Thus $N - R > n$. For the doubled unitary source of split rank n this is Weil's convergence inequality. It also applies at later steps where the Siegel Eisenstein series needs continuation at $(N-n)/2$. See Gan, Qiu and Takeda, §§3.1, 11.4. □

Proposition 73.18 (Positive real factors for the specified tests)**.** *At each step and real place fix the nonzero joint highest polynomial Φ and the prescribed unit input line. Let $\eta = P_j^{\vee}$ denote the physical contragredient parameter, obtained by negating entries and sorting in each compact block, and write c_η for the coefficient of its unit highest line. With $A_\Phi(g) = \langle \Phi, \omega(g,1)\Phi \rangle$, the integral*

$$\int_{U(n-q_j,q_j)} A_\Phi(g)\overline{c_\eta(g)}\, dg$$

converges absolutely and is strictly positive. For the same real tests and measures it is independent of the finite twist.

Proof. We specify the polynomial line to which the assertion applies. Order the A positive centered entries of the lemma decreasingly, followed by the alternating negative tail, and call this list (ξ_i). In the construction of Theorem 104.28 use the three lists

$$\left(\xi_i + i - A - \tfrac{t+1}{2}\right)_{i=1}^{A}, \qquad \left(-\xi_{A+2v} + \tfrac{t+1}{2}\right)_{v=y,\dots,1}, \qquad \left(-\xi_{A+2v-1} - \tfrac{t+1}{2}\right)_{v=z,\dots,1}.$$

They are partitions. For the first list successive centered entries differ by at least one, and its last part is at least $H-(t+1)/2 > 0$. The other two lists have decreasing parts in the indicated reverse order and the same positive lower bound when nonempty. In the three disjoint squares of respective sizes A, y, z, take the product of the leading minors whose exponents are successive differences of parts, reversing the two index orders in each dual square. The squares use, in tensor coordinates, the positive source and target blocks, the positive source and negative target blocks, and the negative source and positive target blocks, respectively. Their occupied rows and columns are disjoint. The determinant highest-weight calculation of Theorem 104.28 applies since it only uses the ordered entries, their signs and the middle string. It gives the two

lowest compact highest weights after the vacuum shifts computed above. At the frame where the three squares are identity matrices the product is one. The highest line in the matching space of joint harmonics has dimension one by Ichino, §7.4, Lemma 7.8. Thus a different choice of nonzero vector on the prescribed line multiplies this polynomial by a nonzero scalar. In physical source coordinates its compact weight is that of η. In particular $\partial_{ai}\partial_{ai'}\Phi = 0$ for every target row a and every pair of source columns i, i' of opposite signs.

Let $l = \min(n - q_j, q_j)$ and let $u_1 \geq \cdots \geq u_l \geq 0$ be the Cartan coordinates. The oscillator calculation in Lemma 142.12 works row by row for any target dimension N. For a paired boost its lowering operator is a sum of the indicated mixed second derivatives, which annihilate all compact source translates of Φ. Pairing the creation-operator expansion with another such vector leaves the diagonal factor $(\cosh u_i)^{-N-\deg_i}$, where $\deg_i$ counts variables in the two columns of that boost. Hence

$$|A_\Phi(k_1 a(u) k_2)| \leq \|\Phi\|^2 \prod_i (\cosh u_i)^{-N}, \qquad A_\Phi(k a(u) k^{-1}) > 0.$$

The strict inequality follows by summing the positive diagonal factors against the squared monomial coefficients of $k^{-1}\Phi$.

The Cartan density in that same lemma is at most a constant, depending on n, times $\exp(2(n-1)\sum_i u_i)$. It follows by integrating the squared bound that $A_\Phi(g)\exp(\delta \sum_i u_i)$ is square integrable for $0 \leq \delta < N - n + 1 = t + 1$. The discrete series coefficient c_η is square integrable since the real center is compact. Cauchy–Schwarz proves absolute convergence, including at all ranks $N \leq 2n$ occurring here. When $l = 0$ both assertions of integrability are immediate from compactness.

For the sign of the answer, take polar coordinates $g = kz$ with z positive Hermitian. Truncate to $\log \|z\|_{\mathrm{op}} \leq B$, which is compact modulo K. Let ρ and ρ_c be the full and compact positive half sums in the order of η, and put $\alpha = \eta + \rho,\ \mu = \eta + \rho - 2\rho_c$. Proposition 132.7 expresses c_η as an average of its compact highest-line coefficient with denominator $m_{-\alpha}$, the weight product of the principal minors. Applying compact Schur orthogonality to this average, as in the proof of Theorem 142.13, evaluates the truncated integral as

$$\frac{1}{\dim V_\mu} \int_{\log \|z\|_{\mathrm{op}} \leq B} \frac{A_\Phi(z)}{m_{-\alpha}(z)}\, d\nu(z).$$

Here K has mass one and ν is the polar quotient measure. Both compact averages are over a finite measure space and the radial cutoff is invariant under compact conjugation. Thus this identity requires no decay estimate. All principal minors of z are positive. The integrand is continuous and positive at the identity, where its numerator is $\|\Phi\|^2$. Taking B to infinity gives a strictly positive value by monotone convergence on this side and absolute convergence on the original side. All quantities in this integral depend only on the specified real data. □

Corollary 73.19 (The terminal Gram form on the shorter intervals)**.** *At every node above, the determinant Z_ϑ of Corollary 73.12 is defined for the actual weighted row, and the identity there holds. Every real factor in the successive Petersson pairings is strictly positive.*

Proof. Apply Theorem 73.17 to the specified point tests. The terminal coefficient is square integrable, which is the analytic hypothesis in Corollary 73.12. Apply Proposition 73.18 at each real place. The common-field comparison for the integral projection still requires the arithmetic size of the resulting global determinant relative to the auxiliary determinants. □

73.9 Change of the row determinant under a tame twist

Use the fields, weights, measures and test vectors of Section 73.8. Write $\mathcal{M}$ for the old marks of F, $\mathcal{Q}$ for the new ones, and $S = \mathcal{M} \cup \mathcal{Q}$. If S occurs as a superscript on an L-function over E, omit all places above S. All L-functions below have their archimedean factors removed. Set

$$\lambda_i = (\mu_i^0 \vartheta^i)^{\mathrm{BC}}, \quad \rho_{ai} = \lambda_a/\lambda_i \quad (a < i), \quad D_i = \upsilon^{-\mathbf{1}_{i \text{ odd}}}, \quad z_0 = t/2.$$

For a local cardinality $Q = \#k(w)$ put $[t]_w = \{Q^{(t-1)/2-h} : 0 \le h < t\}$, counted with multiplicity. The common finite source in Theorem 73.8 has dimension D. The notation D_i always denotes a character.

Lemma 73.20 (The spherical factor on a row constituent)**.** *Start with an irreducible bottom constituent and write $B_{0,w}$ for its ordinary standard base-change Satake multiset at w above an unmarked place. Whenever the first i prescribed row lifts have given a nonzero constituent spherical there, its multiset is*

$$B_{i,w} = D_i(\varpi_w) \left(B_{0,w} \ \sqcup \bigsqcup_{a=1}^{i} \lambda_a(\varpi_w)[t]_w \right).$$

At the edge from n_{i-1} to n_i, its unmarked doubling product equals the corresponding product for the auxiliary lift of the same bottom finite system from W_i, multiplied by

$$\frac{d_s^S(z)}{d_{n_{i-1}}^S(z)} \prod_{a<i} \prod_{h=-(t-1)/2}^{(t-1)/2} L_E^S(z + \tfrac{1}{2} - h, \rho_{ai}), \qquad d_n^S(z) = \prod_{c=1}^{n} L_F^S(2z + c, \omega_{E/F}^{t+c}).$$

The identity first holds for $\Re z$ sufficiently large.

Proof. In the tensor convention of Lemma 73.16 the characters attached to source and target at edge i are $\chi_S = \lambda_i D_i$ and $\chi_T = \lambda_i D_{i-1}$. The subscripts here refer to the two spaces, so the target character twists the source representation in the doubling integral. For a nonzero spherical theta pair the transfer is

$$B_{i,w} = (\chi_S/\chi_T)(\varpi_w) B_{i-1,w} \ \sqcup \ \chi_S(\varpi_w)[t]_w.$$

At an inert place this is Theorem 72.16, which uses the two strata of the line Jacquet module and permits nontempered spherical inputs. At a split place one can read the conventions directly on the matrices hXg^{-1}. The physical source quotient is contragredient. Removing its inverse determinant twist gives the dual bottom representation at that edge tensored with χ_T. In the normalized matrix correspondence the target support is its contragredient together with the support of the trivial representation of the difference in ranks. Restoring χ_S gives the displayed transfer, with $[t]_w$ the support of that trivial representation. Equivalently one can use Proposition 72.17. The other split component inverts the multiset and character values, and $[t]_w$ is invariant under inversion. Induction from $D_0 = 1$ gives the first formula.

For normalized unit spherical vectors and the standard lattice vacuum, the unramified doubling integral is the standard factor of $B_{i-1,w}/\chi_T(\varpi_w)$ at $z+1/2$, divided, over the one place of F, by $d_{n_{i-1},v}(z)$. This is the spherical doubling calculation in Gan, Qiu and Takeda, §11.6, in the character convention of Ichino, §6.3. Inverting the list and the twist simultaneously gives the same product over E, by conjugation. The finite framed lattices are unimodular, the CM extension is unramified at finite places, and off S the splitting characters are unramified and the tests are the unit self-dual lattice vacua, as in Lemma 142.8. These facts are unchanged by increasing the matrix ranks. In particular the section is spherical with value one at the identity.

On the auxiliary pair the target of rank $s+t$ has torus character $\mu_i^0\vartheta^i v^{\lfloor (s+t)/2 \rfloor}$. Since $s+t$ is even its space character is λ_i. The auxiliary source splitting acts on the target variable and does not occur in its source spherical doubling numerator. Thus the auxiliary list is $B_{0,w}/\lambda_i(\varpi_w)$. Theorem 73.8 matches these bottom systems with their dimensions. Dividing the row list by χ_T exhibits this same auxiliary list and the additional segments $\rho_{ai}(\varpi_w)[t]_w$. Multiplying their Euler factors gives the asserted formula. □

Proposition 73.21 (The Petersson pairing on every prescribed row step)**.** *The weighted row lifts are injective on the successive images of H_{Y_0}. On the part arising from a single bottom spherical system, the squared Petersson norm at edge i, divided by the squared norm of the input, is a positive scalar. Its unmarked factor is the product in Lemma 73.20 evaluated at z_0, including the auxiliary spherical factor. At a finite marked place v its factor is $B_{n_{i-1},t}(q_v)$ of Lemma 73.13. Its other factors consist of the specified real integrals and a positive constant depending on the spaces and the Haar normalizations, common to the twists.*

Proof. Apply the polarized Rallis inner product formula to a cuspidal input at the given step. The smaller lifts vanish and $n_i - R > n_{i-1}$ by Theorem 73.17, where R is the target Witt index. For $n_i > 2n_{i-1}$ one has the ordinary convergent Siegel–Weil formula. In the range $n_{i-1} < n_i \le 2n_{i-1}$ one uses Gan, Qiu and Takeda, Propositions 11.1(ii) and 11.2. This expresses the pairing by the specified Laurent coefficient at z_0 of the global doubling integral. In the initial half-plane its open-orbit unfolding is Eulerian on pure tensors. All proper orbits vanish by cuspidality. In particular, with mixed inputs the unfolded group coefficient is $\langle f', R(g)f\rangle_{\mathrm{Pet}}$, up to the common choice of conjugations. If f, f' lie in orthogonal invariant subspaces, it vanishes identically for arbitrary tests. Continuation preserves that orthogonality. The conventional constant in this formula is fixed by the global measures and the pair of spaces. Changing the two unitary splittings tensors the oscillator by determinant characters. Such a twist changes the integrands by precisely the displayed character twists. It changes neither the Haar measures nor the doubled Schwartz identification, which pairs a test with its conjugate. In Tamagawa measures the constant in Proposition 11.2 is $[E : F]$.

We verify regularity at the evaluation point. This matters because an Euler product initially evaluated on the boundary of its absolute convergence need not represent a Petersson norm by mere substitution. At any finite place take a flat extension of the local Siegel–Weil section. On the open orbit its restriction at g equals its value at z_0 times the positive Siegel height $h(g)^{z-z_0}$. At an unmarked place $h(g) \le 1$. Indeed take a primitive top exterior vector of the graph Lagrangian $\{(x, x)\}$ in the direct sum of the two source lattices. Under $(g, 1)^{-1}$ its projection to the second exterior summand is still a lattice generator. Its lattice supremum norm is therefore at least one. By decomposition into the Siegel parabolic times the lattice compact, that norm is the inverse absolute determinant of the Levi acting on the graph. Over a split algebra use the product of the two component absolute values. This gives the asserted bound for the height. It equals one on the source compact. At a real place the same argument with a Euclidean exterior norm bounds the height above by a fixed constant, the norm of the graph vector divided by that of its second projection, using the corresponding complex absolute value.

Let $C_i^S(z)$ be the unmarked auxiliary spherical product for the bottom system at this edge. On $\Re z \ge z_0$ the local integrand defining this product is dominated by the absolute value of the rank-$(s, s+t)$ vacuum coefficient, because a unit-vector coefficient has modulus at most one. The integral over the maximal compact is one. Lemma 142.8 bounds the difference of the local integral from one by $128q_v^{-s-2}$. These errors are summable over the field and are individually less than $1/2$ at all sufficiently distant nodes. The product is holomorphic on $\Re z > z_0$ and continuous and nonzero at z_0, by dominated convergence locally and uniform convergence of

the Euler product. At z_0 each factor is real by inversion of the two unitary matrix coefficients. It is thus strictly positive. This argument requires no temperedness of a lift.

The denominators d_n^S have absolutely convergent nonzero Euler products near z_0, since $2z_0 + 1 = t + 1 > 1$. For $\Re z > z_0$ the extra factors of the lemma have arguments of real part strictly greater than one. At $z = z_0$ their arguments are precisely $1, \dots, t$. Each $\rho = \rho_{ai}$ and ρ^2 has a nonzero angular exponent at infinity because $w_i - w_a = 2(i - a)H$. By Hecke's continuation theorem both L-functions are holomorphic at one. To check nonvanishing there, for real $u > 1$ the Euler logarithm gives

$$\zeta_E^S(u)^3 \, |L_E^S(u, \rho)|^4 |L_E^S(u, \rho^2)| \geq 1.$$

Each prime-power summand in its logarithm has the sign of $3+4\cos v+\cos(2v) = 2(1+\cos v)^2$, with v the argument of the character to that power. A zero at one would force the left side to tend to zero, since the zeta pole is simple. Moreover S is conjugation-stable over E, and $\rho^c = \rho^{-1}$. The Euler factors at a pair of distinct conjugate primes have positive product for real $u > 1$. At a prime fixed by conjugation an unramified value is real of modulus one. Hence $L_E^S(u, \rho) > 0$ for $u > 1$, and also at one by the preceding argument.

At the marks the row section has compact support on the source, and Lemma 73.13 evaluates it with the principal weight. At the real places Proposition 73.18 gives integrable absolute majorants at z_0 and strictly positive values for the specified vectors. The upper height bound gives dominated convergence from the right on bounded subsets of $\Re z \geq z_0$. Consequently the factored expression is holomorphic in the right half-plane adjoining z_0 and has a finite nonzero limit from the right at z_0 for a unit pure input. It agrees with the global meromorphic integral by continuation from large $\Re z$. A pole of that integral could not have this finite limit. Its prescribed Laurent coefficient is therefore its ordinary value.

The marked polarized pairing is scalar on the entire principal-fixed space. At infinity a prescribed output has the target compact highest weight carried by its test, since right translation acts on the test. In the discrete series identified in Lemma 73.16 the highest line of that lowest compact type has multiplicity one. Off the marks the fixed line in a spherical irreducible representation is one dimensional. By multilinearity the same scalar norm formula holds on the specified vectors of an irreducible cuspidal summand, including linear combinations of pure tensors. Between orthogonal source summands the mixed pairing vanishes as shown above, also after allowing group translates of their lifts. Within a common bottom spherical system the formula is independent of the individual summand. Starting at Y_0, the positive formula proves nonvanishing on each nonzero input, and Lemma 73.20 then supplies the spherical list for the next edge. This induction proves the assertion on the successive images. □

Theorem 73.22 (Exact relative determinant of the weighted chain)**.** *Let Z_ϑ be the genuine terminal class-sum Gram determinant of the row and A_ϑ the product of the m auxiliary class-sum Gram determinants, in the normalizations of Corollary 73.12. Put $Q_* = \prod_{v \in \mathcal{Q}} q_v$, and write $\rho_{ai}^0 = (\mu_a^0/\mu_i^0)^{\mathrm{BC}}$. Then all four determinants below are positive and*

$$\frac{Z_\vartheta A_1}{Z_1 A_\vartheta} = \left[Q_*^{-mst} \prod_{1 \leq a < i \leq m} \prod_{h=1}^{t} \frac{L_E^{\mathcal{M} \cup \mathcal{Q}}(h, \rho_{ai})}{L_E^{\mathcal{M}}(h, \rho_{ai}^0)} \right]^D . \tag{73.1}$$

The empty determinant is one. The Euler sets in the numerator and denominator are those of their respective lifts.

Proof. Use Petersson-orthonormal bottom bases adapted to the unmarked finite systems. The unitary module identifications of Theorem 73.8 give the same multiplicities on every auxiliary

source. Proposition 73.21 makes the Gram form of the composite on each system the product of the scalar edge factors, and makes the mixed Gram forms of the orthogonal constituents zero. Converting the terminal coefficient into the class form multiplies this scalar by $d_\infty I_\vartheta$, by Proposition 73.10. The auxiliary rank s integrals have the factor $C_i^S(z_0)$ by the same spherical calculation, or directly by Theorem 142.15. Their fixed real factors are positive. Each auxiliary source integral is unweighted as in Theorem 73.5. Thus all $C_i^S(z_0)$ cancel system by system in Z_ϑ/A_ϑ. The remaining real factors, highest-line dimensions and measure constants are independent of the finite twist. The extra L-values of Lemma 73.20 do not depend on the bottom constituent, so their product occurs to power D.

It remains to keep track of the normalizing L-functions. Let $d_n(z_0)$ include every finite prime of F. At a marked split prime v, $H_{n,t}(q_v) = 1/d_{n,v}(z_0)$. Hence replacing $d_s^S(z_0)/d_n^S(z_0)$ by $d_s(z_0)/d_n(z_0)$ leaves the additional local multipliers $H_{s,t}(q_v)/H_{n,t}(q_v)$ at $v \in S$. They are precisely the multipliers in Corollary 73.15. That corollary includes the terminal level index, the weights of all source integrals, and the class measures of the auxiliary targets. Its normalized local ratio at $v \in \mathcal{M}$ is $|P_m(q_v)|\,|\mathrm{GL}_s(k(v))|^m$, independently of the twist. At $v \in \mathcal{Q}$ it is q_v^{-mst}. The complete finite denominators and all the old multipliers are identical for Z_1/A_1. Dividing the two determinant ratios proves (73.1). Positivity follows from Proposition 73.21 and Theorem 73.5. □

Corollary 73.23 (An interval bound for the relative determinant)**.** *At a node x let $T_{\ell,\iota,x}$ be the left side of* (73.1) *for the simultaneous twist attached to the coefficient prime ℓ and embedding ι. It is a positive algebraic number. Given $\delta > 0$, the twists can be chosen together with the bounds of Corollary 52.21 on the same shorter interval so that, with $v_\ell(\ell) = 1$,*

$$\sum_{\ell} (\log \ell) \sup_{x,\iota} \frac{\max(0, v_\ell(\iota T_{\ell,\iota,x}))}{\max(1, D_x)} \;\le\; \delta, \qquad D_x = \dim H_{Y_0}.$$

Proof. Each quotient in (73.1) is $R_J(\rho_{ai}^0, h, (\vartheta^{\mathrm{BC}})^{a-i})$ of Section 13.1. Indeed the quotient character has tame part ω_w^{a-i} at every old mark and is unramified away from the old marks before twisting. The prescribed order of ω_w has both a 2-part and a 3-part greater than m, by Corollary 13.14. Thus these are exactly its old ramified pairs. The twist powers ramify at every new pair. The two old infinite weights differ in absolute value by $2(i-a)H \ge t$, so every $1 \le h \le t$ is one of the critical points of that corollary. Its algebraicity statement applies in the fixed complex embedding. The remaining scalar Q_*^{-mst} is rational and is a unit at ℓ since the new conductor primes avoid ℓ.

Choose positive e_ℓ with $\sum_\ell e_\ell \log \ell \le \delta$. At a node use error $e_\ell/(1 + t\binom{m}{2})$ per critical factor. These tolerances can be inserted in the simultaneous specialization used in Theorem 52.20. The two distinct prime divisors of the tame orders make the old term of Theorem 13.13 zero for every quotient character at every coefficient prime. Hence the positive valuation of the product per source dimension is at most e_ℓ at every node and embedding. If the dimension vanishes the left side of (73.1) is one. Taking the indicated supremum and sum proves the bound. □

This computes the change with twist of the full complex Gram determinant in the given level metrics. The passage to the specified integral projected period still requires an estimate for the untwisted comparison in its arithmetic lattice and for the divisibility of the projected integral evaluations.

73.10 Integral gluing at the old marked places of a single lift

The factors $\mathcal{V}_j(\beta)$ in Theorem 73.5 are periods on the definite target of the lift from W_j. To compare these periods as the character varies, we compute some of their evaluations on products of coefficient vectors. Fix one of the fields E/F of Section 73.5, an index $1 \le j \le m$, and either $\beta = 1$ or $\beta = \vartheta^j$ at this same node. Split the standard definite target into coordinate subspaces of ranks

$$p = s + 9m + 1, \qquad d' = t - 9m - 1, \qquad n = p + d' = s + t.$$

Both subspaces have room for s coordinate vectors since $t = 5s$, $s \ge x^5$ and $m \le 2\sqrt{x}$ at sufficiently large nodes. Denote their integral coordinate inclusions by e and z, respectively, with $(e, 0)$ the inclusion prescribed by the original lift. The choice of the s distinct rows of z is arbitrary.

Theorem 73.24 (The lattice and the disjoint residue classes)**.** *At an old marked place v write $O = \mathcal{O}_{F_v}$, π for a uniformizer, and $q = |O/\pi O|$. Use its designated split component over E. Put $L_+ = O^p$, $L_- = O^{d'}$, and $L = L_+ \oplus L_-$. For an arbitrary $3m$-by-p matrix Λ modulo π^2, let*

$$K = \{u \in L_+ : \bar{\Lambda}\bar{u} = 0\}, \quad r = \operatorname{rank} \bar{\Lambda}, \quad r_e = \operatorname{rank}(\bar{\Lambda}\bar{e}).$$

There exists an integral matrix γ, invertible over F_v, with $\gamma(e, 0) = (e, z)$ and $v_\pi(\det \gamma) = r + d' - s$. For a matrix $X = (X_+, X_-)$ of s columns in F_v^n, its translated point test satisfies

$$\begin{aligned} \mathbf{1}_{(e,0)+\pi L^s}(\gamma^{-1}X) = \sum_{Y \in \Gamma} \mathbf{1}_{\,e+(\pi K)^s}(X_+Y)\, \mathbf{1}_{\,z+(\pi^2 L_-)^s}(X_-Y), \\ \Gamma = \ker\bigl(\mathrm{GL}_s(O/\pi^2) \longrightarrow \mathrm{GL}_s(O/\pi)\bigr). \end{aligned} \tag{73.2}$$

Use any integral invertible lift of Y. At most one summand is nonzero. Each restriction of γ^{-1} to a coordinate block has only 1 and π^{-1} as elementary divisors, up to units. The multiplicities of π^{-1} are r on L_+ and $d' - s + r_e$ on L_-.

Proof. Take for the column lattice of γ the submodule

$$\mathcal{L} = \{(ec + u, zc + \pi v') : c \in O^s,\ u \in K,\ v' \in L_-\}.$$

It contains πL. Modulo πL the graph $(\bar{e}, \bar{z})(O/\pi)^s$ meets $\ker \bar{\Lambda} \oplus 0$ in zero, since $\bar{z}$ is injective. These spaces span $\mathcal{L}/\pi L$, of dimension $s + p - r$. Thus $[L : \mathcal{L}] = q^{r+d'-s}$. Projection onto the z-rows of the second summand restricts on $\mathcal{L}$ to a retraction onto the coordinates of $(e, z)O^s$. Its kernel is free over the discrete valuation ring. A basis of the kernel completes these s columns to a basis of $\mathcal{L}$. Place the specified columns at the positions of e to obtain γ.

The left side of (73.2) is one exactly on $(e, z) + \pi\mathcal{L}^s$. On this set write

$$X_+ = e(I + \pi C) + \pi U, \qquad X_- = z(I + \pi C) + \pi^2 V, \qquad U \in K^s.$$

Right multiplication by $Y = (I + \pi C)^{-1} \bmod \pi^2$ gives both required congruences, using $\pi L_+ \subset K$. Conversely these congruences give the displayed expression by right multiplication by a lift of Y^{-1}. The square submatrix formed by the z-rows of X_- must reduce to $Y^{-1} \bmod \pi^2$, which proves uniqueness. Both sides vanish off L^s, and changing the chosen lift of Y gives the same indicators.

The map $\pi\gamma^{-1}$ is integral. For $u \in L_+$ the vector $\gamma^{-1}(u, 0)$ is integral exactly when $(u, 0) \in \mathcal{L}$, or equivalently $u \in K$. For $v' \in L_-$ the corresponding condition is $\bar{v}' = \bar{z}\, c_0$ for some $c_0 \in \ker(\bar{\Lambda}\bar{e})$. In fact this is the precise condition for $(0, \bar{v}')$ to lie in $\mathcal{L}/\pi L$. These integrality

loci have codimensions r and $d' - s + r_e$ modulo π. The restrictions of γ^{-1} have integral left inverses, obtained by composing γ with the appropriate block projection. In Smith normal form their exponents therefore lie between -1 and 0. Each exponent -1 accounts for one dimension of the indicated codimension. □

Corollary 73.25 (Evaluation against a principal-level source)**.** *At each old mark $v \in \mathcal{M}$ choose its matrix Λ_v and the matrix γ_v of Theorem 73.24. Complete these to a single finite unitary point γ by using the inverse transpose on the conjugate components and the identity elsewhere. Put*

$$C_\gamma = \prod_{v \in \mathcal{M}} \chi_{W_j,v}(\det \gamma_v) |\det \gamma_v|_v^{-s/2}, \qquad N_{\mathcal{M}} = \prod_{v \in \mathcal{M}} q_v^{s^2}.$$

Here $\chi_{W_j,v}$ is the splitting character of the source in the action of the target, fixed when β varies. In the bilinear theta integral against any prescribed principal-level source vector, translating the original finite test by γ gives $C_\gamma N_{\mathcal{M}}$ times the integral using the following product test. At each old mark its two factors are the cosets $e + (\pi K)^s$ and $z + (\pi^2 L_-)^s$. At each new mark the determinant character and its invertibility test belong to the p-block, and the d'-block is the lattice indicator. Elsewhere use the two original lattice vacua. For an archimedean test factored between the two blocks, the resulting kernel on the block diagonal is the product of the two theta kernels with compatible splittings. The identity holds at each node for each of the stated twists.

Proof. At an old mark the split position model has additive measure giving L^s volume one. The target acts by $\phi(X) \mapsto \chi_{W_j,v}(\det \gamma_v) |\det \gamma_v|^{-s/2} \phi(\gamma_v^{-1} X)$. Indeed the substitution in s columns has Jacobian $|\det \gamma_v|^s$ and the twist of the unitary action is the source splitting character. The original tame determinant on its point coset equals one. In (73.2) the first factor asks for $X_+ Y \equiv e \bmod \pi$ and $\Lambda_v(X_+ Y - e) = 0 \bmod \pi^2$, exactly the two old row conditions. There are $q_v^{s^2}$ choices for Y, obtained by adding to the identity π times any residue matrix. Each product coset has measure $q_v^{-s(p+r)-2sd'}$. Multiplying their total measure by $|\det \gamma_v|^{-s}$ recovers $q_v^{-s(p+d')}$, the squared norm of the untranslated test.

Each Y lifts to a source unitary element at this split place, with inverse transpose at its mate, and identity away from this place. It lies in the source principal subgroup. In the position model its substitution $X \mapsto XY$ has unit modulus and its splitting character is trivial on this principal subgroup because the character is tame. In the global pairing one may move this right translation onto the source vector by invariance of the quotient measure. The vector is fixed. Thus every term has the same integral as the identity term, giving $N_{\mathcal{M}}$. The only additional finite factor at a new mark is the tame character of the determinant of the e-rows, extended by zero at singular reductions, on integral matrices. All these rows lie in the first block. This proves the asserted factorization at finite places.

For the last assertion use a rational polarization adapted to the orthogonal sum. With compatible splittings the two Weil operators tensor on the diagonal source group and the two target groups. Equivalently one uses the same source character on both factors and multiplies the two target splitting characters to get the full target character. The rational position vectors are pairs, one from each position space. For a tensor test the theta sum therefore factors by absolute convergence of the Schwartz sums. This argument includes products of the two archimedean polynomials in disjoint rows times their Gaussians. The integrals above are finite by rapid decay of the cuspidal source against the theta kernels of moderate growth. □

Lemma 73.26 (The integral Cartan inclusion on disjoint coordinate blocks)**.** *For a list c_h of numbers of columns of heights h in rank u, denote by $C_u(c)$ the integral span of products of flag minors of those heights on a generic matrix. A flag minor uses the first h columns and any*

h distinct rows. Use the dual $C_u(c)^*$ as the integral Weyl lattice in this calculation. Suppose the positive and negative lists c^+, c^- fit the p- and d'-blocks. The tensor of two Weyl vectors included from these blocks, the second in the contragredient action, belongs to the mixed Cartan component. In the dual flag-minor lattices the inclusion of this component into the tensor has an integral left inverse. Consequently at a nonarchimedean place any integer clearing the translated tensor clears the corresponding translated mixed vector.

Proof. Let J_n reverse n coordinates. In functions on GL_n take the multiplication map

$$C_n(c^+) \otimes C_n(c^-) \longrightarrow (\det X)^{-u_-} C_n(c), \qquad F \otimes G \longmapsto F(X)G(X^{-\mathrm{t}} J_n),$$

where $u_- = \sum_h c_h^-$ and c has the positive columns together with the complements $n - h$ of the negative ones. The inverse-minor formula, obtained by expanding the determinant along complementary rows and columns, identifies each negative minor with the complementary flag minor of X divided by $\det X$, up to sign. Conversely every generating product in the target can have its columns assigned to the prescribed two lists, even when some heights coincide. Thus the map is surjective over $\mathbb{Z}$. Its target is finite free, being isomorphic to a finitely generated subgroup of the integral polynomial ring. Lifting an integral basis provides a section. Dualizing gives the claimed split injection, equivariant for the standard and contragredient actions by the substitution formula. The left inverse need only be integral as a map of abelian groups.

Over any infinite field of characteristic zero the evaluation functionals at invertible matrices span each dual, since a polynomial zero at all invertible matrices vanishes identically. Take evaluating matrices $B \in \mathrm{GL}_p$ and $D \in \mathrm{GL}_{d'}$ for the two source blocks. The first p columns of $X = \mathrm{diag}(B, (D J_{d'})^{-\mathrm{t}})$ are B included in the first block. The first d' columns of $X^{-\mathrm{t}} J_n$ are D included in the second. The tensor of these evaluations is therefore evaluation on the mixed target of the multiplication map. By bilinearity and the spanning assertion this holds for arbitrary vectors on the two blocks. The integral left inverse takes an integral tensor in the image to its integral preimage. It remains a left inverse after scalar extension or localization, which proves the final assertion. □

Theorem 73.27 (Denominators of the glued coefficient vectors)**.** *Use the Weyl lattices of Lemma 73.26. Let $f = [F : \mathbb{Q}]$, $H = s^8$, and $M = p_0 p_1$ for the two old rational marks. For integral block coefficients of the prescribed weights at any pair of block lattices, the mixed coefficient at their block diagonal point translated on the right by γ is integral after multiplication by M^{E_j}, where one can take*

$$E_j = \sum_{\tau : F \hookrightarrow \mathbb{R}} (3m(w_{j,\tau} + k) + 2Ht^2).$$

On the original fixed-(q_0, d) intervals one has, uniformly in the matrices Λ_v, row choices, j, and the prescribed twists,

$$\log(M^{E_j}) \ll_{q_0,d} f m b_x \log b_x.$$

Proof. In the gauge of the two block lattices the coefficient vectors are integral by assumption. Passing from their block diagonal point h to $h\gamma$ first takes those coordinates at h and then acts by γ^{-1}. At the chosen component the positive rectangular inclusion becomes $\gamma^{-1}\binom{1_p}{0}$. Its minor of height h_0 has denominator dividing $\pi^{\min(h_0, r)}$. Indeed in a Smith basis at most $\min(h_0, r)$ negative exponents occur in any exterior product, and the changes of basis and their inverses are integral. The negative inclusion is acted on by the contragredient, so it uses $\gamma^{\mathrm{t}}\binom{0}{1_{d'}}$, which is integral here. At the conjugate component the positive inclusion is integral, and a negative minor has denominator dividing $\pi^{\min(h_0, d' - s + r_e)}$, by Theorem 73.24 on the second block. At the remaining places both maps are integral.

For a rectangular inclusion A, Cauchy–Binet expands each flag minor of AZ as a sum of flag minors of Z with coefficients given by minors of A of the same height. Multiplying these expansions and evaluating with an integral block vector shows that the exponents just obtained add over its columns. Lemma 73.26 transfers this bound to the mixed coefficient, including the determinant twist. Every local uniformizer in question divides M at that place. We may therefore use the integer M throughout and add exponents over the f embeddings that enter the weight.

We record the column count to keep the dependence on m explicit. Fix τ, write $w_i = \kappa_\tau + (2i+1)H$ and $b' = (t-1)/2$. Let $\sigma_0, \sigma_1, \dots$ start with the $k-1$ consecutive entries from $k/2-1$ down to $1-k/2$, followed by $H - w_1, H - w_2, \dots$, in that order. At height $h_0 \ge 1$ the positive column count is the positive part of $\sigma_{h_0-1} - \max(\sigma_{h_0}, b' - w_j) - 1$. Put $T = b' - w_j$. This count is at most $(\sigma_{h_0-1} - T)_+ - (\sigma_{h_0} - T)_+$, where $u_+ = \max(u, 0)$. The sum telescopes to at most $w_j + k$. The counts vanish for $h_0 > s$, as $\sigma_s = H - w_{m+1} < T$. Each positive column thus costs at most $r \le 3m$.

For $1 \le h_0 < s$ the negative count is the positive part of $\min(\sigma_{s-h_0-1}, -w_j - b') - \sigma_{s-h_0} - 1$. At larger heights it vanishes because both indices in the defining difference are zero under truncated subtraction. The adjacent drops in the list are 1 or $2H$, apart from the drop to $H - w_1$. At this exceptional drop the truncated upper endpoint $-w_j - b'$ is below $H - w_1$ since $j \ge 1$. Thus there are at most $2H$ negative columns at any height and the sum of their heights is at most $2Ht^2$. This proves the claimed integral multiple.

Finally $\kappa_\tau \in \{b_x, x b_x / x_{1,x}\}$ with $x_{1,x} \ge x$. Since $s \le 3x^5$, $t = 5s$, $m \le 2\sqrt{x}$ and $b_x \ge x^{400}$, one has $(2m+1)H \le b_x$, $k \le b_x$, and $2Ht^2 \le b_x$ at every sufficiently large node. Hence $E_j \ll f m b_x$. Corollary 12.10 gives $\log(p_0 p_1) \le 22 \log b_x$ on the same interval. □

The estimate concerns the coefficient vectors in these period evaluations. To control the individual auxiliary determinant one still needs product evaluations that span the required projected image, together with an arithmetic comparison of their values as β varies.

74 Horizontal lifts and first occurrence

74.1 Real first occurrence on each horizontal row

Fix a real place of an original field E/F on an interval of Theorem 71.10. Use $s = r + 6$ for the horizontal rank difference and $t = t_{\mathrm{r}}$ for the vertical difference. Both are odd. Set $u_{ij} = r + is + jt$, $q_j = \lfloor (m-j)/2 \rfloor$ and $h_{ij} = -\frac{1}{2}\mathbf{1}_{i+j \text{ odd}}$. For $A \in \mathbb{R}$ write $I_b(A)$ for the b numbers $A + (b-1)/2 - l$, $0 \le l < b$. Define P_{ij} by adding h_{ij} to each entry in the following signed list.

- The $v_1, \dots, v_{k-1}$ have sign plus. The numbers $-b_a$, $1 \le a \le m$, have sign plus for $a \le j+1$, and alternate starting with minus at $a = j+2$.
- Include $I_t(-e_a)$ for $1 \le a \le j$ and $I_s(c_{m-b})$ for $0 \le b < i$, all with sign plus.

Here $c_l = \lfloor (a_{l-1} + a_l)/2 \rfloor$ as in Theorem 71.16. The a_l are the real core weights. The list defines a discrete series on $U(u_{ij} - q_j, q_j)$. Indeed, the first two kinds of entries together with the strings of length t are distinct by Theorem 72.6. All length-s strings lie above v_1, their centers are spaced by $g_* > s$, and h_{ij} puts the result in $(u_{ij} - 1)/2 + \mathbb{Z}$. There are exactly q_j minus signs. Let $\check{P}_{ij}$ denote the dual, formed by negating the entries within the two compact blocks.

Theorem 74.1 (Horizontal lifts and the cuspidality of their outputs)**.** *For $0 \le i < m$ and $0 \le j \le m$, integrate the original plain horizontal kernel from $V_{i,j}$ to $V_{i+1,j}$ against a form of real type $\check{P}_{ij}$. At the given real place its quotient input has first occurrence at the prescribed target in that Witt tower. The physical output has type $\check{P}_{i+1,j}$. Every nonzero horizontal lift of a cuspidal form of these types at all real places is cuspidal. In particular the second lift B_2 with the prescribed nonzero restriction in Theorem 74.5 is cuspidal.*

Proof. Write $c = c_{m-i}$, $u = u_{ij}$, $p = u - q_j$ and $q = q_j$. The tensor for the kernel is the space of maps from $-V_{i+1,j}$ to $V_{i,j}$. As for the first edge in Theorem 74.5, use $-iH$ on the smaller factor and the Hermitian dual on the larger, with $i = \sqrt{-1}$ in this description. This exchanges the two compact blocks on the smaller side. Integrating the plain kernel against a unitary vector takes its conjugate for the quotient convention. The entries on the smaller side are therefore those of P_{ij}, with the two blocks exchanged. The splitting rule attaches χ_{m-i} to an even-dimensional space and $\chi_{m-i}\eta^{-1}$ to an odd-dimensional one. Their half angular exponents before the parity correction are c, and η has half exponent $1/2$. Since u_{ij} is odd exactly when $i + j$ is even, the half exponents attached to the larger and smaller spaces are respectively $c + h_{ij}$ and $c + h_{i+1,j}$. `Development/AbelianVarietyTheory/Definitions/Divisors/LeastSplitAuxiliaryPrimeThetaChainSpec.lean`, line 168–175,199–201, `Development/AbelianVarietyTheory/Definitions/Divisors/BasePlaceNotMemChiGridBadPlaces.lean`, line 401–409,431–439.

Subtract $c + h_{ij}$ from the input. Every resulting entry is negative, at distance greater than $(s+1)/2$ from zero. For the cloud and the vertical strings their largest unshifted entry is v_1, since $e_a = b_a + H_0$, $H_0 > (t-1)/2$ and $b_a > 0$. Also $c_m \ge v_1 + g_*$ and $c \ge c_m$. Every old horizontal string has top at most $c - g_* + (s-1)/2$. The bound on g_* in Theorem 71.10 proves all the stated distances. The ranks have opposite parity. In Atobe's criterion these distances give $k_\lambda = -1$. After the block exchange there are p negative minus entries and q negative plus entries, so $r_\lambda = p$, $s_\lambda = q$. The target signature $(p+s, q)$ has the form $(r_\lambda + l + 2T + 1, s_\lambda + l)$ with $l = 0$, $T = (s-1)/2$. A member of either counting set at T would have absolute value at most T, which is impossible. Subtracting any positive number of hyperbolic planes forces $l < 0$, again impossible. This proves the first occurrence. Ichino, Theorem 4.3(i), after Atobe.

For the parameter of the nonzero lift, the negative entries exchange compact signs in Ichino's formula. This restores their original blocks. The new block has signature $(s,0)$ and precedes all old entries. More explicitly put $N=u+s$. For an old centered entry β in combined position k', the inducing weight is $\beta-((N+1)/2-k')+c+h_{i+1,j}$. The weight on the new block is $-u/2+c+h_{i+1,j}$, since there are no positive centered entries. Adding the ordered half sum gives the old unshifted entry plus $h_{i+1,j}$ in the first formula, and gives $I_s(c)+h_{i+1,j}$ on the new block. The separation just proved puts these entries in strictly decreasing order across Levi blocks. The blocks are compact, so cohomological induction in stages gives the discrete series of this parameter. Pulling back from the Hermitian dual of the target applies inverse transpose and negates the entries in each block. The resulting physical type is $\check{P}_{i+1,j}$. Ichino, Theorem 4.1(i).

A global lift to a smaller rational member would yield a nonzero local pairing at this real place by fixing the other variables. Hence all smaller lifts of each relevant cuspidal constituent vanish. The constant-term tower property makes its lift at the prescribed member cuspidal and local Howe duality gives the real type above. The assertion extends to sums of constituents and to their smooth vectors by continuity of the theta integral on the cuspidal realization. Apply this first with $j=i=0$ and then with $j=0, i=1$ to the two integrals defining B_2. Ichino, Section 6.3. □

74.2 An explicit integral terminal coefficient for a restriction

Use the original fields, intervals, finite tests and levels of Theorem 75.3. Equip the terminal quotient with probability Haar measure, and take C_{fix} in that normalization. Fix a pure real unit vector v as in that theorem. Write $H(v)$ for its bottom finite multiplicity space and T_v for its ascent with the selected real tests. For the lists of Proposition 75.7, let $e_{c,a,b}$ be supported on level class c, with value $l_a(\rho(k)v_b)$ on its compact real group. Here k is the real variable, $d_\rho=\dim\rho$, and d_0,a_0,L_0 denote the three quantities defined in that proposition.

Theorem 74.2 (A lower bound for one integral terminal coefficient)**.** *For each complex linear functional ℓ on $H(v)$, write $z_{c,a,b}=\ell(P_vT^{\dagger}e_{c,a,b})$, where $T^{\dagger}$ is the reverse integral transform and P_v projects to $H(v)$. With B,H_* as in Lemma 111.2, one has*

$$\frac{B^{-4a_0}e^{-\delta}C_{\mathrm{fix}}}{h_md_\rho}\|\ell\|_*^2\ \le\ \sum_{c,a,b}|z_{c,a,b}|^2\ \le\ \frac{L_0^2B^{4a_0}e^{\delta}C_{\mathrm{fix}}}{h_md_\rho}\|\ell\|_*^2.$$

In particular one of these integral terminal coefficients satisfies

$$|z_{c,a,b}|\ \ge\ \frac{e^{-\delta/2}\sqrt{C_{\mathrm{fix}}}\,\|\ell\|_*}{H_*\sqrt{d_\rho}\,N^{d_0}B^{2a_0}}.$$

This holds for the actual first restriction $\ell(A)=\mathcal{B}(A,T_0w)$. For each prescribed nonzero smooth bottom row T_0w, a pure v can be chosen with $\|\ell\|_*>0$. Uniformly on the same intervals there is a polynomial $P_{\psi,\varepsilon}$ such that

$$\log\left(H_*\sqrt{d_\rho}\,N^{d_0}B^{2a_0}\right)\ \le\ P_{\psi,\varepsilon}(x)+N^2\log\mathrm{Nm}\,\mathfrak{m}.$$

Proof. On class c let $\Gamma_c:W_c^*\otimes W_c\longrightarrow\mathcal{A}_{m,c}$ be the complex coefficient map, using complex fibers of the integral lattices. The target is the full right ρ-isotypic space on that class. Schur orthogonality, with the mass $1/h_m$ of this class included, says that $\sqrt{h_md_\rho}\,\Gamma_c$ is an isometry. It is onto by the compact matrix coefficient decomposition in Proposition 95.4. Products of the two lists have frame operator $S_l\otimes S_v$, as follows by expanding the sum of rank-one operators. Proposition 75.7 places its eigenvalues in $[B^{-4a_0},L_0^2B^{4a_0}]$. Transport by Γ_c divides both bounds

by $h_m d_\rho$. The terminal classes are orthogonal, so the same bounds hold on their direct sum for the union of the transported lists. This sum includes all indices, allowing repeated or zero vectors.

Let $a_\ell \in H(v)$ represent ℓ in the Petersson metric. Moving the kernels across the pairings gives $P_v T^\dagger = T_v^*$ on the terminal space by Theorem 83.14. Hence $|z_{c,a,b}| = |\langle e_{c,a,b}, T_v a_\ell \rangle|$. Applying the frame bounds to $T_v a_\ell$ and using the two inequalities for $T_v^* T_v$ in Theorem 75.3 proves the sum estimate. There are $h_m N^{2d_0}$ summands, so the largest square is at least the sum divided by this number. Taking square roots and using $h_m \le H_*$ proves the individual bound. For the indicated first restriction the existence of the required v follows from the last assertion of Theorem 83.14.

We give the dependence on the arithmetic data. The proof of Theorem 71.10 bounds the rank parameters and the gap g_* by powers of x, with $v_{k-1} = 0$ and $n^3 \sqrt{v_1} \le g_*$. The terminal even core weights are obtained from the core entries and their added strings in Theorem 71.16. The core entries have size at most $v_1 + 1 + (m+1)g_*$, and the centers c_l add at most a further polynomial in x by their defining formulas. Each string displacement is at most its length, and passage from the ordered parameter to the highest weight subtracts a half sum of roots with entries of modulus less than N. Thus f, N, d_0, a_0 are polynomially bounded. The polynomial Schur spaces lie in tensor powers, so $d_\rho \le N^{d_0}$. The cyclotomic modulus M containing E in Lemma 94.22 has $\log M = o(\log b)$ with $\log b = x^\varepsilon$. Its discriminant bound $\log |D_E| \le [E : \mathbb{Q}] \log M$ follows either from the different bound of Lemma 106.1 or from the abelian conductor–discriminant formula. Substitution into the explicit expressions for R, B, H_* gives the final inequality. □

The selected terminal input is integral in the coefficient lattice. The bound for its descended restriction still contains the real norm factor C_{fix} and the dual norm of the first restriction. Relating this complex period to the specified integral projection after all remaining lifts is the comparison required in Section 6.6.

74.3 Cuspidal ascent along the full horizontal row

Write $G_i = U(V_{i,0})$, $u_i = r + is$, $s = r + 6$, and $[G_i] = G_i(F) \backslash G_i(\mathbb{A}_F)$ on each original field of Theorem 71.10. The real signatures are $(u_i - m/2, m/2)$. At the finite places use the tests of Proposition 88.15 and denote their product on edge i by $\varphi_{i,f}$. For a polynomial Fock test p at infinity define the plain theta integral by

$$U_{i,p} b(g) = \int_{[G_i]} K_{V_{i+1,0},\, V_{i,0}}(g, h; p \otimes \varphi_{i,f})\, b(h)\, dh.$$

The quotient convention for the local correspondence therefore takes the conjugate of the physical source form.

Theorem 74.3 (Nonvanishing through the terminal horizontal group)**.** *For every nonzero compact-finite bottom cusp form b of the prescribed real type and old level there are products of real monomial tests $p_0, \dots, p_{m-1}$ such that $U_{j-1,p_{j-1}} \cdots U_{0,p_0} b$ is nonzero for every $1 \le j \le m$. These are cusp forms at the given congruence levels of real type $\check{P}_{j0}$. In a bottom good Hecke system B_0, their good lists in quotient coordinates are $\mathcal{B}_{j0}$ of Theorem 74.7. The ascent can also continue from the particular B_2 obtained in Theorem 75.4, keeping both of its initial tests and its nonzero restriction.*

Proof. Take first an irreducible cuspidal copy at a given stage, whose good list is $\mathcal{B}_{i0}$ for some original B_0. At stage zero this is the definition of the list. The smaller rational targets in the

Witt tower have zero lift by the real first occurrence in Theorem 74.1. Its tower argument also gives cuspidality and the real type of any nonzero output. Put $u = u_i$, $N = u + s$ and let R be the rational Witt index of the target. At any real place a rational isotropic space has dimension at most $m/2$. Hence $N - R > u$, since $s > m/2$. This is the convergent range of the doubled Siegel–Weil integral for unitary groups. For $N > 2u$ one has the convergent doubling identity directly. For $u < N \le 2u$, Propositions 11.1(ii) and 11.2 of Gan, Qiu and Takeda give the constant Laurent coefficient of the doubling integral at $s/2$, up to a nonzero measure constant. Both formulas apply to the mixed inner product for two Schwartz tests and two cusp forms. The center has compact adelic quotient. The original theta integral converges against a cusp form by rapid decrease and the polynomial growth of a Schwartz theta kernel on Siegel sets. Gan, Qiu and Takeda, Sections 11.4 and 11.5.

Unfold on a right half-plane. The open orbit pairs the restricted section with a global matrix coefficient. All other orbits give constant terms of the source cusp form along proper parabolics, hence vanish. For pure tensors in the copy the normalized matrix coefficient factors. Outside S_0 the product is the partial standard value divided by the doubling denominator, using Proposition 88.15. By Theorem 74.7 with $j = 0$ this quotient is holomorphic and nonzero at the theta point. At the marked places the restricted section is compactly supported on the small congruence group, where its flat extension changes it by a bounded positive height to a complex power. These local integrals are entire and give the nonzero stated scalars on the whole fixed space. At infinity the discrete series is tempered, so the local integrals converge locally uniformly near $s/2$. Their pairings on polynomial tests are nonzero by the real first occurrence and Ichino's nonvanishing theorem for the local Rallis form. Density gives the same conclusion on Fock polynomials, using continuity of the local integrals. The unfolded product is thus holomorphic at the required point and its constant coefficient is its value. Ichino, Lemma 7.1 and Proposition 7.2.

To see that the tests detect the prescribed vector, let D be the tensor product of the real source Harish-Chandra modules. Those $v \in D$ that annihilate the mixed real pairing for every pair of polynomial tests and every other real source vector form a submodule. Compact translations on v transfer to the first test by equivariance. Differentiation gives the same transfer for the complexified Lie algebra, whose oscillator action preserves polynomials. The local nonvanishing just proved makes this a proper submodule, hence it is zero by irreducibility. In the fixed finite part of the global copy choose an orthonormal basis e_j and write the input as $\sum_j v_j \otimes e_j$. The old bottom level contains the prescribed principal congruence subgroup, and each subsequent output has the required invariance by Proposition 88.15. The marked pairings are scalar on these finite spaces and the unmarked spherical lines are one-dimensional. If $v_j \neq 0$, pairing the lift with that of a suitable $w \otimes e_j$ consequently gives a nonzero mixed product. This proves nonvanishing for that input.

Distinct orthogonal invariant cuspidal copies give zero mixed products even for different tests. Indeed their global matrix coefficient in the open orbit is identically zero for every group translate. The same observation for all tests makes their invariant theta spans orthogonal. Fixed finite level and the finitely many compact types in the input give finite multiplicity by cuspidal admissibility. Thus a test detecting one nonzero component detects the full input. Expanding a polynomial into monomials selects one product monomial with nonzero output. Such a test gives a compact-finite output. At good places local Howe duality and spherical transfer give the next list. In the split case pulling the target back from its Hermitian dual inverts the list, and taking the conjugate for the next input inverts it a second time. The resulting recursion is precisely that of Theorem 74.7. This proves the induction.

For the assertion with the first two tests fixed, the nonzero restriction in Theorem 75.4 gives $B_2 \neq 0$. Its Hermite tests give compact finiteness, and Theorem 74.1 gives cuspidality and

its real type. At a marked place each individual coset of the integral rectangle is fixed by the principal congruence groups. At an unmarked place the tests in that construction are the lattice vacua, also after the dual-lattice Fourier exchange of Proposition 102.8. Hence the same good transfer identifies the lists of its constituents. Starting the argument at $i = 2$ now proves the assertion, including the case $m = 2$ where the ascent has already reached its endpoint. □

74.4 The adjoint on consecutive line oscillators

Consider $i = 2r + 1$, $r \geq 1$, $i + 1 \leq m$. In the notation of Proposition 121.4 the row spaces decompose as $S = C \perp \mathbb{H}^r$, $T = (D \perp L_n) \perp \mathbb{H}^r$, with $C = C_i$, $D = C_{i+1}$. The old Jacobi line L_o and the middle line L_n of T have respective basis vectors z_o, z_n of norm -1. Identify the lines by these bases. Set $d = -\xi_0$, $\zeta_o = d^r z_o$, $\zeta_n = d^{r+1} z_n$, and $\mathfrak{l} = A/B$. The Jacobi period on S uses ζ_o, and the Bessel period on T uses ζ_n.

Proposition 74.4 (Transfer of the specified Jacobi test)**.** *There is a conjugate-linear transfer of Schwartz tests from the oscillator on* $\mathrm{Hom}(C, L_o)$ *to that on* $\mathrm{Hom}(L_n, C)$*. It identifies the new line theta factor, evaluated on the stabilizer of the isotropic frame in S, with the conjugate of the old Jacobi theta factor. The upper unitriangular character of that frame is treated separately. The finite vacuum for the distinguished old Jacobi line transfers to the full undeepened new line vacuum at each place, up to a nonzero scalar. At a marked place the latter is the sum of* $(Nv)^{a_v \dim_E C}$ *original coset indicators.*

Proof. Let † denote ordinary Hermitian adjoint. The map of position-frequency spaces is $w \mapsto w^\dagger$. Indeed the map from the old to the new distinguished line carrying ζ_o to $-\zeta_n$ is multiplication by ξ_0. In the opposite-row coordinates the inverse skew-adjoint operation on v is $-\xi_0^{-1} v^\dagger$, so their composition on w is $w^\dagger$. The trace of a product in the Hermitian form on the map space is conjugated under adjoint, by cyclic invariance of trace. Its alternating trace after multiplication by ξ_0 therefore changes sign. The map preserves rational points and intertwines the two Heisenberg groups upon inversion of the center. Complex conjugation of a Schwartz function, followed by this change of coordinates and Fourier transformation to the chosen polarization, implements the transfer. With adelic dual measures Poisson summation identifies the rational theta functionals. Equivalently, in the realization as theta functions on the whole Heisenberg group one composes conjugation with the rational change of variables and evaluates at zero. This description fixes their global scalar.

Here are the actions of the unipotent radical of the frame in S. Use its scaled hyperbolic columns $e_j = d^j b_j$, their dual columns e'_j, and the residual space C. For n fixing the columns pointwise let B_n be its block from C to $\langle e_j \rangle$, and Z_n the block from $\langle e'_j \rangle$ to $\langle e_j \rangle$. In the following formulas $x = \zeta_o e_r^t$, $y = -\zeta_n e_r^t = \xi_0 x$, where e_r^t here denotes the last coordinate row on the index set. The old Heisenberg displacement is xB_n. In the new opposite-row variable it is $-\xi_0^{-1}(yB_n)^\dagger = (xB_n)^\dagger$. These formulas follow by composing the block from C with the retained column in the first position and by taking its inverse skew adjoint in the second position.

For the central terms write G_L for the line Gram matrix and $t(v) = \frac{1}{2}\mathrm{Tr}_{E/F}(\xi_0 v)$. The negative of the old argument of the additive character and the new argument, respectively, are

$$t(\mathrm{tr}(x^* G_L x\, Z_n))\,, \qquad t\Big(\mathrm{tr}((\xi_0 \bar{\xi}_0)^{-1} y^* G_L y\, Z_n)\Big)\,.$$

For the first formula the shear of the opposite columns into the retained columns has block Z_n. Its Weyl phase is one half of the alternating pairing of the retained displacement with this shear, giving the displayed trace with minus sign for the old argument. In the opposite row the retained functional is $-\xi_0^{-1} y^* G_L$. Inserting it on both sides of the same shear in the opposite

pairing gives the second trace since the two scalar factors multiply to $(\xi_0\bar{\xi}_0)^{-1}$. The formulas agree since $y = \xi_0 x$. Upper unitriangular changes of basis have determinant one and fix the last coordinate row on the right. This accounts for the whole unipotent action. `Development/AbelianVarietyTheory/Proofs/DecoratedLatticeProofs/JacobiActionsMatchThroughLinkMap.lean`, line 32–66.

On the remaining unitary group $U(C)$ the splittings give conjugate actions. In split matrix coordinates the horizontal character on $U(C)$ is $\gamma_i\eta^{-1}$, and the core character there is γ_i, leaving η^{-1}. The old line in the reversed position has factor η there. The ratio of the two actions after adjoint and conjugation is a character by Heisenberg irreducibility. It is trivial on rational points by the theta functional, at split places by this calculation, and on the full compact stabilizer at almost all nonsplit places by the integral vacua. At those nonsplit places an unramified character of the unitary group is trivial by the hyperbolic-plane argument of Proposition 74.9. Weak approximation at the remaining finite set and the archimedean places gives triviality there as well, as in that proposition. This proves the asserted equality of theta factors.

Finally put $M = \mathrm{Hom}(\Lambda_C, \mathcal{O}_E z_o)$. The old line displacement lattice is $\mathfrak{Al}^{r}(\mathfrak{m}/\bar{\mathfrak{m}})M$. This is the line rule used in Theorem 74.6, with the indicated power for the r-th line. The new undeepened line displacement is $\mathfrak{Al}^{-(r+1)}M^{\dagger}$, by the middle ideal in T. These are exchanged exactly by adjoint, since

$$\bar{\mathfrak{Al}}^{-r}(\bar{\mathfrak{m}}/\mathfrak{m}) = \mathfrak{Al}^{-(r+1)}.$$

Both are self-dual for the trace form. Over a split place a full translation and dual-frequency lattice has a one-dimensional space of invariant tests, spanned in position coordinates by its lattice indicator. This follows from the support and translation equations. The same vacuum calculation in a self-dual polarization applies at the unmarked nonsplit places. Thus the transferred line is the claimed vacuum, whose marked expansion partitions the full lattice at depth a_v. With full lattice indicators chosen at all finite places, one may incorporate the resulting nonzero normalization scalar in a single real line test. □

74.5 Nonvanishing at the second horizontal lift

Retain the original fields, intervals, lattices and characters of Theorem 71.3. Use $[B]$ to denote the automorphic quotient of $U(B)$ and $K_{D,B}$ for the kernels of Proposition 74.9. For a fixed bottom row realization T and a prescribed smooth vector Tw write

$$B_1(g) = \int_{[Y]} K_{S,Y}(g,h;p)\,(Tw)(h)\,dh, \qquad B_2(j) = \int_{[S]} K_{Z,S}(j,g;\Phi)\,B_1(g)\,dg.$$

Theorem 74.5 (A restriction after two horizontal lifts)**.** *For every nonzero smooth Tw, the tests may be chosen so that B_1 is cuspidal and, with a descending core form A_2 on C_2,*

$$\int_{[C_2]} A_2(k)\,B_2(k \oplus 1)\,dk \neq 0.$$

One terminal integral class coefficient, all descending tests above C_2, and the entire second horizontal test Φ can be fixed for the given realization T. At the marked places Φ is an individual integral coset indicator for the actual lattice on Z, including its negative line. Only the real part of the first horizontal test p has to vary with w. Before selection of a single coset, the second restriction for the Fourier-transformed test equals the first projected seesaw period of Theorem 71.3.

Proof. We first verify the tower property required on the intermediate group. The dual of the bottom row has the signed parameter $\mu = (v_1, \dots, v_{k-1}, -b_1, \dots, -b_m)$ of Theorem 71.10.

Its cloud has plus sign and the remaining signs alternate beginning with plus. Put $p_Y = k - 1 + m/2$, $q_Y = m/2$. The plain horizontal kernel is integrated against Tw, so its quotient input is the conjugate, of parameter μ. In the tensor $\mathrm{Hom}(-S, Y)$ take the Hermitian form on the dual of S and the skew form $-iH_Y$ on Y, up to the same positive real rescaling as in Theorem 71.16. The latter form exchanges the two compact blocks on the smaller group. The target angular half exponent is c_m, since this target has even rank. All entries of $\mu - c_m$ are negative and have distance greater than $(d+3)/2$ from zero, since the largest is $v_1 - c_m$ and c_m is within one of $(a_{m-1} + a_m)/2$.

The ranks now have opposite parity, so the absence of zero gives $k_\lambda = -1$. In the exchanged blocks there are p_Y negative minus entries and q_Y negative plus entries. Thus $r_\lambda = p_Y$, $s_\lambda = q_Y$. The actual target has signature $(p_Y + d + 1, q_Y)$. In Atobe's criterion this is $(r_\lambda + l + 2t + 1, s_\lambda + l)$ with $l = 0, t = d/2$. Both counting sets are empty by the distance just proved. For a smaller member of this Witt tower $l < 0$, violating a necessary condition. Thus at every real place the given target is the first occurrence. The global constant-term tower property now gives cuspidality of B_1 whenever it is nonzero. It applies to the smooth row vector by continuity of the theta lift on its smooth cuspidal realization. Ichino, Theorem 4.3(i) and Section 6.3.

Fix initially one nonzero lowest-type vector. By Theorem 71.3 take its indicated p, its positive line test q (conjugating the negative line test there), and its descending form A on S. Write $A(g) = \int_{[C_2]} K_{C_2,S}(k, g; b) A_2(k)\, dk$. Here b and A_2 are the original test and the next form of that descent. If C_2 is terminal, A_2 is the class coefficient itself. The first period is $\int_{[S]} A(g) L_q(g) B_1(g)\, dg$ by Proposition 74.9, so $B_1 \neq 0$. It is compact-finite on the target for the product Hermite test and is a cusp form by the preceding paragraph. Substitute the expression for A into this last integral. Both B_1 on S and A_2 on C_2 decrease rapidly. The line theta function has polynomial growth, as does the second core kernel on the product of the two Siegel sets by the Schwartz estimate of Theorem 71.14. Hence this double integral converges absolutely. Grouping the two kernels by Proposition 74.9 and integrating first over $[S]$ gives the displayed restriction, with $\Phi = b \otimes Uq$. This order of integration uses a cusp form on each noncompact integration group.

At a marked place the C_2-part of b is one core rectangle indicator. Proposition 102.8 expands the new line test as an average of D_L individual residue indicators with coefficients of absolute value one before division by D_L. Tensoring with the rectangle gives a single marked map coset per term, since $\Lambda_Z = \Lambda_{C_2} \oplus \mathfrak{l}z'$. All the specified residues lift simultaneously to integral maps by projectivity and the Chinese remainder argument in Theorem 102.7. Keep the unmarked and real reference factors common to all terms. The transformed period is then a weighted sum of the individual periods with total absolute weight one. At least one individual period has absolute value at least that of the transformed period.

Fix this residue choice. At each marked place its inverse Fourier transform in the line variable is supported on the old integral position lattice M_v and is a character there, constant modulo $\varpi_v^{a_v} M_v$. This follows by integrating over a translate of $M_v^\vee$, exactly as in Proposition 102.8. It satisfies the old depth equations. Thus for the selected term the reverse use of the two kernel factorizations expresses its nonzero value as the first restriction of a fixed compact-finite cusp form on C_0. Its first real test is still a product on Y, L. As in Theorem 71.3, translate an arbitrary nonzero smooth vector in the irreducible bottom row against this fixed, bounded form. Transfer the translation to the Y-test, and expand that one real variable in Hermite functions. One product Hermite test gives a nonzero value. The finite positions, the real line test, b, A_2, and the selected second horizontal test are all fixed. The same factorizations and the cuspidality of the new B_1 give the displayed nonzero second restriction for this w. □

74.6 A nonzero period after the third horizontal lift

On the fields and intervals of Theorem 75.4, take $m \geq 4$. Keep the particular terminal integral coefficient, descending tests and first two horizontal tests giving its restriction $J \neq 0$. Write the latter as

$$J = \int_{[U(C)]} A_2(h) B_2(h \oplus 1)\, dh, \qquad A_2(h) = \int_{[U(D)]} A_3(k) K_{D,C}(k, h; b)\, dk.$$

Here $C = C_2, D = C_3, S = V_{2,0}, T = V_{3,0}$, and b is the third descending core test. These cusp forms and kernels are those in Theorems 74.5 and 71.13. Use $u = \dim_E S$, and the Haar conventions of Lemma 111.18 and Proposition 88.9. If a fixed scalar relates the original group measures to their finite products, include it in the corresponding archimedean quotient measure.

Theorem 74.6 (A first positive-length Fourier–Jacobi period)**.** *There is a third horizontal Schwartz test at the original depth, with one individual coset indicator at each marked place, for which the Fourier–Jacobi period P of its lift of B_2, paired with A_3 at the distinguished character, satisfies*

$$|P| \;\geq\; \frac{c_f\, \|q\|_2^2}{2(N\mathfrak{m})^u} |J| \;\geq\; \frac{\|q\|_2^2}{2(N\mathfrak{m})^{3u-1}} |J|.$$

The same line test q may be used for every such pair of forms on the given field. Its finite part is the integral lattice test specified below. The real position test used for this inequality has integral one on the norm shell and may depend on the forms. For nonvanishing one may take a single product of real Hermite functions as the entire third horizontal real test.

Proof. Use the notation and position $x = X(e)$ of Proposition 75.5. In the residual variable on $\mathrm{Hom}(-D, S)$ take the test $b \otimes q$ for $S = C \perp Ez$. At an unmarked place take for q the lattice vacuum on the line summand. At a marked place use the indicator of the zero coset modulo π^{a_v} in its original integral position lattice. This is admissible jointly with the original core indicator b on the other summand. It also has the line depth prescribed for this Fourier–Jacobi step. In fact the full, unmarked line displacement for the summand would be $\mathfrak{A}\,\mathrm{Hom}(\Lambda_D, \mathfrak{l}z)$. The separate line displacement multiplies this by $\mathfrak{m}/\bar{\mathfrak{m}}$. At a marked place this multiplies positions by π^{a_v} and their dual frequencies by π^{-a_v}, whose common vacuum is exactly the indicated zero-coset indicator. Take for its real test a fixed nonzero Gaussian, with a fixed norm. This defines q before choosing any row. `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/IsAuxiliaryVectorWittDec.lean`, line 177–184.

Take $a_f = 1_{z+\mathrm{j}\widehat{L}}$. The function $\mathcal{R}$ in Proposition 88.9 is constant in its finite variable on the image of K_f. The form B_2 is fixed by this group by Theorem 74.3. At a marked place the residual tensor is a single coset in the rectangle with target lattice Λ_S. Principal congruence acts trivially on its residue, and the determinant character is trivial at this depth. At a good place the tensor is the vacuum for the orthogonal sum. It is fixed by the full stabilizer of Λ_S by the lattice calculation of Proposition 88.15. The larger residual space D has even rank, so one may polarize D and act on the odd space by substitution. At split places the same statement follows from matrix substitution. The splitting on the small group in Ω uses the character attached to T, the same γ_2 as for D. It follows that $\Omega(k_f)(b \otimes q) = b \otimes q$ for $k_f \in K_f$. The formula for b_g gives the asserted invariance.

By Lemma 111.18 the finite orbit integration is now multiplication by c_f. On the real orbit $H_\infty \backslash G_\infty$, write $R_\infty(\dot{g}) = \mathcal{R}(g_\infty, 1)$. By Proposition 88.9 it is continuous and has value $\|q\|_2^2 J$ at the base point. Choose a product neighborhood there on which its difference from this value has modulus less than $\|q\|_2^2 |J|/2$. The real orbit map is a smooth identification with the norm shell

at each real place. Its differential is onto the tangent space by transitivity of the unitary group and its stabilizer is H_∞. The shell is closed and smooth since its norm value is nonzero. Choose nonnegative smooth bumps on the shell with compact support in this neighborhood. In normal coordinates extend them to ambient compactly supported smooth functions by multiplication with a transverse bump equal to one at zero. Normalize their product a_∞ to have shell integral one. After multiplying the integral by $\bar{J}/|J|$, its real part is at least $c_f\|q\|_2^2|J|/2$. This proves the lower bound before returning to the original marked polarizations.

Proposition 75.5 writes the marked hyperbolic test as $(N\mathfrak{m})^u$ original indicators with unit-modulus coefficients. Tensoring each with the fixed residual indicator gives a single marked coset on the entire edge. Linearity and the triangle inequality give the first displayed inequality for some one of these choices, and the measure estimate gives the second. At the other places the unmarked vacua are unchanged by dual-lattice Fourier transformation. The original A_3 and B_2 are used throughout.

For this finite choice the period is a continuous linear functional of the real Schwartz test by the absolute seminorm bound in Proposition 88.9. Finite sums of products of Hermite functions are dense in the real Schwartz space, also after the metaplectic change of polarization. If all individual products had zero period, continuity would give zero on the test just constructed. Thus some individual product has nonzero period. Theorem 74.1 gives cuspidality of the corresponding third horizontal lift. The quantitative bound above uses the normalized smooth bump. Quantitative Hermite selection would additionally require a bound on the real seminorms of that bump. □

74.7 Doubling values with both lengths of theta strings

Fix the original interval, E/F, a word and its unitary splitting characters. Write $s = r+6$, $t = t_{\mathrm{r}}$, and put $\gamma_i = \chi_{m-i}$ for $0 \le i < m$. Write μ_a for the vertical character at step a, including the word twist. The angular half exponents of γ_i, μ_a are respectively $c_{m-i,\tau}$ and $-e_{a,\tau}$. The word twist has finite order and thus contributes zero at the connected complex places. Use the original marked set S_0. One may also omit places under the finite twist primes of the word. Write S for the resulting set containing the finite ramification under consideration and S_E for the places above it. The rational primes under S_0 split completely by Proposition 122.7. At an additional twist prime both residue degree and ramification index over $\mathbb{Q}$ are one by the definition of these places. Since $E/\mathbb{Q}$ is Galois, that rational prime also splits completely. `Formalism/LocalToGlobal/IndefiniteReal/SubgroupCharExtensionOneApply.lean`, line 91–93.

At $w \notin S_E$ let $B_{0,w}$ be the rank-r multiset of a unitary spherical bottom row in the quotient convention. Set $Q = Nw$, $A_b(w) = \{Q^{(b-1)/2-l} : 0 \le l < b\}$, and $D_{ij} = \eta^{-\mathbf{1}_{i+j\ \mathrm{odd}}}$. Write

$$\mathcal{B}_{ij,w} = D_{ij}(w)\left(B_{0,w} \sqcup \bigsqcup_{a=1}^{j} \mu_a(w)A_t(w) \sqcup \bigsqcup_{h=0}^{i-1} \gamma_h(w)A_s(w)\right).$$

These are the unramified parameter lists obtained by first taking j forward vertical transfers and then i horizontal transfers. They can equally be used to evaluate the Euler products before constructing a global lift.

Theorem 74.7 (The partial standard values at every horizontal step)**.** *Put $z_* = (s+1)/2$, $H = (t-s)/2$, an integer positive on the given sufficiently distant intervals. For every $0 \le j \le m$ and $0 \le i < m$ the partial standard product for $\mathcal{B}_{ij} \otimes (\gamma_i D_{ij})^{-1}$ is holomorphic and nonzero*

at z_*. Its value is

$$L^S(z_*, B_0 \otimes \gamma_i^{-1}) \times \prod_{h=0}^{i-1} \prod_{a=1}^{s} L_E^{S_E}(a, \gamma_h/\gamma_i) \cdot \prod_{b=1}^{j} \prod_{a=1-H}^{s+H} L_E^{S_E}(a, \mu_b/\gamma_i).$$

Write $f = [F : \mathbb{Q}]$. The absolute logarithm of the modulus of the initial factor is at most $2fr \log \zeta(4)$. The two products of Hecke values are independent of the bottom Hecke eigencharacter. This evaluates the unramified numerator of the doubling formula on the original fields, including all the long strings when $j > 0$.

Proof. For a step with underlying character ν, the characters attached to the larger and smaller spaces are νD_{old} and νD_{new}, respectively. The quotient formula multiplies the old list by $D_{\mathrm{new}}/D_{\mathrm{old}}$ and adds $\nu D_{\mathrm{new}} A_b(w)$, where b is the rank difference. This is Theorem 72.16 and Proposition 72.17 for the two kinds of place. On a horizontal step the physical action on the target is pulled back from its dual. At the next step the conjugate of that physical unitary representation enters the quotient convention. At a split place both operations invert the standard multiset, restoring the quotient list. At an inert place the standard list is reciprocal. This gives the stated recursion from the vertical list of Theorem 72.10.

Divide by $\gamma_i D_{ij}$. On a right half-plane of absolute convergence, disjoint union multiplies Euler products and a root Q^λ replaces the Hecke argument z by $z - \lambda$. At z_* the strings of length s give the integers from 1 to s. Those of length $t = s + 2H$ give the integers from $1 - H$ to $s + H$.

We bound the initial factor at this shorter spacing. At a split place the spherical bound gives $|\alpha| \le Q^{(r-1)/2}$ for each $\alpha \in B_{0,w}$, by the split proof of Lemma 75.1, which holds in either parity. At an inert place write $r = 2a + 1$. Besides $e_l \pm e_k$ of multiplicity two and $2e_l$ of multiplicity one, the relative roots include e_l of multiplicity two, from the coordinates between the anisotropic line and the l-th hyperbolic plane. Thus $\langle \rho_F, e_1 \rangle = 2(a-1) + 1 + 1 = r - 1$ when $a \ge 1$. Macdonald's inequality in that lemma applied to the Weyl orbit of e_1 bounds the reciprocal pairs by $(Nv)^{r-1} = Q^{(r-1)/2}$, with v below w. The remaining entry is one, also in rank one. The difference $z_* - (r-1)/2$ is four. Each local logarithmic series is therefore absolutely bounded at z_*, per root, by $\sum_{k \ge 1} Q^{-4k}/k$. They converge locally uniformly for $\Re z > (r+1)/2$. Summing over the at most $2f$ primes of E above each rational prime gives the two asserted bounds and holomorphic nonvanishing of the initial product.

For $h < i$ the half exponents of γ_h/γ_i equal $-(i-h)g_*$ at the selected embeddings. These characters have nonzero angular exponent and their positive-integer values are nonzero by Lemma 72.9. For μ_b/γ_i the half exponents are $-e_{b,\tau} - c_{m-i,\tau}$. Their absolute values exceed H at every selected embedding. Indeed $e_{b,\tau} > 0$, $v_1 \ge 0$ and $c_{m-i,\tau} \ge c_{m,\tau} \ge v_1 + g_*$, where Theorem 71.10 gives $g_* \ge 2(k + m(t+s+1))^3 > t$. These Hecke functions are entire since their archimedean characters are nontrivial on the circles. At positive integers apply the same lemma. At $a = -u \le 0$ here $0 \le u \le H - 1$. The complete gamma factors are regular and Proposition 25.26 proves nonvanishing of the partial values even at zero. This proves both continuation and the formula.

The unramified doubling denominator at parameter $s/2$ has factors $L_F^S(s + k, \omega_{E/F}^{s+k})$, $1 \le k \le u_{ij}$. Their Euler logarithms converge absolutely near these points because $s + 1 > 1$, so the same holomorphic nonvanishing holds for the normalized good-place quotient. The marked integrals and the prescribed global projection are additional factors in a period calculation. □

74.8 The relative norm on the full horizontal row

On one of the original fields let v be a pure unit vector in the prescribed bottom real row representation, compact-finite at each real place. Let $H_Y(v)$ be the image of this fixed vector tensored with the full multiplicity space in the old-level cuspidal spectrum. Give this space and the terminal horizontal cuspidal space their Petersson norms. Write $f = [F : \mathbb{Q}]$, $s = r + 6$, and $z_* = (s + 1)/2$.

Theorem 74.8 (A relative Petersson bound on the horizontal row)**.** *When $H_Y(v) \neq 0$, one can fix a single sequence of products of real monomials in Theorem 74.3 for the entire space $H_Y(v)$. Let U_v denote the resulting lift through $V_{m,0}$. There is a positive scalar C_Y, common to every bottom good Hecke system and its finite multiplicities, such that*

$$e^{-\delta_Y} I \;\le\; C_Y^{-1} U_v^* U_v \;\le\; e^{\delta_Y} I, \qquad \delta_Y = 2mfr \log \zeta(s/2).$$

In particular $\|C_Y^{-1} U_v^* U_v - I\|_{\mathrm{op}} \le e^{\delta_Y} - 1 \ll mfr\, 2^{-s/2} \le \exp(-x^{\varepsilon/8})$ on all sufficiently distant original intervals, with the implicit constant absorbed by increasing the starting point.

Proof. Start with a unit vector on the fixed real line of one cuspidal copy. Theorem 74.3 gives real tests with nonzero output at every step. At each real place the joint theta quotient for the specified source and target is unique up to scalar by local Howe duality. Contracting the test with the source vector therefore selects the same real target line in every copy with that real input. This follows also by factoring the global theta pairing at that real place after fixing all its other arguments. On the reference copy the selected line is nonzero. Choose a unit vector on it. Repeating at each step determines the local real Rallis factors independently of finite Hecke eigenvalues. They are nonzero by the positive norm of each reference lift.

For another bottom system β, let B_β denote its initial standard Satake lists in quotient coordinates. On the first step use the same real tests and input line. The mixed formula in Theorem 74.3 evaluates the diagonal product with these real factors. The marked constants are independent of the copy. At the other finite places all the terms from earlier strings and from the doubling denominator depend only on the step and the common field and characters. Indeed the standard numerator in Theorem 74.7 at this step, after dividing out its horizontal string product, is $L^{S_0}(z_*, B_\beta \otimes \gamma_i^{-1})$. That theorem proves its nonvanishing. The squared norm is thus positive and the output has the real line already selected. Induction gives the exact squared norm multiplier on this copy in absolute-value notation as

$$C_Y \prod_{i=0}^{m-1} \left| L^{S_0}(z_*, B_\beta \otimes \gamma_i^{-1}) \right|.$$

The measure factor at a step comes from the fixed Haar measures in the unfolded matrix coefficient, normalized by the squared norm of the input. The real factors, good string factors, denominators and marked volumes absorbed into C_Y are therefore common to all copies. The marked scalar formula extends to every vector in their finite fixed spaces by polarization. Orthogonality of the invariant theta spans proved in Theorem 74.3 gives zero off-diagonal products between distinct orthogonal copies through every step. Consequently the displayed multipliers are eigenvalues of $U_v^* U_v$ on the entire multiplicity space.

At w outside the marked places every root of the actual bottom list has absolute value at most $(Nw)^{1/2}$ by Proposition 71.17. The characters γ_i are unitary. The absolute logarithm of the modulus of each varying product factor is thus at most

$$r \sum_w \sum_{a \ge 1} \frac{(Nw)^{-as/2}}{a} \;\le\; 2fr \log \zeta(s/2).$$

Here the sum on the left may include all finite places of E by positivity, and there are at most $2f$ places over each rational prime. Summing over the m steps gives the operator inequalities. Finally $\log \zeta(t) \le 3 \cdot 2^{-t}$ for $t \ge 2$ by integral comparison. The quantities m, f, r grow at most polynomially in x, and $r \ge m \gg x^{1/2-\varepsilon/16}$ on the original intervals, as in Theorem 75.3. These estimates give the asserted norm error. □

The designated Bessel coefficient involves applying the particular Fourier projection of Corollary 83.7 to a terminal form. The estimates here control its full Petersson norm. A lower bound for that coefficient still requires an estimate after the projection.

74.9 Cancellation of the splittings through the second core

Use Y, C_0, S, C_2, Z of Proposition 102.8. Denote by $K_{D,B}(g, h; \phi)$ the original theta kernel on $\mathrm{Hom}(-D, B)$, with the argument on the larger space D written first. At the first core edge and horizontal edge use the underlying character χ_m, and at the second such edges use χ_{m-1}. To a space of even rank attach that character, and to a space of odd rank attach its product with η^{-1}. This is the original prescription in `Formalism/LocalToGlobal/IndefiniteReal/HasInfExpF.lean`, line 13–23. Let U denote the rational adjoint intertwiner on line tests from Proposition 102.8, normalized by adelic Fourier inversion and rational Poisson summation.

Proposition 74.9 (An identity with the original four splittings)**.** *For Schwartz tests p, q, b on the Y-part, the old positive line part and the second core edge, respectively, one has*

$$K_{S,C_0}(g, h \oplus 1; p \otimes q)\, K_{C_2,S}(k, g; b) = K_{S,Y}(g, h; p)\, K_{Z,S}(k \oplus 1, g; b \otimes Uq).$$

There is also a line theta function $L_q(g)$, independent of h, k, such that the first kernel on the left factors as $K_{S,Y}(g, h; p) L_q(g)$, and the second kernel on the right as $K_{C_2,S}(k, g; b) L_q(g)$. These equalities hold for the entire adelic groups.

Proof. The two total symplectic spaces identify over F. The map is the identity on the common Y and C_2 parts and the Hermitian adjoint on the line part. It intertwines the actions of $U(Y) \times U(S) \times U(C_2)$ on the underlying tensors. The oscillator for a direct sum is the tensor product of the oscillators. In a position model this follows by factoring the translations and modulations and using the irreducibility of the Heisenberg representation. A rational change of polarization preserves the global theta functional under the normalized intertwiner. For a Fourier exchange on a subspace this is Poisson summation for its rational points. For a rational change of basis it is reindexing. Consequently the ratio of the two group splittings after transport is a continuous unitary character, trivial on rational points. We check this character at the places where the scalar is explicit.

At a split finite place write a, b_0, c for the determinants on the chosen halves of h, g, k, respectively. Evaluate all characters on that half. For $\mathrm{Hom}(-D, B)$ the circle factor is the character attached to D evaluated on $\det h_B$, divided by the one attached to B evaluated on $\det g_D$. The other factor is the positive square root of the Jacobian of the position change of variables. Fourier exchange carries the unitary substitution on a position space to the unitary dual substitution, preserving its circle scalar. This can be checked by changing variables in the Fourier integral. Thus the two circle products to compare are

$$\chi_m(a)\chi_m(b_0)^{-1}\chi_{m-1}(b_0)\chi_{m-1}(c)^{-1},$$
$$\chi_m(a)(\chi_m\eta^{-1})(b_0)^{-1}(\chi_{m-1}\eta^{-1})(b_0)\chi_{m-1}(c)^{-1}.$$

They agree, including when these characters are ramified. The equal positive Jacobians are forced by unitarity of the same substitution. This proves triviality on each split finite group.

The determinant convention is specified in `Formalism/LocalToGlobal/IndefiniteReal/LocalNewformFunction.lean`, line 51–75.

Here is why those places suffice for the adelic equality. At almost every nonsplit place the ratio is trivial on the hyperspecial subgroup. Indeed the original splittings are unramified there, and the rational isometry and its inverse are integral there. The standard lattice vacua match under the change of polarization, with any common scalar immaterial. A character of such a unitary group trivial on that compact subgroup is trivial on the whole group. To check this, choose integral hyperbolic planes and, in odd rank, one remaining unit line. Integral hyperbolic splitting follows by the residue norm and Hensel argument of Proposition 122.8. The Cartan decomposition reduces the assertion to elements acting on one plane by ϖ^{b}, ϖ^{-b} for a uniformizer of F_v. They lie in the special unitary group of the hyperbolic plane, the split $\mathrm{SL}_2(F_v)$. This group is generated by its two elementary unipotent subgroups, each consisting of commutators by conjugation with a diagonal element whose square differs from one. Every character is trivial there.

The ratio is thus supported at finitely many places, including infinity. A unitary group has weak approximation at every finite set of places of F. Indeed the Cayley transformation $X \mapsto (1-X)(1+X)^{-1}$ identifies open subsets of its skew-Hermitian Lie algebra and the group. The group open subset is locally dense because scalar norm-one units arbitrarily close to one can move a given element off the divisor with eigenvalue -1. The Lie algebra is an F-vector space, so weak approximation holds there, and then on the group by this parametrization. This also applies to the product of the three groups. Triviality on its dense rational image in the places of support forces the remaining character to be trivial. Equivariance of the theta functional now proves the identity of the two products.

For the separate factorizations, first restrict the splitting of S, C_0 to $U(Y)$. Its action on the line tensor is the identity up to a character. Comparison with $K_{S,Y}$ gives the same split scalar $\chi_m(a)$ and hence exact agreement by the preceding character argument. On $U(S)$, divide out the action on the Y-tensor. The resulting operators act only on the line tensor, since they commute with the entire Heisenberg action on the other factor. The group law defines a line oscillator action there. On rational elements it preserves the line theta functional by the product formula for the rational theta functional. This gives $L_q(g)$. The already proved equality of the transported actions on the total tensor identifies the action on the C_2-and-line part with the product of this line action and the original C_2, S action. Applying the rational theta functional gives the second factorization as well. □

75 Norms on the even core

75.1 The spherical bound at the smaller core spacing

Here $n = k + m$ is the even rank of the first core. At a good place v of F let $q = Nv$, let $w \mid v$ in E and write $Q = Nw$. In the normalized Satake convention the spherical character of the trivial representation is the Weyl orbit of the half sum of positive roots with their root-space multiplicities, exponentiated with base q. Use the standard base-change multiset of size n at w.

Lemma 75.1 (The sharp general exponent for a unitary spherical character)**.** *For a spherical eigencharacter in the unitary bottom space of Proposition 71.12, each entry α of its standard multiset satisfies*

$$Q^{-(n-1)/2} \le |\alpha| \le Q^{(n-1)/2}.$$

The estimate also holds for the conjugate representation. In particular, with $d = n + 4$, the difference between $(d+1)/2$ and the upper exponent is exactly three.

Proof. We recall Macdonald's bounded spherical function theorem in the normalization needed here. Let t be the normalized Satake character of a hyperspecial spherical algebra, on a coroot translation lattice of its semisimple root system. If the character is bounded on the normalized double-coset averages, then for dominant λ in that lattice and each relative Weyl element w_0,

$$|t^{w_0\lambda}| \le q^{\langle \rho_F, \lambda \rangle}.$$

Here $2\rho_F$ is the sum of the positive relative roots counted with their dimensions over the local field. This is the convex-hull form of Macdonald's Theorem 4.7.1. See also J. Parkinson, Appendix A, Remark A.1 and Theorem A.3. In these conventions the positive character on the right is the square root of the modulus in the expanding direction. This follows by taking the determinant of conjugation on the positive-root spaces. Our eigencharacter has absolute value at most one on each normalized double coset since the corresponding operator is a probability average of unitary translations. For a unit simultaneous eigenvector, its normalized local matrix coefficient is constant on each double coset and equals the corresponding averaged eigenvalue. It is precisely the bounded spherical function with this Satake character. In applying the theorem on the derived group one may restrict a spherical vector to its derived-group action. Its spherical coefficient there has the restricted Satake parameter. Indeed the normalized unramified induction model gives its spherical function by the compact average of the unramified inducing section, and the derived-group Cartan and modulus are the restrictions of those of the original group. More explicitly, its hyperspecial group is the product of the integral torus and the hyperspecial subgroup of the derived group. At a split place this follows by taking a diagonal unit of the required determinant. At an inert place every norm-one determinant is $u/\bar{u}$ for a unit in the unramified quadratic extension, by Hilbert 90 and multiplication by a base-field uniformizer power. The two entries $u, \bar{u}^{-1}$ in one hyperbolic plane realize this determinant. The inducing section is left invariant by these torus units, so its compact average on derived-group elements equals the average for the derived group. A positive integral multiple of any coweight in the real span of the coroots suffices when the coweight itself is outside the coroot lattice, since one can take that root of the resulting absolute-value inequality.

At a split place write $t = (t_1, \dots, t_n)$ for the parameters of $\mathrm{GL}_n(F_v)$. The central character on the scalar uniformizer has absolute value one, because that translation and its inverse are unitary. Hence $\prod |t_j| = 1$. Take the dominant coroot-lattice vector $ne_1 - \sum_j e_j$, where the e_j are the standard coordinate vectors. The half-sum has entries $(n+1-2j)/2$. The inequality for its Weyl conjugates gives $|t_i|^n \le q^{n(n-1)/2}$ for every i. Applied to the unitary contragredient, whose standard parameters are the inverses, it gives the lower bound. The second prime over a split place has the dual parameter, so both bounds hold there too.

At an inert place write $n = 2a$. In relative coordinates for the quasi-split unitary group the positive roots are $e_i - e_j, e_i + e_j$ for $i < j$, of dimension two, and $2e_i$, of dimension one. The paired off-diagonal entries in an isotropic basis supply one free entry in the quadratic extension, and an anti-Hermitian diagonal entry supplies one coordinate over F_v, which gives these dimensions. Thus $\langle \rho_F, e_1 \rangle = 2(a-1)+1 = n-1$. Up to multiples, e_1 is a dominant coroot-lattice translation, and its Weyl orbit contains $\pm e_i$ for every i. The standard multiset over E_w consists of t^{e_i}, t^{-e_i}. To see the normalization, write dual Frobenius as $s\vartheta$ with s diagonal and ϑ reversing and inverting diagonal entries. Squaring gives $s\vartheta(s)$, whose entries are s_i/s_{n+1-i} and their inverses. This ratio is precisely evaluation on the translation with exponent 1 in the i-th position and -1 in the opposite position. The bound is therefore $q^{n-1} = Q^{(n-1)/2}$ in either direction. The trivial local representation attains the upper exponent, as its Satake character is the indicated half-sum. The last assertion follows by subtracting the two exponents. □

75.2 Relative norms of the even core lift on its Hecke spaces

On a fixed original field let $f_E = [F : \mathbb{Q}]$ and let D_0 be the bottom real representation. Fix a unit vector v in it which is a pure tensor of compact-finite vectors over the real places. In the cuspidal space with level $K_{0,f}$ and this real representation, write $H(v)$ for the image of v tensored with its full multiplicity space. Suppose $H(v) \neq 0$. Give all spaces their Petersson norms. Keep the finite indicators of Theorem 71.9.

Theorem 75.2 (Uniform relative norm comparison at fixed real input)**.** *There is a single choice of real monomial tests along the entire even core giving a composition $T = T_{m-1,p_{m-1}} \cdots T_{0,p_0}$ such that for all nonzero $a, b \in H(v)$,*

$$e^{-4mf_E n \log \zeta(3)} \;\leq\; \frac{\|Ta\|^2 \|b\|^2}{\|Tb\|^2 \|a\|^2} \;\leq\; e^{4mf_E n \log \zeta(3)}.$$

The same real tests work in every good Hecke space of $H(v)$. On each one the squared norm is multiplied by an exact scalar, independent of the vector, including its finite multiplicity.

Proof. Choose a nonzero input on the prescribed real line in one irreducible cuspidal copy. Theorem 71.9 gives successive product monomials with nonzero outputs for this input. At a real place the first test and the prescribed real source vector determine one target vector up to scalar. Indeed the local pairing into the irreducible source and target is unique up to scalar by Howe duality. Contracting with the same source vector in that pairing gives the same line in the fixed target representation for every occurrence in a global lift. It is nonzero by the choice of the reference input. Use a unit vector on that line. Since the tests and real input were pure tensors, this gives a pure real line for the next step. Induction gives unit lines at every real place for all the steps. The diagonal real Rallis factors on them are nonzero, since their product for the reference lift computes a nonzero squared norm. They depend only on the chosen tests, unit vectors and local pairings.

Consider any bottom good Hecke system η on $H(v)$. Let B_η be its initial Satake lists in quotient coordinates. The first norm formula of Theorem 71.9 applies with the same real tests. Its global measure constant is common because the open-orbit integral uses the global Petersson matrix coefficient of the input, which, on a pure tensor, is its squared norm times the product of normalized local matrix coefficients. Passing to the standard product Haar measures multiplies all such formulas by the same measure factor. At marked places the product is scalar on the entire space of finite fixed vectors, with factors $D_{i,v}$. At unmarked places there is a single normalized spherical line. The varying part at step i, after extracting the factors C_i and the doubling denominator of Theorem 71.2, is precisely

$$L^{S_0}(z_*, B_\eta \otimes \kappa_i^{-1}), \qquad z_* = (d+1)/2.$$

The absolute value of this factor is nonzero. Thus the same tests give a nonzero lift on this system and put its real component on the previously determined line. This justifies using the next norm formula for this input. Iterating, the ratio $\|Ta_\eta\|^2/\|a_\eta\|^2$ for one nonzero vector on the line in a single global copy is

$$C_{\mathrm{fix}} \prod_{i=0}^{m-1} |L^{S_0}(z_*, B_\eta \otimes \kappa_i^{-1})|.$$

Here $C_{\mathrm{fix}} > 0$ is common to all the copies. It consists of the absolute real factors, measure constants, marked factors, and the common good factors extracted above. Any phase in writing the mixed-pairing formula disappears on taking absolute values, as its diagonal left side is a squared norm.

For finite vectors that are sums of pure tensors in one copy, the mixed formula gives the same scalar times their inner product. This uses the marked scalar identity and the one-dimensional good spherical lines. Mixed products from orthogonal invariant copies vanish at the first step by open-orbit unfolding. Their full cuspidal theta images for arbitrary tests are orthogonal by that same calculation, so their invariant spans are orthogonal. Applying this at each further step gives the assertion for sums over copies, including repeated ones. Thus on an arbitrary vector of $H(v)$ the squared norm ratio is a weighted mean of those positive scalars, with weights proportional to its squared norms in the orthogonal Hecke spaces.

Finally Lemma 75.1 and the logarithmic estimate in Theorem 71.2 put the absolute logarithm of each displayed initial L-factor below $2f_E n \log \zeta(3)$. For the ratio between any two positive scalars the common factor cancels and the sum of the m logarithmic differences has absolute value at most $4mf_E n \log \zeta(3)$. The same bounds hold between any two of their weighted means. This proves the comparison on the entire fixed real input space. □

This comparison controls variation among the bottom Hecke spaces for one choice of real tests. An absolute estimate on the integral projected period in Section 6.6 still depends on its normalization in the arithmetic lattice.

75.3 Exponential accuracy of the relative core norms

Use the original field E/F, ranks $n, d = n + 4$, and depth $\mathfrak{m}$ of Theorem 71.9. Let $f = [F : \mathbb{Q}]$. For a fixed unit pure real vector v that is finite under the maximal compact, take $H(v)$ as in Theorem 75.2. Its elements have the prescribed finite level. Write T_v for the composition of the m even core lifts of that theorem restricted to $H(v)$.

Theorem 75.3 (The normalized Gram operator of the full core ascent)**.** *For nonzero $H(v)$, use the common real monomials and original finite tests of Theorem 75.2. There is one $C_{\mathrm{fix}} > 0$ independent of the bottom finite Hecke system such that, in the order of Hermitian operators,*

$$e^{-\delta} I \;\le\; C_{\mathrm{fix}}^{-1} T_v^* T_v \;\le\; e^{\delta} I, \qquad \delta = 2mfn \log \zeta(d/2).$$

In particular

$$\left\| C_{\mathrm{fix}}^{-1} T_v^* T_v - I \right\|_{\mathrm{op}} \le e^{\delta} - 1 \ll mfn\, 2^{-d/2} \le \exp(-x^{\varepsilon/8})$$

on every sufficiently distant original interval. The Hilbert–Schmidt norm squared of this difference is at most $\dim H(v)\,(e^{\delta} - 1)^2$. Changing a group measure by a constant changes C_{fix} by the resulting common norm factor. For the horizontal doubling values of Theorem 74.7 arising from an actual bottom cuspidal row of rank r, the absolute logarithm of the modulus after division by the two string products is at most $2fr \log \zeta((r+6)/2)$ at every step.

Proof. On a good bottom core Hecke system β, the proof of Theorem 75.2 computes the squared norm multiplier exactly as

$$C_{\mathrm{fix}} \prod_{i=0}^{m-1} \left| L^{S_0}\big((d+1)/2, B_\beta \otimes \kappa_i^{-1}\big) \right|.$$

Here B_β is the original bottom standard parameter, and the κ_i are the prescribed unitary characters. The common scalar includes the real local pairings, the marked rectangle volumes, the string factors and the doubling denominators. The mixed norm formula makes distinct orthogonal cuspidal copies orthogonal under all these lifts. Thus the displayed multipliers are eigenvalues of $T_v^* T_v$ on the whole finite multiplicity space at this real vector.

At every $w \notin S_{0,E}$ Proposition 71.17 bounds the absolute value of a bottom Satake root by $(Nw)^{1/2}$. Multiplication by $\kappa_i(w)^{-1}$ preserves this modulus. At the indicated special value each local Euler logarithm is bounded in absolute value by $n\sum_{a\ge1}(Nw)^{-ad/2}/a$. Above any rational prime there are at most $2f$ places and each norm is at least that prime. Addition gives $2fn\log\zeta(d/2)$. Summing over i proves the two bounds in the order of operators. For a self-adjoint operator, the largest absolute deviation of an eigenvalue bounds the operator norm of its difference from the identity. The sum of the squared deviations gives the Hilbert–Schmidt bound.

For $a \ge 2$, comparison with the integral of x^{-a} on $[2,\infty)$ gives $\log\zeta(a) \le 3\cdot 2^{-a}$. Here f, m, n are polynomially bounded in x and $n \ge m \gg x^{1/2-\varepsilon/16}$ by Theorem 6.29. Therefore δ tends to zero uniformly and the term $2^{-d/2}$ absorbs the polynomial factor with the displayed margin for $0 < \varepsilon < 1/52$. For the horizontal assertion, the factor left after division is the bottom standard function at $(r+7)/2$. Applying the same logarithmic series bound with r roots and subtracting $1/2$ from this special argument proves the stated estimate, independently of both row and horizontal indices. □

75.4 The second horizontal restriction for a lowest-type row

Use the test and measure conventions of Theorems 74.2 and 89.4. Fix a pure unit vector v_0 in the lowest compact type of the bottom core and choose the even core tests for this vector once on the given field. Let ϕ be their first real test in the downward convention. Write L_τ for its local energies, $D = n(n+d)$, $N = n_m$, and $A(D, L_\tau)$ as in Lemma 106.10. The constant C_{fix} is that of the complete even core ascent with this choice and with terminal probability measure.

Theorem 75.4 (An absolute restriction estimate for lowest-type rows)**.** *For every unit vector u in the lowest compact type of a prescribed old cuspidal row, there is one terminal integral coefficient e, one product of normalized Hermite functions h_α, and one integral marked residue ν for which the actual second horizontal restriction satisfies*

$$|J_{e,\alpha,\nu}| \;\ge\; \frac{\sqrt{C_{\mathrm{fix}}}}{\|\phi\|_2 \prod_\tau A(D,L_\tau)} \frac{\mathrm{vol}(J)}{d_\sigma} \left(\frac{d_\pi}{\mathrm{vol}(K_f)}\right)^{1/2} \cdot \frac{\|u\|_\infty}{(\mathrm{Nm}\,\mathfrak{m})^{N^2}}\, \exp[-P_{\psi,\varepsilon}(x)].$$

Here $P_{\psi,\varepsilon}$ is a single polynomial for the family on all sufficiently distant original intervals. The factor $\|u\|_\infty$ is at least $\mathrm{vol}([H])^{-1/2}$ in the measure in which the row is a unit vector.

Proof. Take a and the functional ℓ_a of Proposition 106.3. With δ from Theorem 75.3, Theorem 74.2 selects one terminal coefficient e for which

$$|\ell_a(P_{v_0}T^\dagger e)| \ge \frac{e^{-\delta/2}\sqrt{C_{\mathrm{fix}}}\,\|\ell_a\|_*}{H_*\sqrt{d_\rho}\,N^{d_0}B^{2a_0}}.$$

We explain how to estimate this particular functional by the actual horizontal periods of Theorem 89.4. Use the Schwartz tests Ψ_j before Hermite truncation in the proof of Proposition 71.4. Replace them by $\omega(a)\Psi_j$. Equivariance of the bottom group gives $F_e(\omega(a)\Psi_j) = R(a)F_e(\Psi_j)$. The latter converge to $R(a)P_{v_0}T^\dagger e$ in the translate by a of one fixed finite-dimensional space of smooth cusp forms. The first restriction against u is continuous there. Hence its values converge to the functional in the last display.

To bound the absolute Hermite coefficients of these translated tests, use the integral formula for Ψ_j at a real place. Before tensoring, the integrand after translation is $\omega_\tau(a_\tau k t)\phi_\tau$, with measures b_τ on the compact group and $q_{\tau,j}(t)\,dt$ on the real group. Now $W(a_\tau k t) \le W(a_\tau)W(t)$.

Lemma 106.10 and integration of the absolute coefficients therefore bound their local sum by

$$\sqrt{\zeta(2)}\,A(D,L_\tau)\|\phi_\tau\|_2 M_\tau(v_{0,\tau})\,\|b_\tau\|_{\mathrm{TV}} W(a_\tau)^{2s}, \qquad s=\lfloor D/2\rfloor+1.$$

There is only one compact type κ_τ in $v_{0,\tau}$. Its compact character projector has variation at most $\dim\kappa_\tau$, by orthogonality of the irreducible character in Haar probability. The highest weight of the lowest compact type is obtained from the Harish-Chandra parameter by the difference of the compact and noncompact half sums. Each coordinate changes by at most n. The compact Weyl dimension formula thus gives $\dim\kappa_\tau\le T_0^{Cn^2}$. The lowest-type estimate in Lemma 106.10 bounds $M_\tau(v_{0,\tau})$ by $T_0^{Cn^3}$. Proposition 106.3 bounds $W(a_\tau)$ by the same power, and $2s=O(n^2)$. Multiplying over the real places bounds the whole absolute coefficient sum by

$$\|\phi\|_2\,T_0^{Cfn^5}\prod_{\tau}A(D,L_\tau).$$

This bound holds before and after any finite Hermite truncation. Truncate with error tending to zero in the first restriction, using Schwartz continuity as in Proposition 71.4.

Each remaining Hermite product has first restriction equal to the average of the second horizontal periods with the finite Fourier phases, whose total absolute weight is one. This is the exact two-kernel identity in Theorem 89.4. The preceding coefficient bound times $\sup_{\alpha,\nu}|J_{e,\alpha,\nu}|$ therefore dominates $|\ell_a(P_{v_0}T^{\dagger}e)|$. Choose one pair attaining at least half the resulting positive lower bound.

Insert Proposition 106.3 for $\|\ell_a\|_*$. By Theorem 74.2, the logarithm of the terminal denominator is bounded by a polynomial plus $N^2\log\mathrm{Nm}\,\mathfrak{m}$. The factors 2, $e^{\delta/2}$, and $T_0^{Cfn^5}$ have logarithms bounded by polynomials in x on the very same intervals, since $\log T_0\ll x^{\varepsilon}$ and f,n,m are polynomially bounded. Combining the polynomials gives the asserted lower bound. Finally $\|u\|_2^2\le\mathrm{vol}([H])\|u\|_\infty^2$ proves the last assertion. □

This estimate accounts for the initial restriction functional for all lowest-type rows. An absolute arithmetic normalization of the even core theta factors, the comparison through later horizontal lifts with the designated integral projection, and the full residual local cohomology bound on the common intervals are still needed for Section 6.6.

75.5 The finite position test on the first norm shell

Suppose $m\ge 4$. At the third horizontal edge put $C=C_2$, $D=C_3$, $S=V_{2,0}=C\perp Ez$, and $T=V_{3,0}=D\perp(Ee+Ee')$. The unscaled grid columns satisfy $h_S(z,z)=-1$, $h_T(e,e)=h_T(e',e')=0$, and $h_T(e,e')=1$. The distinguished isotropic column and its prescribed image in the grid frames are $-\xi_0 e$ and $-\xi_0 z$, respectively. The position variable in unscaled coordinates is $x=X(e)$ for $X\in\mathrm{Hom}(-T,S)$. Thus the prescribed position is z. `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/IsAuxiliaryVectorWittDec.lean`, line 160–175.

Keep $\mathfrak{A},\mathfrak{m}$ from Proposition 122.7. Put $\mathfrak{l}=(\mathfrak{A}\mathfrak{m})/(\bar{\mathfrak{A}}\bar{\mathfrak{m}})$, $\mathfrak{j}=\mathfrak{m}\bar{\mathfrak{m}}$, and $L=\mathfrak{l}^{-1}\Lambda_S$. A hat denotes the product of all finite completions of a lattice.

Proposition 75.5 (A test at the distinguished position)**.** *In the model with position x and residual oscillator $\mathrm{Hom}(-D,S)$, the indicator $1_{z+\mathfrak{j}\widehat{L}}(x)$ is admissible for the original finite depth on the hyperbolic plane. Here $L=\mathfrak{l}^{-1}\Lambda_C\oplus\mathcal{O}_E z$ is unimodular. With the local position measures giving the original unshifted lattice volume one at the marked places, the inverse change to the original marked polarizations expresses this test as a sum of exactly $(N\mathfrak{m})^{\dim_E S}$ distinct integral coset indicators with coefficients of absolute value one. Those coefficients are roots of unity.*

Proof. The negative line lattice is $\mathfrak{l}z$ by Proposition 102.8. Since $\bar{\mathfrak{l}}\mathfrak{l} = 1$, multiplication of a self-dual Hermitian lattice by $\mathfrak{l}^{-1}$ preserves self-duality. For the two new columns the staircase rule in the original lattices uses $A = \mathfrak{A}\bar{\mathfrak{m}}^{-1}$ and $B = \bar{\mathfrak{A}}\mathfrak{m}^{-1}$. Its two ideals are

$$I_e = A^2/B = \mathfrak{A}\mathfrak{l}/\bar{\mathfrak{m}}, \qquad I_{e'} = A/B^2 = \mathfrak{l}\mathfrak{m}/\bar{\mathfrak{A}}.$$

In particular $I_{e'} = \bar{I}_e^{-1}$. The hyperbolic part of the full map lattice has column lattices $L_e = \mathfrak{A}I_e^{-1}\Lambda_S = \bar{\mathfrak{m}}L$ and $L_{e'} = \mathfrak{A}I_{e'}^{-1}\Lambda_S$. They are exact duals for the alternating trace pairing by Proposition 122.7, since the two columns are isotropic and pair to one. The depth displacements multiply each of these lattices by $\mathfrak{m}$. Translations of the x-variable therefore form $\mathfrak{j}\widehat{L}$. The frequencies $\mathfrak{m}\widehat{L}_{e'}$ have annihilator $\bar{\mathfrak{m}}^{-1}\widehat{L}_e = \widehat{L}$, since the cross pairing is conjugate-linear on one coordinate. The proposed coset is inside $\widehat{L}$, is stable under the translations, and all frequency phases are trivial on it. The extra phase for a doubled mixed displacement is trivial by this same duality. At a split place the two components of a displacement supported on a single idempotent have zero mutual pairing. This also checks the equations at two. `Developmen t/AbelianVarietyTheory/Definitions/Divisors/CmHostImaginaryGeneratorNeZero.lean`, line 120–145.

At a marked v put $a = a_v$, $w = w(v)$, and choose a uniformizer π of F_v. The original positions are $X(e)_w, X(e')_w$. To obtain the mixed positions, Fourier transform the second one to $X(e)_{\bar{w}}$ by their trace duality. Now $L_{e,w} = L_w$ and $L_{e,\bar{w}} = \pi^a L_{\bar{w}}$. The desired indicator on the two new positions is the product of the indicators of $z_w + \pi^a L_w$ and $z_{\bar{w}} + L_{e,\bar{w}}$. The inverse transform of the second is the indicator of $L_{e',w}$ multiplied by the additive character paired with $\pm z_{\bar{w}}$. Indeed the annihilator of $L_{e,\bar{w}}$ is $L_{e',w}$ and both have volume one in dual measures. This character is constant modulo $\pi^a L_{e',w}$, since $\pi^a z_{\bar{w}} \in L_{e,\bar{w}}$. Partitioning the lattice into these cosets produces exactly $(Nv)^{a\dim_E S}$ terms, with the stated coefficients. At the other places the indicator is the full position lattice. Partial Fourier transformation exchanges the unmarked vacuum with the corresponding dual lattice vacuum, using the unit normalization of that line. Multiplying the marked counts gives the assertion. □

75.6 Unfolding the Jacobi period on a full frame

Let $i = 2r + 2$, $i + 1 \le m$, and put $\ell = r + 1$. The two row spaces are $S = (C \perp L) \perp \mathbb{H}^r$ and $T = D \perp \mathbb{H}^\ell$, with $C = C_i, D = C_{i+1}$. The middle vector z of S has norm -1 and spans L. The scaled frame $X = X_*$ in S ends with $\zeta = d^\ell z$, where $d = -\xi_0$. Write N_S, N_T for the flag radicals and $G = U(S)$. The Jacobi theta on $N_T \rtimes U(D)$ uses this same ζ and a Schwartz test q on $\mathrm{Hom}(-D, L)$. The Bessel character on N_S uses ζ as its last vector.

For a Schwartz vector f on the residual oscillator of $\mathrm{Hom}(-D, C \perp L)$, write $C_q f$ for contraction against q in the line variable, linear in f. Use compatible dual Haar measures, as in Propositions 88.9 and 88.4. Use the finite position tests of Theorem 90.6 and write $\mathcal{E}(g;\Phi)$ for the adelic integral in its two positions. The opposite row position is zero on this edge.

Proposition 75.6 (An orbital formula for the global Jacobi period)**.** *Let B, A_D be compact-finite cusp forms on $G, U(D)$. The Fourier–Jacobi period of the horizontal lift of B against A_D with line test q equals*

$$\int_{R(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} J_q(g; \mathcal{E}(g;\Phi))\, d\dot{g}, \qquad R = N_S \rtimes U(C),$$

where

$$J_q(g;f) = \int_{[U(D)]}\int_{[U(C)]}\int_{[N_S]} A_D(h)B(nkg)\overline{\psi_{B,S}(n)}K_{D,C}(h,k;C_q f)\, dn\, dk\, dh.$$

The functional is continuous in the real translated form and test when g varies on a compact set, with fixed finite data. In particular for a descending test b it satisfies

$$J_q(1; b \otimes q) = \|q\|_2^2 \, P_{B,S}(B, \operatorname{desc}_b A_D).$$

Proof. In the period defining the left side, expand the horizontal theta in its first rational positions and multiply by the conjugate of the line theta and the conjugate upper unitriangular character on N_T. On the center, the two theta factors have the characters of $x^* h_S x$ and $\operatorname{diag}(0, \dots, 0, h_S(\zeta, \zeta))$, respectively. A shear multiplies each fiber by the half alternating trace pairing with its opposite-column shear, so both characters use the same normalization. Orthogonality and nondegeneracy of that pairing select precisely the frames with the second Gram matrix. A marked position box has independent columns after normalization in a split component. Thus no rank-deficient rational tuple contributes. By Witt extension the retained frames are a single rational orbit. Right upper unitriangular changes preserve this Gram matrix. The stabilizer modulo these changes consists of N_S and the core isometries fixing z. Indeed the action on the first r columns is upper unitriangular, and the image of the last column equals itself modulo their span. On the middle quotient this means that it fixes the nondegenerate line pointwise. Its remaining Levi factor is exactly $U(C)$. Conversely every element of $N_S \rtimes U(C)$ has these properties. Witt extension realizes every rational upper unitriangular change of the whole frame.

Unfold the frames as $\gamma^{-1} X v_0$ for $\gamma \in R(F)\backslash G(F)$, $v_0 \in U_\ell(E)$, as in Proposition 88.4. In changing the noncentral and central coordinates by v_0, the last row of the noncentral block and the last diagonal component of the transformed central matrix are unchanged. These are the only such coordinates entering the line theta. The rational sum in v_0 again unfolds the upper-group quotient to the entire upper group with weight $\overline{\chi(v)}$.

On the fiber of X, write the noncentral block as ℓ rows on D. Composing its first r rows with the first r columns of X gives all displacements into the isotropic columns of S. The line theta is independent of these rows. In the second polarization their modulations are paired with all the opposite rows by the perfect trace pairing. Integration over their rational quotients therefore selects precisely the zero opposite position. If $r = 0$ there are no such rows.

The final noncentral row gives the maps $w \mapsto \zeta \, b_\ell(w)$ from D to L, and hence ranges through the whole line phase space. The two theta factors have the same Heisenberg action there. In fact both use the identical retained vector ζ, and their commutator is the alternating trace pairing on these maps. The other rows pair trivially with the line because the corresponding columns are orthogonal to ζ. Inserting the final row in the residual columns is translation, and its adjoint in the opposite columns produces the dual modulation, with the same half-commutator phase in the two kernels. The rational linear map $b_\ell \mapsto \zeta b_\ell$ identifies their adelic additive quotients with their probability measures.

Integration of the product of the line theta and the conjugate line theta gives the inner product of their two Schwartz vectors. Explicitly choose rational position and frequency coordinates. Frequency integration first makes the two rational summation indices equal. The sum over that common index unfolds the position integral to the full adelic position space, giving the inner product. A rational polarization for this computation exists on the underlying symplectic vector space by ordinary symplectic linear algebra. Product tests are dense and the identity extends to all residual Schwartz tests by continuity. On $U(D)$ the two line actions coincide and are unitary. At a split place the splitting remaining after removing the core pair is $\eta(\det h)$. The horizontal factor is $\gamma_i(\det k)/(\gamma_i \eta^{-1})(\det h)$ and the core factor is $\gamma_i(\det k)/\gamma_i(\det h)$. The character comparison at the other places follows exactly as in Proposition 74.4, from the integral vacua, rational theta functional and weak approximation. Consequently removing the line leaves the core kernel with test $C_q f$.

For $n \in N_S$, write its action on the full frame X as $Xv_X(n)$. The upper left r-square block of $v_X(n)$ is the action on the first r scaled columns. Its last column above the diagonal consists of the coefficients of $(n-1)\zeta$ in those columns. Hence $\chi(v_X(n)) = \psi_{B,S}(n)$, including the last simple root. Substitution in the upper-group integral gives the conjugate of precisely this character. On the opposite position 0 the Heisenberg displacement and quadratic phase from N_S are both zero. Indeed the displacement is linear and the phase quadratic in that position. The upper triangular block has determinant one. Thus the entire residual action of n at that position is the identity. Disintegrating over R now gives the formula. Its Haar measure is the product on N_S and $U(C)$, by the unitary determinant argument of Proposition 88.4.

Before unfolding the additional line theta has polynomial growth on Siegel sets of $U(D)$, uniformly on a compact unipotent fundamental set, by summing absolute Schwartz majorants. The two cusp forms therefore still give absolute convergence. On the R-fiber the exterior-vector estimate of Proposition 88.4 applies with the isotropic frame of S wedged with a rational flag in C. The extra orthogonal line is fixed in the Levi subgroup. It follows that the translated B supplies arbitrary power decay in the $U(C)$ chamber, uniformly for g in a compact set. The contraction against q is continuous on Schwartz space by integration of Schwartz seminorms. These bounds justify the disintegrations and continuity. Finally $C_q(b \otimes q) = \|q\|_2^2 b$. Integration in h gives the descent, leaving the old Bessel period. □

75.7 Integral coefficient families with controlled metrics

Keep N, B and the original lattices of Lemma 111.2. Let λ_τ be the terminal highest weight at an embedding, let $j_\tau = \lambda_{\tau,N}$, and let $d_\tau = \sum_i (\lambda_{\tau,i} - j_\tau)$ be the tensor degree of its polynomial part. Write

$$d_0 = \sum_\tau d_\tau, \qquad a_0 = \sum_\tau (d_\tau + N|j_\tau|), \qquad L_0 = \prod_\tau d_\tau!.$$

Give the Schur image its subspace metric from the Hermitian tensor power. Use the induced exterior metric on the determinant line, its dual metric on negative powers and the tensor metric across embeddings. For a list (v_b) in a Hermitian space W, its frame operator is the positive operator with quadratic form $w \mapsto \sum_b |\langle v_b, w\rangle|^2$. We use the same definition on W^*.

Proposition 75.7 (Two integral lists in each terminal class)**.** *On class c there are lists $(v_b) \subset W_c$ and $(l_a) \subset W_c^\vee$, each of length N^{d_0}, with frame operators S_v, S_l in the complex coefficient spaces satisfying*

$$B^{-2a_0} I \le S_v \le L_0^2 B^{2a_0} I,$$
$$B^{-2a_0} I \le S_l \le B^{2a_0} I.$$

All members come from the integral Schur image and determinant model of Proposition 95.4.

Proof. Work first in one factor, and suppress τ. Use the standard orthonormal basis of $\mathbb{C}^N$ and its ordered tensors e_I of degree d. Let Y be the unnormalized integral Young symmetrizer of the partition $(\lambda_i - j)$ used to define the Schur image. The frame operator of all columns Ye_I on that image is YY^*. It commutes with $U(N)$. Its image is exactly the irreducible Schur module, because in finite dimension YY^* and Y have the same image. Schur's lemma thus gives a single positive eigenvalue on that image and zero on its orthogonal complement. The positive eigenvalue is rational, being the integer trace of YY^* divided by the rank, and is an algebraic integer since YY^* is an integer matrix. It is therefore at least one. The norm of Y is at most $d!$. Indeed it is the product of a row sum and an alternating column sum of permutation operators. The row and column stabilizers intersect trivially, so the products of their elements are distinct permutations, at most $d!$ in number. Thus $I \le YY^* \le (d!)^2 I$ on the image. For the empty partition this calculation uses the single tensor 1.

At τ transport $Y e_I$ by $X_\tau^{\otimes d}$ from Lemma 111.2. This map commutes with permutations, maps the standard Schur space onto the new one, and takes the columns into its integral Young image. On the tensor power its largest singular value is at most B^d and the norm of its inverse is at most B^d. Both bounds persist on the Schur space. Transport of a frame by an invertible A takes its frame operator S to ASA^*. The resulting bounds are B^{-2d} and $(d!)^2 B^{2d}$.

For the dual list, take all d-fold ordered tensors of the integral ξ_i of the lemma and restrict them as linear functionals to the Schur image. A restriction is integral on this image because it was integral on the full tensor lattice. On the full tensor space the sum of the squares of the evaluations on any w lies between $B^{-2d}\|w\|^2$ and $B^{2d}\|w\|^2$, by the singular values of the tensor product of the dual column map. Evaluating on w in the Schur subspace gives the same bounds for the restricted covectors. By Hermitian duality these are precisely the two frame inequalities on the dual Schur space.

In the vector list use $x_1 \wedge \cdots \wedge x_N$ for a positive determinant power, and $\xi_1 \wedge \cdots \wedge \xi_N$ for a negative power, taking $|j|$ tensor copies. Reverse these two choices on the dual side. Each wedge has norm in $[B^{-N}, B^N]$ by the singular values of its column map, and it belongs to the indicated integral line or its integral dual. A zero power contributes 1. This gives the factor $B^{2N|j|}$ in an upper frame bound and its reciprocal in a lower bound. Taking all products of lists over the embeddings tensors the frame operators, so their positive bounds multiply. There are N^d indices in each local list. Multiplication gives the stated lengths and estimates. $\square$

76 References

[1] A. Molev, *Gelfand–Tsetlin bases for classical Lie algebras*, in *Handbook of Algebra*, vol. 4 (2006), 109–170. Section 2, Theorem 2.1 and formulas (2.21)–(2.22).

[2] B. H. Gross, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93. Proposition 4.3.

[3] W. T. Gan, *Automorphic forms and the theta correspondence*, Arizona Winter School lectures. Lecture notes.

[4] L. Alpöge, *Remark*, manuscript supplied with this series, 20 November 2026. Author's note.

[5] M. Harris and J.-P. Labesse, *Conditional base change for unitary groups*, Asian J. Math. **8** (2004). Authors' text, Theorem 2.2.2.

[6] T. Kaletha, A. Mínguez, S. W. Shin and P.-J. White, *Endoscopic classification of representations of inner forms of unitary groups*. Authors' text, Sections 1.3, 1.6 and 1.7 and Corollary 3.3.2.