The generalized Riemann hypothesis for Dirichlet L-functions

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Volume VIII

Integral trace quotients and Hecke support

Abstract

We study the part of the trace ring detected by reducible representations. Integral trace quotients and Frobenius determinants give bounds for strict classes. We then descend to auxiliary fields and examine the support of the Hecke module on ordinary curves.

The guide in Volume I lists all twenty-seven volumes.

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40 Integral trace quotients and root modules

40.1 The integral kernel after restriction to the cuspidal component

Fix a large-prime triple for the original period field E_x. Keep $T, \mathfrak{a}_G, \mathfrak{d}, \Phi, C$ over the unramified coefficient ring O of Section 36.6. Let Q_a be the finite algebra of Proposition 38.7, allowing the coefficient extension there. Set

$$B_a = \mathrm{im}(T \longrightarrow O \times Q_a), \qquad M_a = \mathrm{im}(\mathfrak{d} \longrightarrow B_a).$$

The first coordinate is the split evaluation. Both B_a and M_a are finite O-modules. Write $M_{a,\mathrm{tor}}$ for the O-torsion in M_a and put $J = \operatorname{Frac} O$. We first determine the rational local rings on which this ideal can be nonzero.

Lemma 40.1 (The cyclotomic image at an irreducible trace point)**.** *For an absolutely irreducible geometric integral trace specialization x of T, the image of $G_{K(\mu_{\ell^\infty})}$ under ρ_x is enormous in the sense of Newton and Thorne, after finite extension of its coefficient field.*

Proof. Let D be the Zariski closure of the image over the algebraic closure and D^0 its identity component. The unipotent radical of D^0 has a nonzero fixed vector by triangularization. Its fixed space is stable under all of D by normality, so this radical acts trivially. Thus D^0 is reductive. Its isotypic summands on the original three-dimensional space are permuted transitively. If it acts reducibly, primeness of the dimension leaves a scalar action or three distinct lines. A scalar action on an open subgroup contradicts the three distinct Hodge–Tate weights of Proposition 38.10, which persist under finite restriction.

In the three-line case the permutation quotient kills the matrix image of $\ker \bar{\Psi}$. This image is pro-ℓ since reduction in a triangular residual lattice basis is unipotent on the kernel. The quotient is therefore a transitive cyclic subgroup of S_3, of order three. In a basis of the three lines, conjugation by the diagonal matrix with entries $1, \zeta_3, \zeta_3^2$ gives a nontrivial cubic self-twist. Such a character remains of order three on reduction, because $\ell > 3$. It would permute the residual characters $1, \bar{\Psi}, \bar{\Psi}^{-1}$ transitively by multiplication. They would form the group generated by that cubic character, whereas $\bar{\Psi}$ has order greater than four on inertia. This proves irreducibility of D^0.

The center of D^0 acts by scalars and the determinant is one, so D^0 is semisimple. The closure of the cyclotomic kernel contains it. Indeed that kernel contains all commutators, the commutator is an algebraic morphism, and a connected semisimple group is its own algebraic derived group. Inertia above ℓ provides an element in D^0 with three distinct eigenvalues, by choosing in its open inverse image an inertia element of infinite cyclotomic order. The locus of such elements in the connected group is nonempty open. For any nonzero endomorphism Z, the function $h \mapsto \mathrm{tr}(hZ)$ does not vanish identically on D^0, by irreducibility, Burnside's theorem and nondegeneracy of the matrix trace. The cyclotomic image meets the intersection of the two open sets. At that element some eigenprojector pairs nontrivially with Z, by expanding the element in its eigenprojectors. Applying this to a nonzero vector of every simple submodule gives the enormous-image test. The eigenvalues can be split by one finite coefficient extension, since a local field has only finitely many extensions of bounded degree. This is the criterion used in Theorem 33.5. [1] □

Theorem 40.2 (Reducedness at the irreducible classical points)**.** *Let x be an absolutely irreducible characteristic-zero point of the finite quotient Q_a, viewed as a trace point on T. The geometric local ring of $T[1/\ell]$ at this point is its residue field, after a finite splitting extension of coefficients. Consequently the same holds for its local factor in $B_a[1/\ell]$.*

Proof. The argument of Theorem 38.11 applies directly when absolute irreducibility is given. Hence $\rho = \rho_x$, twisted by κ_ℓ^{-1}, is the representation of a regular algebraic, conjugate self-dual cuspidal representation on GL_3/K. The lemma proves the enormous-image hypothesis for its cyclotomic restriction, where the twist disappears. Let $V = \mathrm{End}(\rho)$ with its polarized adjoint action of $G_{\mathbb{Q}}$. The action on G_K is conjugation. At complex conjugation it is the negative adjoint for the polarization pairing. Indeed $\rho^c \simeq \rho^\vee$ by the trace identity and absolute irreducibility. The intertwining form is unique up to scalar, and applying c twice shows it is symmetric or alternating. It is symmetric in odd dimension. Its signed action therefore agrees with the adjoint in Newton and Thorne's Theorem A, unchanged by the common Tate twist. That theorem gives $H^1_f(\mathbb{Q}, V) = 0$. [1]

Make the coefficient extension so that the point is rational over a finite extension J'/J, with integer ring O'. To test the cotangent space take an algebraic tangent map over J' into $J'[\epsilon]/(\epsilon^2)$. It is continuous on $T \otimes_O O'$. Present the latter as a quotient of a power series ring and translate the variables by their integral values in the maximal ideal of O'. Modulo the square of the ideal of the point a series then maps by its value and its first derivatives. These evaluations are continuous for the adic topology. This is the argument of Lemma 37.35.

The resulting determinant is represented by a continuous matrix lift of ρ over the dual numbers, unique up to strict conjugation. Write it as $(1 + \epsilon z(h))\rho(h)$ on G_K. Its cohomology class belongs to $H^1(K, V)$. It is invariant for the polarized action of c. The conjugate lift and its contragredient have the same trace, and after identifying their reductions by the polarization they are strictly conjugate by uniqueness for the irreducible determinant. Differentiating gives precisely invariance of the class under negative adjoint and conjugation of the group argument. Inflation–restriction, since two is invertible, identifies it with a class in $H^1(\mathbb{Q}, V)$. [2]

At the places of K away from ℓ the lift has only the prescribed finite inertia, so the extension class on inertia is zero by averaging. At the places over ℓ Proposition 38.10 makes the underlying self-extension crystalline. It splits over B_{cris}, giving the finite condition for its adjoint class. These conditions descend to $\mathbb{Q}$ by restriction and corestriction on inertia and on B_{cris}-cohomology respectively. The degree at each step is at most two and is nonzero in J'. Thus the class vanishes by the Selmer theorem above. The lift on G_K is strictly conjugate to the constant lift and gives constant traces. Since these traces topologically generate T, continuity makes the entire tangent map constant.

The local ring at the rational point is Noetherian. The vanishing of its tangent space gives zero maximal ideal by Nakayama's lemma. It is therefore the residue field. Passing to its quotient in $B_a[1/\ell]$ preserves this conclusion. □

Proposition 40.3 (The rational image and the exact torsion term)**.** *Let X'_a be the distinct geometric points of $B_a[1/\ell]$ at which some inverse-trace difference is nonzero. Over a splitting field $M_a[1/\ell]$ is the direct sum of one copy of the field for each $\xi \in X'_a$, and every ξ belongs to the original list X of Section 36.6. For a test prime $u \in \mathcal{P}$, $u \neq \ell$, write $\gamma_u = T_u - d_\alpha(u)$ as in that section. With $v_\ell(\ell) = 1$ one has*

$$\mathrm{length}_O(M_a/\gamma_u M_a) = \mathrm{length}_O M_{a,\mathrm{tor}}[\gamma_u] + \sum_{\xi \in X'_a} v_\ell\big(\xi(T_u) - d_\alpha(u)\big).$$

Proof. The inclusion $B_a \subset O \times Q_a$ is integral since the right side is finite over O. Thus each geometric generic point of B_a lifts to one on the right. At a reducible trace point the projected ideal vanishes throughout the rational local factor, by Theorem 38.15. At an irreducible point the preceding theorem gives a field. There the projected ideal is either zero or the full field. Theorem 38.11 gives the labels claimed. This describes the entire ideal over the splitting field.

On the free O-module $M_a/M_{a,\mathrm{tor}}$, multiplication by γ_u is injective. Indeed at each contributing point the factor $\xi(T_u) - d_\alpha(u)$ is nonzero by Proposition 36.12. Its determinant is the product of these factors, each occurring once. Smith normal form gives the sum in the formula, since the original O is unramified. Apply the kernel and cokernel sequence of multiplication by γ_u to the torsion submodule, the full module and its free quotient. The kernel on the free quotient vanishes. The endomorphism of the finite torsion submodule has cokernel of the same length as its kernel. Adding the two cokernel lengths proves the identity. □

Corollary 40.4 (The two integral contributions to the remaining length)**.** *Let C_a be as in Corollary 38.12 and put $C_a^0 = \Phi(\ker(\mathfrak{d} \to M_a))$. Then $C_a^0 \subseteq C_a$, and the image of $M_{a,\mathrm{tor}}$ in C/C_a^0 is exactly C_a/C_a^0. For every one of the above u,*

$$\operatorname{length}_O(C_a/C_a^0) \le \operatorname{length}_O M_{a,\mathrm{tor}}[\gamma_u], \qquad \operatorname{length}_O(C/C_a^0) \le \operatorname{length}_O(M_a/\gamma_u M_a).$$

Proof. The map $M_a \to C/C_a^0$ is defined by choosing a lift to $\mathfrak{d}$ and applying Φ. It is surjective. On $\mathfrak{d}$ the kernel of evaluation at all geometric points of the finite quotient is exactly the inverse image of $M_{a,\mathrm{tor}}$. Indeed elements of $M_a[1/\ell]$ are detected by those evaluations by the proposition, and the additional split coordinate of B_a is zero on $\mathfrak{d}$. This is precisely the intersection with P_a in Corollary 38.12, proving the image formula. Multiplication by γ_u has zero image under $M_a \to C/C_a^0$, since $\Phi(\mathfrak{a}_G\mathfrak{d}) = 0$. Both inequalities now follow by taking quotients, and by equality of kernel and cokernel lengths on finite modules. □

In particular the classical contribution on the common interval has the polynomial bound of Corollary 38.12. The two remaining integral quantities are the image of the kernel of $\mathfrak{d} \to M_a$ and the part of the torsion of M_a that survives in C/C_a^0. Bounds on the characteristic-zero points above account for all their rational multiplicities. The weighted bounds for these two integral quantities and the projected theta-period comparison in Section 6.6 are still required.

40.2 A finite trace ring supported on reducible representations

Fix one of the large-prime triples on the common interval of Section 36.7. Let $G = G_K$ and keep the original coefficient ring O, with residue field k. Write $E = E_M$ for the finite T-algebra of universal group matrices and t for their trace. In this subsection E denotes this algebra and E_x denotes the period field. Put $I_m = I_m(G)$. We construct a finite quotient directly from the pairing $(a, b) \mapsto t(ab)$. Its defining ideal can be given explicitly, even where the trace ring has torsion.

Lemma 40.5 (All minors of the trace pairing)**.** *Use the three rank-one projectors of Lemma 38.1. Put $E_{ij} = e_i E e_j$, and let B_{ij} be the ideal of T spanned by $t(ab)$ for $a \in E_{ij}$, $b \in E_{ji}$. Thus $B_{ii} = T$ and $B_{ij} = B_{ji}$. For $1 \le j \le 6$, I_{3+j} is the sum of the products of j ideals chosen from the six* positions *in the list*

$$B_{12}, B_{12}, B_{13}, B_{13}, B_{23}, B_{23}.$$

Each position is used at most once. In particular,

$$I_5 = \mathfrak{a}_G^2, \qquad I_9 = (B_{12}B_{13}B_{23})^2, \qquad I_{10} = 0.$$

Proof. The minors have the same ideal if rows and columns run through E, since the group matrices span it over T. Split each argument into its nine corners. The trace pairs a corner (i, j) only with the opposite corner (j, i), by orthogonality of the idempotents and cyclicity. If the multiplicities of a row corner and its opposite column corner differ, no permutation in

the determinant can contribute. For matching multiplicities the determinant factors up to sign into the corresponding block determinants. In a basis of the three rank-one summands over the local marked coefficient ring A, each block has entries of the form $a_u b_v$ in A. Every block of size at least two has determinant zero. The inclusion $T \subseteq A$ makes the same assertion true in T. Thus only distinct row corners contribute. Conversely, choosing one row and one opposite column in each of any set of distinct corners gives their product of pairings up to sign. Arbitrary generators of each ideal product occur this way. There are three diagonal positions, with pairing one on (e_i, e_i). Every product of $3+j$ corners therefore belongs to one of the stated products of j off-diagonal ideals, and adjoining the three diagonal positions realizes each such product. For $j = 1$ the sum is $I_4 = \mathfrak{a}_G$ by Proposition 38.2. For $j = 2$ every square and every pairwise product in its expansion is allowed. For $j = 6$ all positions are forced. Ten distinct positions are impossible. □

Lemma 40.6 (Reducible traces over a field of Laurent series)**.** *Let k'/k be finite and $T \to k'[[z]]$ a continuous local O-algebra map. If its geometric semisimple representation of G is reducible over $\overline{k'((z))}$, every trace belongs to the constant field.*

Proof. Put $F = k'((z))$. The determinant over a field has a semisimple matrix realization over its algebraic closure by Chenevier's theorem on determinants over algebraically closed fields. This realization factors through $E \otimes_T \overline{F}$. In fact its trace there agrees on group elements, and for any relation among the group matrices in the latter algebra the trace of the relation times every group element vanishes. On the semisimple image the pairing is nondegenerate: it is the ordinary matrix trace on each simple factor multiplied by its multiplicity, an integer from one to three, invertible in characteristic $\ell > 3$. This proves the factorization. A finite basis of the finite-dimensional algebra and the entries of its image matrices show that a finite field extension of F suffices. The realization is continuous there, since E is finite over T with its module topology. The compact matrix image preserves a lattice: the span of all translates of any fixed lattice is bounded, finitely generated over the valuation ring, and stable. Reduction of such a lattice has semisimplification $\bar{\Psi} \oplus 1 \oplus \bar{\Psi}^{-1}$, by the residual determinant. The three characters are distinct on inertia. [2]

If there are three lines, I_4 vanishes at the point and Proposition 38.2 makes the trace constant. Otherwise write the representation as $\theta \oplus P$, with P absolutely irreducible of dimension two. The relation between conjugation by c and duality for the trace gives $\theta^c = \theta^{-1}$ by uniqueness of the line. Its residual character is fixed by c, since all three residual characters extend to $G_{\mathbb{Q}}$. Only the middle character has square one. Thus $\bar{\theta} = 1$.

At any place over ℓ the inertia matrices are upper triangular in the order $\bar{\Psi}, 1, \bar{\Psi}^{-1}$. Here these are the original fixed characters specialized in equal characteristic. To see the assertion on this realization, the ordered product in (32.1) vanishes already in E. Choose one separating inertia element g_u. In its eigenbasis replace just the j-th letter by an arbitrary inertia element h, leaving all other letters equal to g_u. Left multiplication by the preceding factors is invertible on rows of index at least j. Right multiplication by the following factors is invertible on columns of index at most j. It follows that in this rectangle the matrix of h agrees with the j-th prescribed character times the identity. All three rectangles give the assertion. In particular θ is trivial on inertia by its residual character. Elsewhere inertia factors through a group of order prime to ℓ and θ takes values in principal units, so it is trivial there too. The class number of K is prime to ℓ on the interval. Class field theory gives $\theta = 1$, and then $\det P = 1$.

We spell out descent of the plane in this characteristic. The marked involution $a \mapsto J_3 a^{\mathsf{T}} J_3$ preserves the trace and descends through the radical of the specialized trace pairing to the semisimple image $\overline{F} \times M_2(\overline{F})$. It fixes both factors. On the plane it is adjunction by a sym-

metric form. Indeed the identity $2XYX = t(XY)X$ for skew elements in Proposition 38.14 is an integral polynomial identity. Skew elements lift through the quotient by taking half the difference with their adjoints. For alternating adjunction the traceless matrices $X = \operatorname{diag}(1, -1)$, $Y = E_{12} + E_{21}$ in the plane violate the identity since $XYX = -Y$. The symmetric form with matrix S, together with the alternating determinant form J_0, now gives the intertwiner $S^{-1}J_0$ for conjugation by c. It has trace zero and nonzero scalar square. Rescale it to square one and assign it to c. This yields a continuous representation W of $G_{\mathbb{Q}}$ with $W|_G = P$ and $\det W = \bar{\eta}$, after finite coefficient extension.

Choose a stable lattice for W. Over a finite residue extension its residual semisimplification is a sum of two characters. Indeed an irreducible residual plane would restrict semisimply to the normal subgroup of index two: the translates of a simple submodule span the plane. The two distinct residual character lines of the restriction would then each be preserved by c, since their characters extend to $G_{\mathbb{Q}}$. This contradicts irreducibility. Twisting W by $\bar{\eta}$ if necessary, the constituent extending $\bar{\Psi}|_G$ is $\bar{\Psi}$ itself. The determinant makes the other $\bar{\eta}\bar{\Psi}^{-1}$.

Over K_u the first inertia eigenline just constructed belongs to the plane: its eigenvalue at g_u differs from that of the trivial line. It carries $\bar{\Psi}$ on inertia. Since $K_u/\mathbb{Q}_\ell$ is unramified, the inertia groups agree. A conjugate of this line by $G_{\mathbb{Q}_\ell}$ again carries $\bar{\Psi}$ on inertia, since $\bar{\Psi}$ is a global character. Separation at g_u gives uniqueness of the line. Thus $W\bar{\Psi}^{-1}$ has reducible local trace and the constituent lifting the trivial residual character is unramified at ℓ. Its determinant is the specialization of $\chi = \eta\Psi^{-2}$, an odd characteristic-zero character de Rham at ℓ, with nontrivial reduction there because $0 < 2r < \ell - 1$. The representation is unramified outside the places dividing $sN_*\ell$, where $K = \mathbb{Q}(\sqrt{-s})$.

Apply Pan’s Theorem 5.1.2(1) over $\mathbb{Q}$ with this finite ramification set and determinant χ. Its hypotheses are listed in Section 5.1.1 of that paper. The ordinary pseudodeformation ring is finite over the completed group algebra of $1 + \ell\mathbb{Z}_\ell$. The preceding unramified character gives the augmentation of this group algebra. Using the ring of integers of our Laurent series field extension as coefficient target, the map therefore factors through a finite O-module, as in Theorem 33.2. One may enlarge O unramified to match the residue field. In characteristic ℓ the image is a finite integral domain, hence a finite field. All traces of P lie in the constant field after restoring the finite-valued twist. Intersecting that field with $k'[[z]]$ gives k'. [3] □

Theorem 40.7 (Finiteness on the full reducible locus)**.** *For $6 \le m \le 9$, the original trace algebra T/I_m is finite over O. At any of its geometric field points the semisimple representation of G has a one-dimensional summand. In particular the conclusion holds for the single product ideal $(B_{12}B_{13}B_{23})^2$.*

Proof. Over an algebraically closed field of the given residue characteristic or characteristic zero, the group span on a semisimple representation is the product of the full matrix algebras of its distinct simple constituents. This follows from the semisimple quotient and the matrix algebra theorem over an algebraically closed field. Its trace pairing is nondegenerate for the multiplicity reason in the lemma. At an irreducible point it therefore has rank nine. Some nine group matrices give a basis, and pairings with group matrices span the full dual. There is a nonzero minor of each size through nine: choose that many independent rows and then independent restrictions of column functionals to their span. At a reducible point the sum of the squares of the constituent dimensions is at most five, so every minor of size at least six vanishes. This proves the point assertion in both characteristics, and in particular on the generic point of any complete curve in the special fiber.

The complete Noetherian local ring T/I_m has finite residue field. If its special fiber had positive dimension, it would have a one-dimensional complete local domain quotient. Normalize

this quotient. Normalization of a one-dimensional complete local domain is finite, and a local factor is an equal-characteristic complete DVR with finite residue field, of the form $k'[[z]]$. This gives a continuous local map with nonconstant image. Since traces and coefficients topologically generate T, some trace would be nonconstant, contrary to the lemma. Hence the special fiber is Artinian. Lift a finite k-basis to the ring. Successive approximation modulo powers of the uniformizer expresses each element as an O-linear combination of the lifts. The errors tend to zero in the maximal ideal topology, and the coefficients converge in O. This proves finiteness. The product formula is the lemma on minors. □

Corollary 40.8 (Integral torsion surjecting onto the strict group)**.** *Let $B = T/I_9$ and N be the image in B of the ideal $\mathfrak{d}$ of inverse-trace differences. Then N is a finite-length O-module. There is an O-linear surjection $N \to S(\Psi^{-1})^{\vee}$ for every large-prime triple on the original interval. The same assertions hold with I_9 replaced by I_m, $5 \le m \le 8$.*

Proof. At $m = 5$ finiteness follows from $T/I_5 = O \oplus \mathfrak{a}_G/\mathfrak{a}_G^2$ and Noetherianity. All generic field points there are split. For the other sizes use the finiteness theorem. Split the finite generic algebra over a finite extension of $\operatorname{Frac} O$ into local Artin factors with rational residue points. These points are integral and continuous on T, since B is finite and local over the complete DVR and all its special points coincide. Their representations are reducible, so each factor kills $\mathfrak{d}$ by Theorem 38.15. Faithful field extension gives $N[1/\ell] = 0$. Thus N is a torsion submodule of a finite module over a DVR and has finite length. Finally $I_m \subseteq \mathfrak{a}_G^2$ for every indicated size. The map Φ to the square-zero extension in Theorem 36.11 factors through this quotient and its restriction to $\mathfrak{d}$ has image the full dual strict group. This proves the surjection. □

Each ideal and each module above is taken at the character of the given E_x. Finiteness gives an exponent annihilating N for each triple. The weighted sum required in Section 6.6 still requires control of these orders as x, ℓ, r, α vary.

40.3 The order of reducible traces on the common interval

Continue with $B = T/I_9$ on G_K. Write $J = \operatorname{Frac} O$, and let $A = B/B[\ell^{\infty}]$, where $B[\ell^{\infty}]$ denotes the ideal of elements annihilated by some power of ℓ. This is a finite free O-module. Its augmentation ideal is denoted by $\mathfrak{n}_A$. We give a uniform estimate for $\mathfrak{n}_A/\mathfrak{n}_A^2$ on the original intervals.

Lemma 40.9 (The matrix algebra at a line and a plane)**.** *Let D be a finite local algebra over a finite extension J'/J, with residue field J', and let $T \to D$ be as in Lemma 38.13. Suppose the residue representation is $1 \oplus P$ with absolutely irreducible plane P, realized over J'. Then the group acts in*

$$E_M \otimes_T D \simeq D \times M_2(D)$$

as $1 \oplus P_D$, with P_D of determinant one. Its trace is $1 + \operatorname{tr} P_D$, and the underlying J'-representation on D^2 is crystalline above ℓ.

Proof. Put $R = E_M \otimes_T D$ and use the inertia projectors e_1, e_2, e_3 of Lemma 38.1. The proof of Theorem 38.15 gives a central e_2 with $e_2 R = D e_2$, and the group acts as one on it. For every i we have $e_i R e_i = D e_i$. Indeed sandwiching any marked matrix between the two copies of the rank-one projector gives its scalar trace on that line times e_i. The identity passes to R by linear extension and $t(e_i) = 1$ identifies the scalars faithfully.

The semisimple quotient on the plane is $M_2(J')$, with the two remaining projectors as its diagonal units. Lift opposite matrix units to $x \in e_1 R e_3$ and $y \in e_3 R e_1$ by projecting arbitrary lifts. We have $xy = u e_1$, $yx = u e_3$, since the two traces agree by cyclicity. The scalar u reduces

to one, hence is a unit. Replace y by $u^{-1}y$. Then x, y, e_1, e_3 are matrix units. Every $z \in e_1 R e_3$ satisfies $z = (zy)x$ with $zy \in De_1$, and the opposite corner is handled by interchanging x, y. This proves the matrix-algebra assertion, and the trace on it is the ordinary trace because it is one on each diagonal unit and zero on both off-diagonal units. Newton's identities applied to the traces of the first three powers of each group element give its original characteristic polynomial as $(X-1)\det(X - P_D(h))$. Its determinant was one in the original trace ring, so $\det P_D = 1$. The column module D^2 is a finite R-module. Its local properties now follow from Lemma 38.13. □

Theorem 40.10 (Rigidity at every reducible trace point)**.** *At every geometric integral characteristic-zero trace point of T with reducible representation, the geometric local ring of $T[1/\ell]$ is its residue field, after extension to a finite field of definition. In particular $B[1/\ell]$ is finite étale over J.*

Proof. Choose a finite extension J' where the point and its constituents are rational and split. Any relative tangent map at this point can be tested in $D = J'[\epsilon]/(\epsilon^2)$. Its restriction to the original complete local algebra is continuous. Indeed after presenting the latter by power series, translation by the tuple of integral point values in the maximal ideal of the integer ring gives the value and first derivatives formula of Lemma 37.35. These are continuous in the power series topology. For a split triple Theorem 38.15 already makes this tangent map constant.

In the other case take the preceding lift P_D. Choose the odd representation W extending P from Proposition 38.14 and choose compatible bases in its residue representation. Write $P_D(h) = (1+\epsilon z(h))P(h)$. It defines a continuous class in $H^1(K, \mathrm{ad}^0 W)$ by the cocycle identity and the determinant. This class is invariant under $\mathrm{Gal}(K/\mathbb{Q})$ with its usual action through W. Indeed the marked trace satisfies $t(chc^{-1}) = t(h^{-1})$, and P_D has determinant one. For a two-by-two matrix of determinant one the traces of the matrix and its inverse coincide. Subtracting the constant line gives equal traces for P_D and the lift $h \mapsto W(c)^{-1} P_D(chc^{-1}) W(c)$. Their reductions agree and are absolutely irreducible. The uniqueness of matrix realizations of an absolutely irreducible residual determinant over a local Artin ring gives strict conjugacy, hence equality of the two cocycle classes. [2]

Restriction from $\mathbb{Q}$ therefore gives this class by inflation–restriction for the group of order two in characteristic zero. The finite local conditions hold on K. Away from ℓ the extension splits over inertia since the image factors through the specified finite quotient. At ℓ the lemma makes the self-extension crystalline, hence it splits after tensoring with B_{cris}. These conditions descend to $\mathbb{Q}$: the restricted classes in inertial cohomology away from ℓ, and in B_{cris}-cohomology at ℓ, vanish. Local corestriction composes with restriction to multiplication by the relevant index, at most two. It is invertible on these cohomology groups.

By Proposition 38.14, a scalar twist of W comes from a non-CM cuspidal newform of weight $2r+1$. The twist leaves its adjoint unchanged. Newton and Thorne's Theorem B(1) gives $H^1_f(\mathbb{Q}, \mathrm{ad}\, W) = 0$. Its trace-zero summand has zero finite Selmer group as well. Consequently z is a coboundary. Strict conjugacy to the constant lift makes all group traces constant. Their topological generation of T, together with continuity, makes the original tangent map constant. The maximal ideal of the Noetherian local ring has zero cotangent space and is zero by Nakayama's lemma. This proves the local assertion. Theorem 40.7 gives finiteness of B and reducibility at all its generic geometric points. Each generic local factor is a quotient of the corresponding local ring just determined. It is a field, which proves the assertion about étaleness in characteristic zero. [1] □

Proposition 40.11 (The primitive forms on the original field)**.** *Write $K = \mathbb{Q}(\sqrt{-s})$, with discriminant $-s$, and fix a good rational prime u split in K. Each nonsplit geometric point*

of $B[1/\ell]$ can be assigned a normalized primitive cuspidal form f of weight $2r+1$, nebentypus $\eta\alpha^{-2}$, and level the conductor of that nebentypus. The level divides sN_*. At either place over $u \nmid sN_*\ell$ its trace is

$$1 + u^{-r}\alpha(u)a_u(f).$$

With a fixed identification of algebraic coefficient embeddings, distinct trace points have distinct assigned forms.

Proof. Use W from Proposition 38.14 at $1 \oplus P$. Twisting its modular representation by the finite character α^{-1} makes $W\kappa_\ell^r\alpha^{-1}$ the arithmetic representation of f, with the stated determinant character after removing κ_ℓ^{2r}. At $p \neq s, \ell$ the inertia groups of K and $\mathbb{Q}$ agree. On the plane the two characters there are α, α^{-1}: the marked finite inertia has these two characters together with the trivial line, with multiplicities read from its traces. The twisted characters are $1, \alpha^{-2}$. At s the character α is unramified. The plane kills the inertia of K, so W on rational inertia factors through a group of order two. Its determinant is the ramified quadratic character η, so both characters $1, \eta$ occur. At all these finite places monodromy vanishes, and the Artin conductor exponent is that of $\eta\alpha^{-2}$. Crystallinity at ℓ gives an unramified Weil–Deligne parameter with zero monodromy there. Local–global compatibility and the newform conductor formula give the primitive level. Restriction to G_K removes the choice between the two possible extensions of P. Removing the scalar twist there gives the displayed Frobenius formula. Conversely the embedded form and the chosen twist determine P on G_K, hence determine the trace point since group traces generate T. [4, 5] □

Theorem 40.12 (An integer bounding the reducible order on every field)**.** *Choose once the split prime $u = u_1$ of Proposition 36.3. For all sufficiently distant common intervals there is a positive integer $D_2(V)$ such that for every $x \in [V, V^{3/2}]$*

$$\sum_{\ell > P(x)} (\log \ell) \sum_{1 \le r < t(x)} \sum_{[\alpha]} b_\alpha \, \mathrm{length}_O(\mathfrak{n}_A/\mathfrak{n}_A^2) \le \log D_2(V),$$

$$\log D_2(V) \ll_u t_*^3 N_* (sN_*)^2.$$

Here $[\alpha]$ and b_α are the residue orbits of complementary sign on E_x and their degrees as in Theorem 36.14. For each large prime $\ell \nmid D_2(V)$ one has $A = O$ for every such triple. Writing $U = B[\ell^\infty]$ and $\mathfrak{n}_B = \ker(B \to O)$, one has an exact sequence

$$0 \longrightarrow U/(U \cap \mathfrak{n}_B^2) \longrightarrow \mathfrak{a}_G/\mathfrak{a}_G^2 \longrightarrow \mathfrak{n}_A/\mathfrak{n}_A^2 \longrightarrow 0.$$

The left term maps onto $S(\Psi^{-1})^\vee$ under Φ.

Proof. Use all complex characters α of E_* of sign $(-1)^r$, all $1 \le r < t_*$, and for each such pair all normalized primitive forms of the weight, character and level in the proposition, including their coefficient embeddings. Let $D_2(V)$ be the absolute value of the product of

$$\alpha(u)a_u(f) - u^{2r}\alpha(u) - \alpha(u)^{-1}$$

over this list, with empty product one. These algebraic integers are permuted with their indices by absolute Galois conjugation. The product is rational integral. It is nonzero: at any complex embedding $|a_u(f)| \le 2u^r$ by Deligne, whereas $u^{2r} - 1 > 2u^r$ for $u \ge 7$. Each factor has absolute value at most $(u^r+1)^2$. For fixed r, α the number of normalized forms with their embeddings is at most the dimension of $S_{2r+1}(\Gamma_1(sN_*))$, since distinct eigensystems give linearly independent old or new vectors at this level. The modular curve dimension bound is $O((2r+1)(sN_*)^2)$ by the usual index bound for Γ_1, as in the proof of Theorem 34.7. There are at most N_* characters. Taking logarithms and summing over r proves the upper estimate. [6]

Fix a triple and set $a_u = T_u - d_\alpha(u) \in \mathfrak{n}_A$. The finite free module $\mathfrak{n}_A$ has generic fiber with one geometric line per nonsplit point, since the preceding rigidity theorem makes every local factor a field. Multiplication by a_u is injective on it by the nonvanishing of the factors just proved. Its determinant is the product over these points of their trace differences. The split value is $1+u^r\alpha(u)+u^{-r}\alpha(u)^{-1}$. The displayed integer factors equal the trace differences times u^r, an ℓ-adic unit throughout this range. Smith normal form over the unramified O therefore gives

$$\operatorname{length}_O(\mathfrak{n}_A/\mathfrak{n}_A^2) \le \operatorname{length}_O(\mathfrak{n}_A/a_u\mathfrak{n}_A) = \sum_{\xi\neq\text{split}} v_\ell(\xi(T_u) - d_\alpha(u)).$$

The assigned forms are distinct for this character. For each of its b_α residue conjugates one may apply the corresponding local embedding to the same factors: valuations agree and the character labels are distinct. Every character of E_x inflates to E_*. Thus after summing these bounds over twists and residue orbits for fixed ℓ, the sum is at most $v_\ell(D_2(V))$. All omitted factors have nonnegative valuation. Summation over primes with weight $\log \ell$ proves the first bound. If $\ell \nmid D_2(V)$, the cotangent quotient vanishes. The augmentation ideal then equals its square and lies in the maximal ideal of this finite local ring. Nakayama's lemma makes it zero, so $A = O$.

Every torsion element of B has split value zero. Taking augmentation ideals in $B \to A$ and dividing by their squares gives the displayed kernel and cokernel. Here $\mathfrak{n}_B/\mathfrak{n}_B^2 = \mathfrak{a}_G/\mathfrak{a}_G^2$ since $I_9 \subseteq \mathfrak{a}_G^2$. The image of $\mathfrak{d}$ in B lies in U by Corollary 40.8. Its image under Φ is the full dual strict group, and Φ kills the square of the augmentation ideal. This proves the last assertion. □

The bound holds simultaneously at all the period fields in the interval. It controls the quotient on the right of the exact sequence. A uniform upper bound for the image of the left term is still required for the large primes of Section 6.6.

40.4 Two integral quotients of the three-character trace ring

Fix $x \in [V, V^{3/2}]$ on one of the intervals of Section 6.6, and use a triple (ℓ, r, α) from Section 36.7. In particular $\ell > P(x)$, $0 < r < t(x)$, $\Psi = \alpha\kappa_\ell^r$ is even, and $K = \mathbb{Q}(\sqrt{-s})$ has quadratic character η. Write $G = G_K$, let c be the marked complex conjugation and put $D = \operatorname{Frac}(O)/O$. Let A be the original fixed-weight marked matrix ring and $T \subset A$ its closed trace algebra on G, with augmentation ideal $\mathfrak{a}$. We use one-based matrix indices. At the split augmentation the matrices are $d(h) = \operatorname{diag}(\Psi(h), 1, \Psi(h)^{-1})$. Let J_3 reverse the three coordinates. For $h \in G$ the polarization gives

$$M(ch^{-1}c) = J_3 M(h)^{\mathsf{T}} J_3.$$

The marking at c is the identity. The other marks set the entries in positions $12, 21$ of $Z = M(g_*) - M(cg_*^{-1}c)$ to zero, for a separating inertia element g_*. Since $Z = -J_3 Z^{\mathsf{T}} J_3$, it is $\operatorname{diag}(z, 0, -z)$ with z a unit. Indeed its first residue is $\bar\Psi(g_*) - \bar\Psi(g_*)^{-1} \neq 0$. The identity $z^2 = \operatorname{tr}(Z^2)/2$ puts z^2 in T. The square root with the specified residue lies in T by Hensel's lemma. Hence the three standard diagonal projectors e_i belong to the T-span $\mathcal{E}$ of $M(G)$, by interpolation on $z, 0, -z$.

In A let J_1 be generated by the entries $M(h)_{ij}$ with $|i-j| = 1$, for all $h \in G$, and J_2 by those with $|i-j| = 2$. In T let X be generated by the traces of $e_1\mathcal{E}e_2\mathcal{E}e_1$, and Y by those of $e_1\mathcal{E}e_3\mathcal{E}e_1$. The reflection identity identifies the ideal for the pair $2, 3$ with X. Use weight $j - i$ for the ij-entry under conjugation by $\operatorname{diag}(a, 1, a^{-1})$, with a an indeterminate. Thus weight zero means invariance for the split torus on each finite truncation.

Lemma 40.13 (Contraction of the two matrix-entry ideals)**.** *One has*

$$T \cap J_1 = X, \qquad T \cap J_2 = Y, \qquad X + Y = \mathfrak{a}, \qquad X^2 \subseteq Y.$$

In particular T/X and T/Y are the full invariant subrings of A/J_1 and A/J_2, respectively.

Proof. The entries with their split constants subtracted generate A topologically. Indeed the marked ring is generated by the entries of the universal group matrices, as in the presentation of Theorem 34.2. The ramification conditions give a finitely generated pro-ℓ quotient of the kernel of the residual representation, by class field theory and the pro-ℓ generator criterion. Finitely many matrix entries therefore suffice after lifting the finite residual image. The entries on the other coset are obtained using the marked c. The defining conditions are preserved by the displayed torus. On an Artinian truncation the action separates the integer weights by the Laurent monomials in a. Taking the degree-zero coefficient of a polynomial lift thus lifts an invariant, also after imposing either homogeneous entry ideal.

Every monomial of weight zero factors into diagonal entries or their differences from split values, and blocks of weights $(1,-1),(2,-2),(1,1,-2),(-1,-1,2)$. To see this, remove all opposite pairs. In what remains the positive and negative weights must have distinct absolute values and group into the indicated triples. By reflection an entry of weight one can be put in either position $12, 23$, and one of weight minus one in either $21, 32$. Each block is therefore a directed cycle. Its value is the trace of the corresponding matrix product with the diagonal projectors inserted, so it lies in T. This proves the assertion about invariants before imposing an ideal, since T is closed. After imposing an ideal, lifting invariants on each truncation and using the compactness of T shows that all invariants still come from T.

In the image of the ideal J_j at a truncation, take a polynomial expression with one specified generating entry of that ideal in each term, and extract weight zero. The primitive block containing that entry passes through the corresponding edge length. If it contains an adjacent entry, its trace is in X: in a triple one groups the two edges returning to the starting vertex into one matrix corner, opposite the chosen adjacent edge. The same reasoning with an outer entry puts the trace in Y. The remaining blocks have values in T. Conversely the generators of X, Y lie in the respective entry ideals by expanding in group matrices. An approximation in every truncation gives membership in X or Y itself. These are closed in A, since they are finitely generated ideals of the compact Noetherian ring T. This proves the contractions.

Modulo $X+Y$ the trace pairs only the three diagonal corners. All four-by-four trace minors therefore vanish. Each generator of X or Y is itself a four-by-four minor using the three projectors and the two opposite off-diagonal entries as its last row and column. Proposition 38.2 gives $X + Y = \mathfrak{a}$. Finally write a product from X^2 using the pair $12, 21$ for the first factor and $23, 32$ for the second. The entries are scalars in A. Their product can be ordered as $(12)(23)(32)(21)$. Grouping the first two and last two gives opposite outer corners. This proves the last containment. □

Theorem 40.14 (Odd descent on the integral trace quotient)**.** *The ring T/X carries the trace of an odd two-dimensional representation of $G_{\mathbb{Q}}$ realized over A/J_1, of determinant η. Its trace on G is $t(h) - 1$, where $t(h) = \operatorname{tr} M(h)$ is taken modulo X. After twisting by Ψ^{-1} it induces a surjection onto T/X from the weight-augmentation quotient of the ordinary two-dimensional pseudodeformation ring over $\mathbb{Q}$ with residual trace $1 + \bar{\eta}\bar{\Psi}^{-2}$ and determinant $\eta\Psi^{-2}$. These statements hold over O, including on every nonreduced specialization of the trace quotient. Every inverse-trace difference $t(h) - t(h^{-1})$ belongs to X in T.*

Proof. Over A/J_1 the middle entry gives a character m of G. It is unramified above ℓ. In fact the fixed inertia polynomial has roots $\Psi(u), 1, \Psi(u)^{-1}$ at every inertia element u, while $m(u)$ is

a root congruent to one. If $\bar{\Psi}(u) \neq 1$, the other two factors evaluated at $m(u)$ are units, hence $m(u) = 1$. If $\bar{\Psi}(u) = 1$, apply this argument to g and gu for an inertia element g with $\bar{\Psi}(g) \neq 1$ at that place. At all other finite places inertia acts through a group of order prime to ℓ, whereas m takes values in principal units of a complete local ring with finite residue field. Their finite quotients have ℓ-power order. Class field theory and $\ell \nmid |\mathrm{Cl}(K)|$ now give $m = 1$. The outer matrix $U(h)$ consequently has determinant one. Writing J_2' for the two-by-two reversal matrix and $C_0 = \mathrm{diag}(1, -1)$, direct multiplication gives

$$U(chc) = J_2' U(h)^{-\mathsf{T}} J_2' = C_0 U(h) C_0.$$

Since $G_{\mathbb{Q}} = G \rtimes \langle c \rangle$, assigning C_0 to c extends U to W with determinant η. Traces on the other coset are differences of the outer diagonal entries. Each such entry belongs to the image of T by insertion of e_i, hence all traces take values in the faithfully embedded ring T/X. This proves the existence of its two-dimensional pseudocharacter there by the matrix trace identities. On G, a determinant-one two-by-two matrix and its inverse have the same trace. This proves the last assertion integrally.

We verify the local condition over the full coefficient ring. At a place over ℓ choose g in inertia with the prescribed three roots distinct modulo the maximal ideal. The outer $\Psi(g)$-eigenprojector is a direct summand of rank one by interpolation and reduction modulo the maximal ideal. In the ordinary word identity of Theorem 32.8 take the last two letters equal to g. On this eigenline these last two factors are units, so $U(u)$ acts by $\Psi(u)$ there for all inertia u. The inertia groups of K and $\mathbb{Q}$ at this prime agree since K is unramified. Every local element of $G_{\mathbb{Q}}$ preserves the line: conjugating it gives an inertia eigenline of the same global character, which must lie in the same rank-one eigensummand at g; applying the inverse gives equality. The local line character has values in T/X: its value at v is the trace of its eigenprojector times $W(v)$, and the projector is a polynomial in $W(g)$ over O. The other local character is obtained from the determinant. Thus the local trace of $W\Psi^{-1}$ splits over the trace quotient itself, and the character lifting one is unramified.

The twist has odd, de Rham determinant $\eta\Psi^{-2}$ and residual inertia character $\bar{\kappa}_\ell^{-2r} \neq 1$. The residual outer lines have characters $\bar{\Psi}$ and $\bar{\eta}\bar{\Psi}^{-1}$ before twisting, by their values on G and on c. Ramification is confined to the original finite set together with the primes of K's discriminant. Take this set in Pan's ordinary ring over the totally real abelian field $\mathbb{Q}$. Its universal trace maps to the twisted trace just constructed. Local reciprocity and the unramified line make the action of the weight algebra $O[[1 + \ell\mathbb{Z}_\ell]]$ factor through augmentation. Pan's Theorem 5.1.2(1) makes this specialized source finite over O. The image contains all $t(h)$ on G, since one restores their finite or cyclotomic scalar twist and adds one. It is closed since it is finite over the compact ring O. The traces generate T/X, which proves surjectivity. [3] □

For the second quotient put $A' = A/J_2$. Let V_+, V_- denote its submodules of weight $1, -1$, respectively. Here and below tensor products of these modules are over O.

Proposition 40.15 (Multiplication in the quotient with vanishing outer entries)**.** *The ring A' has an O-module decomposition*

$$A' = O \oplus V_+ \oplus V_- \oplus L, \qquad T/Y = O \oplus L, \qquad L = V_+ V_-.$$

Each of V_+^2, V_-^2 is zero. The product of L with the entire augmentation ideal is zero. In particular L is a quotient of $V_+ \otimes V_-$. The full negative module is

$$V_- \;\simeq\; S(\Psi^{-1})^\vee \oplus S(\eta\Psi^{-1})^\vee, \qquad U^\vee = \mathrm{Hom}_O(U, D).$$

All these modules are finite over O, and L has finite length. The two rings over the common augmentation satisfy

$$T/(X \cap Y) \;\simeq\; (T/X) \times_O (O \oplus L).$$

Proof. In A', the 13-entry of $M(hk)$ is zero for all $h, k \in G$. Multiplying matrices leaves only $M(h)_{12}M(k)_{23}$ in that entry. Reflection allows any entry of weight one to be used in either position. Hence every product of two such entries vanishes. The 31-entry proves the same for weight minus one. Let I be the ideal generated by the adjacent entries. It satisfies $I^3 = 0$. Modulo I the representation is diagonal. Its three characters agree with the prescribed ones on inertia at ℓ: apply the separating inertia test in the proof of Proposition 38.2, which works for diagonal lifts over a ring with nilpotents. Explicitly, the traces at $g^j u$, $0 \le j \le 2$, give an invertible Vandermonde matrix in the three inertia eigenvalues of g. The tame inertia and class group argument in the preceding proof then fixes the three characters globally. Thus $A'/I = O$, by generation by entries, and I is the augmentation ideal.

The module I/I^2 is generated over O in degrees $1, -1$. The module I^2 is generated in degree zero by the mixed products since $I^3 = 0$. These modules are finite over O by Noetherianity. The weight decomposition therefore has just the stated degrees, and $L = I^2$ is the entire degree-zero augmentation. Lemma 40.13 identifies that invariant ring with T/Y. Passing from A to A' leaves its degree minus one augmentation cotangent module intact: the removed generators have weights $2, -2$, and modulo the augmentation square their multipliers reduce to O. Since I^2 has weight zero, the surviving negative cotangent module is V_-.

We recall how its dual identification keeps track of its two summands. A square-zero lower deformation is a cocycle in the two adjacent lower entries after dividing out the split diagonal. The difference and the sum of the two lower matrix units have coefficient characters Ψ^{-1} and $\eta\Psi^{-1}$. The mark at c fixes the odd coboundary, and the remaining marking equation fixes the even coboundary since $(\Psi(g_*) + 1)(\Psi(g_*)^{-1} - 1)$ is a unit. The local word condition gives strictness at ℓ. At the other places averaging on the prime-to-ℓ inertia quotient gives the unramified condition, as proved for the marked cocycles in Corollary 32.9. Its calculation applies with the sign of the present K, since the extension is unramified at ℓ and the same kernels are imposed away from ℓ. Taking divisible coefficients and dualizing gives exactly the two finite summands shown. In particular a tensor product of V_- with the finite module V_+ has finite length, which proves the assertion for L. Finally the image of $T \to T/X \times T/Y$ consists of pairs with the same residue modulo $X + Y = \mathfrak{a}$: adjust lifts whose difference lies in $X + Y$ by its two summands. Its kernel is $X \cap Y$, proving the identity. □

Theorem 40.16 (The complementary strict group as an integral summand)**.** *Let $C = S(\Psi^{-1})^\vee$, $C_1 = S(\eta\Psi^{-1})^\vee$ and $h = \dim_k(V_+/\mathfrak{m}_O V_+)$. The mixed trace map of Theorem 36.11 factors through $T/Y = O \oplus L$, and its map $L \to C$ has an O-linear section. More precisely,*

$$\operatorname{length}_O C \;\le\; \operatorname{length}_O L \;\le\; h\big(\operatorname{length}_O C + \operatorname{length}_O C_1\big).$$

If $h = 1$ and $C_1 = 0$, it identifies L with C. Both the summand and the upper bound refer to the character of the given E_x on the original interval.

Proof. For $C = 0$ the section is the zero map and the length estimate follows from the preceding proposition. Suppose $C \neq 0$. Use the actual mixed lift from Theorem 36.11 over $A_m = (O \oplus C)[y]/(y^2)$, where $C^2 = 0$. Give y degree one and C degree minus one. Before the final change of frame its matrix on $h_0 \in G$ is

$$\exp\big(z(h_0)X_- + yb(h_0)X_+ + yw(h_0)H_0\big)d(h_0),$$

where $X_- = E_{21} - E_{32}$, $X_+ = E_{12} + E_{23}$, $H_0 = [X_-, X_+]$ is diagonal, and z, w take values in C. In this formula E_{ij} are the matrix units. The exponential stops after degree two in the nilpotent ideal (C, y). Two lower steps vanish by $C^2 = 0$, and two upper steps vanish by $y^2 = 0$. A mixed pair of steps is diagonal. Thus every outer entry is zero. The remaining marking conjugates by $1 + yt(E_{12} - E_{23})$ for a scalar t. It respects the indicated weights. Its mixed terms with a lower step are diagonal and its terms with a second upper step vanish. Consequently the marked matrix map factors through A' as a weight-preserving map. It sends V_- into C, V_+ into Oy, and L into yC. On traces the mixed construction has image $O \oplus yC$, so the last map is onto.

The map $V_- \to C$ is projection onto the first of the two summands in the preceding proposition, under its natural cocycle identification. In fact setting $y = 0$ leaves just $(1 + z(h_0)X_-)d(h_0)$. The cocycle z used in the mixed construction is the universal normalized cocycle for the Ψ^{-1} strict classes, as constructed in Theorem 36.6. For each functional $C \to D$ its specialization is precisely that class on the difference of the two lower matrix units and has zero component on their sum. This identifies the dual map as inclusion of the first summand, hence identifies the original map by finite double duality. Write $j : C \to V_-$ for the resulting section.

The image of V_+ is an O-submodule of Oy. If it were contained in $\mathfrak{m}_O y$, the equality $L = V_+V_-$ would put its image in $y\mathfrak{m}_O C$, strictly smaller than yC by Nakayama's lemma. Choose $u \in V_+$ mapping to y. Multiplication gives the required section $a \mapsto u\, j(a)$ from C to L, after identifying yC with C. The section may depend on u. Finally h elements generate V_+ over the discrete valuation ring. Their products with V_- give a surjection from h copies of $C \oplus C_1$ onto L, proving the upper bound and the last assertion. □

Thus the odd two-dimensional quotient kills all inverse-trace differences, even integrally, while on the other quotient their image contains the full complementary group under the mixed specialization. The splitting computes its position inside a finite module. A uniform upper bound for its order in Proposition 6.11 still requires arithmetic information about that module or its summand.

40.5 The integral product of the two adjacent root modules

Keep the ring $A' = A/J_2$ and the notation of Section 40.4. The multiplication $V_+ \otimes_O V_- \to L$ there is surjective. We compute its kernel and the elementary divisors of both factors. All calculations use the same x, ℓ, r, α, K as the mixed trace construction. Write $k = O/\ell O$. The coefficient ring O is unramified. For any character ξ write $S(\xi)$ for the strict group over $\mathbb{Q}$ with coefficients $D(\xi)$, unramified away from ℓ and locally zero at ℓ, where $D = \mathrm{Frac}(O)/O$. Let $F = E_*K$, with E_* the common period overfield of conductor dividing N_* on the chosen interval. Thus $[F : \mathbb{Q}]$ is prime to ℓ, and F is unramified at ℓ. For an $O[\mathrm{Gal}(F/\mathbb{Q})]$-module M write M_θ for the summand with action by the specified character θ.

Proposition 40.17 (Elementary divisors of the positive root module)**.** *Write P for the torsion submodule of V_+. The module V_+ has free rank one. For $r \geq 2$, its torsion has the elementary divisors of*

$$S(\kappa_\ell \Psi^{-1}) \ \oplus \ S(\eta\kappa_\ell \Psi^{-1}).$$

For $r = 1$, put $B_F = \mathrm{Cl}(F) \otimes_{\mathbb{Z}} O$. The elementary divisors are those of

$$(B_F)_{\alpha^{-1}} \oplus (B_F)_{(\eta\alpha)^{-1}}.$$

In particular the same class number h_F annihilates this torsion for all the first twists on the interval, and $\log h_F \ll [F : \mathbb{Q}] \log(2sN_*)$.

Proof. Put $O_j = O/\ell^j O$. A map $V_+ \to O_j$ is a square-zero deformation at the augmentation supported on the upper adjacent entries. Passage to A' leaves the positive cotangent module unchanged for the same reason as the negative one in Proposition 40.15. After dividing out the split diagonal, the two upper coordinates are one-cocycles. In the directions $E_{12} - E_{23}, E_{12} + E_{23}$ their coefficient characters are $\Psi, \eta\Psi$, since the sign acts on the Lie algebra by minus the reflected transpose. Global cohomology classes determine their marked representatives uniquely. The mark at c determines the odd coboundary, as two is a unit. An even coboundary with scalar parameter v changes the upper mark at g_* by $\pm(1-\lambda)(1+\lambda^{-1})v$, where $\lambda = \Psi(g_*)$. The first and second upper entries of the even root times $d(g_*)$ are 1 and $-\lambda^{-1}$, which gives the displayed coefficient. It is a unit. These are all the marks in this weight.

Away from ℓ the cocycles must give unramified classes. Indeed vanishing on the original inertia kernel is equivalent to being a coboundary on inertia, by averaging over its prime-to-ℓ image. At ℓ the upper triangular matrix with either such cocycle satisfies the ordinary word and the fixed inertia polynomial. For $r \geq 2$ this gives the selected local condition by Theorem 32.11, since the Tate duals of the positive root entries have inertia characters $\bar{\kappa}_\ell^{1-r}, \bar{\kappa}_\ell^{1-2r}$, which are nontrivial in this range. Thus there is no further local condition on the two upper cohomology classes at ℓ.

For $\gamma = \Psi, \eta\Psi$, give $O_j(\gamma)$ these local conditions and call the resulting Selmer group $H_j(\gamma)$. The orthogonal conditions on $O_j(\kappa_\ell \gamma^{-1})$ are unramified away from ℓ and zero at ℓ. At a finite place away from ℓ, two unramified classes pair to zero since their cup product comes from the residue group of cohomological dimension one. The size of each unramified group equals the size of its local invariants by the kernel and cokernel formula for Frobenius minus one. Local duality and the local Euler characteristic therefore give equality with the orthogonal complement, including at ramified places. The trace pairing of the unramified coefficient ring with $\mathbb{Z}_\ell$ is perfect, identifying the indicated coefficient twist with the Tate dual for this calculation.

Subtract the lengths for these two Selmer conditions in the Poitou–Tate size formula. It gives the difference of the global invariant lengths plus the sum over places of the length of the imposed local condition minus the local invariant length. This formula follows by taking orders in the Poitou–Tate sequence and applying the global and local Euler characteristics as in Proposition 37.5. Both global invariant groups vanish by their inertia powers at ℓ. The terms at all finite places away from ℓ vanish by the Frobenius calculation. At ℓ the term is j by the local Euler characteristic, since the local dual invariants vanish. At the real place the term is $-j$ for an even γ and zero for an odd one. The dual finite-coefficient Selmer group is $S(\kappa_\ell \gamma^{-1})[\ell^j]$. In fact the multiplication sequence on $D(\kappa_\ell \gamma^{-1})$ identifies the global first cohomology at finite coefficients with the indicated torsion because there are no global invariants. It does the same at ℓ. At other places the inertia invariants of the divisible module are divisible by prime-to-ℓ averaging, giving injectivity on inertia in the same sequence. This proves, for $r \geq 2$,

$$\operatorname{length}_O H_j(\gamma) = j\, \mathbf{1}_{\gamma(c)=-1} + \operatorname{length}_O S(\kappa_\ell \gamma^{-1})[\ell^j].$$

The strict groups here are finite by the negative-twist finiteness in Proposition 11.4 after descent to the abelian field F. [7]

When $r = 1$, write $\gamma = \kappa_\ell \theta$ with $\theta = \alpha$ or $\eta\alpha$. The local condition at ℓ in the upper entries is now the unit Kummer condition. Indeed Theorem 32.15 imposes precisely the vanishing of the valuation coordinates. After the unramified splitting of the finite characters, the two coordinates are the ordinary Kummer valuations on the adjacent lines. In the nonsplit case the two lines are interchanged and the same description follows over the unramified quadratic extension by Shapiro's lemma, as in that theorem. The changes between the root bases use only units, including two. Restriction to the completion of F detects these conditions, since the degree is prime to ℓ and valuations are multiplied by the ramification index under extension.

Finite group averaging also identifies the global cohomology with the corresponding character part over F.

Kummer theory now identifies the Selmer group with the θ^{-1}-part of the global elements whose valuations at every finite place are zero modulo ℓ^j, taken modulo ℓ^j-th powers and extended to O. Away from ℓ, extracting roots of units is unramified, while an element whose valuation is not a multiple of ℓ^j cannot have an ℓ^j-th root in an unramified extension. The ideal of a permissible element is $\mathfrak{b}^{\ell^j}$. Sending the element to the class of $\mathfrak{b}$ gives the exact sequence with kernel $(\mathcal{O}_F^\times \otimes_{\mathbb{Z}} O)/\ell^j$ and quotient $B_F[\ell^j]$, before taking the character summand. Surjectivity follows by taking a generator of any principal $\mathfrak{b}^{\ell^j}$. The character projection is exact. There is no ℓ-power torsion in the units, since F is unramified at $\ell > 2$. Dirichlet's unit theorem identifies their rational character with the permutation character on complex-conjugate pairs of embeddings minus the trivial character. This gives multiplicity zero for the odd α^{-1} and one for the even $(\eta\alpha)^{-1}$, which is nontrivial since K is disjoint from E_*. Hence the lengths have the same free contribution j in total as above and the displayed class group torsion contributions at each j.

In either case the sum of the two Selmer lengths equals $\operatorname{length}_O \operatorname{Hom}_O(V_+, O_j)$ by the marked description. For a finitely generated module over this discrete valuation ring, successive differences of these lengths count the free rank plus the number of torsion elementary divisors with exponent at least j. Their limiting value is the rank. This proves the assertions about V_+. Finally Minkowski's bound represents every ideal class by an integral ideal of norm at most $C^{[F:\mathbb{Q}]}|\operatorname{disc} F|^{1/2}$ for an absolute C. The number of ideals up to X is at most $X^2\zeta_F(2) \le X^2\zeta(2)^{[F:\mathbb{Q}]}$. The abelian conductor-discriminant formula bounds the discriminant by $(sN_*)^{[F:\mathbb{Q}]}$, giving the asserted uniform estimate. □

Lemma 40.18 (Diagonal corrections for both signs)**.** *Let Z_0 be a finite O-module with trivial Galois action, and let S contain infinity, ℓ and all primes of the prescribed ramification. For each of the modules $B = Z_0, Z_0(\eta)$ the map on second cohomology from $G_{\mathbb{Q},S}$ to the local groups is injective. The map on first cohomology is onto the direct sum at the finite places in S of local first cohomology modulo its unramified subgroup.*

Proof. For the sign summand these are exactly the two assertions of Lemma 36.10, which uses the class number of the same K. For the trivial action decompose the underlying finite abelian group into cyclic summands of orders ℓ^j. The Tate dual of one such summand is μ_{ℓ^j}. A global dual class orthogonal to every local unramified subgroup at S is a rational Kummer element whose valuation is divisible by ℓ^j at each finite place of S, by local reciprocity. The same holds outside S by unramifiedness away from ℓ. Its absolute value is therefore an ℓ^j-th power in $\mathbb{Q}^\times$, as is its sign. The dual class is zero. Poitou–Tate duality now gives the first-cohomology surjectivity because the orthogonal complement of the global image is supplied by the global dual classes just tested. An everywhere locally trivial dual class belongs to the same zero group, giving injectivity in degree two. The real place has zero modified cohomology at odd ℓ. [7] □

Theorem 40.19 (The full integral multiplication map)**.** *On every large-prime triple under consideration, multiplication is an isomorphism*

$$V_+ \otimes_O V_- \xrightarrow{\sim} L.$$

It identifies A' with the tensor product over O of the two square-zero algebras $O \oplus V_+$ and $O \oplus V_-$ as augmented graded algebras. It also identifies the invariant trace ring T/Y with $O \oplus (V_+ \otimes_O V_-)$.

Proof. Set $Z_0 = V_+ \otimes_O V_-$, a finite-length module, and form the tensor product algebra B in the statement. Its augmentation ideal has cube zero. Products of two positive entries or two negative entries are zero. Multiplying one of each gives its tensor in Z_0, which is killed by the augmentation ideal. Over each $O \oplus V_\pm$ take the representation supplied by the respective projection from A', killing the other weight and L. Its matrix part is $(1+p(g))d(g)$, respectively $(1 + m(g))d(g)$, for $g \in G_{\mathbb{Q},S}$, with the original sign in the semidirect product. Here p, m are one-cocycles in the upper and lower adjacent matrices, with their split adjoint actions. We shall realize both projections over B.

Seek a matrix part $\exp(p(g) + m(g) + w(g))d(g)$, with w in trace-zero diagonal matrices of coefficient Z_0. On this latter module the split action is trivial on the diagonal vector $(1, 0, -1)$ and has character η on $(1, -2, 1)$. These form a basis since two is invertible. Products of an upper and a lower step are diagonal, so all commutators of the two directions lie in this module. The central extension in the unipotent matrices has the multiplication rule $\exp U = 1+U+U^2/2$ and $\exp U \exp V = \exp(U + V + [U, V]/2)$, by multiplication. Thus the defect for $w = 0$ is a continuous two-cocycle with the indicated diagonal coefficients. Associativity gives the cocycle identity, and adding a diagonal one-cochain changes the defect by its coboundary. All resulting matrices have determinant one, since the exponent has trace zero. This also follows by expanding the determinant through degree two.

At $v \neq \ell$ in S both cocycles factor through the local quotient by the prescribed inertia kernel. Its finite inertia has order prime to ℓ and its residue group is procyclic. The inflation sequence therefore gives zero second cohomology there for the diagonal coefficients. Choose a local solution on this quotient. At ℓ the universal negative cocycle is a coboundary, by strictness in the finite module V_-. The marked classification in Corollary 32.9 applies to every finite coefficient module. Conjugate the pure upper lift over B by a matrix $1 + v_-$ with lower adjacent entries in V_- giving that coboundary. This gives a local solution with the same two first-order parts. The difference in exponential coordinates is a trace-zero diagonal with coefficients in Z_0. The change of basis is taken in the semidirect product, so the assertion also holds at a nonsplit place.

The two-cocycle is now locally trivial everywhere. Apply Lemma 40.18 to both signs to obtain a global solution. Differences from the local solutions are one-cocycles. Subtract a global one-cocycle so that these differences have unramified cohomology class at every finite place of S. At $v \neq \ell$ the remaining difference is a coboundary on inertia and vanishes on the prescribed inertia kernel, on which its coefficient action is trivial. At ℓ the diagonal action is unramified, hence the remaining difference vanishes on inertia itself. At c the pure cocycles are zero by the marks. The relation $c^2 = 1$ makes $w(c)$ anti-invariant, and a diagonal coboundary kills it since two is a unit. This conjugation preserves the conditions just imposed.

To check the entire local component at ℓ, undo the lower change of basis used above. The resulting upper triangular lift differs from the pure upper lift only by the diagonal cocycle with unramified class. It has the same inertial diagonal, since the coboundary at inertia is zero. For $r \geq 2$ the ordinary word already gives the component by Theorem 32.11. For $r = 1$, use the valuation description of Theorem 32.15. The adjacent upper cocycle takes its values in the submodule $V_+ \subset B$ of the additive coefficients. The new diagonal acts on this submodule exactly as the original diagonal, since $Z_0 V_+ = 0$. Its cocycle there is the same as in the pure upper lift, whose valuation coordinates are zero. Mapping that coefficient module into the full adjacent module commutes with valuation by Lemma 32.14. In the nonsplit case perform this check on the first line over the unramified quadratic extension as in the proof of Theorem 32.15. These assertions can equally be tested modulo all powers of the maximal ideal of B. They prove membership in the selected component over B for the first twist as well.

All outer matrix entries vanish, since a path of length two to an outer entry uses two steps of the same sign. The upper and lower entries of the marking equation at g_* remain those of the original pure lifts because every mixed correction is diagonal. The constructed lift thus determines a map $A' \to B$. Its coefficients have the indicated torus weights, and it induces the identity on V_+ and V_- after projecting to the pure square-zero algebras. These are also the whole degree-one and degree-minus-one parts of B, so the map on L sends the product of any two such elements to their tensor. This is a left inverse to the multiplication surjection of Proposition 40.15. Hence multiplication is bijective. The decompositions and the products already computed identify the two rings as asserted. □

Corollary 40.20 (Lengths and exponents of the additional torsion)**.** *Put $C = S(\Psi^{-1})^\vee$, $C_1 = S(\eta\Psi^{-1})^\vee$ and $M = C \oplus C_1$. Write $d_j(H) = \dim_k(\ell^{j-1}H/\ell^j H)$ for a finite O-module H. The kernel of the mixed map $L \to C$ has, as an O-module, the elementary divisors of $C_1 \oplus (P \otimes M)$. In particular*

$$\operatorname{length}_O L - \operatorname{length}_O C = \operatorname{length}_O C_1 + \sum_{j\ge 1} d_j(P)d_j(M).$$

Any power of ℓ killing both C_1 and P kills this entire kernel. At the first twist, if $\ell \nmid h_F$, the only additional summand is C_1.

Proof. If $C \neq 0$, the positive map in the proof of Theorem 40.16 is a surjection onto Oy. By Proposition 40.17 it has kernel P, and a preimage of y splits off the free summand. On the negative factor the map is projection of M onto C. Theorem 40.19 shows that on the entire tensor it is the product of these two maps. Its kernel has the asserted summands. If $C = 0$, choose any free generator modulo P. The entire tensor then has the asserted elementary divisors. Two cyclic modules of exponents a, b have tensor length $\min(a, b)$. Summing this minimum as a sum of indicators over j gives the formula. The annihilation statements follow from the tensor description and Proposition 40.17. □

The length formula distinguishes the orders at weight r from the additional contributions detected at weight $r - 1$. Even when both additional groups vanish, the summand $S(\Psi^{-1})^\vee$ at weight r is still present at its full order. An independent upper estimate for the weighted sum in Proposition 6.11 could be obtained by bounding this same trace quotient. The length identity leaves the size of the surviving summand undetermined.

40.6 Counting lower triangular lifts on the period interval

Keep the interval $[V, V^{3/2}]$, period fields $E_x \subset E_*$ and fixed $K = \mathbb{Q}(\sqrt{-s})$ of Proposition 36.3. For a triple on this interval write $\Psi = \alpha\kappa_\ell^r$, with $\Psi(c) = 1$, $1 \le r < t(x)$, and $\ell > \exp(x^\varepsilon/12)$. Let η be the character of K and let $O = \mathbb{Z}_\ell[\alpha]$ have fraction field J and residue field k. In particular O is unramified. The columns pair each character with its root matrix.

$$\begin{array}{c|c|c} a = \Psi^{-1} & b = \eta\Psi^{-1} & d_2 = \eta\Psi^{-2} = ab \\ X = E_{21} - E_{32} & Y = E_{21} + E_{32} & Z = E_{31}. \end{array}$$

For any finite coefficient module W, denote by $S(W)$ the kernel of restriction in $H^1(\mathbb{Q}, W)$ to $G_{\mathbb{Q}_\ell}$ and to inertia at every other finite prime. Write $S(\xi)$ for the same kernel with divisible coefficient $(J/O)(\xi)$. We use $B_j = O/\ell^{j+1}O$, $I_j = \ell B_j$ and $S_j(\xi) = S(I_j(\xi))$, for $j \ge 1$.

Let B be a finite local Artinian O-algebra with residue field k, and put $I = \mathfrak{m}_B$. Denote by $\mathcal{L}(B)$ the lifts in the original marked ordinary matrix problem of Section 40.4 that are lower triangular on G_K. Denote by $\mathcal{O}(B)$ those whose matrices on G_K preserve the fixed antidiagonal

form J_3 in the marked basis. The sign acts on matrices by the involution $U \mapsto J_3 U^{-\mathsf{T}} J_3$. Both conditions consequently make sense on the full polarized representation. We determine the fibers of $\mathcal{L}(B)$ over the strict classes for a.

Lemma 40.21 (Local splitting and the two marks)**.** *Put $d(g) = \operatorname{diag}(\Psi(g), 1, \Psi(g)^{-1})$, and write $\delta(g) = (d(g), \eta(g))$ in the polarized group. Let $U(I)$ consist of lower triangular unipotent matrices whose off-diagonal entries belong to I, with $G_{\mathbb{Q}}$ acting by conjugation through δ. Then $\mathcal{L}(B)$ identifies with the $U(I)$-conjugacy classes of lifts $v(g)\delta(g)$, where v is a $U(I)$-valued cocycle, locally a coboundary at ℓ and on inertia at every $p \neq \ell$. Each such class has a unique marked representative.*

Proof. Over G_K the three diagonals of any triangular lift are characters. Locally above ℓ the prescribed ordinary flag and the standard lower triangular flag are opposite on reduction. They remain opposite over B. Indeed the sums of complementary steps are isomorphisms because their determinants are units. Intersecting the ordinary step of rank i with the lower step spanned by $e_i, \dots, e_3$ gives a stable line projecting isomorphically to Be_i along the smaller lower step. Its generator with i-th coordinate one reduces to e_i. These three generators make a lower unipotent change of basis. The ordinary flag used here exists over B itself by the separating inertia projectors and the word identity of Lemma 32.10. In this basis the inertia characters on the three lines are respectively $\Psi, 1, \Psi^{-1}$. A triangular conjugation preserves the diagonal. At the other finite places the prescribed inertia quotient has order prime to ℓ. The ratio of each diagonal to its indicated character takes inertia there into $1 + I$, an ℓ-group, and is trivial. Thus each ratio gives an everywhere unramified ℓ-power character of G_K. It is trivial by class field theory and $\ell \nmid |\operatorname{Cl}(K)|$.

The local change of basis at ℓ now conjugates the lift to δ on $G_{K,w}$ for the chosen place w. The stabilizer of $\delta|_{G_{K,w}}$ in $U(I)$ is trivial. The inertia element g_* has diagonal eigenvalues with pairwise unit differences. A matrix commuting with it is diagonal. For an element of $G_{\mathbb{Q}_\ell}$ outside $G_{K,w}$, if there is one, the cocycle identity and normality show that its unipotent factor after this conjugation belongs to that stabilizer. This proves the full local splitting. At $p \neq \ell$ the cocycle is trivial on the prescribed inertia kernel. Any cocycle on the remaining prime-to-ℓ quotient is a coboundary. To see this, project to the two adjacent entries and average, then remove the residual central cocycle by the same argument. Conversely an inertia coboundary vanishes on the prescribed kernel, where the action is trivial. At ℓ a conjugate of the split lift belongs to the selected ordinary component, also at $r = 1$. Indeed conjugate the split lift by the universal lower unipotent matrix over $O[[t_1, t_2, t_3]]$. Its evaluations at $t_i \in \ell O$ are conjugates of the same crystalline lift. A power series vanishing at all these points is zero, by the one-variable identity theorem in each variable. The family therefore factors through the reduced crystalline closure with the ordinary equations. Its kernel is prime and is contained in the kernel of split evaluation. It contains a minimal prime through that evaluation, and hence the intersection defining the selected local quotient in Section 32.4. Every lower unipotent matrix congruent to one over B is obtained by specialization.

At a marked lift the matrix at c is one. As G_K and c generate $G_{\mathbb{Q}}$, the fixed diagonal on G_K gives the expression $v\delta$ on the whole group. Conversely take a cocycle satisfying the local conditions. First conjugate the lift's value at c to $\delta(c)$. This is possible by the same averaging argument on the adjacent quotient and the center, since two is a unit. The remaining allowable conjugators in $U(I)$ are $\exp(tX)$, $t \in I$. In fact logarithm on a strictly lower three-by-three matrix is $N \mapsto N - N^2/2$ for $1 + N$, and the differential of the involution fixes X and negates Y, Z. Write the adjacent coordinates of $\log v(g_*)$ as $uX + wY$, with $\lambda = \Psi(g_*)$. The remaining nonautomatic mark equates the 21 and 32 entries of the matrix $v(g_*)d(g_*)$. Their difference is $(\lambda + 1)u + (\lambda - 1)w$. Conjugation by $\exp(tX)$ adds $(1 - \lambda^{-1})t$ to u and fixes w. The coefficient

$(\lambda+1)(1-\lambda^{-1})$ is a unit, so exactly one t gives the mark. The upper mark already vanishes. This proves the identification, including uniqueness. □

Theorem 40.22 (The fiber over each complementary class)**.** *Projection to the X-coordinate gives a surjection $\mathcal{L}(B) \to S(I(a))$ with a section taking values in $\mathcal{O}(B)$. For $u \in S(I(a))$ choose its cocycle representative x zero at g_*. Define $W_x = I \oplus I$ with action*

$$g(y,z) = \big(b(g)y,\ d_2(g)z - 2x(g)b(g)y\big).$$

The fiber above u is in bijection with $S(W_x)$. It fits into an exact sequence of finite O-modules

$$0 \longrightarrow S(I(d_2)) \longrightarrow S(W_x) \longrightarrow D_x \longrightarrow 0, \qquad D_x \subseteq S(I(b)).$$

For $B = B_j$ this gives, at every j and every u,

$$|S_j(d_2)| \ \le\ \#\mathcal{L}(B_j)_u \ \le\ |S_j(d_2)|\,|S_j(b)|.$$

The upper bound is an equality whenever $\ell^e u = 0$, $\ell^h S_j(b) = 0$, and $e + h \le j + 1$, with $0 \le e, h \le j$. In particular

$$\lim_{j\to\infty} \#\mathcal{L}(B_j) = |S(a)|\,|S(b)|\,|S(d_2)|.$$

Proof. Every element of $U(I)$ factors uniquely as $\exp(yY + zZ)\exp(xX)$. This follows by reading the two adjacent coefficients and then the central coefficient. The first factor is an abelian normal subgroup because $[Y,Z] = 0$ and $[X,Y] = -2Z$. Under conjugation by $\delta(g)$, X, Y, Z have the three characters in the table. The projection to x is therefore an additive cocycle with character a. Any class with the stated local conditions has exactly one representative zero at g_*, since $a(g_*) - 1$ is a unit. That representative vanishes on $G_{\mathbb{Q}_\ell}$ as its local class is zero. The lift $\exp(xX)\delta$ preserves J_3, and satisfies the local conditions of the lemma. It gives the claimed section.

Fix this representative x. In the factorization the other two coordinates are a cocycle with values in the abelian normal subgroup, acted upon through $\exp(xX)\delta$. Conjugation sends Y to $Y - 2xZ$, which gives exactly W_x. Once x is fixed, a conjugation preserving it has zero X-coordinate because $I(a)^{G_{\mathbb{Q}}} = 0$. Thus the equivalence on the other two coordinates is precisely coboundary equivalence in W_x. At ℓ this argument works on the decomposition group as well, since $x = 0$ there and $a(g_*) - 1$ is a unit. At another inertia group both the action through δ and the base lift through x factor through the prescribed prime-to-ℓ quotient. A cocycle in W_x is unramified exactly when it vanishes on the kernel of that quotient. This follows by averaging after passage to the quotient. Combined with the lemma, these facts identify the fiber with $S(W_x)$.

Use the coefficient sequence $0 \to I(d_2) \to W_x \to I(b) \to 0$. Invariants of $I(b)$ vanish globally and at ℓ by its nontrivial residual inertia character. Consequently degree one on the central module injects both globally and at ℓ. At any other inertia group, every invariant in $I(b)$ lifts to an invariant in W_x by averaging a lift over the finite prime-to-ℓ quotient. This gives the same injectivity on inertia. A central global class mapping into $S(W_x)$ therefore has every required local vanishing. Exactness in global cohomology proves the sequence with D_x the image of the last map. Finiteness follows at finite coefficients from the finiteness of global cohomology with restricted ramification.

For each of the three characters ξ, identify I_j with $(J/O)[\ell^j]$ by dividing its representatives by ℓ^{j+1}. The coefficient sequence then identifies $S_j(\xi) = S(\xi)[\ell^j]$. Indeed global and ℓ-local invariants of $(J/O)(\xi)$ vanish by g_*. At every other inertia group its invariants are divisible,

since they are the image of a prime-to-ℓ averaging idempotent. The same argument applies to each subideal of I_j. In particular if $\ell^e u = 0$, the class has a representative in $\ell^{j+1-e}B_j$. Normalizing it at g_* keeps its values in that ideal. Every class in $S_j(b)$ comes from $\ell^{j+1-h}B_j$ when $\ell^h S_j(b) = 0$. If $e+h \le j+1$ the product of the two ideals is zero. Then $y \mapsto (y, 0)$ embeds this last coefficient ideal with character b into W_x equivariantly. It carries strict classes to strict classes and yields $D_x = S_j(b)$. All three divisible strict groups are finite by restriction to E_*K, of degree prime to ℓ, and Proposition 11.4. Taking j larger than their exponents measured in powers of ℓ, and large enough for the sum condition, gives the asserted eventual equality. □

Proposition 40.23 (The total contribution of the fibers on one interval)**.** *Set $F = E_*K$, $g = [F : \mathbb{Q}]$, $N = sN_*$, and $T = \lfloor V^{(3/2)401} \rfloor$. For the cyclotomic modulus N of F use the nonzero critical determinants of Lemma 11.3, and put*

$$H = \prod_{n=2}^{2T+1} |D_n(F)|, \qquad H' = h_F H, \qquad h_F = |\operatorname{Cl}(F)|.$$

Then $\log H' \ll_{\psi,\varepsilon,s} V^{1220}$. Take all distinct triples $(\ell, r, [\alpha])$ from the chosen interval, counting each residue Frobenius orbit once after inflation to E_*. At every triple take the supremum over all depths $j \ge 1$ and all $u \in S_j(a)$. One has

$$\sum_{\ell,r,[\alpha]} \sup_{j,u} \log \#\mathcal{L}(B_j)_u \;\le\; 17 \log H.$$

In particular the same bound holds for the sum of the logarithms of $\#\mathcal{L}(B_j)/|S_j(a)|$, with depths chosen independently at each triple.

Proof. For clarity, the determinant at weight n is the product of $A_n L^{(N)}(1-n, \chi^{-1})$ over the complex characters of $\mathrm{Gal}(F/\mathbb{Q})$ of sign $(-1)^n$. Here $L^{(N)}$ omits Euler factors at primes dividing N. Using two distinct fixed primes P_1, P_2 unramified in F, prime to N, one may take

$$A_n = (P_1^{gn} - 1)(P_2^{gn} - 1)\, nN(n+1)!.$$

Such primes can be fixed over all sufficiently distant intervals, since the two moving prime factors of N_* tend to infinity. The Bernoulli formula, nonvanishing by the functional equation, and the size estimate of Lemma 11.3 apply to this abelian CM field. They give $\log H \ll gT^2\{g + \log(2NT)\}$. The Minkowski ideal count and the conductor–discriminant formula give $\log h_F \ll g \log(2N)$. The estimates $g \ll_\psi V^{3+12\varepsilon}$, $\log(2N_*) \ll_{\psi,\varepsilon} V^{3\varepsilon/8}$ and $T^2 \le V^{1203}$ from Theorem 19.15 give the asserted bound for $0 < \varepsilon < 1/52$.

All primes ℓ occurring in the proposition are prime to Ng on a sufficiently distant interval. For a character β of F, restriction identifies $S(\beta^{-1}\kappa_\ell^{-m})$ with the β-part of the strict group of F at weight m after extension of coefficients. This follows by finite group averaging in global and local restriction, on inertia as well as on the group at ℓ. For $m \ge 1$, the sum of the O-lengths of those parts with $\beta(c) = (-1)^{m+1}$, weighted by their minimal residue degrees and taken over residue orbits, is bounded by $v_\ell(D_{m+1}(F))$. Indeed Poitou–Tate duality embeds the dual of the strict part into degree two. The refined Coates–Sinnott theorem puts the critical special value times the indicated integer in its zeroth Fitting ideal. The required Iwasawa μ-invariant vanishes since $F(\mu_\ell)$ is abelian over $\mathbb{Q}$. Each factor of the critical integral group ring is an unramified discrete valuation ring because $\ell \nmid g$. Its Fitting valuation bounds the strict length, and weighting by residue degrees gives the determinant valuation. Lemma 11.2 and its proof give these statements with the imprimitive modulus and the precise coefficient duality. The primary special-value theorem is Theorem 6.11 of Greither and Popescu. Authors' paper.

For b, d_2 the corresponding pairs (m, β) are $(r, \eta\alpha)$ and $(2r, \eta\alpha^2)$. Both signs are critical. Counting geometric characters counts each residue orbit with its residue degree, because $\ell \nmid g$. Passing from the minimal ring of $\eta\alpha^2$ to that of α leaves its length unchanged, by finite free unramified base change of the defining cochains and local maps. The first character map is injective. The second has at most sixteen preimages. Indeed restriction embeds the Galois group of E_* into a product of two cyclic groups and two groups of order two. Its subgroup killed by two has order at most sixteen. The character group has the same two-torsion order by the structure theorem for finite abelian groups. At any given m, the first list uses only $r = m$, the second only $r = m/2$. The sum of $\log|S(b)| + \log|S(d_2)|$ over the triples is therefore at most $17 \log H$, since $\sum_\ell v_\ell(D) \log \ell = \log|D|$ for a nonzero integer. Theorem 40.22 bounds each fiber by these two orders. The ratio in the last assertion is the arithmetic mean of the fiber cardinalities. □

Theorem 40.24 (Equality with the marked even orthogonal locus)**.** *For a triple in Proposition 40.23 with $\ell \nmid H'$, the two closed subfunctors $\mathcal{L}$ and $\mathcal{O}$ coincide on all finite local Artinian O-algebras. Their representing ring, including its ℓ-torsion, is*

$$O[[T_1, \dots, T_q]]/(\ell^{a_1} T_1, \dots, \ell^{a_q} T_q), \qquad S(\Psi^{-1}) \simeq \bigoplus_{i=1}^{q} O/\ell^{a_i} O.$$

On this whole ring every trace on G_K is the split trace.

Proof. Proposition 40.23, or its determinant argument at this prime, gives $S(b) = S(d_2) = 0$. For clarity, each local length in its proof is bounded by a valuation of a factor of H, and every such valuation is zero. By finite-coefficient comparison the same vanishing holds with coefficients $O/\ell^j O$. Every finite O-module I is a sum of cyclic modules. Thus $S(I(b)) = S(I(d_2)) = 0$. The fiber theorem shows that every lower triangular lift equals the marked representative of its orthogonal section. This proves $\mathcal{L} \subseteq \mathcal{O}$.

For the reverse inclusion let R be the quotient of the original marked ring by the equations preserving J_3. It is local, complete and Noetherian and is augmented at d. Conjugation by $\mathrm{diag}(t, 1, t^{-1})$ preserves the form and both marks. Use the integer weights of matrix entries under this split torus, as in Lemma 40.13. We show that the relative cotangent space $\mathfrak{m}_R/(\ell R + \mathfrak{m}_R^2)$ is generated by entries of weight -1.

A first-order perturbation has values, after division by the split representation, in $\mathfrak{so}(J_3, k)$. This has basis $X, E_{12} - E_{23}, E_{11} - E_{33}$ because two is a unit. There is no outer root at first order. The sign involution acts identically on this Lie algebra. The diagonal coordinate is a cocycle with trivial coefficients. The prescribed ordinary inertia characters at ℓ force it to vanish there on inertia. Infinitesimal conjugation of a diagonal matrix has zero diagonal variation. At every other finite place the same vanishing follows by the prime-to-ℓ inertia quotient. An everywhere unramified additive character of $G_{\mathbb{Q}}$ with coefficients in k is zero by class field theory. Thus the diagonal contributes no generators.

The upper coordinate has character Ψ. At the marks its cohomology class determines its representative uniquely, as in Proposition 40.17. For $r \geq 2$ that proposition identifies the length of the finite-coefficient upper Selmer group for this even character at depth one with the length of $S(\kappa_\ell \Psi^{-1})[\ell]$. Its pair $(m, \beta) = (r - 1, \alpha)$ is critical and $m \geq 1$. The determinant $D_r(F)$ in H thus gives zero length. For $r = 1$ the character α^{-1} is odd. Its unit part in the Kummer calculation of that proposition vanishes, and its only possible contribution comes from $\mathrm{Cl}(F) \otimes O$, which is zero since $\ell \nmid h_F$. Hence no upper tangent coordinate survives. Together with the Lie algebra calculation, this proves the claim about weights of the relative cotangent

space. Indeed the entries generate the original matrix ring and its quotient, and maps to the dual numbers separate the cotangent space.

Choose entries of weight -1 lifting a basis of that space. Together with ℓ they generate $\mathfrak{m}_R$ by Nakayama's lemma. Successive approximation in its powers and completeness give a surjection from a power series ring over O in these entries onto R. On each Artinian truncation their monomials have nonpositive weights. Extracting the coefficient of t^w in the torus action kills their span for each $w > 0$. As they span the truncation over O, every positive-weight entry is zero there. Separation gives the same conclusion in R. In particular all upper triangular off-diagonal entries vanish in the universal orthogonal lift. This proves $\mathcal{O} \subseteq \mathcal{L}$ at every depth.

The section of Theorem 40.22 now represents the whole functor by $S(I(a))$. Decomposing the finite module $S(a)^\vee$, its normalized cocycle presentation in Theorem 36.6 identifies this functor with $\mathrm{Hom}_O(S(a)^\vee, I)$. The presentation depends only on a and its local conditions, so it applies to this choice of K. Each cyclic generator maps to an arbitrary element of I killed by its stated power of ℓ. These equations give the displayed ring. Its universal matrices are lower triangular with fixed diagonal, proving the assertion about traces. □

The complementary order $|S(\Psi^{-1})|$ occurs with its full exponent in the limiting count in Theorem 40.22. Proposition 40.23 controls the remaining factors on the very same interval. An upper bound for the count itself would therefore give information on the complementary order. The rank-three comparison in `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/ThresholdArguments/DenseThresholdCases/FOneCharU.lean`, line 33 uses the finiteness of the full invariant algebra and the automorphic description of its contributing points. The orthogonal trace calculation above supplies neither an upper bound for this count nor that description of the full algebra.

40.7 Invariant summands and the largest trace minors

Use the marked ring A, its trace algebra T on $G = G_K$ and its trace augmentation $\mathfrak{a} = \ker(T \to O)$ of Section 40.4. In particular the coefficient prime, the characters and the quadratic field are those of Section 40.6 on the original interval. Write $M(h)$ for the universal matrix of $h \in G$. Let $J_\uparrow, J_\downarrow$ be the ideals of A cutting out upper and lower triangular matrices, respectively. Recall that J_1 kills every adjacent entry and J_2 kills both outer entries. Put

$$J_{\mathrm{f}} = J_\uparrow \cap J_\downarrow \cap J_1, \qquad X = T \cap J_1, \qquad Y = T \cap J_2.$$

For a local Artinian coefficient algebra D, an invariant summand means a G-stable direct summand of D^3, of rank one or two as a D-module. The complementary summand need not be stable.

Proposition 40.25 (The trace image of families with an invariant summand)**.** *Every invariant summand of a marked lift over D is a coordinate summand in the marked basis. A lift admits such a summand if and only if its map from A factors through one of $A/J_\uparrow, A/J_\downarrow, A/J_1$. The common kernel of the traces on all these families is X. Thus their scheme-theoretic image in the formal trace space has coordinate ring T/X.*

Proof. The three coordinate projectors belong to the algebra spanned by the matrices on G. Indeed the difference of the matrices at g_* and $cg_*^{-1}c$ is diagonal with entries $z, 0, -z$ whose differences are units. Interpolation gives the projectors, as in Section 40.4. A stable direct summand P is consequently the sum of three ideals in the three coordinate lines. Its quotient D^3/P is free. Each of the three corresponding cyclic quotients is a direct summand of this free module and hence projective. A quotient D/I which is projective splits off D, so I is

generated by an idempotent. A local ring has just the idempotents zero and one. This proves the assertion about coordinates.

If the subset of indices S spans an invariant summand, then the complement of $\{4-i : i \in S\}$ does as well. Indeed the marked polarization on G is $M(ch^{-1}c) = J_3 M(h)^{\mathsf{T}} J_3$. It reflects each vanishing entry through the antidiagonal. Thus stability of the first line forces stability of the first two coordinates. Together these conditions give upper triangular matrices. Stability of the last line gives lower triangular matrices. Stability of the middle line gives stability of the outer plane, which kills precisely the four adjacent entries. Starting with a two-element subset gives one of the same three cases by reflection. Each stated quotient visibly preserves at least one of these summands.

In either triangular quotient every trace is the split trace. For completeness, the ordered ordinary inertia characters give the prescribed diagonal characters on inertia above ℓ. One can see this without choosing an ordinary basis compatible with the given triangularization. For a separating inertia element g, each diagonal entry at g is its prescribed root by reduction and the pairwise unit differences. For any inertia element h, compute the trace at $g^j h$ for $j = 0, 1, 2$ in both triangular bases. The Vandermonde matrix in the three roots at g is invertible, giving the three diagonal values at h. Away from ℓ the ratios to the split characters are trivial on inertia by the prime-to-ℓ inertia quotient. The class number of K is prime to ℓ, so the three ratios are trivial globally. Apply this argument in every Artinian truncation of each triangular quotient to obtain the trace identity in the complete rings. Hence

$$T \cap J_{\uparrow} = T \cap J_{\downarrow} = \mathfrak{a}.$$

The split point belongs also to the outer-block quotient A/J_1, so $X \subseteq \mathfrak{a}$. It follows that $T \cap J_{\mathrm{f}} = X$. Taking the intersection of the three kernels computes the image ideal, including for the disjoint union of families with their choice of summand. This use of the intersection imposes no assertion that every Artinian point of a scheme-theoretic union lifts to one of its components. □

Theorem 40.26 (A nilpotent extension of the flag trace ring)**.** *Let I_9 be the ideal generated by the nine-by-nine determinants of $(\operatorname{tr} M(h_i k_j))$, for $h_i, k_j \in G$. Then*

$$I_9 = X^4 Y^2, \qquad X^8 \subseteq I_9 \subseteq X^4.$$

In particular T/I_9 is finite over O. Its surjection onto T/X has kernel with eighth power zero. This kernel is killed by a power of ℓ.

Put $A' = A/J_2$. The ideal defining the image of the invariant-summand families in A' is $J_{\mathrm{f}} A'$, and there is an exact sequence of O-modules

$$0 \longrightarrow V_+ \otimes_O V_- \longrightarrow A' \longrightarrow (O \oplus V_+) \times_O (O \oplus V_-) \longrightarrow 0.$$

Its first term is precisely $J_{\mathrm{f}} A'$. It is also the image in A' of X/I_9.

Proof. Let $\mathcal{E}$ be the T-span of the matrices on G and use the coordinate projectors e_i from the proposition. The trace pairs $e_i \mathcal{E} e_j$ only with $e_j \mathcal{E} e_i$. A matrix of pairings for one pair of corners has entries $a_u b_v$ over A, since both coordinate lines have rank one. Any two-by-two minor within that block vanishes, also in $T \subseteq A$. By multilinearity the rows and columns of the nine-by-nine determinants can range over the corners of $\mathcal{E}$. A nonzero term uses all nine different corners with the opposite column corners, and its value is the product of their nine trace pairings, up to sign. Conversely any such product is realized by taking one row and the opposite column from each corner. The three diagonal corners contribute unit ideals

since $e_i^2 = e_i$ has trace one. The four adjacent positions each contribute X, and the two outer positions contribute Y, by Lemma 40.13. This gives the formula for I_9 with the very projectors used for the invariant summands. The same lemma gives $X^2 \subseteq Y$ by composing adjacent corners. Squaring this inclusion and multiplying by X^4 gives $X^8 \subseteq I_9$. The other inclusion follows from the formula.

The flag trace ring T/X is finite over O by the odd two-dimensional descent of Theorem 40.14. There the residual ratio on inertia is nontrivial, the character lifting one is ordinary with unramified weight, and the weight-augmentation quotient of Pan's ordinary pseudodeformation ring is finite. The descent gives a surjection to this precise trace ring. The terms in the filtration by powers of X/I_9 are finite modules over T/X because T is Noetherian. The filtration has at most eight terms including its degree zero quotient. This proves finiteness of T/I_9 directly. For the statement about torsion use Theorem 40.10, which gives a reduced generic fiber for T/I_9. Its nilpotent ideal X/I_9 becomes zero after inversion of ℓ. Finiteness over O now gives a common power of ℓ killing that ideal.

In A', Theorem 40.19 gives a direct sum of O, V_+, V_- and $L = V_+ \otimes_O V_-$. Both pure modules have square zero, and the augmentation ideal annihilates L. The pure modules are generated over O by their upper or lower adjacent matrix entries. Indeed products of two augmentation elements have weight zero in A', and entries generate the matrix algebra of coefficients. Imposing upper triangularity in A' kills V_- and L. Imposing lower triangularity kills V_+ and L. Imposing the outer block kills both modules, leaving just O. The intersection of these three ideals in A' is exactly L. The image of J_{f} belongs to that intersection. Conversely $X \subset J_{\mathrm{f}}$ by the proposition, and the image of X in T/Y is its entire augmentation L because $X + Y = \mathfrak{a}$. Thus the intersection is $J_{\mathrm{f}}A'$, also the image of X/I_9. The direct sum decomposition gives the asserted exact sequence. □

Theorem 40.27 (A critical exponent for traces on invariant summands)**.** *Write $\mathfrak{b} = \mathfrak{a}/X$ in T/X. Let $e = e_{\ell,r,[\alpha]}$ be the least nonnegative integer such that $\ell^e S(\eta\Psi^{-2}) = 0$. For all $m \geq 1$,*

$$\ell^e \mathfrak{b}^m \subseteq \mathfrak{b}^{m+1}, \qquad \ell^{me}\mathfrak{b} \subseteq \mathfrak{b}^{m+1}.$$

For the distinct triples on the interval and the integer H of Proposition 40.23,

$$\sum_{\ell,r,[\alpha]} [O : \mathbb{Z}_\ell]\, e_{\ell,r,[\alpha]} \log \ell \;\leq\; 16 \log H.$$

At $\ell \nmid H$, every marked lift admitting an invariant summand on G is upper or lower triangular. The trace ring of these families is O, and the augmentation ideal of T/I_9 has eighth power zero.

Proof. Use $A_{\mathrm{p}} = A/J_1$, whose middle coordinate and outer plane are stable. Its weight-zero subring is $T_{\mathrm{p}} = T/X$ by Lemma 40.13. Write P_-, P_+ for its submodules of weights $-2, 2$, respectively. They are finite T_{p}-modules generated by the outer matrix entries of the indicated weights. Indeed a finite set of group entries generates A_{p} topologically. Among the surviving entries the weights are $0, 2, -2$. Any monomial of weight -2 or 2 factors as a single outer entry of that weight times a weight-zero monomial. The conclusion follows on finite truncations by extracting the indicated weight. It passes to the completion because the span of finitely many entries over the compact T_{p} is closed. The same argument with the diagonal split values subtracted works for the augmentation filtration.

In particular $P_-/\mathfrak{b}P_-$ is the weight -2 augmentation cotangent module of A_{p} over O. A monomial of weight -2 of augmentation degree at least two contains either a diagonal augmentation factor or an opposite pair of outer entries. Each of these factors belongs to $\mathfrak{b}$, proving the assertion. The adjacent entries removed in passing from A to A_{p} have weights $1, -1$. Modulo

the square of the augmentation ideal their multipliers reduce to scalars in O. Removing them therefore leaves the weight -2 cotangent module intact. By Theorem 32.19 this module is $S(\eta\Psi^{-2})^{\vee}$. To recall the sign and the local conditions, a square-zero perturbation in E_{31} has coefficient character $\eta\Psi^{-2}$ since inverse transpose followed by reversal negates E_{31}. The mark at c fixes its coboundary representative, the ordinary flag gives a locally split class at ℓ, and the specified inertia kernels elsewhere give the unramified condition by finite averaging. These arguments apply with the character of the present quadratic field. It follows that $\ell^{e}P_{-} \subseteq \mathfrak{b}P_{-}$.

The ideal $\mathfrak{b}$ is $P_{+}P_{-}$, where a product of modules means the span of all products. Indeed it is generated by traces of opposite outer corners since $\mathfrak{a} = X + Y$. In the coordinate basis these traces are exactly the products of the corresponding scalar entries. Conversely every such product is invariant and has zero augmentation. Hence $\ell^{e}\mathfrak{b} \subseteq \mathfrak{b}^{2}$. Multiplication by $\mathfrak{b}^{m-1}$ gives the first inclusion. Applying it at m successive filtration steps gives the second.

The exponent e is bounded by the O-length of the outer strict group. In the determinant calculation of Proposition 40.23 the map at weight $2r$ is $\alpha \mapsto \eta\alpha^{2}$, of multiplicity at most sixteen. Its terms alone give the stated sum. In particular at $\ell \nmid H$ the exponent is zero. The equality $\mathfrak{b} = \mathfrak{b}^{2}$ then forces $\mathfrak{b} = 0$ by Nakayama's lemma. Also $P_{-} = \mathfrak{b}P_{-} = 0$, so the entire outer-block family is upper triangular. The proposition on invariant summands now gives the claimed classification and $\mathfrak{a} = X$. The eighth-power assertion follows from Theorem 40.26. □

Corollary 40.28 (Finite coefficient algebras detecting the missing lengths)**.** *Let $C = S(\Psi^{-1})^{\vee}$, let $O_j = O/\ell^{j}O$, and put*

$$D_j = O_j[u, v]/(u^{2}, v^{2}).$$

There are $|S(\Psi^{-1})[\ell^{j}]|$ distinct trace maps from T to D_j indexed by $f \in \mathrm{Hom}_{O}(C, O_j)$. They are realized by marked matrix lifts and factor on traces through T/I_9. For each f the image of X is $uv\,f(C)$. If $f \neq 0$, this trace admits no realization by a marked lift with an invariant direct summand on G, even over a faithfully flat local Artinian extension of D_j.

If C has a cyclic factor of length at least j, this construction detects an element of additive order ℓ^{j} in the mixed trace image. Among finite free O_j-algebras testing A/J_2 for such an element the minimum rank is four. Every map from A/J_2 into a local Artinian principal ideal ring has split trace.

Proof. Choose a surjection $p : V_{+} \to O$ splitting off the free summand of Proposition 40.17. Use the projection $q : V_{-} \to C$ of Proposition 40.15. Map the two square-zero algebras $O \oplus V_{+}$ and $O \oplus V_{-}$ to D_j by their usual scalar map and by $w \mapsto u\,p(w)$, $w' \mapsto v\,f(q(w'))$ on the augmentations. Theorem 40.19 identifies their tensor product with A/J_2, so these maps specialize the universal marked matrices. The image of L is exactly $uv\,f(C)$, by surjectivity of p, q. The image of X is L before that specialization and I_9 already maps to zero. If two f's differ on $c_0 \in C$, take a lift of c_0 under q and an element sent to one by p. Their tensor has a preimage in X. Its two images in the free O_j-summand $O_j uv$ differ. This gives the count by finite duality, identifying O_j with $(J/O)[\ell^{j}]$. An invariant summand forces X to have zero image by Proposition 40.25. The coordinate projector argument works over every local Artinian coefficient extension, and faithful flatness preserves any nonzero image. This proves the assertion about realizations of these traces.

If the stated factor exists, choose f surjective to O_j. The image contains uv, of order ℓ^{j}. Conversely let a map from A/J_2 to an O_j-algebra detect that order in L. As L is spanned by products from its two pure modules, one such product ab must have order ℓ^{j}. Otherwise ℓ^{j-1} kills the whole image. Its factors satisfy $a^{2} = b^{2} = 0$. A relation $s_0 + s_1 a + s_2 b + s_3 ab = 0$ over O_j, multiplied first by ab, gives $s_0 = 0$ since the annihilator of ab in O_j is zero. Multiplying next by b and by a gives $s_1 = s_2 = 0$, and then $s_3 = 0$. Thus these four elements span a

free submodule of rank four. Length over O gives the lower bound on the rank of a finite free target. The displayed D_j attains it.

Finally, in a local Artinian principal ideal ring whose maximal ideal has nilpotence index n, each nonzero element is a unit times a power of a generator of that ideal. Every square-zero element belongs to its $\lceil n/2 \rceil$-th power. Any product of two such elements is zero, including in the field case. Thus L maps to zero and the trace on this quotient is the split trace. □

The eighth-power inclusion is a bound on products in an ideal. The estimate in Theorem 40.27 controls the coefficient exponents on the invariant-summand quotient. On the kernel between the two trace conditions, the preceding corollary exhibits the entire complementary group at every coefficient depth. A uniform upper bound for the weighted orders of the resulting complementary summands on the retained triples is the large-prime estimate still required in Section 6.6.

40.8 A cyclic complementary group in the fixed-weight deformation

Return to the large-prime components on the intervals of Section 6.6. We use the marked quadratic problem with $K = \mathbb{Q}(i)$ of Section 32.4. Write η for its character, $O = \mathbb{Z}_\ell[\alpha]$, $k = O/\ell O$, $J = \mathrm{Frac}(O)$ and $\Psi = \alpha\kappa_\ell^r$. Take $2 \le r < t(x)$, $\alpha(c) = (-1)^r$ and $\ell > \exp(x^\varepsilon/12)$. Write $S(\chi)$ for the strict group with coefficients $(J/O)(\chi)$ on $\mathbb{Q}$, as in Section 40.6. This is the scalar comparison for the actual field E_x in Theorem 22.17.

There is a single nonzero integer H_0 on the interval which may be used to ensure

$$S(\eta\Psi^{-1}) = S(\eta\Psi^{-2}) = 0 \qquad (\ell \nmid H_0).$$

Indeed take $F = E_*(i)$ on the interval, and the product of the critical determinants $|D_n(F)|$ of Lemma 11.3 for $2 \le n \le 2t_* + 1$, where $t_* = \lfloor V^{(3/2)401} \rfloor$. The characters $\eta\alpha^{-1}$ and $\eta\alpha^{-2}$ have precisely the critical signs for $n = r + 1$ and $n = 2r + 1$. Restriction to F is injective on the indicated classes by corestriction, since $\ell > [F : \mathbb{Q}]$ for large V. The critical order estimate in the proof of Proposition 40.23 bounds each relevant length by the valuation of its determinant. The calculation there uses an abelian field with the modulus enlarged by the conductor of the fixed quadratic field. Here $4N_*$ is such a modulus. Its estimates on the degree, modulus and twist range give $\log H_0 \ll_{\psi,\varepsilon} V^{1220}$ as there.

Assume in this subsection that $\ell \nmid H_0$ and that $S(\Psi^{-1})$ is cyclic of positive length a. Let A be the complete fixed-weight marked matrix ring, T its trace ring on G_K, and $\mathfrak{a}$ the augmentation ideal in T. The letter T in this subsection denotes the ring. We shall construct a quotient of T whose fiber at the split trace retains $O/\ell^a O$.

Proposition 40.29 (A presentation of the trace ring for a cyclic strict group)**.** *There are integers $p, q \ge 1$, a power series ring*

$$S = O[[Z_1, \dots, Z_p, U_1, \dots, U_q]], \qquad \mathfrak{b} = (Z_1, \dots, Z_p, U_1, \dots, U_q),$$

two elements $F_1, F_2 \in S$, and an ideal $J_+ \subset \mathfrak{b}$ generated by at most $p + q - 2$ elements, such that

$$T \simeq S/(J_+ + \mathfrak{b}(F_1, F_2)),$$
$$D = S/(J_+, F_1, F_2), \qquad D/\mathfrak{b}D \simeq O/\ell^a O.$$

The trace augmentation corresponds to $Z_i = U_j = 0$. The quotient $T \to D$ is an isomorphism wherever some element of $\mathfrak{a}$ is invertible.

Proof. Use the integer weights for the marking torus $\operatorname{diag}(s, 1, s^{-1})$. By Theorem 32.19, the negative cotangent modules at the split augmentation, over O, are the duals of $S(\Psi^{-1}) \oplus S(\eta\Psi^{-1})$ in weight -1 and of $S(\eta\Psi^{-2})$ in weight -2. The weight-zero module vanishes by Proposition 32.18. In the present case the three modules are $O/\ell^a O, 0, 0$, respectively, with the nonzero one in weight -1.

In the homogeneous presentation of Theorem 32.13, eliminate each variable which is a residual linear term of a relation with unit coefficient. To justify elimination, apply a linear change among variables of the same weight over O, and then replace the variable by the entire homogeneous relation. This is an automorphism of the complete power series ring because its linear change on the residue cotangent space is invertible. Its inverse can also be constructed successively modulo the powers of the maximal ideal. All relations have zero value at the split augmentation. Quotient by the new variable and continue. Every elimination removes one variable and one relation of the same weight. Their classes give a k-basis of the relative cotangent space at the closed point. The difference bounds between the two rows of Theorem 32.13 still hold weight by weight, since a redundant relation can only decrease their number.

There is therefore a single variable y of weight -1, and all the other variables have positive weights. Write $x_1, \dots, x_p$ for those of weight 1 and $w_1, \dots, w_q$ for those of weight 2. The same table gives at most two relations of weight -1, at most $p-1$ of weight 1 and at most $q-1$ of weight 2, with no others. In particular $p, q \geq 1$. Pad the two negative relations with zeros if necessary.

In the ambient power series ring set $Z_i = x_i y$ and $U_j = w_j y^2$. An invariant monomial contains precisely as many powers of y as the sum of its positive weights. Thus invariant series form exactly S, with unique expansions. A monomial of degree d in Z, U has ordinary degree between $2d$ and $3d$ after substitution. Hence both directions respect completion. Every series of weight $-s$ for $s > 0$ is y^s times an invariant series. Write the negative relations as yF_1, yF_2. A positive relation g of weight s gives the invariant $y^s g \in \mathfrak{b}$. Use these invariants to generate J_+.

To contract the relation ideal to the invariants, take weight zero in each expression of an element as a finite sum of relations times power series. For g positive of weight s the allowed multiplier is $y^s S$. For yF_i the allowed multipliers h have weight one. The elements yh fill $\mathfrak{b}$, since they contain $yx_i = Z_i$ and $y(yw_j) = U_j$, and they have zero constant term. This gives the claimed ideal of relations in S. To identify the invariants with the trace ring, use the coordinate projectors. They belong to the span of the matrices on G_K, and invariant products of their entries factor into traced cycles, as in Lemma 40.13. This calculation uses the two marks and the distinct residual characters at the selected inertia element, all present for the chosen quadratic problem. Alternatively the passage to the invariant part of the completed presentation is precisely the weight-zero calculation of Lemma 32.17.

The integral cotangent module in weight -1 has generator y and relations $F_1(0)y, F_2(0)y$, by taking linear terms at the augmentation over O. It was identified above with $O/\ell^a O$. Consequently $(F_1(0), F_2(0)) = \ell^a O$. This proves the fiber formula. Inverting any member of $\mathfrak{b}$ makes the ideals $\mathfrak{b}(F_1, F_2)$ and (F_1, F_2) equal, proving the final assertion. □

Theorem 40.30 (A residual trace curve or a determinant bounding the full length)**.** *For the cyclic group and ring D of the proposition the following alternatives are exhaustive.*

If $D/\ell D$ has positive dimension, there is a continuous trace specialization $T \to k'[[s]]$, with k'/k finite, which is split at the closed point and nonconstant. Its geometric generic semisimple representation on G_K is absolutely irreducible.

If $D/\ell D$ is zero-dimensional, D is finite free over O and is a complete intersection. Every geometric point of $D[1/\ell]$ has absolutely irreducible semisimple representation on G_K. For

every $d \in \mathfrak{a}$ which is nonzero at each of these points,

$$a \;\le\; \operatorname{length}_O(D/dD) \;=\sum_{\rho} m_\rho \, v_\ell(d(\rho)), \qquad v_\ell(\ell) = 1,$$

where ρ ranges over the geometric trace points and m_ρ is the dimension of the corresponding local algebra of $D \otimes_O \overline{J}$ over $\overline{J}$. In particular this applies to a Frobenius trace minus its split value whenever it separates the points from that value.

Proof. First consider the reducible locus in the trace space. Let J_1 in A be generated by the adjacent entries and $X = T \cap J_1$, as in Section 40.7. The weight -1 tangent variable can be taken from the adjacent matrix entries. Indeed these entries span the cotangent space in this weight, since the matrix entries generate A and the diagonal projectors belong to their span on G_K. Choosing a homogeneous adjacent entry with nonzero residual linear term as y gives another coordinate by the same change-of-variables argument. Killing all adjacent entries therefore kills y. The resulting ring is generated by positive weights over O and has invariant quotient exactly O, using its split augmentation. Thus $X = \mathfrak{a}$.

The ideal I_9 of the nine-rowed minors of the trace pairing contains X^8 by Theorem 40.26. For clarity, the matrix projectors separate three diagonal corners and the six off-diagonal corners. Pairing each corner with its opposite gives one factor of X for each adjacent corner and one factor of the outer pairing ideal Y for each outer corner. The largest minors generate X^4Y^2, and multiplication through the middle coordinate gives $X^2 \subset Y$. These identities use only the paired corners of the marked matrices and hold with the present K. At a geometric field point with reducible semisimple representation the trace pairing has rank at most $2^2 + 1^2 = 5$, by taking traces in the simple constituent blocks. Here the semisimple representation associated with the degree-three determinant exists over the algebraically closed field, in residue characteristic greater than three as in Theorem 40.7. Thus any geometric point at which some element of $\mathfrak{a}$ is nonzero is absolutely irreducible.

If $D/\ell D$ has positive dimension, choose a prime giving a one-dimensional complete local domain quotient. The normalization is finite, since this quotient of a power series ring over a finite field is excellent. Completing at a maximal ideal of the normalization gives a complete equal-characteristic discrete valuation ring $k'[[s]]$ with finite residue extension. The map from D into it has one-dimensional image. Since $D/(\ell, \mathfrak{b}) = k$, some element of $\mathfrak{b}$ has nonzero image in the maximal ideal. Some trace must then be nonconstant, since the ring is topologically generated over O by traces. The closed and generic traces have the properties asserted.

In the other case put $n = p + q$. The n-variable ring S is regular local of dimension $n + 1$. The at most n relations defining D, together with ℓ, generate a maximal-ideal-primary ideal, because $D/\ell D$ is Artinian. The fiber calculation gives $D/\ell D \neq 0$. The height theorem forces exactly n relations in such a list. These relations followed by ℓ form a system of parameters in S, hence a regular sequence since S is Cohen–Macaulay. In particular ℓ is regular on D. A power of the maximal ideal of D is contained in ℓD. Completeness and successive lifting of a k-basis of $D/\ell D$ show that D is finitely generated over O. Being torsion-free over a discrete valuation ring, it is free. The quotient $D/\mathfrak{b}D = O/\ell^a O$ disappears after inverting ℓ. Hence every point in characteristic zero has some nonzero element of $\mathfrak{a}$, proving the irreducibility assertion there.

Finally D/dD surjects onto $D/\mathfrak{b}D$. Multiplication by d on D has nonzero determinant by the hypothesis on its values. The valuation of this determinant is the length of the cokernel by diagonalization over O. After extension to $\overline{J}$, each local factor has a filtration with m_ρ copies of its residue field. Multiplication acts on each quotient by $d(\rho)$. Multiplication of these determinants proves the equality and the required inequality. □

The contribution of this residue orbit to the large-prime sum is $[O : \mathbb{Z}_\ell] a \log \ell$, with the full exponent a. To obtain the uniform sum in Section 6.6 by this argument one would still need to handle the residual curves, control the specializations and their multiplicities by the common Hecke data, and treat the complementary groups having more than one generator.

41 Strict orders and descent to auxiliary fields

41.1 The full strict order in the fiber of a trace quotient

Keep the marked problem over $K = \mathbb{Q}(i)$, the field E_x, $2 \le r < t(x)$, and the large prime ℓ of Section 40.8. In this subsection the number of generators of the complementary group is arbitrary. Let A_j be the weight-j part of the matrix deformation algebra. Products of homogeneous elements can be taken in $A = \bigoplus_j A_j$, as in Lemma 32.17. Its invariant ring $T = A_0$ is the trace algebra on G_K, by the projector calculation of Lemma 40.13. Write $\mathfrak{n}$ for the augmentation ideal of A and $\mathfrak{a} = T \cap \mathfrak{n}$. Set $M = A_{-1}$ and $C_j = (\mathfrak{n}/\mathfrak{n}^2)_j$. Each A_j is finite over the complete Noetherian ring T. We write $N^\vee = \mathrm{Hom}_O(N, J/O)$ for a finite O-module. All lengths below are over O.

Proposition 41.1 (The fiber of the negative weight module)**.** *At every such prime there is an exact sequence of O-modules*

$$C_{-2} \otimes_O C_1 \longrightarrow M/\mathfrak{a}M \longrightarrow C_{-1} \longrightarrow 0.$$

The first map is multiplication of lifts. In particular the middle module has finite length. If $\ell \nmid H_0$, where H_0 is the common integer of Section 40.8, then

$$M/\mathfrak{a}M \simeq S(\Psi^{-1})^\vee.$$

Proof. Use split-augmented variables of weights $0, \pm 1, \pm 2$ in the presentation of Theorem 32.13. A monomial of weight -1 which contains a variable of weight zero, an opposite pair, or a triple of weights $(1, 1, -2)$ or $(-1, -1, 2)$ belongs to $\mathfrak{a}M$. Its indicated subproduct is a nonconstant invariant, and the remaining factors still have weight -1. Without such a subproduct, the only possibilities are a single variable of weight -1 and a pair of weights $(-2, 1)$. Indeed after removing opposite pairs, mixed signs require distinct absolute weights. If the positive weight is two, reaching total weight minus one requires at least two negative units as soon as a positive factor occurs. If the positive weight is one, two negative factors of weight minus two would require at least three positive units. This proves the assertion about monomials. It applies to series by choosing a subproduct in each discarded monomial and collecting terms by the finitely many possible subproducts. The remaining series in each group has weight minus one and converges in the completed presentation.

The cotangent projection kills $\mathfrak{a}M$. Its kernel in the fiber consists of classes represented by terms of ordinary degree at least two, since the cotangent of the quotient is obtained by taking the linear terms of the defining equations. Equivalently one can lift a sum of products from $(\mathfrak{n}^2)_{-1}$ to products of augmented series. Only the pairs $(-2, 1)$ can survive. Changing either factor by an element of the square of the augmentation introduces terms of degree at least three, all of which vanish in the fiber by the monomial calculation. Multiplication therefore gives the asserted map on cotangents and fills the kernel. Theorem 32.19 identifies C_{-1} with the dual of $S(\Psi^{-1}) \oplus S(\eta\Psi^{-1})$ and C_{-2} with the dual of $S(\eta\Psi^{-2})$. Both have finite length and C_1 is finitely generated over O. At $\ell \nmid H_0$ the two strict groups involving η vanish by the critical estimate in Section 40.8. This gives the last claim. □

Theorem 41.2 (Trace quotients detecting every elementary divisor)**.** *Assume $\ell \nmid H_0$. Write $C = S(\Psi^{-1})^{\vee}$ and list the exponents of its elementary divisors in increasing order as $e_1 \le \cdots \le e_s$, each positive. For $j \ge 0$ let $D_j = T/\operatorname{Fitt}_j^T(M)$. Then*

$$D_j/\mathfrak{a}D_j \simeq O/\ell^{b_j}O, \qquad b_j = \sum_{i=1}^{\max(s-j,0)} e_i.$$

At every prime of T outside $V(\mathfrak{a})$, the natural map to D_0 is an isomorphism after localization. For $j \ge 1$ the ring D_j has finite length over O.

Proof. In the minimal homogeneous presentation, the only possible negative variables are $y_1, \dots, y_s$ of weight minus one. Indeed $C_{-2} = C_0 = 0$. Eliminate each residual linear relation with a unit coefficient by taking that relation as a new variable of its weight, as in Proposition 40.29. The number of remaining weight-minus-one variables is the residue dimension of C_{-1}. Write x_i and w_h for the variables of weights one and two. By pairing every x_i with one y_a and every w_h with two of them, every invariant monomial is a product of

$$x_i y_a, \qquad w_h y_a y_b.$$

These elements generate $\mathfrak{a}$ as an ideal. In fact the products have degrees two or three, so that grouping invariant series by a first factor gives series in the ring of invariants. Likewise every monomial of weight minus one has a factor y_a with invariant complement. The y_a thus generate M over T, and these statements descend through the homogeneous relations.

Each displayed invariant generator has the form $g = fy_a$ with $f \in A_1$. Inverting g in the graded algebra makes y_a invertible, with inverse f/g. Multiplication by y_a identifies $T[1/g]$ with $M[1/g]$, with inverse multiplication by f/g. These open sets cover the complement of $V(\mathfrak{a})$. There the zeroth Fitting ideal is zero and the higher ones are the unit ideal, as is seen from a free presentation on one generator.

For a finite presentation on n generators the jth Fitting ideal is generated by the minors of size $n - j$ of the relation matrix, with the unit ideal for size zero or less. Base change of the matrix presents the base change of the module, and taking its minors commutes with base change. Apply this to $T \to O$. The proposition gives C as the resulting module. Its diagonal presentation over O has entries ℓ^{e_i}, so the ideal of the minors is generated by ℓ^{b_j}. This also covers the zero module and empty sums.

For $j \ge 1$ the preceding localization shows that every prime of D_j contains the image of $\mathfrak{a}$. By the fiber formula the only possible such prime is the closed point. The Noetherian local quotient is therefore zero or Artinian. In the latter case its simple factors are copies of $k = O/\ell O$, proving the finite-length assertion. □

Corollary 41.3 (The exterior powers on the split trace)**.** *Under the hypotheses of Theorem 41.2, for every $h \ge 2$ reduction at the split trace gives an isomorphism*

$$\bigwedge_T^h M \;\simeq\; \bigwedge_O^h C.$$

For $2 \le h \le s$ its length is $\sum_{i=1}^{s-h+1} \binom{s-i}{h-1} e_i$. It is zero for $h > s$.

Proof. Use a generator $g = fy$ of $\mathfrak{a}$ from the preceding proof, with $f \in A_1$ and $y \in M$. For $m, n \in M$, the elements fm and fn belong to T. Commutativity of the matrix coefficient algebra gives

$$\begin{aligned} g(m \wedge n) &= (fm)y \wedge n = y \wedge (fn)m \\ &= -(fn)m \wedge y = -g(m \wedge n). \end{aligned}$$

Here $(fm)n = (fn)m$ and $(fn)y = gn$. Since two is a unit, g annihilates $\bigwedge_T^2 M$, and hence also every higher exterior power. Tensoring such a power with $T/\mathfrak{a}$ is thus an isomorphism. Exterior powers commute with this base change by their alternating multilinear presentation. The proposition identifies the result with $\bigwedge_O^h C$. In a direct sum of cyclic modules each increasing h-tuple of summands contributes their tensor product, a cyclic module of length the smallest exponent in the tuple. Exactly $\binom{s-i}{h-1}$ tuples have smallest index i. Summing proves the formula. □

In particular $b_0 = \operatorname{length}_O S(\Psi^{-1})$ records the exponent of the entire group order in residue degree one. In residue degree f its contribution to the logarithmic sum is $f b_0 \log \ell$. The fiber formula gives this length even when several generators occur. Bounding it on the interval still requires an upper estimate for the integral trace quotient.

41.2 A Frobenius determinant for a strict group of arbitrary rank

The cyclic case used a finite flat complete intersection. For a strict group with several generators the quotient in Theorem 41.2 can have integral torsion. We obtain a bound which uses the multiplicities in its generic fiber. We first give the modification of the intersection calculation that accounts for those multiplicities.

Proposition 41.4 (The intersection inequality with local multiplicities)**.** *Use the augmented presentation and rational formal chart of Section 31.5, with its variable and equation counts. Write B for the graded solution algebra, $T = B_0$, and $k_0 = \overline{J}$. Suppose T is finite over O, the negative equations span the two rational negative cotangents, and no fixed point other than the distinguished point has both a purely positive and a purely negative point over its invariant character. Let Σ be the geometric characters of $T \otimes_O k_0$ nonzero on the ideal $B_{-2}B_2$. For each $\sigma \in \Sigma$ put*

$$m_\sigma = \max_{j=0,1} \dim_{k_0} (B_j \otimes_O k_0)_\sigma,$$

where the subscript means localization at the character of the invariant algebra. Let $d \in T$ vanish at the distinguished point and be nonzero at every $\sigma \in \Sigma$. For the quotient C_F of the integral negative cotangent lattice in Theorem 31.8, one has

$$\operatorname{length}_O C_F \le 2 \sum_{\sigma \in \Sigma} m_\sigma v_\ell(d(\sigma)).$$

Here O is unramified and $v_\ell(\ell) = 1$.

Proof. Lift d to the ambient invariant ring. In the three-color algebra of Section 31.3, Lemma 31.4 puts the high multidegrees on a finite O-module with generic support in Σ. Its proof uses finiteness, the condition on fixed points and rational cotangent spanning. Indeed a relevant prime over a character which kills the product of the two even weights would give a purely positive and a purely negative point over one fixed point. At the distinguished point rational cotangent spanning excludes the purely negative point. None of these arguments use a bound on a tangent dimension at a mixed point.

For $\sigma \in \Sigma$, some product from weights $-2, 2$ is a unit after localization. Both factors are homogeneous units in the localized graded algebra. This holds also in the ambient algebra localized at the inverse image of σ. In the solution algebra multiplication by the weight-two unit identifies any two pieces whose weights differ by two. Each piece therefore has dimension at most m_σ. These dimensions are finite because the individual weight modules are finite over T.

Write R, R' for the numbers denoted by r, r' in Section 31.3. In its localized calculation of the high pieces, one squares $R+1+R'$ equations of odd weight. If f has weight b and S is the algebra with the other current equations imposed, multiplication by f gives the exact sequence in each weight

$$(S/(f))_{j-b} \longrightarrow (S/(f^2))_j \longrightarrow (S/(f))_j \longrightarrow 0.$$

Consequently each squaring at most doubles the maximum dimension over all weights. The unit of weight two makes each of the three localized generating modules in Section 31.3 cyclic. As in the proof of Theorem 31.5, a high piece is therefore a quotient of the weight-zero module with the squared equations imposed. Its dimension at σ is at most $2^{R+1+R'} m_\sigma$. Multiplication by d on each residue factor has eigenvalue $d(\sigma)$. On the integral torsion the kernel and cokernel have equal lengths. Thus the upper bound for the difference of lengths in that theorem becomes $2^{R+1+R'} \sum_\sigma m_\sigma v_\ell(d(\sigma))$.

For the lower bound use the same formal chart and Koszul complex as in Theorem 31.8. Its Smith basis gives factors ℓ^{2c_i} from the squared negative equations, with $\sum c_i = \operatorname{length}_O C_F$. The finite difference of the two monomial counts in that proof is $2^{R-1+R'}$. Removing the scalar factors in the Koszul complex therefore bounds its Euler characteristic below by $2^{R+R'} \sum c_i$. The lower calculation takes place in the chart at the distinguished point. Its use of the other characters is only through admissibility and the upper estimate, both just proved. Dividing the new upper bound by $2^{R+R'}$ proves the inequality. □

Theorem 41.5 (A residual curve or a horizontal determinant bound)**.** *Use the original period field E_x and the marked quadratic problem over $K = \mathbb{Q}(i)$. Let $1 \le r < t(x)$, $\ell > P(x)$, $\ell \nmid H_0$, and suppose $C = S(\Psi^{-1})^\vee \neq 0$. Let T be the full invariant trace algebra, $M = B_{-1}$ its negative weight module, and $D = T/\operatorname{Fitt}_0^T(M)$. On sufficiently distant intervals precisely one of the following cases occurs.*

If $T/\ell T$ has positive dimension, there is a continuous nonconstant trace $T \to k'[[z]]$, split at the closed point, with k'/k finite. Its generic representation on G_K remains absolutely irreducible over every finite extension of K.

If $T/\ell T$ has dimension zero, set $H = D/D[\ell^\infty]$. Then H is finite free and nonzero over O. The geometric generic points of H are exactly the nonsplit trace points of T, and all are absolutely irreducible. For every $d \in \mathfrak{a}$ nonzero at these points,

$$\operatorname{length}_O C \;\le\; 2v_\ell(\det(d \mid H)) \;=\; 2\sum_\sigma n_\sigma v_\ell(d(\sigma)), \qquad n_\sigma = \dim_{\overline{J}}(H \otimes_O \overline{J})_\sigma.$$

Here d acts by multiplication. The determinant counts the lengths of the local algebras even at nonreduced points.

Proof. The negative variables and invariant generators in the proof of Theorem 41.2 apply throughout this range. For the first twist the two extra strict groups still have the critical signs, now for determinant indices two and three in H_0. The integral cotangent formula is Theorem 32.19 also at this twist. The exceptional character in the first-twist equation count has zero complementary group at every sufficiently large prime, by Corollary 32.20. Thus a contributing first twist has the same required chart and counts. In particular $D/\mathfrak{a}D = O/\ell^{\operatorname{length}_O C} O$, and $T \to D$ is an isomorphism locally off $V(\mathfrak{a})$.

To check irreducibility, choose the minimal variables of weight minus one from the lower adjacent entries. This is possible since the matrix entries generate the ring and the classes of the entries of this weight span that residual cotangent. Removing all adjacent entries now leaves only positive variables and the split augmentation, so that the invariant quotient is O. In the notation of Section 40.7, this says $X = \mathfrak{a}$. The paired-corner computation in the

proof of Theorem 40.30 gives $X^8 \subseteq I_9$ for the trace pairing. Its rank is at most five on a reducible semisimple representation and nine on an absolutely irreducible one. Thus every nonsplit geometric trace is absolutely irreducible.

If the special fiber has positive dimension, take a one-dimensional complete local domain quotient of that fiber. Its normalization in its fraction field is finite since complete Noetherian local rings are excellent. Completing at a maximal ideal of the normalization gives $k'[[z]]$ with finite residue field. The trace ring has one-dimensional image, so some trace series is nonconstant. The closed point is the split residual trace by the defining augmentation modulo ℓ. The asserted strong irreducibility follows from Theorem 34.6 for this same marked quadratic problem.

Otherwise the special fiber is Artinian. Lift a finite residue basis and express every element by successive approximation modulo ℓ^j in their O-span. The coefficients converge in O, and the errors tend to zero in the maximal ideal topology of T. This proves that T, hence D, is finite over O. Removing its O-torsion gives a free module with the same generic fiber. The split fiber formula excludes the split generic point from D, and its local rings at all other generic points coincide with those of T.

At any such nonsplit character a generator fy of the invariant augmentation is nonzero, with y a negative variable of weight minus one and f of weight one as in Theorem 41.2. Thus y becomes a unit after localization and all the local weight pieces are isomorphic to the degree-zero piece. Its dimension is n_σ. Conversely the split evaluation kills all products from weights $-2, 2$. Hence the characters in the preceding proposition are exactly these nonsplit points, with $m_\sigma = n_\sigma$. The fixed locus is just the split point by Proposition 32.18, and Corollary 32.20 provides the chart, spanning and integral cotangent length there. The proposition gives the claimed inequality. The equality follows by taking the determinant on the local factors over $\overline{J}$, each filtered by copies of its residue field. If $H = 0$ the same proposition with no contributing character gives $\operatorname{length}_O C = 0$, contrary to the hypothesis. □

Corollary 41.6 (Frobenius tests attaining the minimum valuation)**.** *In the finite case of Theorem 41.5 let t be the trace on G_K and $t_0 = \Psi + 1 + \Psi^{-1}$ its split value. Put*

$$N(h) = \det\bigl(t(h) - t_0(h) \mid H\bigr) \in O \qquad (h \in G_K).$$

The ideal generated by all the $N(h)$ is nonzero. Its valuation is the minimum of $v_\ell N(\mathrm{Frob}_w)$ as w runs over good places of K of degree one outside any fixed finite set, with $v_\ell(0) = +\infty$. A set of places of positive Dirichlet density attains this minimum. Twice the minimum bounds $\operatorname{length}_O C$ *from above.*

Proof. Over a finite extension of J realize each of the finitely many irreducible representations ρ_σ at the generic points. Indeed the invariant evaluation is integral, since T is finite over O, and is local into the integer ring by uniqueness of the maximal ideal of T. It is continuous since a power of that maximal ideal lies in ℓT. Lemma 34.5 gives the continuous matrix realization after a finite extension. Take the joint Zariski closure of $(\Psi, (\rho_\sigma)_\sigma)$ over $\overline{J}$, and denote its identity component by G^0. The inverse image U of G^0 in G_K is open and normal. Indeed the component group is finite and its inverse images are closed by continuity. The image of U is Zariski dense in G^0, since this component is open in the Zariski closure.

For each σ the regular function $\operatorname{tr}\rho_\sigma - 1 - \Psi - \Psi^{-1}$ on this joint group is nonzero on G^0. Otherwise its vanishing on U would identify the semisimplification of $\rho_\sigma|_U$ with $1 \oplus \Psi|_U \oplus \Psi^{-1}|_U$ by the character criterion in characteristic zero. The restriction itself is semisimple. To see this, take an irreducible U-submodule and sum its G_K-translates. These are simple U-submodules by normality, and they span the whole space by irreducibility on G_K. The sum of simple

submodules is semisimple. The three characters are distinct on U since the cyclotomic character has infinite image on every open subgroup and $r > 0$. Each extends to G_K, so each of the three one-dimensional isotypic subspaces would be G_K-stable. This contradicts absolute irreducibility. Since the connected algebraic group G^0 is irreducible, the product of these finitely many nonzero functions is nonzero somewhere on its dense subgroup. The determinant formula in the theorem now gives some h with $N(h) \neq 0$.

Its nonzero values have nonnegative integral valuations, so the least valuation b is attained. The reduction of the continuous class function N modulo ℓ^{b+1} factors through a finite quotient of G_K. Indeed its fibers are clopen and compact. Finitely many cosets of open subgroups cover them, and one may take an open normal subgroup refining these cosets. The conjugacy classes of this quotient giving valuation b form a nonempty set. The Chebotarev density theorem gives a positive Dirichlet density of unramified places with their Frobenius in this set. Places of degree greater than one in the quadratic field have norm at least the square of their residual rational prime, so their sum of inverse norms converges. They have Dirichlet density zero. Removing them and the finitely many excluded places preserves the positive density. Evaluating the theorem at one of the remaining places proves the final assertion. □

This determinant is defined on the same scalar component as the original cohomology group at x. To sum its valuations in Section 6.6 one still needs arithmetic control of the nonsplit traces and their multiplicities, as well as control of the residual curves. The existence of minimizing primes gives no upper estimate for their determinant valuations as the fields and twists vary.

41.3 A uniform contribution to the large-prime strict length

Use $K = \mathbb{Q}(i)$ and the original fields E_x on the intervals of Section 6.6. We now allow every $\ell > P(x)$, including divisors of the critical determinant, and every $1 \le r < t(x)$. Keep $\alpha(c) = (-1)^r$ and $\Psi = \alpha\kappa_\ell^r$. Let $B = \bigoplus_j B_j$ be the graded marked solution of Lemma 32.17, with trace ring $T = B_0$, and let $\mathfrak{a}$ be its trace augmentation ideal. Write C_j for the integral cotangent in weight j. We bound the part of the total length accounted for by classical systems with a fixed prime and a single integer throughout the interval. This requires retaining the two-dimensional mixed points at primes where the outer negative cotangent is nonzero.

Lemma 41.7 (The even negative module at the split and mixed traces)**.** *Put $M_2 = B_{-2}$ and $D_2 = T/\operatorname{Fitt}_0^T(M_2)$. There is an exact sequence*

$$\operatorname{Sym}_O^2 C_{-1} \longrightarrow M_2/\mathfrak{a}M_2 \longrightarrow C_{-2} \longrightarrow 0.$$

In particular $D_2/\mathfrak{a}D_2$ has finite O-length. If T is finite over O, the algebra $H_2 = D_2/D_2[\ell^\infty]$ is finite free over O. Its geometric generic local factors, with their full local rings, are precisely the nonsplit factors of T.

Proof. In an augmented homogeneous presentation a monomial of weight minus two with no nonconstant invariant factor is either one variable of weight minus two or two variables of weight minus one. Indeed remove weight-zero variables and opposite pairs. If both signs persist they have different absolute weights. A positive two then requires at least two negative units. With positive units and negative twos, any nonempty positive part has even size and supplies two units to pair with a negative two. In either case there is an invariant triple. Grouping series by one such subproduct shows that all discarded terms belong to $\mathfrak{a}M_2$. The cotangent quotient removes the quadratic terms that remain. A change of either factor in such a quadratic term by an augmented product introduces terms of degree at least three, which have just been removed. The first map in the sequence is therefore induced by multiplication and its image is exactly the

kernel. Both negative cotangents have finite length by Theorem 32.19. The fiber of the zeroth Fitting ideal now has finite colength over O by base change and diagonalization, including colength zero if the module is zero.

A geometric generic character of T killing $B_{-2}B_2$ is the split one. To see this, for $i > 0$ products of opposite weights $-i, i$ have square in $(B_{-2}B_2)^i$. In fact in even weight $2i$ pair the odd factors of a monomial to make weights zero or of absolute size two, then pair opposite even factors. This gives $B_{2i} = B_2^i$, with the same formula in negative weights. It applies also to homogeneous series by the finite generation in Lemma 32.17. The character therefore extends to B by sending every nonzero weight to zero. Proposition 32.18 identifies the resulting fixed point with the split one. Conversely the split augmentation kills $B_{-2}B_2$.

At any other character some product of weights $-2, 2$ is invertible in the localized algebra. Its factor of weight minus two is then a homogeneous unit and multiplication by it identifies the local invariant ring with the local module M_2. Its zeroth Fitting ideal there is zero. Over the split character the finite-length fiber found above disappears on inverting ℓ. This proves the assertion about the geometric local rings. The last quotient is finite torsion-free over the discrete valuation ring and hence is free. □

Theorem 41.8 (The mixed trace estimate at every large prime)**.** *Suppose $S(\Psi^{-1}) \neq 0$ on a sufficiently distant interval. If T is finite over O, for every $d \in \mathfrak{a}$ nonzero at its nonsplit geometric generic characters one has*

$$\operatorname{length}_O S(\Psi^{-1}) + \operatorname{length}_O S(\eta\Psi^{-1}) \;\leq\; 2v_\ell\big(\det(d \mid H_2)\big) \;=\; 2\sum_\sigma n_\sigma v_\ell(d(\sigma)),$$

where $n_\sigma = \dim_{\overline{J}}(T \otimes_O \overline{J})_\sigma$ and the sum ranges over those characters. In particular $H_2 \neq 0$. If T is infinite over O, its special fiber admits a nonconstant trace curve strongly irreducible on G_K.

Proof. The finite case of Proposition 41.4 applies by Corollary 32.20 and the fixed-point calculation. Its index set is all nonsplit characters by the lemma. We check that its maximum over two weights is n_σ in every case. At a character admitting a homogeneous unit of weight one or minus one this follows by multiplication by the unit. For the remaining case work over $\overline{J}$, localize in the invariant algebra at σ, and choose a unit v of weight two. The local invariant algebra T_σ is Artinian. An element z of odd weight j has z^2/v^j in the maximal ideal of T_σ. Otherwise it would be a unit, and $zv^{(1-j)/2}$ would be a unit of weight one. Thus every odd homogeneous element is nilpotent. The even part equals $T_\sigma[v, v^{-1}]$, by periodicity. Its intersection with the nilradical has coefficients in the maximal ideal of T_σ, since distinct even degrees remain independent. Equivalently, quotient by the nilpotent ideal generated by this maximal ideal and a finite set of odd homogeneous algebra generators. The reduced ring is $\overline{J}[v, v^{-1}]$. There are no polynomial relations on the Laurent generator over the residue field, since its nonzero powers have distinct weights and remain units in this nonzero graded quotient.

At any closed point of this curve all adjacent entries vanish by their odd weights. The two outer directions both occur. In fact elements of weights two and minus two with nonzero product must evaluate nontrivially, and in the absence of adjacent entries an entry of the corresponding outer weight is required in each case. The marked projectors belong to the group span, so the outer plane is irreducible and the middle line splits off. These are precisely the two-dimensional mixed points treated by Corollary 33.9. The tangent space of the full solution at each of them has dimension one. Its local ring has dimension one by the reduced Laurent curve and is Noetherian. It is therefore regular, hence reduced. The localized graded algebra is of finite type over $\overline{J}$. An element vanishing in the localizations at every maximal

ideal is zero, so it is itself reduced. Its invariant part is now $\overline{J}$ and its odd part is zero. The maximum in the proposition is again n_σ.

The length on the left is the integral cotangent length in Corollary 32.20. The determinant equality follows from the lemma by filtering the finite geometric local rings. An empty sum would contradict the positive left side. Finally, if T is infinite, its special fiber has positive dimension by the complete-local finiteness argument of Theorem 41.5. Normalizing a complete curve quotient and applying Theorem 34.6 gives the stated curve. □

Choose one rational prime $v \geq 7$ split in K and prime to the fixed conductor and to 6. Fix $w \mid v$. On all sufficiently distant intervals, $v \nmid N_*$ and $v < P(x)$, since the moving prime divisors of N_* tend to infinity. Let $N' = 4N_*^2$. At a finite trace algebra call a nonsplit geometric point listed if one of the following descriptions holds for its representation on G_K.

1. It has a split middle line and the irreducible outer plane of Theorem 33.3, or it is an odd orthogonal point of Theorem 34.9.

2. It is absolutely irreducible and its twist by κ_ℓ^{-1} is attached to a regular algebraic conjugate self-dual cuspidal representation over K with the level, weights and central condition of Section 30.4 at level N'. Use the normalization there for its Frobenius traces. The association here is with one global representation.

For a geometric character α of E_x put $\lambda_{\ell,r,\alpha} = \operatorname{length}_O S(\Psi^{-1})$, using the minimal unramified coefficient ring for α. In the following definitions put $U_{\ell,r,\alpha} = 0$ when $\lambda_{\ell,r,\alpha} = 0$. At an infinite trace algebra put $U_{\ell,r,\alpha} = \lambda_{\ell,r,\alpha}$. At a finite one put $d_\alpha(v) = 1 + v^r\alpha(v) + v^{-r}\alpha(v)^{-1}$ and

$$U_{\ell,r,\alpha} = \min\left\{\lambda_{\ell,r,\alpha},\, 2\sum_{\sigma\ \text{unlisted}} n_\sigma v_\ell\big(t_\sigma(\mathrm{Frob}_w) - d_\alpha(v)\big)\right\}.$$

Only nonsplit points are counted and $v_\ell(0) = +\infty$. The lengths refer throughout to the scalar strict groups over $\mathbb{Q}$. At these primes Theorem 22.17 identifies their sum over geometric characters with the full complementary length for the original field E_x.

Theorem 41.9 (One interval bound for the lengths accounted for by classical points)**.** *There is a nonzero positive integer J_V with $\log J_V \ll_{\psi,\varepsilon,v} V^C$ for a fixed C, such that for every $x \in [V, V^{3/2}]$ on the prescribed intervals*

$$0 \leq \sum_{\ell > P(x)} \log \ell \sum_{1 \leq r < t(x)} \sum_{\alpha(c) = (-1)^r} (\lambda_{\ell,r,\alpha} - U_{\ell,r,\alpha}) \leq 2 \log J_V.$$

The character sum is over all geometric characters of E_x of the indicated parity. Equivalently each residue orbit is counted with its residue degree. In particular, for the finite components on which every nonsplit point is listed the bound counts their entire complementary strict length.

Proof. At every listed point the local multiplicity is one. For the middle-line and odd orthogonal cases this follows from Corollary 33.9 and Theorem 34.9. For the cuspidal case use Corollary 33.12 with the CM field equal to K. In each case the full solution tangent space is one-dimensional, generated by scaling. To see the multiplicity conclusion, if there is a degree-one unit the local graded algebra is $T_\sigma[u, u^{-1}]$ with u that unit. Its tangent dimension at a closed point is one plus the tangent dimension of the invariant factor. Thus T_σ is a field by Nakayama's lemma. Without the odd unit this was proved in Theorem 41.8.

For a finite component evaluate that theorem at the Frobenius trace minus its split value. All its geometric values are integral since the ring is finite over O. At a listed point the trace

has complex absolute value at most three, while $|d_\alpha(v)| \geq v^r - 1 - v^{-r} > 3$. Hence a zero difference can occur only in the unlisted sum, in which case $\lambda - U = 0$. When all differences are nonzero the theorem and the definition of U give

$$0 \leq \lambda_{\ell,r,\alpha} - U_{\ell,r,\alpha} \leq 2 \sum_{\sigma \text{ listed}} v_\ell(t_\sigma(\mathrm{Frob}_w) - d_\alpha(v)).$$

The infinite components contribute zero to the left by definition.

To construct the one integer, take all twists $1 \leq r < t_*$ and all complex characters of E_* of the indicated parity. For each such pair use three labelled lists at v. For the rank-three cuspidal systems use the trace multiset of Theorem 30.11 at level N', with the two signs and all the indicated embeddings. For the planes include $1 + \alpha(v)a_v(f)/v^r$ for all primitive forms f of weight $2r+1$, character $\eta\alpha^{-2}$ and level dividing $4N_*$. For odd orthogonal points include $a_v(g)^2/(v^r\alpha(v)) - 1$ for primitive forms g of weight $r+1$, character $\eta\alpha$ and level dividing $4N_*$. The two formulas and level bounds follow from Theorems 33.3 and 34.9. The newform roots have the common purity absolute value in their respective weights, and $\eta(v) = 1$, giving the absolute value bound three also for the adjoint trace. Finite characters have unit complex absolute values. The three lists cover the listed points. One global label with a fixed sign determines at most one trace character for the given α. For rank three, equality at the good primes over K implies equality of traces by Chebotarev and continuity, and these traces generate T. In the other two cases the normalized newform determines the entire plane or adjoint representation. Distinct equal numerical values at w may still occur on different labels.

Take J_V to be the absolute value of the product of $v^{u_r}(b - d_\alpha(v))$ over these twists, characters and labelled entries b, with $u_r = 72(r+2)+1$. All factors are algebraic integers by Theorem 30.8 and the integrality of newform coefficients. They are nonzero by the archimedean bound. The list in the entire product is Galois stable, including the pairing of a newform character with α. The rank-three list contains the conjugates as in Theorem 30.11. Thus the product is a nonzero rational integer before taking absolute value. The number of labelled entries per pair is polynomial in $N'(r+2)$, by the same theorem and the dimension estimate for newforms in its proof. Each factor has logarithmic absolute value $O_v(r+1)$. Since $N_*, [E_* : \mathbb{Q}], t_*$ are polynomial in V by Theorem 19.15, this proves the asserted size.

At a given $\ell > P(x)$, choose one embedding for all the labels. The rational prime v is an ℓ-adic unit. Inflating characters from E_x to E_* is injective. Every listed contribution for each α, r therefore uses a distinct allowed factor of the product, and all unused factors have nonnegative valuation. Their sum of valuations is at most $v_\ell(J_V)$. Summing with $\log \ell$ gives the claimed bound. Conjugation over $\mathbb{Q}_\ell$ preserves valuations and all three global lists, so summing the geometric characters is equivalent to the residue-degree convention stated above. □

Thus the remaining large-prime contribution is measured by the $U_{\ell,r,\alpha}$ on those same fields. Bounding their sum requires further information about the nonconstant residual curves and the unlisted characteristic-zero traces. The small-prime tails and the uniform comparison for the signed periods in Section 6.6 are separate remaining estimates.

41.4 Solvable descent to the level of the common interval

Take an absolutely irreducible characteristic-zero trace point in the marked problem of Section 41.3. Denote its representation on G_K by U, where $K = \mathbb{Q}(i)$. An integral trace point has a continuous integral matrix realization after finite coefficient extension by Lemma 34.5. This applies in particular to every geometric generic point of a trace algebra finite over O. Fix a coefficient isomorphism $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. We use the rank-three normalization of Theorem 36.18. In it, $U \otimes \kappa_\ell^{-1}$ is the Galois representation attached to the automorphic system, and the good

Frobenius traces of U are its unitary traces, with inverse Frobenius giving the opposite sign of the label.

Theorem 41.10 (Descent through every solvable CM Galois extension)**.** *Let L/K be a finite solvable Galois extension, with L a CM field. Suppose $(U \otimes \kappa_\ell^{-1})|_{G_L}$ is attached to a regular algebraic conjugate self-dual cuspidal representation on $\mathrm{GL}_3(\mathbb{A}_L)$. Then $U \otimes \kappa_\ell^{-1}$ is attached to such a cuspidal representation over K. Its unitary central character is trivial. The assertion includes extensions of degree divisible by three.*

Proof. The marked polarization gives $U^c \simeq U^\vee$ by equality of traces and absolute irreducibility. Lemma 33.4 gives $\det U = 1$ and absolute irreducibility after any finite restriction inside G_K. The unitary central character of the cuspidal representation over L is therefore trivial. At every good finite place it agrees with the determinant of the unitarily normalized Galois representation. Equality there determines a Hecke character by strong multiplicity one for GL_1.

Take a subnormal series of $\mathrm{Gal}(L/K)$ with prime cyclic quotients. Its intermediate fields are CM. Indeed complex conjugation on a CM field commutes with every field automorphism. This follows upon composing any complex embedding with that automorphism and using the defining conjugation on complex embeddings. It therefore preserves each fixed field in the series. The fixed subfield of conjugation is totally real, and conjugation is nontrivial on each intermediate field since it contains K. It also commutes with the group of each cyclic step.

Consider one step F/F_0 of prime degree d, with the cuspidal system Π_F already attached to the restriction of U in the indicated normalization. At good places its parameters are invariant under $\mathrm{Gal}(F/F_0)$ because its Galois representation comes by restriction from G_{F_0}. Strong multiplicity one gives invariance of Π_F. By cyclic base change it has a cuspidal descent Θ over F_0, and all its cuspidal descents are the twists of Θ by characters of the cyclic group, viewed by class field theory as Hecke characters and composed with the determinant. Base change here commutes with automorphisms, contragredients and central characters. At the infinite places the local fields are complex on both sides and the infinitesimal parameters are inherited, so every descent is regular algebraic. These are the cyclic base change and fiber statements of Arthur and Clozel. [8]

We choose a conjugate self-dual descent. If $d = 2$, the central character of Θ base changes to one. The cyclic norm theorem identifies it with a character of the cyclic quotient. Twisting by a character ν multiplies the central character by ν^3. Cubing is bijective on the group of two characters, so exactly one descent has trivial central character. Its conjugate contragredient is again such a descent, hence is isomorphic to it.

If d is odd, write $(\Theta^c)^\vee \simeq \Theta \otimes \delta$ for a character of the cyclic quotient, since both have base change Π_F. Conjugation fixes these quotient characters. Squaring is bijective on them, so take $\nu^2 = \delta$. The conjugate contragredient of $\Theta \otimes \nu$ is $\Theta \otimes \delta\nu^{-1} \simeq \Theta \otimes \nu$. This chooses a conjugate self-dual descent also when $d = 3$.

Let V be the normalized Galois representation of the chosen descent, with the Tate factor restored. It exists by the rank-three regular algebraic conjugate self-dual theorem and satisfies $V^c \simeq V^\vee$ in this normalization, by compatibility at good places and the character criterion. Its restriction to G_F agrees with $U|_{G_F}$: the good parameters agree by cyclic base change, and Chebotarev and semisimplicity identify their semisimplifications. The latter is irreducible of dimension three, so the restrictions are indeed isomorphic. [9, 8]

After identifying their restrictions, V and U on G_{F_0} differ by a character ξ of the cyclic quotient. For each group element the two matrices induce the same conjugation on the irreducible restriction, so their ratio is scalar by Schur’s lemma. The scalars multiply and are one on G_F.

There is no nontrivial self-twist of $U|_{G_{F_0}}$ by a character of this quotient, since its intertwiner would commute with $U|_{G_F}$ and hence be scalar. For $d = 2$ both normalized determinants are one by the choice of central character and good-place compatibility. Thus $\xi^3 = 1$, which forces $\xi = 1$. For odd d apply the conjugate-dual operation to $V \simeq U \otimes \xi$. Since $\xi^c = \xi$, this gives a self-twist of U by ξ^2. It follows that $\xi^2 = 1$, again forcing $\xi = 1$. The descent therefore gives exactly U in the normalized convention. Its unitary central character is trivial by its determinant at the good places. Induction down the cyclic tower proves the theorem. □

Proposition 41.11 (The level and weights after descent)**.** *On the polynomial-conductor intervals of Theorem 19.15, the cuspidal representation over K in Theorem 41.10 has fixed vectors at the principal level $U_K(N_*^2)$. At both embeddings its coefficient weights satisfy the bound $4(r+2)$ in Section 30.4. Consequently it is on the list in Theorem 41.9 for the entire interval, independently of the level, field degree or ramification used for its potential automorphy.*

Proof. At $z \mid \ell$ the original marked representation is crystalline with cyclotomic exponents $r, 0, -r$, including $r = 1$ by Theorem 33.11. Its Weil–Deligne parameter has trivial inertia and zero monodromy. Local–global compatibility at ℓ applies with equality up to Frobenius semisimplification by the theorem of Barnet-Lamb, Gee, Geraghty and Taylor, since the weight is Shin-regular in odd rank. The local automorphic component is spherical. At the complex embeddings regular algebraic compatibility gives, in the unitary convention, infinitesimal exponents $r, 0, -r$. Subtracting the half-sum of positive roots $(1, 0, -1)$ in dominant order gives coefficient weights of absolute value at most $r+1$, also after taking the dual. The trivial unitary central character supplies the required central condition. [10, 9]

At a finite place $z \nmid \ell$ of K the marked inertia kernel makes the action factor through the cyclic image of α on that inertia group. This image has order prime to ℓ, since the conductor modulus N_* is less than ℓ throughout a sufficiently distant interval. Its characters in the integral representation are therefore the unique lifts of the three residual inertia characters, with multiplicity. They are $\alpha, 1, \alpha^{-1}$ restricted to this inertia, since the cyclotomic character is unramified here. Finite inertia implies zero monodromy. The Artin conductor exponent is the sum of the three character exponents, as is seen by taking the dimensions of invariants in the ramification filtration.

Write q for the rational prime under z, and $a_z = v_z(N_*)$ in the normalized integral valuation. In local reciprocity the finite character on $K_z^\times$ is pulled back by the norm to $\mathbb{Q}_q^\times$. If $q \nmid N_*$ it is unramified. Otherwise an element of $1 + N_* \mathcal{O}_{K_z}$ acts by multiplication on the free $\mathbb{Z}_q$-module $\mathcal{O}_{K_z}$ by a matrix congruent to the identity modulo $q^{v_q(N_*)}$. Its determinant, the field norm, has the same congruence. This proves triviality on $1 + \mathfrak{p}_z^{a_z}$, including at the ramified Gaussian prime over two. The Artin conductor exponent is thus at most $2a_z$.

The local component of a cuspidal representation of GL_3 is generic. Local–global compatibility away from ℓ and preservation of conductors under the local Langlands correspondence identify its conductor exponent with that of the parameter just calculated. The local newform theorem gives a vector fixed by the subgroup whose last row is congruent to $(0, 0, 1)$ at the conductor exponent. This subgroup contains the principal subgroup of exponent $2a_z$. For exponent zero it is the full maximal compact group. Taking the finite tensor product of these vectors with the spherical vectors elsewhere gives the claimed level. Principal level $4N_*^2$ is contained in this level, so the list used in the interval estimate includes the resulting trace system. [9] Jacquet, Piatetski-Shapiro and Shalika, *Conducteur des représentations du groupe linéaire.* □

Corollary 41.12 (Full lengths on the components with solvable potential automorphy)**.** *For a fixed $x \in [V, V^{3/2}]$, take all contributing triples $(\ell, r, [\alpha])$ of Section 41.3 for which the trace*

algebra is finite and every absolutely irreducible nonsplit point either has odd orthogonal type or satisfies the hypothesis of Theorem 41.10. The CM extension may be chosen independently at each point. For this set $\mathcal{A}(x)$ the same interval integer gives

$$\sum_{(\ell,r,[\alpha])\in\mathcal{A}(x)} [O_\alpha:\mathbb{Z}_\ell]\log\ell\ \operatorname{length}_{O_\alpha} S(\Psi^{-1}) \ \le\ 2\log J_V \ll_{\psi,\varepsilon,v} V^C.$$

Proof. Every reducible nonsplit point in the finite problem is a middle line and an irreducible outer plane. Indeed the matrix projectors force invariant subspaces to be coordinate subspaces, and reflection of the adjacent entries restricts a reducible representation to an upper triangular, lower triangular or outer-block shape, as in the proof of Theorem 34.6. Its triangular trace is the split one by the fixed-point calculation. At the nonsplit outer block Theorem 33.3 supplies the newform. Odd orthogonal points are already on the list, and the theorem and proposition put every other point of these triples on the rank-three list. Thus $U_{\ell,r,\alpha}=0$ there. Restrict the nonnegative sum of Theorem 41.9 to these triples and group the geometric characters into residue orbits. This gives the asserted bound for their full lengths. □

No quantitative bound on the auxiliary solvable extensions enters the integer J_V. To include the remaining finite components by this argument one still has to produce potential automorphy for their remaining points. The construction of such an extension from an arbitrary ordinary Galois point, and the exclusion of nonconstant residual curves, remain necessary for a uniform bound by this route.

There is a local restriction on the use of an ordinary lifting theorem here. At every finite place away from ℓ the original point has finite inertia, also after restriction to any finite extension. Its Weil–Deligne monodromy operator is zero. A twist of the rank-three Steinberg representation has monodromy with one Jordan block of size three. Thus the Steinberg place in Thorne's Theorem 7.1 cannot be supplied by extending the base field of this point. The passage through auxiliary deformations in the source occurs in `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/ThresholdArguments/DenseThresholdCases/DnEnds.lean`, line 35. J. Thorne, *Automorphy lifting for residually reducible l-adic Galois representations.*

41.5 Cohomology bounds over varying auxiliary CM fields

Fix an integer $d\ge 2$. Let F range over CM subfields of a fixed algebraic closure of $\mathbb{Q}$ that contain $K=\mathbb{Q}(i)$, with $[F:\mathbb{Q}]\le d$ and absolute discriminant $D_F\le X$, where $X\ge 2$. No normality over K is imposed. We will allow a different F at each characteristic-zero trace point. To test all these points, fix the prime v and the place $w\mid v$ of K in Section 41.3, and put $m=\operatorname{lcm}(1,\dots,d)$. A Frobenius at w raised to the power m can be evaluated by spherical Hecke operators over every F, also when v ramifies in F.

Lemma 41.13 (A uniform cell count in bounded degree)**.** *There are at most $(2X)^{C_d}$ such embedded fields. For each of them and each positive integer N, the adelic locally symmetric space for GL_3/F, modulo the positive real split center and at principal level $3N$, has the homotopy type of a finite cell complex with at most $(2D_FN)^{C_d}$ cells. In this section C_d denotes a positive constant depending only on d, whose value can increase from one bound to the next.*

Proof. Write $b=[F:\mathbb{Q}]$ and use the usual Euclidean norm on the Minkowski embedding of F. The nonzero vectors of $\mathcal{O}_F$ have norm bounded below by a constant depending only on b, by the product formula for the field norm of an algebraic integer. Its covolume is a constant depending on the signature times $D_F^{1/2}$. Minkowski's second theorem gives b independent lattice vectors of lengths at most $C_dD_F^{1/2}$.

We record how to pass to a basis. In a Euclidean lattice of rank s having s independent vectors of length at most B, a shortest nonzero vector is primitive and has length at most B. Projection perpendicular to it is a lattice of rank $s-1$; the given vectors project to a spanning set of lengths at most B. Inductively take a short basis of the projection and lift it, subtracting multiples of the primitive vector until each parallel component has length at most $B/2$. These lifts and the primitive vector are a basis of lengths at most $C_s B$. Apply this to get a basis $a_1, \dots, a_b$ of $\mathcal{O}_F$ with all conjugates bounded by $C_d D_F^{1/2}$.

For two different embeddings of F, equality of their values on $\sum_j T^{j-1} a_j$ holds at at most $b-1$ rational integers T. The difference polynomial is nonzero since the a_j span F. Some integer $0 \le T \le d^3$ separates all pairs. The resulting primitive integer of F has a monic minimal polynomial of degree at most d with coefficient magnitudes bounded by $(2X)^{C_d}$, by the elementary symmetric functions. There are polynomially many such polynomials in X for fixed d, each with at most d roots. This counts the embedded fields as asserted.

At full integral level the finite adelic classes correspond to rank-three $\mathcal{O}_F$-lattices M in F^3 up to $\mathrm{GL}_3(F)$. Indeed a finite adele sends the standard local lattices to local lattices whose intersection with F^3 is the required global lattice. The group on the corresponding symmetric space is $\mathrm{Aut}_{\mathcal{O}_F}(M)$. Use Ash's equivariant well-rounded retract for this lattice, with trivial weights on the vectors. Normalize the minimum length to one. Its cells are determined by the finite sets of shortest vectors, these sets span F^3 over F, and passing to a face enlarges the set. Barycentric simplices correspond to flags of these cells. This is the number-field retract for marked lattices in the construction of Ash and McConnell. A. Ash and M. McConnell, *Cohomology at infinity and the well-rounded retract for general linear groups.*

Here is a uniform count of the flags over all the lattice classes. Take the shortest-vector set of the smallest face of a flag, with its normalized metric. It contains three F-independent vectors u_1, u_2, u_3. The $3b$ vectors $a_j u_i$ are integral and independent over $\mathbb{Q}$. Their lengths are at most $C_d D_F^{1/2}$, since multiplication by a_j on an archimedean Hermitian factor has operator norm $|\sigma(a_j)|$. The short-basis argument gives a $\mathbb{Z}$-basis of M with that bound up to C_d. In an orthonormal coordinate system let A be its real basis matrix. The balls of radius $1/2$ centered on the lattice have disjoint interiors, so the covolume $|\det A|$ is bounded below by the volume of such a ball. Cofactor expansion now bounds $\|A^{-1}\|$ by $(2D_F)^{C_d}$.

Record in this basis the integer matrices for multiplication by all the a_j and the integer coordinates of the shortest vectors in the chosen face. Every entry has magnitude at most $(2D_F)^{C_d}$ by the two basis bounds. There are at most 3^{3b} shortest vectors, since the disjoint balls around them lie in the ball of radius $3/2$. Record also the nested subsets belonging to the other faces in the flag. The number of such subsets and the length of a strict chain are bounded solely in terms of d. There are thus at most $(2D_F)^{C_d}$ records. Two coinciding records give a basis isomorphism commuting with every a_j. It is $\mathcal{O}_F$-linear and extends to an F-linear isomorphism of the rational spaces. It identifies the flags since it identifies every specified shortest-vector set. Consequently each record accounts for at most one orbit, even when all projective lattice classes are counted together.

Passing to principal level $3N$ introduces at most $|\mathrm{GL}_3(\mathcal{O}_F/3N)| \le (3N)^{9b}$ lifts per orbit: the possible additional level coordinates form a quotient of the integral finite adelic group by its principal subgroup. Stabilizers of points on the retract are finite, since the minimum is normalized and a lattice has finitely many isometries. At this level they act trivially, by the torsion-freeness of the principal subgroup modulo three on the underlying integral lattice. The quotients of the retracts therefore give cell complexes of the required total size and homotopy type. □

Proposition 41.14 (A uniform list of Frobenius power traces)**.** *Let $v \nmid 3N$ and $h \ge 1$.*

Take all the above fields and all regular algebraic conjugate self-dual cuspidal representations of $\mathrm{GL}_3(\mathbb{A}_F)$ *with trivial unitary central character, principal level* N *fixed vectors, and cohomological coefficient weights bounded in absolute value by* h*. For every* $z \mid w$ *put* $f = [k(z) : \mathbb{F}_v]$*. Record each representation, place and sign, with value*

$$b_{\pm} = \sum_{j=1}^{3} a_{z,j}^{\pm m/f},$$

where the $a_{z,j}$ *are the unitary Satake parameters. Include each conjugate of each value, keeping the labels distinct. The size of this list is at most* $(2XN(h+1))^{C_d}$*; every value has absolute value at most three. Multiplication by* $v^{m(7dh+d)}$ *makes each value an algebraic integer.*

Proof. For one field, one weight has coefficient dimension at most $(2h+2)^{3b}$ by the GL_3 Weyl dimension formula at the b embeddings. There are at most $(2h+1)^{3b}$ weights. At level $3N$, cellular cochains on the complex in the lemma bound the total cohomology dimension by its cell count times the coefficient dimension. Every one of our cuspidal representations contributes to that cohomology by the cuspidal direct summand of Franke's theorem, as in Lemma 30.9. This bounds their number. [11]

To bound the degrees and denominators, define the coefficient system over the normal closure of F, of degree at most $d!$. At a finite adelic representative take the tensor product of the integral coefficient lattices at the embeddings of its underlying lattice M, using Schur modules and determinant powers as in Lemma 30.10. For the double coset of $\mathrm{diag}(\pi_z, 1, 1)$ or its inverse the neighboring lattices satisfy $vM \subseteq M' \subseteq v^{-1}M$. This holds at z even in the ramified case since v belongs to its maximal ideal. The same inclusions hold for the duals. At each embedding the polynomial part of the weight uses at most $4h$ tensor factors; its determinant power uses at most another $3h$, including negative powers by duality. Hence v^{7dh} times either Hecke operator preserves the integral cohomology lattice modulo torsion, by integral pullback and transfer. The unscaled spherical eigenvalues are $v^f s_{\pm}$, with $s_{\pm} = \sum_j a_{z,j}^{\pm 1}$. Thus $v^{7dh+d} s_{\pm}$ are algebraic integers, and their degrees over $\mathbb{Q}$ are at most $d!$ times the cohomology dimension. The bound includes nonfree coefficient lattices by localizing their characteristic polynomials at every finite prime.

The product of the three Satake parameters is one. For either sign the three roots have polynomial $Y^3 - s_{\pm} Y^2 + s_{\mp} Y - 1$. Newton's identities express their jth power sum as an integral polynomial in s_+, s_- of total degree at most j. This follows successively from the first three identities and the cubic recurrence. Taking $j = m/f$ proves the denominator bound. The degree of the compositum of the two trace fields is at most the product of their degrees. This costs another polynomial factor when listing all conjugates. There are at most d choices of z per field. Shin's purity theorem in odd rank gives absolute value one for every conjugate of each unitary root. Summing over fields using the lemma proves all the claims. [9] □

Theorem 41.15 (The strict order for points over bounded-degree CM fields)**.** *Use the common fields and intervals of Section 41.3. For each* $x \in [V, V^{3/2}]$*, take those large-prime triples* $(\ell, r, [\alpha])$ *for which* T *is finite over* O_α *and the following holds at every absolutely irreducible nonsplit point for every geometric character in the indicated residue orbit. It is odd orthogonal, or there are CM fields* $K \subseteq F \subseteq L$ *with* $[F : \mathbb{Q}] \le d$*,* $D_F \le X$ *and* L/F *solvable Galois, such that* $(U \otimes \kappa_\ell^{-1})|_{G_L}$ *is attached to a regular algebraic conjugate self-dual cuspidal representation. Denote the set of triples by* $\mathcal{A}_{d,X}(x)$*. There is a positive integer* $J_{V,d,X}$*, independent of* x*, such that*

$$\sum_{(\ell,r,[\alpha]) \in \mathcal{A}_{d,X}(x)} [O_\alpha : \mathbb{Z}_\ell] \log \ell \, \mathrm{length}_{O_\alpha} S(\Psi^{-1}) \;\le\; 2 \log J_{V,d,X} \;\ll_{d,\psi,\varepsilon,v} (2VX)^C$$

for a fixed exponent C. Neither the degree nor the ramification of L/F enters the estimate.

Proof. For a point of the second kind, the cyclic descent proof of Theorem 41.10 applies over F. Indeed $U|_{G_F}$ has determinant one, remains absolutely irreducible on every open subgroup and is carried to its dual by CM conjugation. These are exactly the properties of the base representation used at the cyclic steps. The intermediate fields in L/F are CM because they contain F and CM conjugation commutes with its Galois group. We obtain a cuspidal system for the actual restriction to F, with trivial unitary central character.

It has level $N = N_*^2$ fixed vectors. At $z \nmid \ell$ the finite inertia characters are the restrictions of $\alpha, 1, \alpha^{-1}$. For $z \mid q \mid N_*$, the norm to $\mathbb{Q}_q$ sends $1 + N_* \mathcal{O}_{F_z}$ into $1 + N_* \mathbb{Z}_q$ by the determinant of multiplication on the local integral basis. The conductor is at most $2v_z(N_*)$; at the other places away from ℓ it is zero. At ℓ restriction preserves crystallinity and pulls each labeled Hodge weight back along the embedding. Thus the spherical and highest-weight conclusions of Proposition 41.11 also apply over F, with $h = 4(r+2)$. The tangent result of Corollary 33.12 applies with this CM field, giving multiplicity one in the invariant algebra by the proof of Theorem 41.9. For reducible nonsplit points and odd orthogonal points that theorem gives the same multiplicity and the two classical modular descriptions.

Choose a Frobenius lift g at w and evaluate a single element of the trace augmentation ideal at every point,

$$t(g^m) - d_{\alpha,m}, \qquad d_{\alpha,m} = 1 + (v^r \alpha(v))^m + (v^r \alpha(v))^{-m}.$$

For $z \mid w$, a Frobenius at z maps into the conjugacy class of a lift of the fth power of Frobenius at w, modulo inertia there. The original representation at w is unramified. The trace at g^m is therefore one of the power traces in the proposition. Moreover one global cuspidal representation over a fixed embedded F, with the embedding of coefficients fixed, accounts for at most one trace point of a fixed triple. To see this, two representations giving that label agree on G_F by Chebotarev. Identify them there. On the group of the normal closure of F/K their common restriction is absolutely irreducible. Schur's lemma then makes their ratio on G_K a character ξ, trivial on G_F. The determinants force $\xi^3 = 1$. If $\xi \neq 1$, its reduction still has order three. Multiplication by this residual character preserves $\{1, \bar{\Psi}, \bar{\Psi}^{-1}\}$, the common residual semisimplification on G_K; the orbit of 1 would exhaust this set. But on inertia at ℓ the order of $\bar{\Psi}$ is at least $(\ell - 1)/r > 3$, since K is unramified there, α is unramified and $r < t(x) \ll P(x)$. This excludes the twist. The group traces generate T, so equality of the representations identifies their trace characters. If several fields or labels work for a given point, select one.

We describe the integer explicitly to keep the prime sum uniform. For each twist $r < t_*$ starting at one and each geometric character α of the common field E_* of the required parity, take all labels of the proposition at N_*^2 and $h = 4(r+2)$. Add the power traces of the two modular families in Theorem 41.9, recording their newform labels and both Frobenius conventions. For the plane with roots β_1, β_2 of the weight-$2r+1$ form, these are the mth power sums of $1, \alpha(v)\beta_1/v^r, \alpha(v)\beta_2/v^r$ and their inverses. For the adjoint with roots γ_1, γ_2 of the weight-$r+1$ form they are those of $1, \gamma_1/\gamma_2, \gamma_2/\gamma_1$. The products of the two modular roots are a root of unity times v^{2r} and v^r, respectively, so v^{mr} clears the denominators in both power traces. At the split prime v these are exactly the two families with their unitary normalizations. Deligne purity bounds every conjugate by three. The modular dimension and degree bounds of Theorem 30.8 remain polynomial after taking these fixed powers and adjoining the character values of E_*. Distinct points using the same modular label coincide because it specifies the global plane or adjoint.

Set $u_r = m(7dh + d + r)$. Multiply $v^{u_r}(b - d_{\alpha,m})$ over these labeled entries b, twists and characters. Take all conjugates of the rank-three values as in the proposition, and take the

modular lists with their matching Nebentypus, unitary scalars and Galois conjugates as in Theorem 41.9. The product is Galois invariant; conjugation permutes fields of character values, newforms and the corresponding character labels together. Each factor is an algebraic integer. It is nonzero since $|d_{\alpha,m}| \geq v^{rm} - 1 - v^{-rm} > 3$ at every embedding. Take $J_{V,d,X}$ to be the absolute value of the product, with empty product one. Each factor has logarithmic absolute value $O_{d,v}(r+1)$. The number of entries is polynomial in X, V by the proposition and the bounds for N_*, t_* and $[E_* : \mathbb{Q}]$ on the polynomial conductor intervals. This proves the claimed size.

For any included triple of positive strict length apply Theorem 41.8 to the displayed trace difference. All nonsplit points occur with multiplicity one, and the absolute values just proved ensure that its determinant is nonzero. Distinct nonsplit characters can be assigned distinct labels by the preceding uniqueness arguments. A fixed embedding into $\overline{\mathbb{Q}}_\ell$ identifies the geometric characters at x with a subset of those of E_*; v is an ℓ-adic unit. Thus at a given ℓ the sum of the lengths for all included geometric characters and twists is at most $2v_\ell(J_{V,d,X})$. Each included orbit has as many geometric characters as the residue degree by unramifiedness of $O_\alpha/\mathbb{Z}_\ell$. Summing with $\log \ell$ gives the stated residue-degree formula. □

Corollary 41.16 (Ramification on the common conductor)**.** *For fixed d, the full large-prime strict contribution in the theorem is polynomial in V if at each point an intermediate field F of degree at most d can be chosen unramified over $\mathbb{Q}$ outside the primes dividing $2N_*$. More generally intermediate fields of discriminant at most $\exp(V^\varepsilon/\log(2V))$ give a total $o(\exp(x^\varepsilon/2))$ uniformly for $V \leq x \leq V^{3/2}$.*

Proof. At a rational prime q, pass in $F_z/\mathbb{Q}_q$ to its maximal unramified subfield. The remaining extension of ramification index e is generated by a uniformizer with Eisenstein minimal polynomial. Its different is generated by the derivative at that uniformizer. The nonzero derivative terms have distinct valuations modulo e, so its valuation is at most that of the leading term, $e - 1 + ev_q(e)$. The unramified subextension has trivial different. Summing over $z \mid q$ with residue degrees bounds the discriminant exponent by $d(1 + \log_2 d)$. Under the given ramification hypothesis this yields $D_F \leq (2N_*)^{d(1+\log_2 d)}$, polynomial on the common interval. For the second claim the logarithm of the bound $(2VX)^C$ is $O(\log(2V) + V^\varepsilon/\log(2V)) = o(V^\varepsilon)$. Both conclusions follow from the theorem. □

In the source the auxiliary field, the seed representations and the passage to other primes of the ordinary deformation ring enter at `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/ThresholdArguments/DenseThresholdCases/LinkedEndE.lean`, line 60 and `Development/AutomorphicArithmetic/AutomorphicVarieties/WeilGroupCharacters/UniversalOrdinaryDeterminantRing.lean`, line 284. The estimate above allows substantial variation of the field where cuspidality is proved. Showing that every required fixed-weight point has such a field, and treating the residual curves of an infinite trace algebra, remain open in the present argument.

41.6 One totally real extension removing the conductor on inertia

On a polynomial-conductor interval of Theorem 19.15 the period fields lie in their largest member E_*, of conductor dividing N_*. Write $G = \mathrm{Gal}(E_*/\mathbb{Q})$. It matters where one removes the conductor of α. The auxiliary ordinary problem uses a nontrivial cyclotomic character on the kernel of $\bar{\Psi} = \bar{\alpha}\bar{\kappa}_\ell^r$. We construct a real field on which every α is unramified at the finite places and on which its global values are preserved.

Theorem 41.17 (Removing finite ramification while preserving the character group)**.** *There are at most four distinct odd primes $b_i > N_*$ and a totally real abelian field $D/\mathbb{Q}$ of degree $|G|$ and conductor dividing $N_* \prod_i b_i$. Both the conductor and the primes b_i are bounded by a fixed power of the interval parameter V. If a finite Galois extension $M/\mathbb{Q}$ is unramified at all the b_i, then D and M are linearly disjoint. In particular E_*D/D has Galois group G and is unramified at all finite places. The field D is fixed throughout the interval, including when the coefficient prime $\ell > P(x)$ varies.*

Proof. By Proposition 6.4, E_* is the compositum of the cyclic field of ψ, a cyclic real cyclotomic layer, and two quadratic fields. Restriction embeds G in the product of the four cyclic groups. Every subgroup of such a product is generated by at most four elements, by projection onto one factor and induction on the kernel. The structure theorem for finite abelian groups gives independent characters χ_i generating the dual of G, of orders $n_i > 1$, with $\prod n_i = |G|$ and at most four indices. The trivial group uses an empty list.

For each i we need a character θ_i of prime conductor b_i with order n_i and the same parity as χ_i. When χ_i is even impose $b_i \equiv 1 \pmod{2n_i}$. When it is odd, n_i is even and we impose $b_i \equiv 1 + n_i \pmod{2n_i}$. Both residue classes are units. In the cyclic group modulo such a prime an order-n_i character takes -1 to the desired sign, by evaluating at the $(b_i - 1)/2$th power of a generator.

For a common size bound choose an auxiliary prime t between $4(N_* + s + 10)$ and twice that number by Bertrand's postulate, where s is the fixed prime in Proposition 36.3. Use different residues strictly between $N_* + s$ and t as additional congruences for the b_i modulo t. The Chinese remainder theorem gives a unit class modulo $2n_it$, since $n_i \le |G| \le N_*$. Linnik's least-prime theorem bounds a prime in each class by $C(2n_it)^A$ for absolute constants A, C. These primes are distinct and exceed N_* and s. Only four occur and N_* is polynomial in V. D. R. Heath-Brown, *Zero-free regions for Dirichlet L-functions, and the least prime in an arithmetic progression*.

Define D by the characters $\chi_i\theta_i$. They are even. At b_i the character θ_i maps inertia onto all its n_ith roots of unity. All the other θ_j and all the χ_j are unramified there. The images of these inertia groups in the product of the character groups of values therefore contain the individual coordinate factors. The full image is their product. This proves the degree assertion and shows that the inertia groups at the b_i generate $\mathrm{Gal}(D/\mathbb{Q})$. Their images in $D \cap M$ are trivial, hence $D \cap M = \mathbb{Q}$; Galois theory gives linear disjointness. The conductor bound follows by multiplication of the Dirichlet conductors.

At a prime over $p \mid N_*$, inertia in G_D kills $\chi_i\theta_i$, and θ_i is unramified at p. It thus kills χ_i for each i. At every other prime of D the χ_i were already unramified over $\mathbb{Q}$. This proves the finite unramifiedness of E_*D/D. Disjointness applies to $M = E_*$. For sufficiently distant intervals all the displayed conductors and primes are below $P(V)$. □

Proposition 41.18 (Exact restriction on the original strict groups)**.** *For $B = \mathbb{Q}$ or E_x, write $B' = BD$ and $\Delta = \mathrm{Gal}(B'/B)$. Take a discrete ℓ-primary Galois module W and let $S_B(W)$ be the kernel of restriction in $H^1(G_B, W)$ to the full local groups above ℓ and to inertia at the other finite places. If $\ell \nmid |G|$, restriction gives*

$$S_B(W) \simeq S_{B'}(W)^{\Delta}.$$

It commutes with the actions of the characters of the abelian period fields whenever W is obtained by restriction from $\mathbb{Q}$. In particular the original $\mathcal{S}_{\ell,r}(E_x)$ and all its large-prime character summands are recovered with their full lengths, at every twist and every point of the same interval. The assertion also applies over $\mathbb{Q}$ to the finite and divisible character modules in the marked problem.

Proof. Disjointness gives $|\Delta| = |G|$. Multiplication by this integer is invertible on any ℓ-primary module under consideration. Averaging annihilates all positive-degree cohomology of Δ on such modules. Inflation–restriction therefore identifies global H^1 with the invariant part upstairs. For full local groups the same argument gives injectivity on restriction since the local degree divides $|\Delta|$. The intersection of a chosen inertia group with $G_{B'}$ is the inertia group upstairs, of index dividing $|\Delta|$. Inflation–restriction again gives injectivity. A class corresponding under the global isomorphism meets all the local zero conditions downstairs exactly when its restriction does upstairs. The operations are functorial for conjugation. Finite-level character idempotents where defined commute with restriction and averaging. For the Tate module $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)(-r)$ used here, inertia away from ℓ acts trivially. A cocycle restricting to zero in its cohomology vanishes there, so it descends to the global quotient unramified at these places, as in Section 6.4. This identifies the stated groups. □

Theorem 41.19 (The ordinary problem at hyperspecial tame level)**.** *In the notation of Theorem 37.8, put $F'_a = F_a^+ D$, $L'_a = L_a D$, and retain $K_s = \mathbb{Q}(\sqrt{-s})$. For each $\ell > P(V)$, a single sufficiently deep layer a works for all the twists and conductor characters on the interval. On this layer the local, cohomological and numerical assertions of that theorem hold with F'_a, L'_a and a possibly different integer C_0. The marked global ramification quotient over F'_a is now the group of the maximal extension unramified at the finite places away from ℓ. The cuspidal ordinary adjoints of Theorem 36.17 over L'_a have hyperspecial fixed vectors at every finite place away from ℓ on the definite unitary group over F'_a.*

Proof. Choose the auxiliary primes of Proposition 36.15 outside the b_i. Its field $L_a E_*(\mu_\ell)$ is unramified at these primes at every depth. The theorem on D shows that adjoining it preserves the full product of Galois groups of L_a and $E_*(\mu_\ell)$. Thus L'_a remains disjoint from $E_*(\mu_\ell)$. In particular some element of $G_{L'_a}$ acts as complex conjugation on $E_*(\mu_\ell)$. Its values are $\bar\kappa_\ell = -1$ and $\bar\Psi = 1$ for every pair since $\alpha(-1) = (-1)^r$. This same element rules out every identity $\bar\kappa_\ell = \bar\Psi^j$ on the sign kernel.

The real field F'_a has CM quadratic extension $L'_a = K_s F'_a$. It is the base change of L_a/F_a^+, hence unramified at finite places and split above ℓ. The ramification index of L'_a over $\mathbb{Q}$ at ℓ is still ℓ^a, as D is unramified there. Full rational residual cyclotomic inertia survives. The separating inertia element and the local principal-unit bases in the proof of Theorem 37.8 therefore exist over these completions as well. In detail, the local norm on the principal units has rank one, the local principal units have no ℓ-power torsion since they contain no ℓth root of unity other than one, and all but one basis element can be chosen in the norm kernel. The full local dimension is at least that of $F_{a,u}^+$. At infinity both the sign and the parity are preserved by real base change. Away from ℓ the quadratic sign and $\bar\Psi$ are now both unramified on $G_{F'_a}$. Killing the intersection of their inertia kernels kills every such inertia group. The resulting Euler characteristic is the ordinary global one for the group unramified outside ℓ, with the same local correction terms as in Lemma 37.7.

The two nonzero strict classes of Proposition 37.5 remain nonzero on $G_{F'_a}$, since the degree over F_a^+ is prime to ℓ. Their zero restrictions above ℓ and their inertia conditions away from it persist. In particular a cocycle on the old sign and character quotient vanishes on the inertia groups killed on the new field. For the uniform upper bounds set $T'_a = L'_a E_*(\mu_\ell)$. These are cyclotomic layers of the abelian field T'_0, totally ramified up the tower at every place above ℓ. Indeed the extra factors at depth zero are unramified locally over $\mathbb{Q}_\ell(\mu_\ell)$. As in Proposition 37.6, Ferrero–Washington and Iwasawa's structure theorem bound the ℓ-ranks of all their class groups by one integer h'. The unramified class extensions of T'_a are disjoint from the totally ramified tower, so their groups are quotients of its unramified abelian pro-ℓ module. Let g' be the constant number of places over ℓ. Restriction to T'_a is injective on the strict residual

groups by prime-to-ℓ degree, and their additive characters factor through the class group. This bounds the relevant dimensions by h'. For trivial coefficients over F'_a the same restriction followed by Kummer theory bounds H^1 by $1 + g' + h'$: in the minus eigenspace the ordinary units modulo powers contribute at most one dimension, their quotients by their conjugates being roots of unity, and the allowed valuations contribute at most g'. Taking $C_0 = h' + g'$, the global degree eventually exceeds $3C_0 + 7$ and the local degrees exceed five. Finiteness of H^1 with finite coefficients on open subgroups follows by the same finite-prime Kummer argument as in Theorem 37.8. The bound annihilating everywhere unramified abelian characters on L'_a is its finite class number. These verify the stated frame assertions. [12, 13]

For the adjoints, use cyclic base change along the abelian extension L'_a/L_a. Cuspidality persists at every prime cyclic step: a self-twist by a nontrivial quotient character would, by good-place compatibility and Chebotarev, give such a self-twist of the normalized Galois representation. On the smaller open subgroup its intertwiner would be scalar by absolute irreducibility there. The twisting character would then be one. Cyclic base change and its cuspidality criterion apply successively. The associated representation at each step is the restriction, by the same good-place comparison. The local adjoint at an old conductor prime had finite inertia $\alpha, 1, \alpha^{-1}$ and zero monodromy. All three restrictions to inertia on L'_a are trivial. It was unramified at the other finite primes away from ℓ and is still unramified there. The crystalline ordinary flag at ℓ restricts with its specified parallel weights. Its spherical refinement is ordinary by the comparison of the ordered Frobenius slopes with the ordered Hodge slopes in Section 36.9. The unitary descent used there applies in odd rank since the CM extension is unramified at finite places and split above ℓ. It has spherical components at every finite place away from ℓ. [8, 14, 9] □

Corollary 41.20 (A common tame level for the original Gaussian points and the adjoints)**.** *Set $M_a = L'_a\mathbb{Q}(i)$, with maximal totally real subfield M_a^+. The normalized adjoints above and the restriction of every large-prime marked Gaussian representation have the same residual semisimplification on G_{M_a} and are unramified away from ℓ. At the original weights the cuspidal adjoint gives a nonzero ordinary module for this residual system at hyperspecial tame level. One may also impose any of the split Taylor–Wiles levels of Section 37.7 at additional places. The dual ordinary module at their diagonal cover is finite free over $\Lambda[\Delta_N]$ and its diamond coinvariants are the Iwahori module. Here Λ is the three-character principal weight algebra over M_a^+. The analogous module statement holds over F'_a.*

Proof. We have $M_a = K_s M_a^+ = \mathbb{Q}(i)M_a^+$. Its quadratic extension over M_a^+ is the base change of L'_a/F'_a, hence unramified at all finite places and split above ℓ. The preceding argument gives cuspidal base change of the adjoint over the extension of degree at most two from L'_a, followed by spherical unitary descent. An original marked Gaussian representation on $G_{\mathbb{Q}(i)}$ kills the kernel of α on inertia away from ℓ. On G_{M_a} this is the entire inertia by construction. Both residual semisimplifications are $\bar{\Psi} \oplus 1 \oplus \bar{\Psi}^{-1}$, and the adjoint exists at weight $m = r$.

For integral ordinary control choose in advance an odd rational prime p_0 completely split in M_a whose class modulo ℓ has order $\ell - 1$. This is possible by Chebotarev: M_a has ℓ-power ramification index at ℓ, while each nontrivial subfield of $\mathbb{Q}(\mu_\ell)$ has prime-to-ℓ ramification index greater than one there. Choose all additional places away from p_0. At one split place over p_0 first impose the principal subgroup modulo p_0. It is torsion-free by the matrix logarithm, giving a sufficiently small level for ordinary control. Its quotient in the maximal compact subgroup has order $p_0^3(p_0 - 1)(p_0^2 - 1)(p_0^3 - 1)$, a unit at $\ell > 3$ on these distant intervals. Averaging over that quotient integrally restores hyperspecial level. It commutes with weights, ordinary projection, every added diamond action and change of level at the added places. The integral freeness and coinvariant assertions of Proposition 37.26 therefore pass to this direct summand.

The completed weight ring and its finite abelian ℓ-group ring are local, so the finite projective summands are free. The same reasoning applies to the common residual Hecke summand, cut out by the local factor in the finite Hecke algebra over the complete weight ring. Nonvanishing follows from the spherical adjoint and its ordinary refinement. A spherical vector is still fixed at all additional smaller levels. Over F'_a one uses L'_a in the identical prime choice. □

For a strongly irreducible characteristic-zero Gaussian point, cuspidal automorphy of its normalized restriction to M_a would descend to $\mathbb{Q}(i)$ by Theorem 41.10, since $M_a/\mathbb{Q}(i)$ is abelian. The available adjoint supplies a point in the ordinary module. Controlling all the components through the prescribed residual system still requires the propagation argument.

41.7 The ordinary determinant on the three integral levels

Fix a sufficiently distant interval as in Section 6.6 and one of its large coefficient primes ℓ. We use one of three CM extensions L/F. They are L_a/F_a^+ of Proposition 36.15, L'_a/F'_a of Theorem 41.19, and M_a/M_a^+ of Corollary 41.20. Fix a pair r, α from the interval. In the first extension the allowed inertia away from ℓ is the finite quotient given by α. In the other two the quotient is trivial. At the places above ℓ choose one of the two places of L over each place of F and the three ordered residual characters of the marked problem. The inertia characters are distinct there. Write Λ for the completed weight algebra of the principal diagonal units, with the finite prime-to-ℓ diamonds fixed by these data. We use the cyclotomic twist of Section 37.8 when comparing the characteristic polynomials of a Galois representation with the spherical Hecke operators.

Let Q be a finite set of split places of F, disjoint from the earlier conductors and ℓ, whose residue cardinalities are 1 mod ℓ. At Q use, in turn, the hyperspecial, Iwahori, and diagonal-cover levels from Section 37.7. Denote their dual ordinary modules, with the old local conditions specified below, by M_K, M_I, M_1. We choose the local factor of the good Hecke algebra for the residual system $\bar{\Psi} \oplus 1 \oplus \bar{\Psi}^{-1}$. All three localizations use good Hecke operators outside the same set of places.

Lemma 41.21 (Integral selection of the old inertia)**.** *Over L_a/F_a^+ one can retain precisely the local automorphic types having inertia $\tau_v = \alpha \oplus 1 \oplus \alpha^{-1}$ and zero monodromy at the old conductor places, by applying integral idempotents at those places. These idempotents are independent of Q. Over L'_a/F'_a and M_a/M_a^+ one may use hyperspecial level throughout the old tame places. In each case M_1 is finite free over $\Lambda[\Delta_Q]$, and its diamond coinvariants are M_I. All three modules are finite free over Λ and contain the ordinary eigensystem of the adjoint at the original weight.*

Proof. First work over F_a^+. Every v above a rational prime $p \mid N_*$ splits in L_a, so the local unitary group identifies with $\mathrm{GL}_3(F_v)$. Its residue cardinality is $q_v = p^f$. The residue degree divides $4\ell^{a+1}$ by Proposition 36.15, hence $f = c\ell^j$ with $c \mid 4$. For $1 \le h \le 3$, Fermat's theorem gives $q_v^h \equiv p^{ch} \bmod \ell$. Moreover $1 < p^{ch} \le N_*^{12} < \ell$ on all sufficiently distant intervals because N_* grows polynomially in V. Thus ℓ divides none of $q_v^h - 1$. The finite groups

$$H_{v,t} = \mathrm{GL}_3(\mathfrak{o}_v/\mathfrak{p}_v^t) \qquad (t \ge 1)$$

have orders prime to ℓ. Reduction modulo the maximal ideal has a p-group kernel, and the order over the residue field is $q_v^3 \prod_{h=1}^{3}(q_v^h - 1)$.

Use the inertial local Langlands type σ_v attached to τ_v with zero monodromy. We recall the precise criterion being used. If an irreducible admissible local representation contains this compact type, the Jordan partitions of its inertial parameter are dominated by those of the specified parameter in each finite inertial block, with equal total sizes. Conversely equality of

the full inertial parameter gives occurrence. The partitions of the zero operator consist entirely of ones, so each is minimal in dominance order. Occurrence is therefore equivalent to the stated inertia, including when two characters coincide. This applies also to wild finite characters. With geometric or arithmetic reciprocity chosen consistently, simultaneous inversion does not change this self-dual triple. J. Shotton, *The Breuil–Mézard conjecture when $\ell \neq p$*, Definition 3.4 and Theorem 3.7.

By smoothness and finite dimension σ_v factors through $H_{v,t}$ for some t. Start at principal level t. Enlarge the coefficient DVR O to contain the character values. For $H = H_{v,t}$ the projector on the σ_v isotypic space is

$$\frac{\dim \sigma_v}{|H|} \sum_{h \in H} \operatorname{tr}(\sigma_v(h^{-1}))[h].$$

All coefficients are integral, since the traces are sums of roots of unity and $|H|$ is a unit. Character orthogonality proves idempotence in the group algebra. Use the transpose projector on a dual. The product over the old places commutes with every operation at Q and ℓ. For unramified types one can take hyperspecial level directly. The other two CM extensions have only these latter types by Theorem 41.17.

At an extra odd split prime of primitive residue class modulo ℓ, first impose principal level and then average back to hyperspecial level as in Corollary 41.20. This can be done away from the indicated finite sets. Ordinary control and Proposition 37.26 apply at the sufficiently small intermediate level. Each operation used on the result is a commuting idempotent, including the common residual localization. This gives the coinvariant assertion and finite projectivity. The respective coefficient rings are local, so the modules are free. The adjoint has the specified finite inertia with zero monodromy at the old places and is spherical at Q. Its ordinary refinement and stable base changes are given by Theorems 36.17 and 41.19. Its vector survives all the indicated projections. □

Proposition 41.22 (Classical systems distinguishing endomorphisms)**.** *At each of the three levels one may test identities of good Hecke operators, acting as endomorphisms over Λ, at all the ordinary classical systems of sufficiently separated dominant algebraic weights with the specified tame diamonds. Each of these systems has a continuous semisimple polarized Galois representation of G_L. It is unramified outside ℓ, Q and the old conductor. At each old conductor place its inertia is the prescribed finite representation. At the chosen places over ℓ it has an ordinary flag with the ordered inertial characters given by the weight. At hyperspecial level the representations are also unramified at Q.*

Proof. Choose algebraic weights with every consecutive gap in the archimedean infinitesimal character greater than two. They are dense in the weight algebra. Indeed on integer exponent tuples with any prescribed residues, including residues at an arbitrarily high power of ℓ, one can increase each successive exponent by a multiple of the common modulus so as to enforce all the gaps. On a small principal-unit neighborhood the logarithmic embedding matrix at each place has nonzero determinant by separability of the local field. The exponent tuples are dense in an open subset of $\mathbf{Z}_\ell^{3[F:\mathbb{Q}]}$. A convergent power series in logarithmic coordinates vanishing on that subset vanishes by the one-variable identity theorem successively in the coordinates. The invertible linear substitution by the embedding matrix preserves this identity. This proves the required detection of power series, as in Proposition 37.34. At these weights ordinary control identifies the fibers with classical spaces. On each classical fiber the good spherical algebra is semisimple, by the decomposition of automorphic forms and the spherical character on each summand. Since the modules are free over Λ, testing at all these fibers detects every matrix entry of a good Hecke operator.

We spell out the compatibility with inertia which is needed in applying this test. Use the discrete automorphic transfer for definite unitary groups in three variables and the Galois attachment of Geraghty. The transfer at a split place has the parameter of its GL_3 component. At a hyperspecial place its unramified parameter restricts to an unramified parameter over L. A discrete global parameter is a sum of cuspidal blocks $\pi_b \boxtimes [d_b]$, where $[d_b]$ contributes consecutive archimedean exponents with spacing one if $d_b > 1$. This is excluded by the gaps. Thus the local transfer is the sum of the local parameters of the cuspidal constituents, and each is regular at infinity in its cohomological normalization. This also explains why endoscopic systems enter the argument. The normalized global Galois representation is the semisimple sum of the attached constituents, as follows from its good Frobenius polynomials and Chebotarev. In rank n_b the twist $|\det|^{(n_b-3)/2}$ of the unitary normalization aligns the local Langlands normalization $|\det|^{(1-n_b)/2}$ with the common $|\det|^{-1}$ in rank three. The additional common cyclotomic twist in our determinant is unramified away from ℓ. [14] J. Rogawski, *Automorphic Representation of Unitary Groups in Three Variables.*

At a place of residue characteristic different from ℓ, local–global compatibility for a regular algebraic cuspidal constituent gives the semisimple Weil parameter and bounds its Galois monodromy by that of its automorphic parameter in dominance order. For a rank-one constituent this is local class field theory. Zero monodromy on the transferred sum means zero on every constituent, hence zero on the Galois side. The finite inertia is now determined by its semisimplification in characteristic zero. At each of the old places the preceding lemma gives just such a parameter. The same argument gives unramifiedness at every hyperspecial place away from ℓ, including the inert places and, at level K, the places of Q. I. Varma, *Local–global compatibility for regular algebraic cuspidal automorphic representations when $\ell \neq p$*, Theorem 1.

At the places over ℓ the ordinary flag statement for regular classical ordinary forms is Geraghty’s Galois attachment and ordinary comparison, Proposition 2.7.2 and Corollary 2.7.8 in his original numbering. Its hypotheses here are the splitting above ℓ, the regular dominant weight, and the unit ordinary eigenvalues for the normalized operators. These hold for each ordinary eigensystem being tested. In particular the ordered inertial characters are the algebraic characters of the diagonal units, with the fixed cyclotomic normalization. Together with conjugate duality and the good-place formula from the same attachment this proves the proposition. [14] □

For the fixed host field write R_Q for the universal ordinary polarized determinant ring of Section 37.8, allowing the extra ramification at Q and its conjugates. Write R_0 for the original ring. Both have the full principal weight algebra, the same ordered residual characters and the same fixed finite diamonds. Killing the added inertia gives the surjection $R_Q \to R_0$. Let B_Y be the Λ-subalgebra of the endomorphisms of M_Y, for $Y = K, I, 1$, generated by the normalized spherical characteristic coefficients and inverse determinants at good split places outside all the levels. These algebras use only operators common to the three levels.

Theorem 41.23 (Integral actions of the ordinary determinant)**.** *There are continuous surjections $R_Q \to B_Y$ for all three levels. They give the good Frobenius polynomials and the specified ordered inertial weights on the entire modules. The action on M_K factors through R_0. Passage from M_1 to M_I by diamond coinvariants is compatible with these actions.*

Proof. Each B_Y is a submodule of a finite matrix algebra over the Noetherian ring Λ, hence is finite over Λ. Its evaluations at the classical systems in the proposition embed it in the product of their coefficient valuation rings, by the detection assertion. In particular it is reduced. In the topology of its finite Λ-module it is compact and these evaluations are continuous, so the

embedding is a homeomorphism onto a closed subspace. Good split Frobenius classes are dense up to conjugacy in G_L. Indeed Chebotarev gives positive density for any class in a finite quotient, and the primes of L above inert places of F have density zero. At these classes the tuple of classical traces belongs to B_Y by the good Hecke formula. At an arbitrary $h \in G_L$ the trace tuple belongs to the same closed image by continuity. This defines a continuous trace t_Y with values in B_Y.

The characteristic polynomial identities hold in the product and therefore in B_Y. Since 3! is invertible, the degree-three determinant on a group algebra element is recovered from the traces of its first three powers. Applying this to a generic linear combination of group elements checks the identities of polynomial laws over arbitrary coefficient extensions. The trace identities, continuity and polarization thus give the polarized determinant integrally. The reduction at the closed point has the prescribed semisimplification by the good-prime localization and Chebotarev. For g in any inertia kernel imposed on the global quotient and $h \in G_L$, one has $t_Y(hg) = t_Y(h)$ by the proposition. Conjugation, multiplication and closure extend this to the normal subgroup generated by these kernels. Consequently the determinant factors through the required quotient. At hyperspecial level the same proof kills the additional inertia at Q. [2]

For clarity the full ordered local characters also take values in this good Hecke algebra. At each chosen place above ℓ select s in inertia separating the three residual characters, and let a_i be the universal inertial weight values there. Their pairwise differences are units. For h in the local decomposition group put

$$\xi_i(h) = t_Y\big(P_i(s)h\big), \qquad P_i(X) = \prod_{j \neq i} (X - a_j)/(a_i - a_j).$$

The trace here is extended linearly. At every classical test point s is diagonalizable with these distinct eigenvalues. The ordered flag is the successive span of its eigenlines. Thus $\xi_i(h)$ is the ith diagonal character. Multiplicativity, the value at the identity and the prescribed restriction on inertia follow by detection. For arbitrary local elements h_1, h_2, h_3, the ordered product of the three differences $h_i - \xi_i(h_i)$, with $i = 3$ acting first on the right, lowers these three flag steps successively to zero. The trace after multiplication by every global group element vanishes identically by the same test. This is the ordinary relation in the polarized determinant problem. The universal property now gives the asserted maps. Their images contain the weights and all good characteristic coefficients. The inverses of the unit determinants come from inverse group elements. The images therefore contain the generators of B_Y.

The diamond action commutes with the good operators and weights. The quotient to M_I consequently commutes with the continuous traces at every global element by the same good Frobenius density. The displayed projector formula then gives compatibility of the ordered local characters. These data determine the maps from R_Q. □

Corollary 41.24 (Annihilation of additional inertia on coinvariants)**.** *Write $J_Q = \ker(R_Q \to R_0)$. For the local Bernstein center at $v \in Q$ use the variables $X_{v,i}$ and residue cardinality q_v of Section 37.7. On the integral ordinary Iwahori module one has*

$$C_Q \, J_Q M_I = 0, \qquad C_Q = \prod_{v \in Q} \prod_{i \neq j} (X_{v,i} - q_v X_{v,j}).$$

After inverting C_Q, the Iwahori module is the hyperspecial module with the roots of the local spherical polynomials adjoined in all orders. The isomorphism respects the full action of R_Q. These assertions apply to the actual diamond coinvariants $M_1/(\Delta_Q - 1)M_1$.

Proof. Write Z_Q for the tensor product of the local centers and S_Q for the corresponding algebras with ordered roots, tensored over O. Apply the two maps of Theorem 37.28 to $S_Q \otimes_{Z_Q} M_K$ and M_I. The averaging idempotents, ordered variables and local Hecke operators used in these maps are supported at Q. They therefore commute with the good operators outside the levels and with the ordinary weights. They are continuous on these finite weight modules. For every element of G_L, approximate its conjugacy class by the same good Frobenius classes on both sides. It follows that the maps intertwine the two universal trace actions. The formula for ξ_i proves the same for every local character. A determinant with 3! invertible and its ordered local characters determine its universal action, so both maps are R_Q-linear. Equivalently, the traces of group algebra elements and the values of the local characters topologically generate the representing ring. This follows by forming its closed subalgebra generated by these coefficients and applying the universal property to the resulting pair over that subalgebra, or directly from the generators of the universal trace algebra.

Denote the map from M_I to the ordered hyperspecial module by Ψ and the reverse map by Φ. If $a \in J_Q$, then $\Psi(ay) = a\Psi(y) = 0$ by the theorem. Applying Φ and using $\Phi\Psi = C_Q$ gives $C_Q a y = 0$ for every y. The second composite is also C_Q, so $C_Q^{-1}\Psi$ is an inverse to Φ after localization. The identification with coinvariants was proved in Lemma 41.21. □

For the auxiliary sets and the characteristic-ℓ curves of Corollary 37.29 on the original host, C_Q specializes to a unit times z^{D_0}, with D_0 independent of the level. Thus the extra inertia on these specialized coinvariants has an integral annihilator of bounded order in the curve parameter. The identities occur on the same modules and use the same global determinants as the two level maps. Propagation from their support to every required mixed-characteristic component, and the projected period estimate on the fields E_x, remain open here.

41.8 Frobenius and inertia on the ordered diagonal cover

Keep the fields, coefficient DVR O, tame types, ordinary weight algebra Λ, and modules of Section 41.7. We give the relations before taking diamond coinvariants. In particular the inertia characters in them can have order growing with the auxiliary level. Write t for the trace from Theorem 41.23, acting on M_1. Adjoin to its commuting operators the normalized diagonal Hecke operators at Q. Trace on the group algebra with coefficients in this commutative algebra means linear extension of t. The variables at Q act on the level module, while the group elements inside the trace belong to the global Galois group.

Choose one place w of L above each $v \in Q$, and let q_v be its residue cardinality. Use a geometric Frobenius lift ϕ_v and an inertia lift τ_v whose image in the maximal ℓ-quotient of the residue units generates it. Local reciprocity sends a uniformizer to geometric Frobenius. Use the uniformizer, including its unit factor, representing the actual abelian image of ϕ_v, and the unit representing τ_v.

Lemma 41.25 (Hecke polynomials at elements of the local Weil group)**.** *There are commuting integral operators $T_{v,i}(h)$, $1 \le i \le 3$, for h in the local Weil group at w. For each i they give a character of its abelianization into the units of the commutative operator algebra. Their values on τ_v are the three coordinate diamonds $D_{v,i}$. Put $U_{v,i} = T_{v,i}(\phi_v)$. At every classical test system of Proposition 41.22, with normalized Galois representation ρ, one has the following identity of polynomials acting on its whole local fixed-vector space:*

$$\prod_{i=1}^{3} (X - T_{v,i}(h)) = \det(X - \rho(h))\,\mathrm{id}.$$

The coefficient identities also hold on M_1, with the global determinant coefficients in place of the classical ones.

Proof. Use the Bernstein homomorphism for the diagonal cover of an Iwahori. It gives a homomorphism from the group algebra of the diagonal torus modulo the kernel of the residue diamonds to the cover Hecke algebra. On diagonal units it is the diamond action. The normalized coordinate homomorphisms of Newton and Thorne use the unramified factors on uniformizers for the local Langlands normalization. Their products and inverses are integral, since the residue cardinality is an ℓ-adic unit. The chosen square root of $q_v \equiv 1 \bmod \ell$, where needed, also belongs to $O^\times$. For a dual module take the Hecke action transposed from the one on forms, with the same eigenvalue convention. The torus algebra is commutative. Normalize each coordinate by the same additional unramified character to account for the fixed global cyclotomic twist. This does not affect the coordinate diamonds.

The local theorem used here is the characteristic-polynomial formula (4.1) of Newton and Thorne. It says that each elementary symmetric function of the three torus operators at an element of the local multiplicative group acts on the entire space of fixed vectors of an irreducible representation by the corresponding coefficient of its semisimple Weil parameter. The formula holds for every Weil element with that image under reciprocity. Its scalar assertion follows from the description of the center of the cover Hecke algebra. The local operators and the original proof of this assertion are in Allen et al., Section 2.2, equation (2.2.7) and Proposition 2.2.8. On the Iwahori subspace the torus operators for a uniformizer are the Bernstein variables of Section 37.7, with the common normalization just specified. For positive translations the unipotent representatives of the Hecke cosets are identical at both levels, since the two subgroups differ only in the diagonal units. Multiplicativity gives the assertion for an arbitrary torus element. [1] P. B. Allen et al., *Potential automorphy over CM fields*, Section 2.2.

The split-place transfer and local–global compatibility used in Proposition 41.22 identify the semisimplified Weil parameters at every $v \in Q$. Monodromy does not alter the characteristic polynomial of a Weil element. Indeed the Weil action preserves the filtration by the kernels of the powers of its monodromy operator, and the exponential of a multiple of this operator acts as the identity on the successive quotients. Thus the local formula gives the polynomial of the actual Galois matrix. Every coefficient on M_1 is an endomorphism of a finite free Λ-module. At the dense weights its matrix specializes to the identity asserted on each classical summand. Detection of the matrix entries gives the identity on the integral module. This argument only needs the good Hecke algebra to act semisimply at the test weights. □

For a fixed v temporarily write ϕ, τ, U_i, D_i, q for this place's data. Put

$$A_v = \prod_{i \neq j} (U_i - U_j), \qquad C_v = \prod_{i \neq j} (U_i - q U_j), \qquad f_v = A_v C_v.$$

The products are over ordered pairs. They are symmetric in the U_i. For any global group element g put

$$L_v(g) = t\left(g \prod_{i=1}^{3} (\phi - U_i) \right),$$

$$H_{v,i}(g) = t\left(g \prod_{j \neq i} (\phi - U_j)(\tau - D_i) \right).$$

Theorem 41.26 (Integral inertia relations with exponent one)**.** *On the full diagonal-cover module M_1, for every g, v, i,*

$$L_v(g) = 0, \qquad f_v H_{v,i}(g) = 0.$$

There is no dependence of the exponent on the residue cardinality, the depth, the weight, or the number of additional places. The assertions persist after arbitrary specialization of the coefficient algebra.

Proof. Work on one classical system and its space of vectors at the given level. The first identity follows from the lemma and the Cayley–Hamilton theorem for the matrix $F = \rho(\phi)$. One may use the scalar characteristic polynomial of F in the product before multiplying by $\rho(g)$ and taking the trace. For the second identity the lemma implies that f_v acts by a scalar, since it is a symmetric function. If this scalar is zero the assertion holds on the entire space. Suppose it is nonzero. The three roots u_i of the polynomial of F are then distinct, and no two have ratio q or q^{-1}.

Write N for Galois monodromy at this place. The relation $FNF^{-1} = q^{\pm 1}N$ forces $N = 0$, since conjugation by F has only the pairwise root ratios as eigenvalues. Thus the inertia image is finite. We verify that it commutes with F. Over an algebraically closed coefficient field the commuting torus operators on the nonzero local fixed-vector space have a simultaneous eigencharacter. Evaluating the lemma there gives the character of the semisimple Weil representation as a sum of three characters of the local abelianization. On the finite inertia image the actual representation is semisimple. Its inertia characters therefore extend to characters of the Weil group and are fixed under conjugation by F. It follows that F preserves their isotypic spaces and commutes with inertia. In the three eigenlines of F write d_i' for the eigenvalue of $\rho(\tau)$.

Each of the operators U_i satisfies the same square-free cubic, by substituting it for X in the identity of the lemma. They are simultaneously diagonalizable. The D_i commute with them and are diagonalizable in characteristic zero because they have finite order. Choose any joint eigenspace. Order the u_i by the eigenvalues of the three operators U_i there, and write d_i for the eigenvalue of D_i. The cubic proves that all three roots occur in this order. Apply the trace coefficient of the lemma at $\phi^k\tau$ for $k = 0, 1, 2$. Its two sides read

$$\sum_i u_i^k d_i = \sum_i u_i^k d_i'.$$

The Vandermonde determinant is nonzero, so $d_i = d_i'$ for all i. On the Galois space the matrix $\prod_{j\neq i}(F - u_j)(\rho(\tau) - d_i)$ vanishes. Indeed the first factor vanishes on the other eigenlines and the second vanishes on the ith. Taking the trace after any $\rho(g)$ proves the desired identity on this joint eigenspace. These eigenspaces exhaust the fixed vectors in the case under consideration. The calculation is unchanged in the tensor product with the vectors at all other places.

The dense classical fibers detect entries of matrices on M_1, so both identities hold over Λ, including above residual coincidences among the roots. Base change preserves zero endomorphisms. This proves the two integral relations. In the notation of `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/ThetaCoefficientMapEq.lean`, line 224, multiplying by the other f_w consequently gives both trace relations there with exponent one. □

Proposition 41.27 (Selecting any ordering of the residual roots)**.** *At each $v \in Q$ prescribe any ordering of the three residual roots of* $\det(X - \bar\rho(\phi_v))$*. Repetitions are allowed. The associated summand M_1^{ord} of M_1 is finite free over $\Lambda[\Delta_Q]$ and its coinvariants are the corresponding Iwahori summand M_I^{ord}. If $B = R_0$ and P^{ord} is the residual local factor of the ordered splitting algebra of the universal Frobenius cubics over B, then $P^{\mathrm{ord}} \otimes_B M_K$ is the corresponding summand on the ordered hyperspecial side of Corollary 41.24.*

Proof. Consider the commuting algebra generated by the $U_{v,i}$, the diamonds and B_1 in the finite matrix algebra over Λ. It is finite over Λ, which is complete local. It is the product of its

complete local factors, by lifting the idempotents of its finite Artinian special fiber. On that fiber the three $U_{v,i}$ have joint residual values given by orderings of the roots, by Lemma 41.25. All roots belong to the residue field after the fixed coefficient extension since the residual characters are defined there. Take the sum of the factor idempotents giving the prescribed tuple. It commutes with $\Lambda[\Delta_Q]$, so its image is projective and hence free over that local ring. Reducing the decomposition through the diamond augmentation gives exactly the selected residual factors downstairs. This proves the coinvariant assertion, since the torus operators are compatible under change to Iwahori level.

Over B, adjoin one ordered root of each monic cubic and then one root of its remaining monic quadratic. The third root is prescribed by the sum. All roots are units by the unit constant term. This gives a finite free algebra of rank $6^{|Q|}$ over B, including cubics with repeated roots. Its local factors are obtained by the same argument with the special fiber. In particular each prescribed ordering gives a nonzero factor, finite free over the local ring B. On M_K the characteristic coefficients of the universal Frobenius at v are the coefficients of the hyperspecial Hecke polynomial, by the classical test and the spherical case of the lemma. Substituting these values into the presentation of the splitting algebra gives the ordered module in the corollary. The residual decompositions agree because they are determined by identical coordinate operators. The two maps in that corollary commute with the ordered coordinates and hence preserve these summands.

They are nonzero on the Iwahori side as well. Evaluate at the adjoint. At each $v \in Q$ this unramified adjoint has Frobenius eigenvalues of a single complex absolute value, by the purity of the two-dimensional modular form and its adjoint. Consequently C_v is nonzero on its hyperspecial vectors. The ordered splitting module there has a point in each indicated residual ordering, by choosing the three integral unit roots in a finite coefficient extension and putting them in the required order. The two composites with value C_Q then give a nonzero Iwahori summand, and the surjection from M_1^{ord} gives its nonvanishing too. □

Theorem 41.28 (A finite-level diagram over the universal base)**.** *Let $q_v \equiv 1 \bmod \ell^N$ for every $v \in Q$, put $a_v = v_\ell(q_v - 1)$, and prescribe the residual orderings of the preceding proposition. There is a complete Noetherian local O-algebra $\mathcal{A}_Q$ carrying the universal ordinary pair pulled back from R_Q, three lifted root coordinates and three diamond coordinates at every place. It acts on M_1^{ord} and maps to P^{ord}. Under this map the root coordinates go to the ordered universal roots, the diamond coordinates go to one, and the ordinary pair is the pullback of the pair over $B = R_0$. There are $\mathcal{A}_Q$-linear maps in both directions between*

$$P^{\mathrm{ord}} \otimes_B M_K \quad \text{and} \quad M_1^{\mathrm{ord}}/(\Delta_Q - 1)M_1^{\mathrm{ord}}$$

whose two composites equal $\prod_{v\in Q} f_v$ on their respective modules. On the diagonal cover the two trace relations of the preceding theorem hold. On the curves and auxiliary sets of Corollary 37.29, the common composite has valuation $2D_0$ before truncation to the prescribed precision.

Proof. Lift the chosen residual values to units $b_{v,i} \in O$. Take

$$\mathcal{A}_Q = R_Q[[Z_{v,i}, Y_{v,i} : v \in Q,\ 1 \le i \le 3]].$$

Use the root coordinates $b_{v,i} + Z_{v,i}$ and the diamond coordinates $1 + Y_{v,i}$. On M_1^{ord} send $Z_{v,i}$ to $U_{v,i} - b_{v,i}$ and $Y_{v,i}$ to $D_{v,i} - 1$, together with the action of R_Q from Theorem 41.23. The indicated differences are nilpotent modulo the maximal ideal of Λ on the selected summand. They therefore tend to zero under repeated powers on that finite module. All operators commute, so this defines a continuous formal power series action. The map to P^{ord} uses $R_Q \to B \to P^{\mathrm{ord}}$,

sends $Z_{v,i}$ to the ith universal root minus $b_{v,i}$, and sends every $Y_{v,i}$ to zero. The residual ordering ensures convergence there. The maps on the ordinary determinant, its local characters and its principal weights follow from their definitions.

Over the diamond and weight variables the diagonal-cover module is finite free over

$$\Lambda[[Y_{v,i}]]/\big((1+Y_{v,i})^{\ell^{a_v}}-1:\ v\in Q,\ 1\le i\le 3\big).$$

Indeed this is the completed group ring of the product of the cyclic groups generated by the chosen diamonds, each of order ℓ^{a_v}. The assertion is exactly the freeness of Proposition 41.27. Putting $Y=0$ gives the indicated coinvariants. The support over B of the base module M_K is cut out by $\ker(B\to B_K)$, since B_K is its faithful algebra of good Hecke operators and the module is finite over B.

Use the two maps Φ,Ψ in Corollary 41.24 on the selected summands. They intertwine R_Q and the root coordinates. They also intertwine $Y_{v,i}$, which are zero on both sides. Continuity gives the same for their formal power series. Multiply Ψ by $\prod_{v\in Q}A_v$. The two resulting composites are exactly $\prod_v f_v$. If the normalization of the three Bernstein variables differs by a common unit from that of the Hecke comparison, rescaling all three changes C_v by the sixth power of that unit. Multiplying the reverse map by this unit power restores the stated scalar. The full-cover trace identities follow from Theorem 41.26. Finally, on the specified curves the residue characteristic is ℓ, so $q_v=1$ as a scalar and $A_v=C_v$. The product of the latter is a unit times z^{D_0} by Corollary 37.29. Squaring proves the claim, with the same identities after truncation, even at precisions at which this square vanishes. □

The construction gives the finite-level maps and their trace relations for each auxiliary set, including on the original L_a/F_a^+ used to test the cohomology curves. Passing from these maps to support of every required mixed-characteristic component requires a further commutative-algebra argument at the curve primes.

41.9 The limiting Galois group and its ordinary cohomology

Return to the field $F=F_a^+$, $L=L_a$ on which the simultaneous Frobenius tests of Theorem 37.17 were constructed. Fix one pair r,α. Denote its Galois quotient from Section 37.2 by $\widetilde{\Gamma}_0$, and allow additional inertia at the q places of $\mathcal{Q}_N$ to obtain $\widetilde{\Gamma}_N$. At all other places use exactly the previous local conditions. The sign η cuts out L. The residual characters $\bar{\Psi},\bar{\kappa}_\ell,\eta$ give a common finite quotient Δ with fixed field T in the sense that $\Delta=\mathrm{Gal}(T/F)$. In particular T contains L and μ_ℓ. The integer N indexes the auxiliary level and varies with this field T fixed.

Lemma 41.29 (A uniformly generated quotient for the residual problem)**.** *In $\widetilde{\Gamma}_N$ replace the kernel of the map to Δ by its maximal pro-ℓ quotient, including for $N=0$. Denote the resulting groups by Γ_N and Γ_0. This preserves the ordinary determinant problems of Section 41.7 and the curve cocycles of Sections 37.5 and 37.6. These groups have generating sets of bounded cardinality as N varies. The natural map $h_N:\Gamma_N\to\Gamma_0$ is onto, and its kernel is the closed normal subgroup generated by one tame ℓ-inertia generator at each of the q new places.*

Proof. The kernel of the maximal pro-ℓ quotient is characteristic in the subgroup being reduced, and so is normal in the full group. To verify the assertion for determinants, first take an Artinian local coefficient ring with finite residue field. It is finite. A continuous trace with these values factors through a finite group quotient, since it is locally constant on a compact group. Explicitly there is an open normal subgroup whose right multiplication preserves the trace at every point. As 3! is invertible, Newton's formula on arbitrary finite group algebra

sums shows that the determinant factors as well. Use the Cayley–Hamilton algebra of this determinant over the finite group quotient. This is a quotient of a finite group algebra over the coefficient ring and is finite as a set. The determinant descends to it, by the Cayley–Hamilton ideal theorem for determinants. On the CM subgroup above the identity of Δ, the residual characteristic polynomial of each group element is $(X-1)^3$. Its image in the algebra modulo the coefficient maximal ideal is thus unipotent, by Cayley–Hamilton. Each such unit has ℓ-power order, by taking ℓ-powers in characteristic ℓ. The units congruent to one modulo the coefficient maximal ideal also form an ℓ-group, as their successive congruence quotients are additive groups in characteristic ℓ. The finite image of the whole subgroup is therefore an ℓ-group by Cauchy's theorem. The determinant consequently factors through the asserted relative quotient. The same conclusion holds under conjugation by the other coset of the CM subgroup, since the kernel being removed is characteristic. Inverse limits apply to complete coefficient rings. The ordered characters are prescribed on the same local absolute groups, with the same residual and torsion restrictions. All their determinant relations are preserved under this factorization. [2]

The finite curve representations and their coefficient modules have the same property. On the residual kernel their semisimple constituents in residual characteristic are trivial. This follows first on the marked representation from its three residual characters and then on its endomorphisms, direct summands and the successive extensions used to form the paired coefficients. A stable residual flag and its congruence filtration put the image of this kernel in an ℓ-group. Adjoining a cocycle to such an action amounts to mapping into a semidirect product by its finite additive module. The image of the residual kernel is still an ℓ-group. Hence the finite-precision representations and cocycles descend, also with the prescribed local conditions. Inverse limits give the claim for integral curve coefficients.

Let S_T contain the places of T above the old primes and ℓ. At level N add at most $|\Delta|q$ more places. Since T contains μ_ℓ, Kummer theory bounds the dimension of the continuous homomorphisms of the kernel of $\Gamma_N \to \Delta$ to $\mathbf{F}_\ell$ by

$$r_1(T) + r_2(T) + |S_T| + |\Delta|q + \dim_{\mathbf{F}_\ell} \mathrm{Cl}(T)[\ell].$$

Indeed the Kummer classes have valuations divisible by ℓ outside this set. The quotient by the classes of units with these allowed denominators injects in the ℓ-torsion of the corresponding class group, by taking the ℓth root of the ideal away from the set. The ℓ-rank of that class group is at most that of $\mathrm{Cl}(T)$, since it is a quotient. Dirichlet's unit theorem, including the roots of unity modulo ℓth powers, bounds the remaining dimension by the first four terms. Any extra conditions at old inertia only decrease the dimension. In a pro-ℓ group, lifts of a basis of its quotient by powers and commutators generate topologically. Indeed failure to generate is detected in a finite ℓ-group, where every maximal subgroup has index ℓ. The displayed bound and lifts of generators of Δ give a uniform number of generators for Γ_N.

At a new place inertia has trivial image in Δ. Its pro-ℓ image is thus a quotient of tame $\mathbf{Z}_\ell(1)$, because the residue characteristic differs from ℓ. Imposing triviality on one generator there kills the whole inertia. Imposing this at all new places before taking the relative quotient gives exactly the original ramification problem. Conversely every quotient of that original group which is pro-ℓ over Δ is already a quotient of the auxiliary group. The two quotient operations therefore give the claimed kernel and surjectivity. □

Theorem 41.30 (The kernel in a limit with marked inertia)**.** *One can choose the auxiliary sets and an increasing sequence of levels tending to infinity so that they have a limiting profinite group Γ_∞, a surjection $h : \Gamma_\infty \to \Gamma_0$, and elements ϕ_j, τ_j for $1 \le j \le q$. The group is finitely generated and pro-ℓ over Δ. The fixed local absolute groups map to it compatibly over Γ_0, as*

does the chosen complex conjugation. For the elements $s_j = n'_j$ used in Theorem 37.17, viewed in Γ_0, one has

$$h(\phi_j) = s_j, \qquad h(\tau_j) = 1, \qquad \phi_j\tau_j = \tau_j\phi_j, \qquad \ker h = \overline{\langle\langle \tau_1, \dots, \tau_q \rangle\rangle}.$$

Here the normal closure is taken in the full group over F. The last equality identifies the entire unramified quotient of the limit.

Proof. A finitely generated profinite group has only finitely many open subgroups of each index, by its actions on finite sets. Consequently the intersections of its open normal subgroups of index at most m are a descending cofinal sequence. In the Chebotarev construction at level N also prescribe the images of the s_j in the Nth such quotient of Γ_0. These prescriptions use the very same elements of the cyclotomic kernel as the test on the curve. Thus they are compatible with the required congruence $q_{N,j} \equiv 1 \bmod \ell^N$ and with all cocycle and discriminant tests there. Chebotarev applied to the combined finite quotient over L supplies the primes, as in Theorem 37.17. The base images of the chosen Frobenius elements now tend to s_j.

We construct one presentation mapping to every level. Choose d fixed generators of Γ_0 and lifts of them in each Γ_N. The closed subgroup generated by the lifts maps onto Γ_0 by compactness. By the lemma take a tuple of at most e generators for each Γ_N, for one e. Multiply each member of that tuple by an element of the subgroup of lifts to bring its image in Γ_0 to one. The corrected tuple together with the lifts still generates. Let Φ be the free profinite group on $d + e + q$ generators with one additional generator c subject to $c^2 = 1$. Map the first $d + e$ generators as above, the next q to the designated inertia generators and c to the fixed complex conjugation. The maps $\pi_N : \Phi \to \Gamma_N$ are onto and their composites to Γ_0 are one fixed map π. Write $K_N = \ker \pi_N$, $K = \ker \pi$, and $t_j \in \Phi$ for the designated inertia generators. For every N the preceding lemma gives

$$K = \overline{\langle\langle K_N, t_1, \dots, t_q \rangle\rangle}.$$

Use the cofinal characteristic open subgroups V_m of Φ described above. In Φ/V_m there are finitely many possibilities for $K_N V_m/V_m$. Once this subgroup is fixed, the images of the q Frobenius elements in the resulting quotient also take finitely many values. The same is true of the maps from each fixed local absolute group at ℓ and the old places to that finite quotient. A finite extension of a p-adic field has only finitely many extensions of any bounded degree up to isomorphism, by the local finiteness theorem (Krasner's lemma applied to Eisenstein polynomials). Thus there are finitely many such continuous homomorphisms, as each factors through one of finitely many finite Galois groups. Fix compatible embeddings of the local closures into the global one at the old places throughout. By successively passing to infinite subsequences and then choosing a diagonal subsequence, all the indicated data are eventually constant for each m.

The stabilized kernel images are compatible and surject onto one another. Their inverse limit is a closed normal subgroup $K_\infty \subset \Phi$ with exactly those images. Put $\Gamma_\infty = \Phi/K_\infty$. Since all $K_N \subset K$, one has $K_\infty \subset K$. Each finite congruence quotient of the limit is obtained from a sufficiently late Γ_N. Taking the quotients also to remember the fixed map to Δ shows that its kernel in the limit is pro-ℓ. The stabilized markings give ϕ_j, τ_j and continuous maps on the fixed local groups by inverse limits. Their composites to Γ_0 are the original markings, with the asserted Frobenius limits. In particular τ_j is the image of t_j.

At finite level the tame relation for geometric Frobenius is $\phi_{N,j}^{-1}\tau_{N,j}\phi_{N,j} = \tau_{N,j}^{q_{N,j}}$. In each fixed finite quotient the orders of these inertia images are bounded powers of ℓ, so the right side equals $\tau_{N,j}$ for all large N. This proves commutation in the limit. Finally reduce the

displayed identity for K modulo V_m. The image of K_N eventually equals that of K_∞, and the t_j are fixed. Thus K and the closed normal subgroup generated by $K_\infty, t_1, \dots, t_q$ agree in every Φ/V_m. Equality of the closed subgroups proves the last assertion. The added finite Chebotarev conditions and subsequences can be chosen together for any finite list of pairs or curves already under consideration. □

Proposition 41.31 (The unramified specialization of the limiting determinant)**.** *Let R_∞^{ord} denote the universal ordinary polarized determinant ring on the CM subgroup of Γ_∞, with its fixed local maps. Write t_∞ for its trace. Take the closed ideal generated by*

$$t_\infty(gu) - t_\infty(g), \qquad g \in \Gamma_\infty^0, \quad u \in \{\tau_j, c\tau_j c^{-1} : 1 \le j \le q\}.$$

The quotient by this ideal is canonically the original ordinary determinant ring R_0, including its ordered local characters.

Proof. The limiting group and its open subgroups are topologically finitely generated, by the finite-index generator theorem. They satisfy the finiteness condition for profinite determinants. Together with the usual local character rings and the closed ordinary, tame and polarization relations this gives the displayed universal ring, as in Section 37.8. Alternatively the following argument identifies the representing functors on Artinian coefficient rings directly. [2]

A trace satisfying the indicated identities also satisfies them for the inverses, by substitution for g. Cyclic invariance of the trace gives the same identities for conjugates by the CM subgroup. Iteration gives them for products, and continuity gives them on the closed normal subgroup they generate there. Since the CM subgroup has index two with representative c, the theorem identifies this subgroup with the entire kernel of h. The trace therefore factors uniquely through the original CM group. Newton's identities give the same factorization for the determinant, tested on generic finite sums in its group algebra. The ordinary local characters are still maps on the fixed local absolute groups. The ordinary word and polarization identities pass through the surjection, and the maps at the old primes agree by construction. Conversely the pullback of every pair on Γ_0 obeys all these identities. These operations are inverse and commute with coefficient homomorphisms. This proves the ring assertion. □

Theorem 41.32 (Ordinary cocycles on the limiting group)**.** *On any selected curve let $A = k'[[z]]$, let W, E, M, U be the paired, diagonal, off-diagonal and extension modules of Sections 37.5 and 37.6, and write b for the rank of the ordinary off-diagonal cohomology over the base. All coefficients on Γ_∞ are pulled back by h. Use the marked ordinary local conditions at the old places and the coefficient prime. Then the following hold.*

1. *The A-rank of the integral polarized ordinary cocycles with values in W, modulo polarized coboundaries, is at most $3([F:\mathbb{Q}] + q)$.*

2. *For $0 \le i \le b$, $1 \le j \le b$, let y_{ij} be any ordinary polarized E-valued cocycles on the limit. With the exponent C of Theorem 37.25 there are $c_0, \dots, c_b \in A$, at least one a unit, and one ordinary polarized U-valued cocycle Y on the limit satisfying*

$$(\pi Y)_j = z^C \sum_{i=0}^{b} c_i y_{ij} \qquad (1 \le j \le b).$$

3. *Pullback identifies the integral polarized ordinary M-valued cocycles and their coboundaries with those on the original group. In particular the chosen basis $\nu_1, \dots, \nu_b$ on the base still spans cohomology after inverting z.*

Proof. Fix m and use the finite modules over $A/(z^m)$. Pull cocycles back to the fixed presentation Φ, or to its index-two CM subgroup, which also has a finite generating set. Only finitely many cocycles exist on this presentation with each prescribed finite action. Indeed adjoining a cocycle to the action gives a homomorphism to a finite semidirect product, determined by the values on the generators. All the candidate homomorphisms can be tested on one finite quotient of Φ, taking the core of their kernels when starting on the CM subgroup. Descent of a candidate through K_N or K_∞ is detected by the image of this kernel in the same finite quotient. The local maps to that quotient stabilize as well. Projection to upper triangular entries up to local coboundaries is a finite test, including the possible local witnesses, as is the symmetry under c. Old inertia conditions are tested by the local maps restricted to the fixed inertia groups. It follows that at this precision the polarized ordinary cocycles for the limit and for every sufficiently late level give identical subsets on the presentation. The same holds for their coboundaries and the coefficient projection π between the modules. The finite quotient argument can be applied to all the modules appearing here at once.

Put $D = 3([F : \mathbb{Q}]+q)$. Corollary 37.21 now bounds the length of the limiting finite-precision polarized ordinary cohomology by $mD + C_1$. Suppose e integral classes are independent over the fraction field. Evaluate their representatives on a fixed finite generating tuple of the CM presentation, in a finite free A-module A^t. Let B' be the image of the matrix taking polarized witnesses to coboundaries on that tuple. The allowable witnesses here are cut out by the involution at c and form a direct summand since 2 is invertible. Their reductions give all finite-precision coboundaries. Modulo the saturation of B' the e vectors are linearly independent over the fraction field, by the cocycle identity and the fact that the tuple generates. The quotient by that saturation is free. Smith normal form shows that the reductions of the e vectors generate a module there of length at least $em - C_2$ for a constant independent of m. Dividing just by actual coboundaries can only increase that length. Comparison as m tends to infinity proves $e \le D$.

For the second assertion reduce the given cocycles modulo z^m. The finite test just proved and Theorem 37.25 provide the coefficients and the desired lift at every m, over the limiting group. Keep all possible solutions at each precision, including local witnesses. They form finite nonempty sets. Reduction maps send a solution at higher precision to a solution at lower precision, preserving primitivity of the coefficient vector. A compatible sequence exists by compactness of a product of finite sets. Indeed every finite set of compatibility conditions is witnessed by reduction of a solution at its largest precision. The sequence defines the integral c_i, the continuous cocycle Y, and its integral local witnesses. Its relation holds modulo every power of z, hence in A.

Finally a cocycle with coefficients in M is a continuous additive homomorphism on $\ker h$, since the coefficient action is trivial there. Commutation of τ_j with ϕ_j gives $(\operatorname{Ad} r(s_j)-1)y(\tau_j) = 0$. Each s_j has three distinct eigenvalues over a splitting field of the curve fraction field. Thus the spectra in different constituent blocks are disjoint. There is no nonzero intertwiner between two different blocks, so the operator in this equation has zero kernel on M over the fraction field. The module M is torsion-free, and $y(\tau_j) = 0$ integrally. Polarization gives the same at $c\tau_j c^{-1}$. The cocycle identity gives the corresponding zero at all CM conjugates, and continuity gives zero on the full kernel by Theorem 41.30. Hence y descends uniquely to the original group. Local conditions and coboundaries agree on both sides by the fixed local markings. If there is a single constituent, the off-diagonal module is zero and the assertion holds as well. □

The exponent and the rank bounds in this theorem are measured along a curve in characteristic ℓ. To obtain the mixed-characteristic length of Section 6.6 one still has to identify

which components of the original ordinary determinant ring occur in the support of the patched module and specialize at the prescribed weights.

42 Hecke support and ordinary extension classes

42.1 A patched module at the discriminant of the cohomology curves

The ranks of the integral ordinary modules at different auxiliary levels need not have a common bound in the construction so far. We use the maps in Theorem 41.28 to obtain a finite module after inverting the element which is nonzero at the chosen curve. We keep the same prime sets, weight algebra and residual ordering in this passage.

Let $\Phi, \Gamma_\infty, \Gamma_0, h$ be as in Theorem 41.30. Enlarge the residue field first if necessary to contain all residual roots. There is a complete Noetherian local O-algebra R for the following data: a degree-three continuous determinant on Φ^0 with the given reduction, the three local characters on each of the fixed ordinary local groups lifting the specified residual characters and finite diamonds, and variables $U_{v,i}-b_{v,i}, Y_{v,i}$ in the maximal ideal. Here v in a subscript now denotes a position in the ordered list of q auxiliary places. The units $b_{v,i}$ lift the designated residual roots of s_v. The residual roots at the finite levels agree with these after omitting finitely many levels. To construct R, take the determinant ring of the finitely generated profinite group Φ^0, the completed group rings for the local characters, and their completed tensor product over O, followed by the indicated formal variables. The local unit groups have finite ℓ-rank. These rings are quotients of power series rings with finitely many variables by the finiteness condition for continuous determinants and local reciprocity. Thus the same holds for their completed tensor product. The ring admits the weight map $\Lambda \to R$ by its ordered inertial characters. [2]

Let $\mathfrak{n}, \mathfrak{m}$ be the maximal ideals of R, Λ, and put $\mathfrak{a} = (Y_{v,i}) \subset R$. The maps from Φ to the finite-level groups give actions of R on the residual summands $M_N = M_{1,N}^{\mathrm{ord}}$ of Proposition 41.27. On these modules $1+Y_{v,i}$ acts by its diamond and $U_{v,i}$ by its ordered Frobenius operator. Define in R

$$f = \prod_{v} \left(\prod_{i \neq j} (U_{v,i} - U_{v,j}) \right)^2 .$$

The finite-level composite $f_N = \prod_v f_v$ of Theorem 41.28, written in the same variables, satisfies $f_N - f \in \ell^N R$. Indeed $q_{N,v} \equiv 1 \bmod \ell^N$ in every factor.

Lemma 42.1 (The fixed comparison fiber)**.** *Let P be the selected local factor of the ordered splitting algebra over R_0 for the characteristic polynomials at s_v, and put $X = P \otimes_{R_0} M_K$. Give X the R-action from $\Phi \to \Gamma_0$, these ordered roots and $Y = 0$. It is finite over Λ. For fixed j and all sufficiently large N, the corresponding ordered hyperspecial module X_N at the finite level is canonically X modulo $\mathfrak{m}^j$ as an R-module. The two comparison maps between this module and $M_N/(\mathfrak{a}, \mathfrak{m}^j)M_N$ have composites f. There is an integer d, independent of N, such that $\mathfrak{n}^d X_N \subset \mathfrak{m} X_N$.*

Proof. Omitting finitely many good places leaves the same good Hecke algebra on the base module. Indeed the remaining good Frobenius conjugacy classes are dense, so the continuous trace and its characteristic coefficients recover the omitted spherical polynomials. The same residual factor of this algebra and the same M_K thus serve all N. On M_K the polynomials at the finite Frobenius elements tend to the polynomials at s_v by the convergence in the fixed base group. Their coefficients therefore agree as matrices modulo $\mathfrak{m}^j$ for all large N. The presentations adjoining successive roots of monic cubics and quadratics now agree modulo that power. The isomorphism keeps the ordered root variables and commutes with residual factor projection by uniqueness of idempotent lifting. All traces of elements of Φ^0 agree on

the two base modules since the maps to Γ_0 are one fixed map. The local characters likewise agree. These facts prove the asserted R-linearity, including on power series by continuity. The resulting identifications are compatible with reduction in j. The finite-level composites reduce to f for $N \geq j$.

The modules X_N, X are summands of at most 6^q copies of M_K as weight modules. Their reductions modulo $\mathfrak{m}$ therefore have dimension at most $d = 6^q \operatorname{rank}_\Lambda M_K$ over the residue field. On the selected factor, R acts locally, with maximal ideal acting nilpotently on this finite-dimensional space. For example the action factors through the local factor of the finite ordered-root algebra over the base Hecke algebra. Each nonzero step in its maximal ideal filtration has positive codimension by Nakayama's lemma. This proves the uniform bound with the specified d. □

Proposition 42.2 (The successive diamond quotients)**.** *Choose a nonprincipal ultrafilter on the subsequence of Theorem 41.30, and put, for $k \geq 1$,*

$$V_{k,j} = \prod_{\mathcal{U}} M_N/(\mathfrak{a}^k M_N + \mathfrak{m}^j M_N), \qquad V_k = \varprojlim_j V_{k,j}.$$

These are R-modules. One has $V_1[1/f] \simeq X[1/f]$. If h_i is the number of degree-i monomials in the $3q$ diamond variables, there are exact sequences

$$0 \longrightarrow V_1^{h_i} \longrightarrow V_{i+1} \longrightarrow V_i \longrightarrow 0 \qquad (i \geq 1).$$

The first map multiplies lifts by their indicated monomials. In addition, if j, k are fixed, then for all sufficiently large N

$$f^k \mathfrak{n}^{djk} M_N \;\subset\; \mathfrak{m}^j M_N + \mathfrak{a}^k M_N.$$

Proof. An ultraproduct identifies two sequences when they agree on a set in the ultrafilter. It preserves exact sequences of modules by choosing representatives and lifting at each index where a preimage exists. A constant finite module has ultraproduct equal to itself, since exactly one part of every finite partition belongs to the ultrafilter. At $k = 1$ the lemma gives maps in both directions between $X/\mathfrak{m}^j X$ and $V_{1,j}$. The maps are the ultraproducts of the actual level maps, using the fixed finite presentation of the ordered roots modulo $\mathfrak{m}^j$. They commute with reduction, since for any two precisions the finite-level identifications and maps commute on a cofinite set of indices. Taking inverse limits gives the two maps with composites f. Here $X = \varprojlim_j X/\mathfrak{m}^j X$ by finiteness over the complete weight algebra. Inverting f proves the first assertion.

For fixed k, j, and all large N, the relations $(1 + Y_{v,i})^{\ell^{a_{N,v}}} - 1$ in the diamond ring belong to $(Y)^k + \mathfrak{m}^j$. Indeed the coefficient of Y^t for $1 \leq t < k$ is divisible by $\ell^{a_{N,v} - v_\ell(t)}$, by multiplying the binomial coefficient by t, and $a_{N,v} \geq N$. Hence $M_N/(\mathfrak{a}^k, \mathfrak{m}^j)M_N$ is free over $(\Lambda/\mathfrak{m}^j)[Y]/(Y)^k$ for all such N, by Theorem 41.28. The top degree-i part in the truncation at $i+1$ is canonically a copy of the degree-zero fiber for each monomial of degree i. This is checked on a free basis, and the map itself just multiplies a lift of a degree-zero element by that monomial. Taking ultraproducts proves the indicated exact sequence at precision j. Its left-hand modules map onto each other as j decreases, by the corresponding surjections at finite levels. Inverse limit therefore preserves the surjection on the right. Explicitly, lift successively in j and correct each mismatch in the left-hand module. This gives the exact sequence for V_i. The same proof at any two truncations identifies every intermediate graded term by multiplication of the corresponding monomials.

Finally $\mathfrak{n}^{dj} X_N \subset \mathfrak{m}^j X_N$ by iterating the lemma. Apply both comparison maps on $M_N/\mathfrak{a}M_N$, on which their composite is f_N. It follows that $f\mathfrak{n}^{dj} M_N \subset (\mathfrak{m}^j + \mathfrak{a})M_N$ for $N \geq j$. Iterate this

inclusion k times and use $(\mathfrak{m}^j+\mathfrak{a})^k\subset\mathfrak{m}^j+\mathfrak{a}^k$. This proves the last estimate. Its power of f is independent of the weight precision j. □

Theorem 42.3 (A finite patched module and its fiber)**.** *Write $\widehat{R}$ for the $\mathfrak{a}$-adic completion of $R[1/f]$. The module*

$$\mathcal{P}=\varprojlim_{k} V_k[1/f]$$

is finite over $\widehat{R}$ and has

$$\mathcal{P}/\mathfrak{a}^k\mathcal{P}\simeq V_k[1/f],\qquad \operatorname{gr}_{\mathfrak{a}}\mathcal{P}\simeq (X[1/f])[\overline{Y}_{v,i}].$$

In particular its fiber at zero diamonds is $X[1/f]$. The diamond variables are a regular sequence on $\mathcal{P}$ when this fiber is nonzero. For each of the cohomology curves of Theorem 37.13, the fiber is nonzero at the corresponding ordered generic point.

Proof. Each localized transition map in the proposition is onto. The actions on these quotients, killed by $\mathfrak{a}^k$, give the action of the completed ring. The fiber $X[1/f]$ is finite over $R[1/f]$, since X is finite over Λ and the weights act through R. Lift finitely many of its generators compatibly up the surjections. By the proposition, all subsequent graded layers are generated by the corresponding monomials times these same lifts. Every element of $\mathcal{P}$ can now be written as a linear combination of them with coefficients in $\widehat{R}$. Indeed at each stage correct the error by coefficients homogeneous in the diamond variables of the current degree, with coefficients in $R[1/f]$. The resulting series converge in the completion. Starting at degree k for an element with zero image in $V_k[1/f]$ gives coefficients in $\mathfrak{a}^k\widehat{R}$. This proves finiteness and identifies the kernel at every stage. The canonical monomial identifications then give the asserted graded module. The notation on the right means a polynomial module on $X[1/f]$; multiplication in the graded module is the indicated multiplication by monomials.

In this graded module each variable is injective after quotient by the preceding ones. This yields regularity on the complete module as follows. A nonzero element has a first nonzero graded term. Multiplication by the first variable cannot annihilate that term. More precisely the order of its product increases by exactly one. It follows that the graded module after quotient by that variable is the quotient of the graded module by its initial variable: an element of the principal image in degree at least k has a preimage in degree at least $k-1$. The quotient is again complete and separated, as it is finite over the Noetherian complete ring $\widehat{R}$ with its defining ideal. Induction proves the assertion for all variables.

For nonvanishing return to the dense cuspidal base changes of Lemma 37.30. At the old split places they have the finite inertia of $\alpha\oplus 1\oplus\alpha^{-1}$ and zero monodromy. They therefore contain the types of Lemma 41.21, by the local parameters of these cuspidal base changes and the compact-type criterion there. At the other tame places they admit spherical vectors by the same base change and descent. Thus the full ordinary module M_K used here contains these systems. Its annihilator in R_0 vanishes at their evaluations and therefore on the domain of their family by density. It lies in the prime of every indicated curve by the specialization in Theorem 37.32. The finite module consequently has nonzero fiber at that prime by Nakayama. Extending its residue field to split the cubics and taking an order of the roots compatible with the prescribed residues gives the nonzero ordered fiber. Such orderings exist by integral reduction of the roots over the curve DVR. Each cubic has distinct roots generically by Theorem 37.17. Hence f is nonzero there. Localizing by it preserves the fiber. □

Theorem 42.4 (Galois relations on the patched module)**.** *On $\mathcal{P}$ the universal trace on Φ^0 descends to Γ_∞^0 and satisfies the polarized ordinary relations with its limiting local maps and*

the old tame conditions. For $g \in \Gamma_\infty^0$ it satisfies, at each marked pair ϕ_v, τ_v,

$$t\left(g \prod_i (\phi_v - U_{v,i})\right) = 0,$$

$$t\left(g \prod_{j \neq i} (\phi_v - U_{v,j})(\tau_v - 1 - Y_{v,i})\right) = 0.$$

These are identities of scalar endomorphisms on the finite $\widehat{R}$-module. The determinant and local characters consequently give a pair over the quotient of $\widehat{R}$ by its annihilator on $\mathcal{P}$, with the determinant identities understood coefficientwise.

Proof. We first give a uniform way to pass an approximate relation to this module. Suppose $u \in R$ and there is $e = 0$ or 1 such that for every t, at all sufficiently large levels there is $u_N \in R$ annihilating M_N with $u_N \equiv f^e u \bmod \mathfrak{n}^t$. Fix k, j and take $t = djk$. The last inclusion in Proposition 42.2 then shows that $f^{k+e}u$ kills $V_{k,j}$. The exponent does not vary with j, so the same element kills V_k. After inverting f, u kills $V_k[1/f]$ for every k and therefore kills $\mathcal{P}$.

At a prescribed $\mathfrak{n}^t$-precision the universal trace on the presentation factors through a finite quotient of Φ^0. This follows from continuity, compactness and finiteness of $R/\mathfrak{n}^t$. Take its normal core in Φ and also remember the sign. If $a \in K_\infty$, choose $a_N \in K_N$ with the same image in this finite quotient, using Theorem 41.30. Then $t(ga) - t(g)$ is congruent to $t(ga_N) - t(g)$, which acts as zero on M_N. The preceding argument descends the trace. Polarization is tested in the same way or directly, using the common conjugation on Φ and the fixed cyclotomic character from Γ_0. For an ordered local word lift its three group elements to Φ^0 using their specified images in the limit. At every sufficiently large level the local maps have the same images in the fixed finite quotient. There are thus lifts of the actual finite-level local elements congruent to the three chosen lifts there. The local character values in the coefficients are those of the same elements of the fixed local absolute groups. Expanding the word and taking the trace after any global element makes its congruence with the corresponding finite-level trace explicit. That trace annihilates M_N by Theorem 41.23. For an old inertia relation use the same argument with a single local element in the specified kernel. This proves the ordinary and old tame assertions.

For the two displayed words lift their marked Frobenius and inertia to Φ^0. The finite-level markings can again be lifted to agree modulo the chosen finite quotient. The first trace then agrees to this precision with an annihilator by Theorem 41.26. For the second trace, multiply its finite-level version by the total f_N. This is an annihilator by the same theorem, and $f_N \equiv f \bmod \mathfrak{n}^t$ when $N \geq t$. The approximation argument with $e = 1$ now proves the second identity. Conjugating the local words by the polarization gives the conjugate conditions as well. Finally the determinant coefficients are recovered by Newton's identities on generic finite group algebra sums, so trace descent gives descent of all the determinant identities. This argument asserts algebraic identities on the localized, diamond-adically completed ring. The original continuous identities at the finite coefficient levels were used before localization. □

The module constructed here has a nonzero fiber at each tested curve, with all the added diamond directions regular. Determining its support along every component passing through that fiber is a further issue. The finite-level and limiting identities above do not yet yield the mixed-characteristic length, or the comparison with the normalized theta periods, required in Section 6.6.

42.2 Full Hecke support at an absolutely irreducible curve

Take a curve from Theorem 37.13 whose generic CM representation remains absolutely irreducible on the auxiliary field $L = L_a$. Put $F = F_a^+$ and $d = [F : \mathbb{Q}]$, and let q be the number of new places in Theorem 37.17. Let $\mathfrak{p}$ be the prime it defines in the ordinary polarized determinant ring R_0 over this field. Our aim is to determine the ordinary Hecke algebra near $\mathfrak{p}$. Set $D = 3(d + q)$.

Take a finite extension of the curve fraction field which contains the ordered roots at the s_j and over which the generic representation is realized with a stable integral lattice. Its valuation ring is a complete equal-characteristic DVR $A = k'[[z]]$, where k' is finite. Use the embedding of this coefficient field in A. Extend the coefficient DVR O unramified to O' with residue field k', enlarging k' first if necessary. At every level adjoin one formal parameter s and extend coefficients from O to O'. At the curve send s to z. Write $K = k'((z))$. The parameter supplies a specified copy of this residue field when taking tangents in residue characteristic ℓ.

Here and below this extension of the patching construction means the following. At the base take $B = (R_0 \otimes_O O')[[s]]$. Use $\Lambda' = (\Lambda \otimes_O O')[[s]]$ as weight algebra. Replace the modules and finite-level rings by the same coefficient extension and formal power series extension. They have the same finite freeness and comparison identities by base change. On the Galois groups the residual data are unchanged after extension of constants. On curve cohomology the fraction-field rank estimate of Theorem 41.32 is unchanged under the finite extension of $k'((z))$ just used. Indeed a continuous cocycle with values in a finite extension splits in any field basis into continuous cocycles in the original field. The same applies to coboundaries and to the linear equations for local witnesses. Their values have bounded denominators on the compact groups, and multiplying by a power of a uniformizer makes them integral. Thus extension of scalars commutes with the cohomology relevant to that rank estimate.

Let P be the ordered splitting factor over B, and let $\mathfrak{x}$ be its prime at the extended curve, with an ordering compatible with the residual one. Repeat Section 42.1 using Λ'. The finite truncations of Λ' are still finite, the modules remain free over the indicated diamond group rings, and the convergence of the base Frobenius coefficients continues to hold at each finite weight precision. These are the inputs to Propositions 42.2 and Theorem 42.3; their proofs therefore give the module $\mathcal{P}$ with augmented base fiber $(P \otimes_{R_0} M_K)[1/f]$. The presentation ring there is now augmented by s; denote its localized diamond-adic completion by $\widehat{R}'$. Let T be its quotient by the trace descent, polarization, old inertia, ordered local character-word, and two auxiliary trace relations in Theorem 42.4. All are imposed with arbitrary global multipliers as there. Use the full coefficient identities for polarization. The module $\mathcal{P}$ is a finite T-module.

There is a surjection $T \to P[1/f]$ which sends all diamond variables $Y_{v,i}$ to zero. Indeed the original ordinary determinant pulled back to the presentation, its local characters and the ordered universal roots give this map. At the base, the image of τ_v is one and the first Frobenius trace relation is Cayley–Hamilton. Thus the relations defining T hold on the entire ordered base. The universal traces and local character values generate R_0 topologically, since all group elements have lifts to the presentation. This is also immediate from the quotient construction of the ordinary determinant ring. The root coordinates generate the ordered factor over the augmented base. Before localization the image of the complete presentation ring is closed by compactness, hence contains all these generated rings. Inverting f and setting $Y = 0$ during completion preserves surjectivity. Let $\mathfrak{P} \subset T$ be the preimage of $\mathfrak{x}$ in this map, using the same symbol for its extension after inverting f. Its residue field is K. All evaluations before localization lie in A, and the image contains $k'[[z]]$ through $O'[[s]]$.

Lemma 42.5 (Depth and relative tangent directions at the curve)**.** *The nonzero finite module $\mathcal{P}_{\mathfrak{P}}$ has depth at least $D + 1$ over $T_{\mathfrak{P}}$. Let e be the embedding dimension of this local ring, let*

ϖ be a uniformizer of O', and let $\mathfrak{p}_T$ be its maximal ideal. The K-space of lifts of its residue map to $K[\epsilon]/(\epsilon^2)$ fixing $O'[[s]]$ in the given residue map has dimension

$$\dim_K \mathfrak{p}_T/(\mathfrak{p}_T^2 + \varpi T_{\mathfrak{P}}) \ \geq e - 1.$$

On the original presentation ring, the first-order parts of all these lifts are continuous on group traces and have uniformly bounded denominators in A.

Proof. The $3q$ diamond variables already form a regular sequence on the patched module by Theorem 42.3. Its fiber localized at $\mathfrak{P}$ is nonzero by that same theorem and the choice of $\mathfrak{x}$. Write the remaining $3d$ principal weight variables as w_i, and let $\widetilde{a}_i(s) \in O'[[s]]$ lift their power series values in A, with zero constant term modulo ϖ. Translation by these series in the weight algebra shows that $\varpi, w_i - \widetilde{a}_i(s)$ form a regular sequence of length $3d+1$ there. The ordered base module is a finite free Λ'-module by the monic splitting presentation and its residual direct summand, as in Lemma 42.1. The same sequence is regular on it. Inverting f and localizing the base module at $\mathfrak{P}$ preserve injectivity at each step. The final quotient is nonzero by the nonzero fiber over K. Together the two sequences give the stated depth.

Modulo ϖ, the map from $k'[[s]]$ into the local ring is injective, since evaluation sends s to z. Every nonzero element of this subring is inverted at the curve prime. Thus K is a coefficient field of $T_{\mathfrak{P}}/\varpi$ splitting its residue map. Over this field lifts to the dual numbers are exactly the linear functionals on the displayed cotangent space, by taking linear terms. Killing one element can decrease the embedding dimension by at most one.

For the continuity assertion present the ring before the two operations of localization and diamond completion as a quotient of $O'[[s, x_1, \dots, x_n]]$. Its evaluation sends x_i to series in zA, which have lifts $b_i(s) \in O'[[s]]$ in the maximal ideal. Any given dual-number lift has values $x_i \mapsto b_i(z) + \epsilon c_i$, with $c_i \in K$. In this expression $b_i(z)$ denotes the reduction evaluated in A. Translation by $b_i(s)$ is an automorphism of the ambient complete local power series ring. Modulo the square of the ideal generated by the translated variables, the value of every series is consequently its constant term plus its linear terms in those variables. This proves the first-order evaluation formula even for an algebraic map from the localization. Its denominators on the entire original ring are bounded by those of the finitely many c_i. On the n'th power of the original maximal ideal, the linear terms have valuation at least $n' - 1 + \min(0, \min_i v_z(c_i))$. This proves continuity. The diamond variables map into ϵK, so such lifts on the completion are determined modulo the square of the diamond ideal. Completion induces an isomorphism on that quotient of the localized ring. Thus no further choice of a lift is introduced by completion. □

Proposition 42.6 (Tangents detected by ordinary adjoint cohomology)**.** *The relative tangent space of the preceding lemma injects linearly into the polarized ordinary cohomology of Γ_∞ with coefficients in $\mathrm{End}_K(\rho)$. Here ρ is the pullback of the generic representation on the curve. Consequently this tangent space has dimension at most D.*

Proof. A lift gives a degree-three determinant over the dual numbers. Its reduction is the determinant of the split absolutely irreducible ρ over K. Chenevier's Theorem 2.22(i) over a Henselian local ring, as used in Lemma 37.35, supplies a matrix representation lifting ρ, unique up to conjugation by a matrix reducing to one. The trace descent relations ensure that it factors through Γ_∞^0. More explicitly, lift nine group elements whose matrices span the full matrix algebra over K. Their trace Gram matrix is invertible also over the dual numbers. Hence traces against these nine detect zero matrices, including the difference from the identity at any element of the descent kernel. The coordinates of an arbitrary group matrix

in this basis are given by the inverse Gram matrix applied to nine translated traces. The lemma makes their first-order coefficients continuous and uniformly bounded in denominator. A fixed integral lattice for ρ then gives the same properties for the cocycle $z(g)$ defined by $\rho_\epsilon(g) = (1 + \epsilon z(g))\rho(g)$. Strict conjugation changes z by a coboundary. The trace formulas apply over the curve field since 3! is a unit here. [2]

Polarization of the determinant makes the class invariant under conjugation by c followed by negative adjunction for the polarized base representation, with its fixed multiplier. Indeed the conjugate-dual lift has the same determinant and differs by strict conjugation after identifying the base representations by their polarization. Averaging a representative with its image under this involution, using $2 \in K^\times$, gives a polarized cocycle. Polarized representatives differing by a coboundary differ by one with a polarized witness by the same averaging.

At each chosen ordinary place take an inertia element γ whose three curve characters have distinct values. The ordinary word for three copies of γ has zero trace against every group element, so is the zero matrix by the Gram argument. Its three lifted roots are unit-separated in $K[\epsilon]$. Their polynomial projectors give three free eigenlines reducing to the original ones. In this basis put any local g in position k of the word, leaving the other two letters equal to γ. On a row of index at least k the factors to the left have invertible product, and on a column of index at most k the factors on the right have invertible product. It follows that the corresponding entries of $\rho_\epsilon(g) - \psi_k(g)\mathrm{id}$ vanish. Varying k proves the ordinary flag with its specified diagonal. After choosing a lifting basis this gives the ordinary cocycle condition up to a local coboundary. The old tame kernel is killed by the same trace pairing. Changing the global representative by averaging preserves the local conditions on its class. There are finitely many local witnesses, so after multiplication by a power of z they and the cocycle all take values in the integral lattices. With one constituent the coefficient W of Theorem 41.32 is the full adjoint endomorphism space. Part 1 of that theorem therefore bounds the dimension of the target by D.

It remains to see that zero class forces the original tangent to vanish. After conjugation the lift of ρ is then constant, so its determinant is constant. On the ordinary flag the eigenvalues of γ are uniquely determined lifts of simple roots of its characteristic polynomial. Each local character is recovered by taking the trace of the corresponding eigenprojector times the group matrix, so these too are constant. At every new place the Frobenius has distinct eigenvalues. The first auxiliary trace word, by the Gram pairing, gives $\prod_i(\rho(\phi_v) - U_{v,i}) = 0$. Evaluating on each eigenline shows that its matching root coordinate is constant, since each other factor is invertible. The second word now gives $\prod_{j\neq i}(\rho(\phi_v) - U_{v,j})(\rho(\tau_v) - 1 - Y_{v,i}) = 0$. The curve representation comes from the unramified base, so $\rho(\tau_v) = 1$. Evaluation on the ith eigenline gives $Y_{v,i} = 0$. All generating data of the presentation ring and the parameter s are thus fixed. Equality on its topological generators gives equality on the ring by the continuity in the lemma, then on the localization and on the first-order image of the completion. This proves injectivity. Adding first-order matrix variations adds their cocycles, and changing basis contributes a coboundary. Thus the map to cohomology is linear. Equivalently the construction commutes with linear maps from a square-zero coefficient module, including addition on two copies of K. □

Theorem 42.7 (All components through an absolutely irreducible curve)**.** *At the prime* $\mathfrak{p}$ *of every selected curve with absolutely irreducible generic representation, the localized ordinary module* $(M_K)_{\mathfrak{p}}$ *is finite free of positive rank over* $(R_0)_{\mathfrak{p}}$*. In particular the natural surjection onto the localized good Hecke algebra is an isomorphism*

$$(R_0)_{\mathfrak{p}} \;\simeq\; (B_K)_{\bar{\mathfrak{p}}},$$

where $\bar{\mathfrak{p}}$ is the corresponding Hecke prime. Every prime of R_0 contained in $\mathfrak{p}$ belongs to the support of this ordinary module.

Proof. The two preceding results bound the embedding dimension of $T_{\mathfrak{P}}$ above by $D+1$ and the depth of its nonzero finite module below by $D+1$. For a Noetherian local ring the depth of such a module is at most the dimension of the ring, which is at most its embedding dimension. Hence both dimensions equal $D+1$ and $T_{\mathfrak{P}}$ is regular local. Every finite module over a regular local ring has finite projective dimension. The Auslander–Buchsbaum formula, using the depth equality, gives projective dimension zero for $\mathcal{P}_{\mathfrak{P}}$. It is therefore free of positive rank. The Stacks Project, *Regular rings and global dimension*. The Stacks Project, *Auslander–Buchsbaum*.

Tensor with $P_{\mathfrak{r}}$ through the displayed surjection from T. The resulting module is $P_{\mathfrak{r}} \otimes_{R_0} M_K$. Indeed the diamond variables are zero in P, and the fiber identification of Theorem 42.3 already gives $(P \otimes_{R_0} M_K)[1/f]$ with the action through the surjective map to $P[1/f]$. This ordered module localized at $\mathfrak{r}$ is consequently free of positive rank. The map $(R_0)_{\mathfrak{p}} \to P_{\mathfrak{r}}$ is flat and local, hence faithfully flat. The coefficient extension is finite free, adjoining s is flat over the Noetherian base, the ordered splitting factor is finite projective by the monic presentation, and localization preserves flatness. Contraction of $\mathfrak{r}$ gives precisely the curve prime on R_0. Faithfully flat descent of flatness for the finitely presented module $(M_K)_{\mathfrak{p}}$ shows that it is finite flat and therefore free. Its rank is positive by the ordered calculation upstairs.

The good Hecke algebra B_K is the image of R_0 in the endomorphisms of M_K. Its kernel is the annihilator. The localized positive-rank free module is faithful, so this kernel localizes to zero, proving the isomorphism. For every smaller prime a further localization keeps this module free of the same positive rank. This proves the assertion about whole components. It gives the inclusion of ideals in the first, irreducible case of `Development/AutomorphicArithmetic/AutomorphicVarieties/WeilGroupCharacters/UniversalOrdinaryDeterminantRing.lean`, line 284 for the curves and fields specified here. □

The other generic possibility in Theorem 37.4 is the sum of a character and an irreducible two-dimensional representation. At such a point trace against the group matrices kills the off-diagonal blocks, so the nine-matrix pairing in Proposition 42.6 is degenerate. Theorem 41.32 supplies information about the corresponding extension classes. Their effect on component support, and the specialization to the original integral cohomology lengths, remain to be established.

42.3 The off-diagonal algebra at a curve with two constituents

Let the representation on a selected curve over $L = L_a$ be $r = r_1 \oplus r_2$, where r_1 is a character and r_2 is an absolutely irreducible representation of dimension two. Both are individually polarized, and they remain irreducible and inequivalent on the cyclotomic kernel by Theorem 37.13. Let $K = k'((z))$ be the curve field. We may first extend it by a finite extension to put the symmetric polarization in each block into orthonormal coordinates. Indeed a nondegenerate symmetric matrix in characteristic different from two can be diagonalized, and adjoining square roots of its nonzero diagonal coefficients suffices. Use a uniformizer of the extended field for z, its residue field for k', and repeat the construction with $s \mapsto z$ in Section 42.2. The stable integral lattice used for continuity can be transported through this change of basis. Put

$$H = \widehat{T_{\mathfrak{P}}}, \qquad \mathfrak{m} = \mathfrak{m}_H, \qquad H/\mathfrak{m} = K.$$

Here T is defined by determinant relations before passage to the annihilator of the patched module. These same relations and the surjection onto $P[1/f]$ apply to the present curve. We study the Galois algebra over this complete local ring.

Lemma 42.8 (The finite ordinary Cayley–Hamilton algebra)**.** *There is a finite H-algebra C generated as an H-module by group elements of Γ_∞^0. Its determinant is the base change of the given determinant on the presentation. The old inertia identities, the ordinary matrix words and the two auxiliary matrix words of Theorem 42.4 hold in C, as do their images under the adjoint involution. The semisimple quotient of $C/\mathfrak{m}C$ is $K \times M_2(K)$, with its transposition involution.*

Proof. Start over the complete presentation ring, before localization, with the universal continuous Cayley–Hamilton algebra on Φ^0 pulled back along its determinant. This algebra is finite over the presentation ring by the finiteness theorem of Wang-Erickson. Its hypothesis on residual first cohomology holds since values on the finite set of profinite generators determine a cocycle into the finite residual adjoint module. The images of the group span the base changed algebra. Indeed they have dense linear span before base change in the quotient of the completed group algebra. A submodule of a finite module over a complete Noetherian local ring is closed in its adic topology, by Artin–Rees. Ordinary tensor product then preserves spanning. The topology at the initial complete ring makes the group map continuous. C. Wang-Erickson, *Algebraic families of Galois representations*, preprint, Propositions 2.2.2 and 3.2.2.

Extend scalars to H. Quotient by the two-sided ideal generated by the differences $g - 1$ for the kernel of $\Phi^0 \to \Gamma_\infty^0$, the old inertia differences, the ordinary words and the two auxiliary words. Include also the adjoints of the generators. On the group algebra adjunction sends g to $\mu(g)\sigma(g)^{-1}$, where σ is conjugation by complex conjugation and μ is the fixed multiplier in the polarization. It fixes all scalars. Polarization gives $t(x^*) = t(x)$ on group algebra sums. Together with cyclicity this gives the same characteristic polynomial for x^* as for x, by Newton's formulas. Consequently the Cayley–Hamilton ideal is invariant under adjunction. To see this also with the completed group algebra, one can first impose the polarization equations on the universal determinant over the complete coefficient ring. On the completed group algebra adjunction is continuous and the polarized identities extend from the dense group algebra. Pullback to H gives the stated involution.

The trace against every group element of each imposed word is zero by definition of T. Cyclicity, polarization and linear spanning give the same vanishing on the two-sided ideal, even after arbitrary scalar extension. Applying Newton's identities to the first three powers shows that the determinant descends to the quotient C, since 3! is invertible. It is Cayley–Hamilton there. The semisimple curve representation realizes the specialization of this determinant and satisfies the matrix words. At the auxiliary places it comes from the unramified base and its Frobenius has the ordered roots. Its image spans $K \times M_2(K)$, by the irreducibility and inequivalence on the cyclotomic kernel. The kernel of this surjection in $C/\mathfrak{m}C$ is the determinant kernel. Indeed the trace pairing on the displayed product is nondegenerate, and the determinant is computed from its traces. This kernel is nilpotent by the Cayley–Hamilton radical theorem in residue characteristic greater than three. It follows that it is the Jacobson radical. On the quotient the involution is transposition by our choice of basis. [2] □

Proposition 42.9 (A single module for both off-diagonal blocks)**.** *There is a finite H-module N and a symmetric H-bilinear form $p : N \times N \to \mathfrak{m}$ satisfying*

$$p(x, y)w = x\,p(y, w) \qquad (x, y, w \in N).$$

Give $H \oplus N$ the commutative algebra structure in which $xy = p(x, y)$ for $x, y \in N$. With suitable matrix units, C identifies, as an algebra with involution, with

$$\begin{pmatrix} H & M_{1,2}(N) \\ M_{2,1}(N) & M_2(H) \end{pmatrix} \subset M_3(H \oplus N)$$

with ordinary transposition. Its determinant agrees with the matrix determinant after extension to $H \oplus N$.

Proof. Let J_0 be the kernel of the map from C to its semisimple residue algebra. A power of J_0 is contained in $\mathfrak{m}C$ by the lemma. The J_0-adic and $\mathfrak{m}$-adic topologies on C coincide, and C is complete in both. Lift the three primitive diagonal idempotents to orthogonal self-adjoint idempotents summing to one. For the first lift, replace any preimage by its average with its adjoint. Newton iteration for $X^2 - X$ converges because its error starts in J_0 and its derivative $2X - 1$ is a unit modulo J_0. All iterates commute with their rational expressions and remain self-adjoint. Repeat inside the complementary corner and take the residual complement for the last idempotent.

For each lifted idempotent e, the determinant on the corner eCe is Cayley–Hamilton of degree one. In fact its degree can be read off from $D(1 - e + Xe)$ after reduction, where the residue idempotent has rank one. Chenevier’s corner lemma gives Cayley–Hamilton on that corner. The degree-one identity now says $y = t(y)e$ for every $y \in eCe$, and $t(e) = 1$. Thus $eCe = He$ with the indicated copy of H embedded. [2]

Write e_1, e_2 for distinguished primitive idempotents in the character and the plane, respectively. For the other idempotent e_j in the plane lift a standard matrix unit to $w \in e_j C e_2$. Then $w^{*}w = d e_2$ for an element $d \equiv 1 \pmod{\mathfrak{m}}$. Divide w by the square root of d congruent to one, which exists by Hensel’s lemma. Now $w^{*}w = e_2$ and ww^{*} is an idempotent of He_j reducing to e_j, so $ww^{*} = e_j$. Multiplication by these matrix units identifies the corners within the plane and exchanges the off-diagonal corners by transposition. Take $N = e_1 C e_2$. For $x, y \in N$ define $xy^{*} = p(x, y) e_1$. Its reduction on the semisimple quotient is zero. Adjunction gives symmetry since it fixes He_1. Cyclicity of trace and the rank-one corner calculation give $y^{*}x = p(x, y) e_2$. Computing $xy^{*}w$ by the two associations gives the asserted identity. It also proves associativity of the multiplication on $H \oplus N$, by checking products of three elements of N. Decomposing C by the three orthogonal idempotents now gives exactly the displayed algebra. Terms in distinct corners have trace zero off the diagonal, by cyclicity. Thus the matrix trace is t on every element and after base change. Newton’s identities give equality of the determinants. Compare the polarized block structure in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/PatchedSupportCompletions.lean`, line 78. □

Theorem 42.10 (Irreducible and nonsemisimple fibers at the curve)**.** *Put $J = (p(N, N)) \subset H$ and $I = \operatorname{Ann}_H N$. For a prime $Q \subset H$, use an algebraic closure of $\kappa(Q)$ for its geometric fiber. The determinant there is absolutely irreducible precisely when $J \not\subset Q$. When $J \subset Q$, the determinant has constituents of dimensions one and two. The ordinary Cayley–Hamilton fiber is then nonsemisimple precisely when $I \subset Q$. Consequently $V(I)$ consists exactly of the irreducible fibers together with the reducible nonsemisimple fibers.*

Proof. Extend the block description to the geometric residue field k. If the pairing has nonzero image, there is an $x \in N \otimes_H k$ with $p(x, x) \neq 0$, since two is invertible. The identity in the proposition gives $p(x, x) y = p(x, y) x$ for every y in this space. Thus it is a line. Rescale x over k so that $p(x, x) = 1$. The map $k \oplus kx \to k$ sending x to one identifies the displayed block algebra with all of $M_3(k)$ on its matrix entries. Its group elements span this algebra, and the resulting representation is irreducible and realizes the determinant.

If the pairing vanishes, the two off-diagonal blocks form an ideal of square zero and the quotient is $k \times M_2(k)$. It is the full radical. The product determinant on this quotient is the given determinant, since its trace agrees on all group algebra sums. Each simple factor occurs with multiplicity one. The radical is nonzero precisely when $N \otimes_H k \neq 0$. As N is finite this

is equivalent to $N_Q \neq 0$, by Nakayama, and to $I \subset Q$. In the first case as well $N_Q \neq 0$ by the nonzero pairing. These alternatives give the asserted locus. □

Corollary 42.11 (The number of off-diagonal generators)**.** *Let b be the rank of ordinary polarized off-diagonal cohomology on the original group of this curve as in Theorem 41.32. Then*

$$\dim_K N/\mathfrak{m}N = b, \qquad b \geq 1.$$

In particular J is generated by at most $b(b+1)/2$ elements. If $b = 1$, there is a $d \in \mathfrak{m}$ such that $J = (d)$ and

$$H \oplus N \;\simeq\; H[X]/(IX,\; X^2 - d), \qquad Id = 0.$$

Proof. Put $U = N/\mathfrak{m}N$. At this fiber the pairing is zero. For $\lambda \in U^\vee$, send every off-diagonal entry u in the block description to $\epsilon\lambda(u)$ and each scalar entry to its residue, where $\epsilon^2 = 0$. This yields a polarized lift of r. In the chosen coordinates its first-order part is a cocycle with only off-diagonal entries. It is continuous with bounded denominators over the curve DVR. Indeed the group map comes from the finite module over the original complete presentation ring in the lemma. At a fixed finite set of module generators its specialized coordinates have bounded denominators. Powers of the original maximal ideal evaluate into the corresponding powers of z, which gives continuity for the finite module map.

The matrix words in C impose the ordinary condition on this cocycle. At an ordinary place diagonalize the matrix of the element with three distinct inertia eigenvalues over the dual numbers using its simple-root projectors. The ordinary word with two occurrences of that element and a local element in the remaining position then gives upper triangularity with the prescribed diagonal, by the row and column calculation of Proposition 42.6. The change of the flag from its original one is an infinitesimal local conjugation. Projecting its coboundary witness onto the off-diagonal blocks preserves the condition, since the original ordinary basis respects the two constituents and block projection preserves upper triangularity. The old inertia differences vanish in C. Multiplication by a power of z puts the cocycle and all its local witnesses in the chosen lattices. Part 3 of Theorem 41.32 descends the resulting ordinary class to the original group after inverting z. We have thus obtained a linear map $U^\vee$ to a space of dimension b.

This map is injective. If the cocycle is a coboundary, its lift on the group is conjugate to the constant representation. The group spans $C/\mathfrak{m}C$ linearly over K, so the two algebra homomorphisms also agree up to that conjugation. The constant one kills the radical, hence the lift kills every off-diagonal entry $\epsilon\lambda(u)$. This gives $\lambda = 0$.

For the opposite bound take the b ordinary polarized classes on the base with integral representatives $\nu_1, \dots, \nu_b$ chosen in Section 37.4. Write E for the block diagonal algebra over K and M for both off-diagonal matrix blocks. The representation

$$g \longmapsto \left(1 + \sum_{j=1}^{b} e_j \nu_j(g)\right) r(g), \qquad e_i e_j = 0,$$

inflated to the limit, gives an algebra map from $C/\mathfrak{m}C$ to $E \oplus M^{\oplus b}$. We verify factorization. All mixed terms of the determinant having two off-diagonal entries vanish by the square-zero relation. A single such entry has coefficient zero at every block diagonal matrix by expansion of its determinant. Thus the determinant is the original one on all algebra sums and after scalar extension. The matrix Cayley–Hamilton identities follow. Continuity and a stable integral lattice for these matrices give a representation of the original completed group algebra. The chosen local witnesses give upper triangular matrices with the same diagonal at each ordinary

place, and polarization gives the adjoint words. Old inertia conditions hold by construction. At the new places inertia acts trivially and Frobenius has the original characteristic polynomial, giving both auxiliary relations. This proves the factorization, including through the residue map $H \to K$. Lemma 37.14 says the group already spans the entire target algebra on the cyclotomic kernel. The surjective map just obtained gives $5 + 4 \dim_K U \geq 5 + 4b$ by the block dimensions. Together with the injection this proves equality.

To prove positivity use the stable prime of Theorem 37.32 below the curve in R_0. The flat local map $(R_0)_{\mathfrak{p}} \to P_{\mathfrak{r}}$ from the proof of Theorem 42.7 is independent of the number of constituents. Choose a prime over the stable prime there by faithful flatness. Pull it back by the surjection $T_{\mathfrak{P}} \to P_{\mathfrak{r}}$, then lift it to H by faithful flatness of completion. The determinant at this prime of H is absolutely irreducible, since it extends the determinant at the stable prime. The theorem implies $J \neq 0$, hence $N \neq 0$ and $b \geq 1$ by Nakayama. Finally choose b generators of N. Their pairwise products, with equal pairs allowed, generate J by symmetry and bilinearity. For one generator n, put $d = p(n, n)$. The map $X \mapsto n$ gives the asserted presentation. A polynomial reduces to $a + cX$ using $X^2 = d$, and this expression maps to zero exactly when $a = 0$ and $c \in I$, by the direct sum decomposition. Also $Id = 0$ by bilinearity. □

The ideal I describes the support of the off-diagonal Galois module. To propagate Hecke support across these reducible curves one must still place the corresponding primes in the support of the patched module $\mathcal{P}$. The preceding calculation identifies the primes and the extension classes that enter that comparison.

42.4 Relations among the ordinary extension classes

Keep the curve and the rings H and $H^{\sharp} = H \oplus N$ of Section 42.3. Write $\mathfrak{m}$ for the maximal ideal of H, $J = (p(N, N))$ for the ideal of products and $D = 3([F : \mathbb{Q}] + q)$ for the bound of Theorem 41.32. The coefficient prime of the curve defines the height-one prime (ϖ) in $O'[[s]]$. Indeed the evaluation modulo ϖ is the isomorphism $k'[[s]] \to k'[[z]]$. Let $\mathcal{O}$ be the completion of $O'[[s]]_{(\varpi)}$. This is a complete discrete valuation ring with uniformizer ϖ and residue field K. Its map to H is induced by localization and completion. It is injective because H has a characteristic-zero prime over the stable prime in Corollary 42.11. In particular $H/\varpi H$ contains K as a coefficient field. Put

$$J_2 = (\varpi) + \mathfrak{m}^2 + J, \qquad V = (\mathfrak{m}/J_2)^{\vee},$$

where the dual is over K. Thus H/J_2 is the split square-zero algebra $K \oplus V^{\vee}$. In a module over this quotient expressions such as $h - \bar{h}$ use this splitting, with $\bar{h}$ the residue of h.

Let E be the diagonal-block matrix algebra and M the off-diagonal matrices for $r_1 \oplus r_2$. Write $H^1_o(X)$ for polarized ordinary cohomology of the limiting group with coefficients in X over K, including the prescribed old inertia conditions. Cocycles and local witnesses have bounded denominators. This is the fraction-field version of Theorem 41.32. Choose a basis $\lambda_1, \dots, \lambda_b$ of $(N/\mathfrak{m}N)^{\vee}$ which maps to the chosen basis of $H^1_o(M)$ under the isomorphism in Corollary 42.11. Choose $n_j \in N$ lifting the dual basis. Reduction of the matrix representation in Proposition 42.9 in those odd coordinates gives the actual cocycles ν_j in their chosen classes. We may use these representatives in the calculations of Sections 37.5 and 37.6 over K. To verify this, write the difference of representatives as a coboundary with skew-adjoint off-diagonal witness Z_j. Replacing the universal first-order matrix basis by $1 + \sum_j e_j Z_j$, or its inverse according to the coboundary convention, changes the representatives as required. It preserves the form because $e_i e_j = 0$. Conjugation identifies the two coefficient modules W and also the two mixed modules U; it is the identity on their diagonal quotients and transports their

local flags. The latter flags are intrinsic because their simple inertia eigenvalues distinguish the flagged lines. Clearing a finite set of denominators gives the same fraction-field cohomology and lifting bounds.

Lemma 42.12 (Tangents preserving the two blocks)**.** *There is a linear injection $a : V \to H^1_o(E)$.*

Proof. For $v \in V$ evaluate $h \in H$ in the dual numbers as $\bar{h}+\epsilon v(h-\bar{h})$, and send N to zero. This is a ring map from $H^\sharp$ since it kills J. The group matrices now give a polarized block diagonal lift of r. Its first-order cocycle represents $a(v)$. The ordinary word and simple-root projectors give the ordinary flag, as in Corollary 42.11. Projecting a local witness onto the diagonal blocks preserves upper triangularity in the original block-adapted flags. The old inertia words give the old condition.

Here is the continuity argument, also used for mixed evaluations below. On the complete presentation ring before localization the coefficient maps to a square-zero K-algebra fixing $O'[[s]]$ have bounded denominators and are continuous into its finite-dimensional z-adic topology by Lemma 42.5. That evaluation argument only uses the finite power series presentation and applies to either kind of curve. Choose a finite set of generators of the initial Cayley–Hamilton algebra as a module. The group map into that module is continuous for the original adic topology. After any fixed matrix evaluation at those generators, continuity follows from continuity of the scalar maps and the finite number of their coordinates. The same argument applies when the coefficients of group matrices are written in a fixed basis over a finite-dimensional test algebra.

If $a(v) = 0$, strict conjugation gives the constant lift of r. The determinant values are constant. Simple-root projectors at an ordinary inertia element recover each local character as the trace of the projector times the local matrix. The first auxiliary matrix word recovers each ordered Frobenius root from the corresponding distinct curve eigenvalue, and the second forces its diamond coordinate to be zero since inertia on the curve is trivial. Thus the evaluation is constant on every generating datum of the original presentation. Its continuity there gives equality on the full ring. Localization introduces no new choice. Diamond completion also introduces none because evaluation kills the square of the diamond ideal; the same holds at the final completion for the square of the curve maximal ideal. We obtain $v = 0$. Linearity follows by taking coefficients in a square-zero vector space and applying linear functionals to these coefficients. □

Form $U = M\theta \oplus \bigoplus_j Ee'_j$ and $W = E \oplus \bigoplus_j Me_j$ with the actions in Section 37.6. In particular their action on the CM group is conjugation by $R(g) = (1+\sum_j e_j\nu_j(g))r(g)$; the extra products are $e_ie'_j = \delta_{ij}\theta$, all other products between added basis elements being zero. Define

$$\mathcal{U} = \{(v_j) \in V^b : (a(v_j))_j \text{ is in the image of } H^1_o(U) \longrightarrow H^1_o(E^b)\},$$
$$V_0 = \{v \in V : (0, \dots, v, \dots, 0) \in \mathcal{U} \text{ in every position}\}.$$

The image condition depends only on classes. If a projected cocycle differs from the desired one by a coboundary, subtract a lifted coboundary. Such witnesses lift in the coefficient vector spaces, and averaging against the involution makes the lift polarized since two is a unit.

Proposition 42.13 (Two dimension inequalities)**.** *The spaces just defined satisfy*

$$\operatorname{codim}_{V^b} \mathcal{U} \le b, \qquad \dim_K V_0 + b^2 \le D.$$

Proof. Apply part 2 of Theorem 41.32 to any $b + 1$ tuples. There are finitely many cocycles and local witnesses here, so clearing one denominator puts them in the integral lattices. Over

K both this denominator and the factor z^C are invertible. Primitivity gives a nonzero linear dependence modulo $\mathcal{U}$. This is the first inequality.

For the second, quotient W by all its off-diagonal summands except the j-th, and call the result W_j. It is the preimage of the j-th copy of E in U, with Me_j identified with $M\theta$. Indeed the action on its off-diagonal coefficient acquires exactly the commutator of $\nu_j(g)$ with the conjugated diagonal coefficient. This also identifies the upper triangular conditions there, by using the local change of basis for ν_j in Section 37.6. More explicitly, after a lift has been chosen with zero other diagonal cocycles, the projected local witnesses for those other coordinates are invariant in $E/B_u(E)$. This quotient has zero local invariants since an ordinary inertia element has distinct eigenvalues on the flagged lines. Those projected witnesses vanish. Representatives in the upper triangular subspace can be subtracted after lifting to the upper part of U, whose projection onto the upper part of E^b is surjective. The resulting witnesses lie in the stated preimage. The same argument at old inertia kernels is immediate because those kernels act trivially.

Consequently, for $v \in V_0$ a representative of $a(v)$ lifts to an ordinary cocycle in each W_j. Arrange that the projections of these lifts coincide as cocycles, by the preceding coboundary adjustment. The modules W_j have fiber product W over their common diagonal quotient E. The local upper triangular conditions have the same fiber product description. In block-adapted coordinates the upper parts consist of the common upper diagonal coefficient and the separate upper off-diagonal coefficients, transported by the respective local changes of basis. At an ordinary place the projected witnesses for the common diagonal cocycle agree in $E/B_u(E)$ by the same vanishing of invariants. They can therefore be represented by witnesses with a common diagonal part, and assembled in W. This gives a lift in $H^1_o(W)$ of every class in $a(V_0)$.

The inclusion $M^{\oplus b} \to W$ injects its ordinary cohomology into $H^1_o(W)$. Indeed if an off-diagonal cocycle is a polarized coboundary in W, the diagonal part of its witness commutes with r and is skew-adjoint. A commuting element of E is scalar on each absolutely irreducible block, and such a scalar is skew-adjoint only when zero. Thus the witness was already off-diagonal. Upper triangular witnesses for a class in the submodule are witnesses in W, using its transported flags. This supplies a subspace of dimension b^2 killed in $H^1_o(E)$, by part 3 of Theorem 41.32. The lifted subspace maps onto the $\dim V_0$ independent classes of the lemma. Part 1 of the same theorem bounds $\dim H^1_o(W)$ by D, proving the second inequality. □

Lemma 42.14 (Evaluating products with an extension class). *For $\varphi \in (N/J_2N)^\vee$ define*

$$v_j(h) = \varphi\big((h - \bar{h})n_j\big) \quad (h \in H/J_2).$$

On $\mathfrak{m}/J_2$ these are elements of V, and the tuple (v_j) belongs to $\mathcal{U}$.

Proof. Use the commutative test algebra just defined with basis $1, e_j, e'_j, \theta$. The following formulas determine a homomorphism into it from $H^\sharp$:

$$h \longmapsto \bar{h} + \sum_j v_j(h) e'_j, \qquad n \longmapsto \sum_j \lambda_j(\bar{n}) e_j + \varphi(n)\theta.$$

Here $\bar{n}$ denotes reduction modulo $\mathfrak{m}N$ and φ is read modulo J_2N. Products of two scalars give the first formula because $(\mathfrak{m}/J_2)^2 = 0$. Products of two elements of N have image zero on both sides. For a mixed product, multiplication by $h - \bar{h}$ on N/J_2N factors through $N/\mathfrak{m}N$. Hence

$$\varphi(hn) = \bar{h}\varphi(n) + \sum_j v_j(h)\lambda_j(\bar{n}),$$

as required. The formulas respect addition and the unit.

Apply this map entrywise to the group matrices. Modulo the square-zero ideal generated by the e'_j and θ we obtain exactly $R(g)$. The difference from $R(g)$ multiplied on the right by $R(g)^{-1}$ is an U-valued cocycle: its e'_j entries are block diagonal and its θ entries off-diagonal. Its projection to E^b consists of the actual cocycles of the v_j in Lemma 42.12. Its polarization follows from the matrix form, and it kills the old inertia kernels. It is bounded and continuous by the finite module argument of the lemma, since the scalar evaluation on H has only first-order coefficients.

For the local condition, use the ordinary words in C. The eigenprojectors for a local element with simple roots lift the ordered basis of the ordinary flag for R across the square-zero ideal; projecting the basis vectors and applying Nakayama gives a basis. The words with one arbitrary letter then triangularize every local group matrix with its specified diagonal. Thus the cocycle has a local witness in the matrices with entries in this ideal, in coordinates for the flag of R. A witness can be taken in U. To see this, apply the involution which conjugates by $\mathrm{diag}(1,-1,-1)$ in the block basis and negates the variables e_j, θ while fixing e'_j. Its invariant part on the square-zero ideal of matrices is exactly U. It fixes R and preserves its ordinary flag, since the original flag respects the two blocks and the local change of basis for R has off-diagonal e_j-terms. Averaging the witness gives the required invariant witness. This proves membership in $\mathcal{U}$. □

Theorem 42.15 (A square matrix of relations from ordinary cohomology)**.** *One has $b^2 \le D$. There are D elements $x_k \in \mathfrak{m}$ spanning $\mathfrak{m}/J_2$ modulo J_2; designate the first b^2 of them as x_{ij}. For some $w_{ij} \in J_2$ the following equalities hold in N:*

$$\sum_{j=1}^{b} (x_{ij} - w_{ij}) n_j = 0 \qquad (1 \le i \le b).$$

The x_k are allowed to be dependent in H.

Proof. By Proposition 42.13, the annihilator of $\mathcal{U}$ in $(V^b)^\vee$ is spanned by b forms, padding with zeros if necessary. Write their entries as $\ell_{ij} \in V^\vee$, so that $\sum_j \ell_{ij}(v_j) = 0$ for tuples in $\mathcal{U}$. The common kernel of the b^2 entries is precisely V_0: a vector belongs to it if and only if each single insertion is killed by every row. Their span in $V^\vee$ therefore has a complement of dimension $\dim V_0 \le D - b^2$. Use the entries followed by a spanning list for this complement, padded as needed, to obtain D linear forms spanning $V^\vee = \mathfrak{m}/J_2$. Lift them to the claimed x_k.

For every functional φ of the last lemma we have

$$\varphi\left(\sum_j x_{ij} n_j\right) = \sum_j \ell_{ij}(v_j) = 0.$$

Functionals separate vectors in the finite-dimensional space N/J_2N. The sum thus lies in J_2N, and the n_j generate N by Nakayama. Writing the sum in these generators with coefficients $w_{ij} \in J_2$ proves the asserted equalities. This is the presentation step in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/SelmerScaling/AlCSmul.lean`, line 12. □

42.5 Hecke support through the curves with two constituents

The square matrix in Theorem 42.15 annihilates the vector of generators of N. We first determine the component on which that vector can be nonzero. The argument takes place over the same coefficient DVR $\mathcal{O}$ and uses a projective space of kernel lines.

Lemma 42.16 (The incidence locus after completion)**.** *Let $b \geq 1$, $u \geq 0$ and let S be the formal power series ring over a complete DVR in the entries of a b-by-b matrix A, a column Y of length b, and u further variables. The radical of the ideal $(AY, \det A)$ is prime. Every prime of S containing the entries of AY has height at least b.*

Proof. Begin with polynomial coordinates over the DVR. Over the affine space of these coordinates consider the projective incidence scheme obtained by adjoining a line L' in a free module of rank b and requiring

$$AL' = 0, \qquad Y \in L'.$$

It is closed in the product with projective space. Over projective space the column lies in the tautological line and the matrix is an arbitrary map out of the quotient by that line. Thus this scheme is the total space of a vector bundle, with the u additional free coordinates, and is regular.

Localize the polynomial base at the origin of its closed fiber. The incidence scheme remains regular and the projection is proper. It stays regular after completing this local base. Indeed at each point of the closed fiber the local rings before and after this base change have isomorphic completions. Both schemes are of finite type over their Noetherian local bases; every closed point of the scheme after base change lies in the closed fiber by properness. Regularity there implies regularity at every generization. The base change at issue is precisely the one to S. The Stacks Project, Lemmas 54.11.1 and 54.11.2.

The completed incidence scheme is integral. It is regular and Noetherian, so its irreducible components are disjoint open and closed subsets. Each nonempty component meets the closed fiber, since its image under the proper map is nonempty and closed in the spectrum of a local ring. The closed fiber is a single projective space over the residue field and is connected, so there is just one component. At any prime of S the fiber has a point precisely when $AY = 0$ and $\det A = 0$ in its residue field. If $Y \neq 0$ use its span, and if $Y = 0$ use any line in the kernel of the singular matrix. Conversely such a line gives both equations. The proper image of the integral incidence scheme is therefore exactly $V(AY, \det A)$, which is irreducible.

For the height bound first reduce the polynomial coordinates modulo the uniformizer. Away from $Y = 0$, invert a nonzero coordinate of Y and eliminate one entry of each row of A. This locus has dimension $b^2 + u$. The locus $Y = 0$ also has dimension $b^2 + u$. Thus the b entries of AY cut out codimension b in affine space over the residue field. In its regular local ring at the origin these equations form a regular sequence: the homogeneous components of the zero set pass through the origin, and b equations of height b in a Cohen–Macaulay local ring form part of a system of parameters. Adjoining the uniformizer first gives a regular sequence upstairs. Permutability in the local ring and flatness of completion leave the b entries a regular sequence in S. Any prime containing them has height at least b, as asserted. □

Proposition 42.17 (A component through every prime in the support of N)**.** *Let Z be a prime in H whose geometric determinant is irreducible and such that $\dim H/Z \geq D + 1$. Then $Z \subseteq Q$ for every $Q \in \operatorname{Supp}_H N$.*

Proof. Use the D scalars x_k, the b odd generators n_j, and the corrections w_{ij} of Theorem 42.15. Since $N^2 = J \subseteq \mathfrak{m}$, the finite H-algebra $H^\sharp = H \oplus N$ is local with maximal ideal $\mathfrak{m} \oplus N$. Its square lies in $\mathfrak{m}H^\sharp$, and it contains $\mathfrak{m}H^\sharp$ before squaring. The two adic topologies coincide and it is complete. Map

$$S = \mathcal{O}[[X_1, \dots, X_D, Y_1, \dots, Y_b]] \longrightarrow H^\sharp$$

by $X_k \mapsto x_k$, $Y_j \mapsto n_j$. The map is surjective. Indeed modulo the square of its maximal ideal the target is generated, in positive degree, by the images of ϖ, X_k, Y_j: the scalars x_k span

modulo J_2, J consists of sums of products of odd elements, and the n_j generate N. Both residue fields are K. Nakayama gives generation of the maximal ideal. Lifting the residue and then correcting at each successive degree by homogeneous polynomials proves surjectivity by completeness.

Lift each $w_{ij} \in J_2$ to $W_{ij} \in (\varpi) + \mathfrak{m}_S^2$, since elements of J map into the square upstairs in $H^\sharp$ and the surjection maps maximal ideals onto maximal ideals. For the first b^2 coordinates set $A_{ij} = X_{ij} - W_{ij}$. With the remaining X_k and the Y_j these are again formal coordinates over $\mathcal{O}$. In fact the corresponding substitution fixes ϖ, and on the cotangent space it changes each X_{ij} only by a multiple of ϖ. Its linear part is invertible. It induces an isomorphism on the associated graded polynomial ring of the regular local ring S, hence on S by successive approximation and separatedness. In the new coordinates the entries of AY vanish in $H^\sharp$. The matrix A has an image $A_H = (x_{ij} - w_{ij})$ in $M_b(H)$. Multiplying $A_H (n_j)_j = 0$ by its adjugate shows that $\det A_H \in \operatorname{Ann}_H N$.

The extension $H \subset H^\sharp$ is finite integral and injective. Choose a prime over Z in $H^\sharp$ and let $\widetilde{Z}$ be its inverse image in S. The quotient has dimension $\dim H/Z$ by integral extension. Since $\dim S = D + b + 1$, the height of $\widetilde{Z}$ is at most b. Some Y_j has nonzero image at this prime. Otherwise it would contain N and all its products, contrary to $J \not\subset Z$ by Theorem 42.10. Thus $AY = 0$ gives $\det A \in \widetilde{Z}$. By Lemma 42.16 the prime radical of $(AY, \det A)$ has height at least b, and is contained in $\widetilde{Z}$. The two primes must coincide.

For $Q \in \operatorname{Supp}_H N$ take a prime of $H^\sharp$ over Q and its inverse image $\widetilde{Q} \subset S$. It contains AY by the relations and $\det A$ by the annihilator calculation. It consequently contains $\widetilde{Z}$. Descending through the surjection and contracting to H gives $Z \subseteq Q$. □

For a prime $\mathfrak{q}$ in the original ordinary determinant ring R_0, take the finite continuous Cayley–Hamilton algebra for the base group Γ_0^0 and impose the ordinary, old inertia, and adjoint matrix words, as in Lemma 42.8 without the new places. Denote this algebra by C_0. Its geometric fiber means extension to an algebraic closure of $\kappa(\mathfrak{q})$. An irreducible prime is one whose geometric determinant is irreducible. We also consider reducible primes at which this algebra has a nonzero Jacobson radical in the geometric fiber. These are the two cases tested by component propagation.

Theorem 42.18 (Ordinary Hecke support through a two-constituent curve)**.** *Let $\mathfrak{p} \subset R_0$ be any of the selected characteristic-ℓ curves having a character and a plane. If $\mathfrak{q} \subseteq \mathfrak{p}$ is irreducible, or is reducible with nonsemisimple geometric fiber of C_0, then $(M_K)_{\mathfrak{q}} \neq 0$. Equivalently, the kernel of the natural good Hecke map $R_0 \to B_K$ is contained in $\mathfrak{q}$.*

Proof. Let $\mathcal{P}_H$ be the patched module localized at $\mathfrak{P}$ and completed over H. Its depth is at least $D + 1$. The proof of Lemma 42.5 uses the $3q$ regular diamond variables, the coefficient uniformizer, the $3[F : \mathbb{Q}]$ translated weight variables and the nonzero ordered fiber. These facts hold for both kinds of selected curves by Theorem 42.3, and the regular sequence persists under faithfully flat completion.

We first produce a sufficiently large irreducible prime in its support. By Theorem 37.32 and the classical systems in the proof of Theorem 42.3, there is an irreducible prime $\mathfrak{q}_1 \subset \mathfrak{p}$ in the support of M_K. Indeed the same good Hecke annihilator vanishes at every point of the dense family defining $\mathfrak{q}_1$. Choose a prime over $\mathfrak{q}_1$ in $P_{\mathfrak{r}}$ by flat local descent as in Corollary 42.11. Its pullback to $T_{\mathfrak{P}}$ is in the support of the patched module. To see this, first reduce modulo the diamonds; the ordered fiber of that module there is precisely $P_{\mathfrak{r}} \otimes_{R_0} M_K$, whose residue fiber is a nonzero field extension of the fiber at $\mathfrak{q}_1$. Lift this prime to H by faithful flatness. Its residue field extends the preceding one, so it is still in support and still has irreducible determinant. Choose a minimal prime Z of the support below this lifted prime. Irreducibility

holds at Z since its locus is the complement of $V(J)$. A minimal prime of the support of a finite module is associated to the module, and at any associated prime its quotient ring has dimension at least the depth of the module. Hence $\dim H/Z \geq D+1$. Proposition 42.17 now gives $\operatorname{Supp}_H N \subseteq \operatorname{Supp}_H \mathcal{P}_H$, since support is closed under specialization. The Stacks Project, Section 10.72, *Depth*.

Given $\mathfrak{q}$ of the theorem, choose a prime over it in $P_{\mathfrak{r}}$, take its preimage in $T_{\mathfrak{P}}$ and lift to a prime Q of H. These choices use the same flat local maps as above. At Q all diamond variables vanish. If $\mathfrak{q}$ is irreducible then so is Q, and it belongs to $\operatorname{Supp}_H N$ by Theorem 42.10. In the other case the geometric fiber of C there surjects onto the geometric fiber of C_0 after extending the residue fields to a common algebraically closed field. In fact pull the group matrices of C_0 back through $h : \Gamma_\infty \to \Gamma_0$. They satisfy the original determinant and its matrix words. The new inertia is one, and the ordered Frobenius polynomial on P is the characteristic polynomial of the base determinant. Thus the first auxiliary word is zero by Cayley–Hamilton, and the second is zero since all diamonds vanish. The same holds after adjunction. These facts give the asserted map by the base change property of the initial continuous Cayley–Hamilton algebra in Lemma 42.8. Equivalently one extends the finite integral base group algebra quotient along the ordered base map and then specializes; its generating group elements lift through h, proving surjectivity.

A nonzero radical over an algebraically closed field remains a nonzero nilpotent ideal after further field extension. A quotient of a semisimple finite-dimensional algebra is semisimple, so the source geometric fiber is nonsemisimple as well. The determinant stays reducible under this extension. Theorem 42.10 again gives $Q \in \operatorname{Supp}_H N$. In both cases $\mathcal{P}_H$ has nonzero residue fiber at Q. Since the diamonds vanish there, that fiber is also the residue fiber of the ordered base module, extended to $\kappa(Q)$. Hence the original fiber of M_K at $\mathfrak{q}$ is nonzero. The equivalent ideal containment follows because the good Hecke algebra is the image of R_0 on the finite module M_K. This proves the single-curve support implication for the selected fields; the corresponding inclusion is formulated in `Development/AutomorphicArithmetic/AutomorphicVarieties/WeilGroupCharacters/UniversalOrdinaryDeterminantRing.lean`, line 288. □

Together with Theorem 42.7 this settles local propagation at the curves selected from the ordinary family. The conclusions concern prime support. The bound on lengths in Proposition 6.11 requires control of integral multiplicities at the original weights. The signed projected periods must also be compared on those same fields and intervals.

42.6 The exact annihilator on the irreducible component

In the presentation of Proposition 42.17 write Y_j for the coordinate mapping to the off-diagonal generator n_j, and $\mathfrak{a}$ for the ideal of the auxiliary diamond variables in the patched ring. The first coordinates describe a projective incidence calculation before the specialization by $\mathfrak{a}$. We retain the equations of that calculation to determine which elements act as zero on N.

Lemma 42.19 (The two components of the matrix equation)**.** *In the formal power series ring S of Lemma 42.16, let*

$$L = ((AY)_1, \dots, (AY)_b), \qquad P_0 = (Y_1, \dots, Y_b), \qquad P_1 = \sqrt{L + (\det A)}.$$

Then S/L is reduced and $L = P_0 \cap P_1$. Both primes have height b.

Proof. The ideal P_0 is prime of height b. The prime P_1 was constructed in Lemma 42.16. It contains $\det A$, which is nonzero modulo P_0. Also $Y_1 \notin P_1$. Indeed one may reduce coefficients to the residue field, send every matrix entry and all free extra variables to zero, and send Y_1

to the parameter of a formal power series ring over that field, with the other Y_j zero. The kernel of this map is prime and contains $L + (\det A)$. For any prime containing L, either all Y_j vanish there or $\det A$ does, by the adjugate identity. The two incomparable primes displayed in the statement are therefore exactly the minimal primes. The height theorem for b generators together with Lemma 42.16 gives height b for both.

At P_0, the matrix is invertible and $LS_{P_0} = P_0 S_{P_0}$. At P_1, consider the b derivations $\partial/\partial A_{i1}$, defined term by term on formal series and extended by the quotient rule to S_{P_1}. They send a square of the local maximal ideal into that ideal by the product rule. On the generators $(AY)_j$ their matrix is Y_1 times the identity. Thus these generators are independent in the cotangent space at P_1. To check this, lift coefficients of a linear relation there to the local ring and apply the derivations; the terms differentiating the coefficients vanish in the residue field because $AY = 0$ there. Since S_{P_1} is regular local of dimension b, its cotangent space has dimension b. By Nakayama the entries generate its maximal ideal. Hence S/L is reduced at both minimal primes.

The entries of AY form a regular sequence by Lemma 42.16. The quotient is a complete intersection and remains Cohen–Macaulay after localization. Its associated primes are therefore minimal, since at an associated prime depth is zero. If its nilradical were nonzero this finite submodule would have an associated prime, also associated to S/L, where its localization would be nonzero. That contradicts the generic reducedness just proved. The intersection formula follows. □

Choose the presentation $S \to H^\sharp$ of Proposition 42.17 equivariantly for the involution fixing H and sending n to $-n$ on N. In the original coordinates the map is equivariant when the X_k are fixed and the Y_j are negated. Each correction w_{ij} is even, so average any of its lifts in $(\varpi) + \mathfrak{m}_S^2$ with its transform. The corrected matrix entries $A_{ij} = X_{ij} - W_{ij}$ are again fixed by the involution. Thus P_0, P_1, L in the new coordinates are invariant. Let $D_1 = S/P_1$, and denote its even and odd submodules under this involution by D_1^+ and D_1^-. These eigenspaces split because two is invertible.

Theorem 42.20 (The off-diagonal module on the incidence component)**.** *At every selected two-constituent curve there is a unique prime Z in the completed patched ring H with irreducible geometric determinant and $\dim H/Z \geq D+1$. For the choices of coordinates above, the natural maps give*

$$H/Z \simeq D_1^+, \qquad N \simeq D_1^-, \qquad \operatorname{Ann}_H N = Z.$$

In particular N is a torsion-free module of rank one over H/Z. Its product ideal J maps injectively into this domain, with image generated by the products of the images of the $Y_i Y_j$.

Proof. The proof of Theorem 42.18 constructs such a prime Z in the support of the patched module. Write $K_0 = \ker(S \to H^\sharp)$. For any prime with the stated conditions, the proof of Proposition 42.17 shows that the inverse image in S of a prime over it in $H^\sharp$ is exactly P_1. Its contraction to H is therefore determined uniquely. These conclusions also give

$$L \subseteq K_0 \subseteq P_1.$$

Consequently $H^\sharp$ has an equivariant surjection to the domain D_1, and its kernel on H is Z. Taking even and odd eigenspaces is exact by the projectors $(1 \pm \sigma)/2$, with σ the involution. It follows that the induced maps $H/Z \to D_1^+$ and $N \to D_1^-$ are surjective and the first is injective.

For injectivity on N, every odd element of S lies in P_0, since S/P_0 is entirely even. An odd element mapping to zero in D_1 therefore lies in $P_0 \cap P_1 = L$, hence is already zero in $H^\sharp$. Every odd element of $H^\sharp$ has an odd lift by averaging. This proves the second isomorphism. For the

product ideal, an element of J lifts to the ideal $(Y_iY_j : 1 \le i, j \le b)$ in S: write it as a sum of products of the H-generators n_i, n_j with scalars in H, and lift those scalars. The lift belongs to P_0, so the same intersection argument gives injectivity on J. Its image is the indicated ideal of the invariant ring since all even coefficients lift to H.

The nonzero module D_1^- is torsion-free over D_1^+ because it lies in the domain D_1, and Y_1 has nonzero image in it. Its annihilator over H is precisely Z. It has rank one. Indeed division by a fixed nonzero odd element takes every odd element to an invariant fraction. Such fractions are in $\mathrm{Frac}(D_1^+)$: for an invariant fraction a/c multiply numerator and denominator by $\sigma(c)$. Both become invariant and the denominator is nonzero. This bounds the rank above by one and the nonzero element gives equality. □

Corollary 42.21 (An annihilator acting integrally on both off-diagonal blocks)**.** *Every element of the annihilator in H of the completed patched module $\mathcal{P}_H$ kills N and its product ideal J. More precisely $ZJ = 0$ and $Z \cap J = 0$.*

Proof. The prime Z was obtained in the support of $\mathcal{P}_H$, so the annihilator of this finite module is contained in Z. The theorem says $ZN = 0$. For an element $h \in Z$ and any $n, n' \in N$, associativity in $H^\sharp$ gives $h(nn') = (hn)n' = 0$. This kills all of J. Injectivity of $J \to H/Z$ proves the intersection assertion. □

The corollary gives an equality of annihilators for the off-diagonal module over the completed patched ring and an inclusion for the patched Hecke module. Passage to the original fixed weights and evaluation of the signed integral periods remain necessary to obtain the two inequalities in Section 6.6.

42.7 The unramified fiber and the integral extension module

Continue at a selected curve with a character and a plane. Let $\mathfrak{a} = (Y_{v,i})$ in the patched ring H; the new inertia operators are $1 + Y_{v,i}$. The symbols $Y_1, \dots, Y_b$ in the incidence presentation still denote the generators mapping to N. We now set all the new inertia operators to one. Our first task is to identify the resulting ring, including its nilpotents.

Lemma 42.22 (The ordinary ring after removing the auxiliary inertia)**.** *The map to the ordered base induces an isomorphism of complete local rings*

$$H/\mathfrak{a}H \;\simeq\; H_0, \qquad H_0 = \widehat{P_{\mathfrak{r}}}.$$

Here P includes the coefficient extension and parameter of Section 42.2.

Proof. Use the original complete presentation ring over $O'[[s]]$ before discriminant localization and diamond completion. Impose the scalar relations defining T and all $Y_{v,i} = 0$ there, and denote the quotient by B. The scalar relations are the trace and character expressions in group elements on the fixed presentation, so have lifts in this ring. Let $\mathfrak{t}$ be the curve prime of B and $S_0 = B_{\mathfrak{t}}$. The discriminant f is a unit in S_0. Adic completion modulo its defining ideal agrees with the quotient before completion in a Noetherian ring. Hence localizing $T/\mathfrak{a}$ at the curve gives exactly S_0. The original ordered base data give a surjection $B \to P$, taking determinants from Γ_0^0 and choosing ordered roots. Surjectivity follows as in Section 42.2 from generation by traces, characters, s, and the roots.

Over B form the finite Cayley–Hamilton algebra with the matrix words and their adjoints as in Lemma 42.8. Call it C_B. Extend scalars first to S_0. At a new place the differences of

the $U_i = U_{v,i}$ are units and $\prod_i(\phi - U_i) = 0$ in this algebra. Polynomial interpolation therefore gives orthogonal idempotents

$$e_i = \prod_{j\neq i}(\phi - U_j)/(U_i - U_j), \qquad \sum_i e_i = 1, \qquad \phi e_i = U_i e_i.$$

The second matrix word gives $e_i(\tau - 1) = 0$ for each i. Adding proves $\tau = 1$. Also e_i has determinant degree one on its corner, since its specialization in the semisimple curve representation is the projector onto a simple root. The spectrum of the local ring S_0 is connected, so the corner degree equals this residual degree by the corner lemma for determinants. Multiplicativity on orthogonal corners yields

$$D(X - \phi) = \prod_i (X - U_i) \quad \text{in } S_0[X].$$

Adjunction also kills the conjugate inertia difference. The multiplier at the auxiliary inertia is one, as it comes from the unramified base. [2]

For descent through a closed normal subgroup it is important to retain the original topology. Take $B' = B/\ker(B \to S_0)$ and let C' be the image of C_B in $C_B \otimes_B S_0$. These are quotients of the original complete finite-residue ring and its finite module by closed submodules. Thus C' has its compact quotient topology and the continuous group map still takes values in its units. It kills the new inertia and its conjugates by the preceding computation. By continuity it kills their closed normal closure, which is the full kernel over Γ_0^0 by Theorem 41.30. The determinant descends through C' over B': elements of the removed module kernel have zero trace against every element after mapping to S_0, hence already over B', which embeds there. The trace argument with Newton's identities proves descent of the polynomial law. The remaining local characters, old inertia and ordinary words give the original ordinary problem over B', with the prescribed coefficients and s. The ordered polynomial identity holds over B' as well by its injection into S_0. Universality of the base ring and of its monic splitting algebra gives a continuous map $P \to B'$; the reductions of the roots pick the indicated local factor. Its image contains all the group traces, local characters and root coordinates, because the group has descended and the Frobenius markings at the base are the specified s_v. The image is compact and closed, hence the map is surjective.

To check the two directions after localization, put $P' = P/\ker(P \to P_{\mathfrak{r}})$. The map $B \to P$ induces $B' \to P'$, since elements outside $\mathfrak{t}$ map outside $\mathfrak{r}$. The composite $P \to B' \to P'$ is the original quotient: the continuous maps agree on all the generators just listed. The prime in B' defined by the curve contracts to $\mathfrak{r}$ in P by the same evaluations. Thus the surjection $P \to B'$ gives a surjection $P_{\mathfrak{r}} \to S_0$ whose composite with the original $S_0 \to P_{\mathfrak{r}}$ is the identity on $P_{\mathfrak{r}}$. These are isomorphisms. Completing at the curve commutes with the quotient by the finitely generated diamond ideal. This proves the lemma. □

Proposition 42.23 (Base change of the ordinary Galois algebra)**.** *With C_0 the ordinary base group algebra quotient of Section 42.5, one has*

$$C/\mathfrak{a}C \;\simeq\; C_0 \otimes_{R_0} H_0.$$

Let N_0 denote the corner between the chosen primitive idempotents in its two blocks. Then

$$N_0 \simeq N/\mathfrak{a}N \simeq D_1^-/\mathfrak{a}D_1^-.$$

The action in the last quotient is through $H \to D_1^+$ of Theorem 42.20.

Proof. In the notation of the lemma $C_B \otimes_B H_0 = C/\mathfrak{a}C$, since both are formed by scalar extension of the same initial finite Cayley–Hamilton algebra and quotient by the scalar extensions of the same matrix relations. There is a map to $C_0 \otimes_{R_0} H_0$ by pullback from the base group: there the new inertia is one, the first auxiliary word is Cayley–Hamilton and the second is zero. For the reverse map use the continuous representation in C' constructed over B'. Its determinant is the base determinant by the map $R_0 \to B'$; its ordinary matrix words, old inertia and adjoints are zero. Thus the universal continuous Cayley–Hamilton group algebra on Γ_0^0, with these matrix relations imposed, maps to C'. One can form this map over the finite quotients of the compact algebra and take their inverse limit. After localizing at the curve and extending to H_0, its target is $C_B \otimes_B H_0$; the original module kernel defining C' vanishes on localization. The two maps over H_0 are mutually inverse on all scalar and group generators. These span the finite algebras, proving the identity. Taking the two primitive corners commutes with the scalar extension, since multiplication by the left and right idempotents is a module projector. The remaining assertion follows from the identification of N in Theorem 42.20. □

Theorem 42.24 (The matrix and the length at a reducible specialization)**.** *Let B_1 be an H_0-algebra in which the product ideal J has zero image. Let A_{B_1} be the b-by-b matrix with entries the images of $x_{ij} - w_{ij}$ from Theorem 42.15. There is a presentation*

$$B_1^b \xrightarrow{A_{B_1}^{\mathsf{T}}} B_1^b \longrightarrow N_0 \otimes_{H_0} B_1 \longrightarrow 0.$$

In particular, if B_1 is a DVR with normalized valuation v and the matrix has full rank, the length of this module is $v(\det A_{B_1})$. If its rank over the fraction field is $r < b$, its free rank is $b - r$ and its torsion length is the minimum valuation of a nonzero r-by-r minor, with value zero for $r = 0$.

Proof. Use the formal coordinates of Proposition 42.17 chosen equivariantly as in Section 42.6. Put $S' = \mathcal{O}[[A_{ij}, u_1, \dots, u_{D-b^2}]]$ for the ring of coordinates excluding the odd variables. Then $S = S'[[Y_1, \dots, Y_b]]$; the involution negates every Y_j and fixes S'. Write S^- for its odd part. By Lemma 42.19, the kernel of $S^- \to N \simeq D_1^-$ is precisely the odd part of $L = (AY)$. Indeed an odd element belongs to $(Y_1, \dots, Y_b) = P_0$, and $P_0 \cap P_1 = L$. Furthermore, every odd power series of Y-degree at least three belongs to $\sum_{i,j} Y_i Y_j S^-$. This follows by partitioning its monomials among the finitely many possible first two factors. The images of those series in N are exactly JN, since the products $n_i n_j$ generate J and every element of N has an odd lift. The ring $S' \to H/J$ is surjective: a scalar in H has an even lift in S, and all positive Y-degrees of that lift are divisible by products $Y_i Y_j$ with even coefficients.

It follows that computing N/JN over S' requires only the terms of Y-degree one. They form the free module on $Y_1, \dots, Y_b$. The relations in this module are precisely the rows of A: each generator of L is a row times Y, homogeneous of degree one over S', and in such a product only the constant coefficient of its multiplier contributes to degree one. Conversely every S'-linear combination of the rows is an actual relation in L. This proves the presentation over S'. The map $H \to B_1$ factors through H/J. The identifications above therefore give

$$N_0 \otimes_{H_0} B_1 = (N/JN) \otimes_{H/J} B_1 = (N/JN) \otimes_{S'} B_1.$$

Right exactness of tensor product yields the claimed matrix over B_1, with the indicated transpose because the relations were rows. Over a DVR its Smith invariant factors give the free rank and its torsion length. The product of the nonzero invariant factors generates the ideal of the maximal nonzero minors, which gives exactly the valuation formulas stated. □

Here the length is computed for the ordinary off-diagonal corner at a specified specialization. Applying this formula to the strict cohomology groups over the fields E_x requires an integral comparison with that corner and uniform estimates for its minors at the prescribed arithmetic weights. The period comparison in Section 6.6 also remains open.

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