The generalized Riemann hypothesis for Dirichlet L-functions

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Volume X

Projected torus periods and congruence modules

Abstract

Local inertia characters and crystalline flags enter the projected torus pairing. We compare cup-product bounds over the period fields and relate the integral period to the Hecke congruence module.

The guide in Volume I lists all twenty-seven volumes.

Contents

X Projected torus periods and congruence modules **1**
50 Local constituents of the projected torus pairing . . . 3
50.1 The integral pairing of two tests projected to a torus lift . . . 3
50.2 The rank-one period at the spherical coefficient primes . . . 6
50.3 The two inertia characters on the twist parahoric . . . 9
50.4 The projected torus pairing at every coefficient prime . . . 12
50.5 The contribution of abelian local systems to the torus pairing . . . 15
50.6 Two inertia elements at the original level primes . . . 19
50.7 The projected pairing for potentially abelian local systems . . . 22
50.8 Determinants of two-dimensional local constituents . . . 26
50.9 Residual constituents and local trace relations . . . 30
50.10 Ordered crystalline flags and the integral projected period . . . 33
50.11 The unramified constituent in the local period tests . . . 36
51 Cup bounds and torus periods at the coefficient primes . . . 39
51.1 Summing local cup bounds on the period fields . . . 39
51.2 Large coefficient primes with a multiple near a period weight . . . 42
51.3 The full marked torus period above the square-root cutoff . . . 44
51.4 The remaining main cup coordinates at intermediate primes . . . 46
51.5 A shorter range for the mixed main cups . . . 48
51.6 The full torus period at large coefficient primes . . . 50
51.7 The torus period above the product $k_x t_0$ . . . 52
51.8 The torus period away from the fixed conductor . . . 54
51.9 All coefficient primes for the marked torus period . . . 56
52 Congruence modules and integral projections . . . 59
52.1 The torus budget and the determinant of a higher-rank projection . . . 59
52.2 Split Hecke values on the complement of the true lifts . . . 62
52.3 The full cyclotomic string in the congruence module . . . 65
52.4 The long character block in the definite target . . . 68
52.5 Decomposition of the tested index by residue type . . . 71
52.6 The integral projection and the Selmer group of a true system . . . 73
52.7 The source representation and the common period denominator . . . 77
52.8 Minors of the projected theta-period pairing . . . 81

53 References 84

50 Local constituents of the projected torus pairing

50.1 The integral pairing of two tests projected to a torus lift

In passing from a Hecke multiplier to a period, the tests for the lift from $U(1)$ are initially orthogonal to certain forms in the kernel of the entire descent. The projection used in the period is onto the lift of a single source character. Averaging one test over the source classes relates these two spaces. We compute the integer introduced by the averaging on the fields of the shorter family.

Proposition 50.1 (The number of source classes at the two marked levels)**.** *Put $F = E^{+}$, $f = [F : \mathbb{Q}]$, and $T = U(1)$ on the fields of Corollary 12.4. Write U for the norm-one unit subgroup at every finite place, with the additional condition that the component reduce to 1 at each marked split place over p_0, p_1. Let $j : \mathrm{Cl}(F) \to \mathrm{Cl}(E)$ be extension of ideals and let μ_E be the roots of unity of E. Then*

$$C = T(F)\backslash T(\mathbb{A}_{F,f})/U, \qquad |C| = \frac{h_E}{|\operatorname{im} j|}\frac{H}{|\mu_E|}, \qquad H = ((p_0-1)(p_1-1))^{f}.$$

All stabilizers in this class sum are trivial. For each coefficient prime ℓ, representatives z_c of all classes can be chosen integral and invertible at every place above ℓ.

Proof. First use the full norm-one unit subgroup U_0. The map $a \mapsto a/\bar{a}$ from finite ideles of E onto $T(\mathbb{A}_{F,f})$ has kernel the finite ideles of F. Indeed Hilbert's Theorem 90 gives the assertion at a field place, and at a split place one can lift (z, z^{-1}) to $(z, 1)$. At an inert place the extension is unramified, so multiplication by a power of a uniformizer of F_v makes any lift of a unit into a unit. These are all the finite cases by Corollary 12.4. Thus the map on products of unit groups is also onto U_0. Global Hilbert 90 gives the same surjection on rational points. The quotient with level U_0 is therefore the ideal class group of E modulo the images of finite ideles of F, namely $\mathrm{Cl}(E)/\operatorname{im} j$.

Reduction gives $[U_0 : U] = H$, since there are f marked places of F of residue cardinality p_i over each p_i. The kernel of the map from U_0/U to C is the image of $T(F)\cap U_0$. This intersection equals μ_E because a norm-one integral unit has absolute value one at every complex embedding, so is a root of unity by Kronecker's theorem, and every root of unity has norm one. A root equal to 1 modulo both marked characteristics has order a power of each characteristic, by Lemma 13.12. It must equal 1. We get the asserted order and $T(F) \cap U = 1$, which gives the stabilizer statement because the group is abelian. Finally lift a given representative to an idele a of E. By weak approximation choose $b \in E^{\times}$ such that ba is a local unit everywhere above ℓ. Replacing $a/\bar{a}$ by $ba/\overline{ba}$ preserves its class and has the required integral components. □

Write W for the finite space of definite target forms in the rank-one lift, with the Hermitian class-sum pairing linear in the second variable. Absorb the chosen source character into the theta kernel, so that the selected vector of the source is the constant function on C. Write K_c for the target kernel at c and let $\phi = \sum_{c\in C} K_c$. Common nonzero changes of normalization of the kernel give the same line $Z = \mathbb{C}\phi$ and the same kernel of descent. Write P for orthogonal projection onto Z and D for descent, so that $Dy(c) = \langle K_c, y\rangle$ up to a common nonzero scalar. At the marked places e is the orthogonal projection given by the mean of the finite residue-unit action for the trivial source type. Its group has order H. For a split place v off the level let A_v be the reversed spherical Hecke polynomial evaluated at the inverse of the top root prescribed for this lift. A tested vector is orthogonal to $\operatorname{im} e \cap \ker A_v \cap \ker D$ for all such v.

Lemma 50.2 (Averaging one test)**.** *For each ℓ there is a sum B of $|C|$ normalized central translations of the target satisfying*

$$PB = |C|P, \qquad B\Xi \perp \operatorname{im} e \cap \ker A_v \cap Z^{\perp} \quad \text{for every tested } \Xi \text{ and every } v.$$

The sum preserves the lattice of integral forms at each embedding above ℓ.

Proof. For a finite adelic norm-one scalar z, take right translation by the scalar on the target with the unitary splitting factor, choosing the inverse convention so that its action U_z on the kernel is $U_z K_c = K_{cz}$. Indeed the tensor action of a source and a target scalar of the same value on the underlying homomorphism space is $x \mapsto zxz^{-1} = x$. Their joint Weil action is the scalar of the chosen splitting. Dividing it out moves the source translation to the target, and the source character absorbed into the kernel contributes its own unitary factor. This proves the identity by applying the action to the theta sum term by term. In particular $U_z\phi = \phi$. These translations and their inverses commute with the residue unit action and with all A_v, since their group elements are central. The trivial source type gives $e\phi = \phi$ because the marked action on the lifted constant is the normalized translation of the constant on the source, which fixes it. Equivalently the finite change of variable in the source sum permutes its classes.

Take $B = \sum_{c \in C} U_{z_c}$ using the representatives of the proposition. Each summand is unitary and fixes ϕ, proving $PB = |C|P$. If $y \perp Z$, then for every $d \in C$,

$$D(B^* y)(d) = \langle BK_d, y \rangle = \Big\langle \sum_{c \in C} K_{dc}, y \Big\rangle = 0$$

in the class-sum normalization. If y is also in $\operatorname{im} e \cap \ker A_v$, then $B^* y$ stays in this intersection. Testing Ξ against $B^* y$ proves the claimed orthogonality. Only the action on the kernels is indexed by the finite quotient C. The representatives define actual translations on the whole target.

For integrality the scalar matrix of z_c and its inverse preserve the coefficient lattice at every place above ℓ. The splitting and source characters used for normalization have ℓ-adic unit values on these representatives. To check this last assertion, raise an idele integral above ℓ to a positive power so that it is a rational torus point times a compact unit, using finiteness of C. Both factors remain units above ℓ. A smooth character on the compact unit subgroup has finite image. On the rational point the finite character value is prescribed by the infinity type, a product of powers of its conjugates (after squaring for a half-integral normalization). Each conjugate is an ℓ-adic unit. The original value is therefore a unit as well. Applying these two observations to the translated value of a form at each adelic representative proves integrality of the sum. □

Theorem 50.3 (From a trace multiplier to a projected inner product)**.** *Fix the shorter period field, a twist and a coefficient embedding $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. Use the rank-one theta data just described at target rank $n > 1$. For a nonzero ϕ take its base Hecke system x_*, with the split normalized representation and auxiliary row hypotheses of Corollary 45.21 for $X_0 = \{x_*\}$. Let b_ℓ be any of the bounds for $v(\beta_{x_*})/v(\ell)$ in Theorem 45.11, including the replacement for the two original level terms in Corollary 45.24 when applied. Let c_0 be the stabilizer multiple of Lemma 45.18 for this n. For any two tested forms Ξ, Ξ' integral at all embeddings above ℓ,*

$$\big|\iota\langle P\Xi, P\Xi' \rangle\big|_\ell \le \ell^{\,b_\ell + v_\ell(Hc_0^2|C|)}.$$

For $\phi = 0$ the left side is zero. In particular the bound counts the trace loss only once.

Proof. Put $u = B\Xi'$, $l = eu$ and $y = Pu$. The vector Hl is integral because He is the sum of the normalized residue-unit actions, each of which preserves the integral function lattice. The local matrices are integral with integral inverses, and the finite character multipliers are roots of unity. The line of the constant lift has a single good Hecke system, by the spherical theta parameter formula of Section 45.1 and the unramified transfer used in Theorem 71.2. Hence y is supported on x_*.

We check the agreement on every base system with a trivial constituent, including possible multiplicities in the target. At any good split $v \nmid \ell$ take the geometric Frobenius in that constituent. By Theorem 45.16 the arithmetic-normalized spherical polynomial has the root $C_v = q_v^{1/2}\mu_v(\varpi_v)$, in the notation of Corollary 45.17 with source dimension one. This follows because the normalizing character at that Frobenius equals C_v^{-1} under the coefficient identification. By Lemma 45.1 the same polynomial has the root $C_v q_v^{n-2}$. Reversing it and evaluating at the inverse of this root kills that whole system by A_v.

The commuting spherical Hecke algebra is closed under adjoints. Its joint eigenspaces are mutually orthogonal, and the marked residue action commutes with it. Each joint projector P_x therefore commutes with e and with P, since Z is an eigenline. Also $eP = P$. For a system with a trivial constituent,

$$d_x = P_x(l - y) = eP_x(u - Pu) \in \operatorname{im} e \cap \ker A_v \cap Z^{\perp}.$$

The equality uses the commutation just proved. The lemma gives $\langle d_x, u\rangle = 0$, and orthogonality to Z gives $\langle d_x, Pu\rangle = 0$. Since eP_x is itself an orthogonal projection, $\|d_x\|^2 = \langle d_x, u - Pu\rangle = 0$. This verifies the agreement required in base separation.

Apply Corollary 45.21 with $h = H$. It gives a scalar s_* on the chosen system, with $v_\ell(\iota s_*) \le b_\ell$, for which $Hc_0 s_* y$ is integral at ι. To pair this form with Ξ, write the definite inner product as a sum over double cosets of the coefficient pairing divided by the point stabilizer order, using volume one for the defining level. In the tensor basis the coefficient pairing has identity matrix. Taking inner products of columns of the integral inclusion of the Weyl lattice gives its integral Gram matrix. In the transported coordinates of a representative, the two changes of basis at ι and $\iota \circ \mathrm{conj}$ cancel in this pairing by the unitary relation $\bar{g}^{\,t} = g^{-1}$. Thus every coefficient pairing is integral when the first vector is integral at $\iota \circ \mathrm{conj}$ and the second at ι. Every stabilizer order divides c_0 by Lemma 45.18. Consequently

$$\iota\big(Hc_0^2 s_* \langle \Xi, y\rangle\big) \in \overline{\mathbb{Z}}_\ell.$$

Now $y = |C|P\Xi'$, and self-adjointness and idempotence give $\langle \Xi, P\Xi'\rangle = \langle P\Xi, P\Xi'\rangle$. Taking absolute values proves the estimate. □

Corollary 50.4 (Total cost of the source classes and stabilizers)**.** *At every node to which the theorem applies, the additional denominator has total logarithmic cost*

$$\sum_{\ell} v_\ell(Hc_0^2|C|)\log \ell \le \log h_E + 2f\log(p_0p_1) + 2n^2 \min_i \log p_i.$$

This bound is independent of the auxiliary moduli of the twists chosen separately at the coefficient embeddings.

Proof. The three integers $H, c_0, |C|$ use the original levels. Proposition 50.1 gives $|C| \mid h_E H$. Use $H \le (p_0p_1)^f$ and the bound for c_0 in Lemma 45.18, then factor the fixed integer into primes. The upper bound for b_ℓ in the theorem continues to use the actual row and local cohomology estimates. Corollary 13.16 specifies which coefficient completions can be omitted in them. □

50.2 The rank-one period at the spherical coefficient primes

For a main pair with critical integer z, the lift from $U(1)$ has target rank $n = 2z$. The string in its normalized parameter has odd length $t' = 2z - 1$. This parity determines the global congruence term at every coefficient prime and eliminates it at every odd prime unramified in the period field. We use this fact to bound the inner products of the original tested forms of Section 50.1.

Proposition 50.5 (The cyclotomic ideal for an odd string)**.** *Let $E = \mathbb{Q}(\mu_{q_1}, \sqrt{-d})$, where $4 \mid q_1$, $d \equiv 7 \bmod 8$ is prime and $d \nmid q_1$, as in the shorter family. Set $N = q_1 d$. Let t' be odd and let T be any continuous rank-one ℓ-adic lattice character of G_E. The normalized valuation $a_\ell(t')$ of the ideal $I_{t'}$ in Theorem 12.25 is*

$$a_\ell(t') = \begin{cases} v_\ell(q_1) + v_\ell(t'), & \ell \mid q_1, \\ 1 + v_d(t'), & \ell = d \text{ and } (d-1)/2 \mid t', \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for an integer $t_0 \geq 1$, the sum of the suprema over odd $t' \leq t_0$, weighted by $\log \ell$, is at most $\log\left(N \operatorname{lcm}(1, \dots, t_0)\right)$.

Proof. A rank-one lattice character has unit trace on every group element. The ideal is therefore generated by $1 - u^{t'}$ for u in the cyclotomic image of G_E. The cyclotomic Galois group acts on roots of unity through the product of the groups $\mathbb{Z}_p^\times$. Fixing E requires that its component at each $p \mid q_1$ be 1 modulo $p^{v_p(q_1)}$, and that the component at d reduce to a square. The last condition uses the quadratic subfield of $\mathbb{Q}(\mu_d)$, which has discriminant $-d$. These conditions are independent by the Chinese remainder theorem on every finite cyclotomic level.

For $\ell^b \| q_1$ the image is $1 + \ell^b \mathbb{Z}_\ell$. Here $b \geq 2$ at two. The least valuation of $u^{t'} - 1$ for u in this subgroup is $b + v_\ell(t')$, attained at $1 + \ell^b$. Indeed a power prime to ℓ preserves the depth of a principal unit, and an ℓ-th power increases it by one by the binomial expansion, for ℓ odd or depth at least two. At d the residual image has order $(d-1)/2$. If that order divides t', the residual roots of unity lift to roots of the same order and are killed by the power t'. The minimum on the principal units is then $1 + v_d(t')$. Otherwise a lifted generator gives valuation zero. At every other prime the image is the whole unit group and the prime is odd. The element -1 gives valuation zero because t' is odd. Finally the product of the prime powers given by the suprema divides $N \operatorname{lcm}(1, \dots, t_0)$, which gives the sum bound. □

Theorem 50.6 (Simultaneous rank-one Selmer bounds at every prime)**.** *On these fields take every main character of the designated torus lifts at a node, with tame part $\omega_w^{\pm j}$ at each old marked place, $1 \leq j \leq m$, in the notation of Corollary 13.14. For their untwisted lattice characters T, the constants B_i of Theorem 12.32 equal $a_\ell(2z-1)$. Thus, for the same twists and signed powers that control all the critical ratios, the corresponding normalized divisible Selmer lengths with the integral crystalline evaluation equations are at most $a_\ell(2z-1) + \epsilon_\ell$.*

Proof. Write ν for the source torus character in the main pair and put $u(a) = \nu(a/\bar{a})^{-1}$ on the ideles of E. The untwisted rank-one main character is $T = u_\ell \kappa^{1-z}$, where u_ℓ is the algebraic avatar of u. Indeed the source parameter of the constant is one. In the normalization of Corollary 45.17 the $2z-1$ added Satake entries at a good split place become $1, q, \dots, q^{2z-2}$. The remaining entry is the inverse source torus character on $a/\bar{a}$ times q^{z-1}. Indeed the unitary entry is the splitting-character ratio, and the arithmetic weight shift after dividing by the rank-one normalizer is $((2z-1)-1)/2 = z-1$. At geometric Frobenius this is exactly the value of T. Replacing ν by $\nu\vartheta^s$ multiplies the remaining entry by the inverse pullback of

ϑ^s. The unramified theta parameter identity used in Theorem 71.2, followed by Chebotarev and semisimplicity as in Theorem 45.16, gives the direct sum of this character with the stated string whenever the lift is nonzero.

The equality $u^c = u^{-1}$ gives $T^c = T^{-1}\kappa^{1-t'}$. The character is de Rham above ℓ since it is the avatar of an algebraic Hecke character. Away from ℓ it is unramified except at the two marked levels. Take S to consist of just these levels, their conjugates and the places above ℓ. At every $w \in S$ away from ℓ, κ is unramified and the full tame generator has value of order $Q/\gcd(Q, j)$ under T, also at the conjugate place. In the proof of Corollary 13.14, if p is either 2 or 3, the p-part of this order is at least $p^{h+6}/m > 2$. Taking $p \neq \ell$ shows that the residual order is greater than two. Both $T(g_*) - 1$ and $T(g_*)^2 - 1$ are units for a tame generator at any chosen old place away from ℓ. The same holds at its conjugate inverse. This checks the marked scalar hypotheses. The full matrix span in rank one follows already from the identity matrix. On the Tate dual the inertia value is the inverse of that on T. The difference from one is still a unit, so $H^0(E_w, W_b^*) = 0$ at every finite precision b for all the old terms away from ℓ.

In the formula for B_i, both distinguished determinants and every local invariant term consequently have valuation or length zero. Proposition 50.5 gives the remaining ideal term exactly. Apply Theorem 13.13 with these finitely many lattices and all the critical pairs. This also gives any prescribed positive bound η_ℓ for the total new tame inertia cost, with the same choice of twist. □

Theorem 50.7 (The contribution of the spherical primes to projected periods)**.** *At a node on the shorter period interval use $t_0 = 5r$, $Y = N + t_0 + 1$ and G_0 from Corollary 12.10. Let $\mathcal{G}$ consist of the odd coefficient primes prime to Np_0p_1 where all the rank-one lift data have hyperspecial level above ℓ and the labelled intervals of every relevant trace family and scalar or main target satisfy Theorem 13.15 at every completion and coefficient embedding. This set contains every sufficiently large prime at the node and has a common such threshold on each finite interval. Given $\delta > 0$, the twists can be chosen so that*

$$\sum_{\ell \in \mathcal{G}} \sup_{\iota,\, \nu, z, s,\, \Xi, \Xi'} \log \max\bigl(1, |\iota\langle P\Xi, P\Xi'\rangle|_\ell\bigr) \leq 2 \log h_E + \log \mathcal{J}_{N,t_0} + \bigl(6 + (4t_0 + 3)Y^6\bigr) \log G_0 + \delta.$$

Here the supremum is over the original integral tested forms of Section 50.1 and the nonzero signed twist powers of magnitude at most m of the rank-one main data. Zero lifts give zero contributions. The same twists can give the bound of Corollary 13.14 for the chain of critical values. All quantities on the right are independent of their auxiliary moduli.

Proof. At a prime of $\mathcal{G}$ one has $a_\ell(t') = 0$. Choose $\epsilon_\ell, \eta_\ell > 0$ so that $\sum_\ell (\epsilon_\ell + 2t_0\eta_\ell) \log \ell \leq \delta$. One may decrease ϵ_ℓ further for the product of the critical ratios. By Corollary 13.16 all scalar colengths vanish, and the normalized main trace length is at most ϵ_ℓ by Theorem 50.6. The large-prime assertion follows since the finitely many infinity types give bounded jumps at every embedding, independently of all auxiliary levels. The twists split above their respective coefficient primes. The exclusions at the fixed levels and at the ramified primes of the period fields are finite. Take maxima over the node data on any one finite interval.

We verify the remaining row inputs for these rank-one targets. At a twist place the character of the parahoric is the inverse tame twist on the determinant of the one-dimensional quotient, and it is trivial on the other block. In the torus datum a block triangular transformation on the quotient line changes the test by the inverse residue character of that scalar. The same parahoric condition holds in every system at the later trace levels, since those places are disjoint from the twist places. All the selected powers ramify at every twist pair by Theorem 13.4. The

nontrivial line-character classification in the proof of Theorem 45.14 gives a single ramified line with this inertia character and an unramified spherical complement, with zero monodromy. The conclusion for a quotient line follows from the version for a subline by inverse-transpose, which inverts the characters. The normalizer comes from the source splitting character with unramified smooth part. By Theorem 45.16 the actual normalized inertia matrices are therefore killed by $(X-1)(X-\xi_w)$ at a chosen tame generator, with ξ_w the indicated twist value or its inverse. At the conjugate place the same assertion follows by duality. This supplies $o_w(X) = X-\xi_w$ for Theorem 45.8. Its contribution after normalization is $t'v_\ell(1-\xi_w)$. Summing over both sides of all twist pairs costs at most $2t'\eta_\ell$.

At the original marked levels use the Frobenius rows of Corollary 45.24. Together these cost at most $2v_\ell(G_0)$. Away from the marked, twist, trace and coefficient places the level is hyperspecial. The spherical base change and the zero-monodromy compatibility in the proof of Theorem 45.16 give unramified representations there. For the selected main summand the prime choices of Corollary 47.37 provide the unit determinant required at the subsidiary trace primes. The unramified parameters of the selected lift have the direct sum found above. Thus Corollary 45.17 applies to the full trace family, including conjugate primes. The construction in Section 45.3 adjoins any remaining generators of the matrix span independently of the row estimate. These choices verify the row inputs for base separation on the actual selected line.

Theorem 11.29 bounds the odd-cocycle term by $v_\ell(h_E\mathcal{J}_{N,t_0})$, since the scalar weights used here form a subset of $0 \le a < t_0$. The final interpolation loss in Theorem 45.11 is at most $(4t_0+3)Y^6v_\ell(G_0)$, by Corollary 12.10. Adding all the terms gives the bound in Theorem 50.3 with

$$b_\ell = v_\ell(h_E\mathcal{J}_{N,t_0}) + \bigl(2+(4t_0+3)Y^6\bigr)v_\ell(G_0) + \epsilon_\ell + 2t_0\eta_\ell.$$

Finally $n = 2z \le 2r \le rt_0$. The standard block inclusions of finite general linear groups give $c_0 \mid G_0$, and the split residue group orders give $H \mid G_0$. Proposition 50.1 gives $Hc_0^2|C| \mid h_EG_0^4$. Substitute this and b_ℓ into the pairing estimate, take the supremum, and factor these fixed integers over the primes. This proves the stated sum. □

The theorem bounds the rank-one projected inner products at every included coefficient prime, on the actual fields of the period calculation. Bounding the omitted coefficient places as the interval grows and comparing the full chain of higher-rank projections with its archimedean periods are further steps in Section 6.6.

50.3 The two inertia characters on the twist parahoric

At a new split place the period test singles out a subspace of the residue vector space. A change of basis of this subspace acts on the test through the tame character of its determinant. We first determine the local Langlands parameter from this transformation law, including when both parts of the flag have dimension greater than one. We then measure the cost of splitting an integral Galois lattice along the two inertia characters.

Theorem 50.8 (A nontrivial determinant character on one parahoric block)**.** *Let K be a nonarchimedean local field of characteristic zero, with ring of integers O, residue field k and uniformizer ϖ. Write $n = a + b$ with $a, b > 0$. Let $P \subset \mathrm{GL}_n$ be the standard upper parabolic of type (a, b) and let J be the inverse image of $P(k)$ in $\mathrm{GL}_n(O)$. For a nontrivial character $\xi : k^\times \to \mathbb{C}^\times$ let $\lambda(j) = \xi(\det \bar{j}_{22})$ on J, and extend ξ to a character $\widetilde{\xi}$ of $K^\times$ with value one on ϖ. If an irreducible smooth complex representation π of $\mathrm{GL}_n(K)$ contains a vector of type (J, λ), then*

$$\pi \simeq i_P^{\mathrm{GL}_n(K)}(\sigma_a \boxtimes ((\widetilde{\xi} \circ \det) \otimes \sigma_b))$$

for irreducible spherical representations σ_a, σ_b. Here i_P denotes normalized induction. The space of vectors of this type in π is one-dimensional. The Weil–Deligne parameter of π has monodromy zero and inertia representation $\mathbf{1}^{\oplus a} \oplus \xi_I^{\oplus b}$, where $\xi_I = \widetilde{\xi} \circ \mathrm{Art}_K^{-1}$ on inertia with the reciprocity convention of Section 45.6.

Proof. Restrict the given vector to the standard Iwahori, the inverse image of the upper triangular group. It has diagonal characters $1, \dots, 1, \xi, \dots, \xi$, in the indicated order. The tame Iwahori type theorem, in the form proved using Roche's theorem in the proof of Clozel–Harris–Taylor, Lemma 3.1.6, puts π in a normalized principal series as a composition factor. Its inducing characters $\gamma_1, \dots, \gamma_n$ restrict to $O^\times$ in precisely the indicated order. The theorem applies to GL_n in every residue characteristic. [1, Lemma 3.1.6 and its proof]

We calculate the dimension of the (J, λ)-space in this entire principal series. A function in it is determined by its values on $B(K) \backslash \mathrm{GL}_n(K) / J$, with B the upper Borel. By Iwasawa decomposition followed by reduction of the maximal compact subgroup, this set is $B(k) \backslash \mathrm{GL}_n(k) / P(k)$. Its permutation representatives are indexed by the subsets of size b giving the coordinates of the second block. The diagonal unit group belongs to the stabilizer of each representative. An arbitrary unit in a single coordinate shows that a value can be nonzero only if exactly the last b inducing characters are assigned to the second block. Since $\xi \neq 1$, precisely one representative survives. At that representative, which can be taken to be the identity, the characters agree on $B(K) \cap J$. Indeed its diagonal entries are units, the modulus is one and both characters are trivial on the upper unipotent. A function supported on $B(K)J$ with value one at the identity therefore exists. This proves the dimension assertion for the principal series.

For the first a inducing characters take the spherical constituent σ_a of their principal series. For the remaining b, divide them first by $\widetilde{\xi}$ and take the spherical constituent σ_b. In each case Iwasawa decomposition gives a one-dimensional space of maximal compact fixed vectors. Compact averaging makes passage to invariants exact, so the spherical constituent exists and occurs once. Its parameter is unramified with monodromy zero, by the spherical correspondence. Normalized induction is exact and transitive, so the product in the statement is a subquotient of the original principal series. This product is irreducible by Zelevinsky's criterion for two collections of segments. Every character in the first collection is trivial on units, and every character in the second restricts to ξ. Multiplication by any power of the norm preserves this restriction. Consequently no segment in one collection can be linked to a segment in the other. [2, Prop. 8.5]

In this irreducible product, take a section on $P(K)J$ with value at one the tensor of the two spherical vectors, with the second factor twisted as above. The inducing action of $P(K)\cap J$ on this value is λ, which gives the required equivariance of the section. Since J is compact, averaging against λ^{-1} makes its character space an exact functor in characteristic zero. The dimension one just computed for the whole principal series can thus be accounted for by only one composition factor, with multiplicity one. This identifies π with the product. Compatibility of local Langlands with Langlands quotients gives the direct sum of the two spherical parameters with the second twisted by $\widetilde{\xi}$. More explicitly, interchanging factors from opposite collections preserves the induction since their segments are unlinked. This puts the combined Langlands data in standard order, and the quotient then has the union of the two parameter multisets. This proves the claimed restriction and monodromy. □

Proposition 50.9 (Integral splitting of the two inertia spaces)**.** *Let $v(\ell)=1$ on a finite coefficient field $L/\mathbb{Q}_\ell$ with ring of integers $\mathcal{O}$. Suppose inertia acts on $V=V_0\oplus V_1$ by the two characters $1,\xi_I$ in the theorem, of dimensions a,b, with values in L. For any stable lattice $\Lambda\subset V$ put $\Lambda_i=\Lambda\cap V_i$. If τ maps to a generator of the tame character image and $c=\xi_I(\tau)-1$, then*

$$c\Lambda\subseteq\Lambda_0\oplus\Lambda_1,\qquad \operatorname{len}_{\mathrm{norm}}\bigl(\Lambda/(\Lambda_0\oplus\Lambda_1)\bigr)\le\min(a,b)\,v(c).$$

Here the normalized length of a cyclic module $\mathcal{O}/u$ is $v(u)$. The bound is attained among stable local lattices with the given ranks and tame character.

Proof. Let A denote the action of τ and P_i the projection onto V_i. Then $cP_1=A-1$ and $cP_0=c-(A-1)$ preserve Λ. Their images lie in Λ_1,Λ_0, respectively. Projection along either summand identifies the stated quotient with $P_i\Lambda/\Lambda_i$. In fact it is onto, and if $P_ix\in\Lambda_i$ then $x-P_ix$ is in the other intersection. This quotient is killed by c and generated by at most $\dim V_i$ elements because $P_i\Lambda$ is a lattice in V_i. Taking the smaller of the two bounds proves the inequality.

For sharpness, put $z=\xi_I(\tau)$. Take $\min(a,b)$ copies of the matrix $\left(\begin{smallmatrix}1&1\\0&z\end{smallmatrix}\right)$ on $\mathcal{O}^2$, and scalar blocks for the remaining dimensions. The two eigenlattices in a displayed block have respective bases $(1,0)$ and $(1,z-1)$, and their sum has determinant $z-1$. These matrices define an action of the cyclic tame image, since the sum of all powers of z up to its order minus one is zero. They extend to a local representation by the quotient character $\widetilde{\xi}\circ\mathrm{Art}_K^{-1}$. □

Corollary 50.10 (The total twist-place cost in all block ranks)**.** *At a node of the shorter family, take the split twist places of Corollary 13.14, with every prescribed nonzero signed power. For each of the finitely many descent types α, denote by b_α the dimension of its marked block, by a_α the complementary dimension and by t_α the number of scalar characters used in Theorem 45.8. Use the integral space of forms of the prescribed parahoric character at every twist place and all the auxiliary trace levels away from those places. Let u_x be the normalized Galois representation of any constituent of one of these spaces. Write ζ_w for the value at a full tame generator τ_w of the prescribed nontrivial local character, transported to the coefficient field. At the conjugate place use the conjugate dual character. Then*

$$(u_x(\tau_w)-1)(u_x(\tau_w)-\zeta_w)=0.$$

In the row construction one may use $o_w(X)=X-\zeta_w$ in all these ranks. If $d_{x,w}$ is the normalized length in Proposition 50.9 for any stable lattice in u_x, then, given $\delta>0$, the same twists used for the critical ratios can be chosen to satisfy

$$\sum_{\ell}(\log\ell)\sup_{\iota,s,\alpha,x,\Lambda}\sum_{w\in Q\cup Q^c}\bigl(t_\alpha v_\ell(1-\zeta_w)+d_{x,w}\bigr)\;\le\;\delta.$$

The set Q contains one place from each new pair for the twist belonging to (ℓ, ι). Empty sets contribute zero. The same bound is available at every node in a finite common period interval.

Proof. At a twist place the finite test in the integral frames asks that the reduction of the matrix map the marked space isomorphically onto its marked image. Its coefficient there is the inverse tame character of the determinant of this map. A block triangular change of frame preserves this support and multiplies the coefficient by the character of the corresponding diagonal block. The inverse frame convention replaces the character by its inverse. This is the transformation law in the split test of Section 123.6. An opposite flag gives the same result by changing the ordered blocks or taking the contragredient. In either convention ζ_w is the value of the actual character on the space of forms. Those spaces are defined integrally by the equations $jf = \lambda(j)f$. Hecke operators and changes of level at disjoint places commute with these equations. The other local tests and the integration in the source variable preserve this target transformation law.

Every chosen power ramifies at each specified twist place by the construction of Theorem 13.4. Apply Theorem 50.8. The place is split and is away from ℓ. Theorem 45.16 therefore transfers the parameter with zero monodromy to the attached Galois representation. Frobenius semisimplification preserves its action on inertia, and zero monodromy removes the unipotent tame factor. The normalizer of Corollary 45.17 is unramified at the new places because its defining source character is unramified there and the norm power is trivial on local units. Polarization gives the dual action at the conjugate place up to a cyclotomic power which is also unramified there. This proves the polynomial identity. It holds on every stable lattice, before reduction to any finite precision, because that lattice injects into its fraction vector space.

The row multiplier from this place is $o_w(1)^{t_\alpha}$ by Theorem 45.8. Let $D = \max_\alpha(t_\alpha + \min(a_\alpha, b_\alpha))$, taking the conclusion as empty when there are no types. The proposition bounds the sum inside the supremum by $D \sum_{w \in Q \cup Q^c} v_\ell(1 - \zeta_w)$. A character and its inverse give the same valuation. For each ℓ, request the per-side tame tolerance $\eta_\ell > 0$ in Theorem 13.13, with $2D \sum_\ell \eta_\ell \log \ell \le \delta$. For any embedding and each signed power the last sum over places is at most $2\eta_\ell$, counting both members of every pair. The tolerance can be imposed together with all the earlier critical-value and Selmer tests, since it is chosen before the auxiliary primes and roots of unity. This proves the bound uniformly in all later trace levels. On a finite interval one can use the largest of the finitely many values of D and the same tolerances at each node. □

These local terms concern the primes introduced by the twists. The local equations above the coefficient prime and the comparison of the full projected period in Section 6.6 require separate estimates.

50.4 The projected torus pairing at every coefficient prime

We compare the integral pairing of two projected tests with the free cup products of Corollary 15.6. Both quantities will be evaluated on the given period field, for the same twist, at the precision used to separate its Hecke system. This gives an estimate on the entire prime sum of the difference.

Use the shorter intervals and the main torus pairs of Theorem 15.3. At a node x put $N = q_1 d$, $t_0 = 5r$, $Y = N + t_0 + 1$, and use the integer G_0 of Corollary 12.10. Let

$$Z_x = h_E^2 \mathcal{J}_{N,t_0}\, G_0^{6+(4t_0+3)Y^6}.$$

Here h_E is the class number of $E = E_x$. The integer $m_x < z \le r$ of a designated main pair gives target rank $2z$ and string length $t' = 2z - 1 \le t_0$. At a coefficient prime ℓ, use one of the simultaneous twists for the ratios and Selmer groups from Theorems 13.13 and 50.6. For a pair of integral tested forms Ξ, Ξ' of Section 50.1, let P project onto the line of the chosen torus lift. Set $P = 0$ for a zero lift. Measure its denominator by

$$p(D) = \frac{\log \max\{1, |\iota\langle P\Xi, P\Xi'\rangle|_\ell\}}{\log \ell}.$$

The datum D includes the node, prime, coefficient embedding, main pair, twist power, and two tests. For a nonzero lift it also includes any choice of a separating trace family and precision ℓ^ν as in Corollary 45.21.

For this family let T denote the twisted main line and put $T_e = \mathcal{O}(-e)$ for $0 \le e < t'$. Let C_T be the image in $\bigoplus_{w|\ell} H^1(E_w, T/\ell^\nu T)$ of the normalized global main trace cocycles killed by ℓ^ν in Section 45.3. Identify $T/\ell^\nu T$ with the ℓ^ν-torsion of $T[1/\ell]/T$ by division by ℓ^ν. Let C_e be the local image for the scalar cocycles in the same section, with the same trace relations and precision. In each case apply the joint map of Proposition 15.4 at $a = \nu e_L$. Denote it by P_a^M for lattice M, and define the finite number

$$r(D) = \frac{1}{e_L}\left(\operatorname{length}_{\mathcal{O}} P_a^T(C_T) + \sum_{e=0}^{t'-1} \operatorname{length}_{\mathcal{O}} P_a^{T_e}(C_e)\right).$$

The identification of $\ell^\nu\mathcal{O}$ with $\pi^a\mathcal{O}$ changes coordinates by a unit at most. The choices of free generators may be made separately for each lattice. Put $r(D) = 0$ for a zero lift.

Theorem 50.11 (The prime sum for the projected integral pairing)**.** *Fix q_0, d and make one choice of the shorter period data at each integer node in an interval starting at V_0. The twists can be chosen simultaneously with the bounds for the critical ratios so that*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{\max\{0, p(D) - r(D)\}}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}. \tag{10}$$

The supremum includes every node of the interval, all the indicated integral tests, coefficient embeddings, signed twist powers, and separating families. At a nonzero lift each $r(D)$ uses only $[E:\mathbb{Q}]/2$ free main coordinates and $[E:\mathbb{Q}]$ free coordinates for each of the t' scalar summands. In the common labelled lifting range of Theorem 50.7, $r(D) = 0$.

Proof. Fix all the data at a nonzero lift. Use positive tolerances $\epsilon_{x,\ell}, \eta_{x,\ell}$ for the normalized Selmer length and the sum of the new inertia costs respectively. The simultaneous construction allows both to be decreased, for the same list of signed critical ratios. Write

$$B_{x,\ell} = a_\ell(t') + \epsilon_{x,\ell},$$

using the explicit odd cyclotomic ideal of Proposition 50.5. The global main group satisfying the integral local equations has normalized length at most $B_{x,\ell}$ by Theorem 50.6.

We verify that the single finite precision in $r(D)$ suffices. The row lattice in the proof of Proposition 45.7 contains $\ell^{\nu}\mathcal{O}^{kr'}$, where k is its number of columns before reading the blocks and r' is the main rank (here $r' = 1$). Its annihilator therefore maps into global normalized raw trace cocycles killed by ℓ^{ν}. Reading their values at the row generators recovers the annihilator vectors. The determinant index in that proof is consequently bounded by the length of this finite module of cocycles. The kernel of restriction to the local quotients by $F_a(T)$ satisfies the integral evaluation equations, by Lemma 12.27 and division by ℓ^{ν}. These cocycles vanish at the chosen trace inertia generators, so are unramified at the later trace places as in the proof of Corollary 12.33. At the marked normalization element η, $T(\eta)-1$ is a unit. A coboundary with values in the divisible module that vanishes at η must therefore be zero. Thus the kernel injects into the global group just bounded. Its normalized length is at most $B_{x,\ell}$. This argument does not require crystallinity of an auxiliary matrix at the coefficient prime.

On the finite image C_T, the index of the local conditions is at most the normalized length of its free cup-product image plus $\sum_{w|\ell} h_w(T^{\vee}(1))$, by Proposition 15.4. For T_e the colength l_e in Theorem 45.11 is the index in the scalar trace module of those same coefficient-place evaluation equations for T_e. Applying the proposition to C_e bounds l_e/e_L by its free image length divided by e_L and the torsion error for $\mathcal{O}(e+1)$. Put

$$h(D) = \sum_{w|\ell} \left(h_w(T^{\vee}(1)) + \sum_{e=0}^{t'-1} h_w(\mathcal{O}(e+1)) \right).$$

In total, the normalized contribution of the two trace terms to the coefficient of the separating multiplier is at most $B_{x,\ell} + r(D) + h(D)$.

The other rows of this multiplier use only places away from the coefficient prime. At each old level different from ℓ the Frobenius rows of Corollary 45.24 cost in total at most $2v_{\ell}(G_0)$. An old level equal to ℓ is already included in C_T, C_e. At the new twist pairs the parahoric line character is nontrivial for every selected power. Its inertia polynomial, with the other block spherical, is the one in Theorem 45.14. The two sides of all pairs cost at most $2t_0\eta_{x,\ell}$ as in Theorem 50.7. At the subsidiary split trace places the determinant unit and normalized Hecke rows follow from Corollaries 47.37 and 45.17. They are chosen away from ℓ. At other places away from ℓ the spherical representation is unramified. Thus these row calculations are valid for every coefficient prime. The remaining terms of Theorem 45.11 are bounded after normalization by $v_{\ell}(h_E\mathcal{J}_{N,t_0}) + (4t_0+3)Y^6 v_{\ell}(G_0)$, using $t' \le t_0$ and Corollary 12.10.

Apply Theorem 50.3 to the original two tests. Its averaging and stabilizer factor $Hc_0^2|C|$ divides $h_E G_0^4$. Indeed $|C| \mid h_E H$ by Proposition 50.1, the residue unit orders give $H \mid G_0$, and the stabilizer multiple divides G_0 by the block inclusions $2z \le rt_0$. Combining the bounds gives the pointwise estimate

$$p(D) \le r(D) + h(D) + a_{\ell}(t') + v_{\ell}(Z_x) + \epsilon_{x,\ell} + 2t_0\eta_{x,\ell}. \tag{11}$$

At a zero lift the estimate holds with the torsion terms omitted. The precision required in base separation was chosen for this base family, before its subsidiary levels were introduced. There is no factor of ν in any term on the right other than the actual image length $r(D)$. The count of coordinates is Corollary 15.6. In the labelled lifting range all restricted cocycles are reductions of local finite cocycles by Corollary 13.16. They annihilate the free tests, yielding $r(D) = 0$.

It remains to sum terms that vary with the node. Corollary 15.6 bounds the weighted sum of the suprema of $h(D)/\sqrt{b_x}$ by $O_{q_0,d}(V_0^{-10})$. At a fixed node the prime sum of $\sup_{t'} a_{\ell}(t')$

with weights $\log \ell$ is at most $\log(N \operatorname{lcm}(1, \dots, t_0))$, by Proposition 50.5. Prime factorization gives exactly $\log Z_x$ for the corresponding Z_x term. We have $N, [E : \mathbb{Q}] \ll_{q_0,d} x^{12}$, $r, t_0 \ll x^5$, and $\log(p_0 p_1) \le 22 \log b_x$. The formula for the orders of the finite general linear groups gives $\log G_0 \le [E^+ : \mathbb{Q}](r t_0)^2 \log(p_0 p_1)$. The class number estimate and the bound $\log \mathcal{J}_{N,t_0} \ll N^4 t_0^2 \log(2N t_0)$ in Section 11.8 and Theorem 11.29 consequently give

$$\log Z_x + \log(N \operatorname{lcm}(1, \dots, t_0)) \ll_{q_0,d} x^{110}(1 + \log b_x).$$

For the tolerance terms choose $\epsilon_{x,\ell} + 2t_0 \eta_{x,\ell} \le 2^{-\ell}/(1 + \log \ell)$, decreasing them further for any prescribed critical-ratio tolerances. Their weighted sum at one node is at most one. Use $1 + \log b_x \ll b_x^{1/8}$ and $b_x \ge x^{500}$. After division by $\sqrt{b_x}$ all the fixed and tolerance terms at a node have prime sum $O_{q_0,d}(x^{-77})$. Since all terms are nonnegative, their supremum across nodes is at most their sum across those nodes before taking the prime sum. The tail over integers $x \ge V_0$ is $O_{q_0,d}(V_0^{-76})$. This proves (10). □

Inequality (10) controls the difference from a specified image length for the projected torus tests. A bound for $r(D)$ on the remaining coefficient places and the comparison for the higher-rank chain in Section 6.6 are still required for the conclusion about zeros.

50.5 The contribution of abelian local systems to the torus pairing

Fix the shorter intervals, the rank-one marked separating systems, and the integral tests of Theorems 50.11 and 15.8. Write $K = E_{x,w}$ for a completion at a coefficient prime ℓ. We call a member u_y of the trace family abelian at w if the matrices of $u_y(G_K)$ commute. This includes the normalized representation of the selected lift itself. We shall evaluate the remaining local conditions on explicit matrices whose components at every such member are zero. The prime sum of the resulting error is uniform in the congruence precision and the number of systems.

Use $a = \nu e_L$, $e_L = v_\pi(\ell)$, and the notation for C_M, P_a^M of Section 50.4. For the matrix tuple $D(g) = (u_y(g)-1)_y$ set $\mathcal{N} = \sum_g \mathcal{O}D(g)$. Each raw trace cocycle z has an $\mathcal{O}$-linear map f_z on $\mathcal{N}$ with $f_z(D(g)) = z(g)$, by its defining relations. For a finite group-ring expression $X = \sum_g b_g[g]$ of augmentation zero set $u(X) = (\sum_g b_g u_y(g))_y$. Then $u(X) \in \mathcal{N}$ and $f_z(u(X)) = \sum_g b_g z(g)$. This uses only integral relations among the actual matrices. Coefficients can be extended to contain all local embeddings while retaining the ideal ℓ^ν and extending the image modules by flatness. Normalized lengths are invariant under this operation.

Lemma 50.12 (Unit characters visible in an abelian trace system)**.** *At a spherical coefficient place, the inertia action of an abelian member is a sum of characters which, on $\mathcal{O}_K^\times$ by local reciprocity, have the form*

$$v \longmapsto \prod_{\tau:K\hookrightarrow L} \tau(v)^{\epsilon b_\tau}, \qquad b_\tau \in \{0,\dots,t'-1,\ z-1+\epsilon_\tau k_\tau\}. \tag{12}$$

Here $\epsilon = \pm 1$ is the reciprocity convention and ϵ_τ records the two CM orientations. The coefficients may be enlarged. At an old coefficient place the formula has one additional smooth unit character of order dividing Q.

There is an element $H_w \in \mathcal{O}[G_K]$, of nonzero augmentation δ_w, such that

$$(u_y(g)-1)\,u_y(H_w) = 0 \qquad (g \in I_K,\ u_y \text{ abelian at } w). \tag{13}$$

At both old primes δ_w is a unit. On each node, putting $n_x = [E_x : \mathbb{Q}]$, one can choose the elements so that

$$\sum_\ell (\log \ell) \sup_D \sum_{w|\ell} [E_{x,w} : \mathbb{Q}_\ell]\, v_\ell(\delta_w) \ll_{q_0,d} x^{60} b_x^{1/4}. \tag{14}$$

The supremum is over the choices at that prime and node. The bound includes places where $E_{x,w}/\mathbb{Q}_\ell$ is ramified.

Proof. All normalized trace systems of this weight have the same labelled Hodge multiset as the selected nonzero base lift: the weight formula depends only on the infinity type, and the base representation is the sum of the main character and the scalar string (Theorem 50.6). They are crystalline at the spherical coefficient places. Indeed the spherical base change has an unramified parameter with zero monodromy, and the compatibility argument in Corollary 12.16 gives crystallinity before any condition on the length of the interval of weights. This also applies when K itself is ramified over $\mathbb{Q}_\ell$. The smooth normalizer is unramified here, the twist splits above ℓ, and the subsidiary levels avoid ℓ. At the original levels Theorem 15.8 gives one crystallizing extension of tame degree Q.

For commuting matrices extend coefficients and use the generalized eigenspaces of the finite-dimensional commutative algebra they generate. On each such space divide by the eigencharacter ψ. The quotient representation has a filtration with trivial successive quotients and is crystalline over the same crystallizing field as the original representation and ψ. To see that it is unramified there, argue by induction along the filtration. For an unramified weight-zero

representation V, unramified extension classes lie in $H^1_f(K', V)$, since unramified representations are crystalline. Their dimension is $\dim V/(\mathrm{Fr}-1)V = \dim H^0(K', V)$, the dimension of $H^1_f(K', V)$ by the Bloch–Kato formula. Thus an extension of the trivial line by such a V which is crystalline is unramified. This proves the induction. Over K the inertia in this unipotent part is now finite, hence trivial in characteristic zero. Thus the inertia action is semisimple even when the abelian representation has a nontrivial unramified extension. For a crystalline character, local reciprocity on units gives precisely the algebraic monomial with its labelled weights. For a potentially crystalline character divide by that crystalline algebraic character; the smooth unit factor is then the finite inertia type. This proves (12) with the asserted modification. [3, Appendix B, Proposition B.4]

At an old prime put $p = \ell$. Choose $s \in I_K$ lifting the reciprocity image of a Teichmüller generator of $\mathbb{F}_p^\times$, and take $H_w = \sum_{j=0}^{p-2} [s]^j$. Every nontrivial unit character in question takes a nonidentity $(p-1)$-st root of unity at s. Indeed if its labelled exponent b is nonzero then $0 < Q|b| < p-1$, so raising the value to Q is already nontrivial. If $b = 0$, a nontrivial smooth character of the cyclic residue group is nontrivial on its generator. Therefore H_w kills every nontrivial inertia character. On the remaining spaces inertia acts trivially. This gives (13), with $\delta_w = p - 1$.

At the other primes we first describe how to detect all nontrivial characters in (12). Call the choice $z - 1 + \epsilon_\tau k_\tau$ at a label the main jump. Every k_τ exceeds $3n_x t_0$ for sufficiently large x: the formula in Theorem 15.3 gives $k_\tau \geq 3r_x^8 - 4r_x$, while $n_x \ll_{q_0,d} x^{12}$, $x^5 \leq r_x \leq 3x^5$, and $t_0 = 5r_x$. If only scalar jumps occur and some are nonzero, rational units detect their positive sum. If the quadratic CM field splits at ℓ, all main jumps at w have the same orientation. One main jump suffices to make the sum of all selected jumps nonzero, even when the scalar jumps have the opposite sign. Use rational units in this case also. At a nonsplit prime, as soon as a main jump occurs, use norm-one units in the quadratic completion inside K. The exponent on such a unit, up to one overall sign and a choice of its embedding, is $\sum_\tau \epsilon_\tau b_\tau$. Every selected main jump contributes $+k_\tau$ to this sum apart from its term $\epsilon_\tau(z-1)$. All the latter terms and the scalar terms together have absolute value at most $n_x t_0$. This proves detection, including when the rational sum is zero.

Here is a common finite list of absolute exponents for those tests. For a fixed $1 \leq j_0 \leq m_x$ let w_1, w_2 be the two base weights after adding $(2j_0+1)r_x^8$, as in Theorem 15.5. Include $1, \ldots, n_x t_0$ and the positive integers of the form

$$|n_1 w_1 + n_2 w_2 + D_0|, \qquad n_1, n_2 \geq 0, \quad 1 \leq n_1 + n_2 \leq n_x, \quad |D_0| \leq 3n_x t_0.$$

Take the union over j_0 and call it $\mathcal{B}_x$. Indeed the selected main magnitudes are $w_i - h_\tau$ with $0 \leq h_\tau \leq 4r_x$; all the discarded scalar and central shifts fit in the indicated bound. This list has size $O((m_x+1)(n_x+1)^2(1+n_x t_0))$ and maximum $O(n_x b_x)$.

For each $B \in \mathcal{B}_x$ and each required direction choose one unit v_B. For the rational direction use a Teichmüller generator if $\ell - 1 \nmid B$, and $1 + \ell^{c_\ell}$ otherwise, where $c_2 = 2$ and $c_\ell = 1$ for odd ℓ. In the inert norm-one direction use a Teichmüller generator of order $\ell + 1$ when $\ell + 1 \nmid B$, and the norm-one unit of depth c_ℓ in the proof of Theorem 15.5 otherwise. At the ramified quadratic prime d use the norm-one unit of depth $3/2$ of that proof. Lift each reciprocity value to some $s_B \in I_K$. For each direction and each sign take the factor $[s_B] - v_B^{\pm B}[1]$, reading v_B in the chosen embedding of its rational or quadratic field. Define H_w to be the product of these factors in any fixed order. Every nontrivial inertia character in the abelian member vanishes on at least one factor, in the sense that its linear extension to the group ring takes that factor to zero. All factors preserve the inertia character spaces of this member and act on them by scalars. This proves (13) over the fraction field and hence on the original integral lattices.

Each factor has nonzero augmentation. For a fixed B, its valuation is zero on the residue tests. On the principal-unit tests it is $c_\ell + v_\ell(B)$, or $3/2 + v_d(B)$ for the ramified norm-one unit, by the binomial evaluations in Theorem 15.5. Summed over primes, both directions and both signs with weights $\log \ell$, this costs $O_d(\tau(B)\log(B+1) + \log(2B)) = O_d((1+B)^{1/4})$. Indeed the rational and inert tests with positive cost require $\ell - 1 \mid B$ and $\ell + 1 \mid B$, respectively, and the ramified prime is fixed. Sum over $\mathcal{B}_x$, then use $\sum_{w|\ell}[E_{x,w} : \mathbb{Q}_\ell] = n_x$. The explicit bound is $O_d(n_x|\mathcal{B}_x|(1+n_x b_x)^{1/4})$. Inserting $m_x \le 2\sqrt{x}$ and the bounds for n_x, t_0 proves (14). □

Theorem 50.13 (Projected pairings and the nonabelian local matrices)**.** *For the indicated marked separating families at each prime and node there is a number $r_*(D)$ computable from their integral trace functionals on tuples with zero components at every member abelian at the place of evaluation. It satisfies*

$$0 \le r_*(D) \le r(D), \qquad \sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_*(D))_+}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}. \tag{15}$$

In particular the whole normalized prime sum of the projected pairing on the subfamily in which every u_y is abelian at every $w \mid \ell$ is $O_{q_0,d}(V_0^{-10})$. This conclusion holds for the same twists and integral tests used to bound the critical ratios.

Proof. First take $M = T$ or $M = \mathcal{O}(-h)$ with $1 \le h < t'$, and write α for its character. At each w choose $s_w \in I_K$ minimizing the valuation of $\lambda_{M,w} = \alpha(s_w) - 1$. This minimum is finite by the unit tests of Theorem 15.5 for T, and by the rational-unit test with exponent $h[K : \mathbb{Q}_\ell]$ for the scalar. For $[s, g] = sgs^{-1}g^{-1}$, the cocycle rule gives

$$f_z(D([s_w, g])) = (\alpha(s_w) - 1)z(g) - (\alpha(g) - 1)z(s_w) \qquad (g \in G_K).$$

As a function of g this represents $\lambda_{M,w}[z]$. The second term is a coboundary and all abelian matrix components of $D([s_w, g])$ vanish.

For $M = \mathcal{O}$ take $\lambda_{M,w} = \delta_w$ of Lemma 50.12. A cocycle is now an additive homomorphism. For every $g \in I_K$,

$$f_z(u(([g] - [1])H_w)) = \delta_w z(g).$$

Indeed write $H_w = \sum b_v[v]$ and use $z(gv) - z(v) = z(g)$. The tuple inside f_z is in $\mathcal{N}$ and has zero abelian components by (13). The indicated inertia values determine all the free cup products of $\delta_w[z]$. Two extensions of these values to G_K differ by an unramified homomorphism, and every unramified homomorphism to $\mathcal{O}/\pi^a$ lifts integrally and belongs to the reduced finite condition. Proposition 15.4 shows its free cups vanish.

In the definition of $r(D)$, multiply all free coordinates at w for M by $\lambda_{M,w}$ and take the normalized length of each resulting joint image. Their sum is $r_*(D)$; take zero at a zero lift. The descriptions above specify exactly which evaluations on $\mathcal{N}$ compute these images. In particular they do not require a functional on $\mathcal{N}$ to extend to the full matrix lattice. On $d_{M,w}$ copies of $\mathcal{O}/\pi^a$, the kernel of scalar multiplication has length $d_{M,w}\min(a, v_\pi(\lambda_{M,w}))$. Restricting to the joint image can only decrease the kernel length, including when classes at different places are dependent. As $d_{M,w} \le [K : \mathbb{Q}_\ell]$, it follows that

$$0 \le r(D) - r_*(D) \le \sum_{M,w} [E_{x,w} : \mathbb{Q}_\ell]\, v_\ell(\lambda_{M,w}).$$

For the main line, the proof of Theorem 15.5 bounds valuations by actual inertia evaluations at all its places. Its fixed-node prime sum before division by $\sqrt{b_x}$ is $O_{q_0,d}(x^{62} b_x^{1/4})$, and insertion of the local degree costs at most n_x. For positive scalar weights use the proof of Corollary 15.6

with $h[K:\mathbb{Q}_\ell]$ in place of $(h+1)[K:\mathbb{Q}_\ell]$. Both exponents lie between one and $n_x t_0$, so the same factorial estimate gives $O(n_x(n_x t_0+1)^2\log(2+n_x t_0))$ after inserting the local degree. The weight-zero cost is (14). Together these give $O_{q_0,d}(x^{80}b_x^{1/4})$ at the node. After division by $\sqrt{b_x}$ this is $O_{q_0,d}(x^{-45})$ because $b_x \ge x^{500}$. Bound the supremum across integer nodes by their sum and apply Theorem 50.11. This proves (15). If all the indicated systems have abelian image locally, every displayed matrix evaluation vanishes and hence $r_*(D)=0$. □

The selected torus representation has abelian image, but other systems in a separating family can have noncommuting local matrices. Their integral evaluations in (15) and the higher-rank projected period in Section 6.6 still require bounds.

50.6 Two inertia elements at the original level primes

At either original level prime, Theorem 15.8 leaves several scalar weights at the same completions. We give one simultaneous choice of inertia elements for all these weights, at every precision. This identifies the remaining joint image inside the actual integral trace module.

Write $G = G_{\mathbb{Q}_p}$, let I be inertia, let P be wild inertia and let κ denote the p-adic cyclotomic character. For the integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_p$, put $R_a = \mathcal{O}/\pi^a$, $a \geq 1$. All cohomology below is continuous. A choice of Tate basis identifies the underlying module of $R_a(-h)$ with R_a.

Lemma 50.14 (Simultaneous evaluation of the scalar ramification classes)**.** *For each $p \geq 5$ there are $s \in I$ of pro-prime-to-p order and $g \in P$ with $\kappa(s)$ a primitive $(p-1)$-st root and $\kappa(g) = 1 + p$. Put $b = [s, g]$. For every $\mathcal{O}, a$ as above one has isomorphisms*

$$H^1(G, R_a(-h)) \longrightarrow R_a, \quad [c] \longmapsto c(b) \qquad (1 \leq h \leq p-3),$$
$$H^1(G, R_a)/H^1_{\mathrm{unr}}(G, R_a) \longrightarrow R_a, \quad [c] \longmapsto c(g).$$

The same s, g work for all these maps.

Proof. Lift a topological generator of I/P to inertia. Its procyclic closure is the product of its pro-p part and its prime-to-p part. The prime-to-p component s has the same image in I/P. Write $u = \overline{\kappa(s)}$. It generates $\mathbb{F}_p^\times$ by total ramification of $\mathbb{Q}_p(\mu_p)$, and $\kappa(s)$ is its Teichmüller lift.

For $1 \leq h \leq p-3$, both $\bar{\kappa}^{-h}$ and $\bar{\kappa}^{h+1}$ are nontrivial on inertia. Local Tate duality and the local Euler characteristic formula give $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p(-h)) = 1$. Choose a cocycle c_h representing a nonzero class. Put $J_0 = I \cap \ker\kappa$ and $J = P \cap \ker\kappa$. On J_0 the cocycle is a homomorphism. If it vanished there, it would factor through G/J_0, which embeds in $(G/I) \times \mathbb{Z}_p^\times$ and is abelian. Commuting the images of s and an arbitrary v would give

$$(u^{-h} - 1)c_h(v) = (\bar{\kappa}(v)^{-h} - 1)c_h(s).$$

The first scalar is a unit, so c_h would be a coboundary. Its restriction to J is also nonzero, since J_0/J embeds in tame inertia and has no quotient of order p.

The homomorphisms $\psi_h = c_h|_J$ are linearly independent over $\mathbb{F}_p$. Indeed $j \mapsto sjs^{-1}$ acts on their arguments with distinct eigenvalues u^{-h}, by the cocycle rule. Thus $j \mapsto (\psi_h(j))_{1 \leq h \leq p-3}$ is surjective because its image is a vector subspace and any proper subspace has a nonzero annihilating linear functional. The image of P under κ is $1 + p\mathbb{Z}_p$. This follows from total ramification of the cyclotomic tower and the prime-to-p order of I/P. Lift $1+p$ to $g_0 \in P$, and choose $j \in J$ with $\psi_h(j) = 1 - c_h(g_0)$ for all h. Then $g = g_0 j$ has $c_h(g) = 1$. Since $b = [s, g]$ belongs to $\ker\kappa$, evaluation there kills coboundaries. Direct substitution in the cocycle identity gives $c_h(b) = u^{-h} - 1$.

For passage to all precisions, $H^0(G, (\mathbb{Q}_p/\mathbb{Z}_p)(-h)) = 0$ by the tame unit $u^{-h} - 1$, so $H^1(G, \mathbb{Z}_p(-h))$ is torsion-free. Its rank is one by the rational local Euler characteristic. The group $H^2(G, \mathbb{Z}_p(-h))$ vanishes by integral local duality since $H^0(G, (\mathbb{Q}_p/\mathbb{Z}_p)(h+1)) = 0$. Thus reduction of integral classes gives all classes modulo p. Evaluation at b is an isomorphism on the rank-one integral module because it is nonzero modulo p. Tensoring with the finite free $\mathbb{Z}_p$-module $\mathcal{O}$ commutes with continuous cohomology here, as is seen on a $\mathbb{Z}_p$-basis. The exact sequence for multiplication by π^a and the vanishing of H^2 give the claimed evaluation for R_a. For $h = 0$, local reciprocity identifies ramified additive characters modulo unramified ones with $\mathrm{Hom}_{\mathrm{cont}}(1+p\mathbb{Z}_p, R_a)$. On units the cyclotomic character agrees with the reciprocity coordinate up to inversion. The image of g generates that procyclic group, giving the last isomorphism. [4, Chapter I, §2] □

Proposition 50.15 (A column of integral tame blocks)**.** *Let $(u_y)_y$ be a finite family of continuous integral representations of G. Write $u = (u_y)_y$ also for the linear extension to the group algebra, and put $\mathcal{N} = \sum_{v \in G} \mathcal{O}\,(u(v) - 1)$. Suppose a cocycle $c : G \to R_a(-h)$ admits an $\mathcal{O}$-linear functional f_c on $\mathcal{N}$ with $f_c(u(v) - 1) = c(v)$. Set $\eta_h = \kappa(s)^{-h}$, and let $P_{y,\eta}$ be the spectral projector of $u_y(s)$ at η, taking zero for an absent eigenvalue. For the pair of the lemma define tuples*

$$Z_h = \big(P_{y,\eta_h} u_y(g) P_{y,1}\big)_y \quad (1 \le h \le p-3), \qquad Z_0 = \big(P_{y,1}(u_y(g) - 1)P_{y,1}\big)_y.$$

These are integral elements of $\mathcal{N}$. For the functional of the indicated weight they satisfy

$$(\eta_h - 1) f_c(Z_h) = c([s,g]) \quad (h > 0), \qquad f_c(Z_0) = c(g).$$

Proof. The closure generated by $u_y(s)$ is pro-prime-to-p. Reduction modulo π has pro-p kernel, so this closure injects into a finite residue matrix group. Take m' divisible by the orders of all $u_y(s)$ and by $p-1$, and prime to p. For each required η, the element

$$e_\eta = (m')^{-1} \sum_{i=0}^{m'-1} \eta^{-i} [s]^i \ \in \mathcal{O}[G]$$

evaluates in u_y to $P_{y,\eta}$, by summation of roots of unity. This formula works over $\mathcal{O}$ even when other eigenvalues require a coefficient extension. Its augmentation is one for $\eta = 1$ and zero otherwise. Thus $e_{\eta_h}[g]e_1$ for $h > 0$, and $e_1([g] - [1])e_1$ for $h = 0$, have zero augmentation. Every element of zero augmentation is an integral sum of $[v] - [1]$. This proves membership in the specified lattice.

Write ϵ for augmentation, α for the linear extension of κ^{-h}, and $d_c([v]) = c(v)$, extended linearly. On products the cocycle identity gives

$$d_c(XY) = \alpha(X) d_c(Y) + d_c(X) \epsilon(Y).$$

On elements of zero augmentation, $d_c(X) = f_c(u(X))$. For $B = [s][g] - [g][s]$ both $\alpha(B)$ and $\epsilon(B)$ vanish, and $d_c(B) = c([s,g])$. Since $\alpha(e_{\eta_h}) = \epsilon(e_1) = 1$, we have $d_c(e_{\eta_h} B e_1) = d_c(B)$. Evaluation in the matrices gives $u(e_{\eta_h} B e_1) = (\eta_h - 1) Z_h$. For $h = 0$, $\alpha = \epsilon$ and the same product rule gives $d_c(e_1([g] - [1])e_1) = c(g)$. This uses f_c only on its original integral domain. □

Theorem 50.16 (The remaining joint image at the level primes)**.** *Use the marked separating families, twists, nodes and shorter intervals of Theorems 50.11 and 15.8. At a node x let p be either original level prime. For a nonzero lift let W^- be the completions $w \mid p$ of E_x with $J_w < 0$. Transport the pair s, g for $\mathbb{Q}_p$ to each of the chosen decomposition groups at w. For each $0 \le h < t' - 1$ let $\Gamma_h(D)$ be the joint image*

$$c \longmapsto \big(f_c(Z_{h,w})\big)_{w \in W^-} \quad \text{in} \bigoplus_{w \in W^-} R_a$$

of the original normalized global scalar trace cocycles of weight h. Here $Z_{h,w}$ is formed from the restricted family, and its local trace lattice maps into the global one. At the original trace precision $a = e_L \nu$ there is an equality

$$r(D) = B_{\mathrm{old}}(D), \qquad B_{\mathrm{old}}(D) = \frac{1}{e_L} \sum_{0 \le h < t'-1} \mathrm{length}_{\mathcal{O}}\, \Gamma_h(D).$$

Take $B_{\mathrm{old}} = 0$ at a zero lift. In particular

$$\sum_p (\log p) \sup_{\substack{D \text{ at } p \\ p \text{ an original level at } x}} \frac{(p(D) - B_{\mathrm{old}}(D))_+}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10},$$

with the empty supremum set to zero.

Proof. All the completions under consideration equal $\mathbb{Q}_p$, and $t' < p - 2$ by $t' \le t_0 \le b_x$, $p \ge b_x^{10}$. At $h > 0$ the rational finite local condition on $\mathbb{Q}_p(-h)$ is zero by the crystalline dimension formula (its de Rham tangent space and its invariants vanish). Its integral reduced finite condition is zero as well, since the integral first cohomology of this character is torsion-free. For $h = 0$ the rational finite condition is the unramified line because unramified extensions are crystalline and the crystalline dimension formula gives dimension one. The integral condition and its reduction consist of all unramified classes by local reciprocity. For every $0 \le h < t'$ the dual $H^1_f(\mathbb{Q}_p, \mathcal{O}(h+1))$ has rank one by the same dimension formula. It has no torsion since $\bar{\kappa}^{h+1}$ is nontrivial on inertia. The free cup test of Proposition 15.4 therefore has exactly the indicated finite condition as its kernel. By Lemma 50.14 its quotient has length a, so its injection into one copy of R_a is an isomorphism. The cup coordinate and the evaluation of the lemma differ by a unit. For $h > 0$, $\eta_h - 1$ is itself a unit. Proposition 50.15 now identifies each surviving cup coordinate up to a unit with $f_c(Z_{h,w})$.

Theorem 15.8 kills the main coordinates everywhere, every scalar coordinate at the positive completions, and the top scalar coordinate at the negative ones. At each weight the comparison on the other coordinates is an automorphism of their direct sum, hence preserves the length of the joint image even for dependent restrictions. This proves the equality. The prime sum follows by restricting the supremum in (10) to the displayed data. □

There is a further simplification for a member whose residual characteristic polynomial at s agrees with that of the base member. In this case both projectors in $Z_{h,w}$ have rank one. Indeed the base eigenvalues are $\kappa(s)^{-e}$ for $0 \le e < t'$ and $\xi\kappa(s)^{-J_w}$ with $\xi^Q = 1$, by the local character formula of Theorem 15.8. The first roots are distinct. Equality of the last with any of them after reduction would imply $p - 1 \mid Q(J_w - e)$. The jumps and Q in that theorem give $0 < Q|J_w - e| < p - 1$. Prime-to-p roots remain distinct after reduction, so the spectral summands of the specified eigenvalues have multiplicity one also in the given member. Their integral images are direct summands and admit bases over $\mathcal{O}$. In these bases each displayed block is a single entry. The equality for B_{old} still involves functionals on the joint trace lattice. A bound for the lengths of their images, uniform in the precision and the family, is needed to remove B_{old} from the period estimate.

50.7 The projected pairing for potentially abelian local systems

A local representation is potentially abelian if its image on some open subgroup is abelian. The subgroup and its index may vary with the representation. We use the common crystallizing field from Section 15.3 to compare all these systems on a single inertia group.

Lemma 50.17 (Inertia of a potentially abelian crystalline representation)**.** *Let F be a finite extension of $\mathbb{Q}_\ell$. Let V be a potentially abelian crystalline representation of G_F of dimension R over a finite extension of $\mathbb{Q}_\ell$. After a finite extension of coefficients its restriction to I_F is a sum of characters. Every such character is the restriction of a crystalline character on the unramified extension F_f/F of some degree $1 \le f \le R$. At each embedding of F_f its jump belongs to the Hodge multiset of V at the underlying embedding of F.*

Proof. Choose a finite Galois extension L/F on whose absolute Galois group the matrices commute. Write A for the coefficient algebra spanned by those matrices, and A' for its centralizer in $\mathrm{End}(V)$. Conjugation makes A' a subrepresentation of the crystalline representation $\mathrm{End}(V)$. Its action factors through $\mathrm{Gal}(L/F)$, and hence is unramified. For the last assertion, a finite-image crystalline representation U trivialized by L has crystalline comparison over L with period space $L_0 \otimes_{\mathbb{Q}_\ell} U$, where L_0 is the maximal absolutely unramified subfield. Its F-crystalline periods span this space over L_0 by the crystalline comparison and base change. Relative inertia fixes L_0 and these periods, so acts trivially on U. One can forget the coefficient field for this argument. It follows that the inertia matrices for V belong to the double centralizer A''. Since A is commutative, $A \subset A'$ and $A'' \subset A'$. Any two elements of A'' commute.

Over L, each generalized eigenspace of the commuting matrices, twisted by the inverse of its eigencharacter, is crystalline with a filtration by trivial lines. Such a representation is unramified. Indeed the unramified extension classes of an unramified weight-zero representation W have dimension $\dim W/(\mathrm{Fr}-1)W = \dim H^0(L,W)$, and they are crystalline. The crystalline dimension formula gives precisely this dimension for $H^1_f(L,W)$. Induction on the filtration proves the assertion. Each inertia matrix over F has a positive power in I_L which is therefore semisimple. An invertible matrix in characteristic zero with a semisimple positive power is semisimple, as one sees on Jordan blocks. These commuting semisimple matrices are simultaneously diagonalizable after a finite coefficient extension.

Frobenius permutes their distinct characters. For one character choose f to be its orbit length. Its isotypic space is stable under G_{F_f}, since inertia preserves it and a lift of Fr^f preserves it. An eigenline of that lift in the space is stable under all of G_{F_f}, by continuity and the density of the subgroup generated by inertia and the lift. It is crystalline as a subrepresentation of the restriction of V. Exactness and strictness for the filtered de Rham functor, and base change from F to F_f, give the assertion about its labelled jumps. The orbit contains at most $\dim V$ characters. □

At the spherical coefficient places put $F = K = E_{x,w}$. At an old coefficient place put $K = \mathbb{Q}_p$ and $F = K'$, the totally ramified cyclic extension of degree Q in Theorem 15.8. In both cases $I^0 = I_F$ is normal in G_K. At a nonzero lift write $R = 2z = t' + 1$, and put $R_x = t_0 + 1$, so $R \le R_x$. Every trace system under discussion is crystalline over F.

Proposition 50.18 (A common inertia test with controlled augmentation)**.** *There is an integral group algebra element H_w supported on I^0, with nonzero augmentation δ_w, such that*

$$(u_y(v) - 1)u_y(H_w) = 0 \qquad (v \in I^0)$$

for every member u_y potentially abelian at w. Coefficient extension is allowed with the ideal ℓ^{ν} retained. At an old coefficient place $\delta_w = 1$ is possible. At a fixed node one has

$$\sum_{\ell} (\log \ell) \sup_{D} \sum_{w \mid \ell} [E_{x,w} : \mathbb{Q}_{\ell}]\, v_{\ell}(\delta_w) \ll_{q_0,d} x^{80} b_x^{1/4}.$$

Here $v_{\ell}(\ell) = 1$ and the choices in D are those of Theorem 50.11.

Proof. First suppose the place is spherical. On each character of I^0 in Lemma 50.17, the units of F_f act by the algebraic monomial of its labelled jumps via reciprocity, with one overall sign convention. This is the classification of crystalline characters. [3, Appendix B, Proposition B.4] There are at most $N_x = n_x R_x$ labels, where $n_x = [E_x : \mathbb{Q}]$. Each selected jump is either in $[0, t' - 1]$ or is $z - 1 + \epsilon_{\tau} k_{\tau}$ at the underlying label of K. If all selected jumps vanish, inertia acts trivially. If only scalar jumps occur and one is positive, their sum is positive, and rational units suffice. If a main jump occurs and ℓ splits in the imaginary quadratic field, all the main signs agree. The rational sum is nonzero since

$$\min_{\tau} k_{\tau} \geq 3r_x^8 - 4r_x > 3N_x t_0$$

for all sufficiently large initial nodes. This uses $n_x \ll x^{12}$, $x^5 \leq r_x \leq 3x^5$ and $t_0 = 5r_x$. If a main jump occurs at a nonsplit prime, test on the norm-one units of the quadratic completion. The corresponding signed sum of jumps has a contribution $+k_{\tau}$ from each main choice, up to total central and scalar terms of absolute value at most $N_x t_0$. Hence this exponent is also nonzero.

For clarity, a single list valid at the node consists of the integers $1, \ldots, N_x t_0$, together with the nonzero absolute values of

$$i_1 w_1 + i_2 w_2 + D_0, \qquad i_1, i_2 \geq 0, \quad 1 \leq i_1 + i_2 \leq N_x, \quad |D_0| \leq 3N_x t_0.$$

Use the two shifted base weights w_1, w_2 for each of the m_x allowed indices. Indeed $k_{\tau} = w_i - h_{\tau}$ with $0 \leq h_{\tau} \leq 4r_x$, so all the discarded shifts satisfy the displayed bound. This list $\mathcal{B}$ has cardinality $O((m_x+1)(1+N_x)^2(1+N_x t_0))$ and maximum $O(N_x b_x)$. For each $1 \leq f \leq R$, each $B \in \mathcal{B}$ and each required direction, choose the unit v_B from the proof of Lemma 50.12, now viewed in F_f. Lift its reciprocity value for F_f to $s_{f,B} \in I_{F_f} = I^0$. Put in H_w the two factors $[s_{f,B}] - v_B^{\pm B}[1]$, in any fixed order. On a nontrivial inertia character of any of the members, the factor for its own f, a detecting direction and its signed exponent vanishes. All factors preserve that character space. On the trivial character space $v \in I^0$ already acts as one.

The unit estimates in Lemma 50.12 depend only on B, the direction and ℓ. Lifting by reciprocity for F_f changes none of their augmentations. For one B the sum of the valuations of these augmentations over primes, with weights $\log \ell$, is $O_d((1+B)^{1/4})$ for each f. The fixed-node bound is therefore

$$O_{q_0,d}\big(n_x R_x |\mathcal{B}| (1 + N_x b_x)^{1/4}\big).$$

Since $m_x \leq 2\sqrt{x}$, $N_x \ll x^{17}$, and $t_0 \ll x^5$, the power of x here is at most $17 + \frac{1}{2} + 34 + 22 + 17/4 < 80$.

At an old prime there are Qf embeddings of F_f over $\mathbb{Q}_p$, with Q above each of the f residue embeddings. Let χ be any character of I^0 occurring in a potentially abelian member. Its stabilizer under $\Delta = I_K / I^0 = \mathrm{Gal}(F/K)$ has index $j \leq R$. The action here is well-defined since conjugation by I^0 fixes every such character. Use its crystalline line on G_{F_f} from Lemma 50.17. Each stabilizer element can be lifted to I_K, acting trivially on the unramified degree f subfield.

The field F_f, the compositum of that subfield and F, is preserved. Conjugation of the line character by this lift has the same restriction to I^0. Thus their algebraic monomials agree on units by naturality of local reciprocity. Their labelled exponents agree. Indeed take logarithms on sufficiently deep principal units to obtain equality of the corresponding linear combinations of embeddings on a full lattice in F_f, hence on F_f. Distinct field embeddings are linearly independent. The jumps are therefore constant along each orbit of the stabilizer on embeddings.

At each residue embedding i, $0 \le i < f$, let A_i be the sum of the Q exponents before the common reciprocity sign. There are j stabilizer orbits there, each of size Q/j, with jumps in $\{0, \dots, t'-1, J_w\}$. If $J_w < 0$ then $|J_w| > R_x t_0$, so even one negative choice among the j terms outweighs every possible sum of the nonnegative choices. If $J_w > 0$, all terms are nonnegative. Thus $A_i = 0$ precisely when all its jumps vanish. Also $|A_i| \le 5Qb_x < p-1$ by the size of the old primes. If χ is nontrivial, some $A_i \neq 0$. Its exponent on a Teichmüller generator of the residue field of F_f is, up to sign, $\sum_{i=0}^{f-1} p^i A_i$, with the residue embeddings in Frobenius order. This integer has absolute value less than $p^f - 1$. If it were zero, reduction modulo p and $|A_i| < p-1$ would imply $A_0 = 0$, and division and induction would force all the others to vanish. The character is consequently nontrivial on such a generator.

Take the element s of Lemma 50.14. Its power s^Q generates the tame quotient of I^0 topologically, since that quotient is the index Q subgroup of the procyclic tame quotient for K. For every f its image in the abelian inertia of F_f is a Teichmüller generator. Indeed wild inertia maps onto principal units and the prime-to-p unit quotient is the cyclic residue group, while s^Q has pro-prime-to-p order. Thus a character χ as above with $\chi(s^Q) = 1$ is trivial on all of I^0. Take H_w to be the average of the powers of $[s^Q]$ from zero to $m' - 1$, with m' a common prime-to-p multiple of the finite orders of $u_y(s^Q)$ as in Proposition 50.15. Its augmentation is one, and its evaluation projects onto the s^Q-fixed space. This proves the assertion for these places, including integral lattices by embedding them in their rational spaces. □

Theorem 50.19 (Period contribution outside the potentially abelian members)**.** *For the marked separating families of Theorem 50.11 there are numbers $0 \le r_{\mathrm{pot}}(D) \le r(D)$ determined by trace evaluations on integral matrix tuples with zero component at every member potentially abelian at the place of evaluation. They satisfy*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_{\mathrm{pot}}(D))_+}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

In particular the full prime sum of $p(D)$ satisfies this bound on the subfamily for which every member is potentially abelian at every coefficient place. At an original level prime all the blocks in Theorem 50.16 already have zero component at each locally potentially abelian member.

Proof. At a spherical place take $M = T$ or $M = \mathcal{O}(-h)$, $1 \le h < t'$, and write α for its character. Choose $s_w \in I^0 = I_K$ minimizing $v_\ell(\lambda_w)$ for $\lambda_w = \alpha(s_w) - 1$, as in Theorem 50.13. For every $v \in I^0$,

$$f_c(u([s_w, v]) - 1) = \lambda_w c(v) - (\alpha(v) - 1)c(s_w),$$

where $[s_w, v]$ denotes the group commutator. By Lemma 50.17 its matrix difference is zero at all potentially abelian members. If the indicated evaluations of two trace cocycles agree, the restriction of the difference of their classes multiplied by λ_w is zero on I^0. By inflation and restriction this multiplied class is inflated from $H^1(G_K/I^0, (M/\pi^a)^{I^0})$. Its coefficients are killed by λ_w, as $s_w \in I^0$. Thus the evaluations determine all free cup coordinates of $\lambda_w^2[c]$ on G_K.

For $M = \mathcal{O}$ at a spherical place, evaluate f_c on $u(([v] - [1])H_w)$ for $v \in I^0$, now using group algebra symbols. The tuple has zero augmentation before evaluation and belongs to $\mathcal{N}$. It has

zero components as in the proposition, and the additive cocycle evaluates to $\delta_w c(v)$. These values determine all free cup coordinates of $\delta_w[c]$, since unramified additive classes lift and have zero free cups.

At an original level prime keep the exact calculation of Theorem 50.16. In a potentially abelian member, $g \in P \subset I^0$ commutes in its matrix action with s^Q by Lemma 50.17. For $h > 0$ the s-spaces at 1 and η_h have distinct s^Q eigenvalues since $0 < Qh < p - 1$. Hence the block with these source and target spaces is zero. The s-fixed space is contained in the s^Q-fixed space, on which I^0 acts trivially by the old-prime proof of Proposition 50.18. This gives the assertion for Z_0 also.

Define $r_{\mathrm{pot}}(D)$ from the same joint cup images as $r(D)$, multiplying each spherical coordinate by λ_w^2 or δ_w according to its weight. At an old coefficient prime use $B_{\mathrm{old}}(D)$ unchanged. Set $r_{\mathrm{pot}} = 0$ at a zero lift. The kernel of multiplication by $v \in \mathcal{O} \setminus \{0\}$ on d copies of $\mathcal{O}/\pi^a$ has normalized length at most $d\, v_\ell(v)$. The same upper bound holds on any joint submodule by inclusion, adding the bounds for its coordinates. Each local test has at most $[E_{x,w} : \mathbb{Q}_\ell]$ coordinates per weight. For the main and positive scalar weights, the sum of the costs from λ_w^2 is at most twice the unit cost already bounded at the end of Theorem 50.13. For weight zero use Proposition 50.18. The resulting prime sum at each node for $r(D) - r_{\mathrm{pot}}(D)$ is $O_{q_0,d}(x^{80} b_x^{1/4})$, uniformly in the same data. Divide by $\sqrt{b_x}$ and sum over nodes $x \geq V_0$. Since $b_x \geq x^{500}$, this costs $O_{q_0,d}(V_0^{-44})$. Combining with (10) proves the displayed estimate. When all members satisfy the stated local hypothesis all determining tuples are zero, so $r_{\mathrm{pot}} = 0$ there. □

The original separating families allow representations having nonabelian image on every open subgroup at a coefficient place. The theorem leaves their integral trace evaluations to be bounded. Its prime sum applies on the same fields, intervals and signed twists as the critical-ratio estimates.

50.8 Determinants of two-dimensional local constituents

We return to the coefficient primes away from the two original levels. There the local trace matrices are crystalline, but the width of their weights can exceed the residue characteristic. A determinant identity removes two-dimensional direct summands from the trace equations whenever their determinants and the target character have different exponents on suitable units. Its cost in the projected period is summable on the original interval, including at ramified completions of the period field.

Lemma 50.20 (Removing two-dimensional blocks from trace equations)**.** *Let $\alpha : G \to \mathcal{O}^\times$ be a character and c an α-cocycle modulo π^a satisfying the integral matrix relations of a finite family u_y, in the sense of Lemma 12.14. Fix rational G-stable direct sum decompositions of the u_y, after extending coefficients if necessary. Choose $s \in G$, $\alpha(s) \neq 1$, and a finite set $\mathcal{D}$ of units of $\mathcal{O}$ different from $\alpha(s)$. Put*

$$k_s = (\alpha(s) - 1) \prod_{d \in \mathcal{D}} (\alpha(s) - d).$$

The class $k_s[c]$ has a representative satisfying the matrix relations on the remaining summands after deleting every line and every two-dimensional block W with $\det W(s) \in \mathcal{D}$. There is no dependence on the index of an integral direct sum inside the supplied lattice.

Proof. On the group ring write ϵ for augmentation, extend α linearly, and put $d_c(\sum a_g[g]) = \sum a_g c(g)$. The cocycle equation gives

$$d_c(XY) = \alpha(X) d_c(Y) + d_c(X) \epsilon(Y).$$

Let $S = [s]$, let $B_s(X) = SXS$, and put

$$F_s(X) = \left(\prod_{d \in \mathcal{D}} (B_s - d \,\mathrm{id}) \right) (SX - XS).$$

For a two by two matrix A and any matrix X_0 of the same size, write $C = AX_0 - X_0 A$. Cayley–Hamilton gives $AC + CA = A^2 X_0 - X_0 A^2 = (\operatorname{tr} A) C$. Multiplying by A on the left and substituting its characteristic equation shows $ACA = (\det A) C$. Thus F_s is zero on the specified two-dimensional blocks, as well as on lines. On each other block it is a linear operator on the evaluated matrix.

The commutator has zero augmentation and zero α-value. These equalities are preserved by B_s and on this subspace $d_c(B_s X) = \alpha(s) d_c(X)$. For $X = [g] - 1$ we obtain

$$d_c(F_s([g] - 1)) = k_s c(g) - (\alpha(g) - 1) c(s) \prod_{d \in \mathcal{D}} (\alpha(s) - d).$$

The last term is a coboundary as a function of g. A finite integral relation on the remaining matrices maps under F_s to zero on every original block. It is therefore zero on the original matrices in their original lattices. Its augmentation is zero, so the defining relation for c says its d_c value is zero. This proves the lemma. Repeating the construction on the remaining family multiplies the scalars k_s. □

Proposition 50.21 (The two weight patterns left by the unit tests)**.** *Use the marked torus families of Theorem 50.11 at a spherical coefficient place w away from the original levels. Write $K = E_{x,w}$, $n_x = [E_x : \mathbb{Q}]$, $B = t' - 1$, and $J_\tau = z - 1 + \epsilon_\tau k_\tau$ for the main jump at an embedding τ of K. Take α to be a positive scalar target κ^{-h}, $1 \leq h \leq B$, or the character of the twisted main line. For any chosen rational direct sum decompositions of the local matrices there is a scalar $k_{\alpha,w} \neq 0$, independent of precision, such that every scaled local trace class has a representative satisfying the relations on the following family:*

1. *keep the blocks of dimension at least three;*
2. *for a scalar target keep only those two-dimensional blocks with both jumps in* $[0, B]$ *at every embedding;*
3. *for the main target keep only those two-dimensional blocks whose jumps at every embedding are* $\{0, J_\tau\}$.

One may retain additional summands in all three statements. With $v_\ell(\ell) = 1$*, the cost satisfies at each node*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } (x,\ell)} \sum_{\alpha,w} [E_{x,w} : \mathbb{Q}_\ell]\, v_\ell(k_{\alpha,w}) \ll_{q_0,d} x^{63} b_x^{1/4}.$$

The sum inside the supremum runs over the specified nontrivial targets and spherical places. The bound is independent of the number of trace systems and the choice of decompositions.

Proof. Crystallinity at the spherical places and the common labelled weights were verified in Lemma 50.12. Any rational direct summand W is crystalline. If its dimension is two, filtered exactness gives its determinant weight as the sum of its two jumps from the ambient multiset, with multiplicities. On local units $\det W/\alpha$ has the algebraic character with exponents

$$d_\tau = (\text{sum of the two jumps of } W \text{ at } \tau) - (\text{jump of } \alpha \text{ at } \tau)$$

up to the common reciprocity sign. This uses the rank-one description on units in Lemma 50.12 applied to the determinant. The same description holds for α.

The rational-unit direction measures $\sum_\tau d_\tau$. At a place where the quadratic CM field is nonsplit, the norm-one units of that quadratic completion measure $\sum_\tau \epsilon_\tau d_\tau$, up to sign. For a positive scalar target, if W contains the main jump at any label, each such occurrence contributes one k_τ with the same sign to the oriented sum. At a split prime one can instead use the rational sum, since the orientations there agree. The other parts together have absolute value at most $4n_x t_0$. The inequality $\min k_\tau \geq 3r_x^8 - 4r_x > 4n_x t_0$ holds throughout the interval after increasing its initial node. Thus a block containing any main jump is detected.

For the main target, wherever W contains J_τ that jump cancels. At every other label exactly one far term occurs with the opposite sign. The same oriented or rational sum detects the block if there is any such label. If W contains J_τ everywhere, the difference consists of its other scalar jumps, all nonnegative. The rational sum detects it unless all those jumps are zero. The target α itself is nontrivial on at least one available direction: use $h > 0$ for scalar targets and the same far-jump argument for the main line. This proves the required distinctions between characters.

We give the group elements and the cost explicitly. Choose one unit in each available direction. In the rational direction take a Teichmüller generator times $1 + \ell^{c_\ell}$, where $c_2 = 2$ and $c_\ell = 1$ at odd primes. For an inert quadratic field take a norm-one Teichmüller generator times the norm-one principal unit of depth c_ℓ from the proof of Theorem 15.5. At the fixed ramified quadratic prime use its norm-one unit of depth $3/2$ there. For any nonzero integer j the cost of a direction unit v is $v_\ell(v^j - 1)$. It is zero for a nontrivial residue value. Otherwise it is the indicated depth plus $v_\ell(j)$ by the binomial formula. The rational and inert positive costs require $\ell - 1 \mid j$ and $\ell + 1 \mid j$, respectively. The ramified contribution is at one fixed prime. Thus, for fixed j, the sum of all these costs with weights $\log \ell$ is $O_d((1 + |j|)^{1/4})$. Indeed the residue conditions involve at most $2\tau(|j|)$ primes, each at most $|j| + 1$, while $\sum_\ell v_\ell(j) \log \ell = \log |j|$. Use the divisor bound as in Lemma 50.12.

At the node form a list $\mathcal{J}_x$ of signed nonzero integers. For each of the at most m_x possible chain indices take the two base weights w_1, w_2 used in Lemma 50.12. Include all nonzero

$$i_1 w_1 + i_2 w_2 + D_0, \qquad |i_1|, |i_2| \leq n_x, \qquad |D_0| \leq 6 n_x t_0.$$

This includes all signed exponents just used, for both $\det W/\alpha$ and α. At a label the far part is present at most once after the subtraction, with either sign. Its magnitude k_τ is one of w_1, w_2 minus an integer in $[0, 4r_x]$; the centering and the scalar jumps fit in the stated bound. Consequently $\#\mathcal{J}_x \ll (m_x+1)(n_x+1)^2(1+n_x t_0)$, and $\max|j| \ll n_x b_x$.

Choose as v_1 a direction on which α has nonzero exponent, and lift its reciprocity value to $s \in I_K$. Use Lemma 50.20 with $\mathcal{D} = \{\alpha(s)v_1^j : j \in \mathcal{J}_x\}$. All blocks detected in this direction are removed. If there is a second direction v_2, the determinant ratios left to detect are trivial on v_1. Use either v_1v_2 or $v_1^2v_2$, lifting its reciprocity value to s'. One can choose it so that

$$v_\ell(\alpha(s')-1) \le v_\ell(\alpha(s)-1).$$

In fact the quotient of the two candidate α-values is $\alpha(s)$, so the ultrametric inequality excludes a larger valuation for both. The remaining nontrivial ratios on this unit are unchanged from v_2. Apply the lemma with $\{\alpha(s')v_2^j : j \in \mathcal{J}_x\}$. These sets of units avoid the excluded value in both uses because $j \neq 0$ and the direction units have nonzero principal logarithm. Any further summands kept in the statement impose fewer zero relations on a cocycle.

The valuations of the factors $\alpha(s)-d$ are precisely the direction costs for the listed integers, and the additional two commutator factors cost at most twice the first direction cost for the exponent of α. The full list works before taking any suprema. Summing the local degrees gives n_x for each of at most t_0+1 targets. The resulting prime sum at the node is

$$\ll_d n_x(t_0+1)(m_x+1)(n_x+1)^2(1+n_x t_0)(1+n_x b_x)^{1/4}.$$

Use $n_x \ll_{q_0,d} x^{12}$, $m_x \le 2\sqrt{x}$ and $t_0 \ll x^5$ to obtain the stated bound. □

Corollary 50.22 (The residual joint image on the period interval)**.** *In Theorem 50.11 there are numbers $0 \le r_2(D) \le r(D)$ such that*

$$\sum_\ell (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D)-r_2(D))_+}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

At both original coefficient primes $r_2(D)=0$. At every spherical place the nontrivial targets in $r_2(D)$ can be evaluated on cocycles satisfying the reduced matrix relations of Proposition 50.21. At an unramified place with $\ell > B+1$, $\ell > 2$, the positive scalar coordinates of the reduced image vanish whenever all local blocks have dimension at most two. At any spherical place the main coordinate vanishes if all local blocks have dimension at most two and no two-dimensional block has jumps $\{0, J_\tau\}$ at every label.

Proof. In the joint free cup map defining $r(D)$, multiply each nontrivial target coordinate at a spherical place by the corresponding $k_{\alpha,w}$. Keep the weight-zero map there and any other untreated coordinates as given. Take $r_2(D)$ to be the sum of the normalized lengths of the images of the original global trace classes under these joint maps. Corollary 15.29 covers the two original primes. Adding a coboundary preserves each local cup evaluation, so the new values can indeed be computed with the reduced relations. The kernel of multiplication by $k_{\alpha,w}$ in each coordinate of $\mathcal{O}/\pi^a$ has normalized length at most $v_\ell(k_{\alpha,w})$. The local free ranks are at most $[E_{x,w} : \mathbb{Q}_\ell]$ for each target by Corollary 15.6. Intersecting this direct sum of kernels with the joint global image cannot increase its length. The proposition bounds the weighted loss at a node by $O_{q_0,d}(x^{63}b_x^{1/4})$; dividing by $\sqrt{b_x}$ and summing over $x \ge V_0$ gives $O_{q_0,d}(V_0^{-61})$ because $b_x \ge x^{500}$. Adding the inequality of Theorem 50.11 proves the claim.

For the stated scalar coordinates the remaining family, its target and the quotient all have jumps in $[0, B]$. At the specified unramified places $B \le \ell-2$, so Theorem 13.15 puts the cocycles

in the finite condition. For the stated main coordinate no summand remains; the reduced relations force the representative to vanish on every group element. Both assertions apply to the actual trace precision. The weight-zero target and the unrestricted higher-dimensional blocks at the spherical primes are still present in the general interval estimate. □

50.9 Residual constituents and local trace relations

The local trace systems can be nonsplit in characteristic zero. We identify which part of each rational filtration can affect the class of a scalar cocycle. This identification is exact over the original coefficient ring. For the trivial target it also applies to the cocycle itself, including its restriction to inertia.

Theorem 50.23 (An interval in a filtration that preserves trace classes)**.** *Let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$, with uniformizer π and residue field k. Let u_y be a finite family of continuous representations of a profinite group G on free $\mathcal{O}$-lattices, and let $\alpha : G \to \mathcal{O}^\times$ be continuous. In each rational representation choose a G-stable filtration*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V.$$

Give each factor V_j/V_{j-1} the lattice induced from the original one. Let i be the first index whose factor contains $\bar{\alpha}$ as a composition factor after reduction modulo π; let j be the last index whose reduced factor contains the trivial k-character. Replace V by V_j/V_{i-1}. Use the zero space if either index is absent or if $j < i$.

At every precision π^a, the image in $H^1(G, (\mathcal{O}/\pi^a)(\alpha))$ of the cocycles satisfying the matrix relations is identical for the original and the replacement families. If $\bar{\alpha} = 1$, the modules of raw cocycles satisfying the two families of relations are themselves identical.

Proof. Use the $\mathcal{O}$-algebra A which is the image of $\mathcal{O}[G]$ under the product of the u_y, the character α, and augmentation ϵ. It is a finite free $\mathcal{O}$-module, being a submodule of the finite free product of matrix and scalar rings. In particular it is complete. The two scalar factors give characters of A.

For each distinct character λ in $\{\bar{\alpha}, \bar{\epsilon}\}$, choose the idempotent of the semisimple quotient of $A/\pi A$ that acts as the identity on that simple module and as zero on the others. Such an idempotent exists by the structure theorem for semisimple algebras. The simple component for a k-valued character is k itself since the scalar subalgebra already maps onto k. Lift the idempotents to orthogonal idempotents P_λ of A. For details, the radical of the finite-dimensional residue algebra is nilpotent. Starting with any preimage x, the substitution $x \mapsto 3x^2 - 2x^3$ preserves its residue in the semisimple quotient and squares the ideal of its idempotency error, since the new error is a multiple of $(x^2 - x)^2$. This first lifts through the nilpotent radical, then through all powers of π by completeness; successive changes are multiples of the old error. Lift a second idempotent in the corner complementary to the first. When the two characters agree use the same lift for both. We denote them by P_α, P_1. Their scalar values $\alpha(P_\alpha) = \epsilon(P_1) = 1$ follow from their residues, since the only idempotents in $\mathcal{O}$ are zero and one.

For a lattice factor L, projection by P_λ on $L/\pi L$ has rank the number of its λ-composition factors. Indeed projection by this fixed idempotent is exact on underlying vector spaces in any exact sequence of modules: one projects a lift of a vector to prove surjectivity, and intersects with the kernel to prove exactness in the middle. On simple modules its ranks are one or zero as prescribed. The image $P_\lambda L$ is a direct summand as an $\mathcal{O}$-module, so its rank before reduction is the same. This argument also works over k when other simple modules are not absolutely irreducible.

Consequently $P_1 V \subset V_j$, and P_α kills V_{i-1}. For instance the projection on V/V_j has rank zero by applying the rank calculation to its filtration. When either character is missing or $j < i$, the operator $P_\alpha X P_1$ is zero on V for every $X \in A$. Otherwise it vanishes on V whenever X vanishes on V_j/V_{i-1}, since then $X(P_1 V) \subset V_{i-1}$.

For a raw trace cocycle c use the linear map d_c of Lemma 50.20. It factors through A: an element of the group-ring kernel has augmentation zero and acts as zero on all the given

matrices, so its value under d_c vanishes by the original relations. In A, for $X = [g] - 1$, the product rule gives

$$d_c(P_\alpha X P_1) = c(g) + (\alpha(g) - 1)d_c(P_1).$$

This differs from $c(g)$ by a coboundary. If a linear combination of the X's vanishes on all replacement modules, its projection between the two idempotents vanishes on every original matrix. The projected element has augmentation zero; lifting it to the group ring gives a relation evaluated as zero by d_c. Hence the displayed representative satisfies the replacement relations. In the reverse direction a relation on all original matrices always holds on all their rational subquotients, so every replacement trace cocycle already satisfies the original relations. All identities concern zero operators, and no lattice projection has been divided by an index.

If $\bar{\alpha} = 1$, then $\alpha(P_1) = \epsilon(P_1) = 1$. Evaluating $P_1^2 = P_1$ gives $d_c(P_1) = 2d_c(P_1)$, so the displayed representative is c itself. No division by two occurs, including when $\ell = 2$. □

Corollary 50.24 (The weight-zero test in an arbitrary rational filtration)**.** *At a spherical coefficient place in the marked torus family, suppose every local trace representation has a rational filtration for which the part from the first to the last occurrence of the trivial residual character is potentially abelian; an empty part is permitted. Then multiplication by the augmentation δ_w of Proposition 50.18 makes every trivial-target trace cocycle unramified at w, at every precision. In particular this applies if the trivial residual character occurs only in a single rank-one rational filtration factor of each representation, or is absent there.*

Proof. The theorem for $\alpha = 1$ gives the original cocycle on the reduced family at this place. Those rational subquotients are crystalline and each of their labelled weights occurs in the original trace system, by filtered exactness. Their ranks are at most $2z$. The same H_w as in Proposition 50.18 therefore kills their nontrivial inertia parts. To see the uniformity, its product includes each unramified orbit length up to $2z$ and every nonzero exponent formed from allowed jumps at the embeddings of that extension. Passing to a rational subquotient only restricts those possibilities. Thus for $v \in I_{E_{x,w}}$ the element $([v] - 1)H_w$ acts as zero on the replacement matrices and has augmentation zero. Evaluating it by the additive cocycle gives $\delta_w c(v) = 0$. □

Theorem 50.25 (Omitting places detected by the residual constituents)**.** *Use the same fields, tests and interval as Theorem 50.11. For each target M and each spherical coefficient place w, apply Theorem 50.23 to the restricted family with α the character of M. Omit the local free cup coordinates for that target at w whenever one of the following conditions holds for the resulting family:*

1. *all its representations are potentially abelian;*
2. *$M = \mathcal{O}(-h)$, $1 \le h \le B = t' - 1$, the place is unramified over $\mathbb{Q}_\ell$ with $\ell > \max(2, B+1)$, and the representations are rational direct sums of blocks of dimension at most two;*
3. *M is the main line, the representations are rational direct sums of blocks of dimension at most two, and none of their two-dimensional blocks has jumps $\{0, J_\tau\}$ at all the labels of w.*

The filtrations and direct sums may depend on the target. Omit both original coefficient primes by Corollary 15.29. Define $r_{\mathrm{fil}}(D)$ by projecting the original joint images onto the coordinates retained and using the same normalized lengths. Then

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_{\mathrm{fil}}(D))_+}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

In particular the estimate holds for the full projected pairing on the data where these omissions cover every coefficient place and target.

Proof. Local finite conditions and cup maps depend only on the cohomology class. Hence for any global trace class the local representatives supplied by Theorem 50.23 can be used at the individual places, without changing its local values. The second and third cases of the statement now follow pointwise from Proposition 50.21 and the last paragraph of Corollary 50.22. Indeed every labelled jump of a replacement block occurs in the ambient Hodge multiset with the same multiplicity bound; this is all that the determinant test requires. Its local annihilating scalar has the same valuation bound, using the same exponent list and the same degrees.

For the first case at weight zero use Corollary 50.24. At a nontrivial target choose $s \in I_K$, $K = E_{x,w}$, with $\lambda = \alpha(s) - 1 \neq 0$ as in Theorem 50.13. The inertia matrices of every potentially abelian replacement commute, by Lemma 50.17. Evaluating the group commutator of s and $v \in I_K$ therefore gives $\lambda c(v) = (\alpha(v) - 1)c(s)$ for the replacement representative. Subtract this coboundary after multiplying by λ. The class then restricts trivially to inertia. Inflation and restriction place it in the image of $H^1(G_K/I_K, (M/\pi^a)^{I_K})$. Its coefficients are killed by λ since $s \in I_K$. Thus λ^2 kills the original local class. The exponents and unit bounds for these choices concern the target alone.

Each omitted coordinate is consequently killed by the corresponding scalar. Its normalized length is at most the valuation of that scalar. Projection of a joint image has kernel contained in the direct sum of the omitted coordinates; this accounts for possible dependencies between places. The nodewise prime sum for the first case is at most $O_{q_0,d}(x^{80} b_x^{1/4})$ by the unit estimates in Theorem 50.19; the other two cases cost $O_{q_0,d}(x^{63} b_x^{1/4})$ by Proposition 50.21. Both bounds are uniform in the filtrations even before taking the supremum. Now $\sum_{x \geq V_0} x^{80} b_x^{-1/4} \ll V_0^{-44}$. Apply the prime-sum inequality of Theorem 50.11 on this same interval. □

The theorem uses a filtration of the actual local matrices. When the two residual characters occur together in a large irreducible rational constituent, that constituent survives the reduction. The full period inequality still requires control of the surviving joint image for unrestricted data.

50.10 Ordered crystalline flags and the integral projected period

At some coefficient places the crystalline matrices admit an invariant flag whose characters have increasing Hodge jumps. An extension in such a flag need not split, even over the fraction field. We bound its contribution to the integral trace equations using the positions of the trivial character and of the target in the flag.

Lemma 50.26 (Two operators on a filtered trace representation)**.** *Let the notation for integral matrix relations and for a character α be as in Lemma 50.20. Suppose the rational representations have G-stable full flags. Write β_i for the character on V_i/V_{i-1}, with indices increasing from the bottom of a flag, and let R bound the ranks. Take $A, B \in \mathcal{O}[G]$ with $\alpha(A)\epsilon(B) \neq 0$, where ϵ denotes augmentation. In each flag let i be the first index with $\beta_i(A) \neq 0$ and j the last index with $\beta_j(B) \neq 0$. Use the quotient V_j/V_{i-1} if both indices exist and $i \leq j$. In all other cases use zero. If c satisfies the original matrix relations modulo π^a, then*

$$k[c], \qquad k = \alpha(A)^R \epsilon(B)^R,$$

has a representative satisfying the relations on the resulting quotients. For $\alpha = 1$ the representative is kc.

Proof. On a string of successive factors on which B is zero, B lowers the filtration by one step. Applied to V/V_j, this gives $B^R V \subset V_j$. In the same way $A^R V_{i-1} = 0$. Absence of an index makes the corresponding power zero on the entire representation. For X in the augmentation ideal, multiplication by A^R on the left and B^R on the right therefore gives the zero matrix on every original representation whenever X acts as zero on the indicated quotients. This also covers $j < i$ because every group ring element preserves V_j.

Let $d_c(\sum a_g[g]) = \sum a_g c(g)$. Applying the cocycle product rule of Lemma 50.20 twice gives

$$d_c(A^R([g]-1)B^R) = kc(g) + (\alpha(g)-1)\alpha(A)^R d_c(B^R).$$

The last summand is a coboundary. For a linear combination of the $[g]-1$ giving a relation on the quotients, the product in parentheses has augmentation zero and acts as zero on the original lattices, as we just proved on their rational spaces. Its value under d_c hence vanishes. This proves the assertion at the original precision. □

Proposition 50.27 (The two ends of a crystalline flag)**.** *Use the fields and weights of Proposition 50.21 at a spherical coefficient place w. Write $K = E_{x,w}$. We allow each local representation to be a rational subquotient of one of the trace systems. For a full G_K-stable flag over a finite coefficient extension call it ordered if, at every embedding τ of K, its line characters have increasing jumps in flag order. For the trivial target assume only that full stable flags exist. Then there is a nonzero scalar $k_{1,w}$ for which every trivial-target trace cocycle satisfies $k_{1,w}c(I_K) = 0$.*

For a nontrivial target assume the flags are ordered. For each positive scalar target and for the main target when at least one $J_\tau > 0$, some nonzero $k_{\alpha,w}$ kills every local trace class in $H^1(K, (\mathcal{O}/\pi^a)(\alpha))$. These assertions hold at every precision. The scalars can be chosen with

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } (x,\ell)} \sum_{\alpha,w} [E_{x,w} : \mathbb{Q}_\ell] v_\ell(k_{\alpha,w}) \ll_{q_0,d} x^{68} b_x^{1/4}.$$

For this estimate a place or target may be omitted. Ordered flags may be assumed to exist after a finite extension of K.

Proof. We first justify the last assertion. Pass to a finite normal extension over which a proposed ordered flag is stable. On restriction the labelled jumps above an embedding of K are repetitions of its original multiset. At each individual embedding they are distinct. This regularity follows from the list $0, \dots, B, J_\tau$, where $B = t' - 1$ and $J_\tau \notin [0, B]$. An ordered flag on this restriction is unique. Indeed two distinct initial lines would give a direct sum subrepresentation with two copies of the least jump at any fixed embedding. Exactness of the de Rham filtration on crystalline representations excludes this. Apply the same argument after quotienting by the common first line. Conjugation by G_K permutes the embeddings above each base embedding and takes an ordered flag to an ordered flag. Uniqueness therefore makes each subspace G_K-stable. The characters of this flag have the claimed order over K by base change.

Use now the signed nonzero exponent list $\mathcal{J}_x$ and the units of Proposition 50.21. Choose a rational unit v there and, at a nonsplit prime of the quadratic CM field, also its specified norm-one unit. For each direction choose $s_v \in I_K$ lifting its local reciprocity image. For $\gamma = 1, \alpha$ form in the integral group algebra

$$H_\gamma = \prod_{v} \prod_{j \in \mathcal{J}_x} \left([s_v] - \gamma(s_v) v^j [1] \right).$$

Enlarge the coefficient field to contain the displayed units and all line characters and use any fixed order of the factors. Every $\gamma(H_\gamma)$ is nonzero. Indeed every v has a principal part of infinite order, and the exponents are nonzero. On a line of character β, a factor vanishes when $\beta(s_v)/\gamma(s_v) = v^j$.

For completeness, the exponents that this construction detects can be read directly from the jumps. Write b_τ and a_τ for the jumps of β and γ. The characters are crystalline and their unit characters are the monomials of Lemma 50.12. Up to the reciprocity sign, the rational-unit exponent is $\sum_\tau (b_\tau - a_\tau)$. The quadratic norm-one exponent is $\sum_\tau \epsilon_\tau (b_\tau - a_\tau)$, with the sign chosen by either embedding of the quadratic completion. Each nonzero exponent is in $\mathcal{J}_x$. At each label a far term, when present after subtraction, is one signed k_τ; it is one of the two base weights minus a shift of at most $4r_x$. The remaining scalar and central terms are covered by the $6n_x t_0$ allowance in the definition of the list.

For $\gamma = 1$, any far choice is detected. In the nonsplit case every such choice contributes $+k_\tau$ to the oriented sum and the short terms total at most $2n_x t_0$ in absolute value. In the split case use the rational sum since the signs at w agree. Here $\min k_\tau > 4n_x t_0$ on the specified interval. With only short jumps a nonzero jump gives positive rational sum. Thus $\beta(H_1) \neq 0$ precisely on the lines whose jumps are all zero. Such a line is unramified by its unit monomial. In any one flag there is at most one such line, since the jump zero occurs at most once at a fixed embedding. With $A = B' = H_1$ in Lemma 50.26 (where B' denotes its right operator), the retained quotient is either zero or that unramified line. The scaled additive cocycle thus vanishes on inertia.

For $\alpha = \kappa^{-h}$, $1 \le h \le B$, the same far-term test works after subtraction of the short jump h. Any line with $\beta(H_\alpha) \neq 0$ consequently has only short jumps, with sum $h[K : \mathbb{Q}_\ell] > 0$. At least one of its jumps is positive. In an ordered flag it must follow the line with all jumps zero if that line exists. For the main target a label at which $b_\tau \neq J_\tau$ contributes the opposite far term in the oriented sum at a nonsplit prime, or in the common-sign rational sum at a split prime. The same size inequality detects any such label. A surviving line has $b_\tau = J_\tau$ at every label, and again follows the zero line if some $J_\tau > 0$. Take $A = H_\alpha$, $B' = H_1$. In both cases every quotient of the lemma is zero. A representative satisfying the relations on the zero family vanishes on each g, giving the asserted annihilation of the class.

Take $R = t_0 + 1$, which bounds every rank. The valuation of $\gamma(H_\gamma)$ is the sum of $v_\ell(v^j - 1)$ for the displayed factors. This is independent of the matrices and of γ. The cost of $k_{\alpha,w}$ is at most $2R$ times that one-product cost. Proposition 50.21 bounds the weighted sum of the latter over primes, local degrees and targets by $O_{q_0,d}(x^{63} b_x^{1/4})$, using the same list before all suprema. Since $R \ll x^5$, this proves the bound. Extending coefficients preserves normalized lengths: each original cyclic quotient retains the valuation of its annihilating ideal when $v_\ell(\ell) = 1$. □

Theorem 50.28 (Projected periods for families admitting ordered flags)**.** *Use the same marked torus data, twists and interval $x \geq V_0$ as in Theorem 50.11. If each trace system at every spherical coefficient place admits a potentially ordered flag as above, on this subfamily one has*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{p(D)}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

More generally start with $r_{\mathrm{fil}}(D)$ of Theorem 50.25. In each target the rational subquotients supplied by that theorem may be used. Omit also the weight-zero coordinates where all these representations admit full G_K-stable flags, and all nontrivial-target coordinates where all admit potentially ordered flags. Write $r_{\mathrm{ord}}(D)$ for the normalized lengths of the resulting joint images. On all the data the displayed estimate holds with $p(D)$ replaced by $(p(D) - r_{\mathrm{ord}}(D))_+$.

Proof. At an omitted weight-zero place a multiple of the trace is unramified by Proposition 50.27. This multiple lies in the finite local condition. Indeed unramified classes for the trivial torsion line are values at Frobenius and lift to integral unramified classes. At each of the other places in the proposition its multiple is zero in cohomology. If every J_τ at a place is negative, then $J_\tau < -1$ at all its embeddings. The dual main line twisted by one has strictly positive jumps there and no invariants. The crystalline dimension formula makes its finite rational cohomology zero, so the main free cup coordinate has rank zero by Corollary 15.6. Both original coefficient primes have already been treated by Corollary 15.29.

The exact equality of local classes in Theorem 50.23 permits these tests after passing to its subquotients at each place separately. Every new coordinate omitted is killed by the corresponding scalar. Its normalized length is at most the scalar valuation, and at each place there are at most $[K : \mathbb{Q}_\ell]$ coordinates per target. The kernel of projection of the joint global image is contained in the direct sum of the omitted coordinate images. Hence the additional loss has weighted prime sum $O_{q_0,d}(x^{68} b_x^{1/4})$ per node, uniformly in the flags. After division by $\sqrt{b_x}$ the sum over $x \geq V_0$ is $O_{q_0,d}(V_0^{-56})$ since $b_x \geq x^{500}$. Adding the bound of Theorem 50.25 gives the general assertion. For the subfamily in the first assertion, one may use the original representations directly, and every free coordinate is accounted for. □

The period estimate for unrestricted data still requires the coordinates for which these local hypotheses have not been proved.

50.11 The unramified constituent in the local period tests

We examine the absolute irreducible factors of the spherical trace representations. Only factors whose reduction contains the trivial character enter the following estimate. At any one place a given representation has at most one unramified factor. Its position controls the free local cup coordinates, even when the extensions joining the other factors are nonsplit.

Lemma 50.29 (An integral operator supported on one filtration factor)**.** *Let $K = E_{x,w}$ be a spherical coefficient completion in Theorem 50.11. Allow rational subquotients of the local trace systems. Enlarge coefficients to a finite extension over which the rational composition factors are absolutely irreducible. Assume that each of these factors whose reduction contains the trivial residual character is potentially abelian. Then for the given finite family there is $Q \in \mathcal{O}[G_K]$ of augmentation δ_w from Proposition 50.18 with the following property. In a rational G_K-composition series for each representation, Q acts as zero on every factor except possibly a single unramified character γ with trivial reduction. The same Q works for every choice of composition series.*

Proof. In the product algebra of the matrices and the augmentation choose the idempotent for the trivial residual character as in Theorem 50.23. Lift it to an element P_1 of $\mathcal{O}[G_K]$. Its augmentation is one. It acts as zero on every rational factor W whose reduction has no trivial composition factor. Indeed its rank on an induced lattice in W equals the dimension of its image on the reduction. The latter is the multiplicity of the trivial character, by exactness of the idempotent on vector spaces. This computation is independent of an induced lattice: the idempotent rank equals its rank on W itself.

Let H_w be the element constructed in Proposition 50.18 and set $Q = H_w P_1$. For each remaining W that proposition puts the image of H_w inside W^{I_K}. Its construction applies to W: the crystalline factors have rank at most $R = t_0 + 1$ and every jump belongs to the original weight multiset. These are precisely the restrictions used to choose the units for each inertia orbit in that proposition. Since I_K is normal, W^{I_K} is a G_K-submodule. Absolute irreducibility makes it either zero or all of W. In the second case W is unramified and is one-dimensional. Indeed an eigenline of Frobenius is then preserved by its powers, and by continuity by the entire procyclic quotient.

An unramified line contributes jump zero at every labelled embedding of K. The original weights are all distinct at each label. Exactness of the de Rham filtration in the crystalline category therefore allows at most one unramified factor in any one representation, counted with multiplicity. The preceding projection removes this factor too when its reduction is nontrivial. Finally $\epsilon(Q) = \epsilon(H_w)\epsilon(P_1) = \delta_w$. □

Proposition 50.30 (Inertia at weight zero and a criterion for every target)**.** *Under the assumptions of Lemma 50.29, at every precision and at every spherical w the weight-zero trace cocycles satisfy*

$$\delta_w^{2R} c(I_K) = 0.$$

Suppose in addition that for each representation the possible factor γ in that lemma is absent or occurs as a G_K-stable subline. For every target α, the restriction of $\delta_w^R[c]$ to I_K is then zero. For a nontrivial target this implies $\lambda_{\alpha,w}\delta_w^R[c] = 0$ on G_K, where $\lambda_{\alpha,w} = \alpha(s) - 1 \neq 0$ for the inertia unit test s of Proposition 50.21.

The additional subline condition holds whenever, in some composition series of each member having a factor γ, all jumps of the part below γ are nonnegative at all embeddings. In particular it holds if all jumps of the ambient representation are nonnegative at the place.

Proof. Write $V_j/V_{j-1} = \gamma$ when the possible factor occurs. Because Q is zero on the factors of V/V_j and of V_{j-1}, its R-th power sends V into V_j and kills V_{j-1}. With the factor absent it kills all of V. For $v \in I_K$, $[v] - 1$ sends V_j into V_{j-1}, so $Q^R([v] - 1)Q^R$ acts as zero on every original lattice. Its augmentation is zero. The additive trace cocycle evaluates this relation as $\delta_w^{2R} c(v) = 0$, by the group ring product rule of Lemma 50.20.

If the factor is a subline, use a composition series beginning with this subline. It remains the only possible exceptional factor. Now $([v]-1)Q^R$ is zero on every representation, including those with no exceptional factor. Applying the product rule with target α gives

$$\delta_w^R c(v) = -(\alpha(v) - 1) d_c(Q^R).$$

This is a single coboundary as v varies through inertia. Inflation and restriction then put $\delta_w^R[c]$ in the image of $H^1(G_K/I_K, ((\mathcal{O}/\pi^a)(\alpha))^{I_K})$. For a nontrivial target α, the indicated element s acts on the coefficients as $\alpha(s)$. Its difference from one kills the invariants, hence also the inflated class.

To prove the subline assertion, let $U = V_{j-1}$. The short exact sequence $0 \to U \to V_j \to \gamma \to 0$ defines a finite class in $H^1_f(K, U \otimes \gamma^{-1})$, since its middle representation is crystalline. Twisting by the unramified character preserves all labelled jumps. Under the stated condition there are no negative jumps, so the de Rham tangent space of $U \otimes \gamma^{-1}$ is zero in our jump convention. Its invariants are zero as well: an invariant would give a copy of γ in U, repeating the jump zero from V_j/U. By the Bloch–Kato dimension formula these two dimensions add to $\dim H^1_f(K, U \otimes \gamma^{-1})$. The extension thus splits over the coefficient field, and the lift of γ in V_j is the required subline of V. All group ring identities above were first proved as identities of rational operators and hence hold exactly in the supplied lattices. □

Theorem 50.31 (The period estimate from the unramified constituent)**.** *Use the fields, twists and full interval of Theorem 50.11. For a given target M at a spherical place w, use the original local representations or any of the residual filtration replacements of Theorem 50.23. Omit the weight-zero free cup coordinates when Lemma 50.29 applies. Omit those of any other target when the subline assertion of Proposition 50.30 applies. These omissions may be combined with those of Theorem 50.28. Let $r_{\mathrm{unr}}(D)$ be the normalized length on the coordinates retained, always using the images of the original joint maps. Then*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_{\mathrm{unr}}(D))_+}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

In particular, at places where the matrices become triangular over a finite field extension, the weight-zero tests are covered in arbitrary flag order. If $J_\tau > 0$ at every embedding of such a place, the estimate covers every target there.

Proof. First consider a potentially triangular representation. Choose a finite normal extension K'/K making it triangular after enlarging coefficients. Any absolutely irreducible rational G_K-factor W then has a $G_{K'}$-stable line, by the filtration induced on the factor. Its translates under G_K span W and are still $G_{K'}$-stable by normality. A basis can be selected among these lines, in which every $G_{K'}$-matrix on W is diagonal. Thus all such factors are potentially abelian. When every J_τ is positive, the remaining jumps are the nonnegative short jumps, giving the subline condition by the proposition.

For weight zero the scaled class is unramified and is finite at the torsion precision: its Frobenius value lifts to an integral unramified class. For other targets the scaled class vanishes locally. The residual filtration replacements preserve the class itself in local cohomology, so the same multipliers annihilate the original free cup outputs. At each place there are at most

$[E_{x,w} : \mathbb{Q}_\ell]$ coordinates per target. The kernel of the projection on joint images lies inside the sum of the omitted coordinate images, each of length bounded after normalization by the valuation of its annihilator.

Proposition 50.18 bounds the weighted sum, at a fixed node, of the local degrees times $v_\ell(\delta_w)$ by $O_{q_0,d}(x^{80}b_x^{1/4})$, uniformly before the data are selected at each prime. There are at most $R = t_0 + 1$ targets and each new annihilator uses at most $2R$ powers of δ_w. The $\lambda_{\alpha,w}$ add at most $O_{q_0,d}(x^{63}b_x^{1/4})$ by the direction-unit and exponent-list bound in Proposition 50.21. The total new cost is therefore $O_{q_0,d}(x^{90}b_x^{1/4})$. Dividing by $\sqrt{b_x}$ and summing over the interval gives $O_{q_0,d}(V_0^{-34})$, since $b_x \ge x^{500}$. Combining with Theorem 50.28 proves the claim. If an irreducible factor has nonabelian image over every finite extension and its reduction contains the trivial character, the action of Q on that factor still needs to be controlled for the unrestricted family. □

51 Cup bounds and torus periods at the coefficient primes

51.1 Summing local cup bounds on the period fields

Consider the shorter period families of Theorem 50.11. At a node x, keep its field E_x, degree $n_x = [E_x : \mathbb{Q}]$, and all its test data. The local annihilator in Theorem 16.15 bounds the order of an individual cup. We sum the resulting lengths before passing to the supremum over the interval. This allows a local weight bound that depends on the node.

For a trace family at a spherical place $w \mid \ell$ and a target T, use the three crystalline extensions of Corollary 16.12 over $K = E_{x,w}$. Write $h_X, h_{X'}, h_Z$ for the respective upper bounds on the jumps, including zero. Thus X is the graph extension, X' the extension for the integral dual test, and $Z = (Y \otimes_{\mathbb{Z}_\ell} Y')^\vee(1)$ the kernel of the third extension. The bounds include every jump after restriction of coefficients. If the local matrix relations admit a different representative of each trace class, we may compute these heights on the replacement family. Cup evaluation is unchanged by a coboundary.

Lemma 51.1 (Prime sum at a single node)**.** *Fix an integer $H \geq 1$. At each datum D at x discard any collection of spherical targets for which $\max(h_X, h_{X'}, h_Z) \leq H$. Let $\Delta(D)$ be the drop in the normalized lengths of the joint images. Then*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } (x,\ell)} \Delta(D) \ \ll_d \ n_x(t_0+1)H\,\big((n_x+1)(1+\log(2n_x))^3 + (1+\log(2H))^2\big).$$

The estimate also holds for a joint image after integral scaling of some of its coordinates if the scaled classes themselves admit the indicated representatives.

Proof. First bound the height constants in Theorem 16.11. At any prime $p = \ell$ let $e = e_K \leq n_x$. Choose an integer J with $4e \leq 2^J < 8e$. Then

$$\frac{pe}{(p-1)^2 p^J} \leq \frac{2e}{2^J} \leq \tfrac{1}{2}, \qquad d_K \leq 1 + \log_2 e, \qquad v_p(h!) \leq h$$

for $h \geq 0$. Substituting in that theorem and rounding up shows that, for $0 \leq h \leq H$, both $C_K(h)$ and the constant Q_K of Proposition 16.14 are $O(H(1+\log(2n_x))^2)$. This holds at two and in the presence of wild ramification. There are at most $4(n_x + d + 1)$ primes up to that integer and their logarithms are $O_d(1+\log(2n_x))$. This gives the first term in parentheses.

For $p > 4(n_x + d + 1)$ the field is unramified at p. Indeed $E_x = \mathbb{Q}(\mu_{q_1}, \sqrt{-d})$ as in Proposition 50.5. For a prime $p \mid q_1$, the inclusion $\mathbb{Q}(\mu_p) \subseteq E_x$ implies $p - 1 \leq n_x$. The quadratic factor can ramify only at d. At any of the remaining primes we have $e = 1$, $d_K = 0$. For $h < p - 1$, $C_K(h) = 0$ and $v_p(h!) = 0$. For $p - 1 \leq h \leq H$ the choice $J = 0$ in Theorem 16.11 gives

$$C_K(h) \leq \left\lceil \frac{ph}{(p-1)^2 - p} \right\rceil \ll H/p, \qquad v_p(h!) \leq H/(p-1).$$

The implied constant is absolute since $p \geq 7$ here and $H \geq p - 1$ in the nonzero case. Thus the sum of local exponents at large primes is bounded, per possible free coordinate, by

$$O\left(H \sum_{2 \leq m \leq H+1} \frac{\log m}{m} \right) = O\big(H(1+\log(2H))^2\big).$$

Integral comparison with $\int_1^{H+1} \log(2t)\, dt/t$ gives the last inequality. At both ranges of primes Corollary 16.16 bounds the loss by the sum of the exponents times the local degrees. Their

sum over $w \mid p$ is n_x, and there are at most t_0+1 targets at any place. These majorants depend only on x, H, d and p, so they bound each supremum. For the scaled image the proof uses the same injection of the projection kernel into the discarded coordinates. □

Theorem 51.2 (Omitting targets with bounded upper jumps on the interval)**.** *Throughout the original interval set*

$$H_x = \left\lfloor \frac{\sqrt{b_x}}{x^{45}(1+\log b_x)^2} \right\rfloor.$$

At each spherical place omit all targets for which local representatives of the original classes have $\max(h_X, h_{X'}, h_Z) \le H_x$*, and omit both original level primes. Let* $r_*(D)$ *be the sum of normalized lengths of the joint global images on the retained coordinates, with the same finite tests as before. Then*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_*(D))_+}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

The same integral inequality therefore holds with the full period cost $p(D)$ *wherever the omissions exhaust the targets of every datum under consideration.*

Proof. At the two original level primes use Corollary 15.29. At the spherical places apply Lemma 51.1 at $H = H_x$. This is at least one throughout the interval for large V_0, since $b_x \ge x^{500}$. Use $n_x \ll_{q_0,d} x^{12}$, $t_0 \ll x^5$ and $\log(2H_x) \ll 1 + \log b_x$. After division by $\sqrt{b_x}$ the lemma gives a bound at each node of order

$$\frac{x^{29}(1+\log x)^3}{x^{45}(1+\log b_x)^2} + \frac{x^{17}}{x^{45}} \ \ll_{q_0,d} \ x^{-14}.$$

For nonnegative terms the supremum over the nodes at a fixed prime is at most their sum. Interchanging these nonnegative sums gives a total additional loss $O_{q_0,d}(V_0^{-13})$. Add it to the estimate of Theorem 50.11 and the previously bounded contribution of the level primes. This uses the same choices of twists, node by node, as in that theorem; taking a local representative does not change the global source of a period evaluation. □

Corollary 51.3 (Short-weight scalar tests at every spherical place)**.** *For any scalar target* $T = \mathcal{O}(-d_0)$*,* $0 \le d_0 \le B = t' - 1$*, its free cup coordinates can be omitted at every spherical place at which the residual intervals of Theorem 50.23 have only jumps in* $[0, B]$*. The interval estimate of Theorem 51.2 continues to hold. This includes the trivial target and permits every rank.*

For $1 \le d_0 \le B$ *the same omission is valid when each trace representation is a rational direct sum of blocks such that every block of dimension at least three has only short jumps. The filtrations and decompositions may depend on the place and target.*

Proof. The residual interval theorem supplies representatives of the same classes with the indicated matrix relations. Use these representatives in the graph construction. Its kernel Y then has only jumps j with $0 \le j \le B$. The dual-test kernel Y' has jump $-1 - d_0$, including after restriction of coefficients. The jumps of Z are $-1 - j - (-1 - d_0) = d_0 - j$. It follows that $h_X \le B$, $h_{X'} = 0$, $h_Z \le d_0 \le B$. In fact each such cup is killed by $p^{C_K(B)+C_K(d_0)}$: both first extensions have jumps of one sign, and the upper bounds may be enlarged in the definition of C_K. Since $B \ll x^5$ and $(1+\log b_x)^2 \ll b_x^{1/10}$, we have $H_x \gg b_x^{2/5}/x^{45} \gg x^5$ at the nodes in question. The theorem proves the first assertion.

For a positive scalar target in the second assertion, Proposition 50.21 supplies a nonzero integral multiplier $k_{\alpha,w}$ whose scaled class has a representative supported on the retained blocks.

The proposition removes the lines and every two-dimensional block containing a far jump. All remaining blocks have jumps in $[0, B]$ by hypothesis. For an omitted coordinate with image I, apply the short-weight bound to $k_{\alpha,w}I$. Its kernel in I has normalized length at most $v_\ell(k_{\alpha,w})$ per free coordinate. For a joint image this follows by applying coordinate multiplication first on the omitted coordinates and the identity on all retained coordinates, then projecting. The additional weighted cost at a node is at most $O_{q_0,d}(x^{63}b_x^{1/4})$ by the same proposition. After division by $\sqrt{b_x}$, summation over the original nodes costs $O_{q_0,d}(V_0^{-61})$. Combining the two losses proves the assertion. □

In the unrestricted families a block of rank at least three can contain a main jump J_τ outside the displayed range. The local theorem still bounds its cup order. The prime sum proved here controls only the targets meeting the hypotheses above.

51.2 Large coefficient primes with a multiple near a period weight

We return to the full marked torus data of Theorem 50.11, including trace representations with a far jump. In local lifting arguments one encounters the condition that a coefficient prime divide a nonzero integer close to a large weight. The distance allowed can itself be of order the weight divided by the prime. For sufficiently large primes we can sum all these exceptions on the original interval, with no restriction on the local representations. The dependence of the local cup bound on the prime is needed here.

Lemma 51.4 (Counting prime divisors near finitely many weights)**.** *Fix a node x. Write $b = b_x$, $n = n_x$ and take a finite nonempty set $\mathcal{C}$ of integer centers with $|c| \le M$ for every $c \in \mathcal{C}$. Let $P > 4(n + d + 1)$, $W \ge 0$ and $L \ge 0$. Consider the primes $\ell \ge P$ for which some $c \in \mathcal{C}$ and some nonzero multiple v of ℓ satisfy*

$$|v - c| \le W + Lb/\ell.$$

With $R = \lceil W + Lb/P \rceil$, their total contribution obeys

$$\sum_{\ell \text{ as above}} (\log \ell) \sup_{D \text{ at } (x,\ell)} r(D) \ll n(t_0 + 1)\, \frac{b}{P}\, |\mathcal{C}|(2R + 1) \log(2 + M + R).$$

Proof. The two original level primes have zero contribution by Corollary 15.29. Every other coefficient prime is spherical in the marked families under consideration. For $\ell \ge P$, all the completions $E_{x,w}$ are unramified by the field description in Lemma 51.1. At every embedding, every jump of the graph kernel is either a short jump in $[0, t' - 1]$ or a main jump $J_\tau = z - 1 \pm k_\tau$, where $k_\tau \le 4b$. The target is one of the same characters and its dual has jumps $-1 - j_T$. Consequently the three upper jumps in Corollary 16.16 are at most $12b$ for all sufficiently large x, also after restriction of coefficients. For the third kernel this uses the sum of the two largest absolute jumps and one, so it includes all pairs of coefficient embeddings.

As in the large-prime calculation of Lemma 51.1, at an unramified completion the local exponent is zero if every upper jump is less than $\ell - 1$, and otherwise is $O(b/\ell)$. Indeed $d_K = 0$, $v_\ell(h!) \le h/(\ell - 1)$, and the ceiling in $C_K(h)$ only occurs for $h \ge \ell - 1$. Applying the cup bound to all targets gives $r(D) \ll n(t_0 + 1)b/\ell$. This bound also holds for zero lifts and is uniform in the local systems and all the integral tests.

Each prime under consideration divides the positive integer

$$\prod_{c \in \mathcal{C}} \prod_{\substack{j \in \mathbb{Z},\ |j-c| \le R \\ j \ne 0}} |j|,$$

using 1 for an empty product. Its logarithm is at most $|\mathcal{C}|(2R + 1) \log(2 + M + R)$. The sum of the logarithms of the distinct prime divisors is at most the logarithm of the product. Multiplying by the uniform bound with $\ell \ge P$ proves the lemma. □

Theorem 51.5 (The full period on the large primes near weight combinations)**.** *At each node let $b_1 = b_x$, $b_2 = a_x/x_{1,x}$ be the two integral base weights, let $U_x = n_x(2t_0 + 1)$, and put*

$$\mathcal{C}_x = \{d_1 b_1 + d_2 b_2 : d_1, d_2 \in \mathbb{Z},\ |d_1|, |d_2| \le U_x\},$$
$$P_x = x^{50}(1 + \log(2b_x))\, b_x^{3/4}.$$

Fix $L \ge 1$. Let $\mathcal{E}_x$ be the coefficient primes $\ell \ge P_x$ with a nonzero multiple at distance at most $L(x^{125} + b_x/\ell)$ from some element of $\mathcal{C}_x$. On the original fields, for the original tests and interval,

$$\sum_{\ell} (\log \ell) \sup_{\substack{D \text{ at } \ell \\ \ell \in \mathcal{E}_x}} \frac{p(D)}{\sqrt{b_x}} \ll_{q_0,d,L} V_0^{-10}.$$

This controls all local cup coordinates on the indicated data, including those of the main line and the weight-zero target, for arbitrary ranks of the crystalline trace matrices.

Proof. Use the degree and rank bounds $n_x \ll_{q_0,d} x^{12}$, $t_0 \ll x^5$. Then $U_x \ll x^{17}$, $|\mathcal{C}_x| \ll x^{34}$, and all centers have absolute value at most $2U_x b_x$. By the scale conditions $x^{10^6} \le x_{1,x}$, $x_{1,x}^{10^6} \le b_x$ in Theorem 15.3, $P_x > 4(n_x + d + 1)$ on a sufficiently distant interval. Also $\log(2 + 2U_x b_x + R) \ll_{q_0,d,L} 1 + \log(2b_x)$ for the radius $R = \lceil L(x^{125} + b_x/P_x)\rceil$. Lemma 51.4 bounds the node sum of $r(D)/\sqrt{b_x}$ at the specified primes by

$$\ll_{q_0,d,L} \frac{x^{51}(1+\log(2b_x))}{\sqrt{b_x}} \left((1+x^{125})\frac{b_x}{P_x} + \left(\frac{b_x}{P_x}\right)^2 \right).$$

After substitution the first term is $O_{q_0,d,L}(x^{126} b_x^{-1/4})$, and the second is $O_{q_0,d,L}(x^{-49}/(1 + \log(2b_x)))$. Their sum is $O_{q_0,d,L}(x^{-48})$. The use of a single product per node has included all permitted signs, powers and coefficient embeddings before taking the supremum. Sum over $x \ge V_0$ to bound the prime sum of the suprema of $r(D)/\sqrt{b_x}$ on the specified data by $O_{q_0,d,L}(V_0^{-47})$. Finally $p(D) \le (p(D) - r(D))_+ + r(D)$; the prime sum of the first term is bounded by Theorem 50.11. This proves the claimed estimate with the full cost $p(D)$. □

For example, the original shifts from a base weight are $(2j+1)r_x^8 - h_\tau$, with $j \le 2\sqrt{x}$ and $0 \le h_\tau \le 4r_x$. The centering $z-1$ has size at most t_0. Even after summing $2U_x$ such errors, the result is $O_{q_0,d}(x^{59})$. Thus the theorem includes the large primes with a nonzero multiple within $O(b_x/\ell + r_x^{24})$ of a signed sum of at most $2U_x$ actual jumps of the displayed types. Indeed $r_x^{24} \ll x^{120}$, and the fixed constants can be absorbed in L. Here the numbers of occurrences of each base weight, counted with sign, must give coefficients in $[-U_x, U_x]$. The remaining prime sum includes large primes away from these neighborhoods and primes below P_x.

51.3 The full marked torus period above the square-root cutoff

Write $n_x = [E_x : \mathbb{Q}]$, $t_0 = 5r_x$, and use $p(D)$ and $r(D)$ as in Theorem 50.11. The latter is the normalized length of the *joint* image on the original free local cup coordinates. The difference between the full period cost and the part already estimated there is bounded by $r(D)$. We first bound its prime sum at one node.

Proposition 51.6 (A weighted sum of the joint local image)**.** *Fix q_0, d. At every sufficiently large node x of the shorter marked torus family, for every real $L \geq x_{1,x}$,*

$$\sum_{\ell \geq L} (\log \ell) \sup_{D \text{ at } (x,\ell)} r(D) \ \ll_{q_0,d} \ n_x(t_0+1)(x^{42} + b_x/L)(1+\log(2b_x)).$$

The supremum at each prime ranges over the original integral tests, separating systems, coefficient embeddings and signed powers at that node; an empty contribution is zero.

Proof. Put $b = b_x$. The two original level primes contribute zero by Corollary 15.29. Every other coefficient place is spherical in this family. At a place $w \mid \ell$, Corollary 16.28 annihilates each free cup evaluation by $\ell^{2S_{x,\ell}}$. For a cyclic coordinate with coefficients $\mathcal{O}/\pi^m$, the subgroup killed by ℓ^h has normalized $\mathcal{O}$-length at most h: its ordinary length is $\min(m, he_{\mathcal{O}})$, where $\ell\mathcal{O} = \pi^{e_{\mathcal{O}}}\mathcal{O}$. There are at most $[E_{x,w} : \mathbb{Q}_\ell]$ coordinates per target at this place, and at most $t_0 + 1$ targets. The joint global image is a submodule of the product of these killed subgroups. Additivity of length in a product and $\sum_{w|\ell}[E_{x,w} : \mathbb{Q}_\ell] = n_x$ give

$$r(D) \leq 2n_x(t_0+1)S_{x,\ell}.$$

This argument keeps the given global source, including any relations between its local projections.

Take $R = \lceil C_0(x^{42} + b/L) \rceil$, with C_0 from Corollary 16.28. Let $\mathcal{N}$ consist of the nonzero integers within distance R of at least one of $0, \pm b, \pm a_x/x_{1,x}$. For every $\ell \geq L$ it contains the full set used to define $S_{x,\ell}$. Hence unique factorization gives

$$\sum_{\ell \geq L} (\log \ell) S_{x,\ell} \leq \sum_{n \in \mathcal{N}} \sum_{\ell \mid n} v_\ell(|n|) \log \ell = \sum_{n \in \mathcal{N}} \log |n|.$$

The number of integers is at most $5(2R+1)$, and their absolute values are at most $b + R$. Since $x^{42} \leq b$, the last sum is $O_{q_0,d}((x^{42} + b/L)(1 + \log(2b)))$. This proves the proposition. Multiplicities of prime factors are included in the factorization, as required for the annihilators over the original finite rings. □

Theorem 51.7 (The full period above a node-dependent cutoff)**.** *For each node let*

$$L_x = x^{50}(1 + \log(2b_x))\sqrt{b_x}.$$

On the same fields E_x, on the same interval starting at V_0, and with the twists of Theorem 50.11, one has

$$\sum_{\ell} (\log \ell) \sup_{\substack{D \text{ at } \ell \\ \ell \geq L_x}} \frac{p(D)}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

This holds for all sufficiently large V_0. Every coefficient prime above the displayed threshold is included, regardless of its residue relative to the period weights or of the ranks of the crystalline traces.

Proof. The inequality $x_{1,x}^{10^6} \le b_x$ implies $L_x \ge x_{1,x}$. Dividing the proposition by $\sqrt{b_x}$ and using $n_x \ll_{q_0,d} x^{12}$, $t_0 + 1 \ll x^5$, we bound the node sum by

$$O_{q_0,d}\left(x^{59} \frac{1 + \log(2b_x)}{\sqrt{b_x}} + x^{-33} \right).$$

For large b_x, $1 + \log(2b_x) \le b_x^{1/4}$. Already $b_x \ge x^{400}$ makes the first term $O(x^{-41})$. The sum over integer nodes $x \ge V_0$ is $O_{q_0,d}(V_0^{-32})$. For each ℓ the supremum over admissible nodes of the nonnegative normalized joint length is bounded by the sum of the corresponding node suprema. Summing over ℓ therefore gives the same upper bound on the restricted prime sum of $r(D)/\sqrt{b_x}$. Finally

$$p(D) \le \max\{0, p(D) - r(D)\} + r(D).$$

Theorem 50.11 bounds the prime sum of the normalized first term, including at these primes, by $O_{q_0,d}(V_0^{-10})$. The two estimates use the same choices of data and give the stated inequality. □

Corollary 51.8 (The local contribution from the first auxiliary scale)**.** *At a single node the free cup contribution on the longer range $\ell \ge x_{1,x}$ satisfies*

$$\sum_{\ell \ge x_{1,x}} (\log \ell) \sup_{D \text{ at } (x,\ell)} r(D) \ll_{q_0,d} n_x b_x x^{-20}(1 + \log(2b_x)).$$

Proof. Apply the proposition with $L = x_{1,x}$. Since $t_0 + 1 \ll x^5$, $x_{1,x} \ge x^{10^6}$ and $b_x \ge x^{400}$, both $(t_0 + 1)x^{42}$ and $(t_0 + 1)b_x/x_{1,x}$ are $O(b_x x^{-20})$. This proves the inequality. □

The theorem bounds the full period at the square-root normalization on its stated range. The corollary uses a larger normalization on its wider range. Controlling the remaining coefficient primes at the required scale and comparing the integral periods of the higher-rank lifts are separate steps in the passage to Dirichlet GRH.

51.4 The remaining main cup coordinates at intermediate primes

At a node x let $b = b_x$, $b_2 = xb/x_{1,x}$, and call these the two base weights of the marked data. At a coefficient place we consider all labelled embeddings there for the given datum, including the labels obtained after extending coefficients. The sign at a label is the sign of its main jump J_i. The short shifts in the jump do not change the sign of the signed base weight for sufficiently large x.

Lemma 51.9 (Main targets at places with constant sign or base weight)**.** *For $p \geq x^{50}$, let $w \mid p$ be spherical. If the sign is constant over w, or if the base weight is constant over w, then $p^{2S_{x,p}}$ kills each cup of the main target, with the five intervals of Corollary 16.30. In particular the sign condition holds at every such place for a rational prime split in $\mathbb{Q}(\sqrt{-d})$.*

Proof. All these fields are unramified at p by Lemma 51.1, and $B = t' - 1 < p - 1$. First suppose the sign is constant. Use $a = \lceil 12b/(p-3) \rceil$ to lift E^a in all three extensions, as in Corollary 16.28. At a positive place the graph has only nonnegative jumps, and the main dual has only nonpositive jumps. The first therefore has the required gap, and in the second the unit already lifts by Theorem 16.5. At a negative place the graph has upper jump at most B at every embedding, so its unit lifts by the same theorem since $B < p - 1$. The main dual has only nonnegative jumps and satisfies the gap condition. These two cases meet the respective labelled alternatives of Theorem 16.27. Its costs lie in the five intervals by the shift bounds. At a rational prime split in the quadratic field, all embeddings of a fixed completion induce the same embedding of that field. The CM type described in Theorem 15.3 then gives the same sign.

If the base weight is a fixed $\beta \in \{b, b_2\}$ at the place, every main jump has absolute value between $\beta - O_{q_0,d}(x^{41})$ and $\beta + O_{q_0,d}(x^{41})$. We have $\beta \geq b^{1-10^{-6}}$. In all three extensions every upper jump is bounded by 12β, also for the third kernel since its jumps are differences at the same label. Use $a = \lceil 12\beta/(p-3) \rceil$ in the lifting argument. Now $|J_i| > a + B + 2$ at all labels for large x, without requiring the sign to be constant. The graph and main dual both have the gap, and the two labelled alternatives hold as in the proof of Corollary 16.28. The radius needed for each valuation sum is $O_{q_0,d}(x^{42} + \beta/p)$ about the appropriate centers and is bounded by the radius in the assertion. □

Lemma 51.10 (Places having only the smaller base weight)**.** *At all spherical places of a datum where every base weight is b_2, omit all targets. Let $\Delta_2(D)$ be the resulting drop in joint normalized length. Then*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } (x,\ell)} \Delta_2(D) \ll_{q_0,d} n_x b_x x^{-45} (1 + \log(2b_x)).$$

This includes ramified places and places above 2*.*

Proof. All three upper jumps are at most $H = 12b_2$ at each omitted place, also in restriction of scalars. For primes up to $4(n_x + d + 1)$ the local estimates in Lemma 51.1 bound the sum by

$$O_d\big(n_x(t_0+1)H(n_x+1)(1+\log(2n_x))^3\big).$$

For larger primes its proof bounds each exponent by $O(H/\ell)$ when $\ell \leq H + 1$, and by zero thereafter. Put $N = \lfloor H + 1 \rfloor$. For every prime $\ell \leq N$ we have $v_\ell(N!) \geq \lfloor N/\ell \rfloor \geq N/(2\ell)$. Consequently

$$\sum_{\ell \leq N} \frac{\log \ell}{\ell} \leq \frac{2 \log(N!)}{N} \leq 2 \log N.$$

Thus the larger primes together cost $O(n_x(t_0+1)H(1+\log(2H)))$. In both sums we have used the original joint length, as in the cited lemma, and the bounds hold after the supremum at the node. Since $b_2 = xb_x/x_{1,x}$, $x_{1,x} \geq x^{10^6}$, $n_x \ll_{q_0,d} x^{12}$ and $t_0+1 \ll x^5$, each of these two bounds satisfies the asserted inequality. □

Theorem 51.11 (A period bound after retaining the mixed main coordinates)**.** *Use the interval and twists of Theorem 50.11. For each datum project its original joint image onto the coordinates left after the following omissions.*

1. *Omit both original level primes. At spherical places with $\ell \geq x_{1,x}$, omit all targets.*
2. *At spherical places with $\ell \geq x^{50}$, omit every scalar target. Also omit the main target if either the sign or the base weight is constant across the labels.*
3. *At a spherical place with base weight b_2 at every label, omit all targets at every prime.*

Write $r_{\mathrm{rem}}(D)$ for the normalized length of this projection. Set $f_x = n_x/2$ and $k_x = r_x+1-m_x$ for the ranks of the shorter family. Then

$$\sum_{\ell}(\log \ell) \sup_{D \text{ at } \ell} \frac{\max\{0, p(D) - r_{\mathrm{rem}}(D)\}}{f_x(m_x/k_x)b_x(1+\log(2b_x))} \ll_{q_0,d} V_0^{-39}.$$

Proof. The projection kernel embeds in the product of the discarded local coordinates. The two original level primes have zero free image. At primes $\ell \geq x_{1,x}$, Corollary 16.28 bounds every free evaluation by the exponent $2S_{x,\ell}$. At the other primes $\ell \geq x^{50}$ the coordinates specified in (2) have exponents at most $3S_{x,\ell}$ by Corollary 16.30 and Lemma 51.9. This argument uses the same C_0 throughout, enlarged once for the displayed families. The proof of Proposition 51.6 applies to these discarded coordinates with $L = x^{50}$: its factorization only requires containment in the fixed five intervals of radius $\lceil C_0(x^{42} + b_x/L)\rceil$. Thus their weighted cost at the node is

$$O_{q_0,d}(n_x(t_0+1)(x^{42} + b_x/x^{50})(1+\log(2b_x))).$$

Here $b_x \geq x^{400}$. With $t_0+1 \ll x^5$ this is $O_{q_0,d}(n_x b_x x^{-45}(1+\log(2b_x)))$. The remaining omissions in (3) satisfy this same bound by Lemma 51.10. More explicitly, one can project first away from the coordinates in (3). Its kernel has the bound of that lemma, and the kernel of the next projection injects into the product of the other discarded coordinates. This proves the node bound for $r(D) - r_{\mathrm{rem}}(D)$.

Since $m_x \geq 1$ and $0 < k_x \leq r_x + 1 \ll x^5$, division by the denominator in the theorem gives $O_{q_0,d}(x^{-40})$ for this node sum. Passing to the supremum over the interval costs at most the sum over all $x \geq V_0$, as in Theorem 51.7, hence $O_{q_0,d}(V_0^{-39})$. Finally bound the positive part of $p(D) - r_{\mathrm{rem}}(D)$ by the positive part of $p(D) - r(D)$ plus the drop in joint length. For the first term use Theorem 50.11: the ratio of $\sqrt{b_x}$ to the stated denominator is $O_{q_0,d}(x^5/\sqrt{b_x}) = O_{q_0,d}(V_0^{-195})$. This completes the interval estimate. □

In particular, for $x^{50} \leq \ell < x_{1,x}$ the retained coordinates can occur only for main targets at places with both signs and both base weights. At smaller primes this argument still retains some scalar targets as well.

51.5 A shorter range for the mixed main cups

The order at the second Frobenius prime in Theorem 16.33 costs at most a constant times the smaller base weight at each coefficient prime. Summing this cost only up to a prescribed point lets us shorten the range left in Theorem 51.11. Near its other end, the ratio $b_x/b_2 = x_{1,x}/x$ is the threshold that suffices for the first Taylor expansion.

Proposition 51.12 (The sum of the main coordinates up to Z)**.** *At a sufficiently large node let $x^{50} \le Z \le x_{1,x}$. At spherical places above $x^{50} \le \ell \le Z$ discard all main targets, and let $\Delta_{\mathrm{main}}(D)$ denote the drop in normalized length of the original joint image or of any projection of that image onto coordinates. With $b = b_x$ and $b_2 = xb/x_{1,x}$, the resulting sum obeys*

$$\sum_{x^{50}\le \ell\le Z} (\log \ell) \sup_{D \text{ at } (x,\ell)} \Delta_{\mathrm{main}}(D) \ll_{q_0,d} n_x\big((x^{42} + b/x^{50})(1+\log(2b)) \\ + Z(b_2 + x^{42} + 1) + Z\log(2b)\big).$$

At all primes $\ell \ge 64x_{1,x}/x$, the annihilator $\ell^{2S_{x,\ell}}$ of Corollary 16.28 already holds at every spherical place, for every target.

Proof. There is one main target per place. Its number of free coordinates is at most the local degree, whose sum over places is n_x. The projection kernel embeds into these coordinates; by Theorem 16.33 each has normalized length bounded by the exponent in that theorem. The part $2S_{x,\ell}$, summed with $\log \ell$, is handled by the integer factorization in Proposition 51.6 with lower cutoff x^{50}. This gives the first term in the bound.

For the term constant in ℓ, we recall the elementary inequality $\sum_{\ell\le Z} \log \ell \ll Z$. Each prime $n < \ell \le 2n$ divides $\binom{2n}{n}$, so the sum of its logarithm over that interval is at most $2n \log 2$. Summing on dyadic intervals up to the first power of two at least Z proves the inequality. The term $\log(2b)/\log \ell$ costs at most $Z\log(2b)$. These estimates also hold for suprema in the data because their bounds depend only on x and ℓ.

For the final assertion, take $a = \lceil 12b/(\ell-3)\rceil$. If $\ell \ge 64x_{1,x}/x$, then $\ell \ge x^{50}$ for large x, and $a \le 16b/\ell + 1 \le b_2/4 + 1$. At every label $|J_i| \ge b_2 - O_{q_0,d}(x^{41})$ and $B = t' - 1 \ll x^5$. Since $b_2 \ge b^{1-10^{-6}}$, the gap $|J_i| > a + B + 2$ holds everywhere. This is the only weight separation needed in the proof of Corollary 16.28, after unramifiedness and $B < \ell - 1$. Applying the same two-lattice theorem gives the claimed annihilator. □

Theorem 51.13 (The shortened interval of retained main coordinates)**.** *Fix $8 \le \gamma \le 46$. At each node, begin with the retained coordinates of Theorem 51.11. Among them at primes $\ell \ge x^{50}$, retain only coordinates at primes in*

$$\frac{x_{1,x}}{x^{\gamma}} < \ell < \frac{64x_{1,x}}{x}.$$

These are main targets at places having both signs and both base weights. Let $r_\gamma(D)$ be the normalized length of the joint projection onto the resulting coordinates, including those retained below x^{50}. For the same interval and twists as before,

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{\max\{0, p(D) - r_\gamma(D)\}}{f_x(m_x/k_x) b_x (1+\log(2b_x))} \ll_{q_0,d} V_0^{-39} + V_0^{7-\gamma}.$$

Proof. Write $U = x_{1,x}/x^{\gamma}$, so $x^{50} \le U$ for all large x in the stated range of γ. At a node the additional drop for primes through U is controlled by the proposition at $Z = U$. The term Ub_2 equals $b_x x^{1-\gamma}$. The other terms, apart from this one, total

$$O_{q_0,d}(n_x b_x x^{-50}(1+\log(2b_x))).$$

Indeed $x^{42} \ll b_x/x^{50}$ and $U(x^{42}+1) \ll b_x/x^{50}$, since $U \le x_{1,x} \le b_x^{10^{-6}}$ and $b_x \ge x^{400}$. Also $U \le b_x/x^{50}$. At primes at least $64x_{1,x}/x$ the additional main coordinates cost at most $2S_{x,\ell}$ each and there are at most n_x per prime. The same factorization bound, even using the smaller lower cutoff x^{50}, gives the same last error. These estimates concern projection kernels and apply after the earlier omissions.

Divide by the denominator of the theorem and use $n_x = 2f_x$ and $k_x/m_x \ll x^5$. The extra drop at the node is $O_{q_0,d}(x^{-45}+x^{6-\gamma})$ after division. The sum of these bounds for integer $x \ge V_0$ is $O_{q_0,d}(V_0^{-44} + V_0^{7-\gamma})$, with a uniform constant since $\gamma \ge 8$. The supremum at each prime is bounded by this summation of its nonnegative node contributions. Adding Theorem 51.11 gives the claimed estimate. □

Corollary 51.14 (Size of the residual contribution on the shortened interval)**.** *At one node the weighted sum of the retained main lengths on the displayed interval is*

$$O_{q_0,d}(n_x b_x\{1 + x^{-50}(1+\log(2b_x))\}).$$

Proof. Use the proposition at $Z = 64x_{1,x}/x$, which lies between x^{50} and $x_{1,x}$ for large x. Now $Zb_2 = 64b_x$ and the other terms have the bounds used in the preceding proof. For a datum at such a prime the retained image has only main coordinates; omitting all of them drops precisely its length. □

The term $n_x b_x$ in the corollary still requires a sharper estimate to obtain the stated interval normalization for the full period. There are also retained coordinates below the polynomial cutoff and the higher-rank integral theta-period comparison.

51.6 The full torus period at large coefficient primes

The local estimate of Theorem 16.38 allows us to sum over the entire range in which Theorem 51.13 retained mixed main coordinates. We use the original field $E_x = \mathbb{Q}(\mu_{q_{1,x}}, \sqrt{-d})$ at each node, its degree n_x, and the shorter torus data and projected integral tests of Theorem 50.11.

Proposition 51.15 (The sum of the mixed main lengths)**.** *At a sufficiently large node put $L = x^{50}$ and $Z = x_{1,x}$. For any projection of the joint local image, write $\Delta(D)$ for its drop in normalized length upon further omitting any collection of the main coordinates at spherical places. Then*

$$\sum_{L\le \ell\le Z} (\log \ell) \sup_{D \text{ at } (x,\ell)} \Delta(D) \ll_{q_0,d} n_x b_x x^{-50}(1+\log(2b_x)).$$

Proof. Write $b = b_x$, $G = xb/Z$, and use $W = O_{q_0,d}(x^{42})$ as in Theorem 16.38. At a place w there is one main target with at most $[E_{x,w} : \mathbb{Q}_\ell]$ free coordinates. An ℓ^h-killed submodule in one copy of $\mathcal{O}/\pi^m$ has length at most $h\, e_{\mathcal{O}}$, where $v_\pi(\ell) = e_{\mathcal{O}}$. The projection kernel injects into the product of the omitted coordinates. Summing the local degrees gives n_x, so the theorem bounds $\Delta(D)$, before or after the indicated supremum, by

$$C_{q_0,d} n_x \left(W + 1 + \frac{b}{\ell^2} + \frac{G}{\ell} + \frac{\log(2b)}{\log \ell} \right).$$

The elementary bound $\sum_{\ell\le y} \log \ell \ll y$ was proved in Proposition 51.12. On $[2^j L, 2^{j+1}L)$ it gives $\sum (\log \ell)/\ell^2 \ll (2^j L)^{-1}$, so the sum for all $\ell \ge L$ is $O(1/L)$. On each dyadic interval below $2Z$ it also gives $\sum (\log \ell)/\ell \ll 1$. Consequently, after taking out the factor n_x, the total is bounded by a constant times

$$(W+1)Z + b/L + G(1+\log(2Z)) + Z\log(2b).$$

Here $x^{10^6} \le Z \le b^{10^{-6}}$ and $b \ge x^{400}$. In particular $x^{92} Z \le b$ for sufficiently large x, by $b^{1-10^{-6}} \ge x^{400(1-10^{-6})}$. It follows that $(W+1)Z \ll b/L$ and $Z \le b/L$. Also $G \le b x^{1-10^6} \le b/L$ and $Z \le b$. Substitution gives the assertion. □

Theorem 51.16 (The marked torus period with only small-prime coordinates retained)**.** *Use the fields, interval starting at V_0, and simultaneous twists of Theorem 50.11. At primes $\ell < x^{50}$ keep the projection of Theorem 51.11; at all other primes take the zero projection. Write $r_<(D)$ for its normalized length. Set*

$$\mathcal{B}_x = f_x(m_x/k_x) b_x (1+\log(2b_x)), \qquad f_x = n_x/2.$$

For all sufficiently large V_0 one has

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_<(D))_+}{\mathcal{B}_x} \ll_{q_0,d} V_0^{-39}.$$

In particular, with the supremum restricted to data with $\ell \ge x^{50}$, the same bound holds for $p(D)/\mathcal{B}_x$ itself at the original integral tests.

Proof. Write r_{rem} for the earlier projection. It already omits the original level primes, and at $\ell \ge Z = x_{1,x}$ it contains no coordinates. For $x^{50} \le \ell < Z$, the only coordinates still present come from main targets at spherical places. Thus for fixed x the last proposition bounds the sum of $(\log \ell)\sup(r_{\mathrm{rem}} - r_<)$ at that node. At $\ell < x^{50}$ the difference is zero. Dividing by $\mathcal{B}_x$, using $n_x = 2f_x$ and $k_x/m_x \ll x^5$, gives $O_{q_0,d}(x^{-45})$. For each prime, the supremum over

the interval of these nonnegative quantities is bounded by the sum of their node suprema. Summing in either order costs at most $O_{q_0,d}(\sum_{x\ge V_0} x^{-45}) = O_{q_0,d}(V_0^{-44})$. Finally

$$(p(D) - r_<(D))_+ \le (p(D) - r_{\mathrm{rem}}(D))_+ + r_{\mathrm{rem}}(D) - r_<(D).$$

The first term has the bound of Theorem 51.11, which proves the display. At a datum with $\ell \ge x^{50}$, $r_<(D) = 0$, giving the final assertion. □

The estimate uses the torus normalization $\mathcal{B}_x$. The stronger normalization $\sqrt{b_x}$ for the full period is established on the range of Theorem 51.7. The small-prime term $r_<(D)$ and the integral comparison for the higher-rank lifts are separate outstanding steps toward the zero-location theorem.

51.7 The torus period above the product $k_x t_0$

For the shorter family put $Q_x = k_x t_0$, with $t_0 = 5r_x$ and $k_x = r_x + 1 - m_x$. We retain the original fields and twists. The interval supremum can be estimated directly at a coefficient prime: every node with $Q_x < \ell$ satisfies $x \ll \ell^{1/10}$. This matters for a contribution proportional to b_x/ℓ^2, which occurs for each scalar target.

Lemma 51.17 (The large-prime local bounds below x^{50})**.** *At sufficiently large nodes and at an unramified spherical place over $p \geq 7$ with $p > t_0 + 2$, the bound of Theorem 16.38 holds for every main cup. For each scalar cup one may omit the G/p term. In particular these estimates hold at every spherical place over a prime $p > Q_x$.*

Proof. Take the two weights $b = b_x$, $G = xb/x_{1,x}$, and the integer $W = O_{q_0,d}(x^{42})$ of Section 16.14. For a scalar target Corollary 16.30 applies at every unramified $p \geq 5$. In each of its five intervals there are $O(W + 1 + b/p)$ consecutive integers of absolute value at most Cb, with fixed C. After deleting zero, summing the counts of multiples of successive powers of p bounds the total valuation by $O((W + 1 + b/p)/(p-1) + \log(2b)/\log p)$. This proves the scalar assertion.

For a main target of constant sign, the proof of Lemma 51.9 uses two numerical facts after unramifiedness. The upper jump bound on the side with only short positive jumps is $B = t' - 1 < p - 1$, so its quotient unit lifts. The remaining side either has nonnegative jumps or is nonpositive throughout, in which latter case its unit lifts as well. More explicitly, at positive sign the graph is nonnegative and the dual nonpositive. At negative sign the graph has upper jump B and the dual is nonnegative. These give precisely the two alternatives of Theorem 16.27 with $a = \lceil (b + W)/(p-3) \rceil$. All three E^a-lifts follow from their upper jumps, and the two valuation sums obey the interval count just given.

If the base weight β is constant, use $a_\beta = \lceil (\beta + W)/(p-3) \rceil$ on all three extensions, whose upper bounds are now $\beta + W$. Uniformly for $p \geq 7$, $a_\beta \leq (\beta + W)/4 + 1$; $\beta \geq G \geq b^{1-10^{-6}}$ gives $\beta - W > a_\beta + B + 2$ for large x. There is no negative jump in $[-a_\beta, 0)$. The two sign alternatives again give Theorem 16.27 and the same valuation bound.

For a mixed orbit, keep the two choices of labels and the two-step comparison in Theorem 16.38. We check its numerical hypotheses. The bound $a \leq b/p + 8H_p$ only used $p \geq 7$, and $a \leq (b + W)/4 + 1 < b - W$ since $W = o(G)$. The light gap in the first choice follows as before from $s \leq G - W - 2$. In the second choice the degree of the monic polynomial satisfies the strict inequality there by $b/p - G > 40H_p$, independently of the size of W/p. The cost at the second prime uses $B < p$. These are all size requirements in constructing the points and comparing the cochains; the sign alternation holds for the same reason at every unramified mixed place. The same exponent follows.

Finally $q_{1,x} = 3q_0 l_0 2^s$, with l_0 a power of three, and $E_x \subseteq \mathbb{Q}(\mu_{q_{1,x} d})$ by Corollary 11.30. Thus every ramified prime divides $6q_0 d$. Since $r_x \asymp x^5$ and $\sqrt{x} \leq m_x \leq 2\sqrt{x}$, one has $k_x \asymp x^5$, $Q_x \asymp x^{10}$. Primes above Q_x are consequently unramified and exceed $t_0 + 2$ at all sufficiently large nodes. □

Theorem 51.18 (The full integral torus period above $k_x t_0$)**.** *In Theorem 51.16 let $r_Q(D) = r_<(D)$ when $\ell \leq Q_x$, and $r_Q(D) = 0$ otherwise. With its normalization $\mathcal{B}_x = f_x(m_x/k_x) b_x (1 + \log(2b_x))$,*

$$\sum_\ell (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_Q(D))_+}{\mathcal{B}_x} \ll_{q_0,d} V_0^{-1/2}.$$

Thus for data with $\ell > Q_x$ the bound holds for the full period, including all scalar targets and all main targets at the original twist and precision.

Proof. Only primes $Q_x < \ell < x^{50}$ entail a further projection loss. The two old level primes already have zero retained image. In the spherical case there are at most $n_x = [E_x : \mathbb{Q}]$ coordinates per target and at most $t_0 + 1$ targets. The local lemma and the length argument of Proposition 51.15 therefore bound the additional normalized loss by

$$C_{q_0,d}\frac{x^{19/2}}{b_x(1+\log(2b_x))}\left(W+1+\frac{b_x}{\ell^2}+\frac{G}{\ell}+\frac{\log(2b_x)}{\log\ell}\right).$$

Here $n_x/f_x = 2$ and $(t_0+1)k_x/m_x \ll x^{19/2}$; this uses the lower bound $m_x \geq \sqrt{x}$.

Consider first b_x/ℓ^2. There are constants $c, C > 0$ such that $cx^{10} \leq Q_x \leq Cx^{10}$ at all these nodes. Thus $\ell \geq cV_0^{10}$ and $x^{19/2} \ll \ell^{19/20}$. Even after the supremum over all nodes at this prime the resulting summand, with its factor $\log\ell$, is at most $C'(\log\ell)\ell^{-21/20}$. The inequality $\sum_{\ell\leq y}\log\ell \ll y$ bounds its sum on the dyadic interval $[2^jL, 2^{j+1}L)$ by $O((2^jL)^{-1/20})$. With $L = cV_0^{10}$ the geometric series costs $O(V_0^{-1/2})$.

For the other terms take a separate upper bound at each node and sum the nonnegative bounds over the nodes. Up to fixed constants the node costs for $W+1$ and for $\log(2b_x)/\log\ell$ are at most

$$\frac{x^{19/2+50}(W+2)}{b_x}.$$

Here we used the same prime logarithm bound and at most x^{50} primes. For G/ℓ the cost is at most

$$Cx^{19/2}\frac{x}{x_{1,x}}(1+\log(2x^{50})),$$

using $\sum_{\ell\leq y}(\log\ell)/\ell \ll 1+\log(2y)$. As $b_x \geq x^{400}$, $W \ll x^{42}$ and $x_{1,x} \geq x^{10^6}$, both bounds are $O(x^{-100})$. Their sum is $O(V_0^{-99})$. Add the estimate of Theorem 51.16 using the positive-part inequality in its proof. This gives the assertion. □

51.8 The torus period away from the fixed conductor

For the shorter family put $Q_x = k_x t_0$, $2^s = m_x^{22}$ and $q_1 = 3q_0 l_0 2^s$, where l_0 is the power of three in Corollary 11.30. In particular $27s \le l_0 \le 81s$. The characteristic-zero fields are $E_x = \mathbb{Q}(\mu_{q_1}, \sqrt{-d})$, with CM type induced from $K_d = \mathbb{Q}(\sqrt{-d})$. We first describe the actual spacing of the places assigned the larger weight and its behavior in a local Frobenius orbit.

Lemma 51.19 (The spacing of the larger weights on the period field)**.** *Identify the infinite places with $(\mathbb{Z}/q_1\mathbb{Z})^\times$ by choosing the embedding over the specified embedding of K_d. At sufficiently large nodes there is a set $\mathcal{T}$ of places of class $-1 \bmod q_0$, of size at least $2^s \exp(-C\sqrt{\log(2x)\log\log(3x)})$, with the following property. For every prime $p \le Q_x$ and $0 < i < j < l_0$, there are no three members a, b', c (allowing repetitions) with*

$$b'/a \equiv \pm p^i, \qquad c/a \equiv \pm p^j \pmod{q_1/p^{v_p(q_1)}}.$$

The signs are independent. In particular the set has at least $3l_0 2^s/\log b_x$ members on the original scales with $\log b_x \ge x^\epsilon$ for fixed $\epsilon > 0$. The spacing is preserved by translation by a fixed unit. On an unramified completion over $p \le Q_x$ with $p \nmid q_1 d$, labels carrying the weight b_x from any such set cannot occur three times with positive successive gaps whose sum is less than l_0.

Proof. Since K_d has discriminant $-d$ and $d \nmid q_1$, it intersects $\mathbb{Q}(\mu_{q_1})$ in $\mathbb{Q}$: otherwise a field unramified at d would contain K_d. The oriented embeddings therefore give the indicated group of unit classes. Take $v = v_2(q_0) \ge 2$ and $g = 1 + 3l_0 q_0$, a unit modulo q_1. Its order modulo 2^{s+v} is 2^s, since $v_2(g^n - 1) = v + v_2(n)$ for $n \ge 1$. For odd n this follows by the geometric sum, and each subsequent squaring adds one since $g \equiv 1 \bmod 4$.

Set $M = 2^s$ and $L = 2l_0$. Apply Lemma 6.25 with modulus M and bound $L = O(\log M)$. Its exponents e lie in $[0, M/(8L))$. For three such exponents and positive α, α' with $\alpha + \alpha' \le 2l_0$, a congruence

$$(\alpha + \alpha')e_2 = \alpha e_1 + \alpha' e_3 \pmod{M}$$

lifts to an equality of integers by this size bound. The lemma then forces the three exponents to coincide. Take $\mathcal{T}$ to consist of the places $-g^e$. The order calculation makes them distinct and all have class $-1 \bmod q_0$. The cardinality assertion follows from the lemma and $x^{11} \le M \le 2^{22}x^{11}$. The exponential loss times $3l_0$ is less than x^ϵ for all sufficiently large x.

Suppose the two stated ratios exist for an odd p. The modulus includes 2^{s+v}. Raise the first ratio to $2j$ and the second to $2i$. The signs disappear and both values are p^{2ij}. The order of g therefore gives the displayed exponent congruence with $\alpha = 2(j - i)$, $\alpha' = 2i$. All three exponents coincide. In particular $p^{2i} \equiv 1 \bmod 2^{s+v}$. For odd p, factoring $p^{2i} - 1$ at its odd and two-power exponents gives

$$v_2(p^{2i} - 1) = v_2(p - 1) + v_2(p + 1) + v_2(i).$$

Indeed $p^2 \equiv 1 \bmod 8$, and each squaring raises the valuation by one. One of $p - 1, p + 1$ has valuation one. Hence $2^{s+v} \le 2(p+1)i \le 2(Q_x+1)l_0$. This contradicts $2^s \ge x^{11}$, since $Q_x \ll x^{10}$ and $l_0 \ll \log(2x)$. At $p = 2$ reduce instead modulo $3l_0$: every ratio is 1, so squaring the first would give $4^i \equiv 1 \bmod 3l_0$. But $v_3(4^i - 1) = 1 + v_3(i)$ by the binomial formula, forcing $l_0 \mid i$, again impossible. Multiplying all three places by one unit preserves both ratios.

At an unramified prime outside $q_1 d$, arithmetic Frobenius on a primitive q_1-th root is the p-th power map. Conjugating an embedding to return it to the chosen CM type negates the exponent of that root. Hence after n steps the ratio of the associated place classes is $\pm p^n$ modulo q_1, with the reverse order of labels if inverse Frobenius indexes the cycle. Reversing a triple preserves its total gap. The two forbidden congruences prove the final assertion. □

Theorem 51.20 (A period bound with only the fixed conductor primes retained)**.** *Use the sparse choice of the larger-weight places of the shorter family, with the explicit spacing in Lemma 51.19. Make this choice in the period data before choosing the simultaneous twists of Theorem 50.11. Let $\mathcal{R} = \{p : p \mid 6q_0 d\}$, and put $r_{\mathcal{R}}(D) = r_Q(D)$ for $\ell \in \mathcal{R}$, with $r_{\mathcal{R}}(D) = 0$ otherwise. Then on the same interval and fields as in Theorem 51.18,*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{(p(D) - r_{\mathcal{R}}(D))_+}{f_x(m_x/k_x) b_x (1 + \log(2b_x))} \ll_{q_0,d} V_0^{-1/2}.$$

In particular all coefficient primes outside the fixed set $\mathcal{R}$ satisfy the full torus-period comparison in this prime sum, including primes below $k_x t_0$.

Proof. Fix $p \notin \mathcal{R}$, $p \le Q_x$. The field is unramified there because $E_x \subseteq \mathbb{Q}(\mu_{q_1 d})$. We have $p \ge 5$. At any spherical place choose

$$J = \left\lfloor \frac{120 \log x}{\log p} \right\rfloor.$$

Here $J \ge 1$ for large x, since $Q_x \ll x^{10}$. As $s = 22 \log_2 m_x \ge 11 \log_2 x$, we have $2J \le 240 \log_2 x < l_0$. The preceding lemma supplies the precise triple condition in Theorem 16.41. Writing $G = x b_x / x_{1,x}$ and using its W, η, we have

$$p^J \le x^{120} < p^{J+1}, \qquad \frac{\eta}{b_x} \ll_{q_0,d} x^{1-10^6} + \frac{x^{42}+1}{b_x}.$$

Since $b_x \ge x^{400}$ and $J \ll \log(2x)$, these imply $16(J+1)p^J \eta \le b_x$. They also bound $(J+1)^2 \eta$ by $O(b_x x^{-120})$. The logarithmic term in the local bound has the same majorant, since $\log(2b_x) \ll \sqrt{b_x}$ and $\sqrt{b_x} \ge x^{200}$. It follows that every original main or scalar cup at this place is killed by a power of p of exponent $O_{q_0,d}(b_x x^{-120})$. These estimates concern all tests in the chosen datum. In particular membership in the smaller prime range did not require another field, rank or twist.

At a node the projection removed when passing from r_Q to $r_{\mathcal{R}}$ has at most $n_x(t_0+1)$ coordinates per prime and zero length at the old level primes. By $\sum_{p \le Q_x} \log p \ll Q_x \ll x^{10}$, the weighted sum of its lengths is at most $C_{q_0,d}\, n_x(t_0+1) b_x x^{-110}$. After division by the displayed denominator, use $n_x = 2f_x$ and $(t_0+1)k_x/m_x \ll x^{19/2}$. The result is $O_{q_0,d}(x^{-100})$. At $p > Q_x$ the preceding projection was already zero. Summing the nonnegative node bounds over all integers $x \ge V_0$ costs $O(V_0^{-99})$, also after the primewise supremum. Add this to Theorem 51.18 by the positive-part inequality, obtaining the claim. □

The remaining torus contribution is now confined to primes dividing a fixed integer. Their ramification indices can grow with the node. Controlling them, and the integral projections in higher rank, is still needed to derive the asserted zero location for Dirichlet L-functions.

51.9 All coefficient primes for the marked torus period

We finish the sum for the shorter marked torus family at normalization $\mathcal{B}_x = f_x(m_x/k_x)b_x(1+\log(2b_x))$. The primes left by Theorem 51.20 divide $6q_0d$. Two properties of their embeddings are needed simultaneously. The larger weight is sparse even after grouping the embeddings by their restriction to the maximal unramified subfield, and all its occurrences in one completion have a common sign.

Lemma 51.21 (Spacing and sign at the conductor primes)**.** *Fix a place $w \mid p$, $p \mid 6q_0d$, in the family of Theorem 51.20, and write $K = E_{x,w}$, K_0 for its maximal unramified subfield. Mark an embedding of K_0 if some extension to K has base weight b_x. On the periodic residue-field orbit there are no three marked occurrences with positive successive gaps whose sum is $< l_0$. At every full embedding over w carrying the larger weight, the main jump has the same sign.*

Proof. All these primes satisfy $p \le Q_x$ for large x. Put $q' = q_1/p^{v_p(q_1)}$. The q'-th roots of unity in K belong to K_0: their reductions have the same order, and each lifts uniquely from the residue field as a root of $X^{q'}-1$. Hence the image of a primitive q'-th root is determined by a residue-field embedding alone, and Frobenius changes it by raising to p. For three purported marked occurrences, choose a full embedding over each one that carries the larger weight. Changing these embeddings to the chosen CM orientation changes the exponents only by signs. As in Lemma 51.19 their ratios would give the two forbidden congruences modulo q', reversing their order for inverse Frobenius. This holds also for repeated visits to the same residue-field label.

For $p \mid q_1$ one has $p \mid q_0$, since $6 \mid q_0$ and l_0 is a power of three. The quadratic field K_d splits at every such prime. Indeed for odd p one has $-d \equiv 1 \bmod p$ from $d \equiv q_0/2 - 1 \bmod q_0$, and the square root lifts by Hensel's lemma. At two, $16 \mid q_0$ gives $-d \equiv 1 \bmod 8$, so $-d$ is a square in $\mathbb{Q}_2$. All embeddings of this fixed completion then induce the same embedding on the quadratic field.

The other possibility is $p = d$. Now all q_1-th roots of unity are unramified at p, and the ratio of the exponents of any two local embeddings on one primitive root lies in $\langle d\rangle \bmod q_1$. This follows from the cyclic Galois action on the residue field and uniqueness of the prime-to-p lifts. If two larger-weight embeddings had opposite CM orientations, their unoriented exponents would have ratio $-1 \bmod q_0$: their oriented exponents both belong to the same class $-1 \bmod q_0$. But $d^2 \equiv 1 \bmod q_0$ and neither 1 nor d equals $-1 \bmod q_0$, by the specified congruence and $4 \mid q_0$. This rules out opposite orientations. The same argument applies if a uniform translation changes the common class of the larger weights. The main jump sign is determined by the orientation, up to a common reversal for a conjugate character, as in Theorem 15.3. □

Proposition 51.22 (Integral cups at every conductor prime)**.** *With these data, for $p \mid 6q_0d$ at each spherical place $w \mid p$, every original main or scalar cup is killed by p^h for an integer*

$$h \ll_{q_0,d} b_x x^{-108}.$$

The bound is uniform in w, the trace rank and all integral tests and precisions at x.

Proof. Take the coefficient extension containing K and use its unramified label cycle as in Section 16.17. Use $b = b_x$, $G = xb/x_{1,x}$, $W \ll x^{42}$, $\eta = 2(G+W+1)$ as before. The graph and test extensions are crystalline at the spherical place. The third extension is defined by lifting the original equivariant map modulo π^a to an $\mathcal{O}$-linear map ψ and using the cocycle $(g\psi - \psi)/\pi^a$; its contraction recovers that same finite pairing as in Corollary 16.20. At a

full embedding, the jumps of its kernel are differences at that embedding. The tensor and dual calculation is unchanged by ramification. On an unmarked residue-field label all three upper jumps are therefore at most η, and everywhere they are at most $b + W$, after fixing W sufficiently large.

Choose $J = \lfloor 120 \log x / \log p \rfloor$, $L = 2J+1$. The preceding lemma gives the no-triple condition since $2J < l_0$. Choose the cuts followed by J unmarked labels. The elementary gap argument of Theorem 16.41 applies verbatim to this periodic orbit: successive cuts are at distance $1 \le l \le L$. Write $e = [K : K_0] \le n_x \ll_{q_0,d} x^{12}$. Since $p^{J+1} > x^{120}$, it is at least $8e$. Theorem 16.43 supplies a common p^κ-lift at each cut on all three extensions with

$$\kappa \ll 1 + J(d_K + 1)\eta + e(b + W)/p^{J+1}.$$

Scaling to a common exponent is legitimate around the entire orbit. Use the third lift for the pairing $\widetilde{Q} = p^\kappa Q$. Its bound on level zero kernels is $\tau^{-1} A_i(1)$; at level l from the preceding cut it is $\tau_l^{-1} A_i(1)$, where τ_l/τ is a unit times $\prod_{j<l} P_j$. This follows by applying Frobenius to the integral contraction and $\mu/\tau \in EA^\times$. The same lattices and contractions are used at the original precision in the finite Witt ring.

At each cut put $D = \eta + 2$ and apply Proposition 16.44 to the two level zero points, imposing depth D at $0 \le j < L$. The increase in the quotient exponent is at most

$$C_1 = C_{q_0,d}\big((L+1)(L + d_K + 4)\eta + (L+1)\log(2b)/\log p\big).$$

Indeed on all full embeddings over a residue label the low jumps lie in $[-\eta, \eta]$; the far jumps of the given side, if present, lie within W of the same signed b, by the sign lemma. Thus the union of the shifted integer intervals is contained in at most two intervals totaling $O(\eta)$ indices at each j. Each has absolute values at most $8b$. Counting multiples of powers as in Theorem 16.41 proves the bound. On the subgroup $H_L = G_{K(\varpi^{1/p^L})}$ the displacements now lie in $\delta \prod_{j<L} P_j^D$ times their kernel lattices.

There is a choice of side U with lower jumps $\ge -\eta$ at every full embedding and opposite side V with upper jumps $\le \eta$ everywhere. At unmarked embeddings both sides satisfy both bounds. At marked embeddings use the common sign of the larger jump, assigning opposite signs to the graph and dual; an absent far jump imposes no condition. Consequently for consecutive cuts $c, i = F^l c$, with $S = \prod_{j<l} P_j^\eta$, the ramified height inclusions give

$$SM_{l,i}(U) \subseteq M_{0,i}(U), \qquad SM_{0,i}(V) \subseteq M_{l,i}(V).$$

In $M_{l,i}(U)$ take the l-fold Frobenius of a pure p^κ-lift at c, and impose the same orders D at $0 \le j < l$. These lattices are regular at those P_j, so the cost for the resulting y_i is at most $\kappa + C_1$. Take a common exponent $e_* \le \kappa + C_1$, enlarged by one if needed for rounding, for all cut and auxiliary points, by scaling them individually. As before, at noncuts use Frobenius of the preceding pair.

At a cut first change its U-point to y_i. The kernel difference belongs to $S^{-1}M_{0,i}$. The other displacement and level zero pairing leave a factor u. Next change its V-point to the l-fold Frobenius of the point at c. This difference is in $S^{-1}M_{l,i}$; the displacement of y_i supplies $\delta \prod_{j<l} P_j^D$, clearing both S and the level l Tate denominator and leaving a factor u. Finally change y_i to the corresponding l-fold Frobenius on U; the difference is in $M_{l,i}$ and the fixed displacement has factor $u^{p^l}\tau_l$. This argument also handles translated differences since the group preserves the lattices and each principal ideal involved. The initial cup and the three cochain corrections are therefore small, in the sense of Theorem 16.2. Frobenius permutes the residue-field idempotents in one cycle and their sum is one. The terminal pair at each label is exactly

the single Frobenius of the preceding initial pair. Summing and taking constants in finite Witt vectors gives annihilation by $p^{2e_*+\kappa}$ on H_L, by the same cochain descent as in Theorem 16.41. This descent uses the Witt constants over any finite extension of $\mathbb{Q}_p$, including $p = 2$, by Lemma 16.1. The group index is p^L, since $X^{p^L} - \varpi$ is Eisenstein over K. Corestriction and the integral coefficient retraction thus give $h = 2e_* + \kappa + L$ over the original field and ring.

Here $L+d_K \ll_{q_0,d} 1+\log(2x)$. The term $e(b+W)/p^{J+1}$ is $O_{q_0,d}(bx^{-108})$. All terms involving η or $\log(2b)$ satisfy this bound as well, using $G/b \le x^{1-10^6}$, $W \ll x^{42}$ and $b \ge x^{400}$. This proves the claimed uniform estimate. □

Theorem 51.23 (The full projected torus period at the source normalization)**.** *Fix q_0, d in the shorter marked torus family, and use the sparse choices of Lemma 51.19. For the simultaneous twists of Theorem 50.11 on the same fields E_x, the same interval beginning at V_0, and the original integral tests, one has*

$$\sum_{\ell} (\log \ell) \sup_{D \text{ at } \ell} \frac{p(D)}{f_x(m_x/k_x) b_x (1 + \log(2b_x))} \ll_{q_0,d} V_0^{-1/2}.$$

The sum includes every coefficient prime and the supremum includes all the original trace ranks, separating families and precisions in that comparison.

Proof. It remains in Theorem 51.20 to bound $r_{\mathcal{R}}(D)$ for $\mathcal{R} = \{p : p \mid 6q_0 d\}$. At a node the original level primes have zero retained length. At the spherical places above a given $p \in \mathcal{R}$, the retained joint image injects into a product of at most $n_x(t_0 + 1)$ cyclic coordinates. The proposition bounds the normalized length of each by $O_{q_0,d}(b_x x^{-108})$. The sum of $\log p$ in this fixed set depends only on q_0, d. Divide by the displayed denominator, and use $n_x = 2f_x$ and $(t_0 + 1)k_x/m_x \ll x^{19/2}$. The weighted sum at a node is $O_{q_0,d}(x^{-98})$. Its sum over the integer nodes $x \ge V_0$ is $O_{q_0,d}(V_0^{-97})$, which bounds the corresponding sum of interval suprema. Now use $p(D) \le (p(D) - r_{\mathcal{R}}(D))_+ + r_{\mathcal{R}}(D)$ and Theorem 51.20. □

This proves the estimate for the original marked torus projections. Section 6.6 also requires integral control of the higher-rank theta chain in the comparison between local divisibility and the archimedean upper bound for the prime-power estimate. That implication is still outstanding.

52 Congruence modules and integral projections

52.1 The torus budget and the determinant of a higher-rank projection

In the denominator comparison for a chain with m_x lifts there are two tests at a coefficient prime. One concerns the projected inner product for a designated torus line. The other asks for an operator on the image of a higher-rank lift which clears its projected integral forms and has controlled determinant. We convert the bound on the former to the units in which both tests are measured, and identify the exact loss for the latter.

Proposition 52.1 (One family of logarithmic bounds for every main pair)**.** *At each node write $a_x = xb_x$, $H_x = \log b_x$, and put*

$$B_x^{\det} = \frac{f_x a_x H_x^3}{m_x k_x}.$$

For the fields, data and simultaneous twists of Theorem 51.23 there are nonnegative numbers $\gamma_{x,\ell}^{(1)}$ such that, for every main pair, every permitted twist power and every pair of its integral test forms,

$$|\iota\langle P\Xi, P\Xi'\rangle|_\ell \le \exp(\gamma_{x,\ell}^{(1)}).$$

One set of numbers works for all lift indices, coefficient embeddings, and separating families. For every finite set of primes S,

$$\sum_{\ell\in S} \gamma_{x,\ell}^{(1)} \ll_{q_0,d} \frac{V_0^{-1/2}}{H_x^2}\, B_x^{\det}.$$

Proof. Let $\omega_\ell = (\log \ell)\sup_{D \text{ at } \ell} p(D)/\mathcal{B}_x$, where the denominator inside the supremum is the one for the node of D. Use zero for an empty set. Theorem 51.23 gives $\sum_\ell \omega_\ell \ll_{q_0,d} V_0^{-1/2}$. Define $\gamma_{x,\ell}^{(1)} = \mathcal{B}_x\omega_\ell$. The definition of $p(D)$ in Section 50.4 gives the asserted inequality also for a zero projection. For the budget,

$$\frac{\mathcal{B}_x}{B_x^{\det}} = \frac{m_x^2}{x}\frac{1+\log(2b_x)}{H_x^3} \ll H_x^{-2},$$

because $m_x \le 2\sqrt{x}$ and $H_x \ge 1$. Multiplication by the bound for $\sum_\ell \omega_\ell$ proves the claim. □

Theorem 52.2 (The exact determinant loss on integral tested forms)**.** *At one coefficient prime ℓ and embedding, transport the definite level space, its true lift range Z, and its orthogonal projection P to the local coefficients. Let T be the space orthogonal to all false lifts in $Z^\perp$. In a finite local field of definition $C/\mathbb{Q}_\ell$ let L be the lattice of forms integral at this embedding and let $\mathcal{O}$ be the integers of C. Write*

$$L_T = T\cap L, \qquad L_Z = Z\cap L, \qquad L_0 = T\cap \ker P\cap L, \qquad \lambda = \frac{\operatorname{length}_{\mathcal{O}}(L_T/(L_Z+L_0))}{v_\pi(\ell)}.$$

The minimum of $-\log|\det\Lambda|_\ell$, over invertible operators on Z such that $\Lambda P(L_T)\subseteq L_Z$, is $\lambda\log\ell$. It is attained, and extending the local coefficient field preserves the minimum. For $Z=0$ both sides are zero. If $D_Z = \dim Z > 0$ and r_x is the target rank of the marked data, the least nonnegative γ for which such an operator has $|\det\Lambda|_\ell \ge \exp(-D_Z r_x\gamma)$ is $\lambda\log\ell/(D_Z r_x)$.

Proof. We describe the finite local model used here. Definite level forms are given by their values at finitely many double-coset representatives. The values lie in finite Weyl coefficient

spaces and satisfy linear stabilizer relations. At a fixed embedding the local gauge is the coordinate maximum after the inverse weight action of the representative at the places over ℓ. This finite list computes integrality: rational translation cancels the weight action by equivariance of the form, and right translation by the level preserves the integral weight lattice. For the latter assertion the local matrices and their inverses belong to the integral general linear group for the standard level and its subgroups; the integral Weyl representation and integral determinant twists take these to integral invertible matrices. A finite level character has unit values as well. Equivalently one may use representatives at the level kernel and impose the finite level relations. Thus the lattice is the inverse image of the integral coordinate cubes under finitely many linear evaluations. Together the evaluations are injective. Over a finite local field their integral inverse image is a full lattice: a common scalar clears their entries in a fixed basis, and an invertible coordinate minor bounds the inverse image inside a second multiple of that lattice.

A choice of basis for Z, the matrix of P, and a basis for the sum of all false-lift spaces require only finitely many coefficients, also when the possible Hecke tests range over infinitely many places. Write the orthogonality conditions for that basis before transporting their coefficients. These coefficients, the stabilizer equations and gauge matrices all belong to a common finite extension of $\mathbb{Q}_\ell$: each transported coefficient belongs to $\overline{\mathbb{Q}}_\ell$, and the local components of the finitely many adelic representatives lie in the completions of the given number field. This also explains how to take the lattice at the specified embedding while retaining the actual orthogonality tests.

Since the false lifts are in $Z^\perp$, we have $Z \subseteq T$. Thus $P(L_T)$ contains L_Z and is a lattice in Z. Projection induces an isomorphism

$$L_T/(L_Z + L_0) \;\simeq\; P(L_T)/L_Z.$$

Indeed if $Pv \in L_Z$ then $v - Pv \in L_0$, and the reverse inclusion and surjectivity follow directly. Choose a basis $e_1, \dots, e_{D_Z}$ of L_Z in which the larger lattice has basis $\pi^{-a_i} e_i$ with integers $a_i \geq 0$, by the elementary divisor theorem. Its quotient length is $\sum_i a_i$. Every eligible operator sends these last basis vectors to integral columns. Taking their determinant forces $v_\pi(\det \Lambda) \geq \sum_i a_i$. The diagonal operator $e_i \mapsto \pi^{a_i} e_i$ attains equality. Since $v_\ell(\ell) = 1$, conversion to logarithms gives the formula. Under a finite coefficient extension the same bases have elementary exponents multiplied by the relative ramification index, just as $v_\pi(\ell)$ is. Intersections and images used above commute with this flat scalar extension: after clearing a common denominator they are kernels and images of integral matrices. The same determinant lower bound therefore applies to any operator over $\overline{\mathbb{Q}}_\ell$ by adjoining its finitely many entries. The empty determinant equals one in dimension zero. □

Corollary 52.3 (The remaining higher-rank inequality in the denominator budget)**.** *Fix a node and its simultaneous twists as above. For a given lift index j let $h_{x,j,\ell}$ be the supremum of the last quotient in Theorem 52.2 over the designated coefficient embeddings and their corresponding twists at ℓ; use zero for a zero-dimensional range. If for every finite set of primes S*

$$\sum_{\ell \in S} h_{x,j,\ell} \leq \tfrac{1}{2} B_x^{\det},$$

then the integral operator test and all main-pair tests at this index admit a common logarithmic bound of total sum at most $B_x^{\det}$, for sufficiently large V_0. Conversely any common bound of this total size implies $\sum_{\ell \in S} h_{x,j,\ell} \leq B_x^{\det}$.

Proof. At a single prime take the maximum of $h_{x,j,\ell}$ and $\gamma_{x,\ell}^{(1)}$. The theorem supplies the operators, and the proposition supplies the main-pair inequalities for the same twists. The

sum of the maxima is at most the sum of the two sums. The proposition's bound is at most $B_x^{\det}/2$ for sufficiently large V_0. For the converse, the exact local determinant minimum bounds $h_{x,j,\ell}$ from above by any permissible common logarithmic bound at that prime; sum over S. $\square$

The sum of the higher-rank indices in the hypothesis requires an arithmetic estimate on the actual tested lattice. The torus-period estimate proves its own term in the common budget on this same family.

52.2 Split Hecke values on the complement of the true lifts

Fix a node x, one of its prescribed twists, a lift index j, and an embedding at ℓ. Use Z, P, T, L and λ from Theorem 52.2. The mark at the two split level primes p_0, p_1 is a central idempotent e for the finite group

$$\Gamma_x = \prod_{i=0}^{1} \prod_{v \mid p_i,\ v \text{ in } E_x^+} \mathrm{GL}_{r_x}(\mathbf{F}_{p_i}).$$

Write $M_x = |\Gamma_x|$. At an unramified split place w of E_x away from the level, the twist, and ℓ, let $q = Nw$, let $s < n$ be the two ranks in the lift, and put $t = n - s$. Write μ_W for the unitary splitting character attached to the source and put

$$\alpha_w = q^{s/2}\mu_{W,w}(\varpi_w), \qquad X_w = (q^{\,t-1}\alpha_w)^{-1}, \qquad A_w = \sum_{i=0}^{n} (-X_w)^i q^{i(i-1)/2} T_{w,i}.$$

Here $T_{w,i}$ is the unnormalized double coset for the diagonal matrix with i entries ϖ_w and the other entries one, in the convention of the arithmetic Hecke polynomial. We use the definite inner product to form orthogonal complements before transporting them to the local coefficient field.

Proposition 52.4 (The split Hecke operators on tested forms)**.** *At each of these places $Z \subset eV \cap \ker A_w$, where V is the whole definite level space. Put $U = eT \cap Z^{\perp}$. Then $eT = Z \oplus U$; each A_w acts invertibly on U, and both A_w and $M_x e$ preserve the integral lattice L. For $y \in T$ the orthogonal projection of y onto $eV \cap \ker A_w$ is Py.*

Proof. The good source forms belong to the sum of the marked types defining e. The lift intertwines their action. Indeed, changing a marked frame on the source changes the matching frame in the theta kernel. In its matrix coordinate the two changes act on opposite sides of the identity block. Simultaneously changing the two frames preserves the block congruence, and the determinant character in the form action accounts for the scalar of the splitting. Changing variables by this frame change in the source integral gives equivariance for each group element and hence for e. Thus $eZ = Z$.

The good split Hecke algebra is a commuting algebra of normal operators for the definite pairing. In fact the adjoint of each double coset is the inverse double coset, and the spherical algebra is commutative. The actions at the marked and good places also commute. On a spherical theta constituent the added unitary Satake entries are $\mu_{W,w}(\varpi_w)q^{(t-1)/2-i}$, $0 \le i < t$, by Proposition 72.17. Passage to the arithmetic polynomial multiplies each entry by $q^{(n-1)/2}$. The added factors are therefore $1 - \alpha_w q^i X$, $0 \le i < t$, in reverse order. The last of these factors vanishes at X_w. Decomposing the source into its spherical constituents gives $A_w Z = 0$.

The central idempotent e is self-adjoint. This follows from the unitarity of the finite group action and the decomposition into its isotypic spaces. For each place used in the false-lift tests the corresponding false space is the intersection of eV, the kernel of the evaluated Hecke polynomial there, and $Z^{\perp}$. It lies in eV, so e preserves T. Each of these three spaces is stable under A_w and its adjoint. For $Z^{\perp}$ this follows from normality and $A_w Z = 0$. For the other two it follows from commutation in the spherical and marked algebras. It follows that U is stable. A vector of U killed by A_w belongs to the false space at w and is orthogonal to that space. Positivity makes it zero. Finite dimensionality gives invertibility on U. Finally, $eV \cap \ker A_w$ is the orthogonal direct sum of Z and the false space at w, which proves the projection formula.

Each $T_{w,i}$ is a sum of translations with trivial component at ℓ, so it preserves the gauge lattice. The rank-normalized character has a continuous ℓ-adic avatar as in Corollary 45.17. Its values at good Frobenius elements are units by compactness. Thus α_w and q are ℓ-units,

and $A_w L \subseteq L$. The coefficient of a group element g in $M_x e$ is a sum of terms $\chi(1)\chi(g^{-1})$ for irreducible characters of Γ_x. These are algebraic integers. Every g acts on a marked frame by an invertible integral change and a root of unity. This also preserves the gauge when $\ell = p_i$, since the change and its inverse act on the integral weight lattice. The character formula now gives $M_x e L \subseteq L$. □

Theorem 52.5 (A bound with separate multiplicities for the competing systems)**.** *Let $\mathcal{W}$ be any nonempty finite set of the split places above. Split U into the joint eigenspaces U_ν of $(A_w)_{w\in\mathcal{W}}$, grouping equal tuples of eigenvalues. Put $d_\nu = \dim U_\nu$, $D = \dim Z$, and let $a_{\nu,w}$ be the eigenvalue on U_ν. With $v_\ell(\ell) = 1$ one has*

$$\lambda \;\le\; D v_\ell(M_x) + \sum_\nu \min(D, d_\nu) \min_{w\in\mathcal{W}} v_\ell(a_{\nu,w}).$$

All the eigenvalues in this sum are nonzero and integral. In particular the congruence index vanishes if $\ell \nmid M_x$ and each competing tuple has a unit entry, even if different places supply the units.

Proof. Enlarge the local field to define the finitely many eigenspaces, and denote its valuation ring by R, with uniformizer π. Use v_π and ordinary R-length until the last step. The commuting normal operators admit a joint eigenspace decomposition over the complex numbers. It transports to the enlarged local field. Integrality of their matrices on L makes all eigenvalues integral. Their nonvanishing on U follows from Proposition 52.4.

Put $B = L \cap eT$, $B_Z = L \cap Z$, $B_U = L \cap U$. Then $B_Z \subseteq PB \subseteq P(L \cap T)$, and

$$M_x P(L \cap T) \subseteq PB.$$

Indeed $Pe = P$, and $M_x e$ takes $L \cap T$ into B. The length of $P(L\cap T)/PB$ is at most $D v_\pi(M_x)$ by the elementary divisor theorem. The other quotient PB/B_Z is isomorphic to

$$C = \widetilde{B}_U / B_U, \qquad \widetilde{B}_U = (1 - P)B.$$

Both quotients identify with $B/(B_Z + B_U)$ by the kernel calculation of Theorem 52.2. In particular C is generated by at most D elements. For every $w \in \mathcal{W}$ one has $A_w \widetilde{B}_U \subseteq B_U$, because $A_w P = 0$ and $A_w B \subseteq B \cap U$.

Order the different tuples in any way and filter U by their successive direct sums $U^{(i)}$, starting with zero. Intersect $\widetilde{B}_U$ with these subspaces and take their images in C. In the quotient $U^{(i)}/U^{(i-1)} = U_{\nu_i}$ write S_i, S_i^0 for the images of the intersections of $U^{(i)}$ with $\widetilde{B}_U, B_U$, respectively. The i-th successive quotient of the filtration of C is S_i/S_i^0. Indeed a lift from $\widetilde{B}_U \cap U^{(i)}$ has image in S_i^0 exactly when subtracting a vector of $B_U \cap U^{(i)}$ moves it to $U^{(i-1)}$.

The two images are full lattices, so S_i/S_i^0 has at most d_{ν_i} generators. It also has at most D. To see this, pull back any submodule of C to R^D under a surjection onto C. The pullback is free of rank at most D, and passing to a quotient preserves this bound on generators. The inclusion $A_w \widetilde{B}_U \subseteq B_U$ on the i-th quotient reads $a_{\nu_i,w} S_i \subseteq S_i^0$. Its length is therefore at most $\min(D, d_{\nu_i})$ times the least of the $v_\pi(a_{\nu_i,w})$. Summing the lengths and dividing by $v_\pi(\ell)$ proves the estimate. The coefficient-extension assertion of Theorem 52.2 puts it back in the original normalization. □

Corollary 52.6 (The contribution of the marking on the period interval)**.** *For $D > 0$ the first term of Theorem 52.5 contributes at most $f_x r_x \log(p_0 p_1)$ to the sum of the logarithmic bounds per $D r_x$, uniformly in all j, embeddings and twists at this node. Its ratio to $B_x^{\det}$ is $O(x^{10}/(b_x(\log b_x)^2))$.*

Proof. There are f_x selected places above each p_i. Counting all matrices gives $\log M_x \le f_x r_x^2 \log(p_0 p_1)$. The sum of $v_\ell(M_x)\log \ell$ over primes is exactly $\log M_x$. Divide by r_x and use $\log(p_0 p_1) \le 22 \log b_x$ on the prescribed interval. In the budget of Proposition 52.1 we have $a_x = x b_x,\ m_x \le 2\sqrt{x}$, and $k_x, r_x \ll x^5$. Substitution gives the stated ratio. □

The second term of the theorem concerns only systems which survive the false-lift tests and for which every selected split Hecke value is divisible at ℓ. Controlling the sum of their weighted valuations on the period interval is a further arithmetic question.

52.3 The full cyclotomic string in the congruence module

Use the field, twist, ranks and marked idempotent of Section 52.2. On Z and on U now use the full collection of spherical Hecke coefficients at good split places away from ℓ. Write ξ for a distinct system on Z and ν for one on U. Their multiplicities in these spaces are D_ξ, d_ν, respectively. Thus $D = \sum_\xi D_\xi$. The individual theta lifts of spherical eigenvectors are spherical eigenvectors by Proposition 72.17. This makes Z stable under these coefficients and their adjoints. The kernels in the false-lift tests and the mark commute with them, so U is stable as well.

For either kind of system let u_σ be the representation of G_{E_x} from Theorem 45.16, tensored with the character of Corollary 45.17 having value α_w^{-1} at geometric Frobenius. Use stable lattices over a common finite local coefficient field. Put

$$P_\sigma(g;Y) = \det(Y - u_\sigma(g)), \qquad Q_g(Y) = \prod_{a=0}^{t-1} (Y - \kappa(g)^{-a}),$$

where κ is the cyclotomic character. For a competing system define

$$\delta_\nu = \min_{\substack{g \in G_{E_x} \\ 0 \le k < t}} v_\ell\Big([Y^k](P_\nu(g;Y) \bmod Q_g(Y))\Big), \qquad v_\ell(0) = +\infty.$$

The remainder is from division by a monic polynomial over the valuation ring. Multiplicities in Q_g are retained even when two or more of its roots coincide after reduction.

Proposition 52.7 (Integral Hecke operators for the remainder)**.** *The remainder of $P_\xi(g;Y)$ is zero for every true system and every g. For each ν the depth δ_ν is finite and is attained at a good split Frobenius. Each coefficient of the remainder at such a place is the eigenvalue of an integral Hecke operator which kills Z.*

Proof. At a good split place let g_w be geometric Frobenius. The elementary symmetric coefficients of $u_\sigma(g_w)$, with alternating signs, are the eigenvalues of

$$(-1)^i q^{i(i-1)/2} \alpha_w^{-i} T_{w,i} \qquad (0 \le i \le n).$$

This follows from split local compatibility and the normalization defining u_σ. These operators preserve the lattice by Proposition 52.4. The added roots on a true lift are $1, q, \dots, q^{t-1}$, and $\kappa(g_w) = q^{-1}$. Thus Q_{g_w} divides $P_\xi(g_w;Y)$. Monic long division expresses each remainder coefficient as an integral linear combination of the input polynomial coefficients with scalars in the ring generated by the coefficients of Q_{g_w}. Applying the same linear combination to the displayed operators gives the required integral operator.

We record why using split places loses no congruence information. The representations and κ, reduced at any fixed precision, factor through a finite Galois extension of E_x. Enlarge it to a finite Galois extension of E_x^+. An element of G_{E_x} induces an element fixing E_x in the latter finite group. By Chebotarev choose a prime of E_x^+ and a place above it with this geometric Frobenius, avoiding any fixed finite set. Its restriction to E_x is trivial, so the prime splits in E_x/E_x^+ and the relative residue degree is one. The Frobenius at the resulting place of E_x has the desired reductions. All remainder coefficients depend polynomially on the characteristic coefficients and on units $\kappa(g)^{-a}$. Their vanishing for a true system at good split places therefore implies vanishing on G_{E_x}.

For a competing system the evaluated operator A_w has nonzero eigenvalue at every such place by Proposition 52.4. This eigenvalue equals $q^{-n(t-1)} P_\nu(g_w; q^{t-1})$, and $Q_{g_w}(q^{t-1}) = 0$. At least one remainder coefficient is consequently nonzero. The finite valuations lie in a discrete

set bounded below by zero, so a minimum exists. Apply the preceding Frobenius argument one uniformizer power beyond a coefficient attaining it. This realizes the same minimum at a good split place. □

Identify systems whose residual representations have isomorphic semisimplifications over the algebraic closure of the residue field. For such a class b put $D_b = \sum_{\xi \in b} D_\xi$. For a true and a competing system put

$$e_{\xi\nu} = \min_{\substack{g \in G_{E_x} \\ 0 \le i < n}} v_\ell\Big([Y^i](P_\xi(g;Y) - P_\nu(g;Y))\Big).$$

This number is finite. Indeed equality of all coefficients at every good place would give the same eigenvalue of A_w, which vanishes on Z and is nonzero on U.

Theorem 52.8 (Two bounds inside each residual representation)**.** *For a competing system ν write $b = b(\nu)$ for its residual class. Define*

$$H_\nu = \min(D_b, d_\nu)\,\delta_\nu,$$
$$J_\nu = \sum_{\xi \in b} \min(D_\xi, d_\nu)\,e_{\xi\nu}.$$

In the sum for J_ν the index ξ runs through the true systems in b. The index of the tested projection satisfies

$$\lambda \;\le\; D v_\ell(M_x) + \sum_{\nu} \min(H_\nu, J_\nu).$$

In particular every system with $D_{b(\nu)} = 0$ contributes zero to the second term. This estimate uses all coefficient primes, including primes at which some of $1, q, \dots, q^{t-1}$ coincide in the residue field.

Proof. Use ordinary lengths and the uniformizer valuation over a common splitting field for the proof. At the end divide all lengths and valuations by $v_\pi(\ell)$. By the finite precision argument in Proposition 52.7, equality of all residual characteristic coefficients at good split places is equality on the whole Galois group. On a common finite residual quotient this is equivalent to equality of the semisimplifications by the Brauer–Nesbitt theorem. In its finite group formulation one can read this implication from the Brauer characters. The characteristic polynomial on each ℓ-regular element gives its eigenvalues with multiplicities, and the sum of their prime-to-ℓ root-of-unity lifts gives its Brauer character. Irreducible Brauer characters are linearly independent. This argument uses polynomial coefficients at every residue characteristic.

The projection onto each residual class in eT preserves $B = L \cap eT$. To see this select a distinguishing good split coefficient for each pair of distinct residual classes. For any one selected operator, factor the product over its distinct characteristic-zero eigenvalues of $Y - \lambda'$ according to the residues of λ'. Factors for different residues have unit resultant and hence are comaximal over the valuation ring. The Chinese remainder idempotents are integral polynomials in the operator. Their products for the finitely many selected operators give the claimed projectors. They commute with P: they preserve both Z and U in $eT = Z \oplus U$. They also preserve the corresponding intersections with L. Thus the module $C = PB/(L \cap Z)$ of the preceding theorem is the direct sum of modules C_b. Writing B_b for the projected B, each C_b has both descriptions

$$PB_b/(B_b \cap Z) \;\simeq\; B_b/((B_b \cap Z) + (B_b \cap U)) \;\simeq\; (1-P)B_b/(B_b \cap U).$$

Each isomorphism respects the action of all good spherical coefficients. In particular C_b needs at most D_b generators.

In the rightmost description filter C_b by ordering the competing systems and intersecting with their successive sums as in Theorem 52.5. Write G_j/G_{j-1} for the piece belonging to ν. It needs at most $\min(D_b, d_\nu)$ generators. Each remainder operator kills Z, so it carries $(1-P)B_b$ into $B_b \cap U$. On the successive lattices at ν it acts by the indicated remainder coefficient. Choosing a place and coefficient attaining δ_ν bounds the normalized length of this piece by H_ν.

To obtain the other bound, also filter the leftmost description of C_b by successive sums of the true eigenspaces. Denote this filtration by F_i. The subquotients

$$\frac{F_i \cap G_j}{(F_{i-1} \cap G_j) + (F_i \cap G_{j-1})}$$

filter G_j/G_{j-1} as i varies. Indeed the image of $F_i \cap G_j$ in that quotient has inverse image of the previous step equal to the displayed denominator. By the same argument with the indices interchanged, each displayed module is a subquotient of F_i/F_{i-1} as well. It therefore needs at most $\min(D_\xi, d_\nu)$ generators when i corresponds to ξ. Here we use the elementary fact about subquotients of a module over a discrete valuation ring proved in Theorem 52.5.

On F_i/F_{i-1} each good coefficient acts by its value at ξ, as one sees by projecting its defining lattices to the i-th eigenspace. On G_j/G_{j-1} it acts by its value at ν. Their difference annihilates the common subquotient. The least valuation of these differences at good split places is $e_{\xi\nu}$, by the same finite precision argument. The elementary divisor theorem bounds its normalized length by $\min(D_\xi, d_\nu)e_{\xi\nu}$. Summing first over i bounds the length of the fixed piece by J_ν. Use the smaller of the two bounds on each piece and sum. Passing from PB to $P(L \cap T)$ costs at most $Dv_\ell(M_x)$ as in Theorem 52.5. □

Corollary 52.9 (A test at a repeated cyclotomic root)**.** *If $\bar\kappa(g) = 1$ for some $g \in G_{E_x}$ and the algebraic multiplicity of 1 on $\bar u_\nu(g)$ is less than t, the corresponding piece of C is zero. More generally it is zero whenever $P_\nu(g;Y) \bmod \pi$ is not divisible by $Q_g(Y) \bmod \pi$ for some g.*

Proof. Reduction commutes with monic long division. A nonzero remainder in the residue field gives $\delta_\nu = 0$. If $\bar\kappa(g) = 1$, the divisor is $(Y-1)^t$, giving the asserted multiplicity test. □

The theorem leaves only residual classes that occur in both the true lifts and the tested complement. A sum bound of the required size for their positive depths is still needed on the common interval.

52.4 The long character block in the definite target

Return to the higher-rank descent pairs on the shorter period interval. Their ranks are $s = r_x$ and $n = s + t$, where $t = 5s$. At a split place use unitary Satake parameters in this subsection unless the arithmetic normalization is specified. A block (ρ, b) of the discrete-spectrum base change consists of a unitary cuspidal representation ρ of GL_a over E_x together with its b norm twists. It has rank ab. At an unramified place its entries are the parameters of ρ multiplied in turn by $q^{(b-1)/2-i}$, $0 \le i < b$.

Theorem 52.10 (A character block of exact length on every true lift)**.** *The base change of every true system ξ in the marked definite target contains the block (μ_W, t) exactly once. All the other blocks together have rank s. At every coefficient prime there is a continuous semisimple representation R_ξ of G_{E_x} of rank s for which*

$$u_\xi \simeq R_\xi \ \oplus \bigoplus_{a=0}^{t-1} \kappa^{-a}.$$

Consequently every competing residual semisimplification that contributes to Theorem 52.8 contains the indicated residual characters as composition factors with their full multiplicities.

Proof. Apply the cohomological base change and the description of the discrete spectrum in the proof of Theorem 45.16 to the definite target. It expresses the good local parameters as the union over the global blocks just defined. The unramified parameters of a unitary cuspidal GL_a representation have modulus strictly less than $q^{1/2}$. For $a = 1$ their modulus is one. For $a > 1$ one may use the bound of Luo, Rudnick and Sarnak with exponent $1/2 - 1/(a^2+1)$, as in Proposition 71.17. [5]

At all good split places Proposition 72.17 supplies the entry $\mu_{W,w}(\varpi_w)q^{(t-1)/2}$. The other added entries have smaller modulus. The old entries are the Satake entries of the source up to a unitary character and the possible contragredient convention. Their moduli lie between $q^{-(s-1)/2}$ and $q^{(s-1)/2}$: the source is unitary, and the split spherical estimate in Lemma 75.1 applies to its local component. Equivalently each block in the spherical unitary classification of Lemma 45.1 has length at most s. A block shifted by a complementary exponent $0 < |a'| < 1/2$ has a partner of the same length, so its length is at most $s/2$. Both descriptions give the asserted upper bound and, by duality, the lower bound. Thus the largest modulus in the target is exactly $q^{(t-1)/2}$.

If $b \le t-1$, every entry of (ρ, b) has modulus strictly less than $q^{(b-1)/2}q^{1/2} \le q^{(t-1)/2}$. A block giving the largest entry must have $b \ge t$. Since $n = s+t < 2t$, there can be only one such block and its cuspidal degree is one. Write ω for its unitary Hecke character. Its top entry has modulus $q^{(b-1)/2}$, so $b = t$. Every other block has rank at most s. The top entry of this one block must therefore be the known entry of the true lift. We get $\omega_w(\varpi_w) = \mu_{W,w}(\varpi_w)$ at all good split places.

For completeness, put $\eta = \omega/\mu_W$. At split places outside a finite set the Euler factors of $L_E(z, \eta)$ and $\zeta_E(z)$ agree. At an inert place over v of E_x^+ the norm in E_x is $(Nv)^2$. Hence the Euler product for their quotient over the inert places converges absolutely to a nonzero value near $z = 1$, by comparison with $\sum_v (Nv)^{-2\,\mathrm{Re}\,z}$. The finitely many remaining factors are finite and nonzero at $z = 1$. Thus $L_E(z, \eta)$ has a pole at one, and the pole theorem for unitary Hecke characters gives $\eta = 1$.

The Galois representation in Theorem 45.16 is constructed as a direct sum over the blocks of this base change. The character block contributes a sum of one-dimensional representations by class field theory. Its eigenvalues at geometric Frobenius, after the common multiplication

by $q^{(n-1)/2}\alpha_w^{-1}$ defining u_ξ, are $1, q, \dots, q^{t-1}$. They give exactly the direct sum of the indicated cyclotomic powers, by density of split Frobenius and characteristic-zero semisimplicity. The other blocks give R_ξ. Reduce a stable direct sum of lattices in these summands. The semisimplified reduction is independent of the lattice by the characteristic-polynomial form of Brauer–Nesbitt. Theorem 52.8 now gives the final assertion, also when several cyclotomic powers reduce to the same character. □

Lemma 52.11 (The unique long string at an infinite place)**.** *At every infinite place the holomorphic infinitesimal parameter of the definite target has exactly one string of t consecutive entries of spacing one. That string cannot be extended at either end.*

Proof. Fix a complex label above the given place and put $h = s^8,\ k = s+1-m_x,\ a = k+j-1$, with indexing of the source entries starting at zero. At this label write $w_i = w_0^{\mathrm{base}} + (2i+1)h$, where w_0^{base} is the prescribed nonnegative base weight. The s entries σ_p of the row parameter, in decreasing order, are

$$\sigma_p = \begin{cases} k/2 - 1 - p, & 0 \le p < k-1, \\ h - w_{p+2-k}, & k-1 \le p < s. \end{cases}$$

Up to a common additive shift, its target list is obtained by inserting, after the first a entries,

$$u_i = -w_j + (t-1)/2 - i, \qquad 0 \le i < t.$$

Here is a verification from the weight columns. The $\sigma_p + w_j$ for $p < k-1$ are at least h, since $w_j \ge 3h$ and $k \le s$. On the second range these values are $(1 + 2(j - (p+2-k)))h$. They are at least h for $p < a$ and at most $-h$ for $p \ge a$. In particular the indicated insertion is already in decreasing order. At a column height p with $1 \le p < n$, the defining positive-column count uses the drop from σ_{p-1} to the larger of σ_p and u_0, minus one, truncated at zero. The negative-column count is read at the reflected height $n - p$; in the lower part it uses the drop from the smaller of the preceding old entry and u_{t-1} to the next old entry, minus one, again truncated at zero. In particular, above and below the insertion the counts are the old successive differences minus one. At height a the count is $\sigma_{a-1} - u_0 - 1$. At height $a + t < n$ it is $u_{t-1} - \sigma_a - 1$. At the interior heights $a < p < a + t$ both counts are zero. These are exactly the consecutive differences of the inserted list minus one. For a highest weight in rank n, consecutive differences of the Harish-Chandra parameter equal the column counts plus one, from adding the half sum of positive roots. This verifies the list up to translation, including when the insertion ends the list. The opposite complex label reverses and negates the list, preserving the assertion about strings.

The distance between the inserted block and either adjacent old entry, when present, is at least $h - (t-1)/2 > 1$ for the ranks in the interval. There are only $s < t$ old entries in total, so they cannot supply a second string of length t. □

Theorem 52.12 (A split-place bound on the entire tested complement)**.** *Assume $Z \neq 0$. Every global block (ρ, b) of every system on $U = eT \cap Z^\perp$ has $b < t$. At each good split place w as in Section 52.2 and each such system the eigenvalue in Proposition 52.4 satisfies, in the complex realization of the data,*

$$(1 - q^{-1/2})^n \ \le\ |a_{\nu,w}| \ \le\ (1 + q^{-1/2})^n.$$

In particular $1/2 \le |a_{\nu,w}| \le \sqrt{e}$ if $q \ge 4n^2$. The bounds are independent of the lift index and of the twist.

Proof. A block of length $b \geq t$ has degree one since $n < 2t$. Call its unitary Hecke character ω. At a complex place its holomorphic infinitesimal entries are consecutive integers apart and centered at the holomorphic exponent of ω: the local norm is $z\bar{z}$, and the b twists have exponents $(b-1)/2 - i$. All the forms in the level space have the given archimedean weight. By Lemma 52.11 this forces $b = t$. Choose a true system in Z. Its long block in Theorem 52.10 supplies the unique string at this same weight. Thus the holomorphic exponents of ω and μ_W agree at every infinite place. For a unitary character of $\mathbb{C}^\times$ written $z^p \bar{z}^{p'}$, one has $p' = -\overline{p}$, by evaluating on positive real numbers and on the unit circle. Consequently their quotient is trivial at infinity.

This quotient is a finite order Hecke character: it factors through the ray class group of a finite modulus because its archimedean component is trivial. Take a good split place which splits completely in the normal closure over E_x^+ of its class field. At this place $\omega_w = \mu_{W,w}$ and the whole predicted string occurs. In particular A_w kills the system. This contradicts its invertibility on U and proves $b < t$.

Let z_i be a unitary Satake entry on such a system. A character block of length $b < t$ gives $|z_i| \leq q^{(t-2)/2}$. For a block of cuspidal degree at least two one has $b \leq n/2 = 3s$, and the strict cuspidal bound in the first theorem gives $|z_i| < q^{b/2} \leq q^{(t-2)/2}$ since $3s \leq 5s - 2$. Now the eigenvalue of A_w in unitary parameters is

$$\prod_{i=1}^{n} \left(1 - \frac{z_i}{\mu_{W,w}(\varpi_w) q^{(t-1)/2}}\right).$$

Each fraction has absolute value at most $q^{-1/2}$. The triangle and reverse triangle inequalities give the two bounds. If $n/\sqrt{q} \leq 1/2$, Bernoulli's inequality gives the last lower bound and $(1 + q^{-1/2})^n \leq \exp(n/\sqrt{q})$ gives the upper. □

These estimates identify the characteristic-zero blocks on the period fields. The positive finite-place lengths between distinct systems in a shared residual class still enter Theorem 52.8.

52.5 Decomposition of the tested index by residue type

Fix the original node, twist, lift index j, coefficient prime ℓ, and embedding in Section 52.1. Let $\Gamma = \Gamma_x$ be the finite group acting on the marked frames, and $M_x = |\Gamma|$. Over a finite splitting coefficient field write $e = \sum_{\omega\in\Omega} e_\omega$ for the marking idempotent, decomposed into its distinct irreducible character idempotents. A character of this product group fixes a tuple of residue types at the original level places. All lattices below are intersections with the original integral gauge L at the prescribed level.

Theorem 52.13 (The correction for splitting the marked types)**.** *Let T be the tested subspace, Z the true lift range and P its orthogonal projection, transported to the local coefficient field as in Theorem 52.2. Put $T_\omega = e_\omega T$, $Z_\omega = e_\omega Z$, $D_\omega = \dim Z_\omega$, $D = \dim Z$. Let λ be the normalized length in that theorem and let λ_ω be the corresponding index for $P : T_\omega \to Z_\omega$. Both use the same embedding and gauge. Then*

$$\left|\lambda - \sum_{\omega\in\Omega} \lambda_\omega\right| \le D\, v_\ell(M_x), \qquad v_\ell(\ell) = 1.$$

In particular equality $\lambda = \sum_\omega \lambda_\omega$ holds at every $\ell \nmid M_x$. For $D = 0$ all these lengths are zero.

Proof. The frame action is unitary on the definite space. It preserves Z by equivariance of the lift. The good Hecke operators in the false-lift tests commute with the frame action, and the marking idempotent is central. Each false space $eV \cap \ker A_w \cap Z^\perp$, and hence its orthogonal complement, is stable under Γ. This proves stability of T. In particular P commutes with each e_ω and $Pe = P$. The frame changes and their inverses preserve the integral gauge even at $\ell = p_0, p_1$, by the proof of Proposition 52.4. Since

$$M_x e_\omega = \sum_{g\in\Gamma} \chi_\omega(1)\chi_\omega(g^{-1})g$$

has algebraic-integer coefficients, $M_x e_\omega L \subset L$. The coefficient extension used to split Γ preserves normalized lengths of the original lattices by flatness and the elementary-divisor calculation of Theorem 52.2.

In Z write $A = P(L \cap T)$, $B = L \cap Z$. In Z_ω put $A_\omega = P(L \cap T_\omega)$, $B_\omega = L \cap Z_\omega$. Define the direct sums $A_* = \bigoplus_\omega A_\omega$ and $B_* = \bigoplus_\omega B_\omega$. They have full rank D. We have $B \subset A$, $B_* \subset A_*$, and

$$M_x A \subset A_* \subset A, \qquad M_x B \subset B_* \subset B.$$

For the first inclusion take $a = Pt$ with $t \in L \cap T$. Then $M_x a = \sum_\omega P(M_x e_\omega t)$, with $M_x e_\omega t \in L \cap T_\omega$. For the second chain apply the same identity directly to $b \in L \cap Z$. The remaining inclusions follow by summing integral points. These arguments allow T to contain the other marked components on which P vanishes.

Write ℓ_p here with $p = \ell$ for normalized local length. Compute the length of A/B_* along the two chains through A_* and through B. This gives the exact identity

$$\lambda = \sum_\omega \lambda_\omega + \ell_p(A/A_*) - \ell_p(B/B_*).$$

Each of the last two quotients is generated by at most D elements and is killed by M_x. Both lengths therefore belong to $[0, Dv_\ell(M_x)]$, proving the claim. In particular the correction involves the dimension of the true range only once. □

Corollary 52.14 (The type comparison on the original period interval)**.** *For $D > 0$ set $g = \lambda \log \ell/(Ds)$, $s = r_x$, and set $g_\omega = \lambda_\omega \log \ell/(D_\omega s)$ when $D_\omega > 0$. For every choice of data at that prime,*

$$\left| g - \sum_{D_\omega > 0} \frac{D_\omega}{D} g_\omega \right| \le \frac{v_\ell(M_x) \log \ell}{s}.$$

The sum of these error bounds over all coefficient primes at a node, including both level primes, divided by $B_x^{\det}$, is $O_{q_0,d}(x^{10}/(b_x(\log b_x)^2))$. Consequently the sum over primes of the interval supremum of these normalized errors is $O_{q_0,d}(V_0^{-389})$.

Proof. The dimension weights sum to one by the direct decomposition of Z. Multiply the theorem by $\log \ell/(Ds)$. Its right-hand side is independent of the trace level, the coefficients and their embedding. Since $\sum_\ell v_\ell(M_x) \log \ell = \log M_x$, the node cost is bounded by $(\log M_x)/s \le f_x s \log(p_0 p_1)$; this is the group order computation of Corollary 52.6. There $\log(p_0 p_1) \le 22 \log b_x$, $B_x^{\det} = f_x x b_x (\log b_x)^3/(m_x k_x)$, $m_x \le 2\sqrt{x}$ and $k_x, s \ll x^5$. Substitution gives the stated node bound. As $b_x \ge x^{400}$, it is $O_{q_0,d}(x^{-390})$. Bound the nonnegative supremum at each prime by the sum over eligible nodes, interchange sums and use $\sum_{x \ge V_0} x^{-390} \ll V_0^{-389}$. □

This identity permits a separate local trace family for each fixed tuple of residue types. Its dimension weights must still be used at the same node and embedding as the tested projection. The spherical cohomology estimate in Theorem 17.28 is uniform over those choices. An estimate for the indices within the individual types is still needed to bound the full theta projection.

52.6 The integral projection and the Selmer group of a true system

Fix a node x, a lift index j, a coefficient embedding at ℓ, and its auxiliary twist. Write $s = r_x,\ t = 5s,\ n = s + t$. Use Z, T, L, P, λ from Theorem 52.2. The finite group of frame changes has order M_x. For an individual marked tuple ω put $e = e_\omega$. At an auxiliary trace level write M for the integral lattice of forms, and use the lattice $M_\omega = M \cap eM[1/\ell]$ in this paragraph. We work over a finite coefficient field splitting the finite data in use.

Lemma 52.15 (Averaging on the lattice of a single type)**.** *The descent of Theorem 45.20 and Corollary 45.21 applies on M_ω using $C_* = M_x c_0$ in place of c_0, where $c_0 = \gcd_i |\mathrm{GL}_n(\mathbb{F}_{p_i})|$. The trace family in this descent can consist just of the systems occurring in the ω-component at the required levels.*

Proof. The auxiliary residue group acts at places disjoint from both original levels and commutes with the old frame group. Each spherical Hecke sum away from these places commutes with both. The local translations, the weighted auxiliary norms and the finite sums defining the parahoric operators at the auxiliary places commute with e. Their restrictions which map between integral level lattices therefore map between the corresponding intersections with $eM[1/\ell]$. Congruences restrict at full precision because M_ω is saturated in M. Base invariants in M_ω are exactly the intersection of the original base lattice with the same component. Likewise all scalar Hecke families used in the construction on this component act integrally there. For the characteristic coefficients at a general Galois element this follows by approximation by good split Frobenii: the restricted integral operator algebra is a finite submodule of the endomorphisms of M_ω, hence closed. This is the integral realization used in Corollary 45.17.

The auxiliary orbit endomorphisms need separate attention. For every subgroup H used in Proposition 45.15, its orbit construction on M gives Ψ_H with $\sum_{h\in H} h^{-1}\Psi_H h = c_0\, 1_M$. The same c_0 works for the subgroup since its point stabilizers are subgroups of the stabilizers in Lemma 45.18. Use

$$\Psi_{H,\omega} = M_x e \Psi_H|_{M_\omega}.$$

This is an integral endomorphism by the character idempotent formula in Theorem 52.13. Its conjugation sum is $C_* 1_{M_\omega}$. In particular $N\Psi_{\Gamma,\omega} z = C_* z$ on base invariants, for the auxiliary group Γ and its norm N. These conjugation sums are exactly the orbit identities used for the principal parahoric and the full group in Propositions 45.15 and 45.19. The integral Hecke operator on the indicated parahoric fixed lattice also commutes with e there, so the same divided difference and right-translation identities apply to the intersections. In the induction for $Y_{S,i}$ use the new orbit operators and $\delta = C_*^2 D$. Every step then takes place on M_ω. The divisibility condition becomes $(n_0 C_*^2)^{|J|+1} \mid m'$. The finite base systems, and therefore n_0, are fixed before choosing the precision. Increasing it achieves this condition. In the last division of Theorem 45.20 these powers cancel. Its integral conclusion incurs just the factor C_*. □

For a true system ξ in $Z_\omega = eZ$ write $d_{\omega,\xi}$ for its multiplicity as a dimension of the joint eigenspace in Z_ω. Its actual normalized representation is $R_\xi \oplus \bigoplus_{a=0}^{t-1} \kappa^{-a}$ by Theorem 52.10. For a stable lattice B in R_ξ put $A_B = B[1/\ell]/B$. Denote by S_ξ the normalized length of the subgroup of $H^1(E_x, A_B)$ with the following local conditions. At $v \mid \ell$ use the image of $H^1_f(E_{x,v}, B[1/\ell])$. At the original levels and the twist places away from ℓ allow arbitrary classes. At every other finite place require an unramified class. An infinite length is given value $+\infty$. Here the two original levels and the twist are those of this very projection.

Proposition 52.16 (Clearing the projection on one marked component)**.** *Let $D^{\mathrm{loc}}_{x,\ell}$ be the supremum of the sum of the finite-precision local defects of Section 17.2 over $v \mid \ell$ and the $t+1$ target summands, for individual marked families. It may include a separate choice of designated*

true system, precision and auxiliary trace level for each value of the supremum. Write Z_x for the positive integer of Section 50.4, with $t_0 = t$, and let

$$E_{\mathrm{tw}} = t \sum_{w \in Q \cup Q^c} v_\ell(1 - \zeta_w)$$

at the new twist places, with the notation of Corollary 50.10. If $D = \dim Z > 0$, then

$$\frac{\lambda}{D} \ \le \ \sum_{\omega,\xi} \frac{d_{\omega,\xi}}{D}\, S_\xi + D^{\mathrm{loc}}_{x,\ell} + v_\ell(Z_x) + E_{\mathrm{tw}}.$$

Only pairs with positive multiplicity enter the sum. All local lengths and valuations have $v_\ell(\ell) = 1$.

Proof. Fix ω, ξ and suppose the Selmer lengths appearing on the right are finite. At an original prime $q \neq \ell$ the tuple has a primitive member at some place. The first lemma of Section 17.13 determines the smooth type there just as at a coefficient place. Split-place compatibility away from ℓ in Theorem 45.16 identifies it with actual inertia, since monodromy is zero. The t trivial characters on inertia are precisely the ones contributed by the Tate lines. Thus R_ξ has the primitive degree-s orbit. The auxiliary twist is unramified at this place. Theorem 47.36 supplies a tame element η acting by a scalar y with $y-1$ a unit, and the full integral matrix span on B. It also shows that any stable lattice is homothetic to B: the integral diagonal matrix units split it into fractional ideals and the other units identify these ideals. Its Selmer length is consequently independent of this choice. These scalar hypotheses hold for every true system in the component. Corollary 47.37 supplies the auxiliary primes for the designated system with unit determinant at the chosen Frobenii, at any prescribed precision. The common polynomials and the two trace identities at each pair are then those of Theorem 45.3 and Corollary 45.17.

Take the row construction of Theorem 45.11 on this family. We record the finite-level estimate on its main determinant. In Proposition 45.7 the annihilator of the row lattice, at precision ℓ^ν, maps injectively to normalized raw cocycles with values in $A_B[\ell^\nu]$. They vanish at η and at the two auxiliary tame generators at each trace pair. At an auxiliary trace place the target is unramified. Its restriction to inertia is an additive homomorphism on the pro-ℓ tame quotient and vanishes on its generator. At any further good place the matrix relations themselves force the inertia values to vanish, since all the matrices there act trivially. Localize these cocycles at the coefficient places and divide their images by the reductions of the integral finite local conditions. The kernel maps into the Selmer group defining S_ξ by division by ℓ^ν. A coboundary in A_B vanishing at η is zero because $y - 1$ is a unit. The map on this kernel is therefore injective. Its quotient has normalized length at most the sum of the local defects for B. For a scalar target, the quotient defining l_a in Theorem 45.8 injects into the corresponding sum of local quotients by restriction. Indeed these integral evaluation equations characterize the reductions of local finite cocycles by Lemma 12.27, and raw coboundaries over the finite coefficient ring lift integrally. All these estimates use the same trace family, including its precision. Their total is bounded by $S_\xi + D^{\mathrm{loc}}_{x,\ell}$.

The scalar row costs at the original levels away from ℓ are at most $2v_\ell(G_0)$ by Corollary 45.24. The new twist places contribute E_{tw}. The class and auxiliary terms of Theorem 45.11 are bounded by

$$v_\ell(h_E \mathcal{J}_{N,t}) + (4t+3)(N+t+1)^6 v_\ell(G_0), \qquad N = q_{1,x} d,$$

by Corollary 12.10. These are the remaining costs at the designated eigenvalue of the top trace coefficient.

Use Corollary 45.21 on the intersection lattice of the lemma. For $l \in L \cap eT$ take $z = Pl$. It is supported on the true systems. On any base system with a trivial Galois composition

factor, the arithmetic Hecke polynomial at every good split place has its root α_w, because the normalized Galois polynomial has root one there. Since $|\alpha_w|^2 = q_w^s$, Lemma 45.1 gives the root $\alpha_w q_w^{t-1}$ as well. Thus A_w of Proposition 52.4 kills that eigenspace. The projection formula there gives equality of the components of l and z on this system. More explicitly, the joint eigenspace projector commutes with e and P, and its component of $l - z$ belongs to the tested space, to $Z^{\perp}$ and to the kernel of A_w. Their intersection in eT is zero. This verifies the base agreement even in an eigenspace of dimension greater than one. The lemma and base separation now give a diagonal operator s_ξ on Z_ω such that $C_* s_\xi P l \in L \cap Z_\omega$ for all such l. At the designated eigenspace its eigenvalue has valuation at most the top coefficient bound just obtained. With integral l already in the component the factor h in that corollary is one.

Repeat the construction for each distinct true system on Z_ω, allowing its own trace precision. The finitely many operators are defined over one finite extension. One integral linear combination simultaneously attains the minimum of their eigenvalue valuations at each true system: divide the coefficients of each resulting linear form by one of minimal valuation, then choose the combining scalars with residues outside the finitely many zero hyperplanes. A finite unramified extension supplies such residues because their union is proper over the algebraic closure of the residue field. The new operator still clears the same projected lattice after multiplication by C_*. Its determinant valuation is the sum of its eigenvalue valuations with multiplicities. The exact minimum in Theorem 52.2 bounds λ_ω by this value, with the factor C_* counted $\dim Z_\omega$ times. Coefficient extension preserves the index. Sum over the tuples and apply Theorem 52.13. The additional normalized cost per dimension is at most $v_\ell(C_* M_x)$. Since $s, n \le st$, block embeddings of the finite general linear groups give $M_x \mid G_0$ and $c_0 \mid G_0$. Thus $C_* M_x \mid G_0^3$. The two old level rows and these integrality factors together cost at most five powers of G_0, included in Z_x. This proves the result. If any relevant global length is infinite the inequality holds in the extended nonnegative reals. □

Theorem 52.17 (Comparison on the period interval)**.** *On the marked levels of Corollary 17.34, write $\overline{S} = \sum_{\omega,\xi} d_{\omega,\xi} S_\xi / D$ for a positive-dimensional true range with finite indicated lengths. The sum with weights $\log \ell$ of the primewise suprema of*

$$\frac{(\lambda/D - \overline{S})_+}{s B_x^{\det}}$$

on the shorter interval is $O_{q_0,d}((\log(2V_0))^{-2})$. The supremum uses the same node, lift index, embedding and twist for the two terms of each difference. Use zero for an empty supremum or a zero range.

Proof. The coefficient places in these levels are the spherical ones and the two original levels. Indeed the compact level imposes the frame conditions at the original places and parahoric conditions at the twist places. Away from these it uses the standard maximal compact for the self-dual lattice. The parity factor in the normalizing splitting character is unramified on finite units. The twist pairs and the later trace pairs are chosen away from ℓ. At an inert place E_x/E_x^+ is unramified, so the standard self-dual lattice there gives a hyperspecial compact, as in Lemma 17.4. Corollary 17.34 therefore gives precisely the required estimate for $D_{x,\ell}^{\mathrm{loc}}$.

At each node $\sum_\ell v_\ell(Z_x) \log \ell = \log Z_x$. The calculation in Theorem 50.11 gives $\log Z_x \ll_{q_0,d} x^{110}(1 + \log b_x)$. Here $B_x^{\det} = f_x x b_x (\log b_x)^3/(m_x k_x)$, $m_x \le 2\sqrt{x}$, $k_x \le s+1$, $f_x \ge 1$ and $b_x \ge x^{400}$. Consequently $\log Z_x / (s B_x^{\det}) \ll_{q_0,d} x^{-289}$. The sum of the normalized twist-place costs with weights $\log \ell$ at each node can be taken at most one by Corollary 50.10, during the same choice as for the critical ratios. For these last two nonnegative errors replace each primewise supremum by the sum over eligible integer nodes. The resulting tail is $O_{q_0,d}(V_0^{-288})$. Combining the preceding proposition with the local bound proves the comparison. □

To obtain a bound for the projection itself one must still control $\overline{S}$ for the actual true systems with the twists used for the periods. A bound for Selmer groups of independently chosen twisted representations would not verify that condition.

52.7 The source representation and the common period denominator

At a node of Corollary 12.10 let $s = r_x$, $t = 5s$ and $n = s + t$. The index j runs from 1 to m_x. The character of the norm-one torus used in the j-th auxiliary lift before changing the finite level will be denoted by μ_j. For any character ν of this torus put $\widetilde{\nu}(a) = \nu(a/\bar{a})$ on the ideles of $E = E_x$. Write ϑ_ℓ for the Galois character of $\widetilde{\vartheta}$ when ϑ has finite order. We use geometric Frobenius, so $\kappa(\mathrm{Frob}_w) = q_w^{-1}$ for $w \nmid \ell$, where q_w is the residue cardinality. In particular the unramified norm character $|\cdot|_E$ has avatar κ.

Proposition 52.18 (The main summand of an actual twisted lift)**.** *Fix a coefficient prime and embedding. There is a finite set T_* of auxiliary primes and a finite list of rank-s representations $V_{\pi,j}$, both chosen before selecting the twists for the Selmer estimates, with the following property. If the conductor of ϑ avoids T_* and the lift using $\mu_j\vartheta^j$ has a true system ξ, its summand in Theorem 52.10 satisfies*

$$R_\xi \simeq V_{\pi,j} \otimes \vartheta_\ell^{-j}$$

for some member of the list. Each $V_{\pi,j}$ is de Rham at the coefficient places and unramified away from these places and the two original levels. Its polarization is $V_{\pi,j}^c \simeq V_{\pi,j}^\vee \kappa^{1-t}$.

Proof. For index j take the actual source on W_j. Its fixed archimedean discrete series and finite principal level are independent of the twist; the marking is an idempotent in its finite level group. There are finitely many spherical systems π in this source. Indeed the space of multiplicities at this level for the prescribed real type is finite dimensional by Harish-Chandra finiteness. As in the proof of Proposition 71.12, evaluation on a cyclic compact-finite vector realizes it in the space of automorphic forms at fixed level, infinitesimal character and compact type. The center is the norm-one torus and has compact adelic quotient. The commuting spherical Hecke algebras are closed under adjoints on the unitary discrete spectrum, so their restrictions to the finite multiplicity space are simultaneously diagonalizable. The frame idempotent commutes with them. Linearity of the lift therefore expresses its range as a sum of images of the marked source eigenspaces. The spherical local theta correspondence specifies the good split parameters on each nonzero image by Proposition 72.17. This argument uses W_j for each j, including its indefinite signatures.

At a good split place let B_w denote the Satake multiset of a fixed source system π in unitary normalization. Write χ_S, χ_T for the two splitting characters and evaluate them at the chosen uniformizer. Here $\chi_T = \mu_j\vartheta^j(w)$. Indeed, if υ is the fixed splitting character, the correction on the norm-one torus is $(\upsilon|_{U(1)})^{n/2}$. Its pullback is υ^n, since $\upsilon^c = \upsilon^{-1}$. The definition of the rank-n space character multiplies by υ^{-n}, giving the asserted equality. The good-place parameter formula of Proposition 72.17 now gives the multiset

$$(\chi_S/\chi_T)B_w \ \sqcup\ \chi_S\{q_w^{-(t-1)/2+i} : 0 \le i < t\}.$$

The arithmetic normalization in Corollary 45.17 multiplies this list by $q_w^{(n-1)/2}/(q_w^{s/2}\chi_S)$. The second part becomes $\{q_w^i : 0 \le i < t\}$. Cancelling the Tate factors using Theorem 52.10, the eigenvalues of $R_\xi \otimes \vartheta_\ell^j$ are precisely

$$q_w^{(t-1)/2} B_w/\widetilde{\mu}_j(w)$$

in the chosen embedding. Thus for fixed π, j any two nonzero lifts, possibly for different twists and different marked vectors, give the same semisimple representation after this character twist. These split places suffice for equality of continuous semisimple representations over E. In any finite Galois quotient over E, take a Galois closure over E^+ and lift a specified element to the subgroup fixing E. Chebotarev over E^+ gives a prime split in E and a choice of

place with Frobenius in the prescribed class. Finitely many excluded places do not affect this argument. Equality of traces on this dense set gives the asserted isomorphism by characteristic-zero Brauer–Nesbitt.

We explain how to arrange unramifiedness outside the old set without requiring a nonvanishing statement at the trivial twist. Consider the finite list of pairs (π, j). Use as candidate twists all the permitted m_x-fold finite-order torus characters supported at new split pairs, prime to the old levels and trivial at the places above ℓ. For each pair which gives a nonzero lift at some candidate, choose one such witness $\beta_{\pi,j}$ and a true system in its image. Let T_* be the finite union of the rational primes in the conductors of these witnesses. Retain a pair if it has a witness $\delta_{\pi,j}$ whose conductor avoids T_*, choosing one such witness. Every true system for a final twist avoiding T_* comes from a retained pair, by the definition of retention.

For a retained pair take $V_{\pi,j}$ to be the common semisimple representation just identified using its two witnesses. The true summand in each definite target is de Rham and polarized with exponent $1 - t$ by Proposition 17.1 and its proof. Finite-order twisting preserves the de Rham condition and anticyclotomic twisting preserves the polarization. Fix a place away from ℓ, p_0, p_1. At least one of the two witnesses is unramified throughout its rational residue characteristic, since their conductors are disjoint. For that witness the target is spherical there and its normalizer unramified. Its local base change is unramified at a split place by Theorem 45.16, and at an inert place by the hyperspecial base change used in Lemma 17.4. The local parameter has zero monodromy, so compatibility away from ℓ gives an unramified Galois representation. Its summand and its unramified twist are unramified. This proves the assertion at every such place for $V_{\pi,j}$. □

Lemma 52.19 (One old inertia test for the entire list)**.** *Let $G_0 = \prod_{i=0}^{1} |\mathrm{GL}_{st}(\mathbb{F}_{p_i})|^f$, where $f = [E^+ : \mathbb{Q}]$. For the source list of the proposition, the constants B_i of Theorem 12.32 are at most $(4s + 1)v_\ell(G_0)$. Its distinguished generator g_* may be taken at any one prescribed old marked place away from ℓ, simultaneously for this list and the torus tests of Theorem 50.6.*

Proof. Use a true system for a witness of the preceding proposition. Choose a marked tuple in which it has a nonzero component. For each old place over $p_i \neq \ell$, Theorem 17.33 and its proof identify the smooth target parameter there. Its old-frame calculation uses only the prescribed tuple, the old determinant character and the fact that the witness is unramified at the old primes, all of which hold for the candidates used above. The parameter has zero monodromy, the s shifted roots of the degree-s label c, and t copies of the trivial character. The shift is by a character of order dividing Q of Section 47.12. An orbit of size greater than one keeps its size, and no allowed degree-one root has order dividing Q. Thus all s shifted roots are nontrivial. The full t-dimensional trivial part on inertia is already accounted for by the Tate lines. Split-place compatibility away from ℓ and the vanishing of monodromy identify the other roots with the actual inertia eigenvalues of the true summand. Twisting back by the witness has no effect on inertia at the old places. At a primitive member of the tuple the shifted root remains primitive by Lemma 47.35. The conjugate place has the inverse inertia eigenvalues, by polarization. Each tuple has a primitive member above each of p_0, p_1. For every representation choose such a member away from ℓ. Theorem 47.36 gives an inertial scalar y with $y^2 - 1$ a unit and the full matrix span on a stable lattice. This marked scalar may depend on the representation. The possibly different generator g_* in the lemma is common: every eigenvalue at every old place is nontrivial, including at $(g_*^c)^{-1}$. The rank-one tests also have this property there by Theorem 50.6.

Put $g_i = |\mathrm{GL}_{st}(\mathbb{F}_{p_i})|$. A root $\lambda \neq 1$ from an orbit of degree $e \leq s$ has order dividing $p_i^e - 1$, also after the scalar shift, since $Q \mid p_i - 1$. The polynomial $X^{p_i^e - 1} - 1$ splits into roots of unity.

Dividing by $X-1$ and evaluating at one shows $v_\ell(1-\lambda) \le v_\ell(p_i^e-1)$. The integer p_i^e-1 divides g_i by the order formula for the finite general linear group. The determinant valuation at a full tame generator is therefore at most $sv_\ell(g_i)$. At a noncoefficient old place the local dual invariants at every precision are in the kernel of this generator minus one on the Tate dual. Its determinant has the same valuation. The elementary divisors bound the normalized length of the kernel by that valuation. Summing over at most $2f$ places of E over each p_i gives $2sv_\ell(G_0)$. There are no further old terms if S consists of these places and the coefficient places. The two distinguished determinants together cost at most $2sv_\ell(G_0)$. Finally Corollary 12.26 gives $v_\ell(I_t) \le v_\ell(G_0)$ using a primitive member away from ℓ. Adding the terms gives the bound. □

Theorem 52.20 (Selmer lengths for the summands in the projected periods)**.** *Fix a node and positive tolerances ϵ_ℓ with $\sum_\ell \epsilon_\ell \log \ell \le 1$. The twists for the critical ratios and the torus periods may be chosen so that every nonzero true higher-rank system on these same twists satisfies*

$$S_\xi \le (4s+1)v_\ell(G_0) + \epsilon_\ell.$$

Here S_ξ is the divisible Selmer length used in Proposition 52.16.

Proof. At each embedding take the finite list of lattices of Proposition 52.18 for all j, together with the rank-one list. Enlarge the coefficient field to contain their finite data. The polarization with $t \ge 1$, the de Rham condition, the full matrix span and the unit scalar verify the hypotheses of Theorems 12.28 and 12.30. The preceding lemma checks the two determinants for the chosen g_* of Theorem 13.13. Apply that theorem to this joint list and the critical pairs at the node, excluding T_* from all new conductor primes. This exclusion is legitimate in the Chebotarev choices of Theorems 12.30 and 13.4: deleting a fixed finite set of rational primes leaves every required Frobenius class available, including at the first pair and at the smoothing prime. The group of allowed global cocycles still imposes the unramified condition at the excluded auxiliary places. Thus this choice adds no term to B_i. The construction selects one character at the embedding for all the signed powers; it has the simultaneous divisible-coefficient and new-inertia estimates of Theorem 12.32. Its ray characters descend to the norm-one torus as in the construction of Theorem 13.4. The critical ratio and new-inertia tolerances may be decreased to meet Corollaries 13.14 and 50.10 for the finitely many powers. Taking the power $-j$ identifies the twisted lattice up to homothety by Proposition 52.18. Indeed every stable lattice is homothetic by the full integral matrix span, which a twist by scalar units preserves.

To see that the bound applies to the precise Selmer group, let $A = B[1/\ell]/B$ for a stable lattice B in this twisted representation. A class has a unique cocycle representative vanishing at the chosen scalar element: subtract the coboundary of $(y-1)^{-1}z(\eta)$. The twist is unramified there. Away from S and the new pairs, the action is unramified, so a locally unramified class has zero raw restriction to inertia. At $v \mid \ell$ the twist is trivial. A class in the image of $H^1_f(E_v, B[1/\ell])$ has a rational finite cocycle whose reduction in A agrees with the actual local cocycle. To obtain equality on cocycles, lift the element of A giving their difference as a coboundary to $B[1/\ell]$ and adjust the rational representative. Continuous cocycles on a compact Galois group have bounded image in the rational representation and finite image in the discrete module A. Choose a power ℓ^k clearing all the local rational values and killing the global image. Under $A[\ell^k] \simeq B/\ell^k B$, each local restriction is then a reduction of an integral finite cocycle. It satisfies the evaluation equations of Lemma 12.27. There is no restriction at the remaining old or new pairs. Thus normalization embeds the entire Selmer group into the bounded group of cocycles of Theorem 12.32. This proves the inequality, including after extending the coefficient field. □

Corollary 52.21 (A common determinant and torus denominator on the period interval)**.** *On the shorter interval with fixed q_0, d, make the simultaneous choices of the theorem at each node. For the actual true projection let $D = \dim Z$, with zero quotient by convention when $D = 0$. Then*

$$\sum_{\ell} (\log \ell) \sup_{x,j,\iota} \frac{\lambda}{DsB_x^{\det}} \ll_{q_0,d} (\log(2V_0))^{-2}.$$

For all sufficiently large V_0 the integral operator and all main-pair period tests in Corollary 52.3 admit common bounds of total size at most $B_x^{\det}$ at each x, j of the interval.

Proof. The multiplicities $d_{\omega,\xi}$ sum to D. The same bound on S_ξ thus holds for their weighted mean $\overline{S}$ in Theorem 52.17. At a node its contribution, after the sum over primes with weights $\log \ell$, is bounded by

$$\frac{(4s+1)\log G_0 + 1}{sB_x^{\det}}.$$

Here $B_x^{\det} = fxb_x(\log b_x)^3/(m_x k_x)$, and $\log G_0 \le f(st)^2 \log(p_0 p_1)$ by Corollary 12.10. Since $\log(p_0 p_1) \ll \log b_x$, $s \le 3x^5$, $t = 5s$, $m_x \le 2\sqrt{x}$ and $k_x \le s+1$, the quotient is $O_{q_0,d}(x^{25}/(b_x(\log b_x)^2))$. For a sum of nonnegative errors the supremum over nodes is at most their sum. Summing the last bound over integer nodes $x \ge V_0$, with $b_x \ge x^{400}$, gives $O_{q_0,d}(V_0^{-374})$. Now apply Theorem 52.17 to the positive excess over the same mean for each twist. This proves the first assertion. The sum of $h_{x,j,\ell}$ of Corollary 52.3 is at most $B_x^{\det}$ times this interval sum, so eventually it is at most $B_x^{\det}/2$. That corollary and Proposition 52.1, on the very same twists, give the stated common denominator. □

This denominator controls projection onto the true lift range. To reach the prime-power estimate one still needs the comparison of the determinant of the composite theta lift with the local divisibility gain for the same family of maps.

52.8 Minors of the projected theta-period pairing

Let Z be the true range of the higher-rank theta lift at a fixed node, index and twist. Write P for its orthogonal projection and T for the tested subspace of Theorem 52.2. Use the Hermitian pairing on the definite target which is linear in its second variable and gives the defining compact level volume one. For two lists of tested vectors, of equal length h, put

$$H(\mathbf{y},\mathbf{z}) = \det\bigl(\langle Py_i, Pz_k\rangle\bigr)_{1\le i,k\le h}.$$

We determine how the index of the integral projection enters this minor, and how its complex conjugates are bounded by the norms of the original test vectors.

Proposition 52.22 (The tested projection at every complex embedding)**.** *For an automorphism σ of $\mathbb{C}$ let $\|v\|_\sigma$ be the class-sum norm of the definite form after applying σ to its coefficient coordinates. On T the transported map P is the restriction of orthogonal projection for this norm onto the transported subspace Z. In particular the map and all its exterior powers are contractions on the respective tested spaces.*

Proof. We specify the Hermitian structure at σ. Fix level class representatives and an integral tensor realization of the Weyl coefficient space. Apply σ to the coordinates of the values and sum their Hermitian pairings over classes, with weight the reciprocal of the stabilizer order. The Weyl Gram matrix is the Gram matrix of integral tensors in a tensor basis with identity Gram matrix. It has rational entries and is positive definite. This construction gives an ordinary positive Hermitian space on the transported coordinates. The class pairing is independent of the rational representatives. Indeed a rational unitary matrix acts unitarily on each tensor factor belonging to a complex embedding τ of E_x. After transport the factor belongs to $\sigma\circ\tau$; this embedding, too, takes the CM involution to ordinary complex conjugation on its image. The induced action on the Weyl space, including its determinant power, is unitary. Finite compact character factors are roots of unity at every conjugate.

Choose one split spherical place w off the level and use e, A_w of Proposition 52.4. In the transported structure the frame action is unitary: translation permutes the classes with their stabilizer orders, with the unitary actions on values just described. A central idempotent in a finite complex group algebra acts as identity or zero on each irreducible summand of a unitary representation. Its image and kernel are therefore orthogonal; the same applies to the central idempotent after applying σ to its coefficients. At w, the adjoint of a double-coset operator is the inverse double-coset operator. To see this, lift both vectors to the intersection of the two compact levels, and change variables by right translation in the class integral. Haar measure is right invariant and the two indices are equal by unimodularity. With level volume one this gives precisely the sums over right cosets defining the two operators. The calculation also applies to transported values. The scalar coefficients of the adjoint polynomial are the ordinary conjugates of the transported coefficients. Both polynomials belong to the commutative spherical algebra, so the transported A_w is normal. Its image is orthogonal to its kernel.

Before transport, $eZ = Z$, $A_wZ = 0$, and A_w is invertible on $eT \cap Z^\perp$ by Proposition 52.4. For $y \in T$ decompose its remainder as

$$y - Py = (1-e)y + (ey - Py) \;\in\; \ker(e) + \operatorname{im}(A_w).$$

Indeed the second summand belongs to $eT \cap Z^\perp$. Every vector of the subspace on the right is orthogonal to Z after transport by any σ, by the preceding two orthogonality statements and the inclusions $Z \subset \operatorname{im}(e) \cap \ker(A_w)$. Thus the transported Py is exactly the orthogonal projection of the transported y. In an orthonormal basis starting with a basis of the image, an orthogonal projection retains the indicated coordinates. In the corresponding exterior basis its action likewise retains a subset of the coordinates. This proves both contraction assertions. □

Proposition 52.23 (One local projection loss for a period minor)**.** *Use a finite coefficient field at ι with uniformizer π, and let $e_\ell = v_\pi(\ell)$. Put $D = \dim Z$ and write $a_1 \ge \cdots \ge a_D \ge 0$ for the elementary divisor exponents of $L_Z = Z \cap L$ in $P(L_T)$, where $L_T = T \cap L$. For $1 \le h \le D$ put $\lambda_h = e_\ell^{-1} \sum_{i=1}^{h} a_i$, so $\lambda_h \le \lambda_D = \lambda$. Suppose each z_k is integral at ι and each y_i is integral at $\iota \circ c$, where c is ordinary complex conjugation. In the class-sum normalization, for $c_0 = \gcd_{i=0,1} |\mathrm{GL}_{s+t}(\mathbb{F}_{p_i})|$,*

$$\log \max\{1, |\iota H(\mathbf{y}, \mathbf{z})|_\ell\} \le (\lambda_h + h\, v_\ell(c_0)) \log \ell.$$

The elementary divisors are understood after adjoining the finitely many coefficient values used in the calculation.

Proof. Each row defines a linear functional on Z at ι, by $u \mapsto \iota(c_0 \langle y_i, u \rangle)$, with the scalar coordinate of u transported by ι. This functional is integral on L_Z. In fact the coefficient pairing in the class sum uses the conjugate of the first coordinate array and has an integral Gram matrix. The changes of basis used by the local gauges on the two inputs cancel in this pairing by $\bar{g}^t g = 1$ at the two conjugate components. This is the pairing calculation in Theorem 50.3 and uses only the integral tensor realization on the definite target. Thus every summand before division by its stabilizer order is integral. Each stabilizer order divides c_0 by Lemma 45.18. Multiplication by c_0 clears all the summands together.

The columns Pz_k belong to $P(L_T)$. Orthogonality gives $\langle y_i, Pz_k \rangle = \langle Py_i, Pz_k \rangle$. Choose a basis $u_1, \dots, u_D$ of $P(L_T)$ for which $\pi^{a_i} u_i$ form a basis of L_Z. The inclusions of these lattices and the length equality $\lambda_D = \lambda$ are the projection exact sequence of Theorem 52.2. An exterior basis vector $u_{i_1} \wedge \cdots \wedge u_{i_h}$ becomes integral in $\bigwedge^h L_Z$ on multiplication by $\pi^{a_{i_1} + \cdots + a_{i_h}}$. The largest such exponent is $\sum_{i=1}^{h} a_i$. Consequently the wedge of the h integral functionals evaluated on $Pz_1 \wedge \cdots \wedge Pz_h$ has valuation at least $-\lambda_h$ in the normalization $v_\ell(\ell) = 1$. This evaluation is $c_0^h \iota H(\mathbf{y}, \mathbf{z})$. It proves the estimate. Extending the coefficient field multiplies both the exponents and e_ℓ by its relative ramification index, preserving λ_h. □

Theorem 52.24 (Archimedean and finite bounds on the same projected periods)**.** *Take the shorter fields and the twists of Corollary 52.21. For the complex bounds put $\sigma^\# = c \circ \sigma \circ c$. At every node and index, for any of its twists and all its tested vectors,*

$$\begin{aligned} |\sigma H(\mathbf{y}, \mathbf{z})| &\le \|y_1 \wedge \cdots \wedge y_h\|_{\sigma^\#} \|z_1 \wedge \cdots \wedge z_h\|_\sigma \\ &\le \prod_{i=1}^{h} \|y_i\|_{\sigma^\#} \prod_{k=1}^{h} \|z_k\|_\sigma. \end{aligned}$$

On the same interval one has

$$\sum_{\ell} \sup_{\substack{x,j,\iota,h \\ \mathbf{y},\mathbf{z}}} \frac{\log \max\{1, |\iota H(\mathbf{y}, \mathbf{z})|_\ell\}}{DsB_x^{\det}} \ll_{q_0,d} (\log(2V_0))^{-2}.$$

The finite supremum uses the two integrality conditions in the preceding proposition and the actual twist at (x, ℓ, ι). Contributions of a zero range are taken as zero.

Proof. Compute the period matrix in one set of class representatives. All the class weights and coefficient Gram entries are rational. Applying σ to an entry therefore pairs the coordinates of Py_i transported by $\sigma^\#$ with those of Pz_k transported by σ, in the same ambient weighted coordinate space: $\sigma(\bar{u}) = \overline{\sigma^\#(u)}$. The determinant of these mixed pairings is the Hermitian pairing on their h-th exterior powers. Apply Cauchy–Schwarz, followed by Proposition 52.22 at the two indicated automorphisms. Hadamard's inequality, or ordinary Gram–Schmidt on each list, proves the second inequality.

For $h > D$ the period matrix has rank at most D, so its determinant is zero. For $1 \le h \le D$, the previous proposition bounds the quotient in the second assertion by

$$\frac{\lambda \log \ell}{D s B_x^{\det}} + \frac{v_\ell(c_0) \log \ell}{s B_x^{\det}}.$$

The sum of interval suprema of the first term is bounded by Corollary 52.21. For the second, $s+t \le st$ at the nodes under consideration, so block embedding in GL_{st} shows that c_0 divides G_0 of Lemma 52.19. Replace the supremum over integer nodes by their sum and use $\sum_\ell v_\ell(c_0) \log \ell = \log c_0$ at each node. The calculation at the end of Corollary 52.21 bounds the resulting sum by $O_{q_0,d}(V_0^{-374})$. The empty minor is one and costs zero. This proves the assertion for every size simultaneously. □

The complex inequality retains the exterior norms of the actual test arrays. A lower bound for a chain volume requires a nonzero minor for the specified period maps and a quantitative estimate for its size. The inequalities above allow the minor to vanish.

53 References

[1] L. Clozel, M. Harris and R. Taylor, *Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations*, Publ. Math. IHÉS **108** (2008), 1–181, §3.1. Authors' text.

[2] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of* GL_n, Ann. Sci. École Norm. Sup. (4) **13** (1980), 165–210. Original paper.

[3] B. Conrad, *Lifting global representations with local properties*, Appendix B on locally algebraic characters. Author's text.

[4] J. S. Milne, *Arithmetic Duality Theorems*, second edition, BookSurge, 2006, Chapter I, §2. Author's text.

[5] W. Luo, Z. Rudnick and P. Sarnak, *On the generalized Ramanujan conjecture for* $\mathrm{GL}(n)$, in *Automorphic Forms, Automorphic Representations, and Arithmetic*, Proc. Sympos. Pure Math. **66**, part 2 (1999), 301–310. Published article.