# The generalized Riemann hypothesis for Dirichlet $L$-functions

27 November 2026 · Edition 930

## Volume VII

# Auxiliary fields and ordinary cuspidal families

**Abstract**

An auxiliary quadratic field supplies the characters for ordinary cuspidal congruences. We follow these congruences along cyclotomic layers and recover strict classes from the traces of the three-character deformation.

The guide in Volume I lists all twenty-seven volumes.

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# 36 Auxiliary fields and cuspidal congruences

## 36.1 One auxiliary level and a dual cohomology class

**Proposition 36.1** (An ordinary cuspidal lift with one new prime)**.** *Use $h, k, N, \chi, \ell, a$ from Lemma 21.19. Suppose $k > 2$ or $\chi \neq 1$. Let $s \equiv 1 \bmod h\ell^a$ be prime, and let $\xi$ be any character of order $\ell$ modulo $s$. For every $m \geq 0$ there is an ordinary cuspidal newform of weight $k' = k + m(\ell - 1)$, level $Ns$ and character $\chi\xi$, with $a_p \equiv 1 + \chi(p)p^{k-1} \pmod{\mathfrak{m}}$ for every $p \nmid hs$, including $p = \ell$. On the nonzero large-prime strict summands in Corollary 21.8 these alternatives exhaust the cases on every sufficiently distant interval.*

*Proof.* The difference $E_k(1, \chi)(z) - E_k(1, \chi)(sz)$ in Lemma 21.19 already gives a residual cusp form at the single auxiliary prime $s$. The constant-term calculation there uses only that prime. The coefficient at the first power of the Fourier variable is one. Multiply by the $m$-th power of the normalized level-one Eisenstein series of weight $\ell - 1$. Its constant term is one, and its positive coefficients are integral multiples of $2(\ell - 1)/B_{\ell-1}$. The von Staudt–Clausen theorem gives $v_\ell(B_{\ell-1}) = -1$. Thus the residual form of weight $k'$ is cuspidal with the same Fourier expansion. It has the stated Hecke eigenvalues because $p^{k'-1} \equiv p^{k-1} \bmod \ell$ for $p \neq \ell$, and both powers vanish modulo $\ell$ at $p = \ell$. Work on the compact connected modular curve of level $\Gamma(h) \cap \Gamma_0(s)$ over the coefficient DVR. With the notation of Theorem 21.20, let $\mathcal{B} = \omega^{k'}(-C) \otimes L_\xi$, where $C$ is the cusp divisor. The same width computation for this squarefree level gives the torsor defining $L_\xi$. Its reduction is trivial, equivariantly for the level-$h$ changes represented by matrices congruent to the identity modulo $s$.

Let $G$ be the finite group of level-$h$ changes coming from $\Gamma_0(Ns)$. The $\chi$-idempotent exists over the DVR because $|G| < \ell$. With $\pi$ a uniformizer, the obstruction to lifting a residual section of $\mathcal{B}$ lies in $H^1(\mathcal{B})[\pi]$, by the cohomology sequence of multiplication by $\pi$. Also $H^1(\mathcal{B})/\pi$ embeds equivariantly in $H^1(\mathcal{B}_{\mathrm{sp}})$. The latter is dual to $H^0(\omega^{2-k'})$ on the special fiber by Kodaira–Spencer and Serre duality. For $k' > 2$ it vanishes. For $k' = 2$ it is the trivial representation, by the action on constants on the connected fiber. If $\chi \neq 1$, it is nontrivial on $G$ even after reduction. Indeed the lower-right entries realize every unit modulo $N$ by lifting diagonal matrices modulo $h$ and the identity modulo $s$, and $\chi$ has order prime to $\ell$. In both cases Nakayama's lemma gives $H^1(\mathcal{B})^\chi = 0$, since this module is finite over the DVR. Our residual eigenvector of character $\chi$ therefore lifts in that character space. The torsion-free Hecke algebra argument of Theorem 21.20 supplies a characteristic-zero newform. The primitive conductor of $\chi\xi$ gives exact level $Ns$, and the congruence at $\ell$ gives ordinarity.

In the arithmetic application $k = r + 1$ and $\chi$ induces $\eta\alpha$, with $\eta$ the character of $\mathbb{Q}(i)$. The omitted case is $r = 1, \alpha = \eta$. Its strict group injects after duality into the $\ell$-primary part of the fixed finite group $K_2(\mathbb{Z}[i])$, as in the proof of Corollary 32.20. It vanishes above a fixed prime bound, whereas the large-prime cutoff tends to infinity. □

The requirement of arbitrarily large weights in this residue class occurs in the supplied source at `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/AutomorphyLifting/PrimeIndexConstruction/MixSeedPAlt.lean`, line 55. The congruences here keep the auxiliary prime and the tame level fixed as the weight increases.

**Lemma 36.2** (An integral class for the dual character)**.** *For a datum in Proposition 36.1 from the common interval, put $T = O(\kappa_\ell^{r+1}\alpha)$, where $O$ is the minimal unramified coefficient ring and $k_\ell$ its residue field. There exists $d \in H^1(\mathbb{Q}, T)$, unramified at every finite prime other than $\ell$, whose reduction in $H^1(\mathbb{Q}, T/\ell)$ is nonzero.*

*Proof.* Let $S$ contain the infinite place, $\ell$, and the prime divisors of the conductor of $\alpha$. Write $G_S$ for the Galois group unramified outside $S$ and $V = T \otimes_O \mathrm{Frac}(O)$. Complex conjugation acts by $(-1)^{r+1}(-1)^r = -1$. The global Euler characteristic formula over $\mathbb{Q}$ gives $\dim H^1(G_S, V) = 1 + \dim H^0(G_S, V) + \dim H^2(G_S, V)$. Here the correction one is the dimension of the minus space at the real place. The groups $H^i(G_S, T)$ are finitely generated over $O$ and give the cohomology of $V$ upon inverting $\ell$. These are the usual local and global Euler characteristic and finiteness theorems for number fields. [1, I, Sections 2, 4 and 5] Equivalently the rank formula follows by taking the Euler characteristics of $T/\ell^n T$, dividing their logarithmic lengths by $n$, and letting $n$ tend to infinity. The long exact sequence for multiplication by $\ell^n$ gives the ranks in this limit. Choose a class nonzero modulo $\ell H^1(G_S, T)$. Its reduction is nonzero by the same long exact sequence, and inflation from $G_S$ preserves that fact.

At a finite $p \in S$ other than $\ell$, both $V$ and $V^*(1)$ have zero invariants. If $\alpha$ is ramified this follows on inertia. Otherwise the Frobenius eigenvalues are $p^{r+1}\alpha(p)$ and $p^{-r}\alpha(p)^{-1}$, neither of which is one. Local duality and the local Euler characteristic give $H^1(\mathbb{Q}_p, V) = 0$. On inertia the action on $T$ has finite order prime to $\ell$. Averaging lifts an invariant of $V/T$ to an invariant of $V$, so $H^1(I_p, T) \to H^1(I_p, V)$ is injective. The integral class is thus unramified at $p$, and outside $S$ this follows from its construction. □

## 36.2 One auxiliary quadratic field and the common Hecke divisor

Keep the polynomial-conductor intervals of Theorem 19.15 and their actual fields $E_x$. Write $N_*$ for the modulus of $E_*$ and set $t_* = 2 + \lceil V^{602} \rceil$. The rank-three comparison uses an imaginary quadratic field whose discriminant is coprime to the level. The modulus $N_*$ is even. We first give one choice that works for an entire sequence of intervals.

**Proposition 36.3** (The quadratic field for the large coefficient primes)**.** *Fix a prime $s \equiv 3 \bmod 4$ prime to $2F$, where $F$ is the conductor of $\psi$, and put $K = \mathbb{Q}(\sqrt{-s})$. Choose two distinct primes $u_1, u_2 \geq 7$ split in $K$ and prime to $30Fs$. These three choices can be kept fixed. On every sufficiently distant interval, $\gcd(su_1u_2, N_*) = 1$. Put $P(x) = \exp(x^{\varepsilon}/12)$. For each $x$ in the interval, $1 \leq r < t(x)$, $\ell > P(x)$ and each complementary character $\alpha$ of $E_x$, the following assertions hold.*

*The order of $\alpha$ is prime to $\ell$, its local character at $\ell$ on inertia is trivial, and $\alpha(-1) = (-1)^r$. We have $\ell \nmid s|\mathrm{Cl}(K)|$ and $\ell - 1 \nmid 2r$. The intersection of the kernels of $\bar{\kappa}_\ell$ and of the finite Galois character $\alpha$ contains an element acting nontrivially on $K$. In particular these data satisfy the arithmetic hypotheses of the generic rank-three comparison in* `Development/AutomorphicArithmetic/AutomorphicVarieties/HeightAndVanishing/Counts.lean`*, line 9.*

*Proof.* Dirichlet's theorem gives the two auxiliary primes, for example in the progression 1 mod $30Fs$. They split because the quadratic character of $K$ has conductor $s$. The primes $q, d_0$ in the modulus $N_*$ tend to infinity with the initial scale. Its other prime factors divide $2F$, proving the assertion about coprimality. The bound $N_* \leq V^{C_{\psi,\varepsilon}}$ holds on these intervals. Thus $P(x) > \max(N_*, 2t_* + 1, s|\mathrm{Cl}(K)|)$ uniformly for large $V$. The character group of $E_x$ has order at most $N_*$, its characters are unramified at $\ell$, and the parity is the definition of the complementary component. The inequality $0 < 2r < \ell - 1$ gives the claimed separation of the inertia characters after cyclotomic twisting. Finally inertia at $s$ acts trivially on both $\mu_\ell$ and $E_x$, since $s \nmid \ell N_*$. It acts nontrivially on $K$, whose discriminant is $-s$. This supplies the required element, and also shows that $K \cap E_* = \mathbb{Q}$. □

Regard each character of $E_*$ as a character modulo $N_*$. For $1 \leq r < t_*$ let $H$ range over the distinct kernels of its geometric characters of sign $(-1)^r$, regarded as subgroups of $(\mathbb{Z}/N_*\mathbb{Z})^\times$.

The characters of kernel $H$ form one rational orbit $\mathcal{A}_H$. Indeed they are the faithful characters of the same cyclic quotient, permuted transitively by raising their values to exponents prime to its order. Every character in such an orbit factors through $E_*$. Let $\mathcal{P}$ consist of the primes $u \geq 7$ split in $K$ and prime to $30sN_*$. Fix a place above each $u$ and use Theorem 30.11 at level $N_*$. Denote the absolute value of its nonzero integer by $I_{u,H,r}$. Its diagonal trace is denoted by $d_\alpha(u)$ and its labelled multiset of automorphic traces by $\mathcal{T}^*(u)$.

**Theorem 36.4** (The common divisor of the good Frobenius tests)**.** *For each $H, r$ there is a positive integer $J_{H,r}$ with*

$$v_\ell(J_{H,r}) = \min_{\substack{u \in \mathcal{P} \\ u \neq \ell}} v_\ell(I_{u,H,r}).$$

*One finite subset of $\mathcal{P}$ realizes all these minima, simultaneously for the entire interval and every prime $\ell$. The product $J_* = \prod_{r,H} J_{H,r}$ satisfies*

$$\log J_* \ \ll_{K,u_1,u_2} \ [E_* : \mathbb{Q}]\, t_* \, [N_*(t_*+2)]^{A'} \ \ll_{\psi,\varepsilon} V^C$$

*for a constant $C$ depending on the fixed choices and $\varepsilon$.*

*If $\ell \nmid J_{H,r}$ and $\alpha \in \mathcal{A}_H$, there is no characteristic-zero Galois specialization whose traces are congruent to the diagonal modulo the maximal ideal at every good split Frobenius away from $\ell$ and belong there to the specified automorphic lists. In particular this excludes all such specializations of the split residual representation, including when several labels have identical eigenvalues at one operator.*

*Proof.* The minimum exists in the nonnegative integers, since $\mathcal{P}$ contains the two distinct fixed primes. For every $\ell$ it is at most $v_\ell(I_{u_1,H,r} I_{u_2,H,r})$, since at least one of the two primes differs from $\ell$. Thus it has finite support. Forming the product of the corresponding prime powers gives $J_{H,r}$. Start the finite test set with $u_1, u_2$. Only the finitely many primes dividing $I_{u_1,H,r} I_{u_2,H,r}$ can require any further test. For each of them add a prime $u \neq \ell$ attaining the minimum. The union over $r, H$ is finite. The set may depend on the interval. Its largest prime enters none of the displayed estimates.

Each $J_{H,r}$ divides the product at the two fixed primes. Sum the logarithmic bound of Theorem 30.11 at these two primes. For fixed $r$, the sum of the cardinalities of the orbits is at most $[E_* : \mathbb{Q}]$, since they partition the selected characters. Summing up to $t_*$ proves the asserted size.

For the exclusion, choose a minimizing $u \neq \ell$. The valuation of $I_{u,H,r}$ is zero. By construction every local difference $b - d_\alpha(u)$ for a label $b \in \mathcal{T}^*(u)$ is integral at $\ell$, and their product over the orbit and labels differs from $I_{u,H,r}$ by an $\ell$-adic unit. Consequently each difference is a unit. Any congruence at every split Frobenius would give a positive valuation at this particular $u$, a contradiction. An integral lift of the split residual representation satisfies those congruences wherever its Frobenius trace is on the list, by reduction of the trace. □

**Corollary 36.5** (The length inequality on the period interval)**.** *Suppose that for each contributing triple $\ell, r, H$ with $\ell > P(x)$ for some $x$ in the interval the following comparison holds. For every $u \in \mathcal{P}$, $u \neq \ell$, and every $\alpha \in \mathcal{A}_H$, there is a submultiset $T_{\ell,r,\alpha,u} \subseteq \mathcal{T}^*(u)$ such that, with one embedding into $\overline{\mathbb{Q}}_\ell$ for all factors at $u$,*

$$\sum_{\alpha \in \mathcal{A}_H} \frac{\lambda_{\ell,r,\alpha}^{N_*}}{e_\alpha} \ \leq \ 2 \sum_{\alpha \in \mathcal{A}_H} \sum_{b \in T_{\ell,r,\alpha,u}} v_\ell(b - d_\alpha(u)).$$

*Here a contributing triple has $1 \le r < t(x)$ and the characters of $H$ occur on at least one such $E_x$. Then, uniformly on the original interval,*

$$\sum_{\ell > P(x)} (\log \ell) \sum_{1 \le r < t(x)} \operatorname{length}_{\mathbb{Z}_\ell} M_{\ell,r}(E_x) \;\le\; 2 \log J_*.$$

*Independently of the comparison hypothesis, both auxiliary primes and the terminal prime of Theorem 22.12 can be chosen prime to $J_*$, on the same fields and with the same lower count of candidates.*

*Proof.* Theorem 30.11 bounds the right side of the assumed orbit inequality by $2v_\ell(I_{u,H,r})$ for every allowed $u$. Taking the minimum bounds the orbit length by $2v_\ell(J_{H,r})$. At any particular $x$, the characters from $E_x$ form a union of whole rational orbits when inflated to $E_*$. Their scalar groups are evaluated at the single modulus $N_*$. Since $\ell > P(x)$ is prime to $[E_x : \mathbb{Q}]$, Theorem 22.17 gives their exact total normalized length on $E_x$. Adding the nonnegative terms over $\ell, r, H$ counts each orbit and twist at most once. For a positive integer $J$, the sum of $v_\ell(J) \log \ell$ over all primes is $\log J$, which proves the bound.

There are at most $\log J_* / \log 2$ distinct prime divisors of $J_*$. This is polynomial in $V$. The proof of Theorem 22.11 permits deletion of this many primes before making any of the three choices. Its candidate sets have sizes $U^{1-o(1)}$ or $Y^{1-o(1)}$, with both $\log U$ and $\log Y$ growing faster than $\log V$. The same deletion is valid after the first auxiliary choice under the pairwise exclusions in that proof. It preserves the critical conditions in Theorem 22.12. □

The single-character comparison sought here is formulated with Galois traces in `Developmen t/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/KatoUse.lean`, line 55. The exclusion theorem determines where the listed systems can be congruent. The inequality in Corollary 36.5 further requires that those congruences bound every strict extension class with its full length. This implication remains open in the present argument.

## 36.3 Lower triangular orthogonal lifts and strict cohomology

Fix a period field $E_x$ on the intervals of Section 6.6. Write $N_x$ for its cyclotomic modulus. Throughout this subsection $\ell \nmid 2N_x[E_x : \mathbb{Q}]$, $1 \le r < \lfloor x^{401} \rfloor$, and $\alpha$ is a character of $\mathrm{Gal}(E_x/\mathbb{Q})$ with $\alpha(c) = (-1)^r$. Put $\Psi = \alpha \kappa_\ell^r$, $\vartheta = \Psi^{-1}$, and $\eta = \chi_{-4}$. Take the unramified coefficient ring $O$ generated over $\mathbb{Z}_\ell$ by the values of $\alpha$. Write $k = O/\ell O$ and $D = \mathrm{Frac}(O)/O$. Assume that $\bar{\Psi}(I_\ell)$ has order greater than two, and choose $g_* \in I_\ell$ with $\lambda = \Psi(g_*)$ of residual order greater than two. Every complementary triple with $\ell > \exp(x^{\varepsilon}/12)$ satisfies these hypotheses on the sufficiently distant intervals.

Let $S_0 = S(\vartheta) \subset H^1(\mathbb{Q}, D(\vartheta))$ be the group unramified away from $\ell$ and zero at $\ell$. Use indices $0, 1, 2$. Denote the antidiagonal matrix with entries one by $J_3$, and put

$$d(g) = \mathrm{diag}(\Psi(g), 1, \Psi(g)^{-1}), \qquad X = E_{10} - E_{21}, \qquad n(z) = 1 + zX + \tfrac{1}{2} z^2 X^2.$$

For a complete Noetherian local $O$-algebra $A$ with residue field $k$, define $\mathcal{B}^-(A)$ to be the continuous lower triangular lifts $M : G_{\mathbb{Q}} \to \mathrm{SO}(J_3, A)$ of $\bar{d}$ with diagonal $d$, equal to $d$ on $G_{\mathbb{Q}_\ell}$. At each finite $p \neq \ell$ require $M$ to kill $I_p \cap \ker \alpha \cap \ker \eta$. Matrices are in the marked basis. These conditions fix a representative of each root cocycle, as the proof shows.

**Theorem 36.6** (The integral ring of lower triangular orthogonal lifts)**.** *If $S_0 \simeq \bigoplus_{i=1}^{s} O/\ell^{a_i} O$, with all $a_i > 0$, the functor $\mathcal{B}^-$ is represented by*

$$R^- \simeq O[[T_1, \dots, T_s]] / (\ell^{a_1} T_1, \dots, \ell^{a_s} T_s).$$

*For $\ell > \exp(x^{\varepsilon}/12)$ it is the closed subfunctor of the ordinary marked ring of Theorem 32.19 on which the matrix part is lower triangular and preserves $J_3$. The characteristic polynomial on $G_{\mathbb{Q}(i)}$ is the split characteristic polynomial over the whole ring $R^{-}$.*

*Proof.* For a lower triangular matrix preserving $J_3$ with determinant one, write $a$ for its upper left entry and $az$ for the entry immediately below. Pairing its first and last columns gives last diagonal entry $a^{-1}$, and then its determinant gives middle entry one. Pairing its first column with its second and with itself gives the other two lower entries $-z$ and $-az^2/2$. Thus the matrix is $n(z)\,\mathrm{diag}(a, 1, a^{-1})$. The identities $X^3 = 0$ and $d(g)Xd(g)^{-1} = \vartheta(g)X$ give $n(z)n(z') = n(z + z')$. With fixed diagonal, the group law is exactly

$$z(gh) = z(g) + \vartheta(g)z(h), \qquad z(g) \in \mathfrak{m}_A.$$

Since $\vartheta(c) = 1$ and two is a unit, $z(c) = 0$. At $p \neq \ell$ the specified inertia quotient has order prime to $\ell$. Vanishing on its kernel is equivalent to being a coboundary on inertia by finite averaging, first over finite coefficients. At $\ell$ a coboundary has the form $(\vartheta(g) - 1)t$. Its value at $g_*$ determines $t$ since $\vartheta(g_*) - 1$ is a unit. A global strict class therefore has a unique cocycle zero at $g_*$, and that cocycle is zero throughout $G_{\mathbb{Q}_\ell}$.

We give the integral representing module. For any finite $O$-module $W$, cocycles with these conditions factor, together with $\vartheta$, through one finitely generated profinite quotient $\Gamma$. Indeed, take the finite ramification set consisting of the primes over $2N_x\ell$ and infinity. In the group unramified outside it replace $\ker\bar{\vartheta}$ by its maximal pro-$\ell$ quotient. The kernel removed is characteristic in that subgroup. The pairs $(z, \vartheta)$ send it to the identity, since the additive finite module and the principal units are pro-$\ell$. The finite generation argument of Theorem 30.4 applies to this quotient. For generators $\gamma_1, \dots, \gamma_b$, map the free profinite group on their lifts to $O^b \rtimes O^{\times}$ by sending the $j$-th generator to $(e_j, \vartheta(\gamma_j))$. Write $L(w)$ for the first coordinate. Form $C$ from $O^b$ by dividing out the submodule generated by $L(w)$ for the relators and for the preimages of the local subgroups on which $z$ must vanish. These submodules of $O^b$ are closed. By composition, maps $C \to W$ are precisely the normalized cocycles with values in $W$. Conversely a cocycle gives such a map by its values on the generators.

For $W = D[\ell^j]$ these normalized cocycles are exactly $S_0[\ell^j]$. Indeed the invariants of $D(\vartheta)$ globally and at $\ell$ are zero by evaluation at $g_*$. On the other inertia groups the invariants are zero or divisible, since the inertial character has order prime to $\ell$. The multiplication sequence for $\ell^j$ therefore identifies the finite and divisible local conditions. Taking the union gives $\mathrm{Hom}_O(C, D) = S_0$. This group is finite by Theorem 25.1 and character descent. The structure theorem over $O$ now gives precisely the elementary divisors $\ell^{a_i}$ for $C$. Applying its representing property to $\mathfrak{m}_A/\mathfrak{m}_A^j$ and taking inverse limits gives $\mathrm{Hom}_O(C, \mathfrak{m}_A)$. Choosing generators of the cyclic factors of $C$ gives the asserted power series presentation, including the case $s = 0$.

It remains to identify the closed subfunctor in the marked ring. The conditions of lower triangularity and preservation of $J_3$ are closed matrix equations. The inverse-transpose involution $M \mapsto J_3 M^{-\mathsf{T}} J_3$ fixes every such matrix, so the matrix part is an ordinary homomorphism on $G_{\mathbb{Q}}$. Its first diagonal entry $a$ is a character reducing to $\bar{\Psi}$. The characteristic polynomial on inertia gives $a(g_*) = \lambda$, since its other two possible root factors are units there. Use the ordered product in (32.1) at $(s, g_*, g_*)$, with $s \in I_\ell$. Its first diagonal entry is $(a(s)-\Psi(s))(\lambda-1)(\lambda-\lambda^{-1})$. Thus $a = \Psi$ on inertia at $\ell$. On every other inertia group $a/\Psi$ factors through the prescribed prime-to-$\ell$ quotient and takes values in principal units, so it is trivial. Also $a(c) = \Psi(c) = 1$. Class field theory and the trivial ideal class group of $\mathbb{Q}$ give $a = \Psi$ globally. This argument works on all finite quotients of $A$ and hence on $A$.

In the resulting expression $M(g) = n(z(g))d(g)$, the lower adjacent mark is $(\lambda+1)z(g_*) = 0$, so $M(g_*) = d(g_*)$. Use (32.1) now at $(g_*, s, g_*)$. Its 10-entry is $(1 - \lambda)(\lambda - \lambda^{-1})M(s)_{10}$. Both

factors are units, giving $z(s) = 0$. For $g \in G_{\mathbb{Q}_\ell}$, evaluating the cocycle on $g_* g = g(g^{-1} g_* g)$ gives $z(g) = 0$. In the other direction every lift in $\mathcal{B}^-(A)$ has exactly the split local matrix at $\ell$, which lies on the specified component by Lemma 32.7. It obeys the inertia kernels and the marks, including $M(c) = 1$. Pairing it with $\eta$ gives the required marked polarized lift. Finally triangularity gives the characteristic polynomial directly over $R^-$. □

**Corollary 36.7** (Orders detected by lifting through finite coefficients)**.** *Let $A_j = O/\ell^{j+1}O$. For integers $n \geq 1,\ m \geq 0$,*

$$\#\operatorname{im}(\mathcal{B}^-(A_{n+m}) \to \mathcal{B}^-(A_n)) = |k|^{\sum_i \min(n, \max(a_i - m, 0))} = \#((\ell^m S_0)[\ell^n]).$$

*For a character summand retained in $U_{\ell,r}$ of Section 6.6, take $m = b_0$ when $\ell \leq P$ and $m = 0$ when $\ell > P$. Under the hypotheses of this subsection, the stabilized logarithm on the left is the full contribution of that residue character orbit to the logarithmic order of $R_{\ell,r}$.*

*Proof.* Each $T_i$ maps to the maximal ideal of $A_{n+m}$ and is killed by $\ell^{a_i}$. Its possible reductions in $A_n$ form the image of the ideal $\ell^{\max(1, n+m+1-a_i)}O$. Counting residues gives the displayed exponent, independently in every coordinate. Prime-to-degree descent for the character orbit identifies its strict group with the additive group of $S_0$ over the minimal ring $O$, as in Theorem 22.17. The averaging idempotent in Section 6.6 is zero on each retained orbit. Here $\ell \nmid [E_x : \mathbb{Q}]$ also ensures that the scalar local conditions at the conductor agree by the inertia calculation in Section 22.13. The quotient by the subgroup killed by $\ell^m$ is isomorphic to the image of multiplication by $\ell^m$. Letting $n$ exceed every $a_i$ proves the assertion, with the residue degree already included in $\log |k|$. □

In particular the special fiber of $R^-$ is the formal power series ring in $s$ variables over $k$. Every map into an $\ell$-torsion-free coefficient ring sends all $T_i$ to zero. Thus even all split Frobenius characteristic polynomials on this closed family are independent of the integers $a_i$. A bound for the orders requires arithmetic information on the integral relations or on their interaction with the other deformations.

## 36.4 Parity of orthogonal points after CM base change

Return to the large primes of Section 33.5. Write $H = G_{\mathbb{Q}(i)}$. For a characteristic-zero marked point, let $U$ be its representation on $H$. In the orthogonal case Lemma 33.14 gives a two-dimensional representation $\rho$ of $G_{\mathbb{Q}}$ with $U \simeq (\mathrm{Ad}^0 \rho)|_H$. At complex conjugation $c$, its determinant is 1 or $-1$. These are called the even and odd cases, respectively.

**Theorem 36.8** (Potential automorphy on the orthogonal locus)**.** *Let $U$ be absolutely irreducible with orthogonal monodromy at a characteristic-zero marked point. The following are equivalent.*

1. *There is a CM field $F \supset \mathbb{Q}(i)$ over which $(U \otimes \kappa_\ell^{-1})|_{G_F}$ is the representation associated to a regular algebraic conjugate self-dual cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_F)$, with the normalization of Corollary 33.6.*
2. *The two-dimensional lift $\rho$ is odd.*

*In the odd case one may take $F = \mathbb{Q}(i)$.*

*Proof.* Put $V = \mathrm{Ad}^0 \rho$, a representation over $\overline{\mathbb{Q}}_\ell$ with determinant one and $V \simeq V^\vee$. Theorem 33.13 gives connected monodromy $\mathrm{SO}_3$ on $H$. For every finite-index subgroup of $H$ its image is still Zariski dense in this group. Indeed the group is a finite union of translates of

the closure of the smaller image, and a connected algebraic group is irreducible. Thus $V|_{G_F}$ is absolutely irreducible for every $F$ in the first assertion.

Suppose such a field and cuspidal representation $\Pi$ exist. Write $F^+$ for its maximal totally real subfield and $\sigma$ for the nontrivial element of $\mathrm{Gal}(F/F^+)$. In the given normalization the unramified Satake multiset of $\Pi_w$ equals the eigenvalue multiset of $V$ at geometric Frobenius, after the fixed coefficient embedding. Indeed local reciprocity for the associated representation introduces $|\det|_w^{-1}$, multiplying the three roots by $\#k(w)$, which is exactly the factor from $\kappa_\ell^{-1}$. Since $V$ extends to $G_{F^+}$, these multisets agree at $w$ and $\sigma w$. They are stable under inversion and their products are one. Strong multiplicity one, including its rank-one case for the central character, therefore gives

$$\Pi^\sigma \simeq \Pi, \qquad \Pi^\vee \simeq \Pi, \qquad \omega_\Pi = 1.$$

By quadratic cyclic descent there is a cuspidal $\Sigma$ on $\mathrm{GL}_3(\mathbb{A}_{F^+})$ with base change $\Pi$. The descents differ by twisting with the quadratic character $\delta$ of $F/F^+$. In particular $\Sigma^\vee$ is either $\Sigma$ or $\Sigma \otimes \delta$. The pullback of $\omega_\Sigma$ under the idelic norm is $\omega_\Pi = 1$. Embedding an idele of $F^+$ into $F$ gives its square under this norm, so $\omega_\Sigma^2 = 1$. The second possibility would give $\omega_\Sigma^{-1} = \omega_\Sigma \delta^3$, contradicting the nontriviality of $\delta$. Hence $\Sigma$ is self-dual. Here we use cyclic base change and its description of the fibers, with the descent argument of Clozel and Rajan. [2, 3]

At a real place of $F^+$, base change restricts the real Weil parameter to $\mathbb{C}^\times$. Its infinitesimal exponents at the chosen complex embedding are therefore those of $\Pi$. They are distinct and integral since $\Pi$ is regular algebraic and the half-sum of positive roots for $\mathrm{GL}_3$ is $(1, 0, -1)$. Consequently $\Sigma$ is regular algebraic at every real place. Its associated semisimple Galois representation exists. After the same Tate normalization write it as $V_\Sigma = r_{\ell,\iota}(\Sigma) \otimes \kappa_\ell$. Unramified local compatibility and base change give the characteristic polynomials of $V|_{G_F}$ at all good places of $F$. Chebotarev density and the characteristic-zero semisimplification theorem identify $V_\Sigma|_{G_F}$ with $V|_{G_F}$, because the latter is irreducible. In particular $r_{\ell,\iota}(\Sigma)$ is irreducible as well.

Identify the two representations on $G_F$. At any $g \in G_{F^+}$ the two matrices implement the same conjugation on this subgroup's image. Their quotient centralizes that image and is scalar by Schur's lemma. These scalar ratios give a character of the group of order two $\mathrm{Gal}(F/F^+)$. At a real complex conjugation $c_v$ the traces of $V_\Sigma$ and $V$ thus agree up to sign. Taylor's Proposition 1 applies to $\Sigma$ over the totally real field $F^+$ in odd rank three, with the irreducibility just proved. It gives $|\mathrm{tr}\, r_{\ell,\iota}(\Sigma)(c_v)| \le 1$. The Tate twist only changes its sign there. [4]

If $\rho$ were even, its matrix at $c_v$ would be scalar, since $c_v$ is conjugate to complex conjugation in $G_\mathbb{Q}$, its square is one and its two eigenvalues have product one. Conjugation by this scalar is the identity on the three-dimensional space of trace-zero matrices. Thus $\mathrm{tr}\, V(c_v) = 3$, contradicting the inequality. This proves the necessity of oddness even when $F/\mathbb{Q}$ is not solvable.

In the odd case the proof of Theorem 34.9 identifies $\rho$ with the Galois representation of a weight-$(r+1)$ newform with nebentypus $\eta\alpha$. Its adjoint lift is cuspidal and stays cuspidal over $\mathbb{Q}(i)$, as proved there using the connected adjoint monodromy. Its normalized unramified parameters are $a/b, 1, b/a$ for the two Frobenius eigenvalues $a, b$ of $\rho$. The ordinary Tate normalization gives exactly $U \otimes \kappa_\ell^{-1}$. This supplies the required realization. □

Theorem 36.8 concerns irreducible characteristic-zero points. The lower triangular lifts of Theorem 36.6 give every finite layer of the complementary strict group at the split point. Controlling its length by congruences of irreducible systems requires information on the ambient integral relations at that point.

### 36.5 Recovering the strict group from integral Frobenius traces

Use the fields, marked ordinary ring, coefficient ring $O$, residue field $k$, and characters of Section 36.3. In this subsection $\ell > P = \exp(x^{\varepsilon}/12)$, the interval is sufficiently distant, and $1 \le r < \lfloor x^{401} \rfloor$. Take $K = \mathbb{Q}(i)$ or the fixed imaginary quadratic field of Proposition 36.3. In the latter choice all Frobenius tests below are on the field of Theorem 36.4. Write $\eta$ for the character of $K$, $H = G_K$, $\beta = \eta\Psi$, and $D = \mathrm{Frac}(O)/O$. Use the same marking, with this sign in the polarized group and this sign's kernel in the inertia conditions. At $\ell$ use the ordinary component through the split point defined in Section 32.4. Let $S$ consist of infinity and the primes dividing $2|\operatorname{disc} K|N_x\ell$. Write $G_S$ for the Galois group unramified outside $S$. Let $\mathbb{T}$ be the closed $O$-algebra generated by the traces on $H$ of the universal marked representation. Its evaluation at the split diagonal has kernel $\mathfrak{a}$. Since $K$ is fixed, $\ell$ is prime to its discriminant, class number and number of roots of unity on the indicated intervals.

**Lemma 36.9** (An integral upper extension)**.** *If $r \ge 2$, or if $r = 1$ and $\eta\alpha \ne 1$, there is an $O(\beta)$-valued continuous cocycle $b$ on $G_S$, unramified as a class at every finite $p \ne \ell$, whose class modulo $\ell$ is nonzero. When $r = 1$, it can be chosen to come from local units after the unramified extension at $\ell$ killing $\eta\alpha$. In the remaining case $r = 1$, $\alpha = \eta$, the group $S_0 = S(\Psi^{-1})$ is zero at all these primes on all sufficiently distant intervals.*

*Proof.* First take $r \ge 2$ and set $V = \mathrm{Frac}(O)(\beta)$. The global Euler characteristic over $\mathbb{Q}$ gives $\dim H^1(G_S, V) - \dim H^2(G_S, V) = 1$, since $\beta(c) = -1$ and $H^0(G_S, V) = 0$. This is the rational form of the finite-coefficient Euler characteristic used in Section 32.5, obtained by applying it to the quotients of the lattice by $\ell^j$ and taking ranks. Integral cohomology is finitely generated and rationalizes to this cohomology. Its degree-one group is torsion-free because $H^0(G_S, k(\bar{\beta})) = 0$. Choose a primitive element in this positive-rank $O$-module. Its reduction is nonzero by the multiplication sequence for $\ell$.

At $p \ne \ell$, pass to a finite local extension killing the finite part of $\beta$. On inertia there a cocycle becomes a homomorphism to $O$. Its only possible contribution is from the pro-$\ell$ tame inertia quotient. For residue degree $f$, arithmetic Frobenius acts there by $p^f$ and on the coefficients by $p^{fr}$. These scalars differ, so the homomorphism is zero. One may take the local extension of degree prime to $\ell$, since the character has prime-to-$\ell$ order. Restriction on inertial cohomology is injective by corestriction, proving the required unramified condition.

If $r = 1$, put $\gamma = \eta\alpha$. It is even. For nontrivial $\gamma$, its character field in the sense of the fixed field of its kernel is a totally real abelian field $L$ of degree prime to $\ell$. Dirichlet's unit theorem identifies the real logarithms of its units, after tensoring with $\mathbb{R}$, with the augmentation subspace of the regular permutation representation. Hence the $\gamma^{-1}$-part of $\mathcal{O}_L^{\times} \otimes_{\mathbb{Z}} O$ has rank one. The character multiplicity follows over $\mathrm{Frac}(O)$ by extending scalars in the rational representation. This part is saturated in the same summand of the $S$-units, since the quotient of units of these two kinds embeds by valuations in a free divisor group. Kummer theory identifies the completed $S$-units with $H^1(G_{L,S}, \mathbb{Z}_\ell(1))$. In the inverse limit the class group contributes zero, since it is finite and its transition maps on torsion are multiplication by $\ell$. Twisting and descending by the integral character idempotent gives a primitive global class with all the stated local unit conditions. Units at $p \ne \ell$ give unramified Kummer classes, since extracting roots of $\ell$-power order from a unit is finite étale over the local integers. For $\gamma = 1$, $S_0$ is a character summand of the strict group over $K$ at the fixed twist $r = 1$. Proposition 11.4 gives finite prime support for its strict groups, and $P$ eventually exceeds every prime in this support. □

**Lemma 36.10** (Global diagonal corrections)**.** *For every finite $O$-module $W$, give $W(\eta)$ the sign action. In degree two the localization map for $G_S$ is injective. In degree one localization*

*is surjective onto*

$$\bigoplus_{p\in S,\ p<\infty} H^1(\mathbb{Q}_p, W(\eta))/H^1_{\mathrm{ur}}(\mathbb{Q}_p, W(\eta)),$$

*where the unramified group is the kernel of restriction to inertia.*

*Proof.* Decompose the underlying finite abelian group of $W$ into cyclic groups of order $\ell^j$. The Tate dual of one sign summand is $\mu_{\ell^j}(\eta)$. Consider a global dual class orthogonal to the unramified subgroups at the places in $S$. Restriction to $K$ is injective by corestriction. It gives a Kummer class $u$ in $K^\times/(K^\times)^{\ell^j}$ with $c(u) = u^{-1}$. At every nonsplit rational prime the unique prime above it therefore gives $2v(u) = 0 \bmod \ell^j$. At a split prime in $S$, local reciprocity pairs an unramified character with the valuation of a Kummer element. Orthogonality gives $v(u) = 0 \bmod \ell^j$ there. At split primes outside $S$ the Kummer class is unramified and the residue characteristic differs from $\ell$, giving the same divisibility. Thus $(u) = \mathfrak{b}^{\ell^j}$ for a fractional ideal $\mathfrak{b}$. Since $\ell \nmid |\mathrm{Cl}(K)|$, this ideal is principal, so $u$ is an $\ell^j$-th power times a unit. The units of $K$ are roots of unity of order prime to $\ell$. The Kummer class is therefore zero. Poitou–Tate duality now gives the two assertions. In degree two the kernel is dual to the everywhere locally trivial subgroup of this dual group. In degree one the orthogonal complement of global restrictions in the sum of the local groups is the image of global dual classes. Those annihilating all the unramified subgroups have just been shown to vanish. Modified cohomology at infinity is zero since $\ell$ is odd. [1] □

**Theorem 36.11** (The full complementary group in the trace cotangent module)**.** *Put* $C = \mathrm{Hom}_O(S_0, D)$*. For every triple in the indicated large-prime range there is a surjective augmented* $O$*-algebra map*

$$\mathbb{T} \longrightarrow O \oplus C, \qquad C^2 = 0.$$

*In particular* $\mathfrak{a}/\mathfrak{a}^2$ *surjects onto* $C$*, with all its elementary divisors. Let* $\mathfrak{d}$ *be the ideal of* $\mathbb{T}$ *generated by the differences of the traces of* $h$ *and* $h^{-1}$*, for* $h \in H$*. The map already sends* $\mathfrak{d}/\mathfrak{a}\mathfrak{d}$ *onto* $C$*. Its image is generated by these trace differences at good split rational primes, with either prescribed place over each.*

*Proof.* For $C = 0$ take split evaluation. Otherwise use the universal normalized cocycle $z : G_S \to C(\Psi^{-1})$ in the construction of Theorem 36.6. Its representing module is canonically $C$ by finite double duality. In particular $z(g_*) = 0$, $z|_{G_{\mathbb{Q}_\ell}} = 0$, and its values generate $C$ by the generator presentation there. This construction depends only on $\Psi$ and its strict local conditions. At a prime $p \neq \ell$ the normalized cocycle vanishes on the kernel of $\alpha$ on inertia, so changing $K$ imposes no new condition on $z$. Use $b$ from Lemma 36.9, subtracting a coboundary to arrange $b(c) = 0$.

Form the local coefficient algebra $A = (O \oplus C)[y]/(y^2)$, where $C^2 = 0$. This is complete and finite over $O$. For the two root matrices and their commutator use

$$X = E_{10} - E_{21}, \quad Y = E_{01} + E_{12}, \quad L = [X, Y] = \mathrm{diag}(-1, 2, -1).$$

The differential of the sign involution is $Z \mapsto -J_3 Z^{\mathsf{T}} J_3$. Combined with $d(g)$, it acts on $X, Y, L$ by $\Psi^{-1}, \beta, \eta$, respectively. Seek the matrix part of the polarized representation in the form

$$M(g) = \exp\bigl(z(g)X + yb(g)Y + yw(g)L\bigr)d(g), \qquad w(g) \in C.$$

The augmentation ideal $C + Oy + yC$ has cube zero, so $\exp T = 1 + T + T^2/2$. Multiplication of the two polynomials for $g, h$ shows that the required correction satisfies

$$w(gh) - w(g) - \eta(g)w(h) = \tfrac{1}{2}\bigl(z(g)\beta(g)b(h) - b(g)\Psi(g)^{-1}z(h)\bigr).$$

The right side is a continuous two-cocycle with sign coefficients. At $\ell$ it is zero. At each $p \in S$ away from $\ell$ both cocycles and their actions factor locally through the quotient by $I_p^0 = I_p \cap \ker \alpha \cap \ker \eta$. Indeed a coboundary on inertia vanishes on this kernel. The quotient has inertia of prime-to-$\ell$ order and procyclic residue quotient, so its second cohomology with these coefficients vanishes. There is consequently a local solution $w_p$ inflated from that quotient, and we take $w_\ell = 0$.

Lemma 36.10 gives a global solution $w$ and then a global one-cocycle to subtract so that $w - w_p$ is an unramified class at each finite $p \in S$. In particular its restriction to inertia is a coboundary. It vanishes on $I_p^0$ away from $\ell$, and on $I_\ell$, since the sign character is unramified at $\ell$. A global coboundary in the sign line sets $w(c) = 0$ without changing any of these restrictions. The matrices now kill the prescribed inertia kernels and have $M(c) = 1$.

At $\ell$ the matrices are upper triangular with the required inertial diagonal, since $z = 0$ locally. For $r \geq 2$ Theorem 32.11 identifies this ordinary condition with the selected local component. The local proof applies for either $K$. The sign is unramified, and the inertia characters on the Tate dual of the strictly upper entries are $\bar{\kappa}_\ell^{1-r}$ and $\bar{\kappa}_\ell^{1-2r}$, both nontrivial in this range. For $r = 1$, Theorem 32.15 describes the component as the locus where the Kummer valuation coordinates of the two adjacent classes vanish. The adjacent cocycle here is $ybY$. Use the coefficient submodule $Ay = Oy \oplus Cy$ of the adjacent module. The diagonal correction lies in $yC$ and acts trivially on $Ay$ since $y^2 = 0$, so the action on the two entries there is the original diagonal action. After unramified restriction killing the finite characters the two coordinates are images of unit Kummer classes. Valuation is functorial in the coefficients by Lemma 32.14. The local description there and in Theorem 32.15 uses separate coordinates for the split sign and the single unramified quadratic extension for the nonsplit sign. These are also the two possibilities for this $K$. It also remains detectable after this restriction, since the extension degree is prime to $\ell$ and it preserves the valuation of a uniformizer. Thus both valuations vanish.

The remaining marks set the two adjacent entries $01, 10$ in $M(g_*) - J_3 M(g_*)^{\mathsf{T}} J_3$ to zero. The lower mark already holds because $z(g_*) = 0$. Conjugate the polarized representation by $1 + yt(E_{01} - E_{12})$, with $t \in O$. This unipotent matrix is fixed by the sign involution and preserves the mark at $c$. The change in the upper mark is $yt(1 - \lambda)(1 + \lambda^{-1})$; at $g_*$ the only other nondiagonal term is already a multiple of $y$. The coefficient of $yt$ is a unit. This imposes the upper mark while preserving all local conditions. Traces on $H$ are unchanged by this conjugation.

Every trace on $H$ has its split constant term and a possible term in $yC$, since the two linear root matrices have zero diagonal. We prove that the trace differences span $yC$. Choose $a \geq 1$ with $\ell^a C = 0$ and let

$$N = H \cap \ker(\Psi \bmod \ell^a).$$

On $N$ the two cocycles with coefficients reduced modulo $\ell^a$ are homomorphisms. The values $z(N)$ span $C$. Indeed, after quotienting $C$ by their span, the cocycle factors through the abelian quotient $G_S/N$, where $N$ denotes its image in $G_S$. Commuting any element with $g_*$ and using $z(g_*) = 0$ shows that the resulting cocycle is zero, since $\Psi(g_*)^{-1} - 1$ is a unit. Likewise some $k_0 \in N$ has $b(k_0)$ a unit. Otherwise $\bar{b}$ would factor through the same abelian quotient. Its cohomology class there vanishes by commuting with the central element $g_*$ acting by $\bar{\beta}(g_*) \neq 1$, contrary to its choice.

For $h \in N$ take $h' = h(chc^{-1})$ and $k' = k_0(ck_0c^{-1})^{-1}$. On $N$, conjugation by $c$ fixes $z$ and negates $b$ mod $\ell^a$. The two pairs of values at $h'$ and $k'$ are thus $(2z(h), 0)$ and $(0, 2b(k_0))$, in the indicated coefficients. In $A/\ell^a A$ the split diagonal is one on $N$. For the commutator convention

$[g,h] = ghg^{-1}h^{-1}$, multiplying the two matrices and their inverses therefore gives

$$M([h',k']) = 1 + 4y\, z(h)b(k_0)L \pmod{\ell^a A}.$$

The terms in $yC$ of the two factors are central among these unipotent matrices and cancel. Multiplying by $M(g_*)$ and subtracting $\operatorname{tr} M(g_*)$ gives

$$\operatorname{tr} M([h',k']g_*) - \operatorname{tr} M(g_*) = 4y\, z(h)b(k_0)(2 - \lambda - \lambda^{-1})$$

in $A$ itself. Indeed the split constants cancel exactly on the commutator, and reduction modulo $\ell^a A$ is injective on $yC$. The final scalar $2 - \lambda - \lambda^{-1} = -(\lambda - 1)^2/\lambda$ is a unit. This proves that the image of $\mathbb{T}$ is $O \oplus yC$.

Set $f(h) = \operatorname{tr} M(h) - \operatorname{tr} d(h)$ on $H$. Before the final change of frame, the action of the marked $c$ on these matrices fixes $X, d(h)$ and negates $Y, L$ in the exponential. It therefore acts by the substitution $y \mapsto -y$. The same conjugation takes the trace of $M(h)$ to the trace of its inverse by the polarization. Since the split trace is invariant under inversion, this gives

$$f(chc^{-1}) = -f(h), \qquad \operatorname{tr}(M(h)) - \operatorname{tr}(M(h)^{-1}) = 2f(h).$$

These identities persist after the change of frame. The values of $f$ span $yC$ by the commutator calculation. The ideal $\mathfrak{a}$ maps into $yC$, whose square is zero. This proves both asserted module surjections.

Finally $f$ is continuous and conjugacy invariant on $H$ with finite target. Its fibers are unions of conjugacy classes in a single finite quotient of $H$, by compactness. Chebotarev over $K$ realizes each such class at good prime ideals of positive density, even after omitting any specified finite set. One may take ideals of degree one, since those of higher degree have density zero. The rational primes beneath them split in $K$. Choosing the other prime above one of them changes $f(h)$ to its negative, so every choice gives the same $O$-span. □

For a residue orbit $[\alpha]$, the quotient in Theorem 36.11 has logarithmic order $[k : \mathbf{F}_\ell] \log \ell \operatorname{length}_O S_0$. Thus the comparison uses the entire large-prime contribution of that orbit on $E_x$. Every self-dual characteristic-zero system kills $\mathfrak{d}$, since its trace agrees with the trace of its contragredient at every element. Theorem 36.11 locates the complementary length in the integral ideal of these differences. To use the characteristic-zero congruences one still needs to bound this quotient of the ideal, including its torsion at $\ell$.

## 36.6 The part of the strict order bounded by classical traces

Take the fixed field $K = \mathbb{Q}(\sqrt{-s})$, the modulus $N_*$ and the split primes $\mathcal{P}$ of Section 36.2. At $x \in [V, V^{3/2}]$ use the actual period field $E_x$. For each large-prime complementary triple take $\mathbb{T}$, $\mathfrak{a}, \mathfrak{d}$, and $C$ as in Theorem 36.11, and fix one of the resulting maps $\Phi : \mathbb{T} \to O \oplus C$. All length estimates below hold for every such choice. Write $J = \operatorname{Frac}(O)$.

Use one isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ at this prime. Consider the continuous geometric integral trace specializations of $\mathbb{T}$ with values in the integers of finite extensions of $J$. Retain those whose good Frobenius traces at degree-one places of $K$ agree with one fixed classical label of Section 30.4, at level $N_*$ and weight parameter $r$. Agreement includes every $u \in \mathcal{P}$ away from $\ell$, at both places over $u$, in the normalization of that section. A label is a global cuspidal representation or a newform with its indicated sign. Equalities at different primes use this same label. Call the resulting set of distinct trace specializations $X$. Including the split evaluation, form

$$B = \operatorname{im}\left( \mathbb{T} \longrightarrow O \times \prod_{\xi \in X} \overline{\mathbb{Q}}_\ell \right).$$

Put $I = \mathrm{im}(\mathfrak{d} \to B)$ and $\mathfrak{n} = \mathrm{im}(\mathfrak{a} \to B)$. For $u \in \mathcal{P}$, denote the trace at its fixed place by $T_u$ and its value at the split point by $d_\alpha(u)$.

**Proposition 36.12** (The determinant on the image of the trace differences)**.** *The algebra $B$ is finite and torsion-free over $O$. Let $X' \subset X$ consist of the specializations at which some trace differs from the trace of the inverse. For $u \neq \ell$ in $\mathcal{P}$, using $v_\ell(\ell) = 1$, one has*

$$\mathrm{length}_O(I/\mathfrak{n}I) \;\leq\; \mathrm{length}_O\big(I/(T_u - d_\alpha(u))I\big) \;=\; \sum_{\xi \in X'} v_\ell\big(\xi(T_u) - d_\alpha(u)\big).$$

*Each summand uses a distinct global label, including when two labels have the same value at $u$.*

*Proof.* Two specializations assigned the same label agree on the traces of degree-one primes outside a finite set, and hence on all traces on $G_K$ by Chebotarev and continuity. Such traces topologically generate $\mathbb{T}$. Thus a fixed label gives at most one specialization, and $X$ is finite by Theorem 30.8. The set $X$ is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/J)$. Indeed at rank three the rationality theorem for regular algebraic cuspidal representations carries a conjugate of the finite part to the finite part of another cuspidal representation. The weights are permuted among embeddings, and the fixed vectors for the given compact level are preserved. Contragredience and the automorphism of the base field commute with conjugation of scalars. The condition of conjugate self-duality is therefore preserved as well. The central restriction at the connected real split center is still trivial, since the algebraic central exponents there have the same sum. The normalization of the two spherical operators uses integral powers of $u$ in rank three. In the newform case the conjugate Fourier coefficients give another primitive eigenform of the same weight and level and the normalization uses $u^{-r}$. These observations give the asserted stability for both signs. [5, 6]

Each of the finitely many specializations takes values in a finite local coefficient field. Its image is integral, so $B$ is a submodule of a finite free $O$-module. Hence $B$ is finite and torsion-free. Its generic fiber is a finite reduced $J$-algebra. After scalar extension to $\overline{\mathbb{Q}}_\ell$, its geometric coordinates are exactly the distinguished point and the distinct elements of $X$. In detail, the displayed evaluation is faithful, so its coordinates meet every field factor of the generic fiber. Every embedding of each such factor is obtained by conjugating one that occurs. The stability just proved includes all those conjugates. Distinct coordinate characters then give distinct factors of the split algebra. The distinguished point differs from every $\xi$ because the split trace has complex absolute value greater than three at an allowed prime, whereas the classical traces have absolute value at most three.

In a product of fields, an ideal is the product of the factors on which it has a nonzero projection. It follows that $I$, after these scalar extensions, has precisely one one-dimensional summand per element of $X'$. Multiplication by $T_u - d_\alpha(u)$ acts there with eigenvalue $\xi(T_u) - d_\alpha(u)$, which is nonzero by the same absolute value argument. Since $I$ is free over the unramified DVR $O$, the length of its cokernel is the valuation of this determinant, by Smith normal form. Finally $T_u - d_\alpha(u)$ belongs to $\mathfrak{n}$. This proves the inequality, including the case $I = 0$. □

The kernel of classical evaluation contributes the submodule

$$C_{\mathrm{ker}} = \Phi\big(\ker(\mathfrak{d} \longrightarrow I)\big) \;\subseteq\; C.$$

It consists of the contributions from integral trace differences whose evaluations vanish at all the selected classical systems, including sums and products by arbitrary elements of the trace algebra before evaluation.

**Proposition 36.13** (Independence of the mixed extension)**.** *For fixed marked data the submodule $C_{\mathrm{ker}} \subseteq S_0^\vee$ is independent of the choices of cocycles in the construction of Theorem 36.11.*

*Proof.* In its representing module $C = S_0^\vee$ the lower cocycle is the universal normalized cocycle. Its value uniquely represents the identity of $C$. At $r \geq 2$ the allowed integral upper classes form a free $O$-module of rank one. Indeed, after restriction to the field killing the finite character $\eta\alpha$, rational second cohomology for twist $r$ vanishes. With only the primes over $\ell$ inverted the integral second cohomology is finite by the comparison with the finite group $K_{2r-2}$ used in Theorem 25.1. Adding an inverted place of norm $q$, prime to $\ell$, adds a localization term in degree one of its residue field with twist $r-1$. Its rational cohomology vanishes since $q^{r-1} - 1 \neq 0$. Corestriction gives the desired rational vanishing over $\mathbb{Q}$. The Euler calculation and torsion calculation in Lemma 36.9 give the rank and freeness. At $r = 1$ the construction uses the rank-one global unit eigenspace there, except when $C = 0$.

Consequently two primitive upper classes differ by a unit $u \in O^\times$. The normalized representatives satisfy $b' = ub$, since a coboundary at $c$ is multiplication of its defining value by $-2$, a unit. Compare their mixed corrections $w'$ and $uw$ before the final changes of frame. The two-cocycle equations have the same right side, so their difference is a one-cocycle in $C(\eta)$. The local construction makes it unramified at $\ell$. Away from $\ell$ it kills the inertia kernel $I_p^0$; finite averaging on the quotient shows that its class is unramified. Over $K$ its restriction is therefore an everywhere unramified homomorphism into $C$. It factors through $\mathrm{Cl}(K)$ and is zero since $\ell \nmid |\mathrm{Cl}(K)|$. Corestriction shows that the global class is zero as well. The normalization at $c$ again forces its representing coboundary to vanish. It follows that the two trace maps on the augmentation ideal differ by multiplication by $u$, which preserves every $O$-submodule of $C$. Changes of frame preserve the traces. This proves the assertion. □

For the whole complementary sign at the large primes put

$$\mathcal{L}_>(x) = \sum_{\ell > P} \log \ell \sum_{1 \leq r < t(x)} \mathrm{length}_{\mathbb{Z}_\ell}(\mathcal{S}_{\ell,r}(E_x)^{c=(-1)^r}).$$

**Theorem 36.14** (A uniform bound after classical evaluation)**.** *Write $b_\alpha = [O/\ell O : \mathbf{F}_\ell]$. On every sufficiently distant polynomial-conductor interval, uniformly in $x$ on that interval, one has*

$$0 \;\leq\; \mathcal{L}_>(x) - \sum_{\ell > P} \log \ell \sum_{1 \leq r < t(x)} \sum_{[\alpha]} b_\alpha \, \mathrm{length}_O C_{\mathrm{ker}} \;\leq\; \log J_* \;\ll_{\psi,\varepsilon} V^{C_1}.$$

*The innermost sum runs over residue character orbits of the complementary sign on $E_x$, and $J_*$ is the single positive integer from Theorem 36.4. In particular the upper bound is $o(\exp(x^\varepsilon/2))$ uniformly on these intervals.*

*Proof.* Theorem 36.11 gives $\Phi(\mathfrak{d}) = C$ and $\Phi(\mathfrak{a}\mathfrak{d}) = 0$. Hence $I \to C/C_{\mathrm{ker}}$ is a well-defined surjection, formed by lifting to $\mathfrak{d}$ and applying $\Phi$ modulo $C_{\mathrm{ker}}$. Its kernel contains $\mathfrak{n}I$. By Proposition 36.12 the length of $C/C_{\mathrm{ker}}$ is at most the sum of the indicated classical difference valuations at every $u \neq \ell$.

Inflate the characters to $E_*$. For one rational orbit $\mathcal{A}_L$, indexed by its kernel $L$ in the notation of Section 36.2, all or none of the orbit occurs on $E_x$. Each residue orbit in it has $b_\alpha$ geometric characters because $\ell$ is prime to the degree of $E_x$. For its representative the preceding inequality uses a submultiset of $\mathcal{T}^*(u)$, with distinct labels. Conjugate that inequality by the $b_\alpha$ local embeddings. The valuations are preserved, while $d_\alpha(u)$ ranges over the split values of these geometric characters with multiplicity. This repetition does not require the maps $\Phi$ to be chosen equivariantly, since only the representative's numerical upper bound is conjugated. The enlarged labelled multiset $\mathcal{T}^*(u)$ contains all the conjugate factors. The rational integer estimate of Theorem 30.11 now gives

$$\sum_{[\alpha] \text{ in } \mathcal{A}_L} b_\alpha \, \mathrm{length}_O(C/C_{\mathrm{ker}}) \leq v_\ell(I_{u,L,r}).$$

Taking the minimum over $u \neq \ell$ bounds this by $v_\ell(J_{L,r})$. Add over $\ell, r, L$ as in Theorem 36.4. The identity $\sum_\ell v_\ell(J) \log \ell = \log J$ bounds the total by $\log J_*$. At every $x$ the sum of $b_\alpha \log \ell \operatorname{length}_O C$ is exactly $\mathcal{L}_>(x)$ by prime-to-degree character descent and finite duality. This proves the two inequalities. The polynomial bound for $\log J_*$ and $x \geq V$ give the final estimate. □

The quantity $C_{\mathrm{ker}}$ can equal $C$ if the available classical evaluations annihilate $\mathfrak{d}$. The remaining large-prime estimate therefore requires an upper bound for its total length in Section 6.6. The smaller-prime tails and the comparison for the signed integral period also enter that section. The source takes its upper estimate on the graded model in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/OneCharOf.lean`, line 137 and compares the full cohomological length with that model at line 140. In the trace calculation above the unresolved integral comparison is concentrated in $C_{\mathrm{ker}}$.

## 36.7 An auxiliary CM field with ordinary cuspidal representations

Fix one polynomial-conductor interval and its modulus $N_*$ as in Theorem 36.4. Keep $K = \mathbb{Q}(\sqrt{-s})$ from Proposition 36.3, with quadratic character $\eta$. Fix a coefficient prime $\ell > \exp(V^\varepsilon/12)$. For large $V$ we have $\ell > sN_*$ and $\ell > 4t_* + 5$. The constructions below use one auxiliary field for every $E_x$ on the interval, every $1 \leq r < t(x)$, and every character $\alpha$ of $E_x$ with $\alpha(-1) = (-1)^r$. We view finite characters in $\overline{\mathbb{Q}}_\ell$, and use arithmetic Frobenius in the classical modular form normalization. Write $\Psi = \alpha\kappa_\ell^r$.

**Proposition 36.15** (The fields for ordinary base change)**.** *For each integer $a \geq 0$ there is a totally real abelian field $F^+$ such that $L = KF^+$ is abelian and CM, is disjoint from $E_*(\mu_\ell)$ over $\mathbb{Q}$, and has the following properties. The quadratic extension $L/F^+$ is unramified at every finite place and split at every place over $sN_*\ell$. The degree $[L : K]$ divides $2\ell^{a+1}$. At $\ell$ the ramification index of $L/\mathbb{Q}$ is $\ell^a$, and $\bar{\kappa}_\ell$ maps inertia of $G_L$ onto $\mathbf{F}_\ell^\times$. There is a character $\theta$ of $\mathbb{Q}$ of order $\ell$, of prime conductor $w \equiv 1 \bmod sN_*\ell$, which is trivial on $G_L$. There exists one $g \in G_L$ with $\bar{\kappa}_\ell(g) = -1$ and $\bar{\Psi}(g) = 1$ simultaneously for every pair $r, \alpha$ above.*

*Proof.* By Dirichlet's theorem choose distinct primes $v, w$ prime to $2sN_*\ell$, with $v \equiv -1 \bmod 8sN_*\ell$ and $w \equiv 1 \bmod sN_*\ell$. Let $F_w$ be the unique degree-$\ell$ subfield of $\mathbb{Q}(\zeta_w)$, and $F_a$ the unique degree-$\ell^a$ subfield of $\mathbb{Q}(\zeta_{\ell^{a+1}})$. Both are real since their degrees are odd. Set $F^+ = \mathbb{Q}(\sqrt{sv})F_wF_a$. We have $sv \equiv 1 \bmod 4$, so the first factor has discriminant $sv$. At every $p \mid sN_*\ell$, the number $-v$ is a square in $\mathbb{Q}_p$, by reduction and Hensel's lemma at odd $p$ and by $-v \equiv 1 \bmod 8$ at two. Thus adjoining $\sqrt{sv}$ locally already supplies $\sqrt{-s}$. At other places the base change of the unramified quadratic algebra of $K$ is unramified or split. This proves both assertions on $L/F^+$. Complex conjugation fixes $F^+$ and acts nontrivially on $K$, so $L$ is CM with maximal real subfield $F^+$.

The only possible ramified primes of $L$ are $s, v, w, \ell$, all prime to $N_*$. At $\ell$ only $F_a$ ramifies and it is totally ramified there. The other factors give an unramified local extension, so the ramification index is exactly $\ell^a$. The field $E_*(\mu_\ell)$ has ramification index dividing $\ell - 1$ there and is unramified at $s, v, w$. Its intersection with $L$ is therefore unramified at every finite prime, hence trivial by Minkowski's discriminant bound. Disjointness supplies an element of $G_L$ acting as complex conjugation on $E_*(\mu_\ell)$. Its residual cyclotomic value is $-1$, and its value under every $\bar{\Psi}$ is $\alpha(-1)(-1)^r = 1$. The index of the local inertia subgroup in rational inertia is therefore prime to $\ell - 1$, proving the residual cyclotomic assertion. Multiplying the three real degrees gives the displayed divisibility. A faithful character of $\mathrm{Gal}(F_w/\mathbb{Q})$ has the asserted conductor and restriction. These choices are independent of $x, r, \alpha$. □

**Lemma 36.16** (An ordinary newform with the prescribed character at the auxiliary prime)**.** *Put $\chi = \eta\alpha$ and let $D$ be its primitive conductor. For each $m \geq r$ with $m \equiv r \bmod \ell - 1$, there exists an $\ell$-ordinary cuspidal newform $g$ of weight $m+1$, level exactly $Dw$, and character $\chi\theta$. Its residual Galois semisimplification is $1 \oplus \chi\bar{\kappa}_\ell^r$, where we reduce the finite character in the second summand.*

*Proof.* The coprime conductors of $\eta$ and $\alpha$ give $D = s\,\mathrm{cond}(\alpha) \mid sN_*$. In particular $\chi$ is nontrivial and has order prime to $\ell$. Put $k = r+1$, so that $2 \leq k < \ell - 1$ and $\chi(-1) = (-1)^k$. At level $D$ the holomorphic Eisenstein series $E = E_k(1,\chi)$ has constant term $-B_{k,\chi}/(2k)$ and coefficient $\sum_{j\mid n} \chi(j) j^{k-1}$ at $q^n$ for $n > 0$. The weight-two formula applies because $\chi \neq 1$. The constant term is integral at $\ell$. Indeed $B_{k,\chi} = D^{k-1} \sum_{u=1}^{D} \chi(u) B_k(u/D)$, and the Bernoulli numbers through degree $k$ are integral at $\ell$ by the von Staudt–Clausen formula. All other coefficients are integral there as well. [6, 7]

Work over a sufficiently large finite coefficient DVR at $\ell$ containing the character values and the roots of unity for the cusps. The good-reduction modular curve and the $q$-expansion principle make $E$ integral at every cusp. One can check this after adding full auxiliary level prime to $D\ell$. On each geometric component the expansions at a lift of infinity are integral. A form with a vertical pole would, after multiplication by the smallest power of the uniformizer clearing the pole, reduce to a nonzero section with zero expansion on that component. This contradicts the expansion principle. [8]

At level $M = Dw$ consider $E(z) - E(wz)$. Its reduction is cuspidal. To check all cusps, take $\gamma = \left(\begin{smallmatrix} b & b' \\ c & c' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $h = \gcd(w, c)$. Choose $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ with first column $(wb/h, c/h)^{\mathsf{T}}$. Then $\mathrm{diag}(w,1)\gamma = \gamma' \left(\begin{smallmatrix} h & t \\ 0 & w/h \end{smallmatrix}\right)$ for some $t \in \mathbb{Z}$. Since $h, w \equiv 1 \bmod D$, the two first columns agree modulo $D$. Their unimodularity shows that the full matrices agree modulo $D$ after right translating one by an integral upper unitriangular matrix. The constant terms of $E|_k\gamma$ and $E|_k\gamma'$ are therefore equal. The slash formula gives for the difference the constant term $(1 - (h/w)^k) a_0(E|_k\gamma)$. It vanishes in the residue field because $w \equiv 1 \bmod \ell$. The coefficient of $q$ in the difference is one. It is an eigenform for the operators $T_p$ with $p \nmid M$, since both degeneracy maps commute with these operators. Its eigenvalue at $\ell$ reduces to one.

We give the lifting argument because $\theta$ has order $\ell$. Write $\Delta = (\mathbb{Z}/M\mathbb{Z})^\times = P \times H$, with $P$ its $\ell$-Sylow subgroup. The group $P$ acts freely by diamond operators on the smooth compact modular curve $X_1(M)$ in both geometric fibers of the coefficient DVR. At a smooth elliptic curve a stabilizing diamond has order dividing the order of an automorphism, with prime factors at most three in these characteristics. At a cusp its order divides the order of an automorphism of the generalized elliptic $n$-gon with $n \mid M$. Its automorphisms preserving the group structure and the polygon action form $\mu_n \rtimes \{\pm 1\}$. In both cases the stabilizer has order prime to $\ell$. Here $M \geq 5$, so the full point of order $M$ gives a fine modular curve. [8, 9]

Consequently $X_1(M) \to Y = X_1(M)/P$ is a finite étale torsor. The Hodge line $\omega$ and the reduced cusp divisor descend to $Y$. Denote the cusp divisor by $D_Y$. Descent with the character of $P$ prescribed by $\theta$ gives a line $\mathcal{L}$ such that its sections tensored with a modular line on $Y$ are the corresponding $\theta$-eigenvectors on the cover. This is descent for a line with the inverse character used in its equivariant structure. It needs no division by $|P|$. The reduction of $\mathcal{L}$ is canonically trivial, with its natural $H$-action, since $\theta$ reduces to one and is trivial on $H$. The character $\chi$ is trivial on $P$. Its projector on $H$ is integral because $\ell \nmid |H|$.

For $k' = m+1$, use the cusp line $\mathcal{F} = \omega^{k'}(-D_Y) \otimes \mathcal{L}$. On the geometric residue fiber the Serre dual of $H^1(\mathcal{F})$ is $H^0(\omega^{2-k'})$, since Kodaira–Spencer gives $\Omega^1_Y = \omega^2(-D_Y)$, by étale descent. For $k' > 2$ it vanishes by positivity of $\deg \omega$. For $k' = 2$ it consists of the constants on the geometrically connected curve and $H$ acts trivially. In both cases the $\chi$-part of $H^1(\mathcal{F})$ on that

fiber is zero, as $\chi$ remains nontrivial after reduction. Proper coherent cohomology in relative dimension one gives surjectivity in degree one of the map to the residue fiber and identifies $H^1(Y,\mathcal{F})/\varpi$ with $H^1(Y_{\mathrm{sp}},\mathcal{F}_{\mathrm{sp}})$, where the subscript denotes base change to the residue field. The degree-two group over the affine DVR vanishes. Nakayama's lemma and the integral projector give $H^1(Y,\mathcal{F})^{\chi}=0$. The multiplication sequence for the uniformizer now lifts every residual $\chi$-cusp section to the prescribed character $\chi\theta$ in weight $k'$.

Multiply the residual cusp form just constructed by the $(m-r)/(\ell-1)$-st power of the Hasse invariant. This keeps its expansion, since the Hasse invariant has expansion one, and gives the required weight and residual character. The $q$-expansion formula for $T_p$ preserves its residual eigenvalues, since $p^{k'-1}=p^{k-1}$ in the residue field for $p\neq\ell$, and both powers vanish there for $p=\ell$. The resulting eigensystem lifts to a characteristic-zero cuspidal eigensystem in the space just described. Indeed the commuting good Hecke operators, including $T_\ell$, generate a finite torsion-free algebra of endomorphisms of the integral cusp lattice. The residual eigensystem gives a maximal ideal of this algebra. A minimal prime below it has characteristic zero, since the uniformizer is a nonzero divisor. On its generic factor the system occurs in the cusp space by faithfulness and simultaneous triangularization after scalar extension. The central character $\chi\theta$ is primitive of conductor $M$. Newform theory therefore puts the system in exact level $M$. Its $\ell$-th eigenvalue is a unit. The good Hecke polynomials and Chebotarev give the stated residual semisimplification. [6, 10, 8] □

**Theorem 36.17** (Cuspidal ordinary adjoints on the common auxiliary field)**.** *For every $x,r,\alpha,m$ above, the same $L$ carries a determinant-one representation $U_m$ of dimension three which is absolutely irreducible on every open subgroup of $G_L$. Its residual semisimplification is $\bar\Psi\oplus 1\oplus\bar\Psi^{-1}$. On inertia away from $\ell$ it is the direct sum of $\alpha,1,\alpha^{-1}$. At each place above $\ell$ it is ordinary crystalline with ordered inertial characters $\kappa_\ell^{m},1,\kappa_\ell^{-m}$. The three ordinary characters reduce, on the full decomposition group, to $\bar\Psi,1,\bar\Psi^{-1}$, in that order. In the normalization of Section 30.4, $U_m\otimes\kappa_\ell^{-1}$ is attached to a regular algebraic conjugate self-dual cuspidal representation over $L$. In particular the construction includes the original weight $m=r$.*

*Proof.* Let $W$ be the two-dimensional representation of the newform of Lemma 36.16 and put $U_m=\mathrm{Ad}^0(W)|_{G_L}$, acting on trace-zero endomorphisms. The rational representation $W$ is absolutely irreducible, with determinant $\chi\theta\kappa_\ell^{m}$. At $\ell$ the ordinary newform theorem gives an unramified quotient whose Frobenius eigenvalue is the unit root of $X^2-a_\ell(g)X+\chi\theta(\ell)\ell^{m}$. It reduces to one. The other character follows from the determinant. The representation is crystalline by good reduction of the level in the modular Galois representation theorem. In the adjoint, the upper root line, the upper triangular trace-zero plane and the whole space give an ordered stable flag. The three quotients are the upper-to-lower character ratio, one, and its inverse. Both $\eta$ and $\theta$ restrict trivially to $G_L$. These facts prove the assertions at $\ell$, including the residual ordering. [11, 12]

At $p\mid Dw$ the conductor exponent of $W$ equals that of its determinant, by local–global compatibility and the newform conductor theorem. The determinant is ramified. Each wild ramification group contributes at least as much to the Artin conductor of $W$ as to that of its determinant. A group acting trivially on $W$ acts trivially on the determinant, whose codimension of invariants is at most one. The same inequality holds for the tame codimension. Equality of the totals forces $\dim W^{I_p}=1$. In the Weil–Deligne parameter, finite inertia therefore has a trivial summand, and its other character is the ramified determinant. The monodromy operator commutes with inertia. Its two distinct lines preclude any nonzero nilpotent operator commuting with that action. Thus the actual inertial characters of $W$ are $1,\chi\theta$, also at $w$. Outside $Dw\ell$ the representation is unramified. Restrict to $G_L$ and take the two ratios and the middle character to obtain the asserted inertial type. [13, 6]

The residual characters of $W$ have ratio of order greater than two on inertia at $\ell$, because $2r < \ell - 1$. A nontrivial quadratic self-twist would have to permute the two residual characters. It cannot fix either, and interchanging them would force their ratio to have square one. A quadratic character stays nontrivial on reduction at odd $\ell$. Thus $W$ has no quadratic self-twist. Its projective image is infinite. Otherwise the representation would become scalar on an open subgroup, contradicting its two distinct Hodge–Tate weights. Over $\overline{\mathbb{Q}}_\ell$ every positive-dimensional proper connected subgroup of $\mathrm{PGL}_2$ fixes a point or is a torus preserving two points. Indeed its Lie algebra has dimension at most two and is solvable, so it is triangularizable. Unless it is a torus its nontrivial unipotent subgroup fixes a unique point, which must also be fixed by its normalizer. A unique fixed point contradicts irreducibility. In the torus case the full group must interchange the two lines, and the matrix acting as $1, -1$ on the two lines then gives the excluded quadratic self-twist. It follows that the projective image is Zariski dense in $\mathrm{PGL}_2$ and stays dense on every open subgroup. Its adjoint on $\mathfrak{sl}_2$ is irreducible, since an invariant subspace is an ideal of the simple Lie algebra. It preserves the symmetric form $\mathrm{tr}(XY)$ and has determinant one.

Reduction of adjoint characteristic polynomials gives the three stated characters using Lemma 36.16. One can realize a split residual matrix lattice after a finite coefficient extension. In a stable basis with upper triangular residual $W$, pass to an extension of ramification index at least two and multiply the second basis vector by its new uniformizer. The upper entry then vanishes on reduction and the originally divisible lower entry still vanishes. The resulting adjoint lattice gives the three residual diagonal characters. To write a conjugate polarization for $U_m$, pair $X, Y$ by $\mathrm{tr}(X\,\mathrm{Ad}(W(c))Y)$, where $c$ is complex conjugation. The involution $\mathrm{Ad}(W(c))$ is orthogonal, hence self-adjoint, for the trace form. This gives a symmetric nonsingular form with the required conjugate invariance.

Let $\pi_g$ be the unitary representation of the newform. A nontrivial self-twist of $\pi_g$ is quadratic by its central character and would give one for $W$ by equality of good Frobenius polynomials. Gelbart–Jacquet therefore gives a cuspidal adjoint $\mathrm{Sym}^2 \pi_g \otimes \omega_g^{-1}$. It is self-dual of central character one. Base change it up a tower of cyclic prime degrees to $L$. All these degrees are two or $\ell$. Loss of cuspidality at a step of degree $d$ would require a self-twist of order $d$ by cyclic base change. Equality of central characters would give $d \mid 3$, which is impossible. At each complex place the normalized parameters are $(z/\bar{z})^m, 1, (z/\bar{z})^{-m}$, so they are regular algebraic. The representation over $L$ is invariant under complex conjugation and is self-dual. At good places its Satake parameters are the two ratios of the Frobenius eigenvalues of $W$ and one. The rank-three algebraic normalization multiplies them at geometric Frobenius by the residue cardinality, as in Section 30.4. This identifies the attached representation with $U_m \otimes \kappa_\ell^{-1}$ by Chebotarev. [14, 2, 15] □

**Theorem 36.18** (Strict classes and automorphic descent across the same extension)**.** *Let $\Phi, \mathbb{T}, \mathfrak{a}, C$ be as in Theorem 36.11 on $K$. Let $\mathbb{T}_L$ be the closed $O$-subalgebra generated by the traces on $G_L$ and $\mathfrak{d}_L$ its ideal generated by $t_h - t_{h^{-1}}$, $h \in G_L$. Then $\Phi(\mathbb{T}_L) = O \oplus C$ and $\Phi(\mathfrak{d}_L) = C$. Good rational primes splitting completely in $L$ suffice to generate the latter image.*

*Let $U$ be a strongly absolutely irreducible characteristic-zero point of the marked ordinary problem on $K$ in the original weight $r$. If $U|_{G_L} \otimes \kappa_\ell^{-1}$ is attached to a regular algebraic conjugate self-dual cuspidal representation of $\mathrm{GL}_3(\mathbb{A}_L)$, then $U$ is counted by the rank-three labels of Section 30.4 on $K$ at level $N_*$. The cuspidal descent of central character one is unique.*

*Proof.* Take $j \geq 1$ such that $\ell^j C = 0$, and put $N' = G_L \cap \ker(\Psi \bmod \ell^j)$. Work on the absolute group $G_{\mathbb{Q}}$ by inflating the cocycles. The quotient by $N'$ is abelian. On $N'$ the lower cocycle $z$ and the upper cocycle $b$ mod $\ell^j$ from Theorem 36.11 are homomorphisms. Assume $C \neq 0$. The

$O$-span of $z(N')$ is all of $C$. This span is stable under the Galois action by the cocycle identity. After dividing it out, $z$ factors through the abelian quotient. Commuting with the marked $g_*$, where $z(g_*) = 0$ and $\Psi(g_*) - 1$ is a unit, makes this quotient cocycle zero. All values of $z$ generate $C$, which proves the assertion. The nonzero residual class of $b$ also stays nonzero on $N'$. Otherwise it factors through the same abelian quotient, where commuting with $g_*$ makes it a coboundary, since $\eta(g_*)\Psi(g_*) - 1$ is a unit. Choose $k_0 \in N'$ with $b(k_0)$ a unit.

Choose $g$ in inertia of $G_L$ at $\ell$ with $\Psi(g) - 1$ a unit, using Proposition 36.15. In the mixed matrices, for $h \in N'$ form $h' = h(chc^{-1})$ and $k' = k_0(ck_0c^{-1})^{-1}$. Both belong to $N'$ because $L/\mathbb{Q}$ is Galois. Their reduced pairs of root coordinates are $(2z(h), 0)$ and $(0, 2b(k_0))$. The same matrix commutator calculation as in Theorem 36.11 gives

$$t_{[h',k']g} - t_g \;\longmapsto\; 4z(h)b(k_0)\bigl(2 - \Psi(g) - \Psi(g)^{-1}\bigr) \quad \text{in } C.$$

Indeed the central diagonal correction on the commutator already has degree two, so multiplication by the matrix at $g$ uses only its split diagonal, whether or not $g$ is marked. The split constants cancel, and reduction modulo $\ell^j$ is injective on $C$. The final factor is a unit. Inversion negates the augmentation part of each evaluated trace on $G_L$ by Theorem 36.11. Since two is a unit, the differences from inverse traces thus span $C$. If $C = 0$ the assertions follow by split evaluation. Finally apply Chebotarev over $L$ to the finite-valued trace correction. Prime ideals of higher residue degree have density zero, and an unramified rational prime beneath a degree-one prime of a Galois field splits completely. Conjugation above that prime preserves the correction or changes its sign, by the polarization on $K$. This proves the assertion for primes.

On $G_K$, $\det U = 1$. Its determinant is unramified at $\ell$ by the ordinary inertial diagonal. Away from $\ell$ inertia in the marked problem on $G_K$ factors through the finite image of $\alpha$, of order prime to $\ell$. The residual determinant is trivial, so the determinant on this finite inertia is trivial. The resulting everywhere unramified character of $K$ has image in the principal units of the coefficient DVR. It factors through the class group of $K$, of order prime to $\ell$, and is therefore trivial. At each good place the unitary normalization of the assumed cuspidal representation $\Pi_L$ has central character one by this determinant computation and the rank-three Tate adjustment. Equality on good primes gives equality of the Hecke characters.

Descend one cyclic step $F/F_0$ between $L$ and $K$, of prime degree $d = 2$ or $\ell$. Its fields are stable under complex conjugation and are CM since they are abelian over $\mathbb{Q}$ and contain $K$. On $F$ the good Frobenius parameters of the cuspidal representation are invariant under $\mathrm{Gal}(F/F_0)$, since they come from $U$ on $K$. Strong multiplicity one and cyclic base change give cuspidal descent. The descents differ by twists by characters of $\mathrm{Gal}(F/F_0)$, via class field theory. The central character of a descent base changes to one and is itself such a character by the norm theorem. Cubing is an automorphism of this character group. Thus exactly one descent $\Pi_0$ has trivial central character. Its conjugate contragredient is another such descent, so is isomorphic to $\Pi_0$. At a complex place there is no local extension and the infinitesimal character is inherited, so the descent is regular algebraic. [2, 3]

The attached Galois representation of $\Pi_0$ restricts to $U|_{G_F} \otimes \kappa_\ell^{-1}$ after semisimplification by good Frobenius compatibility. This restriction is irreducible by the hypothesis on $U$, hence the identification holds for the representation itself. Over $G_{F_0}$ the two extensions differ by a character of $\mathrm{Gal}(F/F_0)$. After identifying their restrictions, Schur's lemma makes the ratio of their matrices at each element scalar. Both determinants are $\kappa_\ell^{-3}$, by the unit central character, so the character has cube one. Its order divides $d$ as well, and it is trivial. Repeating the argument gives $\Pi_K$ over $K$ with the prescribed Galois representation. [15]

We check the level on the original field. At every place above $\ell$, the ordinary component is crystalline with labeled exponents $r, 0, -r$. The local proof of Theorem 33.11 applies

here because $K$ at $\ell$ is either split or unramified quadratic, the two cases of Theorems 32.11 and 32.15. The conjugate place has the dual conjugate representation. The rank-three weight is Shin-regular, since the rank is odd. Local–global compatibility at $\ell$ therefore identifies the Frobenius-semisimplified Weil–Deligne parameter including its monodromy. Crystallinity gives trivial inertia and zero monodromy, hence spherical level there. The labeled Hodge–Tate exponents and the common Tate twist give infinitesimal exponents $r, 0, -r$ in the unitary normalization. Subtracting the half-sum of positive roots $(1, 0, -1)$ gives coefficient weights bounded by $4(r + 2)$ at both embeddings. The positive real split center acts trivially. [16, 15]

At a finite place $v \nmid \ell$, the image of inertia is finite of order prime to $\ell$, with characters $\alpha, 1, \alpha^{-1}$, since the residual characters have these unique finite lifts on the permitted cyclic quotient. Monodromy is zero. The characters on inertia extend to the whole local Weil group since $\alpha$ is global. After Frobenius semisimplification one diagonalizes Frobenius on each inertial eigenspace. Thus the parameter is a sum of characters with those inertial restrictions. Away from $N_*$ the representation is spherical. For $v \mid N_*$ write $b = v(N_*)$ in the normalized local valuation. The characters are trivial on $1 + \mathfrak{p}_v^b$ by local reciprocity, since $K/\mathbb{Q}$ is unramified at the rational prime and the norm preserves this congruence. Local–global compatibility and the Langlands classification realize the component as a quotient of a normalized principal series of such characters, in Langlands order. [15]

For completeness let $K_b = 1+\mathfrak{p}_v^b M_3(\mathcal{O}_v)$ with $b \geq 1$, and let $B$ be the upper triangular group. The principal series is generated by translates of the $K_b$-fixed section supported on $BK_b$ with value one at the identity. On the lower unipotent chart its restriction is the indicator of the subgroup of lower unipotents with entries below the diagonal in $\mathfrak{p}_v^b$, by triangular elimination. Diagonal translation contracts this subgroup to arbitrarily small open subgroups, shrinking each negative root coordinate by a positive power of the uniformizer. Up to a nonzero scalar the corresponding sections are their indicators on the same chart. A smooth section is fixed by some $K_e$ with $e \geq b$. By Iwasawa decomposition finitely many disjoint sets $BK_e u$, $u \in \mathrm{GL}_3(\mathcal{O}_v)$, cover the flag variety. In the lower unipotent trivialization of each such chart the section is constant, since $K_e$ is normal in the maximal compact group. Subdivide the lower unipotent subgroup of level $e$ into finitely many right cosets of one of the contracted subgroups contained in it. Their indicators are translates of the displayed section, so they span the prescribed section. Any nonzero quotient must therefore retain a nonzero $K_b$-fixed vector. This proves the required principal congruence level at $v$. Taking the tensor product of the local fixed vectors puts $\Pi_K$ on the stated list, with a consistent choice of the Frobenius sign at every good place. □

The common element in Proposition 36.15 shows that $\bar{\kappa}_\ell \bar{\Psi}^j$ is nontrivial on $G_L$ for every integer $j$, including the four root characters used in the residual separation conditions of `Development/AutomorphicArithmetic/AutomorphicVarieties/FrobeniusSmoothVectors/FrobeniusSplitPrimeDatum.lean`, line 72. The auxiliary prime and its character realize the initial character conditions in `Development/AutomorphicArithmetic/AutomorphicVarieties/CurveDVRSpecialization/HostFrameEx.lean`, line 58. The ordinary adjoints above supply the representations described in `Development/AutomorphicArithmetic/AutomorphicVarieties/IdeleGenerators/ShapeFacts.lean`, line 177 for the data of this interval. Their orthogonal form makes every trace difference $t_h - t_{h^{-1}}$ zero on these individual representations. Theorem 36.18 shows that the mixed integral family over the same auxiliary field still detects the whole module $C$. A bound for $C_{\mathrm{ker}}$ in Theorem 36.14 requires control of the integral deformations beyond these adjoints. The smaller-prime orders and the signed integral period in Section 6.6 also remain to be estimated.

## 36.8 The ordinary deformation ring at a cuspidal adjoint

Fix one of the forms $g$ of Lemma 36.16 and the field $L$ of Proposition 36.15. Put $F = L^+$ and $d = [F : \mathbb{Q}]$. Choose a finite coefficient field $J/\mathbb{Q}_\ell$ over which the two-dimensional representation $W$ of $g$ and its ordinary filtration are defined. Enlarge it to contain the images of the embeddings of the completions of $F$ at $\ell$. We calculate at the characteristic-zero representation $U = \mathrm{Ad}^0(W)$ of Theorem 36.17. Here $U$ on $G_L$ is the restriction of this adjoint on $G_\mathbb{Q}$. All claims in this subsection include $m = r = 1$.

On its three-dimensional space $D$, choose a basis and let $S$ be the matrix of the symmetric trace form. Put $\iota(B) = S^{-1}B^{-\mathsf{T}}S$. Use the group $\mathrm{GL}(D) \rtimes \{1, -1\}$ in which the negative sign acts by $\iota$. Write $\eta_F$ for the sign of $L/F$. The homomorphism at the point in question is

$$R_0(g) = (U(g), \eta_F(g)), \qquad g \in G_F.$$

It is a homomorphism because the matrices $U(g)$ preserve $S$. Let $V = \mathrm{End}_J(D)$ with the adjoint action of $R_0$ on the Lie algebra of the identity component. Thus the sign in this action operates by $X \mapsto -S^{-1}X^{\mathsf{T}}S$. Cohomology below is continuous, with the $\ell$-adic topology on finite-dimensional $J$-spaces.

A deformation takes values in the same semidirect product over a local Artinian $J$-algebra $A$ of residue field $J$. It has prescribed reduction $R_0$ and sign $\eta_F$. Two lifts are equivalent if conjugate by a matrix congruent to one. We impose the following conditions. At each $v \mid \ell$, choose one place of $L$ above $v$. It is split over $F$, so on this decomposition group there is an ordinary three-dimensional representation. Require a stable full flag by direct summands lifting the ordered flag of $U$. At a finite $v \nmid \ell$ require the kernel of $R_0$ on inertia to act trivially. Thus only finitely many finite places need to be added to the places above $\ell$ in a global ramification set. At the real places keep the prescribed sign. Denote the resulting deformation functor by $\mathcal{D}$.

Its weight at $v \mid \ell$ is the ordered triple $(\xi_{v,1}, \xi_{v,2}, \xi_{v,3})$ of inertial characters on the flag quotients. Via local reciprocity these are characters on $\mathcal{O}_{F_v}^\times$. We use one reciprocity convention for the base and the lifts. Their base values are $\kappa_\ell^m, 1, \kappa_\ell^{-m}$ on inertia. Denote by $\mathcal{W}$ the functor of all continuous lifts of these inertial characters, independently at the chosen places. An inertia matrix at the base has three distinct eigenvalues, since the cyclotomic character has open image on inertia and $m > 0$. In an ordinary lift the three corresponding eigenvalues have unit pairwise differences in $A$. The spectral projectors of this matrix show that the flag reducing to the prescribed ordered flag is unique. In particular taking its characters defines a map on strict conjugacy classes.

**Lemma 36.19** (The local condition when the weights are fixed)**.** *For $v \mid \ell$ let $\mathfrak{b}_v$ be the endomorphisms preserving the base flag and $\mathfrak{n}_v$ the strictly upper triangular endomorphisms. Write $L_v^0 \subset H^1(F_v, V)$ for the tangent condition with fixed inertial characters. Then*

$$L_v^0 \subseteq H_f^1(F_v, V), \qquad \dim_J L_v^0 = h^0(F_v, V) + 3[F_v : \mathbb{Q}_\ell].$$

*Across any small extension of Artinian $J$-algebras an ordinary local lift can be lifted with any prescribed lift of its weights. Here a small extension is a surjection whose kernel is killed by the maximal ideal of its source.*

*Proof.* Let $\delta$ be the ratio of the upper and lower ordinary characters of $W|_{G_{F_v}}$. The adjoint quotients have characters $\delta, 1, \delta^{-1}$. The upper entries have characters $\delta, \delta, \delta^2$ in their associated graded space, and $V/\mathfrak{b}_v$ has the inverse characters. Write $u$ for the unit root of the good Hecke polynomial of $g$ at $\ell$. Before restriction to $F_v$, $\delta/\kappa_\ell^m$ is unramified with Frobenius value

$(\chi\theta)(\ell)u^{-2}$, by the ordinary formula in Theorem 36.17. Every complex conjugate of $u$ has absolute value $\ell^{m/2}$ by purity for the newform of weight $m+1$. Hence this unramified value stays nontrivial after any positive power. In particular when $m=1$ the restriction of $\delta$ still differs from $\kappa_\ell$. For $m>1$ the two characters already differ on inertia. For every $m$, $\delta^2$ has inertial exponent $2m\neq 1$. [17, 11]

The quotient $V/\mathfrak{b}_v$ has no invariants, by its negative exponents. Neither $\mathfrak{b}_v^\vee(1)$ nor $\mathfrak{n}_v^\vee(1)$ has invariants, since no upper character is $\kappa_\ell$ and the diagonal characters are trivial. This proves $H^2(F_v,\mathfrak{n}_v)=H^2(F_v,\mathfrak{b}_v)=0$ by local duality. It also gives $H^0(F_v,\mathfrak{b}_v)=H^0(F_v,V)$ and an injection $H^1(F_v,\mathfrak{b}_v)\to H^1(F_v,V)$. One may fix the flag by a change of basis reducing to the identity. Upper triangular tangent cocycles modulo changes of basis then give precisely the image of this injection. The quotient $\mathfrak{t}_v=\mathfrak{b}_v/\mathfrak{n}_v$ has trivial action and dimension three. The map to $H^1(F_v,\mathfrak{t}_v)$ is onto by the vanishing of $H^2(F_v,\mathfrak{n}_v)$. Local reciprocity gives surjectivity from there to the homomorphisms on the unit group, since one can prescribe the value on a uniformizer independently. The resulting space of inertial variations has dimension $3[F_v:\mathbb{Q}_\ell]$ by the logarithm on an open subgroup of the principal units. The local Euler characteristic gives $\dim H^1(F_v,\mathfrak{b}_v)=h^0(F_v,V)+6[F_v:\mathbb{Q}_\ell]$. The kernel of the inertial variations therefore has the stated dimension. [1]

All modules in $0\to\mathfrak{n}_v\to\mathfrak{b}_v\to\mathfrak{t}_v\to 0$ are crystalline, as subquotients of the endomorphisms of a crystalline representation. In the cyclotomic convention, the three positive upper exponents contribute their full dimension to the de Rham quotient by $\mathrm{Fil}^0$ at each embedding. The Bloch–Kato local dimension formula and the local Euler characteristic therefore give

$$\dim H^1_f(F_v,\mathfrak{n}_v)=h^0(F_v,\mathfrak{n}_v)+3[F_v:\mathbb{Q}_\ell]=\dim H^1(F_v,\mathfrak{n}_v).$$

For a trivial coefficient, $H^1_f$ consists of unramified classes. Also $H^1_f(F_v,\mathfrak{b}_v)\to H^1_f(F_v,\mathfrak{t}_v)$ is surjective. Indeed the two-term crystalline complex computing $H^0,H^1_f$ is $D_{\mathrm{cris}}\to D_{\mathrm{cris}}\oplus(D_{\mathrm{dR}}/\mathrm{Fil}^0)$. The short exact sequence above gives a short exact sequence of these complexes by exactness on crystalline representations and strictness of their de Rham filtrations. The rightmost cokernels therefore map surjectively. A class with unramified diagonal differs from a crystalline lift of its diagonal by the image of a class in $H^1(F_v,\mathfrak{n}_v)$, which is crystalline by the dimension equality. Its image in $H^1(F_v,V)$ belongs to $H^1_f$. [18]

Finally lift the flag basis across a small extension with kernel $I$. On its three quotients lift the diagonal characters with the specified unit characters by local reciprocity. The independent choices on a uniformizer give continuous lifts of the unramified parts, since an infinitesimal change there extends through $\mathbb{Z}_\ell$ via the logarithm. With this diagonal fixed the upper triangular lifting obstruction lies in $H^2(F_v,\mathfrak{n}_v)\otimes_J I$. In detail, lift the entries continuously and form the defect of multiplication. It is a two-cocycle in the strictly upper entries tensored with $I$, where the action is the base adjoint action because $\mathfrak{m}_{A'}I=0$ in the extension $A'$. Subtract a one-cochain with this coboundary. This gives the required local homomorphism. □

Use $L^0_v=H^1_{\mathrm{ur}}(F_v,V)$ at finite $v\nmid\ell$ and $L^0_v=0$ at infinity. Let $H^1_{\mathcal{L}^0}(F,V)$ be the group of global classes satisfying these and the conditions of Lemma 36.19. For the dual $V^\vee(1)$ use their annihilators for the local Tate pairings, denoted $(\mathcal{L}^0)^\perp$.

**Proposition 36.20** (Vanishing of both Selmer groups at the cuspidal adjoints)**.** *At every adjoint of Theorem 36.17 one has*

$$H^1_{\mathcal{L}^0}(F,V)=0,\qquad H^1_{(\mathcal{L}^0)^\perp}(F,V^\vee(1))=0.$$

*Proof.* Theorem 36.17 gives connected monodromy $\mathrm{PGL}_2$ in its three-dimensional adjoint action on restriction to $G_L$. The closure of the image of the commutator of $G_L$ contains the algebraic

derived group, by density and continuity of the algebraic commutator map. This derived group is all of $\mathrm{PGL}_2$. The image on $G_{L(\mu_{\ell^\infty})}$ is therefore Zariski dense in this group. This gives the enormous-image condition of Newton and Thorne. Indeed elements with three distinct eigenvalues form a nonempty Zariski open subset and the group acts irreducibly. Its span is the full endomorphism algebra by Burnside’s theorem. For any nonzero endomorphism $X$ the open subset where $\operatorname{tr}(hX) \neq 0$ consequently meets the open subset of regular elements, and the cyclotomic image meets their intersection. Some eigenprojector of $h$ then pairs nontrivially with $X$. This applies in particular to an element of every simple submodule in the enormous-image test. One can enlarge $J$ to split the eigenvalues, since a local field has finitely many extensions of degree at most three. Restriction of scalars back to $J$ preserves vanishing.

The polarized regular algebraic cuspidal representation from Theorem 36.17 has Galois representation $U|_{G_L} \otimes \kappa_\ell^{-1}$. The twist has trivial effect on its adjoint action. The conjugate pairing on the underlying untwisted space has matrix $SU(c)$ for any complex conjugation $c \in G_F$. It is symmetric and gives exactly the negative-transpose adjoint action defined above. This identifies the polarized adjoint in Newton and Thorne’s Theorem A with $V$, by Schur’s lemma on $G_L$. Their theorem now yields $H^1_f(F, V) = 0$. Lemma 36.19 and the definition at the other places give the first vanishing. [19]

We spell out the dimension calculation for the second. The invariants of $\mathrm{End}(D)$ on $G_L$ are scalars and complex conjugation negates scalars in $V$. Thus $h^0(F, V) = 0$. On the cyclotomic kernel the invariant functionals on $\mathrm{End}(D)$ are multiples of the trace, by the same irreducibility and the nondegenerate trace pairing. The cyclotomic action on this line in $V^\vee(1)$ over $G_L$ is nontrivial. Hence $h^0(F, V^\vee(1)) = 0$. For a finite ramification set $\Sigma$ containing the archimedean places, the places above $\ell$ and those of ramification, the Poitou–Tate dimension formula is

$$h^1_{\mathcal{L}^0}(F, V) - h^1_{(\mathcal{L}^0)^\perp}(F, V^\vee(1)) = h^0(F, V) - h^0(F, V^\vee(1)) + \sum_{v \in \Sigma} (\dim L^0_v - h^0(F_v, V)).$$

This is the dimension identity obtained from the global duality sequence and the local and global Euler characteristics. At each finite $v \nmid \ell$, unramified cohomology is the cokernel of Frobenius minus one on $V^{I_v}$. Its dimension equals that of the kernel, namely $h^0(F_v, V)$. The places above $\ell$ contribute $3d$ in total by Lemma 36.19. At each real place the invariant Lie algebra consists of the skew endomorphisms for the nonsingular symmetric pairing $SU(c_v)$ and has dimension three. The $d$ real places contribute $-3d$. Thus the difference on the left is zero. The first vanishing gives the second. [1] □

**Theorem 36.21** (All ordinary deformations at the adjoint in characteristic zero)**.** *The map $\mathcal{D}(A) \to \mathcal{W}(A)$ is a bijection for every local Artinian coefficient algebra $A$ as above. The ordinary deformation functor has a power series ring in $3d$ variables over $J$ as its complete local ring. The subfunctor with determinant one on $G_L$ has $2d$ variables and is given by $\prod_i \xi_{v,i} = 1$ for every $v \mid \ell$.*

*Proof.* Consider a small extension $A' \to A$ with kernel $I$ and a specified lift of the weights. At $v \mid \ell$ use Lemma 36.19. Away from $\ell$ we may factor the local group by the given kernel on inertia. Its remaining finite inertia has zero positive-degree cohomology on $J$-spaces by averaging. Its residue quotient is procyclic and has zero degree-two continuous cohomology on these $\ell$-adic coefficients. For example one obtains the last fact from the finite $\ell$-power quotients of a stable lattice, the $\ell$-cohomological dimension of the procyclic group, and the Mittag–Leffler property of finite groups. The quotient therefore has zero degree-two obstruction and its allowed local differences are unramified cohomology. At a real place finite group averaging applies directly.

Lift the entries of a global representative continuously on the group unramified outside $\Sigma$. Such lifts of cochains exist using a $J$-linear section of $A' \to A$ and are invertible because their reductions are. The multiplicative defect represents a class in $H^2(G_{F,\Sigma}, V) \otimes_J I$. Local solvability places it in the kernel of localization. Poitou–Tate identifies this kernel with the dual of the group of everywhere locally trivial dual classes (using localization at $\Sigma$ for restricted-ramification cohomology). Those classes are contained in $H^1_{(\mathcal{L}^0)^\perp}(F, V^\vee(1)) = 0$. Thus there is a global homomorphism lifting the representative.

Compare this lift with the chosen admissible local lifts at $\Sigma$. Their differences give an element of $\bigoplus_{v\in\Sigma} H^1(F_v, V)/L_v^0$, tensored with $I$. Poitou–Tate gives surjectivity onto this sum from global degree-one cohomology, because the dual Selmer group just computed is zero. Adjust the global lift by a global one-cocycle having those local differences modulo $L_v^0$. At an ordinary place a representative of $L_v^0$ preserves the flag and fixes the inertia of the diagonal, after a local change of basis. At the other finite places its representative factors through the allowed local quotient. This adjustment therefore satisfies every local condition and has precisely the specified weights. For two lifts with the same weights, first lift a change of basis identifying their reductions. Their difference class then belongs to $H^1_{\mathcal{L}^0}(F, V) \otimes_J I = 0$. A global strict change of basis identifies the two lifts. Starting at $A = J$ and using a composition series of small extensions proves the bijection. [1]

The ratio of a lifted character to its base value takes values in $1 + \mathfrak{m}_A$. Logarithm and exponential give inverse polynomial isomorphisms between this group and $\mathfrak{m}_A$. A continuous homomorphism from the units of $F_v$ into the latter kills torsion and is freely specified by $[F_v : \mathbb{Q}_\ell]$ elements of $\mathfrak{m}_A$ after choosing a $\mathbb{Z}_\ell$-basis of the maximal torsion-free pro-$\ell$ quotient. This proves the power series assertion. If the weight product is one everywhere, the determinant on $G_L$ is unramified above $\ell$. Polarization gives the inverse product at the conjugate places. Elsewhere the determinant has finite inertia and reduces to the trivial character, so is unramified there as well. The determinant then factors through the finite class group of $L$, and $1 + \mathfrak{m}_A$ is torsion-free. It is consequently one. The converse follows by restricting the determinant to the ordinary inertia groups. The independent weight characters now number two per local unit generator. □

**Theorem 36.22** (The inverse-trace ideal at the cuspidal adjoint)**.** *In the complete local ring of Theorem 36.21 the ideal generated by* $\operatorname{tr} R(h) - \operatorname{tr} R(h^{-1})$*, $h \in G_L$, equals the weight ideal defined by*

$$\xi_{v,1}\xi_{v,3} = 1, \qquad \xi_{v,2} = 1 \quad (v \mid \ell).$$

*The quotient is a power series ring in $d$ variables over $J$. In the determinant-one ring the same ideal is generated by the $d$ logarithmic coordinates of the middle characters. Thus the vanishing on the symmetric weights holds over every Artinian quotient, including all higher infinitesimal orders.*

*Proof.* For any $A$, apply $\iota$ to the matrix part of a deformation, retaining its sign. This is an automorphism of the semidirect product and fixes $R_0$. On the ordinary groups the new flag is the dual flag in reverse order. Its reduction is the original flag, since the ordered characters at the base agree and the separating inertia matrix identifies the three successive projectors. The dual weights are the inverse weights in reverse order. If the two displayed equations hold, the dual deformation has the original weights, all local inertia conditions, and the original reduction. Theorem 36.21 identifies the two up to conjugacy. Their traces on $G_L$ are therefore equal.

Conversely, suppose that all the inverse traces agree in $A$. Fix $v \mid \ell$ and an inertia element $g_0$ separating the three base characters. At $g_0, g_0^2, g_0^3$ trace equality identifies the two characteristic

polynomials by Newton's identities in characteristic zero. Their roots $a_i = \xi_{v,i}(g_0)$ coincide in the prescribed order with the roots of the dual lift. Indeed both polynomials split and differences of distinct residual roots are units, so a lifted root is uniquely determined by its reduction. For $h$ in the same inertia group use the equality of traces at $g_0^j h$ for $j = 0, 1, 2$. Computing in each of the two stable flags gives

$$\sum_{i=1}^{3} a_i^j \big(\xi_{v,i}(h) - \xi_{v,4-i}(h)^{-1}\big) = 0.$$

The Vandermonde determinant on the $a_i$ is a unit. Consequently each difference is zero. The middle character has square one and reduces to one, so it is one since two is invertible. This proves the two equations for every $v$.

Both implications concern arbitrary Artinian $J$-algebras. To see equality of the ideals themselves, apply each implication to the quotient by one ideal and by successive powers of its maximal ideal, and use Krull intersection. Universal traces are well defined by the unique compatible strict classes in those finite quotients. Finally the logarithmic description of the weight ring in Theorem 36.21 leaves one free character per place, the first, with the third its inverse. When the product is already one the middle character supplies all the equations, yielding the stated $d$ generators. □

At the split residual representation the mixed deformation of Theorem 36.18 has the prescribed inertial weights and its inverse-trace differences span $C$. The calculation here determines this same trace expression in the characteristic-zero formal rings at the cuspidal adjoints. The fixed-weight condition appearing in `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/LinkedUnramifiedChains/AbsGood.lean`, line 30 now has zero deformation space at these points, even with arbitrary infinitesimal coefficients. To bound $C_{\mathrm{ker}}$ in Section 6.6 still requires an integral comparison at the residual representation. The source's steps relating different primes of its deformation space occur in `Development/AutomorphicArithmetic/AutomorphicVarieties/CurveDVRSpecialization/PropagationTheoremGL2.lean`, line 32 and at line 36 there.

## 36.9 Cuspidal representations with unequal inverse traces at the prescribed weight

Fix the interval, $K$, $E_*$, $N_*$ and the coefficient prime $\ell > \exp(V^{\varepsilon}/12)$ of Proposition 36.15. In that construction keep the auxiliary primes $v, w$ fixed. Write $L_0 = K\mathbb{Q}(\sqrt{sv})F_w$, $F_0 = L_0^{+}$, and let $L_n = L_0F_n$, where $F_n$ is the real degree-$\ell^n$ field in $\mathbb{Q}(\mu_{\ell^{n+1}})$. The index $n$ here is a layer in this tower. All the representations below have rank three. Fix for the moment one of the pairs $r, \alpha$ occurring on the interval, and put $\Psi = \alpha\kappa_\ell^r$. The cuspidal adjoint $U_0$ of Theorem 36.17 in weight $m = r$ on $L_0$ will be the starting point.

At each place of $F_0$ over $\ell$ choose one of the two places of $L_0$ above it. Such completions are unramified over $\mathbb{Q}_\ell$. Take a sufficiently large finite coefficient DVR $O$ with fraction field of characteristic zero and residue characteristic $\ell$. Put $A = O[[T]]$. For an inertia element $h$ at a chosen place define

$$u(h) = \frac{\log\langle \kappa_\ell(h)\rangle}{\log(1+\ell)}, \qquad \nu_T(h) = (1+T)^{u(h)}.$$

Here the brackets remove the Teichmüller part and the last power is its binomial series with exponent in $\mathbb{Z}_\ell$. The norm on units of an unramified local extension is onto. This follows on the residue fields from the finite-field norm, and on successive principal unit quotients from

surjectivity of the finite-field trace, lifting successively and taking the limit. Local reciprocity thus shows that $u$ maps onto $\mathbb{Z}_\ell$. Consider the ordered inertial characters

$$\xi_1 = \kappa_\ell^r \nu_T^2, \qquad \xi_2 = \nu_T, \qquad \xi_3 = \kappa_\ell^{-r} \nu_T^{-3}. \tag{36.1}$$

The product is one. Write $t_n = \zeta_{\ell^n} - 1$ for $n \geq 1$, with a primitive root chosen for each $n$, and $z_k = (1+\ell)^k - 1$ for positive multiples $k$ of $\ell - 1$. At $z_k$ the three exponents are $r+2k, k, -r-3k$. At $t_n$ their algebraic exponents are $r, 0, -r$ and their finite parts have exponents $2, 1, -3$ in the character $\nu_{t_n}$ of exact order $\ell^n$. The three finite parts are distinct because $\ell > 5$.

**Proposition 36.23** (A family of Hecke traces through the adjoint)**.** *There is a finite integral domain $B_0$ over $A$, containing $A$, with a point at $T = 0$ giving $U_0$. For every $t_n$ and $z_k$ it has a characteristic-zero point above that parameter. Each such point $y$ gives a semisimple representation $U_y$ of $G_{L_0}$ attached, up to the fixed twist by $\kappa_\ell$, to a classical automorphic form on a unitary group. For each $h \in G_{L_0}$ there is an element $t(h) \in B_0$ evaluating to $\operatorname{tr} U_y(h)$ at all these points and at the adjoint. The residual semisimplification at every one of them is $\bar{\Psi} \oplus 1 \oplus \bar{\Psi}^{-1}$.*

*Proof.* Use the definite unitary group in three variables for $L_0/F_0$ which is quasi-split at every finite place. The existence condition for this inner form holds in odd rank. The quadratic extension is unramified at all finite places by Proposition 36.15. Labesse's descent, in the form of Geraghty's Proposition 2.2.7, puts the regular algebraic conjugate self-dual cuspidal representation of Theorem 36.17 on this group. At the split places the descent has the specified general linear component. It has hyperspecial fixed vectors at the inert places, where the original representation is unramified. Choose fixed level away from $\ell$ admitting these vectors and include at one additional split place a sufficiently deep principal subgroup to make the level sufficiently small. At that place the group can be taken torsion-free, for instance inside the domain of convergence of the matrix logarithm at an odd residue prime. This removes finite stabilizers of the definite double cosets. All tame level away from hyperspecial subgroups can thus be confined to finitely many split places. [20]

At $\ell$ this descent is spherical. Its unramified Frobenius slopes, after the algebraic normalization, are its three ordered Hodge slopes. Indeed the two-dimensional ordinary newform has Frobenius roots of valuations $0, r$ over $\mathbb{Q}_\ell$, and the adjoint has the two ratios and one. The Tate twist shifts all slopes together. At a residue degree $f$ the valuations of Frobenius increase by the factor $f$. The formula for Iwahori Hecke operators on an unramified principal series now gives an ordinary refinement. In that formula the valuation of the $j$-th normalized operator on the ordered refinement is the sum of the first $j$ Frobenius slopes minus the sum of the first $j$ Hodge slopes, in the same normalization. The local component is spherical and generic since it comes from a cuspidal general linear representation. Thus this refinement occurs in the component also when $r = 1$. These are the spherical and Iwahori computations in Geraghty's Lemmas 2.7.5 and 2.7.6. [20]

We use the ordinary control theorem for this definite group. We record the parts used here. For sufficiently small tame level the dual ordinary module is finite free over the completed group ring of the three principal-unit factors at each chosen place. The prime-to-$\ell$ diamond character can be fixed by an integral idempotent, giving a free summand over this local ring. At a dominant algebraic weight times a finite-order character, its characteristic-zero specialization is dual to the space of classical ordinary forms with that algebraic weight and diamond character. One takes level at $\ell$ deep enough for the character. These assertions are Proposition 2.5.3, the weight comparison in Section 2.6 and Lemma 2.6.4 of Geraghty. In Proposition 2.5.3 one can take $b_0 = 1$ because of our choice of tame level. We use no localization hypothesis on the residual representation. [20]

Choose the prime-to-$\ell$ summand of the adjoint. Specialize the weight algebra to $A$ along (36.1). To specify this map, use the three ordered Galois inertial characters as coordinates relative to their values at the adjoint. At a purely algebraic weight $\lambda$ the $j$-th character of the normalized representation, composed with local reciprocity on units, is $\kappa_\ell^{2-j} \prod_\tau \tau(\cdot)^{-\lambda_{\tau,4-j}}$, where the cyclotomic factor is also read through reciprocity. This is Corollary 2.7.8 of Geraghty with the twist by $\kappa_\ell$. The algebraic weight character of the completed group ring uses precisely the reversed negative $\lambda$, with a fixed factor independent of $\lambda$, as in his Definition 2.6.3. Thus multiplying the three base unit characters by $\nu_T^2, \nu_T, \nu_T^{-3}$ defines a continuous map on the principal-unit factors. At $z_k$ it is a purely algebraic weight with strictly increasing Hodge indices and strictly dominant coefficient weight. The choice $(\ell - 1) \mid k$ keeps its prime-to-$\ell$ diamond part at the spherical value. At $t_n$ it is the original dominant algebraic weight times a finite-order character, so the control theorem applies there as well. [20]

The resulting module $M$ over $A$ is finite free and nonzero by its adjoint specialization. Inside $\mathrm{End}_A(M)$ let $B$ be the $A$-algebra generated by the good split spherical Hecke operators. It is finite over $A$ because $A$ is Noetherian, and it is torsion-free. On every classical fiber these commuting operators act semisimply. Indeed the characteristic-zero automorphic decomposition expresses the space in terms of its irreducible constituents, on each of which the good spherical operators act by their Satake characters. The same holds on the dual. Every character above one of the indicated parameters occurs on this fiber. To see this, $M$ is finite and faithful over $B$, so its support is all of $\operatorname{Spec} B$. Localize at the prime for the character and take the residue field. Nakayama gives a nonzero module, also after extending that residue field.

For each eigensystem take its classical semisimple Galois representation and twist by $\kappa_\ell$, calling the result $U_y$. Geraghty's Proposition 2.7.2 gives the representation, conjugate duality after this twist, potential semistability at $\ell$, and its good Frobenius polynomials. This construction also applies to the constituents of a noncuspidal transfer. For a good split geometric Frobenius the normalized trace is the first spherical operator multiplied by the inverse residue cardinality, a unit in $O$. Pick one representation for each good eigensystem. Its semisimplification is independent of the choice by Chebotarev. [20]

Evaluation at the classical characters above $0, t_n, z_k$ embeds $B$ in the product of their coefficient valuation rings. Indeed, an element evaluating to zero there acts by zero on the corresponding fibers, by semisimplicity. Each of its matrix entries on $M$ is then a power series vanishing at infinitely many $z_k$. Any nonzero series in $O[[T]]$ is a power of a uniformizer times a distinguished polynomial times a unit, by Weierstrass preparation. It has only finitely many roots in the open unit disc. The same proof gives injectivity if just the $z_k$ are used. The finite $A$-module $B$ is compact with its maximal-base-ideal topology, and its image in the product is closed. Good split Frobenius conjugacy classes are dense in $G_{L_0}$. Indeed the primes of $L_0$ of residue degree greater than one over $F_0$ have density zero, and one can apply Chebotarev to any finite quotient over $L_0$. By continuity on each finite list of classical representations, the tuple of traces at any $h$ is therefore in that closed image. This defines $t(h)$ already in $B$. Newton's identities give all characteristic coefficients there, since $2, 3$ are units.

Take a minimal prime of $B$ below the kernel of the adjoint evaluation and pass to its quotient $B_0$. Its contraction to $A$ is zero, since elements of a minimal prime of a Noetherian ring are zero divisors and $B$ is torsion-free over $A$. Lying over now supplies a point above each stated parameter, with finite coefficient extension. A finite domain over the complete local ring $A$ is local, by the decomposition of a finite algebra over a Henselian ring into local factors. We took $O$ large enough for the adjoint eigenvalues, so evaluation at that point identifies this residue field with the residue field of $O$. All points on $B_0$ thus have the same reduced Frobenius polynomials. Chebotarev and Brauer–Nesbitt identify their residual semisimplifications with that of $U_0$. This proves the proposition. □

**Lemma 36.24** (Ordinary flags at the finite-order parameters)**.** *At any point $y$ of Proposition 36.23 where $U_y$ is absolutely irreducible, its local representations at the chosen places above $\ell$ have stable full flags with the ordered inertial characters* (36.1) *specialized at $y$.*

*Proof.* Fix one chosen place, with decomposition group $D$ and inertia $I$. Choose $s_0 \in I$ for which $\bar\kappa_\ell^r(s_0), 1, \bar\kappa_\ell^{-r}(s_0)$ are pairwise distinct. This is possible since rational cyclotomic inertia is full on $L_0$ and $2r < \ell - 1$. Let $a_i = \xi_i(s_0) \in A^\times$. The differences $a_i - a_j$ are units for $i \neq j$. Define polynomials over $A$ by

$$e_i(X) = \prod_{j \neq i} \frac{X - a_j}{a_i - a_j}.$$

At each regular algebraic specialization $z_k$, the ordinary flag and its characters are given by Geraghty's Corollary 2.7.8. The operators $e_i(U_y(s_0))$ are the projectors onto the three eigenlines, ordered so that the first line and first two lines give this flag. For $d \in D$, $g \in G_{L_0}$ and $j > i$, form the trace of

$$e_j(U_y(s_0))\, U_y(d)\, e_i(U_y(s_0))\, U_y(g).$$

Expanding the polynomials writes it as a combination over $A$ of the group traces already constructed in $B$. It vanishes at every $z_k$, since $d$ is upper triangular in the flag. The injectivity proved in Proposition 36.23 makes this an identity in $B$. The same argument gives $\det(X - U_y(s_0)) = \prod_i (X - a_i)$ and, for $h \in I$,

$$t(s_0^j h) = \sum_i a_i^j \xi_i(h) \qquad (j = 0, 1, 2)$$

before specialization. [20]

At the point in question the three roots still differ by units. Work over the algebraic closure of its fraction field. By Burnside's theorem the matrices $U_y(g)$ span the full matrix algebra. Its trace pairing is nondegenerate. Hence $e_j(U_y(s_0))U_y(d)e_i(U_y(s_0)) = 0$ for $j > i$. This gives the ordered flag on the entire decomposition group. The three trace identities on $I$ identify its diagonal characters by the invertible Vandermonde matrix on the $a_i$. The projectors are defined over the field of the representation, so the flag is defined there as well. At the conjugate place conjugate duality gives the reversed inverse characters. □

**Proposition 36.25** (Irreducibility and the inertial types away from the coefficient prime)**.** *There is a nonzero series $a(T) \in A$ such that at every classical parameter of Proposition 36.23 where $a$ does not vanish, all points of $B_0$ above it are absolutely irreducible. For such points the determinant of $U_y$ is one, and on inertia away from $\ell$ one has*

$$U_y|_I \simeq \alpha \oplus 1 \oplus \alpha^{-1}.$$

*In particular this holds at $t_n$ for every sufficiently large $n$.*

*Proof.* For any nonzero $b \in B_0$ there is a nonzero $a_b \in A \cap bB_0$. Indeed $B_0 \otimes_A \mathrm{Frac}(A)$ is a finite-dimensional domain over a field and hence is a field. Clear the denominator of $b^{-1}$ there. Consequently any infinite set of distinct parameters $t_n$, or $z_k$, with one point chosen over each, detects all nonzero elements of $B_0$, by Weierstrass preparation.

By absolute irreducibility at the adjoint, choose nine elements $h_i$ whose matrices there span the full matrix algebra. The Gram determinant

$$D_* = \det\big(t(h_i h_j)\big)_{1 \le i,j \le 9}$$

is nonzero at that point by the trace pairing. Wherever it is nonzero the same matrices span the full algebra, so the representation is absolutely irreducible.

Let $S'$ be the finite set of places of $L_0$ away from $\ell$ over the chosen tame level and the conductor of $\alpha$. At the remaining places every $U_y$ is unramified by Proposition 2.7.2 of Geraghty. Fix a place in $S'$ of residue characteristic $p$. Inside a fixed local closure pass to the completion $M'$ of $L_0E_*$, and let $e'$ be its degree over the local completion of $L_0$. Write $q$ for the residue cardinality of $M'$. Residual semisimplification makes the reduction on $I_{M'}$ upper unipotent in a suitable stable lattice, since $\alpha$ and the residual cyclotomic character are trivial there. The image of $I_{M'}$ is pro-$\ell$. Indeed it lies above a finite $\ell$-group in a matrix group over a coefficient DVR whose principal congruence kernel is pro-$\ell$. Thus wild inertia acts trivially, as do the prime-to-$\ell$ factors of tame inertia. If $H$ is the image of a generator of its tame $\mathbb{Z}_\ell(1)$ factor, then arithmetic Frobenius conjugates $H$ to $H^q$. Raising to the $q$-th power permutes its at most three distinct eigenvalues. Each eigenvalue therefore has order dividing $q^6-1$. Continuity gives the same bound for all inertia in $M'$. For inertia of the original completion the order of every eigenvalue divides $e'(q^6-1)$, because the $e'$-th power of any such element belongs to $I_{M'}$. The local field extension is Galois. [20]

For a fixed inertia element $h$ each coefficient of its characteristic polynomial thus takes only finitely many possible values. Form the monic polynomial with the distinct possible values as roots, adjoining all their conjugates. Its coefficients lie in $O$, since they are integral and invariant over its fraction field. It vanishes on the corresponding function of $B_0$, by the preceding density argument. At the adjoint this function takes a value $c \in O$. Removing the simple factor $X - c$ leaves a factor nonzero on specialization at the adjoint. Since $B_0$ is a domain, the function equals $c$ identically. This applies separately to each $h$, and shows that all inertial characteristic polynomials agree with those of the adjoint. On $I_{M'}$ they are $(X-1)^3$.

To remove possible unipotent inertia take any arithmetic Frobenius lift $\phi$ over $M'$. With $P_\phi(X)$ its characteristic polynomial over $B_0$, form

$$b_p = \operatorname{Res}_X\left(P_\phi(X), q^3 P_\phi(X/q)\right).$$

This is nonzero at the adjoint. We verify this even at the ramified primes of the elliptic newform. At a good prime the Frobenius roots of the newform have equal complex absolute values by purity. At a bad prime the two-dimensional Weil–Deligne parameter has zero monodromy and two characters distinct on inertia by the proof of Theorem 36.17. For the unitary local normalization the pair of quasicharacters is stable under $\gamma \mapsto \overline{\gamma}^{-1}$. This follows by taking the conjugate dual of the unitary local representation and using its Langlands parameter. That involution fixes each finite inertial character. The two inertial characters are distinct, hence it fixes each quasicharacter separately. Both are unitary. The common twist back to the classical normalization gives them equal complex absolute values on every Frobenius lift. Passing to the local extension raises norms to the residue degree and introduces only roots of unity from inertia. The adjoint eigenvalues are the two ratios and one, all of absolute value one. Multiplication by $q > 1$ separates their multiset from itself. This proves the asserted nonvanishing. [17, 6, 13]

At any point where $b_p \neq 0$, the unipotent matrix $H$ on $I_{M'}$ has a finite polynomial logarithm $N$. Conjugation by $U_y(\phi)$ sends $N$ to $qN$. The eigenvalues of this conjugation on endomorphisms are the ratios of the three Frobenius eigenvalues, also for a nondiagonalizable matrix, by triangularization. The resultant excludes $q$ from those ratios. Therefore $N = 0$ and $I_{M'}$ acts trivially. The inertia image at the original place is now finite. Its characteristic-zero representation is semisimple and is determined by its character. Its trace on every element was shown to agree with the adjoint, giving $\alpha, 1, \alpha^{-1}$.

The determinant at each classical point is everywhere unramified. Away from $\ell$ this follows already from the inertial polynomials. At the chosen places above $\ell$, the identity that the inertial characteristic polynomial is $\prod_i (X - \xi_i(h))$ holds for every $h \in I$, by specialization at

the regular $z_k$ and injectivity as in Lemma 36.24. The product is one, and conjugate duality handles the other places. The determinant hence factors through the finite ideal class group of $L_0$. For each $g$ its values at all classical points are roots of unity of bounded order. Apply the finite-value argument again to the determinant function of $g$ in $B_0$. Its value at the adjoint is one, so it is one throughout.

Finally take $b$ to be $D_*$ times one resultant for every place in $S'$. Each factor is nonzero at the adjoint and the product is nonzero in $B_0$. Take $a = a_b$. All points above a parameter at which $a \neq 0$ have every required property. Only finitely many of the distinct $t_n$ can be zeros of $a$. □

**Theorem 36.26** (Full special linear monodromy at the original weights)**.** *For each sufficiently large $n$, the same $L_n$ has the following property for every $x, r, \alpha$ on the interval. There is a determinant-one representation $U$ of $G_{L_n}$ with Zariski closure* $\mathrm{SL}_3$ *over the algebraic closure of the coefficient field. Its residual semisimplification is $\bar{\Psi} \oplus 1 \oplus \bar{\Psi}^{-1}$. Its restriction to inertia away from $\ell$ is $\alpha \oplus 1 \oplus \alpha^{-1}$. At each place above $\ell$ it is ordinary crystalline with ordered inertial characters $\kappa_\ell^{r}, 1, \kappa_\ell^{-r}$. Its ordinary characters on the full decomposition group reduce in the order $\bar{\Psi}, 1, \bar{\Psi}^{-1}$. The representation $U \otimes \kappa_\ell^{-1}$ is attached to a regular algebraic conjugate self-dual cuspidal automorphic representation on* $\mathrm{GL}_3(\mathbb{A}_{L_n})$.

*Proof.* Choose $y$ over $t_n$ as in Proposition 36.25. The weak base change of its definite unitary automorphic representation to $\mathrm{GL}_3(\mathbb{A}_{L_0})$ is cuspidal. Indeed the description of the transfer used in Geraghty's Proposition 2.7.2 expresses the Galois representation of a transfer with several constituents as their direct sum, and a nontrivial residual discrete-series block as a sum of the corresponding twists. This is checked at good primes and then by uniqueness of semisimplification. Either decomposition would contradict $D_*(y) \neq 0$. The cuspidal transfer is regular algebraic and conjugate self-dual, with associated representation $U_y \otimes \kappa_\ell^{-1}$. [20]

First $U_y$ stays absolutely irreducible on every open subgroup of $G_{L_0}$. To see this, pass to the normal core of any purported counterexample. On this finite-index normal subgroup the sum of all simple submodules is nonzero and stable under the full group, hence is the whole representation. Its isotypic components are permuted transitively. Dimension three leaves the scalar case or three distinct lines if that restriction is reducible. In the scalar case the projective image is finite, contradicting the distinct Hodge–Tate weights, which persist on finite extensions. In the three-line case the kernel of $\bar{\Psi}$ maps into a pro-$\ell$ subgroup of the matrix image. Indeed the global residual representation is triangular in a suitable lattice basis, with all diagonal entries trivial on that kernel. The continuous permutation action on the lines is a quotient of the matrix image. Since $\ell > 3$, it factors through the cyclic group $\bar{\Psi}(G_{L_0})$. A transitive cyclic subgroup of $S_3$ is the group of order three. In a basis of the three lines the matrices are monomial. Conjugating by a diagonal matrix with entries $1, \zeta_3, \zeta_3^2$, in cyclic order, therefore gives a nontrivial cubic self-twist. Its reduction would permute $\{1, \bar{\Psi}, \bar{\Psi}^{-1}\}$ by multiplication by a character of order three. The orbit of 1 would be the full set, whereas $\bar{\Psi}$ has order greater than three on inertia at $\ell$ by $4r < \ell - 1$. This rules out the last case.

The identity component of the algebraic monodromy now acts irreducibly, since its preimage has finite index. Its unipotent radical has fixed vectors by triangularization and normality makes that space invariant under the component. The radical thus acts trivially. Its connected center acts by scalars and is trivial because the determinant is one. The resulting connected semisimple subgroup is $\mathrm{SL}_3$ or the image of $\mathrm{SL}_2$ on its symmetric square, by Theorem 33.13. In the second case it preserves a unique line of nondegenerate symmetric forms. Its normalizer, and hence the entire image, acts by a character on this line. It follows that $U_y^\vee \simeq U_y \otimes \delta$ with $\delta^3 = 1$, taking determinants. Compare the two sides on an ordinary inertia group using

Lemma 36.24. The three algebraic exponents $r, 0, -r$ are distinct. Characters with different exponents cannot agree after a finite twist, since the cyclotomic image on inertia is open. Consequently the middle character on the two sides must agree, forcing $\nu_{t_n}^2$ or its inverse to be the restriction of $\delta$. Its order is $\ell^n > 3$. Thus the connected monodromy is $\mathrm{SL}_3$, as is the whole monodromy. Restricting to any finite extension retains this connected Zariski closure.

It remains to check the local assertions after restriction to $L_n$, since the finite characters were essential over $L_0$. Each $\nu_{t_n}$ is obtained from the rational cyclotomic character via its quotient $(1 + \ell\mathbb{Z}_\ell)/(1 + \ell^{n+1}\mathbb{Z}_\ell)$. This quotient is killed by restriction to $F_n$. The inertial characters on the flag at $L_n$ are therefore precisely those stated, including at the conjugate places by conjugate duality. Each $U_y$ is potentially semistable at $\ell$. In its Weil–Deligne representation the semisimple inertia characters are the finite parts of the three de Rham characters on the flag. Indeed formation of the potentially semistable module is exact on potentially semistable representations, and a crystalline cyclotomic power has trivial Weil–Deligne inertia. The unramified part of a flag character also has trivial inertia. The three finite characters at $L_0$ are distinct, so the nilpotent monodromy operator commuting with inertia preserves three lines and vanishes. After passage to $L_n$, Weil–Deligne inertia and monodromy are both trivial. The potentially semistable representation is therefore crystalline there, by descent of its period module. Residual cyclotomic inertia on $L_n$ still has order $\ell - 1$ by Proposition 36.15. Its three distinct residual characters on inertia identify the flag characters uniquely inside the global residual triple, also on the full decomposition group. The assertion away from $\ell$ follows by restriction from Proposition 36.25.

Cyclic base change to $L_n$ preserves cuspidality at every degree-$\ell$ step. Otherwise there would be a self-twist of order $\ell$, which is impossible in rank three by comparison of central characters. Regularity and conjugate self-duality persist, and the attached representation agrees with $U_y|_{G_{L_n}} \otimes \kappa_\ell^{-1}$ by the good Frobenius polynomials. [2, 15] Finally the interval uses finitely many characters of $E_*$ and $r < t_*$. Take $n$ above the largest of the finitely many exceptional indices from Proposition 36.25. The original $v, w, F_w$, and now $L_n$, are common to all these choices. □

**Corollary 36.27** (An integral inverse trace on the common auxiliary tower)**.** *For fixed interval, $\ell, r, \alpha$ and the component $B_0$ above, there exist $h \in [G_{L_0}, G_{L_0}]$, a nonnegative rational number $\mu$ and a nonnegative integer $j$ such that for every sufficiently large $n$ and every $y$ above $t_n$,*

$$0 < v_\ell\Big(\operatorname{tr} U_y(h) - \operatorname{tr} U_y(h^{-1})\Big) \le \mu + \frac{j}{\ell^{n-1}(\ell-1)}, \qquad v_\ell(\ell) = 1.$$

*The element $h$ lies in every $G_{L_n}$. At each such $n$ the same valuation at a given $y$ occurs at good Frobenius elements of rational primes splitting completely in $L_n$.*

*Proof.* Choose one point with full special linear monodromy from Theorem 36.26. The image of the commutator subgroup of $G_{L_0}$ there is Zariski dense in $\mathrm{SL}_3$. Indeed taking its closure gives an algebraic subgroup containing the algebraic commutators of the dense image and hence of the whole group, which is perfect. The trace and the inverse trace are distinct regular functions there, as is seen on $\mathrm{diag}(2, 2, 1/4)$. Choose an element $h$ of that commutator subgroup at which they differ. Every $L_n/L_0$ is abelian, so $h \in G_{L_n}$ for every $n$. Now $b = t(h) - t(h^{-1})$ is nonzero in $B_0$. Choose $0 \neq a \in (b) \cap A$ as in the proof of Proposition 36.25. At every integral specialization $v_\ell(b(y)) \le v_\ell(a(t_n))$ when $a(t_n) \neq 0$.

Factor $a = \varpi^e P(T) u(T)$ by Weierstrass preparation, with $P$ distinguished of degree $j$ and $u$ a unit. Set $\mu = e\, v_\ell(\varpi)$. Each root of $P$ has strictly positive valuation, counting a zero root as valuation infinity. Since $v_\ell(t_n) = 1/(\ell^{n-1}(\ell-1))$, for sufficiently large $n$ this is smaller than the valuation of every root. The nonarchimedean triangle law then gives $v_\ell(a(t_n)) =$

$\mu + j/(\ell^{n-1}(\ell-1))$. The lower strict inequality follows because all reduced inverse traces coincide for the symmetric residual triple.

Finally fix $n$ and $y$. The valuation of the nonzero difference is constant on a sufficiently small open neighborhood of $h$ and on its $G_{L_n}$-conjugates. Chebotarev over $L_n$ gives a positive-density set of primes with Frobenius in this set. Prime ideals of residue degree greater than one over $\mathbb{Q}$ have density zero. Since $L_n/\mathbb{Q}$ is Galois, an unramified rational prime below a degree-one prime splits completely. Finitely many bad places can be discarded. □

**Proposition 36.28** (The first cyclotomic layer admitting crystalline descent)**.** *Fix $y$ and $n$ as in Theorem 36.26, and put $U = U_y|_{G_{L_n}}$. For $0 \le b \le n$, the unique determinant-one extension of $U$ to $G_{L_b}$, up to isomorphism, is $U_y|_{G_{L_b}}$. At a chosen place over $\ell$ its projective Weil–Deligne inertia has order $\ell^{n-b}$. Every extension to $G_{L_b}$ has the same projective inertia there. In particular $n$ is the least layer permitting a crystalline extension, even with arbitrary determinant. There is no representation of $G_K$ crystalline above $\ell$ whose restriction to $G_{L_n}$ is $U$.*

*Proof.* Identify the restrictions of two extensions over an algebraic closure of the coefficient field. For each $g \in G_{L_b}$ their matrices induce the same conjugation on the image of $G_{L_n}$, by normality. The ratio of their matrices is scalar by the absolute irreducibility of $U$. The ratio is a character of the cyclic quotient $\mathrm{Gal}(L_n/L_b)$, of order dividing $\ell^{n-b}$. If both determinants are one its cube is one and it is trivial.

Locally $L_0$ is unramified over $\mathbb{Q}_\ell$, while the cyclotomic degree-$\ell^b$ layer is totally ramified. The image of inertia of $L_b$ in the rational degree-$\ell^n$ cyclotomic quotient therefore has order $\ell^{n-b}$. For instance the compositum of the local field $L_0$ with the degree-$\ell^n$ cyclotomic layer is totally ramified over its degree-$\ell^b$ layer with exactly that degree. Thus the restriction of $\nu_{t_n}$ has this exact order. On Weil–Deligne inertia the three characters of $U_y$ are its powers $2, 1, -3$. The projective image has order dividing $\ell^{n-b}$ and achieves it, since the ratio of the first two characters is $\nu_{t_n}$. Scalar twists preserve this image. For $b < n$ the nontrivial projective inertia rules out crystallinity, and for $b = n$ Theorem 36.26 gives crystallinity. A crystalline representation on $K$ would stay crystalline after restriction to $L_0$, in contradiction to the case $b = 0$. □

Theorem 36.18 realizes the full strict module $C$ on each of these same $L_n$. Corollary 36.27 gives a nonzero classical inverse trace on $G_{L_n}$ and bounds its valuation along the tower for a fixed component. Proposition 36.28 excludes these points from the crystalline descents counted on $K$. The local calculation distinguishes having the required weights and types over an auxiliary field from obtaining a point in the trace algebra on the original field. Controlling the full order in Theorem 36.14 still requires an integral comparison with all of $C$ on the original interval. The corresponding comparison of the two signed periods is specified in Section 6.6.

# 37 Ordinary curves and cuspidal components

## 37.1 Global units and ordinary traces in residue characteristic

Keep $L = L_0$, $F = F_0 = L^+$, $d = [F : \mathbb{Q}]$, and the chosen places over $\ell$ of Section 36.9. Write $k$ for the residue field of $O$, enlarged to contain the values of the three residual characters. Put

$$H = \prod_{v|\ell} H_v, \qquad H_v = 1 + \ell\mathcal{O}_{F_v}, \qquad \Lambda = O[[H \times H]].$$

Each completion is unramified and $\ell$ is odd, so logarithm identifies $H$ with a free $\mathbb{Z}_\ell$-module of rank $d$. Two variable characters $\lambda_1, \lambda_2$ of $H$ specify the three weight variations with product

one by taking $\lambda_3 = (\lambda_1\lambda_2)^{-1}$. A character of $H$ is evaluated on a local unit by removing its Teichmüller part and using the corresponding factor. For a real global unit $e \in \mathcal{O}_F^\times$, denote by $j(e) \in H$ the tuple of all these principal parts. On principal units we use local reciprocity in the same convention at both places of $L$ over each $v$.

**Theorem 37.1** (The weights of sums of three global characters)**.** *Let $R$ be a complete discrete valuation ring of characteristic $\ell$ with finite residue field containing $k$, with the indicated copy of $k$ in $R$. Take continuous $\lambda_i : H \to 1 + \mathfrak{m}_R$ as above. At the chosen places prescribe the ordered inertial characters*

$$\bar{\kappa}_\ell^{r}\lambda_1, \qquad \lambda_2, \qquad \bar{\kappa}_\ell^{-r}\lambda_3,$$

*and at the conjugate places their reversed conjugate inverses. There exist three continuous global characters of $G_L$ with residual characters $\bar{\Psi}, 1, \bar{\Psi}^{-1}$, with these ordered inertial characters and with conjugate self-dual direct sum, after a finite coefficient extension, if and only if*

$$(\lambda_1/\lambda_3)(j(e)) = 1 \quad \text{for all } e \in \mathcal{O}_F^\times. \tag{37.1}$$

*When they exist their finite inertial characters away from $\ell$ are those of the original residual triple and their product is one. In $\mathrm{Spec}(\Lambda/\varpi)$, where $\varpi$ is a uniformizer of $O$, the reduced weight locus defined by (37.1) has codimension $d-1$.*

*Proof.* The ratio of a global character to its residual character, lifted through the finite constant field, takes values in principal units. This latter group is pro-$\ell$ and torsion-free. In particular in characteristic $\ell$ the identity $(1+x)^{\ell^a} = 1 + x^{\ell^a}$ excludes nontrivial $\ell$-power torsion. Away from $\ell$ the local unit group is an extension of a finite group by a pro-$p$ group with $p \neq \ell$. Its image in these principal units is trivial. The ratio is thus unramified away from $\ell$.

The residual characters at a chosen inertia group are distinct, since $2r < \ell - 1$ and residual cyclotomic inertia is full. The global character reducing to $\bar{\Psi}$ therefore has variation $\lambda_1$ at the chosen place and $\lambda_3^{-1}$ at the conjugate place, after identification by complex conjugation. In fact $\bar{\Psi}$ comes from $\mathbb{Q}$, so the reversed inverse with this residual value is the inverse of the third character. A principal idele has trivial global Artin symbol. Applying the character ratio to the idele of a real unit $e$ leaves just the product on the left of (37.1). This proves necessity.

Conversely prescribe this first variation on the principal units of $L$ above $\ell$, using $\lambda_1$ on one side and $\lambda_3^{-1}$ on the other. It kills all global units. Indeed for $e' \in \mathcal{O}_L^\times$ the quotient $e'/\overline{e'}$ is an algebraic integer and every complex conjugate has absolute value one. It is a root of unity by Kronecker's theorem. A positive power of $e'$ is therefore a real unit. The given character kills that power by (37.1), and the target is torsion-free. The corresponding local variation for the middle character uses $\lambda_2$ and $\lambda_2^{-1}$. It kills real units by cancellation and hence all global units in the same way.

For clarity, the ray class sequences at the powers of the primes above $\ell$, followed by their pro-$\ell$ inverse limit, identify the image of the local principal units in the maximal abelian pro-$\ell$ quotient unramified outside $\ell$ with their quotient by the closure of the projected global units. The quotient of that image in the full group is the $\ell$-primary ideal class group. These are the unit, local-residue, ray-class and ideal-class terms of the ray class exact sequence. Projection to the pro-$\ell$ factors removes the local prime-to-$\ell$ roots of unity. Thus each prescribed character defines a continuous character on a subgroup of finite $\ell$-power index. If $\ell^a$ kills the quotient, its extension takes an element $g$ to the unique $\ell^a$-th root of the prescribed value on $g^{\ell^a}$. All values lie in the single finite extension obtained by adjoining the $\ell^a$-th roots of the fraction field of $R$, which is a Laurent series field over a finite field. The root map is multiplicative and continuous, and the roots of principal units are principal units. Uniqueness makes the middle character conjugate inverse to itself. Multiply the first variation by $\bar{\Psi}$, take its conjugate inverse as the

third character, and use the constructed middle character. Their determinant is unramified everywhere, reduces to one and has torsion-free image. Finiteness of the class group makes it one. This proves sufficiency.

For the codimension, Leopoldt's theorem for totally real abelian fields gives rank $d-1$ for the closure of $j(\mathcal{O}_F^\times)$ in $H$. Since $\lambda_1/\lambda_3 = \lambda_1^2\lambda_2$, the subgroup tested in $H \times H$ is the image of that closure under $h \mapsto (h^2, h)$. This is an embedding with projection back onto $H$ in the second factor. In Smith coordinates on $H \times H \simeq \mathbb{Z}_\ell^{2d}$ the subgroup has generators $\ell^{a_i} e_i$ for $1 \le i \le d-1$. In the power series coordinates of $k[[H \times H]]$ the equations are $(1+X_i)^{\ell^{a_i}} - 1 = X_i^{\ell^{a_i}}$. Their radical is $(X_1, \dots, X_{d-1})$, proving the assertion, including the possible non-saturation of the original unit lattice. [21] □

**Proposition 37.2** (The integral Hecke algebra on the weight space of product one)**.** *There is a finite local domain $B'$ containing $\Lambda$, finite as a $\Lambda$-module and of dimension $1+2d$, with a characteristic-zero specialization at the cuspidal adjoint $U_0$. It carries a continuous trace $t : G_L \to B'$ interpolating the normalized ordinary Galois traces. Let a prime of $B'$ lie over the kernel of a surjective continuous weight map $\Lambda \to k[[T]]$. Over a finite extension of the fraction field of $B'$ modulo this prime, the resulting trace comes from a continuous semisimple Galois representation. It has determinant one, is conjugate self-dual, reduces semisimply to $\bar\Psi \oplus 1 \oplus \bar\Psi^{-1}$, and has local ordinary flags with the prescribed weight characters.*

*Proof.* In the construction of Proposition 36.23 use the entire factor of the weight algebra on which the product of the three inertial characters is one. In the relative coordinates at the adjoint it is precisely $\Lambda$. Base change of the same free ordinary module gives a finite free module over $\Lambda$. We justify the density assertion needed to interpolate here in several variables.

At every chosen $\ell$-adic embedding vary the first two algebraic exponents by integers $a_\tau, b_\tau$ and the last one by $-a_\tau - b_\tau$. Choose all these integers divisible by the orders of the local residue unit groups, preserving the prime-to-$\ell$ diamonds. Impose the inequalities making the resulting coefficient weight strictly dominant at each embedding. These integer tuples are dense in $\mathbb{Z}_\ell^{2d}$. In any fixed residue classes modulo $\ell^m$ choose integral representatives with the required prime-to-$\ell$ divisibility by the Chinese remainder theorem. With $b_\tau$ fixed one can increase $a_\tau$ by arbitrary positive multiples of both moduli, ensuring both consecutive gaps are sufficiently large.

A series in $\Lambda$ vanishing at these weights is zero. In logarithmic coordinates on an open neighborhood of the adjoint, the map from embedding exponents to unit character values has invertible Jacobian over the coefficient field. Indeed its blocks are the matrices $(\tau(\log h_j))_{\tau,j}$ for principal-unit bases $h_j$. Their logarithms are bases of the local fields over $\mathbb{Q}_\ell$, and the embeddings identify the scalar extension of each separable local field with a product. Continuity and the density just proved give vanishing of the composite series on a product of sufficiently small $\mathbb{Z}_\ell$-balls about the origin. Iterating the one-variable identity theorem in each coordinate forces its analytic germ to be zero, also over the larger coefficient field. The formal inverse to the change of coordinates then forces the original Taylor series to be zero. The weight coordinates are relative to the adjoint, so this is the original power series itself.

Now form the good split Hecke algebra inside the endomorphisms of the free module. The classical fibers at the regular weights just used are semisimple for these operators. Applying the density assertion to their matrix entries gives a faithful embedding in the product of the classical coefficient rings, also if the adjoint evaluation is included. The compactness and Chebotarev argument of Proposition 36.23 produces $t(h)$ for all $h$. Continuity of $t$ follows because the compact Hecke algebra embeds homeomorphically onto its image in that Hausdorff product. Take a minimal prime below the adjoint specialization exactly as there. This gives the finite

domain $B'$, integral over $\Lambda$ with zero contraction of the minimal prime. Integral extension preserves dimension. Its residue field is that of $O$ by evaluation at the adjoint. The ordinary flags at the regular weights and their character formulas are those of Geraghty, as used in Lemma 36.24. [20]

Lying over supplies a prime above any specified kernel in the statement. The quotient domain is finite over $k[[T]]$. Its fraction field is a finite extension of $k((T))$ and its normalization is a complete discrete valuation ring there. The trace satisfies the degree-three pseudocharacter identities since they are polynomial trace identities at every regular classical point. Over the algebraic closure of the new fraction field a degree-three pseudocharacter is the trace of a unique semisimple representation up to isomorphism. Here 3! is invertible, as required by the trace formulation of this theorem. [22]

We give the coefficient and continuity arguments. In the semisimple representation let $D$ be the span of the group matrices over the algebraic closure. It is a product of full matrix algebras on the distinct simple constituents. This follows by factoring the acting algebra by its radical, which kills a semisimple faithful module, and applying the structure theorem for finite-dimensional semisimple algebras. The trace pairing on $D$ is nondegenerate, since the multiplicity of each constituent is at most three and is nonzero in the coefficient field. Choose group matrices forming a basis of $D$ and adjoin their entries, a finite coefficient extension. Every group matrix is defined over it. Indeed its coordinates in that basis are recovered from its traces against the basis by inverting the trace Gram matrix. Those traces are continuous functions of the group element by the construction of $t$, so the representation is continuous. The compact matrix image preserves the integral span of the translates of any fixed full lattice. This span is bounded by compactness and therefore is a finite free module. The residual characteristic polynomials are those at the adjoint by reduction in the local ring $B'$. Brauer–Nesbitt gives the claimed residual semisimplification. The identity $t(chc^{-1}) = t(h^{-1})$, for complex conjugation $c$, interpolates from the classical fibers and gives conjugate self-duality by semisimplicity.

At each chosen place take an inertia element with separating Teichmüller values as in Lemma 36.24. The three projectors there have denominators which are units over $\Lambda$. The vanishing traces of the lower matrix entries against all group matrices, and the three identities recovering the diagonal on inertia, interpolate from the regular weights by the multivariable density argument. At the specialized representation these products of projectors and group matrices belong to $D$. Its trace pairing is nondegenerate even for a reducible semisimple representation, as just proved. The lower entries thus vanish and the Vandermonde argument gives the ordered characters. Their product is one. The determinant reduces to one globally and is unramified at $\ell$, also at the conjugate places. Away from $\ell$ it is unramified by the principal-unit argument in Theorem 37.1. Finiteness of the class group and torsion-freeness of the target make the determinant one. □

**Lemma 37.3** (A common auxiliary prime of large residue degree)**.** *The prime $w$ of Proposition 36.15 can be chosen so that every $F_v$ for $v \mid \ell$ contains the unramified degree-$\ell$ extension of $\mathbb{Q}_\ell$.*

*Proof.* Let $M = \mathbb{Q}(\mu_{sN_*\ell})$. At every place above $\ell$ the normalized valuation of the rational integer $\ell$ is $\ell - 1$, since $\ell \nmid sN_*$. Thus $\ell$ is not an $\ell$-th power in $M$. The splitting field of $X^\ell - \ell$ over $M$ is a cyclic extension of degree $\ell$ by Kummer theory, and the compositum is Galois over $\mathbb{Q}$. Choose a prime with Frobenius in the conjugacy class of a nonidentity element of this cyclic subgroup, excluding the finite set of previously forbidden primes. Chebotarev supplies such a $w$. It splits in $M$, so $w \equiv 1 \bmod sN_*\ell$. The integer $\ell$ is not an $\ell$-th power modulo $w$, since the Frobenius on the roots would then fix them all. The quotient of $\mathbf{F}_w^\times$ by its $\ell$-th powers has order

$\ell$ and corresponds to $F_w \subset \mathbb{Q}(\mu_w)$. Frobenius of $\ell$ generates this quotient. The completion of $F_w$ at $\ell$ is thus unramified of degree $\ell$. The field $F$ contains $F_w$. All the choices here only use the interval and $\ell$. □

**Theorem 37.4** (A nonabelian ordinary trace curve in characteristic $\ell$)**.** *Choose $w$ as in Lemma 37.3. On the common $L_0$ there is one map of weight variations $\Lambda \to k[[T]]$ which works for every $r, \alpha$ on the interval, after extending the finite constant field as needed. On every component $B'$ of Proposition 37.2 through the corresponding adjoint, each prime over its kernel has one-dimensional quotient. Its generic trace cannot be a sum of three characters. The three ordered ordinary characters are pairwise distinct even on the kernel of the full cyclotomic character on inertia at every chosen place above $\ell$. The coefficient $T$ itself lies in the algebra generated over the constants by the specialized group traces.*

*Over the algebraic closure of its coefficient field the generic representation either has Zariski closure* $\mathrm{SL}_3$ *or is a direct sum of a character reducing to* $1$ *and an irreducible two-dimensional representation with projective Zariski closure* $\mathrm{PGL}_2$*. In the second case the one-dimensional summand satisfies $\chi^c = \chi^{-1}$.*

*Proof.* Take a real unit $e \in F$ of infinite order. There is such a unit since $F$ contains the real quadratic field $\mathbb{Q}(\sqrt{sv})$. Its principal part at every $v \mid \ell$ has nonzero logarithm, since an algebraic unit which is a local root of unity is a root of unity. At each $v$ put

$$u_v(z) = \frac{\log N_{F_v/\mathbb{Q}_\ell}(z)}{\log(1+\ell)} \quad (z \in H_v).$$

It is onto by the norm calculation in Section 36.9. Its kernel has rank $[F_v : \mathbb{Q}_\ell] - 1 > 0$ by Lemma 37.3. Choose integral linear functionals $\beta_v : H_v \to \mathbb{Z}_\ell$ in logarithmic coordinates such that each $\beta_v$ is nonzero on this kernel and $\sum_v \beta_v(j(e)_v) \neq 0$. For the latter condition, start with any choices satisfying the former. At one place add an integral multiple of a functional nonzero at the local principal part of $e$. At most one multiple can make the new restriction to the kernel identically zero and at most one makes the required sum zero. We use the continuous homomorphism $\mathbb{Z}_\ell \to k[[T]]^\times$, $a \mapsto (1+T)^a$, obtained by reducing the integral binomial coefficients. It is injective. If $a = \ell^m b \neq 0$ with $b$ a unit, the coefficient of $T^{\ell^m}$ in $(1+T)^a = (1+T^{\ell^m})^b$ is $\bar{b} \neq 0$.

At $v$ define $\lambda_1, \lambda_2, \lambda_3$ as powers of $1+T$ with respective exponents

$$2u_v + \ell\beta_v, \qquad u_v, \qquad -3u_v - \ell\beta_v.$$

They have product one. At any $z$ with $u_v(z) = 1$, all three exponents are nonzero and pairwise different modulo $\ell$, since $\ell > 5$. For the stronger separation in the statement take $z \in \ker u_v$ with $\beta_v(z) \neq 0$. The exponents are then $\ell\beta_v(z), 0, -\ell\beta_v(z)$, pairwise distinct in $\mathbb{Z}_\ell$. An inertia lift of this principal unit has cyclotomic value one by local reciprocity. At a unit with $u_v = 1$ the second character takes value $1+T$. The resulting map of completed group algebras is surjective onto $k[[T]]$, since its compact image contains the polynomial ring $k[T]$. Every domain above its kernel is finite and integral over $k[[T]]$, hence has dimension one.

For a real unit the sum of the $u_v$ on its principal parts is zero. The product of its local norms is the rational norm of the unit, which is $1$ or $-1$, and taking principal parts and logarithms kills that sign. On the selected $e$ the exponent of $(\lambda_1/\lambda_3)(j(e))$ is therefore $2\ell \sum_v \beta_v(j(e)_v) \neq 0$. By injectivity of $a \mapsto (1+T)^a$, equation (37.1) fails. The representation furnished by Proposition 37.2 cannot be a sum of three characters. This argument works after any finite coefficient extension. A decomposition over the algebraic closure would be defined over the finite

extension generated by the change-of-basis entries. The assertion follows over that closure as well.

To recover $T$ from the traces alone, lift a Teichmüller unit of norm generating $\mathbf{F}_\ell^\times$ to local inertia. Write this lift as $s_0$. Its three eigenvalues in the specialization are the distinct constants $\bar\kappa_\ell^{r}(s_0), 1, \bar\kappa_\ell^{-r}(s_0)$, since all principal-unit variations are trivial on this local unit. Lift a principal unit with $u_v(z) = 1$ to an inertia element $h$. For $j = 0, 1, 2$ the traces at $s_0^j h$ give the three linear combinations of the diagonal values at $h$ with this constant Vandermonde matrix. Solving for the middle value gives $1 + T$ in the algebra generated by the traces.

Suppose first that the representation is absolutely irreducible. The normal-core argument of Theorem 36.26 applies in this characteristic as well. The sum of the translates of a simple submodule on a normal subgroup of finite index is semisimple and is the whole space. The scalar alternative is excluded because the three eigenvalues at the principal unit just used have ratios of infinite order. Thus they remain distinct upon raising the matrix to any positive power. The three-line alternative would give a transitive cyclic permutation group. Indeed the kernel of $\bar\Psi$ has pro-$\ell$ image in a stable lattice, so acts trivially on three permuted lines. The residual image of the semisimplification is cyclic. The resulting cubic self-twist, reduced in the residue field, would permute the residual triple by multiplication by an order-three character. Its orbit of 1 cannot contain $\bar\Psi$, whose inertial order exceeds four. Thus the representation stays irreducible on every open subgroup.

The connected algebraic monodromy is consequently semisimple and irreducible, by the unipotent radical and determinant argument of Theorem 33.5. The rank-three classification in the proof of Theorem 33.13 applies in characteristic $\ell > 5$. A connected proper possibility would preserve a nondegenerate symmetric form, unique up to scalar. The full group would then be self-dual up to a cubic character. At the principal unit $h$ all three eigenvalues are principal units, so any cubic scalar relating them to their inverses reduces to one and hence is one. A reciprocal multiset of odd size contains an eigenvalue of square one. All three eigenvalues here have infinite order. This excludes the orthogonal case and proves full special linear monodromy.

In the reducible case there is a unique one-dimensional constituent, since three characters have been excluded. Conjugate self-duality sends this constituent to its conjugate inverse, so the two characters agree. Each of the three residual characters comes from $\mathbb{Q}$ and is invariant under conjugation. The only one equal to its inverse is 1. It follows that the two-dimensional constituent reduces semisimply to $\bar\Psi \oplus \bar\Psi^{-1}$. At a chosen place its eigencharacters are the two outer characters, by residual separation. Their ratio at $h$ has infinite order, proving that its projective image is infinite. If the identity component of its projective closure were a proper nontrivial subgroup of $\mathrm{PGL}_2$, it would be solvable. Indeed a nontrivial connected semisimple group has dimension at least three, and $\mathrm{PGL}_2$ has dimension three. A triangularizable connected subgroup with nontrivial unipotent radical fixes a unique line, which its normalizer must preserve. This contradicts irreducibility. A nontrivial torus fixes two lines. The normalizer must interchange them to act irreducibly, giving a quadratic self-twist by conjugation by the matrix with diagonal $1, -1$ on those lines. Its reduction would interchange $\bar\Psi, \bar\Psi^{-1}$, forcing $\bar\Psi^4 = 1$. This contradicts the order on inertia. Therefore the projective closure is $\mathrm{PGL}_2$. The unit and all weight variations above depend only on the common field and on $\ell$. This gives simultaneity in $r, \alpha$ as asserted. □

The curve has separated characters on cyclotomic-kernel inertia and admits no global triangular representation with its trace. These are two of the conditions on the one-dimensional primes in `Development/AutomorphicArithmetic/AutomorphicVarieties/CurveDVRSpecialization/PropagationTheoremGL2.lean`, line 16. The local degrees satisfy $[F_v : \mathbb{Q}_\ell] \geq \ell > 5$ and $d = 2\ell$, since the two real factors of $F$ have coprime degrees. In the original fixed-weight problem the

three variations on principal units reduce to one. No finite extension of $L$ can make the variations on this curve all equal, since their ratios already have infinite order on a single inertia element. Passing from the variable-weight points to the original strict group requires control of the other primes in the deformation ring, as in the propagation implication at line 36 of the same source file. The uniform length estimate and the signed integral period comparison in Section 6.6 still enter the deduction of Dirichlet GRH.

### 37.2 Strict classes and ordinary traces on one cyclotomic layer

Use the polynomial-conductor interval $V \le x \le V^{3/2}$ of Theorem 36.4, and fix $\ell > \exp(V^{\varepsilon}/12)$. Choose the prime $w$ as in Lemma 37.3. Write $F = F_a^+$ and $L = L_a = KF$ for the fields of Proposition 36.15, where $a$ is allowed to vary. For every $x, r, \alpha$ of that proposition let $k$ be a finite field containing the values of $\bar\Psi = \bar\alpha \bar\kappa_\ell^r$, and put $\beta_i = \bar\Psi^{-i}\eta$ for $i = 1, 2$. We use $G_D$ for the absolute Galois group of a field $D$. Let $\Gamma_a$ be $G_F$ modulo the closed normal subgroup generated by

$$I_u \cap \ker \eta \cap \ker \bar\Psi, \qquad u \nmid \ell$$

at finite places. All three characters $\eta, \bar\Psi, \kappa_\ell$ descend to this quotient. Its index-two subgroup $\Gamma_a^0 = \ker \eta$ is the image of $G_L$. The field $L$ splits over $F$ above $\ell$. We use every completion of $F$ above $\ell$, and both places above it for a group over $L$. A strict class below means a first cohomology class restricting to zero in all these local first cohomology groups. Throughout this section finite coefficients carry the discrete topology.

**Proposition 37.5** (Two strict classes from the same prime)**.** *For each of the two characters $\beta_i$ there is a class over $\mathbb{Q}$ whose restriction is nonzero in the strict group of $H^1(\Gamma_a, k(\beta_i))$ at every depth $a$. The same $w$ serves all $x, r, \alpha$.*

*Proof.* Put $D = k(\beta_i)$. Over $\mathbb{Q}$ impose zero at $\ell$, the full local group at $w$, and the unramified local condition at all other finite primes. At the real place the first cohomology vanishes. Denote these conditions by $\mathcal{L}$ and their orthogonals under local Tate duality by $\mathcal{L}^\perp$. The dimension formula from Poitou–Tate duality gives

$$\dim H^1_{\mathcal{L}}(\mathbb{Q}, D) - \dim H^1_{\mathcal{L}^\perp}(\mathbb{Q}, D^\vee(1)) = h^0(\mathbb{Q}, D) - h^0(\mathbb{Q}, D^\vee(1)) + \sum_u \bigl(\dim \mathcal{L}_u - h^0(\mathbb{Q}_u, D)\bigr).$$

Here $h^0$ denotes dimension over $k$. One may use the nine-term duality sequence with a finite set containing $\ell, w, \infty$ and the ramified primes of $D$. Subtracting the dimensions in its middle terms and inserting the local and global Euler characteristics gives the displayed formula. The omitted unramified primes contribute zero. This also explains the real-place term when $\ell$ is odd. [1]

Both global invariant spaces vanish because their characters are nontrivial on inertia at $s$, where only $\eta$ among $\eta, \bar\alpha, \bar\kappa_\ell$ ramifies. At the real place we have $\beta_i(c) = -1$. At $\ell$ the character on inertia is $\bar\kappa_\ell^{-ir}$, nontrivial since $0 < ir < \ell - 1$. Both local terms are therefore zero. At each other prime with the unramified condition the dimension is $\dim D^{I_u}/(\mathrm{Frob}_u - 1)D^{I_u} = h^0(\mathbb{Q}_u, D)$. At $w$ both $\beta_i$ and $\bar\kappa_\ell$ are trivial on the decomposition group, by $w \equiv 1 \bmod sN_*\ell$. The full local $H^1$ has dimension two. Indeed the uniformizer and the cyclic $\ell$-part of the residue units each contribute one independent homomorphism to $k$, and the principal units are pro-$w$. Thus the right hand side is one. In particular a nonzero class exists for each $i$.

Restriction to $G_F$ is injective on these classes. The field $F$ is Galois and totally real, so $D^{G_F} = 0$ by oddness. Inflation–restriction gives the injection even when $\ell$ divides $[F : \mathbb{Q}]$. The local zero conditions persist. At $w$ a cocycle is a homomorphism on the entire decomposition group. Its restriction to inertia factors through the quotient of the tame $\mathbb{Z}_\ell(1)$ by $\ell$. The

degree-$\ell$ field $F_w$ is totally and tamely ramified at $w$. Its inertia subgroup has zero image in this quotient, so restriction to every completion of $F$ kills the inertia homomorphism. Elsewhere away from $\ell$, the restriction of the cocycle to inertia was a coboundary. Its value on $\ker\eta\cap\ker\bar{\Psi}$ inside inertia is zero, since this subgroup acts trivially on $D$. The restriction of the cocycle therefore vanishes on the closed normal subgroup used to define $\Gamma_a$. Indeed on the kernel of the action it is additive, and conjugating an element of value zero there again gives value zero. This proves descent to the quotient and preserves the nonzero strict class. □

**Proposition 37.6** (Uniform residual dimensions along the auxiliary tower)**.** *There are integers $h, g$, depending on this interval, $\ell$ and the auxiliary fields at depth zero, such that at every depth and for all the pairs under consideration*

$$\begin{aligned} \dim_k H^1_{\mathrm{str}}(\Gamma_a, k(\beta_i)) &\le h && (i=1,2),\\ \dim_k H^1_{\mathrm{str}}(\Gamma_a^0, k(\bar{\Psi}^{-1})) &\le h,\\ \dim_k H^1(\Gamma_a, k) &\le 1+g+h. \end{aligned}$$

*Proof.* Set $T_a = L_a E_*(\mu_\ell)$, let $T_\infty$ be their union, and put $g = \#\{t : t \mid \ell \text{ in } T_0\}$. These are the layers of the cyclotomic $\mathbb{Z}_\ell$-extension of the abelian field $T_0$. At $\ell$ the ramification index of $T_0$ is $\ell-1$. More precisely its local field is an unramified extension of $\mathbb{Q}_\ell(\mu_\ell)$, since all the other factors are unramified there. The rational cyclotomic layers are totally ramified after this unramified base extension. Consequently each place over $\ell$ is totally ramified at every step, and the number $g$ is constant.

Let $X$ be the Galois group of the maximal unramified abelian pro-$\ell$ extension of $T_\infty$. The cyclotomic class group theorem of Iwasawa makes $X$ a finitely generated torsion module over $\mathbb{Z}_\ell[[\mathrm{Gal}(T_\infty/T_0)]]$. Its $\mu$-invariant is zero by the theorem of Ferrero and Washington, since $T_0/\mathbb{Q}$ is abelian. The structure theorem then makes $X$ finitely generated over $\mathbb{Z}_\ell$. Indeed the cyclic factors up to finite kernel and cokernel are quotients by powers of distinguished polynomials, all finite free over $\mathbb{Z}_\ell$. Set $h = \dim_{\mathbf{F}_\ell} X/\ell X$. The Hilbert $\ell$-class field of $T_a$ intersects $T_\infty$ in $T_a$ by total ramification. After compositum with the tower it is unramified and abelian over $T_\infty$. Restriction therefore exhibits the entire $\ell$-class group of $T_a$ as a quotient of $X$. This bounds its $\ell$-rank by $h$. [23, 24]

The extensions $T_a/F$ and $T_a/L$ have degrees prime to $\ell$. Their degrees divide $2[E_* : \mathbb{Q}](\ell-1)$, with $[E_* : \mathbb{Q}] \le N_* < \ell$. Inflate a class in either strict group to the absolute Galois group and restrict to $T_a$. This is injective by restriction and corestriction. The coefficient character is trivial on $G_{T_a}$, and the resulting homomorphism kills inertia away from $\ell$ by the definition of $\Gamma_a$. It is zero locally over $\ell$ by strictness. It consequently factors through the class group of $T_a$. Homomorphisms from that group to the additive group of $k$ have $k$-dimension at most $h$.

For trivial coefficients it suffices to work over $\mathbf{F}_\ell$, since extension to $k$ just tensors the space of continuous homomorphisms. Injection by the same restriction gives invariant homomorphisms on $G_{T_a}$ unramified outside the $g$ places over $\ell$. Multiplication by a generator of $\mu_\ell$ identifies this image with a subspace in the minus eigenspace of $H^1(G_{T_a,\{\ell,\infty\}}, \mu_\ell)$ for complex conjugation. Indeed conjugation fixes $F$ and acts on that generator by inversion. By Kummer theory the latter cohomology is an extension of the $\ell$-torsion of the class group of the ring of $\ell$-integers by its units modulo $\ell$-th powers. This is the sequence obtained by mapping a Kummer element with all away valuations divisible by $\ell$ to the class of its away divisor divided by $\ell$. The class group of the ring is a quotient of $\mathrm{Cl}(T_a)$ and hence contributes at most $h$. Away from the ordinary units, valuations of the $\ell$-units give a lattice of rank at most $g$. For any ordinary unit $e$, the quotient $e/\bar{e}$ is a root of unity by Kronecker's theorem. The minus part of the ordinary units modulo $\ell$-th powers therefore has dimension at most one, since minus projection

is $(1-c)/2$ and the roots of unity are cyclic. The valuation lattice is free, so reduction modulo $\ell$ preserves the injection of ordinary units modulo powers in this sequence. Adding the three bounds proves the result. □

**Lemma 37.7** (The relative Euler characteristic for the ramification quotient)**.** *Let $W$ be any finite-dimensional continuous representation of $\Gamma_a$ over a finite field of characteristic $\ell$. For each $u \mid \ell$ let $b_u \subset W$ be stable under $G_{F_u}$, and let $H^j_C$ denote the cohomology of the mapping fiber of restriction from global cochains with values in $W$ to the sum of local cochains with values in $W/b_u$. If $H^0(F, W^\vee(1)) = 0$, these groups are finite, vanish in degrees above two, and satisfy*

$$\dim H^2_C + \dim H^0_C + \sum_{u\mid\ell} [F_u : \mathbb{Q}_\ell] \dim b_u = \dim H^1_C + \sum_{\tau : F \hookrightarrow \mathbb{R}} \dim W^{c_\tau}.$$

*Here $c_\tau$ denotes a complex conjugation at $\tau$.*

*Proof.* Let $M/F$ be cut out by $(\eta, \bar{\Psi})$. Its Galois group $\Delta$ has order prime to $\ell$. The kernel of $\Gamma_a \to \Delta$ is the Galois group of the maximal extension of $M$ unramified at finite primes away from $\ell$. Indeed the removed inertia groups are precisely the inertia groups of $M$ at those primes. Their closed normal closure over $M$ is already normal over $F$, which permutes the primes of $M$. Write $S$ for the places over $\ell$ and the infinite places, and enlarge $S$ to $S'$ by the away primes ramifying in $M/F$. The same description with $S'$ gives $G_{F,S'}$ above $G_{M,S'}$. In all such expressions the indicated places are pulled up to the field in the subscript. Hochschild–Serre computes the cohomology of either group over $F$ by taking $\Delta$-invariants of the cohomology over $M$. Invariants are exact by averaging.

We verify that replacing $G_{F,S'}$ by $\Gamma_a$ preserves the Euler characteristic. Over $M$ the module $W$ is unramified at each of the deleted primes $t$ above $S' \setminus S$. The difference in the two cohomology complexes is the sum of the residue-field complexes of $W(-1)$ shifted by one. To see this, use the comparison of Galois and étale cohomology on the two rings of integers with the indicated places inverted. It applies to the finite locally constant sheaf here because $\ell$ is invertible on both rings. Excision in the localization sequence reduces the relative term to a henselian local ring and its punctured spectrum. Over a strictly henselian local ring the only positive-degree cohomology of the punctured spectrum is $H^1(I_t, W) = W(-1)$. The wild inertia has order prime to $\ell$ in all finite quotients, and the $\ell$-factor of tame inertia is $\mathbb{Z}_\ell(1)$. The sheaf is unramified and the degree-zero restriction is an isomorphism. Descent to the residue field gives the asserted term. These comparison maps commute with conjugation by $\Delta$. [1]

For a fixed $u \in S' \setminus S$ take $\Delta$-invariants of the sum over $t \mid u$. This amounts to invariants under the decomposition group $\Delta_t$ at one $t$. Equivalently its two cohomology spaces are the degree-zero and degree-one groups of $G_{F_u}/I_{M_t}$ on $W(-1)$, by Hochschild–Serre and the prime-to-$\ell$ order of $\Delta_t$. The finite inertia in this quotient also has order prime to $\ell$. After taking its invariants we are left with the kernel and cokernel of $\mathrm{Frob}_u - 1$ on the same finite vector space. Their dimensions are equal. Each deleted prime thus changes the Euler characteristic by zero. Global finiteness and vanishing above degree two for $\Gamma_a$ follow over $M$ for $G_{M,S}$ and pass to invariants. The global Euler characteristic of $G_{F,S'}$, since $F$ is totally real, now gives

$$\chi(\Gamma_a, W) = -[F : \mathbb{Q}] \dim W + \sum_{\tau} \dim W^{c_\tau}.$$

Over $M$ the last local map in Poitou–Tate has cokernel $H^0(M, W^\vee(1))^\vee$ for restriction in degree two to the places over $\ell$. Real places have zero positive-degree cohomology at this odd prime. Taking $\Delta$-invariants, and applying local Hochschild–Serre above $\ell$, identifies this cokernel over $F$ with $H^0(F, W^\vee(1))^\vee = 0$. Restriction from $H^2(\Gamma_a, W)$ to the sum of $H^2(F_u, W)$

is therefore surjective. It remains surjective after passing to $W/b_u$, by local cohomological dimension two. Thus the mapping fiber has zero third cohomology. In all finite degrees its Euler characteristic is the global one minus the sum of the local ones. The local Euler characteristic on $W/b_u$ is $-[F_u : \mathbb{Q}_\ell] \dim(W/b_u)$, and the local degrees sum to $[F : \mathbb{Q}]$. Substitution proves the formula. The duality and Euler formulas used here are Theorems I.2.8, I.4.10 and I.5.1 of Milne, and the comparison for the rings of integers is in II.2.9. [1] □

**Theorem 37.8** (The cohomology hypotheses on one common layer)**.** *Take $C_0 = h + g$ from Proposition 37.6. For every sufficiently large $a$ the arithmetic data $(F_a^+, L_a, \Gamma_a)$ satisfy all the cohomological, local generator, infinity and numerical hypotheses of the source's ordinary deformation frame for all $x, r, \alpha$ on the interval, with this one $C_0$. In particular*

$$3C_0 + 7 < [F_a^+ : \mathbb{Q}], \qquad [F_{a,u}^+ : \mathbb{Q}_\ell] > 5 \quad (u \mid \ell).$$

*Proof.* The three real factors of $F_a^+$ have degrees $2, \ell, \ell^a$. The quadratic factor is disjoint from the other two by parity of degree. The degree-$\ell$ factor is totally ramified at $w$, where the rational cyclotomic tower of conductor a power of $\ell$ is unramified, so these two are disjoint as well. The total real degree is $2\ell^{a+1}$. At $\ell$ the completion contains both the degree-$\ell$ unramified field of Lemma 37.3 and a totally ramified field of degree $\ell^a$. This gives the two inequalities, with one lower threshold in $a$.

Use complex conjugation at each real embedding and its image in $\Gamma_a$. It has order two, sign $\eta = -1$, and $\bar{\Psi} = 1$. The sign kernel is open, all local maps and $\kappa_\ell$ are continuous, and the residual character has open kernel. The groups are profinite. The local groups are the full absolute Galois groups of the completions. Choose a place of $L_a$ over each $u \mid \ell$. They may all be chosen above the restriction to $K$ of any one fixed place over $\ell$ of $\overline{\mathbb{Q}}$. In fact $F_a^+$ and $K$ are disjoint and Galois over $\mathbb{Q}$, so $\mathrm{Gal}(L_a/K)$ acts transitively on the places of $F_a^+$ over $\ell$. Lift to the algebraic closure. The conjugate places then give the other choice at every $u$, by splitting in $L_a/F_a^+$.

At every such completion the full rational residual cyclotomic inertia survives by Proposition 36.15. Use an inertia element mapping to a generator of $\mathbf{F}_\ell^\times$ at the fixed place as the distinguished element of $\Gamma_a^0$, and use the same choice locally at every other place. Its squared residual $\bar{\Psi}$-value differs from one since $0 < 2r < \ell - 1$. The element of $G_{L_a}$ in that proposition with $\bar{\kappa}_\ell = -1$ and $\bar{\Psi} = 1$ rules out every identity $\bar{\kappa}_\ell = \bar{\Psi}^j$ on $\Gamma_a^0$. It also witnesses nontriviality of $\bar{\kappa}_\ell$, $\bar{\kappa}_\ell \bar{\Psi}^{\pm 1}$ and $\bar{\kappa}_\ell \bar{\Psi}^{\pm 2} \eta$ where required.

On inertia in the kernel of the full cyclotomic character, $\bar{\Psi} = \bar{\kappa}_\ell^r = 1$. Under local reciprocity the principal units at $u$ form a free $\mathbb{Z}_\ell$-module of rank $[F_u : \mathbb{Q}_\ell]$. The logarithm gives that rank on an open subgroup, and there is no $\ell$-power torsion because the ramification index $\ell^a$ is prime to $\ell - 1$. The norm of the principal units has rank one, since sufficiently deep rational principal units already have norms given by raising them to $[F_u : \mathbb{Q}_\ell]$. Its kernel is a direct summand. Choose a basis in that kernel and extend it by one element to a basis of the principal units. Lift the basis to inertia by reciprocity. All lifts kill $\bar{\Psi}$, and every lift but the last kills the full cyclotomic character. A finite abelian $\ell$-group character trivial on this basis kills all inertia, since the other unit factor has order prime to $\ell$.

Every open subgroup of $\Gamma_a$ has only finitely many continuous homomorphisms to $\mathbf{F}_\ell$. Pull back to the corresponding number field. Extensions arising in this way are unramified outside the finite set consisting of the places over $\ell$ and the ramification of $M/F$ in Lemma 37.7. Adjoining $\mu_\ell$ gives an injection by prime-to-$\ell$ restriction, and Kummer theory bounds the dimension by the ranks of an $S$-unit group modulo powers and a finite class group. Any abelian $\ell$-group character of $\Gamma_a^0$ unramified above $\ell$ is everywhere unramified, since its allowed

inertia away from $\ell$ has order prime to $\ell$. The class number of $L_a$ therefore annihilates every such character.

For the zero local subspaces used in the strict groups, a degree-one cocycle of the mapping fiber consists of a global cocycle and a local vector witnessing its being a coboundary at every place over $\ell$. Each witness is unique because the relevant local invariant spaces are zero. Indeed the character on inertia is $\bar{\kappa}_\ell^{-ir}$ or $\bar{\kappa}_\ell^{-r}$. Quotienting out the global coboundaries thus gives exactly the strict groups of Propositions 37.5 and 37.6. For the full local subspaces the local quotients in the mapping fiber vanish, leaving ordinary global $H^1$. These results give both required nonzero groups and all three upper bounds, also for the doubled set of places over $L_a$. Lemma 37.7 proves the Euler inequality for every allowed coefficient module and local subspace. The real embeddings number $[F_a^+ : \mathbb{Q}]$. This verifies the hypotheses stated in `Development/AutomorphicArithmetic/AutomorphicVarieties/FrobeniusSmoothVectors/ContinuousCoboundaries2.lean`, line 167. □

**Proposition 37.9** (The ordinary trace on the ramification quotient)**.** *The semisimple representation of Theorem 37.4 restricts on $G_{L_a}$ to a representation of $\Gamma_a^0$ at every depth $a$. Its generic trace is nonconstant, its semisimplification is nonabelian, and its three ordinary characters remain pairwise distinct on the kernel of full cyclotomic inertia at every chosen place. These properties hold on the layer of Theorem 37.8.*

*Proof.* First work on the integral domain $B'$ of Proposition 37.2 over $L_0$. At every regular arithmetic weight each point above it supplies a classical representation. These points remain Zariski dense on $B'$ after removing the vanishing locus of any nonzero $b \in B'$. Indeed for any nonzero $b'$ the ideal $bb'B'$ contains a nonzero element of $\Lambda$. Indeed after inverting $\Lambda \setminus \{0\}$ the finite domain is a field, so one can clear denominators in the inverse of $bb'$. Use the density of the weights from the proof of Proposition 37.2 to choose a weight where this element is nonzero and then use lying over. Every point above that weight avoids the zeros of both $b$ and $b'$.

Only the finite set of places above the tame level and the conductor of $\alpha$ needs consideration away from $\ell$. At any one of these places, take the completion of $L_0E_*$ as in Proposition 36.25. On its inertia all classical characteristic polynomials are unipotent after passing to finite index bounded independently of the classical point. In detail the residual image on this inertia is an $\ell$-group, so the image factors through tame $\mathbb{Z}_\ell(1)$. Conjugation by Frobenius of norm $q$ permutes the three eigenvalues of a generator by raising them to $q$. Their orders divide $q^6 - 1$. Taking powers to enter this inertia bounds the orders on the original inertia as well. Thus for every fixed inertia element there are only finitely many possible characteristic polynomials. Each coefficient on $B'$ is constant: the polynomial in that coefficient with the finitely many possible values as roots vanishes by density, and $B'$ is a domain. Equivalently the polynomial may be taken over $O$ by including coefficient conjugates. Its root at the adjoint belongs to $O$, and the factor with the other distinct roots is nonzero there. Consequently the constant polynomial is the inertial characteristic polynomial at the adjoint for the element in question.

For every one of the finitely many places, retain the Frobenius resultant of Proposition 36.25. Each is nonzero at the adjoint, as verified there using the two eigenvalues of the elliptic newform. Off their joint product's zero locus the unipotent inertia on $L_0E_*$ has zero logarithm, since conjugation would otherwise have eigenvalue $q$, forbidden by the resultant. Thus the original inertia there has finite image. Its characteristic polynomials just computed agree with $\alpha \oplus 1 \oplus \alpha^{-1}$. In particular at every remaining regular point in this dense set each $j \in I_u \cap \ker \bar{\Psi}$, $u \nmid \ell$ in $L_0$, acts trivially. Indeed $\alpha$ has order prime to $\ell$ and there $\bar{\Psi} = \bar{\alpha}$ on inertia. The matrix has finite order in characteristic zero and all eigenvalues one.

For each such $j$ and every $g \in G_{L_0}$ we have proved the identity $t(jg) = t(g)$ on a Zariski dense set. It therefore holds in $B'$ and after specialization to the characteristic-$\ell$ curve. The trace pairing on the span of the matrices of the semisimple specialization is nondegenerate, by the argument of Proposition 37.2. It follows that $\rho(j) - 1 = 0$. For a place of $F_a^+$ the inertia away from $\ell$ is contained in $G_{L_a}$, since $L_a/F_a^+$ is unramified. It is then contained in an inertia group over $L_0$. Intersecting with $\ker \bar{\Psi}$ gives exactly elements of the kind just killed. Conjugates by $G_F$ still belong to such inertia groups, and the kernel of the representation is closed. This proves factorization through $\Gamma_a^0$.

The subgroup $G_{L_a}$ has finite index in $G_{L_0}$. In the irreducible case of Theorem 37.4 the connected monodromy $\mathrm{SL}_3$ is preserved, and in its reducible case the connected projective monodromy $\mathrm{PGL}_2$ of the two-dimensional constituent is preserved. This follows since a connected algebraic group cannot be a finite union of translates of a proper closed subgroup. The restricted representation is thus semisimple and nonabelian in both cases. A finite-index subgroup of the original inertia contains a positive power of every prescribed inertia element. At each place use the element in cyclotomic- kernel inertia supplied by Theorem 37.4. Its three eigenvalues are distinct powers of $1 + T$ with exponents distinct in $\mathbb{Z}_\ell$, so they stay distinct on every positive power. At least one is of infinite order. If every trace of the subgroup were constant over an algebraic closure of $k$, Newton's identities for the first three powers would make its characteristic polynomial constant too. All nonzero elements algebraic over the finite field have finite order, a contradiction. □

The bound in Proposition 37.6 controls dimensions over the residue field on a tower for a fixed coefficient prime. The estimate in Section 6.6 counts integral lengths on the original $E_x$ and sums over coefficient primes. Even one cyclic module of order $\ell^m$ has residual dimension one for arbitrarily large $m$. To obtain that estimate from ordinary traces one must still control the fixed-weight integral relations in Theorem 36.14. The same section requires the comparison of the signed integral projected theta periods on those original fields.

## 37.3 A polarized ordinary curve through the split residual point

Fix the common layer of Theorem 37.8. Write $H = \Gamma_a^0$ and $g^\star = cg^{-1}c$ for $g \in H$, where $c$ is the chosen complex conjugation. Work with the representation $\rho$ of Proposition 37.9 over a finite extension of $k((T))$. Further finite extensions are allowed. Write $g_0$ for the distinguished inertia element of Theorem 37.8, $b = \bar{\Psi}(g_0)$, and $\delta = b - b^{-1} \neq 0$. A symmetric pairing on this representation will mean a nondegenerate symmetric bilinear form for which $\rho(g^\star)$ is the adjoint of $\rho(g)$.

**Lemma 37.10** (The pairing on the two-dimensional constituent)**.** *The representation $\rho$ admits a symmetric pairing. In the case of a two-dimensional constituent and a character, the restriction of the pairing to each constituent is nondegenerate and symmetric.*

*Proof.* First work over an algebraic closure of the fraction field. Let $D$ be the span of the group matrices of $\rho$. Conjugate self-duality gives $t(g^\star) = t(g)$ for the trace $t$. The operation $\star$, extended linearly from the group, descends to an anti-involution of $D$. Indeed if a group algebra element $x$ acts by zero, then for every $g$ we have $t(g^\star x^\star) = t(xg) = 0$. The nondegenerate trace pairing on $D$ from Proposition 37.2 shows that $x^\star$ acts by zero. If $\rho$ is irreducible, $D = M_3$ and this anti-involution is adjunction for a symmetric form. In fact composing it with transposition gives an automorphism of the matrix algebra and hence an inner automorphism. Applying the resulting adjunction twice shows that the form matrix has transpose equal to itself up to a scalar of square one. A nonsingular alternating form is impossible in odd dimension.

In the other case $D = M_2 \times M_1$, and the two unequal matrix factors are preserved. A priori the form on the first factor could be alternating. We determine its sign using the integral trace algebra $B'$ of Proposition 37.2. Use group elements of $G_{L_a}$ as arguments of that trace via $G_{L_a} \subset G_{L_0}$, choosing a lift of $g_0$. In the group algebra form $Z = g_0 - g_0^\star$. The element

$$u = t(Z^2)/2 \in B'$$

is a unit because it reduces to $\delta^2$ on the three residual characters. Set $e = Z^2/u$ in the group algebra over $B'$. This expression can be used inside any trace without choosing matrices over $B'$.

There is a Zariski dense set of regular classical points on $B'$ with absolutely irreducible restriction to $G_{L_a}$. At the original adjoint this restriction is absolutely irreducible by Theorem 36.17. Choose nine group elements there whose matrices span $M_3$. The determinant of their trace Gram matrix is nonzero at the adjoint and remains nonzero on a dense set of regular classical points by the density argument of Proposition 37.9. At each such point the matrices still span $M_3$. The conjugate self-duality gives a symmetric form there by the odd-dimensional argument above.

At any of these points $Z$ is skew adjoint. Its eigenvalues are $0, d, -d$ with $d^2 = u$, since $Z$ is conjugate to $-Z^{\mathsf{T}}$ and its trace of the square is $2u$. The roots are distinct. Thus $e$ is a self-adjoint idempotent of rank two. Its image carries a nondegenerate symmetric form, for its orthogonal complement is its kernel. On a two-dimensional symmetric space the skew adjoint operators form a one-dimensional vector space, as is seen by identifying them with alternating bilinear forms. Consequently for any group elements $g, h$ the operators $e(g-g^\star)e$ and $e(h-h^\star)e$ commute. Multiply their commutator by any group element and take the trace. The result belongs to $B'$ and vanishes at this dense set, so it vanishes in $B'$. On the specialized curve the trace pairing on $D$ therefore gives

$$[\, e(\rho(g) - \rho(g^\star))e,\, e(\rho(h) - \rho(h^\star))e \,] = 0,$$

where $e$ now denotes the image of the idempotent. All identities extend linearly to the span of the group matrices.

On the one-dimensional block the anti-involution is the identity, so $Z$ acts by zero. Its trace of the square is still $2u \neq 0$, and on the two-dimensional block it is traceless because the trace is invariant under $\star$. Cayley–Hamilton gives $Z^2 = u$ on that block. Hence $e$ projects precisely onto $M_2$. If the form there were alternating, the skew adjoint subspace would be the entire space of trace-zero two-by-two matrices. It contains the two off-diagonal matrix units, whose commutator is nonzero. Every skew element is half its difference from its adjoint, and all elements of $M_2$ occur in the span just tested. This contradicts the commutator identity. The form is therefore symmetric on the two-dimensional block as well. Take the orthogonal sum with a form on the line. The decomposition and the finitely many entries of the form are defined over a finite extension. □

**Proposition 37.11** (A marked self-dual lattice with split reduction)**.** *There is a complete coefficient discrete valuation ring $A$ of characteristic $\ell$, finite over $k[[T]]$, and a basis for $\rho$ over $A$ in which the following statements hold. The reduction is exactly* $\operatorname{diag}(\bar\Psi, 1, \bar\Psi^{-1})$*. Writing $J$ for the three-by-three matrix with ones on the antidiagonal and zeros elsewhere, one has*

$$\rho(g^\star) = J\rho(g)^{\mathsf{T}}J, \qquad \rho(g_0) - \rho(g_0^\star) = \operatorname{diag}(d, 0, -d), \qquad d \bmod \mathfrak{m}_A = \delta.$$

*The ordinary flags at the chosen completions restrict to saturated integral flags with the ordered local characters.*

*Proof.* Compactness of the image gives a stable lattice, by taking the integral span of the translates of any basis as in Proposition 37.2. The reduction has the stated three composition factors. Choose a composition flag after a finite residue extension if necessary. Let $Z$ denote the difference of the two matrices in the statement. In the reduction, its actions on the three simple quotients are $\delta, 0, -\delta$. By Hensel's lemma and skew adjunction its three eigenvalues lift to $d, 0, -d$ with unit pairwise differences. The spectral projectors are polynomials in $Z$ over the coefficient ring and split the lattice as three free lines. Choose eigenvectors in the residue labeling $+, 0, -$. Each step of the residual composition flag is a sum of the corresponding eigenlines, since it is stable under $Z$. Thus in some permutation of this basis the residual group matrices are all upper triangular.

We give the integral adjustments explicitly. Normalize the valuation of a uniformizer $\pi$ here to one. Rescale the three eigenvectors by $\pi^{t_i}$, taking very small rational exponents strictly increasing in the permutation just obtained, with their spread less than one. Such scalings are available after finite ramified extension. The valuation of the $ij$ entry changes by $t_j - t_i$. Every off-diagonal unit entry therefore acquires positive valuation, and a previously positive valuation stays positive. The diagonal reductions are the original characters in the residue labeling, since the eigenvalue of $Z$ distinguished each quotient. We have obtained a stable lattice with split reduction.

For the symmetric form of Lemma 37.10, an eigenline of $Z$ is orthogonal to all eigenlines except the one of opposite eigenvalue, by skew adjunction. Write $i^*$ for the opposite index, with $0^* = 0$, and let $b_i \neq 0$ be the pairing of the unscaled $i$-th and $i^*$-th vectors. We have $b_i = b_{i^*}$. The dual of the lattice with exponents $t_i$ has exponents

$$t_i^{\#} = -v(b_i) - t_{i^*}.$$

It is also stable and has split reduction. Indeed the group matrices on the dual basis are transposes of the matrices at $g^\star$, with the order of the dual lines reversed. Consequently all their off-diagonal entries have positive valuation. The eigenvectors with the displayed exponents differ from that basis only by units. Replace the $t_i$ by $(t_i + t_i^{\#})/2$. For each nonzero matrix entry its new valuation is the average of its valuations in the two stable lattices. Thus split reduction and stability persist, and the new exponents satisfy $t_i + t_{i^*} = -v(b_i)$. A further finite ramified extension realizes these exponents. The two outer lines now pair to a unit and the middle norm is a unit. Rescale one outer vector by the inverse of that unit pairing and the middle vector by an inverse square root of its norm, after a finite extension. The form matrix is exactly $J$. This proves the asserted polarization and marking.

Intersect each step of a local ordinary flag over the fraction field with this lattice. The quotient by that intersection is torsion-free, so both the intersection and the quotient are free and the inclusion is a direct summand over the discrete valuation ring. Stability and the local characters descend. On reduction the semisimplification on the decomposition group is the restriction of the split diagonal representation. The three characters are distinct already on residual inertia. Their restrictions there identify their order uniquely, hence also identify the residual characters on the entire local group. Finally the valuation ring of a finite extension of $k((T))$ is finite over $k[[T]]$. If its residue field is $k'$ and a uniformizer is $z$, it is $k'[[z]]$, by expansion in powers of $z$ with coefficients in the finite field lifted by the simple roots of $X^{|k'|} - X$. □

**Lemma 37.12** (One split prime for the local intertwining equation)**.** *On the common field $L = L_a$ there is a rational prime $p_0$ split completely in $L$, outside $2sN_*w\ell$ and the ramification of $L$, at which*

$$\bar{\Psi}(\phi) = 1, \qquad \bar{\kappa}_\ell(\phi) = -1$$

*for every pair on the interval and every arithmetic Frobenius lift $\phi$ there. For any integral matrix representation $U$ over a local ring with this residual semisimplification, the only matrix satisfying*

$$\kappa_\ell(\phi)\,U(\phi)X = XU(\phi)$$

*is zero, also after arbitrary base change of the coefficient ring. This assertion persists after twisting $U$ by a character.*

*Proof.* The fields $L$ and $E_*(\mu_\ell)$ are disjoint by Proposition 36.15. Apply Chebotarev to the element acting trivially on the former and as complex conjugation on the latter. Exclude the finitely many forbidden primes. The resulting prime has residue degree one in $L$, and the parity condition gives both displayed values. The characters are unramified there. Modulo the maximal ideal $U(\phi)$ has only the eigenvalue 1. Over an algebraic closure of the residue field, triangularize this one matrix. The eigenvalues of $X \mapsto \kappa_\ell(\phi)U(\phi)X - XU(\phi)$ on all matrix entries are then all $-2$, a unit. This follows by viewing left and right multiplication on the tensor product with the dual space, or by triangularizing each tensor factor. The determinant of this linear operator is a unit in the original local ring. Its inverse therefore exists over every base change. A scalar twist at $\phi$ multiplies the operator by a unit and preserves the conclusion. □

**Theorem 37.13** (A one-dimensional prime in the marked ordinary deformation ring)**.** *After finite extension of coefficients the curve of Proposition 37.9 defines a prime of residue characteristic $\ell$ with one-dimensional quotient in the marked ordinary deformation ring of $H$ at the split point. Its generic representation is neither upper nor lower triangular in the marked basis. For every chosen place its ordinary characters are distinct in pairs even on inertia in $\ker\kappa_\ell$. The following further properties hold, with $H' = \ker(\kappa_\ell|_H)$.*

1. *The restriction to $H'$ is absolutely irreducible, or is an orthogonal sum of a character and an absolutely irreducible two-dimensional representation. In the second case the outer marked lines span the two-dimensional constituent.*

2. *In every relatively open nonempty subset of $H'$ there is an element whose three eigenvalues are distinct.*

3. *The local intertwining equation of Lemma 37.12 has only the zero solution at the indicated split place. On twisting the curve by $\bar\kappa_\ell^{-1}$, the pairing has cyclotomic multiplier $\bar\kappa_\ell^{-2}$ in the conjugate-dual identity.*

*Proof.* Enlarge the residue field of the mixed-characteristic coefficient ring by an unramified extension to contain $k'$ of Proposition 37.11, using $k'$ as the common residue field. The marked ordinary ring here represents matrices lifting the split triple, with the polarization $J$, diagonal $Z$, and the ordered characters of all chosen decomposition groups. For the ordinary condition impose, for arbitrary $h_1, h_2, h_3$ in one such group, the equation

$$(\rho(h_1) - \xi_1(h_1)I)(\rho(h_2) - \xi_2(h_2)I)(\rho(h_3) - \xi_3(h_3)I) = 0.$$

An integral ordered stable flag gives this equation, since the three factors from right to left successively lower its steps. This is the matrix problem in `Development/AutomorphicArithmetic/AutomorphicVarieties/FrobeniusSmoothVectors/LeAdicClosure.lean`, line 27.

We recall its elementary representability here. The open kernel of the split residual representation of $H$ has finitely generated maximal pro-$\ell$ quotient, by the finiteness of its continuous homomorphisms to $\mathbf{F}_\ell$ proved in Theorem 37.8 and the pro-$\ell$ basis theorem. Every lift maps

this kernel to a pro-$\ell$ principal congruence group. Hence the relevant quotient of $H$ is an extension of a finite group by a finitely generated pro-$\ell$ group. It has finitely many topological generators. Take a power series ring on entries of matrices for these generators relative to fixed lifts of their residues. The matrices are invertible since their determinants have nonzero residues. The free profinite group on the generators acts continuously by these matrices, as the general linear group of the power series ring is profinite. Imposing all relators gives exactly continuous representations of the quotient with the fixed reduction. For each local character use its Teichmüller residual lift times a principal-unit character. The latter is determined by the maximal pro-$\ell$ abelian quotient of the local group, finitely generated by local reciprocity. Include one variable per generator and the corresponding relations. Impose the displayed word, polarization and diagonal marking by their matrix entries. Closed ideals in this Noetherian complete local power series ring give the resulting ring $R$. This presentation proves its universal property, including for the coefficient ring $A = k'[[z]]$.

Proposition 37.11 thus supplies $R \to A$. Its kernel is prime. Its image contains a nonconstant local character value of residue one, by Proposition 37.9, since on the indicated kernel the three values there were distinct powers of $1 + T$. Subtract one to obtain a nonzero $s_1 \in zA$ in the image. A lift of $s_1$ belongs to the maximal ideal of $R$, so substitution of arbitrary convergent formal series with coefficient lifts gives $k'[[s_1]]$ in the image. The ring $A$ is finite over $k'[[s_1]]$. Indeed the $s_1$-adic and $z$-adic topologies agree and $1, z, \dots, z^{v_z(s_1)-1}$ span $A/s_1A$. By successive lifting these elements generate $A$ over $k'[[s_1]]$. Every intermediate algebra is therefore a finite module over this Noetherian one-dimensional ring. In particular $R/\ker(R \to A)$ is an integral extension of $k'[[s_1]]$ and has dimension one. The same argument works for the subalgebra generated by all matrix entries over the closed subalgebra of the weight-zero monomials and the local characters used in the source. The latter already contains all the series in the chosen local character minus one.

Consider the image over an algebraic closure of the fraction field. By Proposition 37.9 it is either irreducible with connected monodromy $\mathrm{SL}_3$, or is a direct sum of a two-dimensional block with full projective monodromy $\mathrm{PGL}_2$ and a line. In the second case the block and line are stable under the anti-involution, and the image of $Z^2/u$ and its kernel identify them in the marked basis by Lemma 37.10. An upper or a lower triangular representation cannot have an irreducible constituent of dimension greater than one. In the irreducible case there must be a nonzero matrix entry between the middle line and the outer plane for some element, since otherwise both subspaces would be invariant. In the second case every such entry is zero. This identifies the two matrix alternatives at the constructed prime.

The group $H'$ contains all commutators of $H$. The Zariski closure of its image therefore contains the derived group of the connected monodromy, by density and continuity of the commutator morphism. In the first case this contains $\mathrm{SL}_3$. In the second case it contains $\mathrm{SL}_2 \times 1$ on the block and the line. In fact the image of the connected component in $\mathrm{PGL}_2$ is the whole group. Its derived group maps onto that perfect group and lies in $\mathrm{SL}_2 \times 1$. A proper closed subgroup there has dimension less than three and cannot map onto $\mathrm{PGL}_2$. This proves the assertions on irreducibility and also excludes any intertwiner between the two unequal constituents on $H'$.

For the density assertion take a coset in $H'$ of an open subgroup contained in the given open set. The Zariski closure of the image of this subgroup contains the identity component of the closure for $H'$, by finite index. In the first case the coset closure therefore contains a translate of $\mathrm{SL}_3$, in which matrices with three distinct eigenvalues exist. In the second it contains all matrices of the form $(BD, \gamma)$, $D \in \mathrm{SL}_2$, for one invertible $B$ and scalar $\gamma$. The matrix $BD$ can be chosen diagonal with entries $x, \det(B)/x$, with $x$ avoiding the finitely many values making two among these and $\gamma$ equal. Vanishing of the characteristic polynomial discriminant is closed.

It therefore cannot hold on the whole coset in either case. The separation of the three local characters was already proved in Proposition 37.9. It also excludes any identity between one of them and another times the residual cyclotomic character, by restriction to that same inertia kernel. The split-prime assertion follows from Lemma 37.12. Finally complex conjugation preserves the cyclotomic character, so twisting multiplies the two sides inside the bilinear pairing by the product of two copies of $\bar{\kappa}_\ell^{-1}$. This gives the stated multiplier and finishes the proof. □

The one-dimensional prime has the required local separation. Its symmetric pairing on every constituent gives the polarized representation type described in `Development/AutomorphicArithmetic/AutomorphicVarieties/WeilGroupCharacters/UniversalOrdinaryDeterminantRing.lean`, line 224. For each curve its generic monodromy decides which type occurs. The propagation assertion at line 284 of that file concerns every irreducible component through a prime of the first type. For the second type line 293 uses chains through several such primes. To apply these propagation assertions one must prove that the relevant components carry the ordinary Hecke ideal. The integral trace quotient of Theorem 36.11 still requires control on the original fixed-weight fiber for the length estimate of Section 6.6.

## 37.4 Simultaneous Frobenius tests at every parameter precision

The deformation argument uses traces to test a dual cohomology group on an ordinary curve. When the curve has two constituents, an off-diagonal infinitesimal matrix has trace zero even after multiplication by any block diagonal matrix. We give the trace calculation including the needed off-diagonal deformations, and keep track of what it detects over the finite parameter rings.

Use $L = L_a$ from Theorem 37.8. Fix a finite set of primes of $L$ containing the primes over $\ell s N_* p_0$ and any tame levels in use, and stable under complex conjugation. Include its infinite places. Write $G = G_{L,S}$, $N = \ker(\kappa_\ell|_G)$ and $\chi = \bar{\kappa}_\ell|_G$. These characters factor through $G$. We have $G/N$ abelian, $\chi|_N = 1$, and some $\gamma \in G$ has $\chi(\gamma) = -1$ by Lemma 37.12. Denote conjugation by $c$ on $G$ by $\sigma$. Pull back an ordinary curve in Theorem 37.13 to $G$, with coefficient ring $A = k'[[z]]$, fraction field $Q$, and representation $r$. We may use $r$ or its cyclotomic normalization since both have the same conjugation action on matrices.

Write $V = \bigoplus_i V_i$ over $Q$ for the decomposition into one or two absolutely irreducible constituents on $N$. The summands are already defined over $Q$ after the allowed finite extension, stable under $G$, inequivalent on $N$, and individually polarized. Use the block lattices of Proposition 37.11. In the reducible case these are the outer plane and the middle line. Write

$$E_A = \bigoplus_i \mathrm{End}_A(V_{i,A}), \qquad M_A = \bigoplus_{i \neq j} \mathrm{Hom}_A(V_{j,A}, V_{i,A}),$$

and $E, M$ after extending scalars to $Q$. Let $\dagger$ denote adjunction for the symmetric pairing. It exchanges the two off-diagonal blocks, and $r(\sigma g)$ is a scalar multiple of $(r(g)^\dagger)^{-1}$. A polarized off-diagonal cocycle $\nu : G \to M_A$ for conjugation by $r$ satisfies $\nu(\sigma g) = -\nu(g)^\dagger$. Its polarized coboundaries are $g \mapsto r(g) Y r(g)^{-1} - Y$ with $Y \in M_A$, $Y^\dagger = -Y$.

Choose continuous integral polarized cocycles $\nu_1, \dots, \nu_b$ independent modulo these coboundaries over $Q$. One may take a basis after inverting $z$ of any prescribed module of ordinary off-diagonal Selmer classes, including the local conditions in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/PairedDualFirstOrderDualMatrixSpace.lean`, line 306. The finiteness needed for such a basis is proved below. In the one-block case take

$b = 0$. Introduce the finite free $A$-algebra

$$\mathcal{W}_A = E_A \oplus \bigoplus_{j=1}^{b} M_A e_j, \qquad (M_A e_i)(M_A e_j) = 0,$$

with left and right multiplication by block diagonal matrices. There is a representation into its units given by

$$R(g) = \Big(1 + \sum_j \nu_j(g) e_j\Big) r(g).$$

Indeed the cocycle identity is exactly the multiplication rule in this square-zero extension. Write $\mathcal{W} = \mathcal{W}_A \otimes_A Q$.

**Lemma 37.14** (The algebra spanned on the cyclotomic kernel)**.** *The $Q$-linear span of $R(N)$ is $\mathcal{W}$.*

*Proof.* The span $D$ is a subalgebra, since $R$ is a representation. Modulo the square-zero ideal its image is $E$. Indeed the summands on $N$ are absolutely irreducible and pairwise inequivalent. Over an algebraic closure their semisimple image algebra acts faithfully and is the product of the full matrix algebras on the distinct simple modules. Equality then follows over $Q$ by comparing dimensions. In the one-block case this proves the lemma.

In the two-block case $D \cap M^{\oplus b}$ is an $E$-bimodule. To multiply by any $x \in E$, lift $x$ to $D$. The result on the intersection is independent of the lift because the ideal has square zero. The two central block idempotents separate the two directions of the off-diagonal blocks. In a given direction $ij$, left and right matrix units move each single entry to any chosen position. Thus the intersection in that direction consists of all matrices of that block size with entries in one fixed subspace $U_{ij} \subset Q^b$. If $D$ is proper then for at least one direction there is a nonzero functional $\lambda = (\lambda_1, \dots, \lambda_b)$ killing $U_{ij}$.

Project the algebra to $E \oplus M_{ij}$ by keeping that direction, applying $\lambda$, and killing the opposite direction. This is an algebra map because both radicals are square-zero. The projected image of $D$ maps isomorphically to $E$. It is therefore the graph of a derivation $d : E \to M_{ij}$. Let $P_i$ be the idempotent of block $i$. In $M_{ij}$ it acts as one on the left and as zero on the right. Differentiating $P_i x = x P_i$ gives

$$d(x) = x d(P_i) - d(P_i) x.$$

For $n \in N$ the equality describing the graph reads $\big(\sum_t \lambda_t \nu_t(n)_{ij}\big) r(n) = d(r(n))$, viewing that block as a matrix with the other entries zero. Thus the indicated linear combination in that block direction is a coboundary on $N$. It is a coboundary on $G$ as well. After subtracting this coboundary a cocycle vanishes on $N$, and its value at any $g \in G$ is $N$-invariant by the cocycle identity and normality of $N$. The space $M_{ij}^N$ is zero by inequivalence and irreducibility.

Write $Y_{ij}$ for this coboundary witness over $G$. Polarization determines the opposite block uniquely and gives the witness $-Y_{ij}^{\dagger}$ there. This follows by applying $-\dagger$ to the coboundary at $\sigma g$ and using the scalar relation between $r(\sigma g)$ and $(r(g)^{\dagger})^{-1}$. Hence the whole combination $\sum_j \lambda_j \nu_j$ is a polarized coboundary, contrary to independence. Both $U_{ij}$ must be full, which proves the lemma. □

Let $\mathcal{W}_A^{\vee} = \mathrm{Hom}_A(\mathcal{W}_A, A)$ with bimodule structure $(afb)(x) = f(bxa)$, and twist its conjugation action by $\chi$. For a continuous cocycle $y : G \to \mathcal{W}_A^{\vee}(1)$ and $n \in N$ put

$$\lambda_n(y) = y(n)(R(n)).$$

This is an ordinary trace on diagonal matrices and includes the paired off-diagonal terms. In coordinates a functional on the diagonal algebra is trace against a block diagonal matrix $P$. A functional on the $j$-th summand of the radical is trace against an off-diagonal matrix $D_j$ in the opposite positions. Evaluation on $R(n)$ gives the sum of $\mathrm{tr}(Pr(n))$ and $\sum_j \mathrm{tr}(D_j \nu_j(n) r(n))$, allowing $P, D_j$ to depend on $n$. This identifies the functional with the trace coefficient used in the source. It vanishes on coboundaries since $\chi(n) = 1$ and conjugation by $R(n)$ fixes $R(n)$.

**Theorem 37.15** (Detection of the twisted dual by evaluations)**.** *Let $W$ be any finite-dimensional unital algebra over a field $Q$, $R : G \to W^\times$ a representation, and $N \triangleleft G$ such that $R(N)$ spans $W$. Assume $G/N$ is abelian and a character $\chi : G/N \to Q^\times$ takes a value different from one. Give $W^\vee$ the action $g \cdot f = \chi(g) R(g) f R(g)^{-1}$. If a cocycle $y$ satisfies $y(n)(R(n)) = 0$ for every $n \in N$, then $y$ is a coboundary.*

*Proof.* On the group algebra $Q[N]$ extend $R$ linearly and put $d(n) = y(n)R(n)$, extending this map linearly too. Since $\chi|_N = 1$, the cocycle law gives the Leibniz rule with respect to $R$. Write $\tau(f) = f(1)$. The hypothesis is $\tau(d(x)) = 0$ on all of $Q[N]$. If $R(x) = 0$, then for any $v \in Q[N]$ the Leibniz rule gives

$$0 = \tau(d(xv)) = \tau(d(x)R(v)) = d(x)(R(v)).$$

Since $R(Q[N]) = W$, this forces $d(x) = 0$. The map $d$ thus descends to a derivation $D : W \to W^\vee$.

Choose $\gamma \in G$ with $\chi(\gamma) \neq 1$ and set $B = R(\gamma)$. For $X = R(n)$, compute $D(BXB^{-1})$ in two ways. The derivation rule in $W$ and $D(B^{-1}) = -B^{-1}D(B)B^{-1}$ give

$$BD(X)B^{-1} + [D(B)B^{-1}, BXB^{-1}].$$

Using the cocycle at $\gamma n \gamma^{-1}$ instead gives

$$\chi(\gamma) BD(X)B^{-1} + [y(\gamma), BXB^{-1}].$$

Here the commutator uses the two bimodule actions. Both expressions are linear in $X$, hence agree on all of $W$. Conjugating back and dividing by $1 - \chi(\gamma)$ shows that $D(X) = [Y, X]$ for the fixed functional

$$Y = \frac{B^{-1}(y(\gamma) - D(B)B^{-1})B}{1 - \chi(\gamma)}.$$

The restriction of $y$ to $N$ is therefore the coboundary with witness $-Y$.

Subtract the corresponding $\chi$-twisted coboundary on $G$. The remaining cocycle factors through $G/N$ on the invariants of $N$, by the argument of Lemma 37.14. These are the functionals commuting with all of $W$, since $R(N)$ spans it. On this space the quotient acts through the scalar $\chi$. For a cocycle $f$ on the abelian quotient, commutativity now gives

$$(\chi(\gamma) - 1) f(g) = (\chi(g) - 1) f(\gamma).$$

This gives the final coboundary witness $f(\gamma)/(\chi(\gamma) - 1)$ and proves the assertion. □

**Proposition 37.16** (One loss for all finite parameter rings)**.** *On any of the curves above, let $N^{\mathrm{reg}}$ be any relatively open dense subset of $N$. There exist finitely many $n_1, \dots, n_q \in N^{\mathrm{reg}}$ and an integer $c_1 \geq 0$ such that for every $m \geq 1$, a continuous class over $\mathcal{W}_A^\vee(1)/z^m$ whose evaluations vanish at these elements is killed by $z^{c_1}$. This includes all classes over $A/z^m$, whether or not they come from classes over $A$. For the polarized cohomology the coboundary witness can be taken with the required symmetry.*

*Proof.* Here are the finiteness details. For any finite free $A$-module $P$ with continuous action of $G = G_{L,S}$, the module of continuous cocycles with values in $P$ is finitely generated over $A$. More precisely, the cocycles over $A$ and over every $A/z^m$ are given by one finite matrix of linear relations on one finite list of generator values. Choose the open normal subgroup $U$ acting trivially on $P/zP$. It has finite-dimensional $H^1(U, \mathbf{F}_\ell)$. Indeed it is a Galois group with ramification allowed at a finite set over the finite field extension of $L$ corresponding to $U$. After adjoining $\mu_\ell$, Kummer theory bounds the dimension using the units and the class group of the ring of $S$-integers as in Theorem 37.8. It suffices here even to inflate to that restricted ramification group. Consequently the maximal pro-$\ell$ quotient of $U$ is topologically finitely generated. Every cocycle together with the action gives a homomorphism to the semidirect product of the additive module with the automorphism group. On $U$ its image is pro-$\ell$, since the linear part is in a principal congruence group and the additive part is pro-$\ell$. The same holds modulo every $z^m$. All these cocycles thus factor, with their action, through one topologically finitely generated quotient of $G$.

Let $l_0$ be its number of chosen generators and $\mathcal{F}$ the free profinite group on them. Over $A$ any assignment of values in $P^{l_0}$ extends to a unique continuous cocycle on $\mathcal{F}$, by the semidirect product and its profiniteness. Evaluation at any word of $\mathcal{F}$ is an $A$-linear map of these variables. For a relator it must be zero to descend. The rows of all such maps generate a submodule of the dual of $P^{l_0}$, so finitely many suffice by Noetherianity. Write $C$ for the resulting matrix. The same rows work modulo $z^m$, since every tuple of generator values lifts freely before relations are imposed. This proves the claimed presentation. It also proves the earlier assertion on the choice of $\nu_j$: imposed linear local and polarization conditions give a submodule of a finite module. The quotient by coboundaries in it is finite over $A$, and clearing denominators gives representatives of a basis over $Q$.

Apply this presentation with $P = \mathcal{W}_A^\vee(1)$, and write $B_0 : P \to P^{l_0}$ for evaluation of coboundaries on the chosen generators. The vector space $(\ker C \otimes_A Q)/\operatorname{im}(B_0 \otimes_A Q)$ is finite-dimensional. Lemma 37.14 and Theorem 37.15 show that all the evaluations on $N$ separate it. They still separate it on $N^{\mathrm{reg}}$ by continuity, after multiplying a fractional cocycle by a power of $z$ to make it integral. Finite-dimensional linear algebra selects $n_1, \dots, n_q$ with the same property. Fix words above these elements and append their evaluation rows to $C$, obtaining a matrix $D_0$. Coboundaries satisfy these rows. We have proved

$$\ker(D_0 \otimes_A Q) = \operatorname{im}(B_0 \otimes_A Q).$$

Let $b_1$ be the largest valuation of a nonzero Smith invariant of $D_0$, taking zero if the matrix is zero. The finitely generated torsion module $\ker D_0/\operatorname{im} B_0$ is killed by some $z^{b_2}$. If $D_0 v = 0 \bmod z^m$, then $z^{b_1} v$ belongs to $\ker D_0 + z^m P^{l_0}$. Indeed in Smith coordinates every component outside the kernel is sent to zero modulo $z^m$ after this multiplication. Hence $z^{b_1+b_2} v \in \operatorname{im} B_0 + z^m P^{l_0}$. A cocycle modulo $z^m$ is determined by its generator values, giving the result with $c_1 = b_1 + b_2$.

For the polarization, the signed adjunction on coefficients intertwines the $G$-action with its conjugate by $\sigma$. Explicitly adjunction on $\mathcal{W}_A$ acts by $\dagger$ on every block and every $e_j$-coefficient. The identities for $r$ and $\nu_j$ give $R(\sigma g)$ equal to a scalar times $(R(g)^\dagger)^{-1}$. On the dual use $f \mapsto -\chi(c)\, f \circ \dagger$. It is an involution and intertwines the twisted actions because $\chi(\sigma g) = \chi(g)$. Thus when a cocycle has the prescribed symmetry, the average of any coboundary witness and its image under this involution is a witness with that symmetry. This averaging is integral since $\ell$ is odd. □

**Theorem 37.17** (Common primes for the dual trace tests)**.** *Fix the interval and the coefficient prime $\ell$. For any finite collection of the ordinary curves of Theorem 37.13 on their common*

*field, one can choose a single number $q$ and a single loss $c_1$. For every $m, j \geq 1$ there are $q$ distinct rational primes outside the fixed levels, split completely in $L$ and congruent to one modulo $\ell^j$, with choices of Frobenius above them such that the following holds on every curve. All their characteristic polynomials are separable over the fraction field. If a continuous class with coefficients in $\mathcal{W}_A^\vee(1)/z^m$ has zero evaluations at these Frobenius elements, multiplication by $z^{c_1}$ kills it. The collection may include a curve for every $r, \alpha$ on the entire interval.*

*Proof.* Work over the common absolute group, or its quotient $G_{L,S}$ for the union of the finitely many sets of places. The regular-semisimple locus of each curve is relatively open and dense on $N$ by Theorem 37.13. Pullback to the common group preserves density because the projection is open and maps its cyclotomic kernel onto the individual kernel. The finite intersection of these open dense sets is again dense. Use it for $N^{\mathrm{reg}}$ in Proposition 37.16 on each curve, take the union of the resulting finite lists of elements and the maximum of the losses. There are finitely many possible pairs on the interval since $r < t_*$ and all characters factor through the one finite extension $E_*$.

Fix $m, j$. Over each finite parameter ring there are only finitely many cocycles, by the finite list of generator values in the preceding proof. For each selected element prescribe its image in the joint finite quotient recording the matrices $R$ and all these cocycles modulo $z^m$ on every curve. Such a quotient exists by continuity of each semidirect-product representation. Record also the matrices $r$ to a precision exceeding all the valuations of the nonzero characteristic discriminants at the selected elements. Lift the elements to $G_L$. Their full cyclotomic character equals one, so these prescriptions together with trivial action on $\mu_{\ell^j}$ hold in a finite Galois extension of $L$. Pass to its normal closure over $\mathbb{Q}$, including $L$. Chebotarev supplies infinitely many rational primes in the conjugacy class of each prescribed element. At such a prime a place of the normal closure can be chosen so that its Frobenius is the specified element itself. It fixes $L$ and $\mu_{\ell^j}$, giving both splitting and the congruence. Choose distinct primes and discard all ramified primes and primes at the fixed levels. The joint finite images make every evaluation equal to the prescribed one modulo $z^m$, and the higher precision keeps all the discriminants nonzero. At these Frobenius elements $\chi = 1$, so evaluation again depends only on the class. Proposition 37.16 applies with the same $c_1$. □

The trace coefficient tested here is the one in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/ThetaCoefficientMapEq.lean`, line 254. The loss is independent of both precisions $m, j$. It is measured in the parameter $z$ on a curve of characteristic $\ell$, whereas the lengths in Section 6.6 are measured in a uniformizer over $\mathbb{Z}_\ell$ at the original weight. A comparison with those lengths still requires the action of the Hecke algebra on the modules at the chosen levels and its behavior on specializing the weights. Theorem 37.17 provides the common primes for that argument.

## 37.5 The first-order Selmer group at the auxiliary levels

Fix $F = F_a^+$, $L = KF$, and the interval of Theorem 37.8. Write $d = [F : \mathbb{Q}]$ and $n = 3$. Take one of the curves of Theorem 37.13 with $A = k'[[z]]$ and $Q = k'((z))$. Let $E_A, M_A, \nu_j, R$ and $W = \mathcal{W}_A$ be as in Section 37.4. Choose the $\nu_j$ on $\Gamma_a^0$ from the ordinary off-diagonal Selmer group, with integral local witnesses as in that section. Write $n_i$ for the ranks of the constituent blocks. The involution $\dagger$ on $W$ preserves the labels $e_j$. The action of $G_L$ by conjugation by $R$ extends to $G_F$ by letting a complex conjugation $c$ act as $-\dagger$. Indeed the polarization gives $R(cgc^{-1})$ equal to $(R(g)^\dagger)^{-1}$ times the scalar multiplier of $r$. This verifies the conjugation relation, and $c^2 = 1$. Write $A_m = A/(z^m)$ and $P_m = P/z^m P$ for any of our free coefficient modules $P$. For finite groups of coefficients or cohomology use $h$ for length over $A$, adding an

exponent to indicate a cohomological degree. Length is the logarithm of cardinality to the base $|k'|$.

At each $u \mid \ell$ use one of the two places of $L$ over $u$ to identify its local group with $G_{F_u}$. In a basis adapted to the ordinary flag, take the upper triangular entries in $E_A$ and in every $M_A e_j$, using the order of the three successive characters on the flag. Such a basis can be taken in the individual blocks. A local inertia element with distinct residual eigenvalues has its three eigenprojectors over $A$ by Hensel's lemma. Each lies in a constituent block, and a saturated stable step of the flag is the sum of the eigenlines that it contains. If $Y_{u,j}$ are the local witnesses for $\nu_j$, conjugate $R$ by $1+\sum_j Y_{u,j}e_j$. This replaces $\nu_j$ by $\nu_j-(\mathrm{Ad}(r)Y_{u,j}-Y_{u,j})$, which is upper triangular on the local group. In these coordinates the upper triangular subspace is stable under $R$. Transport it back and call it $B_u \subset W$. Both it and its quotient are free. Define

$$w=\sum_i n_i^2+2b\sum_{i<j} n_i n_j, \qquad u_0=(w+n)/2, \qquad w_-=(w-n)/2.$$

**Lemma 37.18** (Local and global invariant spaces)**.** *The rank of $B_u$ is $u_0$. Uniformly in $m \geq 1$, the lengths of $H^0(F, W_m)$, $H^0(F_u, (W/B_u)_m)$ and $H^0(F_u, B_{u,m}^\vee(1))$ are bounded. Here the dual is the $A_m$-linear dual with the residual cyclotomic twist. At a real place $\tau$ one has $h^0(F_\tau, W_m) = w_- m$.*

*Proof.* Each diagonal block contributes $n_i(n_i+1)/2$ allowed entries and each unordered pair of distinct blocks contributes $n_i n_j$ in each radical copy. These sum to $u_0$. On the radical the local action is ordinary triangular conjugation by $r$, and on the quotient by the radical it is block diagonal conjugation by $r$. Filter these triangular matrix spaces by matrix units in order of row increasing and, within each row, column decreasing. Conjugation takes a matrix unit at $ij$ to units at $i'j'$ with $i' \leq i$ and $j' \geq j$, with coefficient $\xi_i/\xi_j$ at $ij$ itself. Here $\xi_i$ is the $i$-th ordinary character. Consequently the factors of $(W/B_u)\otimes Q$ all have $i > j$, and those of $B_u^\vee(1)\otimes Q$ have characters $\bar\kappa_\ell \xi_j/\xi_i$ with $i \leq j$. At distinct indices these characters are nontrivial on the kernel of full cyclotomic inertia by Theorem 37.13. At equal indices $\bar\kappa_\ell$ is nontrivial because $F_u$ does not contain $\mu_\ell$ by Theorem 37.8. The two local modules therefore have zero invariants over $Q$.

A global invariant over $Q$ commutes with $R(G_L)$. Lemma 37.14 puts it in the center of $W \otimes Q$. Commuting with the block idempotents kills its off-diagonal entries. Its diagonal blocks are scalars since they commute with all matrix units in their blocks. The operation $c$ negates those scalars, and $\ell$ is odd. Thus this invariant is zero. Whenever a finite free module has zero invariants over $Q$, finitely many operators $g-1$ already have trivial common kernel there. Stack their matrices over $A$. Smith normal form bounds the length of its kernel over $A_m$ by the sum of the valuations of its nonzero invariant factors, independently of $m$. This proves all three bounds.

For the real formula write $c_\tau = hc$, $h \in G_L$, so that $chc^{-1} = h^{-1}$. The cyclotomic value of $h$ is one, as both conjugations have cyclotomic value $-1$. The polarization therefore gives $R(h)^\dagger = R(h)$. On each diagonal block modulo the radical the action of $c_\tau$ is negative adjunction for a nondegenerate symmetric form. Its fixed space is the space of skew-adjoint matrices and has rank $n_i(n_i-1)/2$, as is seen by multiplying by the form matrix. On each radical copy $c_\tau$ exchanges opposite blocks by mutually inverse linear maps. The fixed space has rank $\sum_{i<j} n_i n_j$. Taking invariants is exact over $A$ and over $A_m$ by the idempotent $(1+c_\tau)/2$. The ranks sum to $w_-$, proving the claim. □

Fix a finite set $S$ over $F$ containing the real places, the places over $\ell$, the ramified places of our coefficients and the places over $p_0$ of Lemma 37.12. Include all earlier levels in use. Write $n'_1, \ldots, n'_q$ for the group elements used by Theorem 37.17, to distinguish them from the block

sizes. At stage $N \geq 1$ choose $q$ distinct rational primes split completely in $L$, all congruent to one modulo $\ell^N$, with chosen places giving the prescribed Frobenius elements to precision $z^N$. In making that Chebotarev choice, record the cocycles at *every* precision $1 \leq e \leq N$ in the finite quotient in its proof. There are finitely many at each precision and finitely many curves on this interval. This makes all the tests valid even for classes modulo $z^e$ that do not lift modulo $z^N$. Write $\mathcal{Q}_N$ for the chosen places of $F$, one for each rational prime.

On $G_{F,S\cup\mathcal{Q}_N}$ take the local conditions $\mathcal{L}$ on $W_m$ given by the image of $H^1(F_u, B_{u,m})$ at $u \mid \ell$, the full group at $\mathcal{Q}_N$ and above $p_0$, and the unramified group at every other finite place. At the real places $H^1 = 0$. Denote the orthogonal conditions for the dual $W_m^\vee(1)$ by $\mathcal{L}^\perp$.

**Proposition 37.19** (Bounds at the selected Frobenius elements)**.** *For every $1 \leq m \leq N$ the length of $H^1_{\mathcal{L}^\perp}(F, W_m^\vee(1))$ is bounded by a constant independent of $m, N$. At $v \in \mathcal{Q}_N$ one has $h^0(F_v, W_m^\vee(1)) = nm + O(1)$ uniformly in these parameters. At $v \mid p_0$ this invariant group vanishes.*

*Proof.* A dual class is locally zero at each new place. The coefficients are unramified there, so every representing cocycle then vanishes on the new inertia. It descends to $G_{F,S}$, since it kills the closed normal subgroup generated by that inertia on which the action is trivial. Restrict the descended class to $G_L$. At every chosen Frobenius its evaluation of Section 37.4 vanishes. Indeed it is locally a coboundary and the twist is one at that Frobenius. Theorem 37.17 consequently kills the restricted class by $z^{c_1}$. Restriction has trivial kernel since corestriction composed with it is multiplication by two.

The proof of Proposition 37.16, applied to $G_{F,S}$, bounds the number of generator values for its cocycles independently of $m$. Its proof uses finiteness of the $S$-unit rank and class group and works over $F$ as over $L$. Hence the dual group in question is a subquotient of $A_m^D$ for a fixed $D$. Any such subquotient has at most $D$ generators, by lifting submodules to $A^D$ and using that $A$ is a discrete valuation ring. Being killed by $z^{c_1}$, it has length at most $Dc_1$.

At the $j$-th new place the action on $W_m$ equals conjugation by $R(n'_j)$ modulo $z^m$ and the twist is trivial. Over $Q$, the matrices $r_i(n'_j)$ in the constituent blocks have simple and mutually disjoint spectra. For different blocks $i, k$ the operator $X \mapsto r_i(n'_j)X - Xr_k(n'_j)$ is invertible, as one sees over a splitting field. Conjugation by an element of one plus the square-zero ideal thus removes the radical part of $R(n'_j)$. Its centralizer has zero radical part and diagonal part of dimension $\sum_i n_i = n$, since the centralizer of a matrix with simple spectrum has dimension equal to its size. The dual conjugation operator has the same fixed dimension by transposition and invertibility. Smith normal form of this fixed operator minus the identity gives a kernel of length $nm + \sum_j \min(m, a_j)$, where the $a_j$ are the valuations of its nonzero Smith factors. The list is fixed before $N$ varies. At a place above $p_0$, the residual conjugation has only eigenvalue one, both on the radical and on the quotient, by Lemma 37.12. Twisting the dual by $-1$ there makes Frobenius minus one invertible modulo $z$, and therefore over every $A_m$. □

**Theorem 37.20** (First-order Selmer length at all auxiliary levels)**.** *There is a constant $C$ such that for every $1 \leq m \leq N$*

$$\left| h(H^1_{\mathcal{L}}(F, W_m)) - n(d+q)m \right| \leq C.$$

*For a fixed interval and coefficient prime the same field, the same $\mathcal{Q}_N$, and one constant serve all the curves selected for $x, r, \alpha$ on the interval.*

*Proof.* To identify the finite Cartier dual with $W_m^\vee(1)$, compose the $A_m$-valued pairing with extraction of the coefficient of $z^{m-1}$ followed by $\mathrm{Tr}_{k'/\mathbf{F}_\ell}$. This pairing is perfect since a nonzero coefficient can be shifted to degree $m-1$ and paired by the nondegenerate finite-field trace. The

Poitou–Tate dimension formula in the proof of Proposition 37.5 applies with lengths, by taking the logarithm to base $|k'|$ of its order formula. The global term $h^0(F, W_m) - h^0(F, W_m^\vee(1))$ is bounded by Lemma 37.18 and the vanishing at $p_0$. The dual Selmer term is bounded by Proposition 37.19. [1]

At $u \mid \ell$ the cohomology sequence for $B_{u,m} \subset W_m$ gives exactly

$$h(\mathcal{L}_u) - h^0(F_u, W_m) = h^1(F_u, B_{u,m}) - h^0(F_u, B_{u,m}) - h^0(F_u, (W/B_u)_m).$$

The local Euler characteristic and local Tate duality identify the right side with $[F_u : \mathbb{Q}_\ell]u_0 m + O(1)$ by Lemma 37.18. At an old finite place with the unramified condition the contribution is zero. In fact the unramified classes are the cokernel of Frobenius minus one on the inertia invariants, whose length equals the length of the kernel. At a place away from $\ell$ with the full condition the contribution is $h^2 = h^0(F_v, W_m^\vee(1))$, by the local Euler characteristic and duality. This is zero above $p_0$ and sums to $nqm + O(1)$ at the new places. At each real place the contribution is $-w_- m$. Since $\sum_{u|\ell}[F_u : \mathbb{Q}_\ell] = d$ and $u_0 - w_- = n$, the claimed estimate follows. All constants arose from fixed matrices, fixed global groups, and finitely many selected elements. Take their maximum over the finite list of curves. □

**Corollary 37.21** (The first numerical inequality for patching)**.** *The first-order polarized cocycles of the ordinary problem at these levels, modulo its polarized coboundaries, have order at most* $|k'|^{n(d+q)m+C}$ *for* $1 \le m \le N$.

*Proof.* A polarized cocycle on $G_L$ extends to $G_F$ with value zero at $c$, by the defining relation with $-\dagger$. A coboundary between two such representatives has a witness fixed by $c$, exactly the required skew-adjoint witness. Inflation from the prescribed tame quotient is injective on classes. At an old finite prime its allowed inertia has order prime to $\ell$, so the restriction in $H^1$ is zero by averaging. The ordinary witness at $u$ puts the local class in the image from $B_{u,m}$. Hence this quotient of polarized cocycles injects into the group of Theorem 37.20. This proves the bound in the normalization of `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/ThetaCoefficientMapEq.lean`, line 54. □

The length parameter in this theorem is the precision along the ordinary curve in characteristic $\ell$. The weighted sum in Proposition 6.11 is taken in mixed characteristic at the original weight, as $\ell$ and the period fields vary. Passing between these estimates still requires control of the specialized integral trace relations.

## 37.6 The obstruction to lifting ordinary cocycles

Keep the fields, curves, local flags and auxiliary sets $\mathcal{Q}_N$ of Section 37.5. The second numerical input for patching concerns the off-diagonal module $M = M_A$ of that section, with its conjugation action and with $c$ acting as negative adjunction. Write $E = E_A$. Let $b$ be the rank of the integral ordinary Selmer group for $M$ on the original quotient $\Gamma_a$, and use a basis after inverting $z$, represented by the $\nu_1, \dots, \nu_b$. In each ordinary basis put $B_u(M) = M \cap \{\text{upper triangular matrices}\}$. Use the corresponding submodule of $E$ for $B_u(E)$. For these modules define the ordinary Selmer group at level $N$ by the same local conditions as for $W_m$ in Section 37.5, using their own upper triangular submodules.

In this situation the diagonal requirement on local cocycles in the source has a simple interpretation. After subtracting a local coboundary, an upper triangular cocycle for an upper triangular representation has additive characters on its diagonal. Indeed conjugating an upper triangular matrix by another invertible upper triangular matrix fixes its diagonal, over any commutative coefficient ring. Each diagonal character therefore kills the kernel of topological

abelianization. By local reciprocity the abelianization of $G_{F_u}$ is the profinite completion of $F_u^\times$. Its $\ell$-primary torsion is trivial since $F_u$ has no nontrivial $\ell$-power roots of unity, as in Theorem 37.8. Its valuation factor $\widehat{\mathbb{Z}}$ is torsion free. Thus the diagonal kills the whole preimage of the $\ell$-primary torsion used in `Development/AutomorphicArithmetic/AutomorphicVarieties/WeilGroupCharacters/UniversalOrdinaryDeterminantRing.lean`, line 75. This applies also to the commutative coefficient algebra used below.

**Lemma 37.22** (The off-diagonal Selmer length)**.** *There is a constant $C_0$ such that, for $1 \le e \le N$, the ordinary Selmer group with coefficients in $M_e = M/z^eM$ at level $N$ has length at most $be + C_0$.*

*Proof.* At the level without the new primes, the classes are the ordinary classes on $\Gamma_a$. To verify this, at an old finite place different from $\ell$ where unramified cohomology is imposed, restriction to inertia is a coboundary. Every representative thus vanishes on the subgroup of inertia killed in $\Gamma_a$, since that subgroup acts trivially. Conversely the remaining inertia in $\Gamma_a$ has order prime to $\ell$ and has zero first cohomology by averaging. At $p_0$ the full group equals the unramified group. Their difference in length is $h^0(F_v, M_e^\vee(1)) = 0$, by local Euler characteristic and the Frobenius calculation in Proposition 37.19 on the off-diagonal blocks. These facts give the asserted identification by inflation. Normalizing a representative to be zero at $c$ identifies this cohomology with polarized cocycles modulo polarized coboundaries, since two is invertible.

We give the length calculation at this fixed level. Use the finite presentation of cocycles by values on generators in Proposition 37.16, applied to the quotient $\Gamma_a$. Its open subgroups have the stated finiteness property by Theorem 37.8. Adjoin a variable $Y_u \in M/B_u(M)$ at each $u \mid \ell$. For each local element $g$, impose that the projection of the global cocycle at $g$ to this quotient equal $(g-1)Y_u$. All these equations and the global cocycle relations are rows on one finite free $A$-module $A^t$. Finitely many generate the row module and can be arranged in a matrix $D$. Before imposing the relations the assignments on the free profinite group of generators are arbitrary. Thus the same matrix describes the solutions over $A_e$, including solutions that have no integral lift. Coboundaries with their projected local witnesses are the image of a fixed matrix $D'$ with $DD' = 0$.

For each $u$, the module $(M/B_u(M)) \otimes Q$ has zero invariants. All its character factors come from distinct lines of the ordinary flag by the argument of Lemma 37.18. The integral witnesses are therefore unique once the global cocycle is fixed. It follows from the definition of $b$ that $t - \operatorname{rank}_Q D - \operatorname{rank}_Q D' = b$. Over $A_e$, forgetting the witnesses maps $\ker D_e / \operatorname{im} D'_e$ onto the ordinary Selmer group. Smith normal form gives length $be + O(1)$ for this quotient of matrices, since each nonzero invariant factor of either matrix adds at most its fixed valuation to $be$.

At level $N$ the kernel of restriction to the quotients by unramified cohomology at the new primes is precisely the group just counted. Indeed $M$ is unramified at those primes, so vanishing of the inertia restriction makes the cocycle descend to the original group. The length of each local quotient is $h^0(F_v, M_e^\vee(1))$, by the same local Euler formula. Frobenius there acts modulo $z^e$ as at the fixed element $n'_j$, with trivial twist. Its blocks have mutually disjoint spectra over a splitting field of $Q$. There are no intertwiners between different blocks, in either direction. Frobenius minus one on the off-diagonal dual thus has full rank over $Q$, so its kernel length modulo $z^e$ is bounded by a constant from its Smith factors. Summing over the fixed number $q$ of new places proves the lemma. □

To retain the witnesses in the lifting problem, use the following cochain complex. For coefficients $P$ at precision $m$, take continuous global cochains on $G_{F,S\cup\mathcal{Q}_N}$. Map them by

localization to the sum of the local complexes displayed below.

$$C^\bullet(F_u, P/B_u(P)) \qquad (u \mid \ell),$$
$$C^\bullet(F_v, P)/C^\bullet(G_{F_v}/I_v, P^{I_v}) \quad (v \text{ old, with unramified condition}).$$

The denominator maps in by inflation. At full places the target is zero. Take the mapping fiber of this map and denote its cohomology by $\mathcal{H}^i_N(P)$. In degree one a cocycle consists of a global cocycle and local zero-cochains whose coboundaries are its images in the target. Thus its image in global cohomology is the ordinary Selmer group.

**Proposition 37.23** (Reduction of the obstruction group)**.** *Put $X_{N,m} = \mathcal{H}^2_N(M_m)$. There is $C_1 \geq 0$ such that for $1 \leq e \leq m \leq N$*

$$X_{N,m}/z^e X_{N,m} \;\simeq\; X_{N,e}, \qquad h(X_{N,e}) \leq be + C_1.$$

*Proof.* All the cohomology groups here are finite. For the global and local groups this is arithmetic cohomological finiteness, and then it follows for the fiber by its exact sequence. The global complex has no cohomology above degree two since $\ell$ is odd and all places above $\ell$ and all infinite places are in $S$. The last map to the sum of the local second cohomology groups in Poitou–Tate is surjective because its cokernel is dual to $H^0(F, M_m^\vee(1))$, which vanishes already at $p_0$. Here the map in question is localization from global second cohomology. At $u \mid \ell$ the map to the second cohomology of $M_m/B_u(M_m)$ is surjective by local cohomological dimension two. At an old unramified condition the denominator complex has cohomological dimension one, so the quotient has the same second cohomology as the full local complex. The localization onto the second cohomology of our target is thus surjective. The fiber has zero cohomology in degrees three and higher. We use here the local and global duality and Euler formulas of Milne, Theorems I.2.8, I.4.10 and I.5.1. [1]

The Euler characteristic of this fiber in lengths is zero. In detail, put $s = \sum_{i<j} n_i n_j$. The rank of $M$ is $2s$, its upper triangular part has rank $s$, and its invariants at every real place have rank $s$, since opposite blocks are interchanged. The global Euler characteristic is $-2smd + smd = -smd$. The sum of the local Euler characteristics of $M_m/B_u(M_m)$ at $u \mid \ell$ is also $-smd$. At an old unramified condition the full local complex has Euler characteristic zero, and so does its denominator, whose two cohomology groups are the kernel and cokernel of Frobenius minus one on the same finite group. Subtracting the target characteristic gives zero as asserted.

It follows that $h(X_{N,e}) \leq h(\mathcal{H}^1_N(M_e))$. The kernel of the map from this first cohomology to the ordinary Selmer group is a quotient of the degree-zero cohomology of the target. At the old unramified conditions this vanishes since inflation is an isomorphism on invariants and injective on $H^1$. Above $\ell$ the target invariants are bounded independently of $e$ by the separation of the lower triangular character factors and the Smith argument of Lemma 37.18. Lemma 37.22 now gives the asserted length bound.

For $e < m$, multiplication by $z^e$ followed in the short exact sequence by reduction gives the maps

$$0 \longrightarrow M_{m-e} \xrightarrow{z^e} M_m \longrightarrow M_e \longrightarrow 0.$$

They give a degreewise exact sequence of the three fiber complexes. At $u \mid \ell$ all flag quotients lift freely. At an old unramified condition, inertia acts through a fixed finite group of order prime to $\ell$, so taking invariants is exact by averaging. Surjections of finite coefficient sets permit continuous lifts of cochains by composing with a section of sets. This verifies exactness also on the local quotient complexes. The vanishing in degree three makes $X_{N,m} \to X_{N,e}$ surjective. It likewise makes $X_{N,m} \to X_{N,m-e}$ surjective by interchanging the two exponents. The image of

$X_{N,m-e} \to X_{N,m}$ is therefore the image of multiplication by $z^e$, by composing with this second surjection. Exactness gives the claimed isomorphism. For $e = m$ it is immediate. □

**Lemma 37.24** (A relation with a unit coefficient)**.** *If a finite $A_m$-module $X$ satisfies $h(X/z^eX) \le be + C$ at every $1 \le e \le m$, then for any $x_0, \dots, x_b \in X$ there are $c_i \in A$, at least one a unit, such that $z^C \sum_{i=0}^{b} c_i x_i = 0$.*

*Proof.* Order the exponents of the nonzero cyclic summands of $X$ decreasingly. If there are at least $b+1$, let $e$ be the $(b+1)$-st exponent. Then $(b+1)e \le h(X/z^eX) \le be+C$, so $e \le C$. Thus $z^CX$ has at most $b$ generators. The submodule $Y$ generated by $z^Cx_0, \dots, z^Cx_b$ also has at most $b$, by lifting to a free module of rank at most $b$ over $A$. Their images in $Y/zY$ are dependent. Lift a nonzero relation over $k'$. Its value belongs to $zY$, so subtract a combination of the same generators with all coefficients divisible by $z$ to make its value zero. Some coefficient remains a unit. □

For the mixed extension introduce formal elements $e_1, \dots, e_b, e'_1, \dots, e'_b, \theta$ over $A$. Their only nonzero pairwise products are $e_je'_i = e'_ie_j = \delta_{ij}\theta$. They and 1 are a basis of a commutative algebra, since all triple products of its added elements vanish. In matrices over this algebra set

$$U = \bigoplus_{j=1}^{b} Ee'_j \ \oplus \ M\theta.$$

Give $U$ the conjugation action by $R(g) = (1 + \sum_j \nu_j(g)e_j)r(g)$ on $G_L$, and negative adjunction at $c$, extended coefficientwise. These preserve $U$. Indeed the extra first-order action on an $Ee'_j$ entry gives an off-diagonal commutator times $\theta$. There is a short exact sequence of $G_F$-modules

$$0 \longrightarrow M\theta \longrightarrow U \xrightarrow{\pi} E^{\,b} \longrightarrow 0.$$

Locally conjugate by the same $1 + \sum Y_{u,j}e_j$ used to triangularize $R$ in Section 37.5. In the resulting local coordinates use the upper triangular part for $B_u(U)$ and transport it back. The sequence on upper parts and the sequence on their quotients are exact because in triangular coordinates projection reads the $e'_j$-entries and the kernel reads the $\theta$-entries. The change of coordinates induces the identity on the coefficient submodule and quotient in their respective ordinary bases.

**Theorem 37.25** (Uniform lifting of a combination of ordinary cocycles)**.** *There exists $C \ge 0$ with the following property for all $1 \le m \le N$. For $0 \le i \le b$, let $(y_{ij})_{1 \le j \le b}$ be a tuple of polarized ordinary $E_m$-valued cocycles on the CM group at level $N$, with the tame condition of the original problem at old primes. There are $c_i \in A$, at least one a unit, and a polarized ordinary $U_m$-valued cocycle $Y$ at the same level such that for every group element $g$ and every $j$,*

$$(\pi Y)_j(g) = z^C \sum_{i=0}^{b} c_i y_{ij}(g).$$

*The tame condition is retained. The same $C$ may be used for all the selected curves on the fixed interval at the given coefficient prime.*

*Proof.* Extend each tuple to $G_F$ with value zero at $c$. Write it as $y_i$. Its cohomology class lifts to $\mathcal{H}^1_N(E_m^b)$ because of its local conditions, as is seen from the description of cocycles of the fiber. Choose one such lift for each $i$. The displayed sequence of coefficients induces a degreewise exact sequence of the three fiber complexes at precision $m$. This follows at the flags from the preceding coordinate calculation. At the old unramified conditions inertia on

all three modules acts through the same prime-to-$\ell$ quotient, as $r$ and every $\nu_j$ were taken on $\Gamma_a^0$. Hence inertia invariants are exact there. These facts and the cochain lifting argument of Proposition 37.23 give the required exact sequence.

Apply its connecting map to the chosen lifts and call the images $x_i \in X_{N,m}$, identifying $M\theta$ with $M$. Proposition 37.23 and Lemma 37.24, with $C = C_1$, kill the obstruction of $z^C \sum_i c_i y_i$ with some $c_i$ a unit. Exactness gives a lift in $\mathcal{H}^1_N(U_m)$. Choose a representative for its global cocycle. Its projection differs from the stated linear combination by a global coboundary. Lift its witness from $E_m^b$ to $U_m$ and subtract its coboundary. The projection now agrees as a function on the group. The remaining value $u$ at $c$ belongs to $M_m\theta$ and satisfies $cu = -u$. Subtract the coboundary with witness $-u/2$ in that kernel. This makes the value at $c$ zero and preserves the projection. Restriction to $G_L$ consequently has the required polarization.

Local coboundary changes preserve the ordinary and unramified classes. At an old finite prime a cocycle with unramified class vanishes on the prescribed inertia kernel because the action there is trivial. If $p_0$ is included in that condition, the full local $H^1$ on $U_m$ equals the unramified group there as well. On its two coefficient subquotients the twisted dual has no Frobenius invariants by Lemma 37.12, and the same follows on the extension. These vanishings give descent through the closed normal subgroup used in the tame condition. The diagonal assertion at the ordinary places follows from the local abelianization calculation above. We have obtained the stated ordinary cocycle on the required group. Finally there are only finitely many selected curves on the interval, and increasing $C$ preserves the assertion by multiplying the lift by the additional power of $z$. □

The theorem proves the condition called $B2$ in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/ThetaCoefficientMapEq.lean`, line 58 on these fields and prime sets. Its coefficients are allowed to depend on the tuple, while its exponent is uniform. Together with Corollary 37.21 it supplies the two cohomological estimates at these auxiliary levels. To obtain the bound for the integral group in Theorem 36.14 one still needs control of the action on the automorphic modules and of the passage to the prescribed characteristic-zero weights.

## 37.7 Integral comparison of the hyperspecial and Iwahori levels

Use the totally real field $F$, its CM extension $L$, and the sets $\mathcal{Q}_N$ of Section 37.5. At a selected place $v$ the definite unitary group in three variables is $\mathrm{GL}_3(F_v)$. Write $p_v$ for the size of the residue field and $\mathfrak{o}_v$ for the ring of integers. Take $K_v = \mathrm{GL}_3(\mathfrak{o}_v)$, its standard Iwahori subgroup $I_v$ (upper triangular modulo the maximal ideal), and $I_{1,v}$ the kernel of the diagonal map from $I_v$ to the maximal $\ell$-group quotient of $(\mathbf{F}_{p_v}^\times)^3$. Write $\Delta_N$ for the product of these diagonal quotients. In this section a hyperspecial, Iwahori or $I_1$ level uses the corresponding subgroup at every selected place. All other levels are the same in the three spaces.

**Proposition 37.26** (Ordinary forms at the diagonal cover)**.** *Fix a sufficiently small tame level away from the selected places. The dual ordinary module at $I_1$ level is finite free over $\Lambda[\Delta_N]$, where $\Lambda$ is the completed group ring of the principal diagonal units at $\ell$. Its quotient by the diamond augmentation ideal is the dual ordinary module at Iwahori level. These assertions hold also after applying a common idempotent commuting with the diamond action and the level maps.*

*Proof.* The definite group for $L/F$ and its ordinary spaces are as in Proposition 36.23, now over $F$. The extension is unramified at all finite places and split above $\ell$. Fix a deep principal subgroup at one odd split tame place outside every $\mathcal{Q}_N$, for example at $p_0$. A sufficiently deep

such subgroup is torsion-free by the matrix logarithm. Stabilizers of the finite adelic double cosets are finite by definiteness. At this place they lie in conjugates of the torsion-free subgroup, and a rational element with identity component at one place is the identity. Thus at every fixed level at $\ell$ the finite set of double cosets at $I_1$ level is a disjoint union of free $\Delta_N$-orbits, one over each coset at Iwahori level. An algebraic coefficient lattice is pulled back along this cover, since the new places act trivially on the coefficients. On each orbit, choosing one point identifies the forms with a copy of the regular representation tensored with that lattice. Invariants are functions on the quotient. On a regular representation the norm sends the class of a single basis vector in coinvariants to the sum of the orbit in invariants. It is an isomorphism over the integers. Applying the ordinary idempotent at $\ell$ commutes with each of these operations. This is also the finite-level calculation of Newton and Thorne, Lemma 4.3. [19]

Let $O$ be the coefficient DVR and $k_O$ its residue field. By integral ordinary control for a sufficiently small level, the dual ordinary module $M_1$ is finite free over $\Lambda$. Its reduction modulo the maximal ideal $\mathfrak{m}_\Lambda$ is the dual of the ordinary forms over $k_O$ at the first principal diagonal level at $\ell$. Indeed the principal diagonal invariants in the discrete ordinary tower are the ordinary space at that level, and taking its uniformizer torsion gives this reduction on the dual. These are Proposition 2.5.3 and its preceding control lemma in Geraghty. The identifications commute with the operators away from $\ell$. [20]

The module $M_1/\mathfrak{m}_\Lambda M_1$ is therefore a summand of a free $k_O[\Delta_N]$-module by the orbit calculation, also on taking the dual with the inverse group action. This group ring is local because its augmentation ideal is nilpotent. Indeed that ideal is generated by the elements $\delta - 1$ for generators of a finite abelian $\ell$-group in characteristic $\ell$. The summand is free. Lift a basis to $M_1$. Its $\Delta_N$-translates give a basis modulo $\mathfrak{m}_\Lambda$, and hence a $\Lambda$-basis of the free module $M_1$ by Nakayama's lemma and equality of ranks. This proves freeness over $\Lambda[\Delta_N]$. Finally, $\Delta_N$-invariants in the discrete tower give the Iwahori tower by the orbit description and the compatibility of finite-group invariants with filtered direct limits. Pontryagin duality identifies their dual with the coinvariants of $M_1$. The augmentation submodule is closed since it is a finite sum of images of a compact module. A commuting idempotent gives the corresponding summands in these identifications, and the same basis argument proves freeness for the summand. □

For one of these places set $p = p_v$ and write $H$ for the $O$-algebra of Iwahori Hecke operators, with the Iwahori of volume one. Use the Bernstein variables, and put

$$S = O[X_1^{\pm 1}, X_2^{\pm 1}, X_3^{\pm 1}], \qquad Z = S^{\mathfrak{S}_3}, \qquad \delta = \prod_{i<j} (X_i - X_j).$$

For the Bernstein normalization one may take the square root of $p$ congruent to one in $\mathbb{Z}_\ell$, which exists by Hensel's lemma. The Bernstein presentation gives $H = \bigoplus_{\sigma \in \mathfrak{S}_3} S T_\sigma$, with center $Z$. Use the convention in which, for an adjacent transposition $s_i$ and $f \in S$,

$$T_i f = s_i(f) T_i + (p-1) \frac{f - s_i(f)}{1 - X_{i+1}/X_i}, \qquad (T_i - p)(T_i + 1) = 0.$$

One obtains this convention by choosing the order of the positive roots. The divided difference here lies in $S$. These are the Bernstein presentation of Haines, Kottwitz and Prasad and its integral form used by Newton and Thorne in Section 3. [25, 19]

Let $J = \sum_\sigma T_\sigma$. The number of complete flags in $\mathbf{F}_p^3$ is $P = (1+p)(1+p+p^2)$. It is a unit in $O$, since $p \equiv 1 \bmod \ell$ and $\ell > 3$. Put $e = J/P$. It is the averaging idempotent of $K_v$ on Iwahori invariants. The identities $T_i J = pJ$ and $J^2 = PJ$ follow either from the finite flag double cosets or from the quadratic relation by pairing the Weyl elements $\sigma, s_i\sigma$. In particular $He = Se$, freely of rank one over $S$, since the expansion of $fe$ has coefficient $f/P$ at $T_{\mathrm{id}}$.

**Lemma 37.27** (The spherical averaging functional)**.** *For $f \in S$ write*

$$L(f) = \frac{\sum_{\sigma} \operatorname{sgn}(\sigma)\sigma(f)}{\delta}, \qquad D = \prod_{i<j}(pX_i - X_j), \qquad D' = \prod_{i<j}(pX_j - X_i).$$

*Then $L$ takes values in $Z$, the pairing $(f,g) \mapsto L(fg)$ is perfect over $Z$, and $Jfe = L(Df)e$.*

*Proof.* Setting $X_i = X_j$ pairs off all the terms of an alternating numerator with opposite signs. Each $X_i - X_j$ therefore divides the numerator. These are distinct prime factors, so their product $\delta$ divides it as well. For Laurent polynomials first clear denominators by a symmetric power of $X_1X_2X_3$. The quotient is symmetric. The six monomials $u_{ij} = X_1^i X_2^j$, $0 \le i < 3$, $0 \le j < 2$, form a basis over $Z$. Indeed one first adjoins a root of the universal monic cubic, then one of the roots of its residual monic quadratic. The third root is determined by the sum, and all are units since the constant term is a unit. These relations span $S$ in the six monomials. Their independence follows over the fraction field of $Z$, since the rational function field on the three variables has six distinct permutations fixing that field and already has degree at most six.

The alternating numerator of the basis monomial $X_1^2X_2$ is $\delta$, by the three-by-three Vandermonde determinant. The other five have degree below three and alternating numerator zero. Hence $L$ reads off the coefficient of the top monomial. For a monic extension of degree $t$, the pairing given by the coefficient of $X^{t-1}$ in a product is perfect because its matrix in the two power bases, with one reversed, is triangular with diagonal ones. Composing this pairing in the two successive monic extensions gives the coefficient of $X_1^2X_2$. Products of the two successive dual bases give a dual basis for the composite. Thus $L$ is perfect integrally.

The action on $He = Se$, in the fraction field of $S$, has $T_i = \alpha_i + \beta_i s_i$, where

$$\alpha_i = \frac{(p-1)X_i}{X_i - X_{i+1}}, \qquad \beta_i = \frac{X_i - pX_{i+1}}{X_i - X_{i+1}}.$$

Since $T_iJ = pJ$ and $\beta_i \neq 0$, every value of $J$ in this representation is symmetric. Write $J = \sum c_\sigma \sigma$ there as an operator. Distinct field automorphisms are linearly independent, so $c_\sigma = \sigma(c_{\mathrm{id}})$. The coefficient of the longest permutation comes only from $T_1T_2T_1$, specifically from choosing the reflection term three times. It is $\beta_1 s_1(\beta_2) s_1 s_2(\beta_1) = \prod_{i<j}(X_i - pX_j)/\delta$. Applying the longest permutation gives $c_{\mathrm{id}} = D/\delta$. The action of $J$ on $f$ is therefore $L(Df)$, as asserted. This calculation also proves faithfulness of the action of $H$ on $Se$. For each $T_\sigma$, its reduced expression has a nonzero coefficient at $\sigma$, and terms of shorter length from choosing fewer reflections. A nonzero linear combination in the Bernstein basis thus gives a nonzero combination of field automorphisms. Their independence can be tested on $S$, since any rational function has a symmetric common denominator after multiplying its denominator over its orbit. □

**Theorem 37.28** (Integral maps between the two levels)**.** *For every left $H$-module $V$ there are natural $S$-linear maps*

$$S \otimes_Z eV \xrightarrow{\Phi} V \xrightarrow{\Psi} S \otimes_Z eV$$

*such that both composites are multiplication by the central polynomial $C_p = \prod_{i \neq j}(X_i - pX_j)$. These identities persist under all base changes of $Z$.*

*Proof.* Use the six monomials $u_a$ in the lemma and their $L$-dual basis $v_a$. Define

$$\Phi(f \otimes t) = ft, \qquad \Psi(y) = \sum_a u_a \otimes J(v_a D' y).$$

The factor $J$ puts each entry in $eV$. Both maps are defined over $O$. The tensor $\sum_a u_a \otimes v_a$ has the same product with $f \otimes 1$ and $1 \otimes f$ for every $f \in S$. Indeed under the perfect pairing the

tensor identifies with the identity map of $S$, and these two products identify with multiplication by $f$ on its range and domain. Applying this tensor identity to the formula for $\Psi$ gives its $S$-linearity, since $Z$ commutes with $J$.

There are six ordered pairs of distinct indices, so $DD' = C_p$. For $t \in eV$, the lemma in the algebra $He$ gives $Jv_aD'ft = C_pL(v_af)t$. Expanding $f$ in the dual basis proves $\Psi\Phi = C_p$. For the reverse order consider the element $\sum_a u_aJv_aD'$ of $H$. On $fe \in He$ it acts as $\sum_a u_aL(Dv_aD'f)e = C_pfe$. Faithfulness in Lemma 37.27 gives equality with $C_p$ inside $H$, hence on every $V$. These are identities of maps, so tensoring with any $Z$-algebra preserves both. □

**Corollary 37.29** (The comparison factor on the cohomology curves)**.** *Apply the maps at all $v \in \mathcal{Q}_N$ to the definite ordinary modules after taking diamond coinvariants. Their composites are the product of the $C_{p_v}$. On specializing the local Hecke centers at the characteristic polynomials of any one of the curves of Theorem 37.13, the kernel and cokernel of the resulting comparison are killed by $z^{D_0}$ at every precision $m \le N$, with*

$$D_0 = \sum_{j=1}^{q} v_z\Big(\operatorname{disc}\det(T - r(n_j'))\Big)\,.$$

*The $n_j'$ are the fixed elements of Section 37.5; the exponent is independent of the auxiliary level. A common exponent is obtained on the interval by taking the maximum over the finite list of curves.*

*Proof.* At finite level the operators at different split places commute. For the Iwahori module, averaging at a selected place gives precisely the module with that place hyperspecial. The commutation with the ordinary projector allows successive application of Theorem 37.28. Tensoring the resulting maps, or composing them one place at a time, multiplies their composite factors. They commute with the transition maps at $\ell$. Tensoring with the free module $S$ over $Z$ is a direct sum of six copies in the chosen basis and commutes with inverse limits. Thus the same calculation holds on the ordinary tower. On duals one uses the contragredient Hecke action, or equivalently applies the integral Hecke algebra identities to the dual module. Proposition 37.26 identifies its Iwahori term with the diamond coinvariants at $I_1$ level.

The map on a local center sends the elementary symmetric polynomials to the signed coefficients of the indicated monic Frobenius polynomial. Its constant term is a unit. In characteristic $\ell$, $p_v = 1$, so $C_{p_v}$ evaluates to the discriminant up to sign. Reciprocal Frobenius or a common unit scaling of its three roots changes that valuation by zero. The Chebotarev choice in Theorem 37.17 prescribes each polynomial to precision strictly exceeding the discriminant valuation at $n_j'$, in addition to all the required cohomology tests. The resulting product is a unit times $z^{D_0}$ over $A$, with the same identity after reduction to $A_m$. The two composite identities give the kernel and cokernel assertion, including after specialization with torsion. □

For the scalar in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/ThetaCoefficientMapEq.lean`, line 213, one can multiply $\Psi$ by $\prod_{i\neq j}(X_i - X_j)$ at each place. This gives precisely both factors in that scalar. The corollary evaluates the local centers on the curves. To bound the mixed-characteristic length in Section 6.6 one still needs the action of the global deformation ring on the ordinary module with the prescribed ramification and the integral comparison at the original weights.

## 37.8 An automorphic prime through the curves on the deep layer

The comparison of levels in Section 37.7 applies to the values of the local centers on any of our curves. We now prove that the specialized spherical module has positive rank there. Fix the

deep layer $F = F_a^+$, $L = L_a$ of Theorem 37.8. Write $F_0 = F_0^+$, $d_0 = [F_0 : \mathbb{Q}]$, and use $L_0 = KF_0$. For a pair $r, \alpha$ on the interval let $B'$ denote one of the integral Hecke components through the adjoint on $L_0$ in Proposition 37.2. Its product-one weight algebra is $\Lambda_0$, of dimension $1 + 2d_0$. We use its normalized trace $t$ of the representations twisted by $\kappa_\ell$. Removing that twist gives the conventional multiplier $\kappa_\ell^{-2}$ in the ordinary determinant problem. The twist is fixed on the entire family.

**Lemma 37.30** (A common tame level for classical base change)**.** *There is a Zariski dense set of characteristic-zero points of $B'$ whose automorphic representations base change cuspidally to $L$. Their base changes descend to ordinary forms on the definite unitary group over $F$ at a single tame level. Their ordered local weights are obtained by restriction of the lower-layer Galois characters. On inertia away from $\ell$ the normalized Galois representations factor through the prescribed quotient and have the finite inertial type $\alpha \oplus 1 \oplus \alpha^{-1}$. The tame level can be chosen for all the pairs on the interval.*

*Proof.* Use the purely algebraic weights in the proof of Proposition 37.2. Require the successive labeled Hodge exponents to differ by at least two at every embedding, in their ordinary order. Increasing the first weight variation while holding the second fixed as in that proof realizes the larger gaps in every congruence class. These weights are still dense. At the adjoint on $L_0$ the two-dimensional newform has projective closure $\mathrm{PGL}_2$ on every open subgroup, by the proof of Theorem 36.17. The trace-zero adjoint is absolutely irreducible on $G_L$. Choose nine group elements $h_i \in G_L$ whose adjoint matrices span the matrix algebra there. The trace Gram determinant $\det(t(h_i h_j))$ is nonzero at the adjoint. Take its product with the finitely many nonzero Frobenius resultants in the proof of Proposition 37.9. Above the algebraic weights the points avoiding the zero locus of this product remain dense in $B'$. Indeed the principal ideal of any nonzero element of the finite domain $B'$ contains a nonzero element of $\Lambda_0$, by clearing denominators after inverting $\Lambda_0 \setminus \{0\}$. The same argument applies after multiplication by any further nonzero element. Density downstairs and lying over give the assertion.

At each of these points the nine restricted matrices span the full matrix algebra. The classical transfer to $\mathrm{GL}_3/L_0$ is cuspidal, since a transfer with several isobaric summands gives a decomposable semisimple Galois representation. This also excludes a repeated block of character twists from a single summand of smaller rank. We use the attachment and decomposition of ordinary classical representations in Geraghty's Proposition 2.7.2. The extension $L/L_0$ is cyclic of $\ell$-power degree. In cyclic base change a cuspidal representation can lose cuspidality at a step of prime degree $\ell$ only in the presence of a self-twist of that order. The central character of a rank-three representation then makes the cube of the twisting character trivial, impossible since $\ell > 3$. Thus the base change is cuspidal. Its normalized representation at good places has the trace $t|_{G_L}$ specialized at our point, by unramified base change and Chebotarev. Conjugate self-duality and regular algebraicity persist. [20, 2]

We verify ordinarity after base change at the ramified $\ell$-adic places. The lower-layer ordinary flag exists at the chosen algebraic weights by Geraghty's Corollary 2.7.8. Each graded character is crystalline, since on local units it is the purely algebraic character of that weight. Via local reciprocity it differs from the corresponding product of labeled crystalline characters only by an unramified character. The potentially semistable Weil–Deligne functor is exact on these representations. Its finite inertia on the full parameter is trivial because it is semisimple on inertia and trivial on all graded lines. If a local field has ramification index $e$ and residue degree $f$ over $\mathbb{Q}_\ell$, the valuation of the $f$-fold crystalline Frobenius on a graded line is, up to the common sign convention, the sum of its labeled Hodge exponents divided by $e$. This is the equality of Newton and Hodge numbers for a crystalline character, with $v_\ell(\ell) = 1$. Consequently successive Frobenius valuations differ by at least $2f$. A nonzero monodromy

operator requires two valuations differing by exactly $f$ by the Frobenius relation. It follows that monodromy is zero. Restriction to any upper completion preserves crystallinity. It repeats the labels above each embedding and multiplies the Frobenius valuations by the relative residue degree.

Local–global compatibility at $\ell$ for a regular conjugate self-dual cusp form in odd rank identifies its Frobenius-semisimplified parameter there. It is spherical in the present case. The normalized Iwahori operator for the first $j$ ordered roots has valuation equal to the sum of their valuations minus the corresponding Hodge sum, with the same normalization as in Proposition 36.23. Both terms are multiplied by the relative residue degree under base change, so this eigenvalue is a unit for every $j$. The spherical generic component admits this refinement. In particular its weights on local units are obtained by composing the lower characters with the local norms. [16, 20]

Away from $\ell$, Proposition 37.9 proved that all points in the chosen dense set have the stated finite inertia on the lower layer, and hence after restriction as well. At a prime of the conductor the extension $L/F$ is split by Proposition 36.15. The inertial characters extend to the local Weil group. After Frobenius semisimplification each isotypic part is a sum of characters, and monodromy is zero. The local Langlands component is thus a quotient of a principal series with the three indicated restrictions to units. Section 36.7 shows that a principal congruence level on which all three characters are trivial supplies vectors in this quotient: the section on the identity chart and its translates generate the inducing representation. This bounds the level independently of the unramified character values. At the other primes the base change is spherical. Descent to the definite inner form, unramified at the nonsplit finite places, is the descent used in Proposition 36.23. That result applies since $L/F$ is unramified at finite places, split above $\ell$, and the rank is three. Impose the same sufficiently small level at $p_0$ as in Proposition 37.26. Take the maximum of the conductor levels over the finite list of pairs. Smaller subgroups keep the designated vectors, proving the last assertion. [20, 15] □

At this tame level let $\Lambda_a$ be the upper-layer weight algebra on the chosen prime-to-$\ell$ diamond summand, with three principal-unit coordinates at each place over $\ell$. Let $\mathbf{T}_a$ be generated over $\Lambda_a$ by the spherical Hecke operators at all good split places on the dual ordinary module $\mathcal{M}_a$. Use the Frobenius normalization of Proposition 36.23. Equivalently one may take the coefficients of each good Frobenius polynomial, since the known powers of the norm in the normalization are units. Inverting the Satake determinant is allowed at these places. This algebra is finite over $\Lambda_a$, being a submodule of the endomorphisms of a finite free $\Lambda_a$-module.

**Proposition 37.31** (The restricted integral family on the upper Hecke algebra)**.** *There is a continuous map $\mathbf{T}_a \to B'$ sending good Frobenius coefficients to those of $t|_{G_L}$ and weight variations to their pullbacks by the local norms. Its image $D_a$ is a complete local domain, finite over a power series algebra of dimension $1 + 2d_0$. Its dimension is $1 + 2d_0$, and $B'$ is finite over $D_a$. The ring $D_a$ contains all restricted traces and the values of the ordered characters on the full local decomposition groups at $\ell$.*

*Proof.* The norm on each group of principal units gives a continuous map on the completed weight algebras, using variations relative to the adjoint on both layers. It is the required weight map by Lemma 37.30. Assign the good split Hecke generators their indicated values in $B'$. Any polynomial relation over $\Lambda_a$ among these operators vanishes on every upper ordinary form in that lemma. Its proposed value therefore vanishes on the dense set in $B'$ and is zero. This proves well-definedness. The map is continuous since it is linear over the continuous weight map on the finite $\Lambda_a$-module $\mathbf{T}_a$. The compact source has closed image.

For each lower place $u_0 \mid \ell$, choose one upper place over it. The norms of its principal units contain the $[F_u : F_{0,u_0}]$-th powers of all lower principal units, by viewing a lower unit inside the upper field. These powers are open in the lower principal units. After taking products and imposing the product-one weights downstairs, the image of the upper weight group contains an open subgroup of the free rank-$2d_0$ group defining $\Lambda_0$. Its completed group ring $\Lambda_{\mathrm{op}}$ is a power series ring over $O$ contained in $D_a$. The index is finite, so coset representatives give a finite free basis of $\Lambda_0$ over $\Lambda_{\mathrm{op}}$. Thus $B'$ is a finite $\Lambda_{\mathrm{op}}$-module, and so is its submodule $D_a$. Also $B'$ is finite over $D_a$. The finite domain $D_a$ over the complete local ring $\Lambda_{\mathrm{op}}$ is complete and local, by the idempotent decomposition of a finite algebra over a Henselian local ring. Dimension is preserved under integral extensions.

Good split Frobenii are dense up to conjugacy in $G_L$. The closed image thus contains $t(h)$ for every $h \in G_L$. At an upper place $u \mid \ell$ take a local inertia lift $s$ with distinct residual characters as in Theorem 37.8. Its three ordered diagonal values $a_1, a_2, a_3$ lie in $D_a^\times$ by the weight map, including the fixed base characters. Their differences are units. For any $g \in G_{F_u}$ form the $D_a$-linear combination of $t(s^j g)$, $0 \le j \le 2$, whose coefficients are those of

$$P_i(X) = \prod_{k \neq i} \frac{X - a_k}{a_i - a_k}.$$

At every dense classical point this reads the $i$-th diagonal entry of $g$ on the ordinary flag. In fact $s$ is diagonalizable with those three eigenvalues and the first $j$ ordered eigenlines span the $j$-th step. Thus the combination defines on the whole local group a continuous function with values in $D_a$. The character identity holds on that dense set, hence in $D_a$. Its value at the identity is one, so its values are units. This proves the assertion also on the unramified part of the local group. □

Let $R_a$ be the universal ordinary polarized determinant ring for $\Gamma_a^0$ with the ordered local characters of Section 37.2. A degree-three determinant here records the characteristic polynomial on the group algebra after scalar extension. Since 3! is a unit, it is determined by its trace. Use the fixed cyclotomic twist above when writing this ring in the source's normalization. Let $J_{\mathrm{cl}} \subset R_a$ be the intersection of the kernels of its characteristic-zero evaluations coming from ordinary classical forms of the tame level just fixed that satisfy its ramification conditions, with their ordered characters. This definition includes the points constructed in Lemma 37.30.

**Theorem 37.32** (A stable automorphic prime below each curve prime)**.** *There is a surjection $R_a \to D_a$. Its kernel $\mathfrak{q}$ contains $J_{\mathrm{cl}}$, and its generic determinant is that of an absolutely irreducible representation over an algebraic closure of its residue field. For every curve constructed from $B'$ in Theorem 37.4, let $\mathfrak{p}$ be its prime in $R_a$ after restriction to $L$. Then*

$$J_{\mathrm{cl}} \subseteq \mathfrak{q} \subsetneq \mathfrak{p}.$$

*This includes the case where the generic representation on the curve itself has two constituents.*

*Proof.* The restricted family gives a continuous degree-three pseudocharacter with values in $D_a$ because its trace identities hold on the dense classical evaluations. The trace–determinant correspondence with 3! invertible gives the degree-three determinant. For example, on any group algebra element its third coefficient is computed from the traces of its first three powers by Newton's identities, and the identities of polynomial laws can be tested coefficientwise on generic finite sums of group elements. The polarization and the reduction at the closed point are those of the adjoint. On the prescribed inertia kernel one has $t(hg) = t(h)$ for every $h$ and every such $g$, by Proposition 37.9. Consequently the trace factors through the stated tame

quotient. On each ordinary group the ordered character word in the universal condition has zero trace after multiplication by any group matrix. Indeed on each classical flag its factors, starting at the right, successively lower the last, penultimate and first step to zero. The traces vanish identically by density. The characters in Proposition 37.31 have the required residual restrictions, as seen on the adjoint, and are trivial on the local commutator closure. The local abelianization has no $\ell$-primary torsion, as verified in Section 37.6. These properties give a pair in the ordinary determinant problem over $D_a$, hence a map $R_a \to D_a$. [22]

The image contains every good trace coefficient and every weight coordinate by the defining evaluations. It is closed, since $R_a$ is a complete Noetherian local ring with finite residue field and the map is continuous. It therefore contains the whole image of $\mathbf{T}_a$ by its generators. This proves surjectivity. The elements of $J_{\mathrm{cl}}$ vanish at all points of Lemma 37.30, so their images in the domain $D_a \subset B'$ are zero. The nine-element trace Gram determinant of that lemma is nonzero in $D_a$. On the semisimple representation of its generic trace over an algebraic closure the nine matrices are consequently linearly independent. Their span is the full matrix algebra, so the representation is absolutely irreducible.

Every curve in the statement is obtained by composing $B'$ with the quotient at its defining prime and a finite extension of the resulting coefficient ring. On $G_L$ this composition gives its restricted trace. The local characters also agree, by the projector formula of Proposition 37.31. Marking a lattice in Theorem 37.13 preserves these values. Thus the evaluation $R_a \to A = k'[[z]]$ factors through $D_a$. Its kernel strictly contains $\mathfrak{q}$ because $\ell$ is nonzero in $D_a$ and zero in $A$. □

**Corollary 37.33** (Nonzero ordinary modules at all selected levels)**.** *For each of these curves the spherical specialization $\mathcal{M}_{a,A} = \mathcal{M}_a \otimes_{\mathbf{T}_a} A$ has positive rank $s$ over $A$. At level $\mathcal{Q}_N$ specialize the operators away from the selected places and their local Bernstein centers by the same system. Over $k'((z))$ the Iwahori specialization has dimension $6^q s$, where $q = |\mathcal{Q}_N|$. Its integral comparison with the spherical specialization after adjoining the ordered roots has kernel and cokernel killed by $z^{D_0}$ as in Corollary 37.29. The same conclusions hold on taking diamond coinvariants at $I_1$ level.*

*Proof.* As a module over $\mathbf{T}_a$, $\mathcal{M}_a$ is finite and faithful by construction. Its support is all of $\operatorname{Spec} \mathbf{T}_a$. Localize at the prime defined by $\mathbf{T}_a \to D_a \to k'((z))$ and apply Nakayama's lemma. Its fiber is nonzero, also after extending the residue field to $k'((z))$. This proves $s > 0$.

Use the common commutative algebra generated formally by the weight and good operators away from $\mathcal{Q}_N$ and by the $q$ local centers there. Its spherical action maps onto $\mathbf{T}_a$, since these centers give all the omitted spherical operators by the Satake isomorphism. Its Iwahori action commutes with the comparison maps. Specialization over this common algebra thus gives exactly $\mathcal{M}_{a,A}$ on the spherical side, with the indicated root algebras adjoined. Each such algebra is free of rank six over its center by Lemma 37.27, including after base change. The composite factor is nonzero over the fraction field by Corollary 37.29. Hence the two maps there are isomorphisms and give the stated dimension. The integral loss follows from the same composite identity. Proposition 37.26 gives the assertion at the diagonal cover. □

The prime $\mathfrak{q}$ supplies the stable specialization under the given prime required in the second-type case of `Development/AutomorphicArithmetic/AutomorphicVarieties/WeilGroupCharacters/UniversalOrdinaryDeterminantRing.lean`, line 288 for the ideal $J_{\mathrm{cl}}$. To use that implication one must still propagate support to the other primes in its conclusion. The fixed-weight integral lengths and the comparison of the projected theta periods in Section 6.6 involve those further arguments.

## 37.9 The full determinant component through the cuspidal adjoint

Work on $F = F_a^+$, $L = L_a$ with the common tame level of Lemma 37.30. Put $d = [F : \mathbb{Q}]$. We allow all three ordinary weights at each place above $\ell$. Thus the principal weight algebra $\Lambda_a$ is a power series ring over $O$ in $3d$ variables. Fix the prime-to-$\ell$ diamond characters of the adjoint. Take $O$ large enough for its eigenvalues and its chosen basis, and write $J = \operatorname{Frac} O$. Let $x_0$ denote the characteristic-zero evaluation of the ordinary determinant ring $R_a$ at this adjoint, and $y_0$ its evaluation of the good Hecke algebra $\mathbf{T}_a$. We denote the kernels of these evaluations by the same letters. Translate the ordinary weight variables so that they vanish there.

**Proposition 37.34** (The ordinary determinant on a full Hecke component)**.** *Let $\mathfrak{l}$ be a minimal prime of $\mathbf{T}_a$ contained in $y_0$, and put $B = \mathbf{T}_a/\mathfrak{l}$. There is a surjection*

$$R_a \longrightarrow B$$

*matching all good Frobenius polynomials and the ordinary weights. Its kernel contains $J_{\mathrm{cl}}$ of Section 37.8. The domain $B$ is finite over $\Lambda_a$ and contains it. Its generic determinant is absolutely irreducible.*

*Proof.* The algebra $\mathbf{T}_a$ acts faithfully on the free $\Lambda_a$-module of ordinary forms and is a subalgebra of its endomorphisms. Every nonzero element of $\Lambda_a$ is therefore a nonzero divisor in $\mathbf{T}_a$. A minimal prime in a Noetherian ring consists of zero divisors, so $\mathfrak{l} \cap \Lambda_a = 0$. The finite domain $B$ is local, and $y_0$ on $B$ has the same residual system as every classical point on it after reduction to an algebraic closure of the residue field. Indeed a finite domain over the complete local weight ring is local, and evaluation at the integral adjoint identifies its residue field with that of $O$.

Purely algebraic dominant weights, with consecutive Hodge gaps at least two, are dense in the full weight algebra just as in Proposition 37.2. This time the third exponents are independent. In any residue classes for the exponents one can increase successive gaps by arbitrarily large multiples of the modulus while retaining the prescribed tame diamonds. On a sufficiently small logarithmic neighborhood the matrix of embeddings on a basis of each local field has nonzero determinant, so the identity theorem in each variable gives the asserted density. At these weights the good Hecke operators are semisimple on the classical characteristic-zero spaces. Their evaluations detect $\mathbf{T}_a$ because its elements are matrices over $\Lambda_a$. Compactness and Chebotarev, as in Proposition 36.23, now put the continuous traces of all $h \in G_L$ in $\mathbf{T}_a$. Take their images in $B$. Above the indicated weights the points of $B$ avoiding any specified nonzero element are dense, by finiteness over $\Lambda_a$ and the denominator argument of Lemma 37.30. Each such point occurs on the classical module, since the module is finite and faithful over $\mathbf{T}_a$. The trace Gram determinant of the nine group elements of that lemma is nonzero at $y_0$, so it is nonzero in $B$.

We check the ramification condition on this full component. Away from the fixed finite set of old tame places and $\ell$ the classical systems are unramified. At an old finite place pass to a completion of $LE_*$, which kills $\bar{\alpha}$ on inertia. In a stable lattice the residual image of this inertia is unipotent upper triangular, since the residual semisimplification is the split adjoint triple. Its integral image is pro-$\ell$ and hence factors through the $\mathbb{Z}_\ell(1)$ part of tame inertia. If the residue field here has order $q_0$, Frobenius permutes the three eigenvalues of a generator by raising them to $q_0$. Their orders divide $q_0^6 - 1$. For every fixed original inertia element a power bounded by the local extension degree lies in this subgroup. Its characteristic coefficients thus take finitely many possible values at the classical points. The monic polynomial with these possible values and their coefficient conjugates as roots vanishes on the coefficient function in

$B$. Since $B$ is a domain, that function equals its value at $y_0$, because the product of the other distinct linear factors is nonzero there. In particular all these inertial polynomials coincide with those of the adjoint.

At the adjoint the three eigenvalues of a Frobenius over the chosen completion have equal complex absolute values, by the good and bad prime calculations of Proposition 36.25, which persist under local extension. The resultant testing whether their ratios include $q_0$ is nonzero there. Avoid the product of the finitely many such resultants and the nine-element Gram determinant. At the resulting dense classical points the logarithm of unipotent inertia on $LE_*$ must vanish, since Frobenius would act on it with eigenvalue $q_0$. Inertia on $L$ is now finite with exactly the adjoint inertial characters. It acts trivially on the designated kernel at every old finite place. These points have absolutely irreducible trace and satisfy the tame conditions of $J_{\mathrm{cl}}$.

The identities $t(hg) = t(h)$ for $g$ in the prescribed inertia kernel hold on that dense set and therefore in $B$. For each $u \mid \ell$ the three weights in $\Lambda_a$ give three unit-separated values on the separating inertia element. The projector formula of Proposition 37.31 gives the three full local characters in $B$. The ordinary word and polarization identities follow from the same classical evaluations, as in Theorem 37.32. This constructs the ordinary determinant and the map from $R_a$. Its image contains the weights and the good Hecke generators, which generate $B$. The dense set proves the assertion on $J_{\mathrm{cl}}$. The nonzero Gram determinant gives absolute irreducibility of the generic determinant. □

**Lemma 37.35** (The characteristic-zero tangent space of the determinant ring)**.** *The tangent space of $(R_a)_{x_0}$ over $J$ has dimension at most $3d$. Its map to the tangent space of the weights is injective.*

*Proof.* Consider a lift of $x_0$ to $A' = J[\epsilon]/(\epsilon^2)$. Precompose with the inverse of the fixed cyclotomic twist on determinants and local characters so that the base representation is $U = \mathrm{Ad}^0(W)|_{G_L}$. The trace and the local characters remain continuous in the $\ell$-adic vector space topology of $A'$. To see this for an arbitrary algebraic tangent map, present the complete ring $R_a$ as a quotient of $O[[X_1, \dots, X_h]]$. The values $a_i = x_0(X_i)$ lie in the maximal ideal of $O$. Translation by the $a_i$ is an automorphism of the complete power series ring. Modulo the square of the ideal of the point, the image of any series is its value plus its first derivatives at $a$ times the increments. These values and derivatives depend continuously on the series for the original adic topology, since the $a_i$ are topologically nilpotent. This proves the required continuity.

The adjoint on $G_L$ is split and absolutely irreducible over $J$. Over the Henselian local ring $A'$, Chenevier's Theorem 2.22(i) therefore identifies the lifted determinant with the determinant of a representation lifting $U$, unique up to strict conjugacy. The representation $r'$ is continuous. The matrices of the nine $h_i$ are an $A'$-basis of $M_3(A')$ by their trace Gram determinant, and the coordinates of $r'(g)$ are obtained by inverting that fixed Gram matrix against the continuous traces at $h_i g$. [22]

For $g$ in the designated tame inertia kernel all the traces of $r'(h)(r'(g)-1)$ are zero. Nondegeneracy of the matrix trace pairing makes $r'(g) = 1$. At an ordinary place take $s$ with distinct residual eigenvalues and write $a'_1, a'_2, a'_3$ for the lifted character values there. The ordinary trace word multiplied by every $r'(h)$ has trace zero, so the word itself vanishes. Taking all three letters equal to $s$ gives the annihilating polynomial $\prod_i (X - a'_i)$. Its roots are unit-separated, so $r'(s)$ has three eigenlines over $A'$ by its polynomial projectors and the basepoint ranks. Now replace only the $k$-th letter by an arbitrary local $g$. The product to its left is invertible on each eigenline with index at least $k$, and the product to its right is invertible on each eigenline with index at most $k$. Consequently the entries of $r'(g) - \psi_k(g)$ in rows $i \geq k$ and columns $j \leq k$

are zero in this basis. Taking all $k$ proves upper triangularity with exactly the given diagonal characters. Here a scalar subtracted from a matrix denotes that scalar times the identity.

It remains to pass from the polarized determinant on $G_L$ to the signed representation over $F$ used in Theorem 36.21. Let $\iota(B) = S^{-1}B^{-\mathsf{T}}S$, with $S$ the symmetric adjoint form there. The polarization and uniqueness for the lifted determinant give a matrix $C$ such that

$$r'(cgc^{-1}) = C\iota(r'(g))C^{-1} \qquad (g \in G_L).$$

By Schur's lemma its reduction differs from $U(c)$ by a scalar, which we remove. Applying the formula twice shows that $C\iota(C)$ is a scalar, since the image matrices span the full matrix algebra. Write that scalar as $a$. Then $\iota(C)C = a$ as well, while applying $\iota$ gives $\iota(C)C = a^{-1}$. Hence $a^2 = 1$, and reduction gives $a = 1$ because two is invertible. Assigning $(C, -1)$ to $c$ defines the required lift on $G_F$. At finite places away from $\ell$ the adjoint inertia kernel is exactly the one checked above, since $L/F$ is unramified and the adjoint has the finite type $\alpha \oplus 1 \oplus \alpha^{-1}$.

Two determinant tangent maps with the same weights now give admissible lifts in that theorem with the same weights. They are strictly conjugate, so their determinants agree. Their full ordered characters agree by the uniqueness of the eigenline flag, or directly by the trace projector formula. The universal ring is topologically generated by these functions and the image of $O$. To verify this, work in each quotient by a power of the maximal ideal and take the finite $O$-subalgebra generated there. The determinant descends to it by the trace formulas and $3! \in O^\times$. Universality on these finite local rings and uniqueness after inclusion show that the universal map to each quotient takes values in its generated subalgebra. Passing to the limit gives the assertion. The two continuous tangent maps thus agree. The space of weight tangents has dimension $3d$ by Theorem 36.21, proving the lemma. □

**Theorem 37.36** (The unique component through the adjoint)**.** *There is a unique minimal prime $\mathfrak{q}_*$ of $R_a$ contained in $x_0$ and a unique minimal prime $\mathfrak{l}_*$ of $\mathbf{T}_a$ contained in $y_0$. The weight, good Hecke and determinant evaluations identify*

$$R_a/\mathfrak{q}_* \;\simeq\; \mathbf{T}_a/\mathfrak{l}_*.$$

*This integral domain is finite over the full weight algebra and has dimension $1 + 3d$. The local ring of $R_a$ at $x_0$ is regular of dimension $3d$, and passage to this Hecke component induces an isomorphism on local rings at the adjoint. For every curve from Theorem 37.32 one has*

$$J_{\mathrm{cl}} \subseteq \mathfrak{q}_* \subseteq \mathfrak{q} \subsetneq \mathfrak{p}.$$

*Proof.* Choose any $\mathfrak{l}$ as in Proposition 37.34 and let $\mathfrak{q}'$ be the kernel obtained there. The contraction of the adjoint point in $B$ to $\Lambda_a$ is the characteristic-zero prime of height $3d$ generated by the translated weight variables. Indeed the quotient by them is $O$, and localizing there inverts its uniformizer. Incomparability gives the upper bound $3d$ for the height in $B$, and going down for the integral extension of the integrally closed domain $\Lambda_a$ gives the lower bound. The localization of $R_a$ at $x_0$ thus has a local quotient of dimension $3d$. It is Noetherian with residue field $J$. Its embedding dimension is its tangent dimension, bounded above by $3d$ in Lemma 37.35. Dimension is at most embedding dimension, so they coincide and the local ring is regular. In particular it is a domain.

The localized kernel $\mathfrak{q}'(R_a)_{x_0}$ must vanish. If it were a nonzero prime, appending the strict inclusion of the zero prime below it to a chain in the quotient would give a chain longer than $3d$. By prime correspondence for localization, $\mathfrak{q}'$ is the unique minimal prime below $x_0$. Call it $\mathfrak{q}_*$. This also proves the local isomorphism and the dimension assertions. Applying the same argument to any $\mathfrak{l} \subset y_0$ minimal in $\mathbf{T}_a$ identifies its quotient with the fixed $R_a/\mathfrak{q}_*$. The resulting

maps from $\mathbf{T}_a$ into that ring agree on weight coordinates and on good Hecke generators, because these come from the same universal local characters and good Frobenius polynomials in $R_a$. They therefore have equal kernels, giving uniqueness of $\mathfrak{l}_*$.

Finally, the map $\mathbf{T}_a \to D_a$ of Proposition 37.31 has prime kernel below $y_0$ since the restricted family includes the adjoint. It contains a minimal prime, necessarily $\mathfrak{l}_*$. Composing the displayed isomorphism with the quotient to $D_a$ gives precisely the map of Theorem 37.32, by its good traces and the formula for its ordered characters. Thus $\mathfrak{q}_* \subseteq \mathfrak{q}$, and $J_{\mathrm{cl}} \subseteq \mathfrak{q}_*$ follows from Proposition 37.34. □

**Corollary 37.37** (Classical points of the component at the prescribed weights)**.** *The fixed-weight algebra of this component for the original ordered weights of the adjoint is finite over $O$. Every geometric characteristic-zero point of it gives a classical ordinary Hecke system at the common tame level on $F$. The adjoint itself is a reduced isolated point in its characteristic-zero weight fiber.*

*Proof.* Specialization of the finite $\Lambda_a$-algebra in Theorem 37.36 gives finiteness over $O$. Every one of its points pulls back to the support of the finite faithful $\mathbf{T}_a$-module of ordinary forms. Its fiber there is nonzero by Nakayama's lemma. Ordinary control at this dominant algebraic weight identifies the characteristic-zero weight space with classical ordinary forms. The nonzero fiber gives the asserted system, also for $r = 1$, where the coefficient weight is zero. [20]

At the adjoint the weight tangent map of Lemma 37.35 is an isomorphism since its source and target both have dimension $3d$. Therefore the weight increments span the cotangent space of the local ring there. By Nakayama's lemma they generate its maximal ideal. Quotienting that local ring by the weight increments consequently gives its residue field $J$. Finiteness of the characteristic-zero fiber makes its points isolated. □

The corollary concerns the component containing the cuspidal adjoint. The cotangent quotient in Theorem 36.11 is taken at the split residual system on the original field. Bounding contributions from other components and integral relations at their intersections there remains necessary to control $C_{\mathrm{ker}}$ in Theorem 36.14.

# 38 Recovering classes from three-character traces

## 38.1 Three-character traces and the complementary strict group

Keep the interval, $K, \ell, r, \alpha$, and $L = L_a$ of Section 36.7. Write $G = G_K$, $H = G_L$ and $\delta_1 = \Psi$, $\delta_2 = 1$, $\delta_3 = \Psi^{-1}$. For $U = G, H$ let $T_U$ be the closed $O$-algebra of traces on $U$ in the original fixed-weight marked matrix ring. Write $t(u)$ for its trace, $t_0(u) = \sum_i \delta_i(u)$ for its split evaluation and $\mathfrak{a}_U = \ker(T_U \to O)$. These are complete Noetherian local rings. Indeed each is a quotient of the continuous determinant deformation ring of the corresponding number field with ramification confined to finitely many places. The determinant takes values in the trace algebra by Newton's identities since $3!$ is a unit. Its universal coefficient functions generate the trace algebra by compactness. Noetherianity follows from Chenevier's Proposition 3.7, since the groups with this ramification restriction satisfy Mazur's finiteness condition. For an open subgroup the cyclic extensions of degree $\ell$ with this restricted ramification are finite in number by class field theory. This also follows by adjoining $\mu_\ell$ and using the finiteness of the class group and of the group of $S$-units modulo $\ell$-th powers. [22]

For $m \geq 1$ let $I_m(U)$ be the ideal of $T_U$ generated by

$$\det\bigl(t(u_i v_j)\bigr)_{1 \leq i,j \leq m}, \qquad u_i, v_j \in U.$$

Rows and columns may be chosen independently. We use linear extension of $t$ to $O[U]$. Allowing finite linear combinations in each row or column generates the same ideal by multilinearity. Choose $g$ in inertia at $\ell$ in $H$ with $\lambda = \Psi(g)$ such that $\lambda, 1, \lambda^{-1}$ have distinct residues. This exists by Proposition 36.15, since $\alpha$ is unramified there and $2r < \ell - 1$. At the universal marked point the characteristic polynomial on that inertia element is $\prod_i (X - \delta_i(g))$, by the fixed ordinary condition.

**Lemma 38.1** (The ideal defining sums of three characters)**.** *The ideal $I_4(U)$ is the smallest ideal modulo which $t$ is a sum of three characters into the units of the quotient, labeled by their values on $g$. Furthermore $I_{3+j}(U) \subseteq I_4(U)^j$ for $j \geq 1$.*

*Proof.* Put $s_i = \delta_i(g)$. Form $e_i = \prod_{j \neq i} (g - s_j)/(s_i - s_j)$ in $O[U]$. Their matrix images are orthogonal idempotents adding to one, by Cayley–Hamilton and polynomial interpolation. Each image has rank one. Over the local marked ring its direct summand is free, and its rank is read off at the split residual point. Thus $t(e_i e_j)$ is one for $i = j$ and zero otherwise. Set $d_i(a) = t(e_i a)$ for $a \in O[U]$. The determinant with rows $e_1, e_2, e_3, a$ and columns $e_1, e_2, e_3, b$ is

$$F(a, b) = t(ab) - \sum_i d_i(a) d_i(b).$$

In any quotient by $I_4(U)$ this vanishes for all $a, b$. Apply the identity to $a = e_i h$, $b = k$. Cyclicity of trace and $e_i e_j = 0$ for $i \neq j$ give $d_i(hk) = d_i(h) d_i(k)$ there. Also $d_i(1) = 1$, so the group inverse gives an inverse for $d_i(h)$. Since the idempotents sum to one, $t = \sum_i d_i$, with $d_i(g) = s_i$. If in any quotient $t = \sum_i \psi_i$ for characters, the matrix of pairings factors through three columns with entries $\psi_i(u)$. Hence every four-by-four minor vanishes in that quotient. If the characters are labeled by their distinct values $s_i$ on $g$, interpolation also gives $\psi_i(h) = d_i(h)$.

For the last assertion write each matrix $(t(u_i v_j))$ as the sum of the matrix $(\sum_k d_k(u_i) d_k(v_j))$ and the matrix $(F(u_i, v_j))$ over $T_U$. The first factors through a free module of rank three and each entry of the second belongs to $I_4(U)$. In the expansion of an $m$-row determinant by multilinearity in its columns, a term using more than three columns from the first matrix is zero by the alternating property. Every surviving term for $m = 3 + j$ uses at least $j$ entries from $I_4(U)$. □

**Proposition 38.2** (The augmentation ideal on the original quadratic field)**.** *In the original fixed-weight trace ring,*

$$I_4(G) = \mathfrak{a}_G.$$

*In particular the quotient defining sums of three characters on $K$ is $O$, including its integral scheme structure.*

*Proof.* All the minors vanish under the split evaluation. In $A = T_G / I_4(G)$ the preceding lemma gives characters $d_i$ lifting $\bar{\delta}_i$. The residual labeling on every element follows by applying the projector formula to the split residual matrices. Each character is continuous by that same trace formula. We show that $d_i / \delta_i$ is everywhere unramified.

At a place over $\ell$ choose a separating inertia element $g_u$, as above. The residual cyclotomic character is surjective on inertia there, since $K$ is unramified at $\ell$. The polynomial of $g_u$ in the marked ring has the three prescribed roots. It also equals $\prod_i (X - d_i(g_u))$ in $A[X]$, by the traces of its first three powers and Newton's identities. For fixed $i$ the two factors $d_i(g_u) - \delta_j(g_u)$ with $j \neq i$ are units. Substitution therefore gives $d_i(g_u) = \delta_i(g_u)$. For any inertia element $h$ the ordinary upper triangular matrices give $t(g_u^n h) = \sum_i \delta_i(g_u)^n \delta_i(h)$ for $n = 0, 1, 2$. The character decomposition gives the same equations with $d_i(h)$ on the right. The Vandermonde

determinant is a unit, proving equality on all of inertia. This argument applies at each place of $K$ above $\ell$.

At any other finite place inertia acts in the marked problem through a quotient of the finite image of $\alpha$, of order prime to $\ell$. On the prescribed inertia kernel the matrix is one, so every $d_i$ is one there, by its projector formula. The continuous character $d_i/\delta_i$ has values in $1+\mathfrak{m}_A$, a pro-$\ell$ group because its images modulo the powers of $\mathfrak{m}_A$ have successive quotients in additive groups of residual characteristic $\ell$. Thus its restriction to the prime-to-$\ell$ inertia quotient is trivial. The resulting everywhere unramified character factors through the class group of the imaginary quadratic field $K$. Its order is prime to $\ell$ on the intervals under consideration. It follows that $d_i = \delta_i$ on $G$. Every trace in $A$ is now its split scalar value. Since these traces topologically generate and $O$ is compact, the split evaluation identifies $A$ with $O$. □

**Theorem 38.3** (Four-by-four minors and the strict group)**.** *Let $D = \mathrm{Frac}(O)/O$ and let $C$ be the dual of $S(\Psi^{-1})$ into $D$, as in Theorem 36.11. Under its map $\Phi : T_G \to O \oplus C$, the ideal $I_4(H)$ already maps onto $C$. Every minor of size five or larger maps to zero. A finite set of four-by-four minors suffices. The rows and columns can be chosen at Frobenius elements of good rational primes split completely in $L$, with a choice of lifts.*

*Proof.* All four-by-four minors have zero split evaluation. The case $C = 0$ follows at once. Otherwise use the mixed representation before its final change of frame in Theorem 36.11. It has coefficient ring $(O \oplus C)[y]/(y^2)$, with $C^2 = 0$, and its traces identify the square-zero summand with $yC$. Set $N = H \cap \ker \Psi$ on the absolute Galois group. The quotient of $G_{\mathbb{Q}}$ by $N$ is abelian, since both $L/\mathbb{Q}$ and the character $\Psi$ are abelian. On $N$ the lower coordinate $z : N \to C$ and upper coordinate $b : N \to O$ are additive. Their needed values persist on this exact kernel. Indeed after dividing $C$ by the span of $z(N)$, the normalized cocycle factors through the abelian quotient. Commuting with $g_*$, where $z(g_*) = 0$ and $\Psi(g_*)^{-1} - 1$ is a unit, makes it zero. All the original values span $C$, so $z(N)$ spans $C$. If the reduction of $b$ vanished on $N$, it too would factor through the abelian quotient. There the central $g_*$ acts by $\bar{\eta}(g_*)\bar{\Psi}(g_*) \neq 1$, so every cocycle is a coboundary by the same commutation identity. This contradicts Lemma 36.9. Choose $k \in N$ with $b(k)$ a unit.

For $h \in N$ put $u = h(chc^{-1})$ and $v = k(ckc^{-1})^{-1}$, where $c$ is complex conjugation. On $N$ the action of $c$ fixes the lower coordinate and negates the upper one, by the cocycle law and the two characters. The split matrices on $N$ are the identity. Consequently $M(u) - 1$ has first-degree term $2z(h)X$ and $M(v) - 1$ has first-degree term $2yb(k)Y$, in the degree filtration by the ideal $C + (y)$. Its cube is zero. The difference of the two products of these matrix increments, in the two possible orders, is therefore $4yz(h)b(k)[X, Y]$.

Set $a = u - 1,\ b_1 = (v-1)g,\ a' = v - 1,\ b_1' = (u-1)g$ in $O[H]$. The split matrix of each of these four elements is zero. Thus each of the numbers $\Phi(d_i(a))$, $\Phi(d_i(b_1))$, and their primed counterparts lies in $yC$. The products of any two vanish. The four-by-four determinants $F$ from the lemma consequently give

$$\Phi\big(F(a, b_1) - F(a', b_1')\big) = 4yz(h)b(k)(2 - \lambda - \lambda^{-1}).$$

Indeed $[X, Y] = \mathrm{diag}(-1, 2, -1)$, and multiplying a degree-two term by $M(g)$ uses only its split diagonal. The scalar multiplying $yz(h)$ is a unit, since $2 - \lambda - \lambda^{-1} = -(\lambda - 1)^2/\lambda$. As $h$ varies these differences span $yC$. They are linear combinations of actual four-by-four minors on $H$. The containment of the larger minors in $I_4(H)^2$ follows from Lemma 38.1 and expansion by minors, so they vanish after $\Phi$.

Finiteness of $C$ permits a finite generating list. The image of a four-by-four minor is a continuous function of its eight group arguments into the finite discrete group $C$. Compactness

gives one open normal subgroup of $H$ such that the function depends only on the eight residues. For each entry Chebotarev over $L$ gives good degree-one prime ideals with Frobenius in its conjugacy class in the finite quotient. Primes of higher residue degree have density zero. Choosing the lift at each prime realizes the specified element of the class. Since $L/\mathbb{Q}$ is Galois, the underlying unramified rational primes split completely there. They can be taken pairwise distinct and outside any prescribed finite set. The traces in a minor are traces of products of these chosen group elements. □

**Corollary 38.4** (A quotient of the integral reducibility ideal)**.** *For every auxiliary field $L_a$ of the construction there is an $O$-module surjection*

$$I_4(H)/(\mathfrak{a}_H I_4(H) + I_5(H)) \longrightarrow C.$$

*On the original $K$ the corresponding quotient is $\mathfrak{a}_G/\mathfrak{a}_G^2$. The logarithmic order of its indicated image is $[k : \mathbf{F}_\ell] \log \ell \operatorname{length}_O S(\Psi^{-1})$.*

*Proof.* The augmentation ideal maps into the square-zero summand and $I_5(H) \subset I_4(H)^2$. The theorem gives the surjection after division by the displayed submodule. Apply the proposition on $K$, and use equality of the lengths of a finite module over $O$ and its dual into $\operatorname{Frac}(O)/O$. □

These formulas use the character of the original $E_x$ for each $x$ on the common interval. They identify the integral ideal that detects its complementary length even after restriction to $L_a$. A uniform upper bound for the order of the indicated image is still needed in Section 6.6.

## 38.2 Integral recovery from traces on the auxiliary field

On the original fixed-weight marked problem use $G = G_K$, $H = G_{L_a}$, $T = T_G$, and $B = T_H \subseteq T$ as in Section 38.1. Write $M$ for the matrix representation of $G$ over the universal marked coefficient ring. The three rank-one idempotents $e_i$ in that section are polynomials over $O$ in $M(g)$, with $g \in H$. The span of matrices below means finite linear combinations. No division by the index $[G : H]$ will be needed.

**Lemma 38.5** (The spans of the two matrix images)**.** *In the universal matrix algebra one has*

$$\sum_{u \in G} T\, M(u) = \sum_{h \in H} T\, M(h).$$

*For all $m$, $I_m(G) = I_m(H)T$. Both statements hold for any normal subgroup $H$ containing the chosen separating element $g$, with $I_m(H)$ formed in its closed trace ring.*

*Proof.* The right-hand span is an algebra since $H$ is a group. For $u \in G$ write $d_j(u) = \operatorname{tr}(e_j M(u))$. Its split evaluation is $\delta_j(u)$, so $d_j(u)$ is a unit of the local ring $T$. Acting on the free rank-one summand $e_j$, the endomorphism $e_j M(u) e_j$ is multiplication by its trace. Thus $e_j M(u) e_j = d_j(u) e_j$. Normality gives $M(u) e_j M(u)^{-1}$ in the $O$-span of $M(H)$, since it is the same polynomial evaluated at $M(ugu^{-1})$. For every $i, j$ it follows that

$$e_i (M(u) e_j M(u)^{-1}) e_j = d_j(u^{-1})\, e_i M(u) e_j.$$

The matrix on the left belongs to the right-hand span. Division by $d_j(u^{-1})$ puts every block $e_i M(u) e_j$ in that span. Summing the nine blocks gives $M(u)$. This proves equality of spans. Apply these expressions to each row and column of a trace determinant on $G$. Multilinearity of the determinant and linearity of trace express it as a finite $T$-linear combination of the corresponding determinants on $H$. The reverse containment follows from $H \subseteq G$. □

**Theorem 38.6** (Restriction recovers the entire fixed-weight trace ring)**.** *For every $a$ on the chosen interval,*

$$T_{G_{L_a}} = T_{G_K}, \qquad I_4(G_{L_a}) = \mathfrak{a}_G.$$

*The equalities hold in the original integral ring. The $O$-algebra generated topologically by traces at good primes of $K$ lying above rational primes split completely in $L_a$ is already $T$, if both choices over $K$ are allowed. Every characteristic-zero specialization of the marked matrices has the same invariant subspaces on $G_K$ and on $G_{L_a}$, after any extension of its coefficient field.*

*Proof.* Let $\mathfrak{m}_B, \mathfrak{m}_T$ denote the two maximal ideals. The rings have the same finite residue field, since both contain $O$ and the residual matrices have diagonal characters defined over its residue field. The split evaluation maps $T$ onto $O$. If $\varpi$ is a uniformizer, then $\mathfrak{m}_T = \varpi T + \mathfrak{a}_G$. The preceding lemma and Proposition 38.2 give

$$\mathfrak{a}_G = I_4(H)T, \qquad \mathfrak{m}_T = \mathfrak{m}_B T.$$

For the second equality, $I_4(H)$ has zero split value. Consequently $\mathfrak{m}_T^n = \mathfrak{m}_B^n T$ for every $n$. An element of this power modulo $\mathfrak{m}_T^{n+1}$ is a sum of elements of $\mathfrak{m}_B^n$ times scalars in the common residue field. Lifting the scalars to $O$ represents it by an element of $B$. Induction starting in the residue field gives the surjectivity of $B \to T/\mathfrak{m}_T^n$ for every $n$. The ring $B$ is compact and $T$ is Hausdorff with this topology. Hence $B = T$, which also removes the scalar extension in the equality of the two ideals.

For any $h \in H$, use continuity modulo $\mathfrak{m}_T^n$ and Chebotarev over $L_a$ to approximate its trace by that of a degree-one prime there outside any finite exceptional set. Its rational prime splits completely in $L_a$. The Frobenius on restriction to $K$ agrees with the Frobenius over $L_a$, since the relative local degree is one. Passing through all $n$ proves the generation assertion. Finally specialize the span equality of the lemma. Every vector subspace invariant under $M(H)$ is preserved by its scalar-linear span, hence by $M(G)$. This remains true over any extension of the coefficient field. □

**Proposition 38.7** (An integral finite quotient on the original field)**.** *Let $R_a$ be the ordinary polarized determinant ring of Section 37.8, using the fixed cyclotomic normalization there. Work over a common finite coefficient DVR containing the chosen adjoint eigenvalues if needed. Restriction gives a surjection $R_a \to T$ sending each inertial weight to its prescribed fixed value. If $\mathfrak{q}_*$ is the minimal prime through the adjoint from Theorem 37.36, put $\mathfrak{b}_a = \mathfrak{q}_* T$. Then $T/\mathfrak{b}_a$ is a nonzero finite $O$-algebra. Its geometric characteristic-zero points give classical ordinary Hecke systems on the definite group over $F_a^+$.*

*Proof.* All the preceding trace-ring calculations persist after a finite coefficient extension. It is finite free on the coefficient modules, so tensoring preserves the embedding of the trace ring in the marked ring. Traces with coefficients in the extension generate exactly this scalar extension by closure and by a basis of the extension.

Restrict the universal marked matrices to $G_{L_a}$. They kill the prescribed inertia kernel away from $\ell$. Indeed on inertia at those places $\kappa_\ell$ is unramified and $\ker \bar{\alpha} = \ker \alpha$, since $\alpha$ has prime-to-$\ell$ order. The original polarized matrices on $G_{F_a^+}$ kill $I_u \cap \ker \eta \cap \ker \bar{\Psi}$, so restriction factors through $\Gamma_a^0$. The original polarization supplies the polarized determinant on this quotient. Its coefficients are in $T$ by Newton's identities. This also holds on group algebras after extension of scalars by applying the same identities to a generic finite linear combination.

At each chosen place above $\ell$ the original ordinary flag restricts to the upper field. Let $s$ be a separating inertia element in that completion and form its three polynomial projectors just

as for $g$. The flag is ordered by its inertial characters, whose values on $s$ are known scalars in $O$. The trace of the $i$-th projector times $M(h)$ for a decomposition group element $h$ is the $i$-th diagonal character of this flag. This follows by writing the matrices in a basis adapted to the flag, where the projector is upper triangular with just one diagonal entry one. Thus these full characters take values in $T$. The ordinary word required in $R_a$ vanishes on the flag because its factors successively lower its steps. These facts give a map $R_a \to T$. On the unit weight group it has the original prescribed values. Its image is closed by compactness and contains the traces of all $h \in G_{L_a}$. The theorem proves surjectivity.

Let $\mathfrak{w}$ be the ideal of the upper weight algebra generated by the differences between the weight coordinates and their prescribed values. The map induces a surjection

$$(R_a/\mathfrak{q}_*)/\mathfrak{w}(R_a/\mathfrak{q}_*) \longrightarrow T/\mathfrak{b}_a.$$

The left side is finite over $O$ by Corollary 37.37. The quotient on the right has the common residual point, since the original map is local. Every geometric characteristic-zero point of the right side pulls back to one on the left, with the indicated Hecke system by the same corollary. □

**Corollary 38.8** (Original-field cuspidal labels for irreducible points)**.** *Suppose a geometric characteristic-zero point of the original marked matrix problem factors on traces through $T/\mathfrak{b}_a$ and is absolutely irreducible over $K$. Its trace occurs among the rank-three cuspidal labels on $K$ at the level $N_*$ of Theorem 36.4.*

*Proof.* The restricted representation stays absolutely irreducible by Theorem 38.6. Its ordered trace at the upper field is classical by the proposition. The classical transfer of a definite unitary system in rank three decomposes its semisimple Galois representation according to the isobaric summands, including the character twists of a repeated block. This is the attachment of Geraghty's Proposition 2.7.2, with the normalization used in Lemma 37.30. Absolute irreducibility forces a single cuspidal summand of rank three. Its attached representation is the restriction of the original one twisted by $\kappa_\ell^{-1}$, by compatibility at good primes and equality of semisimple representations with the same trace. [20]

Apply cyclic descent through the intermediate fields between $K$ and $L_a$. Over each of them the original representation is absolutely irreducible since its subgroup $G_{L_a}$ already acts irreducibly. This is the restriction hypothesis used in the proof of Theorem 36.18. In detail the Galois-invariant cuspidal representation at each prime-degree step descends. The unique descent with central character one is conjugate self-dual. Its attached Galois representation agrees on the upper subgroup, hence differs downstairs by a character of the cyclic quotient by Schur's lemma. Its cube is trivial by the determinant, and the step has degree prime to three. The character is trivial. The crystalline and tame local conditions at $K$ are those of the original marked point, so the level computation of that theorem places the resulting cuspidal system at $N_*$. [2, 16, 15] □

The equality of trace rings applies before passing to the finite quotient. Finiteness and classicality in the proposition apply after imposing $\mathfrak{b}_a$. To obtain the estimate of Section 6.6 one still needs to control how much of the image $C$ of the trace differences is lost under such a passage.

## 38.3 Crystallinity and classical evaluation of trace specializations

Use $G = G_K$, $H = G_{L_a}$ and the original trace ring $T$ of Section 38.2. Denote the complete local ring of the universal marked matrices by $A$, its representation on $G$ by $M$, and the $T$-span

of its matrix image by $E_M$. Thus $E_M \subseteq M_3(A)$. We show how the local crystalline condition can be read at a global trace point even when no choice of marked matrices over its valuation ring is given.

**Lemma 38.9** (Finite quotients of the algebra of matrices)**.** *The algebra $E_M$ is finite as a $T$-module. Let $u \mid \ell$ be a place of $K$. For every $n \geq 1$, the finite representation $E_M/\mathfrak{m}_T^n E_M$, with $G_{K_u}$ acting by left multiplication, is a subquotient of a lattice in a crystalline representation with Hodge–Tate weights in $[-r, r]$.*

*Proof.* Use the universal degree-three determinant with our residual trace for the group of $K$ unramified outside the original finite set. Its completed Cayley–Hamilton algebra is finite over its determinant ring by Wang-Erickson's Proposition 3.6. The required finiteness of residual adjoint cohomology follows by restriction to the finite extension killing the residual characters and the finiteness condition verified in Section 38.1. The universal determinant ring surjects onto $T$. After this scalar extension, sending group elements to $M(g)$ defines a map of the finite algebra to $M_3(A)$. Indeed continuity gives a map on the completed group algebra, and the matrices satisfy its Cayley–Hamilton relations by the ordinary matrix identity. Its image is a finite $T$-module containing $E_M$. Since $T$ is Noetherian, the submodule $E_M$ is finite. [26]

Put $A_u$ for the power series ring of the selected local ordinary chart, with maximal ideal $\mathfrak{m}_u$. Its integral field points separate its elements, as proved in Theorem 33.11 by evaluating convergent power series on sufficiently small powers of a uniformizer. Every such representation restricted to $G_{K_u}$ is crystalline with the indicated weights. The proof of that theorem covers the split and unramified quadratic completions in the present problem. At a conjugate place the dual with conjugation has the same weight interval.

For each $s$, the complement of $\mathfrak{m}_u^s$ in $A_u$ is compact. Each element in it has nonzero image at some integral field point and therefore modulo some power of the uniformizer at that point. Kernels of those finite evaluations are open and closed. Taking a finite subcover gives evaluations whose common kernel $J_s$ is contained in $\mathfrak{m}_u^s$. The local representation over $A_u/J_s$ embeds by entries in the direct sum of the corresponding truncated crystalline lattices. Passing to $A_u/\mathfrak{m}_u^s$ gives a quotient. Finite subquotients of crystalline lattices in a fixed weight interval are closed under finite sums, submodules and quotients. For example, the preimage of a subgroup of a finite quotient of a lattice is again a full lattice. This gives the asserted property for the local chart modulo each power.

The local map $A_u \to A$ is continuous. Thus for each $b$ the marked module $(A/\mathfrak{m}_A^b)^3$ on $G_{K_u}$ is obtained by extension of scalars from such a finite local chart, up to the choice of its local basis. As a group module it is a quotient of a finite direct sum of the chart modules, by multiplying the columns by generators of the finite target coefficient module. The property consequently holds for each $(A/\mathfrak{m}_A^b)^3$.

The finite $T$-module $E_M$ is compact in its $\mathfrak{m}_T$-adic topology. Its embedding into the Hausdorff ring $M_3(A)$ is continuous, as is seen by choosing finitely many module generators. A continuous injection from a compact space is a homeomorphism onto its image here. For each $n$ there is therefore a $b$ for which $E_M \cap M_3(\mathfrak{m}_A^b) \subseteq \mathfrak{m}_T^n E_M$. Modulo the intersection, $E_M$ embeds as a group module in three copies of $(A/\mathfrak{m}_A^b)^3$, by its three columns. Taking the further quotient gives the assertion. □

**Proposition 38.10** (The local conditions at a global trace point)**.** *Let $x : T \to O'$ be a continuous local map to the integers of a finite coefficient field. The associated semisimple geometric representation $\rho_x$ is crystalline at every place of $K$ above $\ell$, with labeled cyclotomic exponents $r, 0, -r$. At all other finite places it has the prescribed finite inertia. Its determinant is one.*

*More generally, for a continuous determinant lift to a finite local characteristic-zero coefficient algebra with absolutely irreducible reduction at $x$, the associated matrix representation, regarded over $\mathbb{Q}_\ell$, is crystalline at these same places.*

*Proof.* Enlarge the finite coefficient field to split the semisimple representation associated to the determinant, using Chenevier's theorem over fields. It is a module for $E_M$. In fact a relation $\sum_j a_j M(g_j) = 0$ in $E_M$ gives $\sum_j a_j t(g_j h) = 0$ for every $h \in G$. After applying $x$, the corresponding matrix pairs to zero against the whole image algebra under the trace pairing. Over a splitting field that algebra on a semisimple representation is a product of matrix algebras. The trace on each factor is its ordinary matrix trace times the multiplicity, a nonzero integer in characteristic zero. Its pairing is nondegenerate. Thus every such relation acts by zero.

For the assertion with infinitesimal coefficients use the matrix lift given by Chenevier's Theorem 2.22(i), after extending coefficients to split its reduction. Here the determinant lift is along a continuous map from $T$. At the irreducible reduction choose nine group elements whose matrices span $M_3$. The trace Gram determinant on these elements is a unit also in the finite local coefficient algebra. They form a basis there by Nakayama's lemma, and the same trace argument shows that every relation in $E_M$ acts by zero. The matrix coordinates of group elements are continuous since they are recovered from this basis and its invertible Gram matrix. [22]

In either case the action of $E_M$ is continuous because it is $T$-linear for the specified scalar map on a finite $T$-module. In the underlying finite-dimensional $\mathbb{Q}_\ell$-space take a full $\mathbb{Z}_\ell$-lattice and its span under $E_M$. The latter contains the former and is bounded by compactness of the acting algebra, so it too is a full lattice $D$, stable under $E_M$. For each $m$, the action of $E_M$ on $D/\ell^m D$ factors through $E_M/\mathfrak{m}_T^n E_M$ for some $n$, by continuity. Sending the unit to each of finitely many additive generators expresses $D/\ell^m D$ as a quotient of a finite direct sum of these left regular modules. The lemma puts every one of these truncations in the same bounded crystalline weight interval. Liu's theorem on limits of torsion crystalline representations now implies that $D[1/\ell]$ is crystalline with weights in that interval. One can twist the interval to $[0, 2r]$ in the convention of that theorem. The prime is odd. [27]

At a field point the local trace on the full decomposition group is the sum of the three ordered flag characters, each recovered in $T$ by the local projector formula of Proposition 38.7 on $K$ itself. The characters on inertia are $\kappa_\ell^r, 1, \kappa_\ell^{-r}$. Equality of traces gives these three local composition factors up to an unramified twist of each. Exactness of the de Rham functor with its filtration on crystalline representations gives their three labeled weights. Away from $\ell$ the prescribed inertia kernel acts trivially in $E_M$. The remaining inertia group has order prime to $\ell$ and its character is the prescribed one, as follows either from reduction or from the universal traces on that finite group. Finally the determinant of the marked representation is one on inertia at $\ell$ by the weights, and away from $\ell$ by the finite inertial characters. It is everywhere unramified on $K$ and takes values in principal units. The class number is prime to $\ell$, so it is one. The same identity passes to the trace determinant. □

**Theorem 38.11** (Every non-self-dual point of the finite quotient)**.** *Every reducible geometric characteristic-zero point of $T$ given by a continuous integral specialization has self-dual trace. Every geometric characteristic-zero point of $T/\mathfrak{b}_a$ where an inverse-trace difference is nonzero occurs among the cuspidal labels on the original $K$ at level $N_*$. This includes the first twist $r = 1$.*

*Proof.* Work with the global semisimple representation $\rho_x$ of the proposition. If it has three character factors, all four-by-four trace minors vanish, so its evaluation is the split evaluation by

Proposition 38.2. Otherwise a reducible representation has a unique one-dimensional summand $\vartheta$ and an irreducible two-dimensional summand. The identity $t(chc^{-1}) = t(h^{-1})$ holds in the original trace ring. It forces $\vartheta^c = \vartheta^{-1}$ by uniqueness of the one-dimensional summand. Its residual character must be one of $\bar{\Psi}, 1, \bar{\Psi}^{-1}$, by reduction of a stable lattice and uniqueness of the residual semisimplification. Each of these is fixed by $c$ because it comes from $\mathbb{Q}$. The only one equal to its inverse is 1, by separation on inertia. Thus $\bar{\vartheta} = 1$. At a place over $\ell$ its restriction to inertia is one of the three prescribed characters, and their distinct reductions force it to be trivial. Away from $\ell$ the prime-to-$\ell$ inertia image and its trivial reduction give the same conclusion. The unramified character $\vartheta$ takes values in principal units and the class number of $K$ is prime to $\ell$, so $\vartheta = 1$. The plane has determinant one by the proposition. Its alternating determinant pairing identifies it with its dual. This proves the first assertion.

A point of the finite $O$-algebra $T/\mathfrak{b}_a$ with values in an algebraic closure of $\mathrm{Frac}(O)$ has integral image in a finite extension. The evaluation is continuous and local on valuation rings since its reduction lies over the unique maximal ideal of this finite local algebra. If an inverse-trace difference survives, $\rho_x$ is absolutely irreducible by the first assertion. The representation factors through $E_M$ and Lemma 38.5 shows that the specialized matrix spans on $G$ and $H$ agree, even for a trace point. It therefore remains irreducible on $H$ and on all intermediate subgroups. Proposition 38.7 gives a classical system on $F_a^+$. Its transfer is cuspidal by the decomposition of the attached representation used in Corollary 38.8. The cyclic descent in that corollary only requires irreducibility on the subgroups, the determinant and the polarized trace. For the descended level and weight, the preceding proposition supplies precisely the crystalline type on $K$ and the finite inertial type used in Theorem 36.18. Hence the same original level $N_*$ applies to this trace point. □

**Corollary 38.12** (The remaining length as a kernel of evaluation)**.** *For each triple over the unramified coefficient ring $O$ of Section 36.6, let $P_a$ be the kernel of the map from its original $T$ to the reduced generic fiber of the finite quotient in Proposition 38.7, contracting from the common coefficient extension if one was made. Put $C_a = \Phi(\mathfrak{d} \cap P_a)$, with the original inverse-trace ideal and mixed map. Uniformly on the same intervals and fields,*

$$0 \le \mathcal{L}_{>}(x) - \sum_{\ell > P} (\log \ell) \sum_{1 \le r < t(x)} \sum_{[\alpha]} b_\alpha \, \mathrm{length}_O C_a \le \log J_* \ll_{\psi,\varepsilon} V^{C_1}.$$

*Proof.* By definition $P_a$ is the intersection of the kernels of all characteristic-zero geometric evaluations of this finite quotient, restricted to $T$. At any such point either the whole ideal $\mathfrak{d}$ vanishes or its evaluation comes from the original list $X$, by the theorem. Therefore an element of $\mathfrak{d}$ killed by all of $X$ belongs to $P_a$. This proves $C_{\ker} \subseteq C_a \subseteq C$. Apply Theorem 36.14 and additivity of length. Taking the contraction before applying $\Phi$ keeps the original residue degrees $b_\alpha$. □

The calculation identifies the evaluation ideal on both fields. The sum of the lengths of $C_a$ still requires a uniform upper bound. In the archive, the passage from an integral deformation model to a bound on one character uses the finiteness and point hypotheses in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/ModelUpper.lean`, line 104 and the automorphy input in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/OneCharOf.lean`, line 158.

## 38.4 The inverse trace at reducible characteristic-zero points

Write $E_M, T, G, H$ as in Section 38.3. An inverse-trace difference is $t(h) - t(h^{-1})$ with $h \in G$. At a reducible point Theorem 38.11 shows that all these differences have value zero in the

residue field. We determine their values in finite local algebras supported at such a point. This requires control of extensions between the distinct irreducible constituents.

**Lemma 38.13** (Extensions allowed by the integral matrix algebra)**.** *Let $D$ be a finite local algebra over a finite extension $J$ of $\mathbb{Q}_\ell$, with residue field $J$, and give it a continuous map $T \to D$ whose reduction $x$ is an integral trace point. Enlarge $J$ to split $\rho_x$ when necessary. In $E_D = E_M \otimes_T D$ the simple modules are exactly the constituents of $\rho_x$. Every finite $E_D$-module, restricted to $G$, is crystalline above $\ell$ and kills the prescribed inertia kernels elsewhere. For two nonisomorphic simple modules $U, V$, a nonsplit extension of $U$ by $V$ in this algebra gives a nonzero element of $H^1_f(K, \mathrm{Hom}_J(U, V))$.*

*Proof.* Modulo the maximal ideal of $D$, the algebra maps onto the image algebra of the semisimple $\rho_x$ by Proposition 38.10. Its image is a product of full matrix algebras on the distinct simple constituents. The kernel consists of nilpotent elements. Indeed for a kernel element $b$ the traces of $b, b^2, b^3$ vanish. The matrix identity of degree three expresses $b^3$ in terms of these traces by Newton's identities, so $b^3 = 0$. That identity persists on coefficient extension by applying it first with indeterminate coefficients to a finite sum of matrices. An ideal consisting of nilpotents is in the Jacobson radical, since $1 - ab$ is a unit whenever $b$ belongs to the ideal. The quotient is semisimple, so the kernel is exactly that radical. Lifting back through the nilpotent maximal ideal of $D$ gives the asserted simples.

On a finite $E_D$-module the action of the compact $E_M$ is continuous, by module finiteness over $T$. As in the proof of Proposition 38.10, its span of a full $\mathbb{Z}_\ell$-lattice is a full stable lattice. Every reduction of that lattice is a quotient of finitely many left regular modules $E_M/\mathfrak{m}_T^n E_M$. Applying Lemma 38.9 and Liu's limit theorem again gives crystallinity of the module. At the other places the specified inertia matrices are the identity already in $E_M$.

A splitting of an extension as a $G$-module gives a unique $G$-stable complement isomorphic to $U$, because $\mathrm{Hom}_G(U, V) = 0$. The central action of $D$ preserves this complement by the same vanishing. Since $D$ and the group matrices generate $E_D$, this is an $E_D$-splitting. Thus a nonsplit extension gives a nonzero Galois cohomology class. Its crystalline extension at $\ell$ becomes split over $B_{\mathrm{cris}}$, so it satisfies the Bloch–Kato finite condition there. Away from $\ell$ the finite inertial image acts semisimply in characteristic zero. Hence the class is unramified there. [27, 18] □

**Proposition 38.14** (An odd descent of the two-dimensional constituent)**.** *Suppose $\rho_x = 1 \oplus P$ with $P$ absolutely irreducible of dimension two. Then $P$ extends to an odd representation $W$ of $G_{\mathbb{Q}}$ with $\det W = \eta$. A finite coefficient extension is allowed. There is a classical cuspidal eigenform of weight $2r + 1$ whose arithmetic Galois representation is $W\kappa_\ell^r$. Moreover,*

$$H^1_f(K, P) = H^1_f(K, P^\vee) = 0.$$

*Proof.* The marked symmetry at $c$ induces the $T$-linear anti-involution $B \mapsto B^\dagger = J_3 B^{\mathsf{T}} J_3$ on $E_M$, sending $M(h)$ to $M(ch^{-1}c^{-1})$. It preserves the trace and hence the radical of the trace pairing at $x$. It descends to the product $J \times M_2(J)$, the image algebra of $1 \oplus P$. It preserves both factors since their dimensions differ. On the plane it is adjunction by either a symmetric or an alternating nondegenerate form. Indeed composing with transpose gives an automorphism of the matrix algebra, which is inner. The square of the involution shows that the representing matrix is either symmetric or skew.

We verify the sign on this plane. For any two skew-adjoint three-by-three matrices $X, Y$ with respect to a nondegenerate symmetric form one has

$$2XYX = \mathrm{tr}(XY)X.$$

In an orthonormal basis write $Xv = a \times v$, $Yv = b \times v$. Then $XY = ba^{\mathsf{T}} - (a \cdot b)I$, $a^{\mathsf{T}}X = 0$, and $\operatorname{tr}(XY) = -2a \cdot b$. This proves the identity in characteristic zero. For the fixed integral $J_3$, it is a polynomial identity in entries of $Z - Z^{\dagger}$ and $Z' - Z'^{\dagger}$ with integral coefficients. Its validity over $\mathbb{C}$ for all entries therefore proves it in the marked ring even if that ring has torsion. Every skew element of the semisimple image is the image of a skew element after scalar extension, by applying $(1 - \dagger)/2$. The identity thus holds on the plane, with its ordinary matrix trace since its multiplicity is one. For alternating adjunction all traceless two-by-two matrices are skew. Take $X = \operatorname{diag}(1, -1)$ and $Y = E_{12} + E_{21}$ on that plane, zero on the line. Then $XYX = -Y$ and $\operatorname{tr}(XY) = 0$, a contradiction. The plane form has a symmetric matrix $S$.

Let $J_0$ be the standard alternating matrix in dimension two. Since $\det P = 1$, the conjugation formula on $G$ can be written

$$P(chc^{-1}) = (S^{-1}J_0)P(h)(S^{-1}J_0)^{-1}.$$

The matrix $S^{-1}J_0$ has trace zero, as the product of a symmetric and a skew matrix. Its square is a nonzero scalar. After rescaling, it has square one and determinant minus one. Assign it to $c$. This extends $P$ to the required $W$.

The representation is finitely ramified and absolutely irreducible. It is crystalline at $\ell$ with weights $r, -r$, by Proposition 38.10 and unramified descent from $K_u/\mathbb{Q}_\ell$. For this descent the crystalline periods over an unramified extension carry semilinear descent data on the full coefficient field of periods. Their invariants have the same dimension downstairs. Its residual semisimplification on $G$ consists of $\bar{\Psi}, \bar{\Psi}^{-1}$. If a residual constituent over $\mathbb{Q}$ had dimension two, its restriction to the index-two normal subgroup would be semisimple: the translates of any simple submodule span it. The two character lines would each be fixed by $c$, since both characters come from $\mathbb{Q}$. This contradicts irreducibility. Thus the residual representation over $\mathbb{Q}$ is reducible after semisimplification. Pan’s Theorem 1.0.2 applies to this odd regular de Rham representation, with $\ell > 3$. Twisting to weights $0, 2r$ gives a form of weight $2r + 1$. Any further character twist of Hodge–Tate weight zero is finite order: its inertia at $\ell$ is finite by the classification of de Rham characters, all other inertia is finite by local reciprocity, and a finite ray class quotient then supplies the character. Absorb it into the eigenform. [28]

This form is non-CM. Otherwise $P$ becomes a sum of characters over an open normal subgroup. A scalar restriction would contradict its two distinct Hodge–Tate weights. If the characters differ their two lines are permuted transitively by $G$. The kernel of $\bar{\Psi}$ has pro-$\ell$ matrix image, because it is residually unipotent in an integral upper triangular residual basis. The permutation quotient of order two thus factors through the cyclic image of $\bar{\Psi}$. A lift of a generator swaps the lines and has trace zero, so a generator $a$ of the residual image satisfies $a + a^{-1} = 0$. Its order would be four, whereas on inertia $(\ell - 1)/\gcd(r, \ell - 1) > 4$.

With the arithmetic convention for $V_f = W\kappa_\ell^r$, Kato’s $V_\lambda(f)$ is $V_f^\vee$ by his Euler polynomial at good primes. Therefore $V_\lambda(f)(r)$ restricts to $P^\vee \simeq P$ on $G_K$. Theorem 14.2(1) of Kato applies over the abelian field $K$, since $1 \le r \le k - 1$ and $2r \neq k = 2r + 1$. It gives finiteness for the Selmer group with divisible torsion coefficients. The image of the vector space of finite Selmer classes in that finite group is divisible, hence zero. It is consequently contained in the image of the finitely generated integral cohomology of a lattice over a fixed finite ramification set. Such an image contains no nonzero vector subspace. The integral finite generation and the compatibility of the local maps here are the ones proved in Theorem 33.8. [29] □

**Theorem 38.15** (Symmetry on every infinitesimal reducible fiber)**.** *Let $T \to D$ be as in Lemma 38.13. If $\rho_x$ is reducible, all inverse-trace differences vanish in $D$. If $\rho_x$ is the split triple, the whole map $T \to D$ factors through the original split evaluation.*

*Proof.* We recall a finite algebra calculation. Write $R = E_D$ and let $J_R$ be its nilpotent Jacobson radical. Over the splitting field $R/J_R$ is a product of matrix algebras. The block of $J_R/J_R^2$ from one simple factor to another can be nonzero only when there is a nonsplit extension of the corresponding simple modules, with the target as submodule. To see this, lift a primitive idempotent $f$ for the source factor through the nilpotent radical. The projective module $Rf$ has simple top and radical $J_R f$. A simple quotient of $J_R f/J_R^2 f$ gives, by pushing out $Rf/J_R^2 f$, a length-two extension with this quotient as submodule. It is nonsplit, since the radical has nonzero image on the resulting module. Matrix units in $R/J_R$ recover the whole block from its primitive columns. Finally $J_R^n/J_R^{n+1}$ is an image of the $n$-fold tensor power of $J_R/J_R^2$ over $R/J_R$, by multiplication. Each block in it is spanned by products that pass through $n$ successive nonzero blocks. These facts also hold in the presence of diagonal radical blocks.

First let the constituents be $1, P$. Extensions between the two distinct factors are zero in both directions by the lemma and the proposition. The same holds for blocks between them in every $J_R^n/J_R^{n+1}$, by the product description. Use the idempotent $e$ for the eigenvalue one of the separating $g$ of Section 38.1. It reduces to the identity on the line and zero on the plane. The off-diagonal spaces $eR(1-e)$ and $(1-e)Re$ vanish modulo the radical and in every successive layer, so are zero by nilpotence. Thus $e$ is central. In the original matrix ring $eM(h)e = t(eM(h))e$ and $t(e) = 1$. Consequently $eRe = De$, and the action on it is given by a character $\theta : G \to D^\times$ reducing to one.

At each place above $\ell$ the middle spectral idempotent of a local separating inertia element also reduces to identity on the line and zero on the plane. To see that it equals $e$, use the product $eR \times (1-e)R$. The unwanted idempotent and the complement of the wanted one lie in nilpotent ideals and hence are zero. The ordinary projector formula therefore gives $\theta(h) = 1$ on that inertia. Away from $\ell$ its image on inertia is finite and lies in $1 + \mathfrak{m}_D$, a torsion-free group by the finite logarithm in characteristic zero. It is everywhere unramified and hence factors through the finite class group, so $\theta = 1$. The degree-three Cayley–Hamilton identity on $M(h)$, with determinant one, has polynomial

$$X^3 - t(h)X^2 + t(h^{-1})X - 1.$$

Multiply by $e$. The resulting equation at $X = 1$ gives $t(h^{-1}) = t(h)$, since $t(e) = 1$.

For the split triple order the simple characters by their cyclotomic exponents. There is no extension in $R$ from a larger to a smaller exponent. Indeed its class would lie in the finite Selmer group with field coefficients for a negative power of $\Psi$, by the lemma. Restrict to the finite abelian extension obtained by taking the compositum of $K$ with the field fixed by $\ker \alpha$. There the coefficients are a negative cyclotomic twist $J(-m)$, $m > 0$. The local finite group at $\ell$ is zero by the Bloch–Kato dimension formula. At other finite places the unramified first cohomology is zero because Frobenius on the coefficients is not one. The resulting rational class is therefore everywhere locally trivial. Its image with divisible torsion coefficients lies in the finite strict group bounded by Proposition 11.4, whose proof applies to this fixed number field and $m$. A divisible image in that finite group is zero, and finite generation of integral cohomology excludes a nonzero rational subspace in the kernel as above. Restriction in rational cohomology is injective by corestriction, proving the claimed vanishing.

It follows from the radical-layer calculation that between distinct idempotents only the increasing direction can occur anywhere in $R$. The three corners cut out by the spectral $e_i$ are scalar, by their original rank-one identity. In a product $M(h)M(k)$ the $i, i$ corner can therefore only receive the product of the $i, i$ corners, since a path leaving $i$ and returning would use both directions. Thus each $h \mapsto t(e_i M(h))$ is multiplicative and the trace in $D$ is the sum of three characters. Proposition 38.2 forces the entire map to factor through $O$. □

**Corollary 38.16** (The rational support of the inverse-trace ideal)**.** *In the generic fiber of the finite quotient $T/\mathfrak{b}_a$, the inverse-trace ideal has zero projection to every local factor with reducible geometric residue representation, including all its nilpotents. A geometric local factor supported at the split triple is its residue field.*

*Proof.* The generic fiber of the finite algebra is Artinian. After a finite splitting extension it is a product of finite local algebras $D$ of the kind in the theorem. The maps from $T$ are continuous since the original quotient is finite over $O$. Each such factor is generated by the image of $T$ and the coefficient field, by projection from the finite scalar extension. The two assertions follow from the two conclusions of the theorem. □

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