The generalized Riemann hypothesis for Dirichlet L-functions

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Volume VI

Marked Galois deformations and trace congruences

Abstract

We relate strict extension classes to rank-three Galois representations. We calculate tangent spaces of ordinary deformations, distinguish orthogonal components, and compare their trace congruences on the common interval.

The guide in Volume I lists all twenty-seven volumes.

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30 Extension classes and trace families

30.1 Passing the strict condition to finite coefficients

Fix an odd prime ℓ. Let $E/\mathbb{Q}$ be totally imaginary abelian, let P be the ℓ-Sylow subgroup of $\mathrm{Gal}(E/\mathbb{Q})$ and let H be its prime-to-ℓ factor. Put $K = E^H$. Write $\mathcal{O}$ for a finite unramified extension of $\mathbb{Z}_\ell$ containing the values of all characters of H, $J = \mathrm{Frac}(\mathcal{O})$, $v_\ell(\ell) = 1$. Extend a character α of H trivially across P. For a positive twist r set

$$\vartheta = \kappa_\ell^{-r}\alpha^{-1}|_{G_K}, \qquad D = (J/\mathcal{O})(\vartheta), \qquad D_m = D[\ell^m].$$

When referring to the complementary sign take $\alpha(c) = (-1)^r$, with c complex conjugation. The strict group $S_\vartheta \subset H^1(K, D)$ consists of classes trivial on inertia at every finite place away from ℓ and trivial on the full local group at every place above ℓ. It is the character component of the strict group over E by Lemma 24.8. That lemma also covers $\ell \mid [E : \mathbb{Q}]$. For the fields of the prime count, the components for which $\bar\vartheta = 1$ have already been bounded in Corollary 24.10. We consider every other component here.

Define $S^{\mathrm{f}}_{\vartheta,m} \subset H^1(K, D_m)$ by the same local conditions with coefficient D_m, including at $w \mid \ell$. For such a place set $s_w = 0$ if $\bar\vartheta|_{G_{K_w}} \neq 1$. Otherwise put

$$s_w = \min_{g \in G_{K_w},\ \vartheta(g) \neq 1} v_\ell(\vartheta(g) - 1).$$

The minimum exists since the cyclotomic character has infinite image there and $r \neq 0$.

Theorem 30.1 (The cost of imposing finite local conditions)**.** *Suppose $\bar\vartheta \neq 1$ on G_K. For every $m \geq 1$, finite-to-divisible coefficients give an injection with*

$$0 \leq \mathrm{length}_{\mathcal{O}} S_\vartheta[\ell^m] - \mathrm{length}_{\mathcal{O}} S^{\mathrm{f}}_{\vartheta,m} \leq \sum_{w \mid \ell} \min(m, s_w). \tag{30.1}$$

If the residual character is locally trivial at w, then

$$s_w = a_w + v_\ell(r), \qquad 1 \leq a_w \leq 1 + v_\ell([K_w : \mathbb{Q}_\ell]).$$

In a representative of a strict class killed by ℓ^m, the local coboundary at w can be taken with values of its defining element in D_{m+s_w}.

Proof. The exact sequence for multiplication by ℓ^m on D gives an isomorphism $H^1(K, D_m) \simeq H^1(K, D)[\ell^m]$, since some global element has $\vartheta(g)-1$ a unit and thus $D^{G_K} = 0$. On inertia away from ℓ, the character has finite image of order prime to ℓ. If this image is trivial its invariant module is divisible, and if nontrivial its invariant module is zero, since some Teichmüller value differs from one by a unit. In either case the kernel of the local coefficient map, the quotient of the invariant module by ℓ^m, is zero. The finite lift already satisfies the required inertia conditions.

At $w \mid \ell$ the possible local images of that lift lie in the connecting image of $D^{G_{K_w}}/\ell^m D^{G_{K_w}}$. Its length is $\min(m, s_w)$, since the invariant module is precisely D_{s_w} (with $D_0 = 0$). Simultaneous localization gives an injection of the quotient in (30.1) into the product of these local kernels. All groups in the inequality are finite, for instance by the positive-twist finiteness in Proposition 11.4 and character descent. If a local finite cocycle represents a coboundary in D, its values have the form $(\vartheta(g)-1)y$. An element attaining the minimum shows $\ell^{m+s_w} y = 0$; use a unit difference if $s_w = 0$. This proves the assertion about the local element. More precisely, put $s_* = \max_{w \mid \ell} s_w$. Under the global coefficient isomorphism, $S_\vartheta[\ell^m] = (S^{\mathrm{f}}_{\vartheta,m+s_*})[\ell^m]$. Indeed

the chosen local elements place every lifted class inside this finite strict group after inclusion of coefficients. Conversely all its classes map to strict classes in divisible coefficients, and the global coefficient map is injective.

Let $\langle u\rangle$ be the principal-unit part of $u \in \mathbb{Z}_\ell^\times$. The local cyclotomic character on $G_{\mathbb{Q}_\ell}$ is onto $\mathbb{Z}_\ell^\times$, as follows from the local cyclotomic extension of degree $(\ell-1)\ell^{n-1}$ at level n. Hence the image of $\langle\kappa_\ell\rangle$ on the subgroup G_{K_w} has index dividing $[K_w:\mathbb{Q}_\ell]$ in $1+\ell\mathbb{Z}_\ell$. By the ℓ-adic logarithm it is $1+\ell^{a_w}\mathbb{Z}_\ell$ with the asserted bound on a_w. The finite character α has only Teichmüller values. If $\bar\vartheta$ is locally trivial, its entire Teichmüller part cancels, so that its image is the $(-r)$-th power of this principal-unit group. The logarithm shows that its first nonzero valuation is $a_w + v_\ell(r)$. □

Corollary 30.2 (A bound at all odd primes on the power interval)**.** *Use the fields $E = E_x \subset \mathbb{Q}(\zeta_N)$ and $t = \lfloor x^{401}\rfloor$ of Volume IV. Let $d_E = [E:\mathbb{Q}]$. Independently choose a positive $m_{\ell,r,\alpha}$ for each nontrivial residual component. The sum of the left-hand length differences in (30.1) over all odd primes, all $1 \le r < t$ and all such complementary components, weighted by $\log\ell$, is at most*

$$Cd_Et(t+\log(2N))\log(2td_E) \le \exp(x^{\varepsilon/3}) \tag{30.2}$$

on every sufficiently distant common interval. In particular one can take $m_{\ell,r,\alpha}$ large enough to contain each entire strict group.

Proof. If $\ell \nmid N$, both $K/\mathbb{Q}$ and α are unramified at ℓ. Local inertia over K_w then has full mod-ℓ cyclotomic image, since the local cyclotomic extension is totally ramified. Local residual triviality forces $\ell - 1 \mid r$, hence $\ell \le t$. At any contributing prime and twist, Theorem 30.1 bounds $s_w\log\ell$ by $\log\ell + \log r + \log d_E$. The product of the number of characters $|H|$ and the number of places over ℓ is at most $|H|[K:\mathbb{Q}] = d_E$. Sum first over $\ell \le t$. For the remaining primes dividing N use $\sum_{\ell\mid N}\log\ell \le \log N$ and $\#\{\ell : \ell \mid N\} \le \log(2N)/\log 2$. The displayed bound follows after summing over $r < t$. The estimates $d_E \ll_\psi x^{2+8\varepsilon}$ and $\log(2N) \ll_{\psi,\varepsilon} x^{\varepsilon/4}$ from Volume IV give the second inequality. The last assertion follows from finiteness of each strict group. □

30.2 An integral presentation of the global extension classes

Keep $\bar\vartheta \ne 1$. Take a finite set S of finite places of K containing those over ℓ and the ramification of α, and write $G = G_{K,S}$. Fix $g_* \in G$ with $\vartheta(g_*) - 1$ a unit. At $w \mid \ell$ let H_w be the kernel of ϑ on the full local group. At $w \in S$ away from ℓ let it be the kernel on inertia. Local groups are mapped into G using fixed embeddings. Outside S the finite modules are unramified, and classes vanishing on inertia descend to G.

Lemma 30.3 (Normalized finite cocycles)**.** *The group $S_\vartheta[\ell^m]$ is isomorphic to the module of continuous functions $z : G \to D_m$ satisfying*

$$z(gh) = z(g) + \vartheta(g)z(h), \qquad z(g_*) = 0, \qquad z(H_w) = 0 \quad (w \in S).$$

Proof. The global lift is unique as a cohomology class by Theorem 30.1. Every finite cocycle has a unique cohomologous representative vanishing at g_*, obtained by subtracting $(\vartheta(g)-1)z(g_*)/(\vartheta(g_*)-1)$. Its restriction to H_w is a homomorphism. The map from finite to divisible coefficients on its values is injective there since the action is trivial. A local class in divisible coefficients trivial on H_w comes by inflation from the local character image. Away from ℓ the latter is finite of order prime to ℓ, so its degree-one cohomology vanishes. At $w \mid \ell$ its principal-unit subgroup is open procyclic and its quotient has order prime to ℓ. A topological generator acts by a scalar different from one, whose difference from one is onto on $J/\mathcal{O}$. Matching a

cocycle at this generator by a coboundary and using averaging on the quotient gives the same vanishing. Thus killing the specified kernels is equivalent to the divisible strict condition at all places. □

Theorem 30.4 (The relation module and its determinants)**.** *There are finitely many topological generators in a common quotient of G sufficient for all the cocycles in Lemma 30.3. If their number is n, there is a full-rank relation submodule $B \subset \mathcal{O}^n$ such that, for all m,*

$$S_\vartheta[\ell^m] \simeq \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^n/B, \ell^{-m}\mathcal{O}/\mathcal{O}), \qquad \mathrm{length}_{\mathcal{O}}\, S_\vartheta = \min v_\ell \det(b_1, \dots, b_n).$$

The minimum runs over n relation vectors described in the proof, with valuation $+\infty$ for zero determinants.

Proof. Let K'/K be the field cut out by $\bar{\vartheta}$. In G, factor its kernel by the kernel of its maximal pro-ℓ quotient. This latter kernel is characteristic in the open normal subgroup, so we obtain a quotient Γ of G. The character ϑ factors through it: on $G_{K'}$ its values are principal units. For a finite cocycle the homomorphism $g \mapsto (z(g), \vartheta(g) \bmod \ell^m)$ takes this subgroup into a finite ℓ-group, a semidirect product of the additive finite module by a principal-unit image. Hence every such cocycle also factors through Γ.

The normal pro-ℓ subgroup has finite-dimensional continuous homomorphisms to $\mathbf{F}_\ell$. Indeed these homomorphisms are unramified away from the primes over S. Adjoin μ_ℓ to K', an extension of degree dividing $\ell - 1$, and use injectivity of restriction by corestriction. After identifying $\mathbf{F}_\ell$ with μ_ℓ, Kummer theory bounds the resulting space by the dimension of the units of the ring of S-integers modulo ℓ-th powers and the dimension of its ℓ-torsion class group. Both are finite. Thus the maximal elementary abelian quotient of the pro-ℓ subgroup is finite. Lifts of a basis topologically generate: if their closed subgroup were proper, its image would be proper in a finite ℓ-group quotient, hence contained in a subgroup of index ℓ. That would give a nonzero character killing the basis. Adjoin lifts of generators of the finite residual quotient to obtain generators of Γ. [1, Ch. 4]

Choose these n generators, let F_n be the free profinite group on their lifts u_i, and pull ϑ back to F_n. The free profinite group is the profinite completion of the ordinary free group on this finite basis. There is a unique continuous crossed homomorphism $v : F_n \to \mathcal{O}^n$ with $v(u_i)$ the i-th basis vector: map u_i to $(v(u_i), \vartheta(u_i))$ in the compact profinite group $\mathcal{O}^n \rtimes \vartheta(G)$ and use the defining property of the completion on each finite quotient. Give the vectors $v(h)$ as relations for all h in the kernel of $F_n \to \Gamma$, for a lift of g_*, and for the preimages of the images of all H_w. Their $\mathcal{O}$-span is B. This submodule is finitely generated because $\mathcal{O}$ is Noetherian, and is closed since a finite sum of compact cyclic submodules is compact. A linear map from $\mathcal{O}^n$ to a finite coefficient group sends v to a cocycle. Conversely every finite cocycle on the free group is given by this recipe by its generator values and continuity. It descends and satisfies the conditions of Lemma 30.3 exactly when the linear map kills B. This proves the first formula.

Passing to the union in m identifies S_ϑ with $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}^n/B, J/\mathcal{O})$, since the source has finitely many generators. Finiteness of S_ϑ excludes a free summand in $\mathcal{O}^n/B$. The elementary-divisor theorem over the discrete valuation ring now gives equality of its length with the length of its dual and with the valuation of the ideal generated by the maximal minors of a relation matrix. The originally specified vectors contain a finite generating set for their span, by finite generation. The determinant ideal is unchanged when all of them are allowed, by multilinearity. Its valuation is the claimed minimum. □

The theorem determines precisely which global determinants would suffice. It supplies finite generation but no uniform upper bound for their valuations. The following calculation connects these linear equations to the deformation argument of the supplied source.

30.3 A rank-three deformation and the trace comparison

Put $\Psi = \vartheta^{-1}$, and let $\xi : G \to \{1, -1\}$ be a continuous quadratic character, allowing the trivial character. Write J_3 for the matrix with antidiagonal entries one and all other entries zero. The nontrivial element in $\{1, -1\}$ acts on GL_3 by $h \mapsto J_3(h^{\mathsf{T}})^{-1}J_3$. The diagonal matrix $d(g) = \mathrm{diag}(\Psi(g), 1, \Psi(g)^{-1})$ is fixed by this involution. Thus $\rho_0(g) = (d(g), \xi(g))$ is a homomorphism to the semidirect product. For a finite module $M = \mathcal{O}/\ell^m\mathcal{O}$, denote by $\mathcal{O} \oplus \epsilon M$ the ring in which ϵM is a square-zero ideal.

Proposition 30.5 (An integral root direction for the marked classes)**.** *Let $N_0 = E_{21} - E_{32}$, with E_{ij} the standard matrix units. Identify D_m with M in the chosen coefficient coordinate. For a normalized cocycle of Lemma 30.3 put*

$$\rho_z(g) = (1 + \epsilon z(g) N_0)\rho_0(g).$$

These are distinct homomorphisms lifting ρ_0, with $\rho_z(g_*) = \rho_0(g_*)$. For every $g \in \ker \xi$ their characteristic polynomials agree with that of $d(g)$. On local groups subject to strictness they are conjugate to the base representation by elements of the same root direction after the coefficient enlargement of Theorem 30.1.

Proof. On the square-zero ideal the involution acts by $X \mapsto -J_3X^{\mathsf{T}}J_3$; it fixes N_0. Conjugation by $d(g)$ scales each of E_{21}, E_{32} by $\Psi(g)^{-1} = \vartheta(g)$. Multiplying two displayed matrices therefore gives the cocycle law, since products of two elements of ϵM vanish. The lower triangular matrix part recovers $z(g)$ from its $(2,1)$-entry since $\Psi(g)$ is a unit. Its diagonal equals that of $d(g)$, giving the characteristic polynomial. If locally $z(g) = (\vartheta(g) - 1)y$, conjugate $\rho_0(g)$ by $1 - \epsilon y N_0$ to obtain $\rho_z(g)$. At primes above ℓ the element y has the bound just proved. Away from ℓ any inertia coboundary can be taken in D_m, by the inertia calculation in Theorem 30.1. □

For the quadratic character of discriminant -1208, $\ell = 3$ and odd r, take $E = \mathbb{Q}(\sqrt{-302})$. The complementary character is its nontrivial quadratic character, with nontrivial residual ϑ on inertia at 151. Proposition 24.15, with its proof over this quadratic field, gives a nonzero class already at $m = 1$. Thus the equality of all traces on $\ker \xi$ in Proposition 30.5 actually occurs for different marked deformations arising from the arithmetic problem.

Proposition 30.6 (What an integral cotangent bound detects)**.** *Let A be a graded $\mathcal{O}$-algebra with an augmentation $x : A \to \mathcal{O}$ killing all nonzero degrees. Write $I = \ker x$, and*

$$I^{(2)} = \{z \in I : \text{some } 0 \neq u \in \mathcal{O} \text{ satisfies } uz \in I^2\}.$$

Fix a nonzero degree j and homogeneous equations $F_i \in A_j$. Suppose a finite $\mathcal{O}$-module V injects additively into the augmentation derivations $A \to M$ that vanish outside degree j and kill each F_i. Suppose every derivation in the image lifts to an $\mathcal{O}$-valued augmentation derivation of degree support j. If the module

$$C_j = A_j/((A_j \cap I^{(2)}) + \textstyle\sum_i \mathcal{O}F_i)$$

has finite length, then $\mathrm{length}_{\mathcal{O}} V \leq \mathrm{length}_{\mathcal{O}} C_j$.

Proof. An augmentation derivation obeys $D(ab) = x(a)D(b) + x(b)D(a)$, so kills I^2. An integral lift takes values in the torsion-free module $\mathcal{O}$; it therefore kills $I^{(2)}$. Reducing the lift shows that the original derivation kills this saturation. Its values are determined by its restriction to A_j, which factors through C_j. The resulting additive injection into $\mathrm{Hom}_{\mathcal{O}}(C_j, M)$ bounds cardinalities. A cyclic summand $\mathcal{O}/\ell^b$ contributes length $\min(m, b)$ to this Hom module. Taking

logarithms to base $|\mathcal{O}/\ell|$ gives the inequality even for an injection only known to be additive. The linear presentation in Theorem 30.4 realizes this calculation in a polynomial algebra on n variables of degree -1: the equations are linear forms with coefficient vectors in B. Lifts of derivations on the ambient polynomial ring are obtained by lifting the values on its variables. □

Theorem 30.7 (A trace estimate for a finite order)**.** *Let R be a commutative $\mathcal{O}$-algebra finite as a module, $x : R \to \mathcal{O}$ an augmentation, $I = \ker x$, and T its $\mathcal{O}$-torsion ideal. Suppose the generic fiber $R \otimes J$ is reduced. Write Σ for its geometric points other than x. Let $a \in I$ have $a(P) \neq 0$ at every $P \in \Sigma$. Then*

$$\operatorname{length}_{\mathcal{O}}(I/I^2) \leq \operatorname{length}_{\mathcal{O}}(T/(T \cap I^2)) + \sum_{P \in \Sigma} v_\ell(a(P)). \tag{30.3}$$

In particular if $T = 0$ and for every m the classes of $S_\vartheta[\ell^m]$ inject into augmentation derivations $R \to M$, their full length is bounded by this sum.

Proof. Torsion maps to zero under x. Set $R' = R/T$, $I' = I/T$. The kernel of $I/I^2 \to I'/I'^2$ is $T/(T \cap I^2)$. The ideal I' is finite free over the discrete valuation ring. Over an algebraic closure of J it has one coordinate line for each point of Σ: a finite reduced algebra in characteristic zero is a product of separable fields, and augmentation removes precisely its own coordinate. Multiplication by a on I' is injective and has determinant $\prod_{P \in \Sigma} a(P)$, with the empty product one. The elementary-divisor theorem gives $\operatorname{length}(I'/aI') = \sum_P v_\ell(a(P))$. Since $aI' \subset I'^2$, additivity of length gives (30.3). Augmentation derivations to M are precisely $\operatorname{Hom}_{\mathcal{O}}(I/I^2, M)$, by the decomposition $R = \mathcal{O} \oplus I$ and the product rule. Its length is at most $\operatorname{length}(I/I^2)$. Taking m above the exponent of the finite strict group proves the consequence (also for additive injections). □

The hypotheses on an integral model matter. In a finite torsion-free $\mathbb{Z}$-graded $\mathcal{O}$-algebra with reduced generic fiber every nonzero-degree piece vanishes. Indeed, after extending scalars to an algebraic closure of J, scaling homogeneous elements of degree j by t^j permutes the finite set of geometric points. A nonzero value in nonzero degree would consequently take the infinitely many values $t^j a(P)$ in a finite set. Reducedness then forces each such element to vanish, and torsion-freeness gives vanishing integrally. A comparison preserving the root direction must therefore account for the scaling orbits or supply a different integral model on which the required derivations survive. Traces alone take identical values on the deformations of Proposition 30.5.

Connection with the source. In the supplied archive the normalization and the reflected root occur in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/ShaInjGen.lean`, line 30 and line 101 of that file. The integral lifting condition of Proposition 30.6 occurs in `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/OrbitVanishingLocal.lean`, line 120. Lines 103–160 of `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/ModelUpper.lean` feature a graded model with a finite weight-zero algebra and a sum over scaling orbits. That setting differs from the finite-order hypothesis in Theorem 30.7. To complete the length bound by this route one must construct the relevant integral model for the global local conditions, control its orbits and tangents, and show that its points yield the bounded automorphic systems of Section 30.5 with the required multiplicities. The local bound (30.2) applies in advance of that comparison. The integral theta-period estimate of Volume I remains a separate requirement.

30.4 The two families of traces

Fix an imaginary quadratic field K with discriminant of absolute value d_K, a positive integer N, and $r \geq 1$. Write c for complex conjugation. Choose a rational prime $p \geq 7$ split in K and prime to $3Nd_K$, and fix $w \mid p$. We use the split condition so that the residue cardinality is p. At the end two such primes will be fixed for the entire interval.

Here are the automorphic representations we count. The congruence level in this definition is on the automorphic side. Write $U_K(N) = \ker(\mathrm{GL}_3(\widehat{\mathcal{O}}_K) \to \mathrm{GL}_3(\mathcal{O}_K/N\mathcal{O}_K))$.

1. Take the cuspidal Π of $\mathrm{GL}_3(\mathbb{A}_K)$ with $\Pi^\vee \simeq \Pi^c$, with $U_K(N)$-fixed vectors, and cohomological for a coefficient system of highest weights $\lambda_{\sigma,1} \geq \lambda_{\sigma,2} \geq \lambda_{\sigma,3}$ at the two embeddings of K. Take all such systems with $|\lambda_{\sigma,i}| \leq h = 4(r+2)$ and trivial action of the positive real split center. Cohomological means that relative Lie algebra cohomology of Π_∞ tensored with the coefficient system is nonzero, modulo that split center. At w, let $\alpha_1, \alpha_2, \alpha_3$ be the unitarily normalized Satake parameters of Π_w. Record two labels, with values $b_{\Pi,\pm} = \sum_{j=1}^{3} \alpha_j^{\pm 1}$.

2. Take all normalized primitive holomorphic cusp eigenforms f of weight $2r+1$ and level dividing $M = Nd_K$, with arbitrary nebentypus ω_f. At p write $a_p(f)$ for the Fourier coefficient. Record

$$b_{f,+} = 1 + p^{-r} a_p(f), \qquad b_{f,-} = 1 + p^{-r} \omega_f(p)^{-1} a_p(f).$$

A label records its representation and sign. Repeated values on different labels are retained. In the second case the characteristic polynomial at p has roots β_1, β_2 with sum $a_p(f)$ and product $\omega_f(p)p^{2r}$. The two numbers are the traces on $1 \oplus (\beta_1/p^r, \beta_2/p^r)$ and its inverse. Since p splits in K, restriction from $\mathbb{Q}$ to K_w leaves the residue degree one. Thus these are also the traces for the corresponding restricted, normalized Deligne representation with a trivial line adjoined. In rank three the two signs likewise accommodate either choice of Frobenius convention.

Theorem 30.8 (Size, purity and integrality)**.** *Let $\mathcal{T} = \mathcal{T}(N, r, p)$ be the labelled multiset above. There are constants C_K, A such that*

$$|\mathcal{T}| \leq C_K[N(r+2)]^A, \qquad [\mathbb{Q}(b) : \mathbb{Q}] \leq C_K[N(r+2)]^A \quad (b \in \mathcal{T}). \tag{30.4}$$

Under every complex embedding $|b| \leq 3$. With $u = 18h + 1$, every $p^u b$ is an algebraic integer. The constants and u are independent of the split auxiliary prime.

Lemma 30.9. *For one coefficient system of the specified weights, the total dimension of cohomology of the GL_3/K locally symmetric space of level $U_K(3N)$ with these coefficients is $O_K(N^{36}(h+1)^6)$. The number of distinct cuspidal representations from this coefficient system at level $U_K(N)$ has the same upper bound.*

Proof. The locally symmetric space is the adelic double quotient by the maximal compact at infinity, by the positive real split center and by the indicated finite level. The quotient at level $U_K(3)$ has finitely many components and the homotopy type of a finite cell complex by the Borel–Serre compactification. Here stabilizers at the archimedean symmetric space are trivial. An arithmetic stabilizer preserves a rank-six integral lattice, has determinant of absolute norm one, and is compact at infinity after removing the positive real center. An element fixing a point up to real scaling has scaling one by this determinant condition. The stabilizer is therefore finite and embedded, in a basis of that lattice, into matrices congruent to the identity modulo three. This principal congruence group is torsion free by Lemma 19.2. [2]

The space at level $U_K(3N)$ is a finite cover of this fixed space of degree at most $[U_K(3) : U_K(3N)]$. This index is at most $(3N)^{36}$, by reduction of the action on $(\mathcal{O}_K/3N)^3$, a free module of rank six over $\mathbb{Z}/3N$. Pulling back cells, allowing separate components, gives $O_K(N^{36})$ cells in total. For one embedding the Weyl dimension formula is

$$\dim W_\lambda = \frac{(\lambda_1 - \lambda_2 + 1)(\lambda_1 - \lambda_3 + 2)(\lambda_2 - \lambda_3 + 1)}{2}.$$

The rank of the product coefficient system for both embeddings is therefore $O((h+1)^6)$. Cellular cochains with local coefficients have one copy of the fiber per cell. This proves the dimension bound.

The cuspidal part of automorphic cohomology injects into ordinary cohomology with the corresponding coefficients. In this form one uses Franke’s automorphic cohomology theorem and the cuspidal direct summand: in the complex of automorphic forms the projection to the cuspidal support for the full group splits off the cuspidal cohomology. At finite level its summands are the tensor products of relative Lie algebra cohomology at infinity with finite-level fixed vectors of the cuspidal representations, counted with automorphic multiplicity. Our cohomological assumption makes the first factor nonzero and $U_K(3N) \subset U_K(N)$ makes the second nonzero. Each distinct Π therefore contributes at least one dimension. There are $O((h+1)^6)$ possible ordered weights over the two embeddings, which will allow us to sum this bound. [3] □

Lemma 30.10 (An integral action of the spherical operator)**.** *In the first family $p^{18h+1} b_{\Pi,\pm}$ is an algebraic integer of degree at most $O_K(N^{36}(h+1)^6)$.*

Proof. Let T^+ be the double coset at w of $\mathrm{diag}(p,1,1)$, and let T^- be its inverse double coset. Use hyperspecial volume one. The unramified Satake formula gives as their eigenvalues on a spherical vector $p\sum_i \alpha_i$ and $p\sum_i \alpha_i^{-1}$, in either order if the opposite normalization for right translation is used. The factor p is $p^{(3-1)/2}$. For example, in the normalized unramified principal series, the index-p sublattices in the double coset divide into the three triangular orbits of sizes $p^2, p, 1$. The factors of the square root of the Borel modulus for their diagonal parts are $p^{-1}, 1, p$. This gives p times the sum of the three unramified character values. The inverse calculation gives the inverse values. The Π-summand of Lemma 30.9 contains the spherical action at w since the level there is hyperspecial.

We explain integrality on the entire cohomology lattice. The coefficient system is defined over K: the two factors of $(\mathrm{Res}_{K/\mathbb{Q}}\,\mathrm{GL}_3)_K$ correspond to the two embeddings of K into itself. At a finite-adelic representative g let $\Lambda_g = K^3 \cap g\widehat{\mathcal{O}}_K^3$. Using its images by the two embeddings and the Schur and determinant construction of Section 70.1, gives an integral coefficient lattice at that representative. It transforms equivariantly under $\mathrm{GL}_3(K)$ and is fixed by the finite level on the right. These lattices thus form an integral local system.

On the two legs of either Hecke correspondence, the underlying lattices, compared in the common rational space, obey $p\Lambda \subseteq \Lambda' \subseteq p^{-1}\Lambda$. At w this follows from the elementary divisors of the double coset, and away from w the local lattices agree before the harmless multiplication by the rational integer p. The same two-sided inclusion holds for dual lattices, in the reversed direction. At one embedding W_λ uses tensor degree $|\lambda - (\lambda_3,\lambda_3,\lambda_3)| \le 4h$ and the determinant to the power λ_3, costing at most $3h$ powers of p in either direction. To see the first bound, sum the two differences $\lambda_1 - \lambda_3, \lambda_2 - \lambda_3$. Images under an integral Young symmetrizer preserve the resulting inclusions, since the same symmetrizer is used on the common rational tensor power. The determinant assertion follows from the third exterior power, with its dual for negative exponents. Both embeddings together cost at most $14h$, so p^{18h} suffices.

The cohomological action of a double coset is pullback, change on coefficients, and transfer (the sum over the sheets). Pullback and transfer are integral. The preceding inclusions make the coefficient map integral after multiplication by p^{18h}. Thus $p^{18h}T^{\pm}$ preserves integral cohomology modulo torsion, an $\mathcal{O}_K$-lattice in the cohomology over K. It has a monic characteristic polynomial over $\mathcal{O}_K$, by localization at all finite primes, and its degree is bounded by Lemma 30.9. The eigenvalue $p^{18h+1}b_{\Pi,\pm}$ is therefore integral and has degree over $\mathbb{Q}$ at most twice that dimension. □

Proof of Theorem 30.8. Lemmas 30.9 and 30.10 give the rank-three count and degree bounds, after summing over the weights. For purity use Shin's theorem for regular algebraic conjugate-self-dual cuspidal representations of a CM field. Its infinitesimal-character condition is met by cohomology for our algebraic coefficient system. The extra weight restriction in even rank is absent in rank three. At w the representation is unramified. Shin's Theorem 1.2(iii) says that, in his motivic normalization, all its Frobenius roots have absolute value $p^{(3-1)/2} = p$ under every complex embedding. Dividing by p gives the unitarily normalized roots. Both these numbers and their inverses have modulus one at all embeddings, so $|b_{\Pi,\pm}| \le 3$. [4, Thm. 1.2]

For the second family, put $k = 2r+1$. The newforms embed as independent Hecke eigensystems in $S_k(\Gamma_1(M))$ by the classical newform theorem. Independence also follows by using their Hecke eigenvalues away from M, which distinguish primitive eigensystems by multiplicity one. We recall a dimension bound. If $d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(M)]$, multiply the weight-$k$ translates of a nonzero form over a set of representatives of $\Gamma_1(M)\backslash \mathrm{SL}_2(\mathbb{Z})$. The product is a nonzero level-one modular form of weight kd. Its order at infinity is at least that of the original form because each other translate is holomorphic there in a suitable cusp parameter. The level-one valence formula bounds this order by $kd/12$. Hence the first $\lfloor kd/12 \rfloor + 1$ Fourier coefficients determine a cusp form. Reduction modulo M gives $d \le M^4$. This proves the needed polynomial count.

The Fourier coefficients of a normalized newform are algebraic integers, and conjugating its coefficients by a field embedding produces a normalized newform of the same weight and level with conjugate nebentypus. These facts follow from the arithmetic theory of newforms and their Hecke algebra. Consequently $p^r b_{f,\pm}$ is integral, and the number of conjugates of either trace is bounded by the number of labels just counted. Deligne's purity theorem for holomorphic newforms of weight $k \ge 2$ gives $|\beta_i| = p^{(k-1)/2} = p^r$ at every embedding. Thus their normalized traces with the extra unit have absolute value at most three. Here $p \nmid M$, as required. Multiplying by p^{u-r} gives the stated common normalization. [5, 6, 7] □

30.5 A simultaneous bound for the Hecke congruence ideals

Let $G_N = (\mathbb{Z}/N\mathbb{Z})^{\times}$. For a subgroup H with G_N/H cyclic, write $\mathcal{A}_H = \{\alpha : \ker \alpha = H\}$, the full orbit of complex characters with this kernel. For the chosen p define

$$z_\alpha = \alpha(p)^{-1}, \qquad d_\alpha = 1 + p^{-r} z_\alpha + p^{r} z_\alpha^{-1}.$$

The α's index the factors even if their values at p coincide. To each label $s \in \mathcal{T}$ attach the set of all conjugates of its trace b_s, once per embedding of $\mathbb{Q}(b_s)$. Keep the sets for different labels disjoint. Denote the resulting labelled multiset by $\mathcal{T}^{*}$.

Theorem 30.11 (The rational integer for an orbit)**.** *There is a nonzero integer $I_{p,H,r}$, the same for every prime $\ell \neq p$, with*

$$\log |I_{p,H,r}| \le C_K |\mathcal{A}_H| [N(r+2)]^{A'} \log p.$$

For any common embedding of the algebraic numbers into $\overline{\mathbb{Q}}_\ell$, normalize $v_\ell(\ell) = 1$. For each α choose any submultiset T_α of $\mathcal{T}^*$. Then

$$\sum_{\alpha \in \mathcal{A}_H} \sum_{b \in T_\alpha} v_\ell(b - d_\alpha) \le v_\ell(I_{p,H,r}).$$

Proof. With $u = 18h + 1$ from Theorem 30.8 put

$$I_{p,H,r} = \prod_{\alpha \in \mathcal{A}_H} \prod_{b \in \mathcal{T}^*} p^u (b - d_\alpha).$$

The product is an algebraic integer since $u \ge r$. Both multisets of values are invariant under field automorphisms and the factors are taken over their full product. Hence the integer is rational. As $|d_\alpha| \ge p^r - 1 - p^{-r} > 3$ and $|b| \le 3$, every factor is nonzero. Its absolute value is at most $6p^{u+r}$. By (30.4), adjoining all conjugates increases cardinality by at most another polynomial in $N(r+2)$. Taking logarithms proves the size bound. At $\ell \ne p$ all factors have nonnegative valuation and p^u is a unit. Summing over fewer factors proves the local assertion, including duplicate traces carried by different labels. □

Proposition 30.12 (Congruence length at a Hecke operator)**.** *Let $\mathcal{O}$ be the valuation ring of a finite extension of $\mathbb{Q}_\ell$ containing $d = d_\alpha$ and the trace values under consideration, where $\ell \ne p$. Write $e_\mathcal{O}$ for its ramification index. For any list $b_1, \dots, b_s$ of such values, set $t = (d, b_1, \dots, b_s)$ in $\mathcal{O}^{1+s}$, and let $\mathbb{T}$ be an $\mathcal{O}$-subalgebra of that product containing t and the diagonal scalars. Define the congruence ideal*

$$\eta(\mathbb{T}) = \{a \in \mathcal{O} : (a, 0, \dots, 0) \in \mathbb{T}\}.$$

It measures the finite index along the distinguished eigensystem in the joint algebra of eigenvalues. In the algebra generated by the one operator one has

$$\eta(\mathcal{O}[t]) = \left(\prod_{\beta \in \{b_1, \dots, b_s\}_{\mathrm{distinct}}} (d - \beta) \right) \mathcal{O}. \tag{30.5}$$

In particular $e_\mathcal{O}^{-1} \operatorname{length}_\mathcal{O} \mathcal{O}/\eta(\mathbb{T}) \le \sum_{j=1}^{s} v_\ell(d - b_j)$.

Proof. A polynomial $P \in \mathcal{O}[X]$ represents an element supported in the distinguished coordinate precisely when it vanishes at all the b_j. Since these values lie in $\mathcal{O}$, successive division by the distinct monic factors $X - \beta$ shows that their product divides P in $\mathcal{O}[X]$. At each division any other root remains a root because $\mathcal{O}$ has no zero divisors. Evaluating at d gives (30.5), including the reverse inclusion by taking the product itself. This ideal is nonzero by Theorem 30.11. Enlarging the algebra enlarges the ideal, and $\operatorname{length}_\mathcal{O} \mathcal{O}/(a) = e_\mathcal{O} v_\ell(a)$. The bound follows. □

This construction applies to the algebra generated by the indicated spherical trace on the two automorphic lists and the diagonal comparison parameters. The normalized trace is integral at $\ell \ne p$. On the newform block it is a sum of the identity and the normalized T_p, with the diamond operator for inverse trace. On the rank-three block it is the normalized $T^{\pm}$. Equivalently one can take either orientation in separate lists. The algebra of joint eigenvalues of additional good Hecke operators, when acting on the same lists and integral at ℓ, contains this single-operator algebra. The proposition applies to it as well.

Corollary 30.13 (The two auxiliary primes on the power interval)**.** *Take $N = M_x = 2^{l+2} F q d$, $t_x = \lfloor x^{401} \rfloor$, on the intervals of Volume IV. Fix distinct primes $p_1, p_2 \ge 7$ split in K and prime to $3 F d_K$. For sufficiently large intervals they are prime to N everywhere. Choose the operator*

at p_2 for $\ell = p_1$ and at p_1 otherwise. For every $r < t_x$ and every complex character α of G_N, form any algebra in Proposition 30.12 from a submultiset of the prescribed list at the chosen prime. Denote its ideal by $\eta_{\ell,r,\alpha}$, allowing the local coefficient ring $\mathcal{O}$ to vary. Then

$$\sum_{1\le r<t_x}\ \sum_{\alpha\in\widehat{G}_N}\ \sum_{\ell}(\log \ell)\,\frac{\operatorname{length}_{\mathcal{O}} \mathcal{O}/\eta_{\ell,r,\alpha}}{e_{\mathcal{O}}} \ \le\ C_{K,p_1,p_2}[N(t_x+2)]^{A''} \ \le\ \exp(x^{\varepsilon/3}) \tag{30.6}$$

for every sufficiently large x on these intervals.

Proof. Such fixed primes exist, for instance by Dirichlet's theorem in the progression $1 \bmod 3Fd_K$ after excluding its prime divisors. In Theorem 5.1 the auxiliary primes q, d tend to infinity with the initial scale. Thus they eventually avoid p_1, p_2. For a fixed orbit and twist, Propositions 30.12 and Theorem 30.11 bound the weighted sum over all ℓ by $\log|I_{p_1,H,r}| + \log|I_{p_2,H,r}|$, since $\sum_\ell v_\ell(I)\log\ell = \log|I|$. This is uniform over all the local choices. The orbits partition the characters, so $\sum_H |\mathcal{A}_H| = \varphi(N) \le N$. Sum next over $r < t_x$ and enlarge the polynomial exponent. Finally $\log N \ll_{\psi,\varepsilon} x^{\varepsilon/4}$ and $\log(2t_x) \ll \log(2x)$ on the common family. The logarithm of the middle bound is eventually at most $x^{\varepsilon/3}$. □

31 Orbits in the marked deformation space

31.1 The contributions of the scaling orbits

We calculate the terms on the characteristic-zero side of the proposed cotangent comparison for a graded algebra. Throughout this section $B = \bigoplus_{i\in\mathbb{Z}} B_i$ is a commutative graded $\mathcal{O}$-algebra whose degree-zero algebra B_0 is finite as an $\mathcal{O}$-module. Assume that B is generated over B_0 by finitely many homogeneous elements of degrees $-2, -1, 1, 2$. Let k be an algebraic closure of the fraction field J. Scaling by $t \in k^\times$ multiplies the value of a homogeneous element of degree j at a geometric point by t^j. Call an orbit *mixed* if its points have nonzero values on some positive-degree and some negative-degree elements. We assume the Zariski tangent space of $\operatorname{Spec}(B \otimes k)$ has dimension at most one at each mixed point. The Zariski tangent space at a k-point with maximal ideal $\mathfrak{m}$ is the dual of $\mathfrak{m}/\mathfrak{m}^2$.

Theorem 31.1 (The components at mixed points)**.** *There are finitely many mixed orbits, at most* $\dim_J(B_0 \otimes J)$*. Each is an open and closed reduced subscheme of* $\operatorname{Spec}(B \otimes k)$ *with graded coordinate ring $k[T^g, T^{-g}]$, where $g \in \{1, 2\}$ is the greatest common divisor of the absolute degrees of the generators nonzero at any of its points. Distinct mixed orbits have distinct restrictions to B_0.*

Proof. Put $C = B \otimes k$. Its degree-zero algebra C_0 is Artinian and is a product of finitely many local Artin k-algebras, one for each character of C_0. Their primitive idempotents also split C into open and closed parts. Fix a mixed point P in one part. Its nonzero coordinates among the chosen homogeneous generators include both signs: a homogeneous polynomial of positive (respectively negative) degree must use a generator of that sign in every monomial. Denote these coordinates by x_i and their degrees by d_i, indexed by a set S.

Consider any other closed point Q on the same part. It agrees on C_0 with P. If i, j have opposite signs, a product of positive powers of x_i, x_j has weight zero. Starting with an opposite-sign pair nonzero at P, this shows that pair is nonzero at Q; pairing either of them with each remaining coordinate shows that the nonzero support at Q is exactly S. Write $z_i = Q(x_i)/P(x_i)$ there. For any nonnegative integer vector (b_i) with $\sum b_i d_i = 0$ we have $\prod z_i^{b_i} = 1$, by equality on C_0. There exists such a zero-weight vector strictly positive in every coordinate: add, for each coordinate, a relation with a coordinate of opposite sign using positive exponents on the

pair. Adding a sufficiently large multiple of this vector proves the same multiplicative relation for *every* integral vector of weight zero. Consequently $(b_i) \mapsto \prod z_i^{b_i}$ factors through the degree map $\mathbb{Z}^S \to g\mathbb{Z}$. Choose a g-th root $t \in k^\times$ of the value at g. Then $Q(x_i) = t^{d_i} P(x_i)$ on S, and all remaining coordinates vanish at both points. This identifies Q with the translate of P.

The map from the coordinate ring of this part to $k[T, T^{-1}]$ given by the orbit of P has image $k[T^g, T^{-g}]$. Indeed the monoid generated by the supported degrees equals the group they generate: the strictly positive zero-weight relation just constructed turns every integral representation of a degree into a nonnegative one. The quotient map thus embeds this one-dimensional torus as a closed subscheme. All closed points of the part belong to it by the preceding paragraph. It remains to check reducedness, since equality on closed points alone would only detect the radical. At each closed point the local ring maps onto the local ring of this torus, so its dimension is at least one. Its embedding dimension is at most one by the tangent hypothesis; the local ring is Noetherian, and dimension is bounded by embedding dimension. It is therefore a regular local ring of dimension one, hence a discrete valuation ring. The displayed quotient has dimension one, so its localized kernel is zero, since a quotient of a discrete valuation ring by a nonzero ideal has dimension zero. The global kernel is zero as well: a nonzero module has an element with proper annihilator, surviving at a maximal ideal containing that annihilator. All maximal ideals of our finite-type k-algebra are closed points. This proves the scheme and grading assertions and the bound from the dimension of C_0. □

Let B_J^{mix} be the direct summand of $B \otimes J$ which after extension to k is the union of the mixed parts. It is defined over J: the sum of their primitive degree-zero idempotents is fixed by every automorphism of k/J, since being mixed is defined by homogeneous elements and their signs. For $j \in \mathbb{Z}$ let L_j be the image of B_j in its degree-j summand, and let $K_j = \ker(B_j \to L_j)$. An empty union gives $L_j = 0$.

Proposition 31.2 (The determinant in each weight)**.** *Each L_j is finite free over $\mathcal{O}$. Let $a \in B_0$ be nonzero at every mixed orbit. With one representative P per geometric mixed orbit and g_P as above, one has*

$$\operatorname{length}_{\mathcal{O}}(L_j / aL_j) = \sum_{P:\ g_P \mid j} v_\ell(a(P)). \tag{31.1}$$

There is an exact sequence

$$0 \longrightarrow K_j/aK_j \longrightarrow B_j/aB_j \longrightarrow L_j/aL_j \longrightarrow 0.$$

In particular the sum of the lengths in weights 2 and -2 on the mixed parts is exactly twice the sum of the trace valuations when a is the weight-zero trace difference.

Proof. The component of a finitely generated graded algebra in a fixed degree is a finite module over its degree-zero algebra in this situation. In fact exponent vectors of monomials of a prescribed weight have finitely many minimal elements under coordinatewise comparison. Otherwise their infinite antichain would contain an infinite coordinatewise nondecreasing subsequence, by taking a nondecreasing subsequence one coordinate at a time (an unbounded sequence in $\mathbb{N}$ has such a subsequence, as does a bounded infinite sequence). Subtracting a minimal vector lying below a given vector gives a nonnegative vector of weight zero. The finitely many minimal monomials therefore generate that component over B_0. It follows that B_j is finite over $\mathcal{O}$, and that its image L_j is finite and torsion-free, hence free. Its generic fiber is the entire degree-j component of the mixed summand, since extension to J commutes with the image. By Theorem 31.1 each geometric orbit contributes one basis line if $g_P \mid j$, and zero otherwise. On that line a acts by $a(P)$. Its determinant valuation counts the length of its

cokernel by elementary divisors; the nonzero eigenvalues are integral over $\mathcal{O}$ because $a \in B_0$. This proves (31.1). Finally, $aB_j \cap K_j = aK_j$ since multiplication by a is injective on L_j. This proves the exact sequence. Both 1 and 2 divide each of $2, -2$, giving the last assertion. □

The trace values in this computation refer to genuine generic orbits. The term K_j/aK_j includes contributions supported away from the mixed parts. It can have positive rank. A comparison between a cotangent quotient at the augmentation and the two modules in (31.1) therefore needs control of those other contributions and of the integral map. We next establish that comparison for a class of presentations in which the opposite weight has a single coordinate.

31.2 A graded presentation with one positive coordinate

Let $u_1, \dots, u_n$ have degree -1, let v have degree 1, and let $A = \mathcal{O}[u_1, \dots, u_n, v]$ with augmentation all variables zero. Every homogeneous element F_i of degree -1 can be written

$$F_i = \sum_{j=1}^{n} u_j f_{ij}(vu_1, \dots, vu_n), \qquad f_{ij} \in \mathcal{O}[t_1, \dots, t_n].$$

Indeed each monomial has one more u-factor than v-factors. The coefficients of its linear terms are uniquely defined even though the displayed expression may have more than one choice. Put $B = A/(F_1, \dots, F_s)$, $t_j = vu_j$, $G_i(t) = \sum_j t_j f_{ij}(t)$, and

$$R = \mathcal{O}[t_1, \dots, t_n]/(G_1, \dots, G_s), \qquad C_{-1} = \mathcal{O}^n / \sum_i \mathcal{O}(f_{i1}(0), \dots, f_{in}(0)).$$

The last group is exactly the quotient of the ambient degree--1 cotangent lattice by the differentials of the equations. Saturation of the square of the ambient augmentation ideal changes nothing, because the linear terms of a polynomial form a free $\mathcal{O}$-module.

Theorem 31.3 (An integral comparison in rank one on the positive side)**.** *Suppose R is finite over $\mathcal{O}$ and its generic fiber is reduced. Write T for its $\mathcal{O}$-torsion, $I = \ker(R \to \mathcal{O},\ t_j \mapsto 0)$, and let $a \in B_0$ vanish at the origin but be nonzero at every mixed orbit. Then*

$$\operatorname{length}_{\mathcal{O}} C_{-1} \le \operatorname{length}_{\mathcal{O}}(T/(T \cap I^2)) + \sum_{P \text{ mixed}} v_\ell(a(P)). \tag{31.2}$$

If R is torsion-free the orbit sum alone suffices. In this presentation the cotangent quotient is determined by the integral degree-zero ring, even though a first-order deformation in the individual u_j coordinates has constant values on all t_j at the origin.

Proof. The degree-zero piece of A is precisely the polynomial ring on $t_1, \dots, t_n$, by inspection of monomials. The degree-zero piece of the homogeneous ideal generated by the F_i is generated there by $vF_i = G_i(t)$, since every homogeneous coefficient of degree one is v times a polynomial in the t_j. Thus $B_0 = R$. In R the cotangent module at the origin is generated by the t_j modulo the linear terms of the G_i: multiplying an equation by a polynomial with zero constant term contributes only to the square of the augmentation ideal. These linear terms have coefficient rows precisely $(f_{ij}(0))_j$. This gives an integral isomorphism $I/I^2 \simeq C_{-1}$.

Over k a mixed point necessarily has $v \neq 0$ and some $u_j \neq 0$. Scale uniquely to $v = 1$; its coordinates $t_j = u_j$ solve the equations $G_i(t) = 0$ and are not all zero. Conversely any nonzero geometric point of R gives exactly this normalized mixed point. This is a bijection between mixed orbits and the geometric points of R distinct from the origin. The values of $a \in B_0 = R$ agree under it. Substitution in Theorem 30.7 proves (31.2). The last statement follows directly from $t_j = vu_j$ when the basepoint has $v = 0$. □

For instance take $n = s = 1$ and $F_1 = uf(uv)$ for a monic polynomial $f \in \mathcal{O}[t]$ with distinct roots in characteristic zero and $f(0) \neq 0$. Then $R = \mathcal{O}[t]/(tf(t))$ is finite free and generically reduced, and $C_{-1} = \mathcal{O}/f(0)\mathcal{O}$. For $a = t$ equality holds in (31.2): the mixed orbits are parametrized by the roots λ of f and $v_\ell(f(0)) = \sum_\lambda v_\ell(\lambda)$. Taking $f(t) = t - \ell^b$ realizes every positive length b.

The proposed arithmetic model has several positive coordinates, including weight two, and equations of both signs and weight zero. The hypotheses recorded in `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/OrbitVanishingLocal.lean`, line 103 allow that structure. The orbit conditions and the sum over representatives appear at lines 45–70 of the same file. The determinant formula here accounts for their generic mixed contributions. The special presentation in Theorem 31.3 realizes an integral cotangent comparison in one case. Applying the full comparison to all arithmetic components of Proposition 6.14 requires a smooth chart and equations for the actual local conditions, together with automorphy and trace control for their mixed orbits.

31.3 An upper bound from the mixed orbits

Let $\mathcal{O}$ be a discrete valuation ring with uniformizer π, fraction field J of characteristic zero, and algebraic closure k of J. Use the valuation on k extending that of J with $v(\pi) = 1$, and assume J is complete. Let $A = \bigoplus_{j \in \mathbb{Z}} A_j$ be generated over its Noetherian degree-zero ring $D = A_0$ by finitely many elements of weights $-2, -1, 1, 2$. Fix an $\mathcal{O}$-augmentation $x : A \to \mathcal{O}$ vanishing on nonzero weights. Write $\mathfrak{b} = \ker x$. Take homogeneous equations

equations	$F_1, \dots, F_r, F_*$	$G_1, \dots, G_s$	$E_1, \dots, E_{r'}$	$E'_1, \dots, E'_{s'}$	$Z_1, \dots, Z_m$
weight	-1	-2	1	2	$0.$

They all vanish at x. Write I for their ideal, $B = A/I$ and $I_0 = I \cap D$. Use the following hypotheses.

1. The invariant algebra $B_0 = D/I_0$ is finite over $\mathcal{O}$. In the cotangent space $(\mathfrak{b}/\mathfrak{b}^2) \otimes_{\mathcal{O}} J$, the F's span weight -1 and the G's span weight -2.
2. For every geometric fixed point of B different from x, its invariant character does not support both a purely positive and a purely negative point of B. A purely positive point kills all negative weights and has some nonzero positive value. Purely negative points are defined by reversing signs. A fixed point kills all nonzero weights.
3. At each mixed geometric point of B the tangent space has dimension at most one. A mixed point has a nonzero value of each sign. Fix $a \in D$ with $x(a) = 0$ and $a(P) \neq 0$ on all mixed points.

All geometric points here are over k. Hypothesis 1 makes $B \otimes k$ a finite-type k-algebra. Section 31.1 gives finitely many mixed orbits, with distinct invariant characters.

31.3.1 The three graded modules

Over D, set $U = A_{-2}$, $V = A_2$, $W = DF_*^2 + U^2V$, and let $D_1 = UV$ denote the ideal of D generated by these products. In particular $W \subset U$. Inside $A[T_0, T_1, T_2]$ form the algebra

$$R = \bigoplus_{d \in \mathbb{N}^3} U^{d_0} V^{d_1} W^{d_2} T_0^{d_0} T_1^{d_1} T_2^{d_2}.$$

Module products mean spans of products over D and the zeroth power means D. Each A_j is a finite D-module by the monomial argument in Proposition 31.2. Thus R is a Noetherian

standard multigraded algebra, meaning that its three unit degrees generate it over D. Denote the following list of sections by $\mathfrak{t}$:

$$F_i^2 T_0 \ (1 \le i \le r), \quad G_j T_0, \quad E_i^2 T_1, \quad E_j' T_1, \quad F_*^2 T_2. \tag{31.3}$$

Let H be any multihomogeneous ideal containing these sections and all Z_j. For $d \in \mathbb{N}^3$ put $M_d = (R/H)_d$. We call a prime of R relevant if it contains none of the three ideals generated by a whole unit-degree piece. The product $\mathfrak{h}$ of these three ideals is contained precisely in the irrelevant primes.

Lemma 31.4 (The invariant character at a relevant prime)**.** *If $\mathfrak{p} \supset H$ is relevant, then $I_0 \subset \mathfrak{p} \cap D$. For all sufficiently large d, coordinate by coordinate, M_d is finite over $\mathcal{O}$. Its geometric generic support consists only of invariant characters of mixed orbits.*

Proof. Choose $u \in U, v \in V, w \in W$ whose sections in the three colors survive modulo $\mathfrak{p}$. The identity $(F_*^2 T_0)(wT_2) = (wT_0)(F_*^2 T_2)$ puts $F_*^2 T_0$ in $\mathfrak{p}$. Every negative squared or weight-two equation $h \in U$ in the list therefore has $hT_0 \in \mathfrak{p}$. If $c \in V$, multiplication gives

$$(ch)(uT_0)(vT_1) = (hT_0)((cuv)T_1).$$

The two chosen sections survive in a domain, so $ch \in \mathfrak{p}$. For a negative weight-one equation F and $b \in A_1$, apply this argument to $h = F^2$, $c = b^2$ and take radicals. Interchanging the colors gives the same assertion for positive equations and opposite-weight coefficients. The Z_j already lie in $\mathfrak{p}$. On projecting any expression in the ideal of the equations to weight zero, only these opposite-weight coefficients occur. This proves the assertion for I_0.

It follows that $I_0 \mathfrak{h} \subset \sqrt{H}$. Taking a power in the Noetherian ring gives $I_0^b \mathfrak{h}^b \subset H$ for some b. Every degree d monomial with all $d_i \ge b$ is divisible by a product of b generators of each color. Consequently $I_0^b M_d = 0$. The ring D/I_0^b is finite over $\mathcal{O}$. Indeed its successive factors I_0^j / I_0^{j+1} are finite modules over D/I_0. This proves the finiteness assertion and leaves only finitely many possible geometric invariant characters $\sigma : D \to k$ in the support.

Suppose one such character kills D_1. We show that the fiber of R/H over σ has no relevant prime. Otherwise choose one, and take a field containing the fraction field of its residue domain. The following elementary identity holds for every $n \ge 0$:

$$A_{2n} = V^n, \qquad A_{-2n} = U^n.$$

In a monomial of even weight pair all the factors of odd weight. Each pair has weight 0 or ± 2. Pair off opposite factors of weight two and absorb their products into the coefficients in D. The unpaired factors have the specified sign and number.

The positive even subalgebra $\bigoplus_{n \ge 0} A_{2n}$ maps to the field just chosen by evaluating g of weight $2n$ as $g T_1^n$. It is nonzero on some $v \in V$ and kills E_i^2, E_j'. The full nonnegative subalgebra is integral over its even subalgebra, since the square of every odd homogeneous element has even weight and these elements generate the extension. By lying over, this character extends to the nonnegative subalgebra over an algebraically closed extension field. In particular it kills every E_i. It extends further to A by setting every negative weight to zero. To verify multiplicativity, take factors of weights $-i, j$ with $j \ge i > 0$. The square of their product belongs to $U^i V^j = D_1^i V^{j-i}$; its positive even evaluation is zero, since $\sigma(D_1) = 0$. Products of negative total weight already go to zero. This gives a purely positive point of B with invariant character σ. Applying the construction in the negative color, using $F_*^2 T_0 \in \mathfrak{p}$ in the fiber as above, gives a purely negative point of B. The equations, invariant character, and a required nonzero coordinate define finite-type locally closed conditions in $\operatorname{Spec}(B \otimes_D k)$. Existence over

an extension of k gives existence over k by the Nullstellensatz. Here the tensor product uses σ. The requirements to kill all weights of one sign can equivalently be imposed by their ideal in this Noetherian fiber.

The same invariant character with all nonzero weights set to zero gives a fixed point of B. Multiplicativity follows since $(fg)^2 \in D_1^i$ for factors of opposite weights $i, -i$. Hypothesis 2 now forces this fixed point to be x. A purely negative solution over x is impossible. On such a solution every product of two homogeneous elements of $\mathfrak{b}$ of total weight -1 vanishes, as one factor has nonnegative weight. The weight-zero factors in $\mathfrak{b}$ vanish by equality on D. Rational cotangent spanning by the F's then kills all of A_{-1}. In total weight -2 the only additional pair of negative weights is $(-1,-1)$. These products also vanish, and cotangent spanning by the G's kills A_{-2}. The algebra generators now preclude a nonzero negative value.

Thus in the finite-type fiber over each such σ the product of the three irrelevant ideals is nilpotent modulo H. All sufficiently high pieces of that fiber vanish by standard generation. For each degree, Nakayama's lemma then excludes support over σ on the finite geometric generic module. There are only finitely many σ to exclude. At every remaining character D_1 has nonzero image. Such a character of B_0 extends to a point of B over k: the fiber has nonzero degree-zero piece k and is of finite type. A nonzero product from UV forces both signs there. This proves the lemma. □

Theorem 31.5 (The upper estimate for all the coordinates)**.** *For all sufficiently large d, the kernel and cokernel of multiplication by a on M_d have finite $\mathcal{O}$-length and*

$$0 \le \operatorname{length}_{\mathcal{O}}(M_d/aM_d) - \operatorname{length}_{\mathcal{O}} M_d[a] \le 2^{r+1+r'} \sum_{P \text{ mixed}} v(a(P)). \tag{31.4}$$

Use one representative per geometric orbit. The bound is uniform in the multidegree after its three coordinates pass a threshold depending on the presentation and H.

Proof. At a surviving invariant character σ, localize $D \otimes k$ at its kernel, and call the resulting ring C. Some product uv, $u \in U, v \in V$, is a unit. In the localized ambient algebra u and v are therefore units. The modules U, V, W become Cu, Cv, Cu, respectively. For example any negative weight-two element divided by u has weight zero, and $u^2 v$ generates Cu. Extension from $\mathcal{O}$ to k and this localization are flat, so they preserve the defining inclusions of the graded pieces. The localized ring R is a polynomial ring over C in uT_0, vT_1, uT_2. Each piece of degree at least one in every coordinate, modulo H, is consequently a quotient of the cyclic module over C obtained by imposing

$$F_i^2/u, \quad F_*^2/u, \quad G_j/u, \quad E_i^2/v, \quad E_j'/v, \quad Z_j.$$

This module is the localized weight-zero piece of A modulo the ordinary equations with the $r+1+r'$ weight-one equations squared. To see the equality, take weight-zero parts of the ideal. Coefficients of opposite weight two are precisely multiples of u or its inverse as appropriate.

With the original equations, every localized weight piece has dimension at most one over k, by Theorem 31.1. Replacing one homogeneous equation f of weight b by f^2 at most doubles a common upper bound for these dimensions. Indeed in the algebra S with the other current equations imposed there is an exact sequence

$$(S/(f))_{j-b} \xrightarrow{f} (S/(f^2))_j \longrightarrow (S/(f))_j \longrightarrow 0.$$

Localization and projection to a weight are exact. Induction gives dimension at most $2^{r+1+r'}$ for the localized piece of M_d.

By Lemma 31.4, M_d is finite over $\mathcal{O}$. On its torsion submodule every endomorphism has kernel and cokernel of equal length. On its free quotient the determinant of multiplication by a is nonzero, since all generic eigenvalues are the values at surviving mixed characters. After scalar extension to k, the contribution to this determinant from a character σ is $\sigma(a)$ to the power of its local dimension. Indeed $a - \sigma(a)$ is nilpotent on that finite local summand. The kernel–cokernel exact sequence and elementary divisors now identify the left difference in (31.4) with the determinant valuation. Each $a(P)$ is integral since a acts through the finite invariant algebra on the original solution, so these valuations are nonnegative. The dimension bound and Lemma 31.4 give the assertion. □

The cancellation of the two lengths in (31.4) accounts for torsion supported at the residue characteristic. A lower bound for this same difference in terms of the cotangent lattice requires the formal chart at x.

Locations in the accompanying source. The definitions of the three modules and of the sections are in `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/GradedModulesLength/ToDegInjective.lean`, line 187. See also `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/GradedModulesLength/ExistsSecNotMem.lean`, line 63 for the relevant primes and `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/GradedModulesLength/TwoSigned.lean`, line 17 for the orbit estimate.

31.4 Euler characteristics in high multidegrees

We give the integral argument that will be used at the distinguished point. Let R be a Noetherian standard $\mathbb{N}^3$-graded algebra over a Noetherian $\mathcal{O}$-algebra D, and L a finite graded R-module. A tuple $\mathbf{s} = (s_1, \dots, s_b)$ of homogeneous sections of nonnegative multidegrees is admissible on L if the pieces of $L/(\mathbf{s})L$ have finite $\mathcal{O}$-length in all sufficiently large multidegrees. The Koszul complex of the tuple is the tensor product of the two-term complexes of multiplication $R(-\deg s_i) \to R$. With coefficients in L, write

$$\chi_d(\mathbf{s}; L) = \sum_{j=0}^{b} (-1)^j \operatorname{length}_{\mathcal{O}} H_j(\mathbf{s}; L)_d.$$

Here $L(-b)_d = L_{d-b}$ for a multidegree b. We use this expression only when all lengths are finite.

Lemma 31.6 (Positivity and passage to a quotient)**.** *For an admissible tuple, $\chi_d(\mathbf{s}; L)$ exists and is nonnegative for all sufficiently large d. It is additive on exact sequences when the tuple is admissible on the middle term. For an initial subtuple $\mathbf{u}$ followed by $\mathbf{v}$, one has eventually*

$$\chi_d(\mathbf{u}, \mathbf{v}; L) \le \chi_d(\mathbf{v}; L/(\mathbf{u})L).$$

A regular first section can be removed by passing to its quotient with equality.

Proof. First observe a support fact. If a finite graded module N has finite lengths in high degrees, so does any finite graded module supported on $\operatorname{Supp} N$. Indeed, sending a scalar to its products with homogeneous generators embeds $R/\operatorname{Ann} N$ into a finite sum of shifts of N. The same high-degree property holds modulo any power of $\operatorname{Ann} N$, by the filtration by ideal powers, whose factors are finite graded modules over $R/\operatorname{Ann} N$. A power of this annihilator kills any finite graded module supported on $\operatorname{Supp} N$, by the radical criterion for containment of closed sets in a Noetherian spectrum. A finite homogeneous set of generators proves the claim.

For $N = L/(\mathbf{s})L$ its support is $\operatorname{Supp} L \cap V(\mathbf{s})$ by Nakayama's lemma. Every Koszul homology is supported on this set, since a section that becomes a unit contracts its two-term complex. This proves finiteness and shows that subquotients of L inherit admissibility. The homology long exact sequence proves additivity degree by degree.

We prove positivity by induction on the number of sections. The empty tuple gives a length. A finite graded module over a Noetherian graded ring has a finite filtration with factors shifts of quotients by homogeneous primes. To recall the argument, select a nonzero homogeneous element whose annihilator is maximal among annihilators of such elements. Its annihilator is prime on homogeneous elements, as follows by applying maximality to a nonzero homogeneous multiple. It is prime on all elements by taking extreme nonzero homogeneous terms in the quotient using a translation-invariant lexicographic ordering of the degrees. Iterate in the quotient module. The ascending chain condition stops the construction. Additivity therefore reduces positivity to $L = R/\mathfrak{p}$, allowing a degree shift. An irrelevant prime gives zero in high degrees by standard generation. In the domain of a relevant prime, a nonzero first section is regular. Its complex resolves the quotient, so the induction hypothesis for the remaining sections applies.

If the first section is zero in this domain, the remaining tuple $\mathbf{v}$ is already admissible. For any first section of the same multidegree e, taking homology first in $\mathbf{v}$ shows that the Euler characteristic of the full complex is $\chi_d(\mathbf{v}; L) - \chi_{d-e}(\mathbf{v}; L)$. All lengths on this intermediate page are finite in high degrees and differentials preserve the alternating sum. Relevance supplies a nonzero section of degree e: take a product of surviving generators of the three colors, or one if $e = 0$. Replacing the zero section by this regular section preserves the Euler characteristic and gives positivity by induction.

Finally take homology first in a single initial section s of degree e. The two rows are its quotient and its kernel, the latter shifted by e and by one in homological degree. Hence

$$\chi_d(s, \mathbf{v}; L) = \chi_d(\mathbf{v}; L/sL) - \chi_{d-e}(\mathbf{v}; L[s]).$$

Both tuples on the right are admissible by the support observation. Positivity bounds the left side by its first term. Iteration proves the inequality. The same identity gives equality for a regular section. □

Lemma 31.7 (The contribution of a scalar factor)**.** *Suppose π acts injectively on L and every graded piece of $L/\pi L$ has finite length. Let y be homogeneous and assume $(\pi^h y, \mathbf{v})$ is admissible. Then in high multidegrees*

$$\chi_d(\pi^h y, \mathbf{v}; L) = \chi_d(y, \mathbf{v}; L) + h\, \chi_d(\mathbf{v}; L/\pi L).$$

For a function $f(d)$ let $\Delta_e f(d) = f(d) - f(d-e)$. If $H(d) = \operatorname{length}_{\mathcal{O}}(L/\pi L)_d$, then $\chi_d(\mathbf{v}; L/\pi L) = \prod_{s \in \mathbf{v}} \Delta_{\deg s} H(d)$.

Proof. Replacing $\pi^h y$ by y only shrinks the support of the final quotient, so admissibility follows from the proof of Lemma 31.6. The kernels of the two first sections agree because π is regular. Their quotients fit into

$$0 \longrightarrow L/yL \xrightarrow{\pi^h} L/(\pi^h y)L \longrightarrow L/\pi^h L \longrightarrow 0.$$

Regularity identifies all h successive factors of $L/\pi^h L$ with $L/\pi L$ without a shift in multidegree. Apply additivity and the two-row identity in the previous proof. On $L/\pi L$ the terms of the complex themselves have finite length, so their alternating sum is the Euler characteristic. Writing the terms as direct sums of shifts gives the finite differences. □

31.5 The cotangent length at the distinguished point

Use the presentation and three hypotheses of Section 31.3. For the lower estimate assume $r \geq 1$ and that x has the following homogeneous formal chart. There is an $\mathcal{O}$-algebra map from A to a formal power series ring over J whose constant term is x and which preserves weights monomial by monomial. Taking linear terms identifies $(\mathfrak{b}/\mathfrak{b}^2) \otimes J$ with the span of the variables. Their numbers are

weight	-1	-2	1	2	0
number	r	s	$r'+1$	$s'+1$	m.

The m equations Z_j span the rational cotangent of weight zero. These are hypotheses about the actual augmented algebra, including the existence of the map on all its relations.

Let $\mathfrak{b}^{(2)} = \{b \in \mathfrak{b} : \lambda b \in \mathfrak{b}^2 \text{ for some } 0 \neq \lambda \in \mathcal{O}\}$. The lattice C is the image of A_{-1} in $\mathfrak{b}/\mathfrak{b}^{(2)}$, and C_F is its quotient by the classes of all the F's. The ambient algebra is Noetherian, so $\mathfrak{b}/\mathfrak{b}^2$ is a finite module over $A/\mathfrak{b} = \mathcal{O}$. It follows that C is finite free of rank r. Rational cotangent spanning gives C_F finite length.

Theorem 31.8 (Integral cotangent length and the mixed orbits)**.** *Under these hypotheses one has*

$$\operatorname{length}_{\mathcal{O}} C_F \;\leq\; 2 \sum_{P \text{ mixed}} v(a(P)). \tag{31.5}$$

There is one term for each geometric scaling orbit of the actual solution algebra $B = A/I$.

Proof. Select a basis of C and make invertible $\mathcal{O}$-linear combinations of the $r+1$ negative weight-one equations. Smith normal form gives elements $y_i \in A_{-1}$ lifting a basis of C and integers $c_i \geq 0$ with

$$F_i - \pi^{c_i} y_i \in \mathfrak{b}^{(2)} \ (1 \leq i \leq r), \qquad F_* \in \mathfrak{b}^{(2)}, \qquad \operatorname{length}_{\mathcal{O}} C_F = \sum_{i=1}^{r} c_i.$$

The ordinary equation ideal and the hypotheses of Section 31.3 persist under this change. Use the new equations throughout the construction of R.

Choose rational homogeneous lifts of a linear coordinate basis in every nonzero weight, choosing the y_i in weight -1, and use the Z_j in weight zero. Their images are formal parameters. Indeed a substitution by power series with zero constant terms and invertible linear matrix is formally invertible: after the inverse is chosen up to degree $n-1$, its degree-n error is canceled using the inverse linear matrix. The coefficients stabilize in every degree. Both this substitution and its inverse preserve the weights, as the recursion does. Using these parameters and then setting the zero-weight parameters to zero gives a map ϕ to a power series ring in the nonzero-weight variables. It kills all Z_j. Each remaining variable is the exact image of a homogeneous element of $A \otimes J$. Elements of $\mathfrak{b}^{(2)}$ have zero linear term under ϕ, since after multiplication by a nonzero scalar they lie in a square of the augmentation ideal.

We may suppose $\phi(a) \neq 0$. If it vanishes, choose some F_i of nonzero linear term and an integral positive weight-one element z with nonzero linear term, which exists by the chart and clearing denominators. Replace a by $a + zF_i$. The value at x, all the values on B, and hence the desired bound stay the same. In the formal series domain $\phi(zF_i) \neq 0$.

Apply ϕ coefficientwise to R, retaining the three variables T_i, and write K for its kernel. The ring R/K is a domain, a is regular there, and $Z_j \in K$. Put $H = K + (\mathbf{t})$. Theorem 31.5 makes $(\mathbf{t}, a)$ admissible on R/K. Lemma 31.6, permutation of Koszul factors, and regularity of a give

$$\chi_d(\mathbf{t}; R/(K,a)) = \chi_d(\mathbf{t}, a; R/K) \leq \chi_d(a; R/H) \leq 2^{r+1+r'} \sum_{P} v(a(P)). \qquad (*)$$

Here the one-section characteristic is exactly the difference of lengths in Theorem 31.5.

For a monomial in the formal coordinates, let n_- be the sum of the absolute weights of its negative factors, counted with multiplicities. Define n_+ in the same way with its positive factors. With $d = (d_0, d_1, d_2)$, set

$$b_-(d) = 2d_0 + 4d_2, \qquad b_+(d) = 2d_1 + 2d_2.$$

The power series coefficient of an element of R_d has $n_- \geq b_-(d)$ in every monomial. Indeed a weight -2 term has $n_- \geq 2$. A term of $\phi(F_*)$ has $n_- \geq 2$, since in weight -1 a term with $n_- = 1$ is linear. Thus F_*^2 and U^2V both have $n_- \geq 4$, giving the assertion by products. The original signed weight in degree d is $2(d_1 - d_0 - d_2)$. Consequently the monomials with $n_- = b_-(d)$ have $n_+ = b_+(d)$.

In each degree retain exactly the terms with this pair of weights and carry along T^d. This operation on R is an algebra homomorphism into a polynomial ring over J: the minima add, so a product at the specified pair uses only terms at the respective minima. Let L be its image. In degree zero only constants remain. Thus D acts through $x : D \to \mathcal{O}$, and the map factors through $R/(K, a)$. Each L_d is a finite torsion-free $\mathcal{O}$-module, being the image of a finite D-module in a finite-dimensional J-space on which D acts through x. It spans the entire space of the monomials with that pair of weights over J. To check the last assertion, group the negative factors of such a monomial into blocks of absolute weight two. A variable of weight two is one block and two variables of weight one give one block. The total is even, so every factor is used. Make the same grouping for the positive factors. Use one negative block for each of the d_0 slots, one positive for each of the d_1 slots, and two negative blocks and one positive block for each of the d_2 slots. The last product belongs to $U^2V \otimes J$, and the others lie in $U \otimes J$ or $V \otimes J$, because the individual variables have exact rational homogeneous lifts. This realizes the prescribed monomial.

The images in L of the sections $F_i^2 T_0$ are π^{2c_i} times the images of $y_i^2 T_0$, since the retained terms of F_i in weight -1 are its linear terms. By Lemma 31.6, the quotient onto L and its kernel give $\chi_d(\mathbf{t}; R/(K, a)) \geq \chi_d(\mathbf{t}; L)$. On L strip off the r scalar factors using Lemma 31.7, permuting the complex to put each chosen factor first. The characteristic of the tuple with all factors stripped is nonnegative by Lemma 31.6. For every stripped factor the remaining tuple has the same list of degrees. Thus, writing e_i for the three unit vectors and $H_0(d) = \operatorname{length}(L/\pi L)_d$, we obtain

$$\chi_d(\mathbf{t}; L) \;\geq\; 2\Big(\sum_i c_i\Big)\Delta_{e_2}\Delta_{e_1}^{r'+s'}\Delta_{e_0}^{r+s-1}H_0(d).$$

It remains to compute this integer. For $p \geq 1, q \geq 0$, let $M_{p,q}(n)$ count monomials of total weight $2n$ in p variables of weight one and q of weight two. The finite free rank calculated above gives

$$H_0(d) = M_{r,s}(d_0 + 2d_2)\, M_{r'+1,s'+1}(d_1 + d_2).$$

The generating function by total weight is $(1-z)^{-p}(1-z^2)^{-q} = (1+z)^p(1-z^2)^{-p-q}$. Reading its even coefficients shows that for large n, $M_{p,q}(n)$ is a polynomial of degree $p + q - 1$ with leading coefficient $2^{p-1}/(p+q-1)!$. Indeed the even-indexed coefficients of $(1+z)^p$ sum to 2^{p-1}, and every resulting binomial polynomial has the displayed factorial as denominator of its leading term. Applying $r + s - 1$ differences in d_0 leaves the first factor equal to 2^{r-1}. Applying $r' + s'$ differences in d_1 leaves the second factor linear in $d_1 + d_2$, of slope $2^{r'}$. The final difference in d_2 gives $2^{r-1+r'}$. This also covers a constant first factor when $r = 1, s = 0$. Combining the lower estimate with $(*)$ and dividing by $2^{r+r'}$ yields (31.5). □

Corollary 31.9 (Strict classes represented in a graded model)**.** *Let S be an $\mathcal{O}$-module of finite length. Suppose for every h its subgroup $S[\pi^h]$ injects additively into augmentation derivations*

over $\mathcal{O}$ on this ambient algebra with values in $\mathcal{O}/\pi^h$. For infinite residue field assume these injections are $\mathcal{O}$-linear. Assume the derivations are supported in weight -1, kill the F's and lift to $\mathcal{O}$-valued augmentation derivations on the ambient algebra. Then $\operatorname{length}_{\mathcal{O}} S \le 2\sum_P v(a(P))$.

Proof. An integral lift kills the saturation of $\mathfrak{b}^2$, by the product rule and torsion-freeness of $\mathcal{O}$. Thus the injection factors through $\operatorname{Hom}_{\mathcal{O}}(C_F, \mathcal{O}/\pi^h)$. Its length is at most $\operatorname{length} C_F$ by elementary divisors. For a finite residue field, take cardinalities as in Proposition 30.6. With a linear injection the same inequality follows directly from additivity of length. Take h above the exponent of S and apply Theorem 31.8. □

For the fields of Proposition 6.14, this supplies the commutative algebra implication once one constructs the arithmetic presentation, proves its chart and point hypotheses, and realizes the strict classes as such liftable derivations. The size of the orbit sum must then be bounded using the specified arithmetic trace. The separate estimate on the theta periods requires its own argument on those same intervals.

Locations in the accompanying source. The general algebraic statement occurs in `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/OrbitVanishingLocal.lean`, line 111. The filtration and the integer measuring its monomials occur in `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/GradedModulesLength/ToDegInjective.lean`, line 222. The comparison of the two bounds appears in `Development/AutomorphicArithmetic/AutomorphicVarieties/CommutativeAlgebra/GradedModulesLength/AlgMemCoef.lean`, line 92.

32 The ordinary locus and its tangent spaces

32.1 The marked Galois representation at large primes

We realize the strict classes in an integral deformation algebra on the actual power intervals. In this section $E = E_x$, $N = 2^{l+2}Fqd$, $t = \lfloor x^{401} \rfloor$, and

$$\ell > \exp(x^{\varepsilon}/12), \qquad 1 \le r < t.$$

For large x this implies $\ell > 2t + 3$, $\ell > N$, and $\ell \nmid [E : \mathbb{Q}]$, by the bounds in Proposition 6.14. Let $\mathcal{O}$ be the ring of integers in a finite unramified extension of $\mathbb{Q}_\ell$ containing the character values of $\mathrm{Gal}(E/\mathbb{Q})$. Write J for its fraction field. Take a character α with $\alpha(c) = (-1)^r$, where c is a chosen complex conjugation, and put

$$\Psi = \kappa_\ell^r \alpha, \qquad \vartheta = \Psi^{-1}, \qquad K' = \mathbb{Q}(i).$$

Here K' is fixed even as the prime-count field E varies. Let η be its quadratic character. Fix a decomposition subgroup $G_\ell \subset G_{\mathbb{Q}}$ and its inertia subgroup. Inertia acts trivially on K' and on α. Its mod-ℓ cyclotomic image is $\mathbf{F}_\ell^\times$, since $\mathbb{Q}_\ell(\mu_\ell)/\mathbb{Q}_\ell$ is totally ramified of degree $\ell - 1$. Choose g_* in inertia mapping to a generator. Then $\overline{\Psi}(g_*)^2 \ne 1$ for every r, α under consideration. The same element works throughout that range at a fixed ℓ.

Put $d(g) = \operatorname{diag}(\Psi(g), 1, \Psi(g)^{-1})$. Use the group $\mathrm{GL}_3 \rtimes \{1, -1\}$ of Section 30.3, with involution $M \mapsto J_3 M^{-\mathsf{T}} J_3$. The base representation has value $(d(g), \eta(g))$. A lift keeps second component η and has matrix $M(g)$ reducing to $\overline{d}(g)$. We impose the following conditions on the matrices themselves. We fix $M(c) = 1$ and $M(g) = d(g)$ for every $g \in G_\ell$. For each $p \ne \ell$, choose inertia I_p and kill the subgroup $I_p \cap \ker \Psi \cap \ker \eta$ in the lifted representation. The kernel condition is unchanged by conjugating the choice of inertia. These equations impose a specific basis on the local representation at ℓ. Independent conjugation of the global lift is not taken in defining the functor.

Theorem 32.1 (A graded integral algebra for the marked lifts)**.** *There are an augmented Noetherian $\mathbb{Z}$-graded $\mathcal{O}$-algebra A, generated in weights $-2,-1,1,2$ over its complete Noetherian weight-zero ring, and a homogeneous ideal I of A, with the following properties.*

1. *The completion of $B = A/I$ at the maximal ideal of the residual basepoint represents the marked lifts with the above local conditions to complete Noetherian local $\mathcal{O}$-algebras with the same residue field. Its matrices are continuous for the respective adic topologies.*
2. *For every $h \geq 1$, the classes in $e_\alpha(\mathcal{S}_{\ell,r}(E) \otimes \mathcal{O})[\ell^h]$ are exactly the points of this functor over $\mathcal{O} \oplus \epsilon(\mathcal{O}/\ell^h)$, $\epsilon^2 = 0$, of the form*

$$M_z(g) = (1 + \epsilon z(g)(E_{21} - E_{32}))d(g).$$

These are lifts of the $\mathcal{O}$-valued basepoint under the square-zero projection. Their associated augmentation derivations of A are supported in weight -1 and each lifts to an $\mathcal{O}$-valued augmentation derivation of A.

Proof. All representations in question factor through the Galois group unramified outside the finite set S consisting of ℓ, the prime divisors of $2N$, and infinity. Let G be this group and H the kernel of its finite residual representation $(\overline{d}, \eta)$. Quotient G by the kernel of the maximal pro-ℓ quotient of H, which is characteristic in H. Call the quotient Γ. Over an Artinian coefficient ring the principal matrix congruence kernel is a finite ℓ-group, by its filtration by powers of the maximal ideal. Over a complete coefficient ring it is a pro-ℓ group by passage to the inverse limit. Thus all the lifts factor through Γ, as does the base representation. The proof of Theorem 30.4 applies to the finite residual field extension here. The mod-ℓ characters unramified outside the primes over S have finite dimension, by Kummer theory after adjoining μ_ℓ and finiteness of the class group and the rank of the S-units. Thus the pro-ℓ subgroup and then its finite extension Γ have finitely many topological generators.

Take generators $g_1, \ldots, g_n$ and the free profinite group on their lifts. Set $P = \mathcal{O}[[X_{s,ij}]]$, giving $X_{s,ij}$ weight $k_i - k_j$ for $(k_1, k_2, k_3) = (1, 0, -1)$. Assign the generators matrices $d(g_s) + (X_{s,ij})_{ij}$ and sign $\eta(g_s)$. The matrices are invertible since their residual determinants are units. The target over P is profinite, as P has finite quotients by powers of its maximal ideal. The universal property gives a continuous representation of the free profinite group. Impose matrix equality to one for every relator, along with the conditions at c, G_ℓ, I_p on chosen lifts in the free group. The resulting closed ideal represents the required functor. Indeed any choice of generator matrices gives a unique continuous substitution on P, and the equations are exactly those needed for the continuous homomorphism to descend and have the prescribed specializations.

We construct A explicitly to retain an ordinary grading. For every $j \in \mathbb{Z}$ let P_j consist of the formal series all of whose monomials have weight j, and take $A = \bigoplus_j P_j \subset P$. The ring P_0 is a quotient of a formal power series ring over $\mathcal{O}$ on finitely many zero-weight monomials. In fact the minimal nonzero exponent vectors of weight zero form a finite set by the coordinatewise monomial argument of Proposition 31.2. Every zero-weight monomial factors into these by induction on total degree. Choose one factorization of each monomial. Its length tends to infinity with total degree because the individual factors have bounded degree. This expresses any invariant series as a convergent series in those finitely many monomials. Hence P_0 is Noetherian, complete and local, with residue field that of $\mathcal{O}$. For arbitrary j, partition the monomials of weight j by a chosen minimal divisor in the same weight. Dividing by those finitely many choices leaves invariant series. This exhibits A as a finite-type algebra over P_0 generated by the entry variables of nonzero weight.

The entries of the matrix of an element of the free profinite group are homogeneous of the indicated entry weights. For words this follows by conjugation with $\mathrm{diag}(t, 1, t^{-1})$, which is fixed by the involution. Equivalently, products, inverses of matrices of homogeneous entries with invariant determinant, and the reflected inverse transpose preserve the entry weights. Inverse invariant determinants here are units of P_0. For general elements use continuity and the closedness of each P_j in P. Subtracting the specified diagonal constants gives homogeneous equations in A. Define $I \subset A$ to be their ideal. If $\mathfrak{m}$ is generated in A by ℓ and all entry variables, then $A/\mathfrak{m}^b = P/(\ell, X)^b$ for every b. Truncations give surjectivity. For injectivity, split each pure series lying in $(\ell, X)^b P$ into terms divisible by a chosen one of the finitely many monomials $\ell^{b-i} X^\nu$, $|\nu| = i \le b$, and divide termwise. The resulting coefficients in each group are still of pure weight and integral. This proves the equality of the ideals on contraction. The completion is thus P; quotient and completion commute for a finite ideal in a Noetherian ring. This proves the first assertion.

By character descent in Theorem 24.1, the component in the second assertion is the strict group over $\mathbb{Q}$ with coefficients $(J/\mathcal{O})(\vartheta)$. Here $\ell \nmid [E : \mathbb{Q}]$. The element g_* acts with $\vartheta(g_*) - 1$ a unit. Thus each ℓ^h-torsion class has a unique finite-coefficient cohomology class lifting it and a unique cocycle representative z with $z(g_*) = 0$. Locally at ℓ the invariants also vanish. Strictness therefore means that the finite-coefficient restriction is a coboundary. Evaluating it at g_* forces its defining element to be zero. Consequently $z|_{G_\ell} = 0$. At c the character is even, so the cocycle identity gives $2z(c) = 0$, hence $z(c) = 0$.

At inertia away from ℓ the image of ϑ has order prime to ℓ. Its invariants on $(J/\mathcal{O})(\vartheta)$ are either zero or the entire divisible module, so a strict divisible class again restricts to a finite-coefficient coboundary. In particular z vanishes on $I_p \cap \ker \Psi \cap \ker \eta$. Conversely vanishing there makes the restriction of z factor through the finite image of (Ψ, η) on inertia. This image has order prime to ℓ, so averaging makes its cocycle a coboundary. Together with vanishing at G_ℓ this is exactly the strict condition. By Proposition 30.5, the matrices in the statement obey the group law precisely for a ϑ-cocycle. They therefore give the asserted bijection with the marked root-valued points.

Their first-order changes on the generator variables occur only in weight -1. A derivation at the augmentation of A obtained from these points is the first-derivative functional on formal series at $X = 0$ with the specified changes on the entries. Lift these finitely many values arbitrarily from $\mathcal{O}/\ell^h$ to $\mathcal{O}$, taking zero on all other entries. The same linear-term functional defines the promised integral lift on the ambient algebra. This lifting statement concerns A; the original residue-valued derivation kills the equations in I. □

32.1.1 Scaling an algebraic point to an integral lift

Write $k = \overline{J}$. The following assertion applies to the algebra B just constructed whenever its invariant algebra is finite, and also to its further homogeneous quotients with this property and with the basepoint retained.

Proposition 32.2 (Integral representatives and their restrictions)**.** *If B_0 is finite over $\mathcal{O}$, every k-valued point of B has a scaling translate given by a continuous marked lift over the integers of a finite extension of J, reducing to the specified residual basepoint. On the mixed orbits, isomorphism of the associated three-dimensional representations of $G_{K'}$ identifies exactly one scaling orbit. Each such representation is absolutely irreducible or is the direct sum of a line in the middle coordinate and an absolutely irreducible two-dimensional corner.*

Proof. The invariant algebra is a quotient of the local ring P_0 and has the same residue field via the basepoint. Finiteness makes every invariant value integral. Its maximal ideal has a

power inside ℓB_0 because the local algebra modulo ℓ is Artinian. All invariant monomials in entry variables of positive total degree therefore have strictly positive valuation or vanish at the point. Write z_i for the nonzero values of its finitely many nonzero-weight variables, and b_i for their weights. We can choose a rational number s with $v(z_i) + b_i s > 0$ for every i. Indeed, for $b_i > 0 > b_j$, the invariant monomial $z_i^{|b_j|} z_j^{b_i}$ has strictly positive valuation, which says precisely that the lower endpoint $-v(z_i)/b_i$ is less than the upper endpoint $v(z_j)/|b_j|$. There are finitely many endpoints, and a missing sign imposes only one-sided conditions. Scaling by an element of $k^\times$ of valuation s puts all entries in the maximal ideal, including the weight-zero variables already controlled. All values lie in a finite extension since B_0 is finite and only finitely many variables are used.

Substitution into P now converges. It agrees with the given point on P_0, since invariant series there are series in invariant monomials and the character of B_0 is adically continuous. It agrees on the other generators by construction. The continued map thus kills all the relations of the completed ring and gives a continuous marked lift.

For its restriction $U(h) = M(h)$ on $H = G_{K'}$, one has

$$U(ch^{-1}c^{-1}) = J_3 U(h)^{\mathsf{T}} J_3, \qquad U(g_*) = \operatorname{diag}(\Psi(g_*), 1, \Psi(g_*)^{-1}).$$

The first identity follows from the semidirect product law and the marking at c. The second diagonal has three distinct eigenvalues. Every invariant subspace is therefore spanned by a subset of its coordinate axes, using the spectral projections, which are polynomials in $U(g_*)$. Form the directed graph of possible entries, with an edge from j to i if some $U(h)_{ij} \neq 0$. A coordinate subspace is stable exactly when no edge leaves its subset. The first identity pairs the edges across the antidiagonal with reversed orientation. In particular, each direction along the adjacent axes, if present, occurs on both consecutive pairs. A mixed point has both matrix weight signs: every homogeneous function is a polynomial in the entry variables over invariants, and every element outside H has matrix equal to that of its product with c. With adjacent edges in both directions the graph is strongly connected. With adjacent edges in only one direction mixedness forces the reverse corner edge, again giving strong connectivity. Either case implies irreducibility. Without adjacent edges the middle line separates, and both corner directions occur. The distinct outer eigenvalues then imply irreducibility on the corner.

If two restrictions are isomorphic, an intertwiner commutes with the same regular diagonal $U(g_*)$, so is diagonal, say $\operatorname{diag}(d_1, d_2, d_3)$. Set $\bar{i} = 4 - i$. Applying the intertwining equality to h and to $ch^{-1}c^{-1}$ gives $d_i d_{\bar{i}} = d_j d_{\bar{j}}$ whenever the ij entry is nonzero. If an adjacent edge occurs, $d_1 d_3 = d_2^2$, so the intertwiner differs by the scalar d_2 from a scaling matrix. In the corner case choose $t \in k^\times$ with $t^2 = d_1/d_3$; scaling by $\operatorname{diag}(t, 1, t^{-1})$ has the same effect on all entries. In either case the matrices at c agree, so all global generator matrices agree up to this scaling. Their invariant monomial values agree. Continuity on B_0 makes all invariant series values agree as well, so the two algebraic points differ by exactly a scaling action. The converse follows by conjugation. □

Proposition 32.3 (The arithmetic tangent space modulo scaling)**.** *Under the finiteness assumption in Proposition 32.2, let P be a mixed point with restriction $U : G_{K'} \to \mathrm{GL}_3(k)$. The tangent space at P, modulo the one-dimensional scaling direction, injects into the kernel of*

$$H^1(G_{K'}, \mathrm{End}(k^3)) \longrightarrow \prod_{v \mid \ell} H^1(K'_v, \mathrm{End}(k^3)) \times \prod_{v \nmid \ell\infty} H^1(I_v, \mathrm{End}(k^3)).$$

The coefficient action is conjugation by U. Continuous cocycles here are taken over finite coefficient extensions and then unioned in k.

Proof. At an integral representative an infinitesimal change $\dot{U}$ defines the usual continuous cocycle $h \mapsto \dot{U}(h)U(h)^{-1}$ by differentiating multiplication. To justify continuity in this algebraic tangent space, choose a finite coefficient extension containing the values of the point and of the derivation on finitely many invariant-ring module generators and entry variables. An $\mathcal{O}$-linear map on the finite invariant ring is bounded in this extension. Multiply the derivation by a common nonzero scalar to put its values on the generator variables in the ring of integers. Substitution of the variables plus their infinitesimal changes into the completed power series ring then converges in the ring of dual numbers over those integers, since their constant parts are in the maximal ideal. It agrees on the invariant series by continuity there, and hence on all the finite-weight functions. Undo the scalar change. The universal continuous matrices give the asserted continuous cocycle.

The cocycle vanishes on the chosen decomposition group intersected with $G_{K'}$ by the defining equations. At the other places above ℓ its restriction class also vanishes. Indeed on a conjugate decomposition group the deformation is conjugate to the fixed matrices there by its changed conjugating matrix, with the reflected inverse transpose if its sign is negative. The infinitesimal change of conjugation is a coboundary. Away from ℓ the restrictions to inertia factor through the finite quotient specified by the kernel condition, so their characteristic-zero cohomology class vanishes by averaging.

If the global class vanishes, write $\dot{U} = [X, U]$ for a matrix X. Equality at g_* forces X to be diagonal, with entries x_i. Differentiate the antidiagonal symmetry in Proposition 32.2. At every nonzero entry the resulting equation is $x_i + x_{\bar{i}} = x_j + x_{\bar{j}}$. With an adjacent edge this gives $x_1 + x_3 = 2x_2$, and the scalar part of X has zero commutator. In the corner case the diagonal matrix $\frac{1}{2}(x_1 - x_3)\,\mathrm{diag}(1, 0, -1)$ induces the same commutators. Thus in both cases the infinitesimal changes on the entries are exactly a multiple of the scaling vector. They determine the derivation on the invariant series by the continuity just proved. Every scaling vector is a coboundary, and the scaling vector is nonzero because a nonzero entry of nonzero weight at P has nonzero derivative under scaling in characteristic zero. This proves the injection. □

The construction realizes the strict classes for all primes in the displayed large range and gives their integral lifting property on the ambient algebra. Applying Theorem 31.8 additionally requires an ambient presentation with the prescribed tangent and equation counts, finiteness of its solution algebra in weight zero, and control of its fixed and mixed points. Proposition 32.3 identifies a precise sufficient arithmetic vanishing for the mixed tangent condition.

Locations in the accompanying source. For the universal representation compare `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/UnivEx.lean`, line 342. The two shapes and the isomorphism test occur in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/PtShape.lean`, line 85. The fixed local basis at ℓ in this section is available because the chosen marking belongs to inertia and every class being represented is strict there.

32.2 Excluding the two-dimensional corner

Keep the large-prime range and the fixed local matrices of Section 32.1. We work first with any continuous marked lift whose entries lie in the integers of a finite extension of J, with residual representation $(\overline{d}, \eta)$. Let U denote its restriction to $H = G_{K'}$, where $K' = \mathbb{Q}(i)$. The following exclusion also applies to points that acquire such an integral representative by scaling.

Lemma 32.4 (The determinant and the middle line)**.** *One has* $\det U = 1$. *If the middle coordinate line splits off as a direct summand, its character on* H *is trivial.*

Proof. Both characters reduce to one on H. At inertia away from ℓ their images are finite of order prime to ℓ, by the defining kernel condition. A root of unity of order prime to ℓ reducing to one is one, since its order is a unit and the principal units are pro-ℓ. At the chosen place above ℓ, the prescribed diagonal matrices have determinant one and middle entry one, on the entire decomposition group in H. The same statements hold at every place above ℓ. For conjugation within H this is immediate. Conjugation by c replaces a matrix by its reflected inverse transpose, which inverts its determinant and the character on a separated middle line. Hence the characters are everywhere unramified. Global class field theory makes every continuous everywhere unramified character of H factor through the ordinary ideal class group, since K' has no real places. That class group is trivial because the Gaussian integers are Euclidean. This proves both assertions. □

Theorem 32.5 (There are no integral mixed corner points)**.** *For all sufficiently large x on the common intervals, every mixed continuous marked lift in the range of Section 32.1 is absolutely irreducible on $G_{K'}$. Its restriction cannot be an irreducible representation induced from a character of an index-three subgroup. Moreover its crystalline Frobenius eigenvalues at a place of K' over ℓ have three different complex absolute values.*

Proof. The coordinate and graph argument of Proposition 32.2 applies to the given continuous representation itself. If it is reducible and mixed, write $U = \beta \oplus \lambda$ on the outer two coordinates and the middle line, respectively. Then β is absolutely irreducible. Lemma 32.4 gives $\lambda = 1$ and $\det \beta = 1$. For $h \in H$, with the two outer coordinates in their original order,

$$\beta(chc^{-1}) = J_2 \beta(h)^{-\mathsf{T}} J_2 = C_2 \beta(h) C_2^{-1}, \qquad J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the second equality, an inverse transpose in SL_2 is conjugation by the matrix of the standard alternating form. Multiplying that matrix by J_2 gives C_2 up to sign. Since $c^2 = 1$ and $G_{\mathbb{Q}} = H \rtimes \langle c \rangle$, assigning $c \mapsto C_2$ extends β to a continuous absolutely irreducible odd representation $\widetilde{\beta}$ of $G_{\mathbb{Q}}$. In fact its value at g is the outer block of $M(g)$ times C_2 when $\eta(g) = -1$, and just the outer block otherwise. This follows by writing $g = hc$ in the negative coset and using $M(c) = 1$.

The restriction to G_ℓ is exactly $\Psi \oplus \eta \Psi^{-1}$. The finite-order factors α, η there are unramified, so these are crystalline characters with distinct Hodge–Tate weights, differing by $2r > 0$. The global representation is unramified outside the finite set from Section 32.1. Its residual semisimplification is $\overline{\Psi} \oplus \overline{\eta}\,\overline{\Psi}^{-1}$. On inertia the ratio is $\overline{\kappa}_\ell^{2r}$. Since $2 \le 2r < \ell - 2$, this ratio is neither trivial nor either of the first powers $\overline{\kappa}_\ell^{\pm 1}$. In particular the two residual local characters are distinct. Pan's modularity theorem over $\mathbb{Q}$ for odd irreducible two-dimensional de Rham representations of distinct weights applies in this residually reducible case, with $\ell > 5$. It makes $\widetilde{\beta}$ a Tate twist of the representation of a classical cuspidal eigenform of weight $2r + 1$, the weight being determined by the difference of the two Hodge–Tate weights. A finite-order global twist, if present in the normalization, can be absorbed into the eigenform. [8]

For such a Tate twist, when the representation is crystalline at ℓ, its two crystalline Frobenius eigenvalues have the same absolute value under each complex embedding. To specify the classical inputs, local–global compatibility at the coefficient prime identifies the Frobenius-semisimplified Weil–Deligne parameter with the parameter of the local automorphic representation, in the common normalization. Crystallinity gives trivial inertia and zero monodromy, hence the local component is unramified. The two roots at a good prime for a holomorphic cuspidal newform of weight $k \ge 2$ have absolute value $\ell^{(k-1)/2}$ by Deligne. A common Tate twist and a reversal of Frobenius convention preserve the equality of the two absolute values. [9, 10]

Here the two crystalline characters have eigenvalues which, in either convention, are roots of unity times ℓ^{r} and ℓ^{-r}, in some order. This follows from their explicit expressions on G_{ℓ}: the unramified factors give roots of unity, and the cyclotomic character contributes ℓ^{-1} to crystalline Frobenius in the usual convention. Their absolute values differ. This contradiction excludes the corner.

For the induction assertion, an induced representation from an index-three character has three one-dimensional summands permuted by H. Since $\ell \neq 2$, K' is unramified at ℓ. The inertia inside H contains an element h with $\Psi(h)$ of infinite order, since its cyclotomic image contains $1 + \ell\mathbb{Z}_{\ell}$. Then $U(h^{6})$ fixes all three summands and is diagonal with three distinct eigenvalues. The summands are the three coordinate axes. But every $U(g)$ in the integral representative reduces to a diagonal matrix with three unit entries. A matrix permuting the axes nontrivially has a zero diagonal entry. Thus each individual line is preserved, contradicting irreducibility. This excludes in particular all imprimitive irreducible representations in this dimension.

Finally the restriction of U to a decomposition group in H at the chosen place $v \mid \ell$ has characters $\Psi, 1, \Psi^{-1}$. For residue degree f_{v}, the linearized crystalline Frobenius $\varphi^{f_{v}}$ therefore has absolute values $\ell^{f_{v}r}, 1, \ell^{-f_{v}r}$. This calculation requires no automorphy assumption in dimension three. □

If the invariant algebra in Proposition 32.2 is finite, Theorem 32.5 applies to every geometric mixed point of the algebra of Section 32.1. In particular the contribution of its reducible mixed points to the orbit sum is zero throughout this prime range, for every choice of separating trace. The last calculation of the theorem also excludes any irreducible point with a pure crystalline Frobenius parameter at v. Establishing purity, or an alternative bound for all the remaining irreducible points, would require additional arithmetic input. The fully fixed local representation used in this section is stronger than requiring crystallinity with prescribed Hodge–Tate weights. The cotangent inequality of Section 31.5 also needs the tangent and relation counts for the chosen local conditions. The identification of torsion-valued first-order points alone supplies neither those counts nor a cohomological length bound.

Locations in the accompanying source. The alternative over the imaginary quadratic field is stated in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/PtShape.lean`, line 85, and the passage from a corner to modular representations is treated in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/AdmissibleOfTypeII.lean`, line 103. The exclusion above uses the exact local matrices of Theorem 32.1.

32.3 The algebraic group of an irreducible mixed lift

Keep the fixed-matrix large-prime model of Section 32.1. Write $H = G_{\mathbb{Q}(i)}$, let c be its marked complex conjugation, and take an integral mixed lift as in Theorem 32.5. We work in its marked coordinates. Put $T(t) = \operatorname{diag}(t, 1, t^{-1})$ and let J_{3} be the antidiagonal matrix with entries one. Here the algebraic monodromy group means the Zariski closure of the image over the algebraic closure of the coefficient field.

Theorem 32.6 (The dense case and the even two-dimensional case)**.** *For such a lift U on H, exactly one of the following holds.*

1. *The algebraic monodromy group of U is SL_{3}.*
2. *Its algebraic monodromy group is $\mathrm{SO}(J_{3})$ in the marked coordinates. There is a continuous absolutely irreducible two-dimensional representation ρ of $G_{\mathbb{Q}}$, over the integers of a finite*

coefficient extension, with

$$\det \rho = \Psi, \qquad \rho(c) = I_2, \qquad \bar{\rho} = \operatorname{diag}(\bar{\Psi}, 1), \qquad \rho|_{G_\ell} = \operatorname{diag}(\Psi, 1).$$

It kills $I_p \cap \ker \Psi \cap \ker \eta$ at every $p \neq \ell$. In the explicit basis below, U is the restriction of its conjugation action on trace-zero matrices.

Conversely, a representation with the properties in the second alternative whose trace-zero conjugation action is absolutely irreducible on H gives an integral mixed point of the fixed local model. In particular the parity in the second alternative is forced.

Proof. By Lemma 32.4, the closure G is contained in SL_3. By Theorem 32.5 it acts irreducibly. Its identity component G^0 contains the torus of matrices $T(t)$: the inertia image at ℓ contains infinitely many such matrices, hence is Zariski dense in the one-dimensional torus. The unipotent radical of G^0 is normal in G. It has a nonzero fixed vector by triangularization; the fixed subspace is G-stable, hence all of the representation. Thus this radical is trivial. In characteristic zero the representation of the connected reductive group G^0 is semisimple. Its isotypic components are permuted transitively by G, since the sum over any orbit is invariant. More than one component would give three permuted lines in dimension three. The infinite inertia element and reduction argument in Theorem 32.5 excludes even a permuted system of three lines. If there is just one isotypic component of one-dimensional type, G^0 consists of scalars, contrary to its torus. Thus G^0 acts irreducibly. Its connected center acts by scalars and is trivial in SL_3, so it is semisimple.

Its rank is one or two. In rank two, after a change of basis, its maximal torus is the full determinant-one diagonal torus. Each root space of the ambient Lie algebra is one-dimensional and has a distinct torus character. The Lie algebra of G^0 consists of the diagonal algebra and a collection of these root spaces. The roots of a connected semisimple group occur with their negatives, so the presence of E_{ij} entails that of E_{ji}. The graph of the pairs present must be connected: otherwise a sum of coordinate lines is invariant under the Lie algebra and therefore the connected group in characteristic zero. A connected graph on three vertices gives all the root spaces by taking brackets. Thus $G^0 = \mathrm{SL}_3$. In rank one the semisimple Lie algebra is $\mathfrak{sl}_2$; its irreducible representation of dimension three is the symmetric square of the standard representation. These elementary characteristic-zero structure facts equivalently follow from the rank-one root system and the highest-weight strings for $\mathfrak{sl}_2$. The symmetric square preserves the nondegenerate discriminant form on binary quadratics. Schur's lemma makes its invariant symmetric form unique up to scalar. The connected image is its special orthogonal group, since both have dimension three. It remains to determine the form and how the whole image acts on it.

Write S for a matrix of this form in the marked coordinates. Its line is preserved by G, which normalizes G^0. Each matrix $U(h)$ multiplies the form by a scalar $\nu(h)$ with $\nu(h)^3 = 1$, by taking determinants. The torus $T \subset G^0$ preserves S, so it has only the entries $S_{13} = S_{31} = u$ and $S_{22} = v$, with $uv \neq 0$. After finite coefficient extension and common scaling, S is integral with at least one unit entry. Every $U(h)$ reduces to $T(\bar{\Psi}(h))$, which preserves the reduction of the form. Thus $\nu(h)$ reduces to one. Since $\ell > 3$, the reduction of cube roots of unity is injective, and $\nu(h) = 1$. The group G is equal to $G^0 = \mathrm{SO}(S)$.

The marked polarization now gives, for $h \in H$,

$$U(chc^{-1}) = J_3 U(h)^{-\mathsf{T}} J_3 = (J_3 S) U(h) (J_3 S)^{-1}.$$

Because $c^2 = 1$, the square of $J_3 S = \operatorname{diag}(u, v, u)$ centralizes an absolutely irreducible representation. Thus $v/u = \delta \in \{1, -1\}$. Divide the form by u, so its matrix is $S_\delta = J_3 \operatorname{diag}(1, \delta, 1)$. Put

$T_\delta = T(\delta) = \mathrm{diag}(\delta, 1, \delta)$. It differs from $J_3 S_\delta$ by a scalar, belongs to $\mathrm{SO}(S_\delta)$, and has square one. The semidirect decomposition $G_{\mathbb{Q}} = H \rtimes \langle c \rangle$ therefore extends U to $W : G_{\mathbb{Q}} \to \mathrm{SO}(S_\delta)$ by $W(c) = T_\delta$. If $g \notin H$, this extension has matrix $M(g)T_\delta$, because the original marking was $M(c) = I_3$.

Let $\phi = \Psi$ for $\delta = 1$ and $\phi = \eta\Psi$ for $\delta = -1$. We have $\bar{W}(g) = T(\bar{\phi}(g))$ globally and $W(g) = T(\phi(g))$ on G_ℓ. We lift W to a two-dimensional representation, keeping both equalities in these bases. Map the basis vectors for S_δ to

$$E_{12}, \qquad \mathrm{diag}(1, -1), \qquad (2/\delta)E_{21}$$

in $\mathfrak{sl}_2$. The trace pairing $(X, Y) \mapsto \mathrm{tr}(XY)$ in this basis is $(2/\delta)S_\delta$, and the basis is integral since $\ell > 2$. A determinant-one isometry of this pairing preserves the alternating volume form $\mathrm{tr}([X, Y]Z)$. Nondegeneracy of the pairing shows it preserves the bracket. It extends by fixing the identity to an algebra automorphism of M_2, using $XY = \frac{1}{2}\mathrm{tr}(XY)I + \frac{1}{2}[X, Y]$ for trace-zero X, Y. Every automorphism of the full matrix algebra over a field is conjugation by an invertible matrix, unique up to scalar. This can also be seen by transporting the two matrix-unit idempotents and then choosing generators of their images so that the transported off-diagonal units act as the usual ones.

Use a finite coefficient field containing the entries of the model and the chosen form, with integer ring R. As $W(g)$ preserves the trace-zero lattice and its inverse does too, the conjugating matrix preserves $M_2(R) = RI \oplus \mathfrak{sl}_2(R)$ by conjugation. Its elementary divisors must have equal valuation: in a diagonal elementary-divisor basis the two off-diagonal matrix units acquire opposite valuations. Thus the conjugating matrix has a representative in $\mathrm{GL}_2(R)$. Its residual class is that of $\mathrm{diag}(\bar{\phi}(g), 1)$, by the formula for $\bar{W}$ and the scalar centralizer of the full matrix algebra. Scale to a representative A with exactly this reduction. The scalar $\phi(g)/\det A$ is a principal unit, so it has a unique square root congruent to one since two is invertible in R. Multiplication of A by that root gives the unique representative $\rho(g)$ of determinant $\phi(g)$ and the specified reduction. Uniqueness makes this assignment multiplicative. It is continuous: the compact set of integral matrices with the specified diagonal residual shape and second diagonal residue one maps continuously and injectively under $A \mapsto (\mathrm{Ad}^0 A, \det A)$ to a Hausdorff space, hence has continuous inverse on its image. Injectivity uses that a scalar of square one and residue one is one. Consequently $\rho|_{G_\ell} = \mathrm{diag}(\phi, 1)$ exactly, $\rho(c) = \mathrm{diag}(\delta, 1)$, and ρ kills each of the inertia kernels in the statement, where both W and ϕ are trivial. Its restriction to H is absolutely irreducible: a preserved line in two dimensions gives a preserved line of nilpotent trace-zero endomorphisms with image in that line and kernel containing it.

If $\delta = -1$, this is an odd two-dimensional representation of $G_{\mathbb{Q}}$. It is finitely ramified and is crystalline at ℓ with distinct Hodge–Tate weights differing by $r \geq 1$. The regular odd Fontaine–Mazur theorem over $\mathbb{Q}$ for $\ell \geq 5$, including the residually reducible case, makes it a Tate twist of a classical cuspidal eigenform. Pan's Theorem 1.0.4 applies here also for $r = 1$. [8] Local–global compatibility and purity as in Theorem 32.5 then give equal complex absolute values for the two crystalline Frobenius eigenvalues. The explicit characters $\phi, 1$ at ℓ have absolute values $\ell^{\pm r}$ and one, since all finite-order factors of ϕ are unramified there. This excludes $\delta = -1$. For $\delta = 1$, $S_\delta = J_3$, $\phi = \Psi$ and $\rho(c) = I_2$, giving the second alternative.

Conversely, in the same basis with $\delta = 1$ let $W = \mathrm{Ad}^0 \rho$. Since W takes values in $\mathrm{SO}(J_3)$, the inverse-transpose involution with antidiagonal J_3 acts trivially on its image. Thus $g \mapsto (W(g), \eta(g))$ is a homomorphism to the marked semidirect product. It has $M(c) = I_3$, the correct residual matrices, the fixed local diagonal matrices and the required triviality on inertia kernels. Irreducibility on H forces nonzero entries of both weight signs, since otherwise its matrices would all be upper or all lower triangular. It is therefore mixed. It cannot have monodromy SL_3, completing the converse. □

The non-dense case consequently asks for control of an even, residually split, regular two-dimensional representation with fully split local matrices. Odd modularity supplies no contradiction for this even representation. The dense case is a separate three-dimensional problem. Theorem 32.6 classifies the possible characteristic-zero images of the fixed local model; the integral relations and their weighted lengths in Section 31.5 still require an estimate. These assertions use the stricter fixed local matrices throughout.

32.4 Strict classes on the ordinary component at the coefficient prime

Use the large primes and character summands of Section 32.1. Thus $\ell > \exp(x^{\varepsilon}/12)$, $1 \le r < \lfloor x^{401} \rfloor$, and x lies in a sufficiently distant common interval. Write $\Psi = \kappa_{\ell}^{r}\alpha$ and let η be the quadratic character of the auxiliary imaginary field K' of that section. Both finite characters are unramified at ℓ, their orders are prime to ℓ, $\Psi(c) = 1$ and $\eta(c) = -1$ at complex conjugation. Let $\mathcal{O}$ be the coefficient DVR, J its fraction field, ϖ a uniformizer, and $D = J/\mathcal{O}$. In the notation of this section $d(g)$ denotes $\operatorname{diag}(\Psi(g), 1, \Psi(g)^{-1})$. Our semidirect product has involution $\theta(A) = J_3 A^{-\mathsf{T}} J_3$, where J_3 reverses the three coordinates. The base representation is $r_0(g) = (d(g), \eta(g))$. Choose g_* in inertia at ℓ as in Section 32.1. The three residual diagonal entries of $d(g_*)$ are distinct.

We describe the local ideal used in the source. Index rows and columns by $0, 1, 2$, adding indices on the list of characters modulo three. Write $\chi_0 = \Psi$, $\chi_1 = 1$, $\chi_2 = \Psi^{-1}$. Let U be the complete local $\mathcal{O}$-algebra representing framed lifts of the local residual representation with prescribed sign η, and $x_0 : U \to \mathcal{O}$ the point r_0. The framed construction in the proof of Theorem 32.1 also gives this ring for the local group. The only finiteness needed there is finite generation of the maximal pro-ℓ quotient of the kernel of the finite residual image. Locally its characters of order ℓ form a finite group by local class field theory over the finite extension cut out by the residual representation. The pro-ℓ generator criterion gives the finiteness. Matrix entries of a finite set of generators then give a complete Noetherian power series presentation as in that proof.

Let $M(g)$ be the matrix coordinate of the universal lift. For inertia elements s write $S_a(s) = M(s) - \chi_a(s) I_3$. The ordinary ideal I_{ord} is generated by the coefficients of $\det(X - M(s)) - \prod_a (X - \chi_a(s))$ and all entries of

$$S_0(s_0')S_1(s_1')S_2(s_2'), \qquad s_i' \in I_{\ell}. \tag{32.1}$$

In the source the inertia elements are also required to lie in the kernels of η and of the finite inertial-type character. For these large primes the latter character can be taken unramified, and both kernels contain I_{ℓ}.

Choose $s_0 \in I_{\ell}$ of infinite cyclotomic image and an element γ in the two kernels that induces a positive power of arithmetic Frobenius on the residue closure. Such choices exist because the cyclotomic image on inertia is open and the two finite characters are unramified. Set

$$\begin{aligned} c_a &= (\chi_a(s_0) - \chi_{a+1}(s_0))(\chi_a(s_0) - \chi_{a+2}(s_0)), \\ t_a &= \operatorname{tr}\left(S_{a+1}(s_0) S_{a+2}(s_0) M(\gamma)\right), \\ q_a &= c_{a+1} t_a - \kappa_{\ell}(\gamma) c_a t_{a+1}, \qquad \Delta = \ell q_0 q_1. \end{aligned}$$

Saturate I_{ord} by the powers of Δ, take the radical, and denote the result by I_{cr}. The local ideal I_t is the intersection of the minimal primes of I_{cr} contained in $\ker x_0$. The framed point x_0 is unique with its specified matrices by representability. These are the saturation and component choices of `Development/AutomorphicArithmetic/AutomorphicVarieties/PadicHodgeTheory/IsInertiaInfiniteCyclotomic.lean`, line 93 and lines 112–135 of that file.

Lemma 32.7 (A family on a component through the marked local point)**.** *The set of primes defining I_t is nonempty. The local point r_0, its unramified diagonal variation $\Psi \mapsto \Psi\xi$ through the identity, and the two lower simple-root conjugations of that variation all lie on a common component through x_0.*

Proof. Let $n(g) \in \mathbb{Z}_\ell$ be the unramified exponent given by the projection $\widehat{\mathbb{Z}} \to \mathbb{Z}_\ell$ sending arithmetic Frobenius to one. Over $B = \mathcal{O}[[T, u, v]]$ put $\xi(g) = (1+T)^{n(g)}$, by the binomial expansion. It gives a continuous character: its coefficients are integral and continuous in $n(g)$, and the character identity follows from integer exponents by density. Let E_{ab} denote the matrix units and put

$$X_- = E_{10} - E_{21}, \qquad X_+ = E_{10} + E_{21}, \qquad H = I_3 + uX_- + vX_+.$$

Replace Ψ by $\Psi\xi$ in r_0, then conjugate the result in the semidirect product by $(H, 1)$. The diagonal formula defines a representation because θ fixes $\mathrm{diag}(a, 1, a^{-1})$. Its specialization at $T = u = v = 0$ is r_0. This determines $U \to B$. On inertia and at γ the sign is positive, so all the ordinary equations and traces are invariant under this conjugation. On inertia the diagonal is unchanged. Each diagonal entry of (32.1) has one zero factor. Hence I_{ord} maps to zero.

For the unconjugated family let $d_a(T)$ be its a-th diagonal entry at γ. The two shifted factors of $S(s_0)$ in the definition of t_a have product equal to $c_a E_{aa}$. Thus

$$q_a = c_a c_{a+1}(d_a(T) - \kappa_\ell(\gamma) d_{a+1}(T)).$$

Each $c_a \neq 0$ because s_0 has infinite cyclotomic image. For $a = 0, 1$, the ratio of the two diagonal entries is $\Psi(\gamma)(1+T)^{n(\gamma)}$, a nonconstant series since $n(\gamma)$ is a positive integer. Thus Δ has nonzero image in the domain B, including when its value at x_0 is zero. The prime kernel of $U \to B$ contains the saturated radical and is contained in $\ker x_0$. A minimal prime of I_{cr} contained in that kernel therefore occurs in I_t. The map to B factors through that component, proving the assertions. In particular $I_{\mathrm{ord}} \subseteq I_t \subseteq \ker x_0$. □

Let W be a finite $\mathcal{O}$-module. Use $\mathcal{O} \oplus W$ for the augmented ring in which the ideal W has square zero. A framed deformation of weight minus one has matrices

$$M(g) = (I_3 + u(g)X_- + v(g)X_+)d(g), \qquad u(g), v(g) \in W.$$

Set $\vartheta = \Psi^{-1}$. The involution on the Lie algebra acts by $-J_3 X^{\mathsf{T}} J_3$, with eigenvalues $+1$ on X_- and -1 on X_+. Conjugating by $d(g)$ multiplies these two matrix directions by $\vartheta(g)$. Consequently the representation condition is exactly that u and v be continuous cocycles for $W(\vartheta)$ and $W(\eta\vartheta)$, respectively. We use the convention $z(gh) = z(g) + gz(h)$.

Theorem 32.8 (The local component detects strictness on the two root lines)**.** *Such a local weight-minus-one deformation factors through U/I_t if and only if both cocycles are coboundaries over $G_{\mathbb{Q}_\ell}$. This holds over every finite module W at every indicated prime of the same family.*

Proof. If the cocycles are local coboundaries, the representation is a conjugate of r_0 by $(I_3 + uX_- + vX_+, 1)$ for some $u, v \in W$. Indeed conjugation by $I_3 + Y$ contributes the cocycle $Y - gY$, and its sign can be changed by negating Y. Specialize the family of Lemma 32.7 to $T = 0$ and these two values in W. This proves the required factorization integrally.

Conversely every such factorization satisfies (32.1). Write $\lambda = \Psi(g_*)$; the differences of $\lambda, 1, \lambda^{-1}$ are units in $\mathcal{O}$. On both lines the action of g_* is multiplication by λ^{-1}. By conjugating locally in the two root directions we can therefore make both cocycles zero at g_*. The inertia

equations still hold, since on inertia conjugation acts on all their matrix factors by the same ordinary conjugation.

In (32.1) take $s'_0 = s'_2 = g_*$ and $s'_1 = s$, an arbitrary inertia element. The first and last factors are diagonal. The $(1,0)$ entry of the product is

$$(1-\lambda)(\lambda-\lambda^{-1})M(s)_{10}.$$

The $(2,1)$ entry is $(\lambda^{-1}-\lambda)(1-\lambda^{-1})M(s)_{21}$. The scalar coefficients are units. Thus both entries of $M(s)$ vanish. Its diagonal characters are units and two is invertible, so both normalized cocycles vanish on all inertia. For $g \in G_{\mathbb{Q}_\ell}$, compare their values at $g_*g = g(g^{-1}g_*g)$. The conjugate inertia element on the right has value zero. Hence $(\lambda^{-1}-1)u(g) = 0$ and likewise for $v(g)$. The same unit proves that both cocycles now vanish on the full local group. Undoing the local conjugation proves that the original classes were coboundaries. □

Corollary 32.9 (The global root cotangent quotient at the marked point)**.** *Keep the inertia kernels away from ℓ in Section 32.1, use the local ideal I_t at ℓ and impose the source's two marks described below. Write C_{-1} for the quotient of the augmented cotangent module that represents its deformations supported in the two lower simple-root entries. With $S(\mathbb{Q}, D(\xi))$ the group of global classes unramified away from ℓ and strict at ℓ, there is an $\mathcal{O}$-linear isomorphism*

$$\mathrm{Hom}_{\mathcal{O}}(C_{-1}, D) \simeq S(\mathbb{Q}, D(\vartheta)) \oplus S(\mathbb{Q}, D(\eta\vartheta)). \tag{32.2}$$

In particular the desired complementary character summand on E_x, under the large-prime descent of Theorem 32.1, occurs with its full integral length.

Proof. The marks require $M(c) = I_3$ and the $(0,1), (1,0)$ entries of $M(g_*) - M(cg_*^{-1}c)$ to vanish. These are the equations in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/UnitaryThreeReflectionRight.lean`, line 281. At c the coefficient actions are $+1$ on u and -1 on v. Thus $u(c) = 0$ already, and a unique odd-line coboundary adjustment sets $v(c) = 0$, since two is a unit. With this mark satisfied the second matrix at g_* in the difference is $J_3 M(g_*)^{\mathsf{T}} J_3$. The upper entry vanishes automatically and the lower equation is

$$(\lambda+1)u(g_*) + (\lambda-1)v(g_*) = 0.$$

It determines the remaining even-line coboundary adjustment uniquely: its coefficient is, up to sign, $(\lambda+1)(\lambda^{-1}-1)$, a unit. Thus every pair of global cohomology classes has exactly one marked representative.

At a prime other than ℓ, the marked representation kills the prescribed inertia kernel if and only if the restrictions of both classes are coboundaries on inertia. Indeed the kernel is the simultaneous kernel of η, Ψ on inertia and the quotient has order prime to ℓ. The two characters act trivially on the kernel, and averaging kills the first cohomology of the finite quotient. The converse follows by restricting a coboundary to the kernel. Theorem 32.8 supplies the condition at ℓ. This proves the corresponding assertion for every finite coefficient module W.

For precision, form the global framed ring with these conditions as in Theorem 32.1, taking the closed ideal generated by the image of I_t from the local ring in place of the fixed local matrix equations. Impose the two marks above. If $\mathfrak{a}$ is its augmentation ideal, maps to $\mathcal{O} \oplus W$ over the basepoint are $\mathcal{O}$-linear maps $\mathfrak{a}/\mathfrak{a}^2 \to W$. Quotient by the classes of all entries of $M(g)d(g)^{-1} - I_3$ outside the two allowed positions, for all g. This defines C_{-1} and proves the claimed representation of root deformations. The module is finite type over $\mathcal{O}$, by Noetherianity.

Take $W = D[\varpi^m]$ and pass to the union. Globally and at ℓ, the invariants of $D(\xi)$ vanish for either ξ because $\xi(g_*) - 1$ is a unit. On inertia away from ℓ the invariants are either zero or the

whole divisible group, since the scalar character has order prime to ℓ. The multiplication-by-ϖ^m cohomology sequence therefore identifies finite-coefficient strict classes with the ϖ^m-torsion of the divisible-coefficient strict group. Taking these unions gives (32.2). The character descent in Section 32.1 applies to ϑ on the same E_x and in the same twist range. □

The equation of the ordinary locus is essential in the converse: the proof uses a unit multiplying each prohibited inertia entry over every $\mathcal{O}/\varpi^m$. A characteristic-zero tangent-space dimension by itself would give no such conclusion. The remaining global estimate requires an upper bound for the length of C_{-1} as the field, twist and coefficient prime vary.

32.5 An integral local chart at the large coefficient primes

Continue with U, I_{ord}, I_t and the three diagonal characters of Section 32.4. The residue field is $k = \mathcal{O}/\varpi$. On $\mathfrak{n}$, the strictly upper triangular matrices over k, let $G_{\mathbb{Q}_\ell}$ act by the adjoint action of the residual base representation, including its inverse-transpose involution. Write $\mathfrak{n}^*(1)$ for the dual of this representation tensored with the mod-ℓ cyclotomic character.

Lemma 32.10 (The ordinary inertia word and its flag)**.** *Over any finite Artinian local $\mathcal{O}$-algebra with residue field k, the equations of I_{ord} are equivalent to the existence of a full flag lifting the standard one, stable on inertia with ordered graded characters χ_0, χ_1, χ_2. This ordered flag is unique. After the unique lower unipotent change of flag congruent to the identity, every local matrix is upper triangular in the semidirect representation.*

Proof. Let A be the Artinian ring. At g_* the characteristic polynomial factors with the three roots $\lambda_a = \chi_a(g_*)$ having pairwise unit differences. Cayley–Hamilton and the Chinese remainder theorem split A^3 into the eigensummands: their projectors are $\prod_{b\neq a}(M(g_*) - \lambda_b I)/\prod_{b\neq a}(\lambda_a - \lambda_b)$. The summands are free lines by reduction and Nakayama's lemma. Choose on each a generator reducing to its standard basis vector. In that basis use (32.1) first with its last two elements equal to g_*. The last two factors multiply to a unit times E_{00}. Thus the first column of every $M(s)$ on inertia is $\chi_0(s)e_0$. Next take its first two elements to be g_*. Their product is a unit times E_{22}, so the last row of $M(s)$ equals $\chi_2(s)e_2^{\mathsf{T}}$. The trace supplies the middle diagonal entry. This proves upper triangularity and identifies all three characters. Conversely, for upper triangular inertia matrices with these diagonals, the rightmost factor of the word takes the whole space into the second flag step, the middle factor takes it into the first, and the left factor kills the first step. The characteristic polynomials have the required values.

Any ordered flag with those characters must have as its first step the first eigenline of g_* and as its second the sum of the first two eigenlines. This follows by applying the three polynomial projectors to each invariant flag step, since the characters on it determine the ranks of its summands. A flag lifting the standard one is taken to it, or conversely obtained from it, by a unique lower unipotent matrix reducing to the identity. Indeed normalize a generator of the first step to have first coordinate one, then select the unique vector of the second step with its first two coordinates zero and one, respectively. These two vectors and e_2 are the columns of the required lower unipotent matrix.

Conjugate the representation in the semidirect product to put this flag in standard position. For g of positive sign ordinary conjugation intertwines the inertia matrices with those at gsg^{-1}. For g of negative sign first apply θ to those at s. This preserves upper triangularity and reverses and inverts the three diagonal entries, which gives the same ordered characters. On inertia those characters are also invariant under $s \mapsto gsg^{-1}$. Uniqueness of the ordered flag implies in both cases that $M(g)$ takes the standard flag to itself. This is the assertion about all local matrices. □

Theorem 32.11 (Smoothness of the ordinary component over the integers)**.** *Suppose $H^0(G_{\mathbb{Q}_\ell}, \mathfrak{n}^*(1)) = 0$. Then $I_t = I_{\mathrm{ord}}$ and U/I_t is a formal power series ring over $\mathcal{O}$ in twelve variables. At the point r_0 one may take integral coordinates whose numbers at the five weights are*

j	-2	-1	0	1	2
number of coordinates	1	2	3	4	2.

The weights refer to conjugation by $\mathrm{diag}(z, 1, z^{-1})$*. On the large-prime range above, the hypothesis holds for every $2 \le r < t$. For $r = 1$ it holds if and only if both unramified residual characters $\bar{\alpha}$ and $\bar{\eta}\bar{\alpha}$ are nontrivial on $G_{\mathbb{Q}_\ell}$.*

Proof. Lemma 32.10 identifies the ordinary framed functor with a choice of three lower unipotent flag coordinates in the maximal ideal and an upper triangular lift whose inertial diagonal is fixed. Consider first its diagonal. Its ratio to the base diagonal is a torus cocycle, with the Galois action given by the sign involution on diagonal matrices. It vanishes on inertia and so factors through $\widehat{\mathbb{Z}}$. Its value at arithmetic Frobenius can be any diagonal matrix congruent to one. Indeed this value together with the sign specifies an element of a finite semidirect product over an Artinian coefficient ring and hence a continuous homomorphism from $\widehat{\mathbb{Z}}$ to that product. Thus the diagonal lifts freely in three parameters, all of weight zero.

Take a small extension $A' \to A$ of such coefficient rings with kernel I killed by the maximal ideal of A'. Choose a lift of the given diagonal. Lift the remaining entries of a given upper triangular representation as continuous cochains. This is possible by finiteness of the coefficient rings, choosing lifts on the fibers of the old entries and of the new diagonal. Their defect of multiplication is a continuous two-cocycle with values in $\mathfrak{n} \otimes_k I$: on this square-zero kernel conjugation is the residual adjoint action, and the cocycle identity is associativity. Local Tate duality for finite modules identifies the dual of $H^2(G_{\mathbb{Q}_\ell}, \mathfrak{n})$ with $H^0(G_{\mathbb{Q}_\ell}, \mathfrak{n}^*(1))$, using the nondegenerate trace pairing of $k/\mathbf{F}_\ell$ on coefficients. By hypothesis the defect is a coboundary. Correct the lifted cochain by an upper unipotent one-cochain with that coboundary, with the sign chosen to cancel the defect. The result is a representation with the chosen diagonal and in particular with the prescribed inertia flag. The three flag coordinates lift freely as well. This proves formal smoothness over $\mathcal{O}$. We use the local duality and Euler characteristic theorems for finite Galois modules over a local field. Milne, *Arithmetic Duality Theorems*, Chapter I, Section 2.

For the tangent dimension, split $\mathfrak{n} = W_1 \oplus W_2$ by positive weights. Their dimensions are two and one. With the diagonal fixed, infinitesimal upper triangular lifts are all continuous one-cocycles in these spaces, since the basis is part of the deformation. For $W = W_j$, local Euler characteristic over $\mathbb{Q}_\ell$ gives $\dim H^1 = \dim W + \dim H^0 + \dim H^2$. The exact sequence of coboundaries then gives $\dim Z^1 = 2 \dim W + \dim H^2 = 2 \dim W$. All dimensions are over k. These are four variables of weight one and two of weight two. The unramified diagonal variables contribute three at weight zero, and the lower unipotent matrix contributes two at weight minus one and one at weight minus two. This yields the table.

A formally smooth complete Noetherian local $\mathcal{O}$-algebra with this residue field and relative tangent dimension twelve is $\mathcal{O}[[z_1, \dots, z_{12}]]$. One can see the assertion directly. Choose lifts of a relative cotangent basis to obtain a surjective map from this power series ring by complete Nakayama. Formal smoothness lifts an inverse on relative tangent spaces to a map back into the power series ring: first choose a map to the quotient by the square of its maximal ideal by the small extension lifting property and adjust the coordinate-linear coefficients by a derivation over $\mathcal{O}$. Then lift through the successive powers of the maximal ideal. The composite endomorphism of the power series ring induces an isomorphism on its full cotangent space, since it fixes $\mathcal{O}$ and

is invertible on the relative part. Successive lifting by degree gives an inverse endomorphism. The original surjection was consequently injective.

The matrix entries $M(g)_{ab} - d(g)_{ab}$ span the relative cotangent space: a map to $k[\epsilon]/(\epsilon^2)$ that annihilates them has the base matrices everywhere and is the basepoint. Each entry has weight $b - a$ under conjugation. These weights are distinct as characters on $k^\times$ for the large primes in question. Choose the coordinate lifts just used among these entries, so their values at x_0 vanish and their weights agree with the table. For any $z \in 1 + \ell\mathbb{Z}_\ell$ of infinite order the same transformation holds integrally. Comparison of coefficients of monomials under this action shows that a matrix entry of weight j has only monomials of total weight j in this chart. Since the power series ring is a domain and Δ has nonzero image by Lemma 32.7, saturation and taking the radical leave I_{ord} fixed. Its unique minimal prime belongs to $\ker x_0$, giving $I_t = I_{\mathrm{ord}}$.

Finally, on inertia the characters of $\mathfrak{n}$ are $\bar\kappa_\ell^{r}$ twice and $\bar\kappa_\ell^{2r}$ once. For $2 \le r < t$ and $\ell > 2t + 3$ neither exponent is congruent to one modulo $\ell - 1$. Thus the dual twist has no inertial invariants. For $r = 1$ its only possible invariants come from weight one. On the span of E_{01}, E_{12} the sign involution is minus the swap, and its two characters in $\mathfrak{n}$ are $\bar\Psi, \bar\eta\bar\Psi$. After the dual twist the invariants occur precisely when one of the stated unramified characters is trivial. This proves the criterion. □

The table is the one prescribed in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/ConjStable.lean`, line 22. The local chart is used in the construction of the global presentation in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/ModelEx.lean`, line 172. Here the integral chart has been established in the stated range by the inertia equations and local duality. Bounding the relations in the global representation ring is a further step.

32.6 Numbers of generators and relations over the local chart

Take $2 \le r < t$ and the large primes of Section 32.4. Let $R_\ell = U/I_t$ be the local ring in Theorem 32.11. First allow all global frames at the residual basepoint, with the prescribed sign, the away inertia kernels and this local condition at ℓ. Write B for their complete local ring. Restriction gives an $\mathcal{O}$-algebra map $R_\ell \to B$. We will impose the two global marks after counting the relative relations.

Let S consist of the finite primes where the residual data may ramify together with ℓ, and write G_S for the Galois group of $\mathbb{Q}$ unramified away from those primes and infinity. For $q \in S \setminus \{\ell\}$ put

$$N_q = I_q \cap \ker \eta \cap \ker \Psi, \qquad H_q = G_{\mathbb{Q}_q}/N_q.$$

The inertia quotient in H_q is finite of order prime to ℓ. The residual adjoint action on all matrices over k decomposes into weight spaces W_j for $j = -2, -1, 0, 1, 2$, where the (a, b) entry has weight $b - a$. These are G_S-modules, since the involution $X \mapsto -J_3 X^{\mathsf{T}} J_3$ preserves the weight.

Lemma 32.12 (The relative cohomology count)**.** *Let $C^\bullet(G, W)$ denote continuous group cochains. For $W = W_j$ take the shifted mapping cone $K_j^\bullet$ of the map*

$$C^\bullet(G_S, W) \oplus \bigoplus_{q \in S\setminus\{\ell\}} C^\bullet(H_q, W) \longrightarrow C^\bullet(G_{\mathbb{Q}_\ell}, W) \oplus \bigoplus_{q \in S\setminus\{\ell\}} C^\bullet(G_{\mathbb{Q}_q}, W),$$
$$(z, (y_q)) \longmapsto (z|_{G_{\mathbb{Q}_\ell}}, (z|_{G_{\mathbb{Q}_q}} - \inf y_q)).$$

The shift is by minus one, so its long exact sequence has the cohomology of the domain followed by that of the target. Set $a_j = \dim_k H^1(K_j^\bullet)$. *Then*

$$H^0(K_j^\bullet) = H^3(K_j^\bullet) = 0, \qquad \dim_k H^2(K_j^\bullet) = a_j + \mathbf{1}_{|j|\le 1}. \tag{32.3}$$

Moreover $H^1(K_j^\bullet)$ *is the space of framed infinitesimal global lifts of weight* j *with the local representation at* ℓ *held fixed.*

Proof. A relative one-cocycle consists of a global cocycle, local quotient cocycles at the q's, and degree-zero cochains witnessing their prescribed agreements up to coboundary. Subtract the global coboundary of the witness at ℓ to make that witness zero. Independently subtract a local coboundary at each q to make its witness zero: degree-zero inflation from H_q is the identity on W. No residual change of global representative remains after setting the witness at ℓ to zero. What remains is a global cocycle vanishing on $G_{\mathbb{Q}_\ell}$, whose restriction to each $G_{\mathbb{Q}_q}$ comes from H_q. The local representatives are uniquely determined by surjectivity onto H_q. This is exactly the framed tangent assertion, by the square-zero multiplication law. In degree zero, the map on global invariants to the ℓ-term is injective, and inflation on the other invariants is injective. Thus $H^0(K_j^\bullet) = 0$.

At $q \neq \ell$ averaging over the finite inertia quotient reduces the cohomology of H_q to that of the procyclic unramified quotient on the inertia invariants. Its degrees zero and one have equal dimensions and the higher groups vanish. The local Euler characteristic of $G_{\mathbb{Q}_q}$ is also zero. At ℓ the local Euler characteristic is $-\dim W$. For G_S the global Euler characteristic is $-\dim W + \dim W^{c=1}$. Here ℓ is odd, S contains the coefficient prime and every finite ramified place, and all the modules are finite. These hypotheses also give vanishing above degree two for the local and global groups. The last part of the Poitou–Tate sequence identifies the cokernel of global-to-local restriction in degree two with the dual of $H^0(G_S, W^*(1))$; modified cohomology at infinity vanishes. These are the classical Euler characteristic and duality theorems. Milne, *Arithmetic Duality Theorems*, Chapter I, Sections 2, 4 and 5.

Inertia at ℓ acts on $W_j^*(1)$ by $\bar\kappa_\ell^{1-rj}$. In the stated range $1 - rj \not\equiv 0 \pmod{\ell - 1}$ for all five weights, since $r \ge 2$ and $2r < \ell - 2$. Thus the dual invariants vanish. The long exact sequence of the cone gives $H^3(K_j^\bullet) = 0$. Its Euler characteristic is the Euler characteristic of the domain minus that of the target, hence $\dim W_j^{c=1}$. At c the matrix is the identity and the action on the Lie algebra is minus anti-diagonal transposition. It fixes a line in the three-dimensional diagonal space and a line in each of the two-dimensional weights $1, -1$. It acts by minus one on each weight $2, -2$. This proves (32.3). □

Theorem 32.13 (A weighted presentation of the completed global ring)**.** *Over the integral power series ring* R_ℓ*, the ring* B *admits a presentation with* a_j *additional variables of each weight* j*. Its ideal of relations has a set of homogeneous generators with at most* $a_j + \mathbf{1}_{|j|\le 1}$ *generators of weight* j*, and none of other weights. After imposing the source's global marks, the resulting complete local ring has a presentation over* $\mathcal{O}$ *with the following counts:*

j	-2	-1	0	1	2
variables	$a_{-2}+1$	$a_{-1}+2$	a_0+3	a_1+4	a_2+2
relations, at most	$a_{-2}+1$	$a_{-1}+3$	a_0+3	a_1+3	$a_2+1.$

Here homogeneity refers to the integer weight, and a homogeneous power series may have monomials of more than one ordinary degree.

Proof. The framed global ring is Noetherian by the construction of Theorem 32.1. Its group is a quotient of G_S. For the residual-kernel finiteness used there one may also apply the finiteness

of H^1 with finite coefficients over the finite extension cut out by the residual matrices. The local condition at ℓ and the kernels away from it are imposed by closed ideals. As in the local chart, the differences of the universal matrix entries from their base values span the relative cotangent space. Take a basis from these homogeneous entries, by Lemma 32.12, and choose the corresponding relative variables. Complete Nakayama gives a surjection $P = R_\ell[[X_1, \dots, X_a]] \to B$ with kernel I, where $a = \sum_j a_j$. In particular every relation has zero linear part in the relative variables modulo the residual maximal ideal. Terms linear in elements of the base ring are permitted.

We give the relation bound. For a small extension with kernel a k-vector space L killed by the maximal ideal, suppose we have a prescribed lift at ℓ and a global representation over the quotient ring. Lift its global matrices as a continuous cochain, keeping the sign. At each $q \neq \ell$ choose a lift on the quotient H_q; this can be taken to be a homomorphism since its second cohomology with residual adjoint coefficients vanishes, by the procyclic computation in the lemma. The defect of global multiplication is an adjoint-valued two-cocycle. Comparing the restrictions of the lifted cochain with the chosen local lifts gives local one-cochains whose coboundaries are the restrictions of that defect, with the same sign convention throughout. These data give an obstruction in

$$\bigoplus_j H^2(K_j^\bullet) \otimes_k L.$$

Changing the global cochain, local quotient lifts or local identifications changes the representative by a cone coboundary. For example, two quotient lifts on H_q differ by a cocycle there; its inflation enters as the local component of the boundary of a degree-one cone element. If the obstruction vanishes, subtract cochains realizing its boundary. This kills the global defect and makes its restrictions agree with allowed local lifts up to infinitesimal conjugation. A global infinitesimal conjugation makes the agreement at ℓ exact. At all other primes conjugating a representation still factoring through H_q preserves that condition. This produces a lift over the local base. Conversely such a lift has zero obstruction by taking its restrictions as the local data.

Apply the construction to $P/(\mathfrak{m}_P I) \to B$. Its kernel $L = I/\mathfrak{m}_P I$ is a finite-dimensional k-space. The rings here may be infinite; the continuous cochains used above still exist. The map between the two compact complete local rings is a surjection with finite kernel. On a sufficiently deep maximal-ideal power it is injective, and maps it homeomorphically onto its image, an open ideal. Choosing lifts on the finitely many cosets gives a continuous section. The same construction on matrices supplies the cochains, and the obstruction still takes values in the finite discrete kernel.

Evaluation of the universal obstruction on L^* is injective. Indeed if a nonzero functional killed it, push out the small extension by that functional, or equivalently quotient by its kernel in L. The representation would then lift, giving an R_ℓ-algebra section from B into this pushout. Compare the resulting map from P with the quotient map. They agree on the base and their relative variables differ only by elements of the square-zero residual kernel. On any $f \in I$ their difference is consequently the sum of these differences multiplied by the residual linear coefficients of f in the relative variables. All such coefficients vanish by minimality. The map through B kills I, whereas the quotient map on I induces the chosen nonzero functional. This contradiction proves injectivity.

We record why this bound holds weight by weight. Conjugation by the diagonal torus acts on a universal matrix entry by its integer weight. The matrix product, the group equations, the away inertia kernels, and the local ordinary equations are equivariant under this action. Thus on each finite-degree truncation of these presentations the split torus acts algebraically,

over $\mathcal{O}$, and all maps above respect it. Equivalently one may conjugate by $\operatorname{diag}(z,1,z^{-1})$ with an indeterminate invertible z and compare Laurent coefficients modulo each power of the maximal ideal. This construction gives integral homogeneous projections that are compatible under truncation. It applies to arbitrary group equations expressed by adic limits, since each finite truncation commutes with these limits. The kernels and their closed ideals are therefore stable under projection to each weight.

In particular $I/\mathfrak{m}_P I$ has a finite weight decomposition. Pushing the universal obstruction by the torus on its coefficients has the same effect as conjugating the matrices in its adjoint factor, by naturality of the cochain construction. Thus the component in $H^2(K_j^\bullet)$ pairs only with coefficients of weight j. To check equality of integer weights one may use the finite-degree torus action just described, or pass to finite unramified coefficient extensions of arbitrarily large residue cardinality and vary the torus unit. Finite flat extension commutes with the matrix presentations, their finite-dimensional relation spaces and the continuous finite-coefficient cochains here. It follows that $\dim_k L_j \le \dim_k H^2(K_j^\bullet)$. For a basis of L_j, lift each basis vector in I and take its weight-j component; closedness and the compatible projections give a lift in I of exactly that weight. Nakayama's lemma for the finite P-module I makes these lifts ideal generators. This proves the relative count.

Finally $r(c)^2 = 1$ and the negative sign of c imply $M(c) = J_3 M(c)^{\mathsf{T}} J_3$. Thus the equation $M(c) = I_3$ is imposed by its two independent diagonal differences at $(0,0),(1,1)$, its entries at $(0,1),(1,0)$ and those at $(0,2),(2,0)$. Their weights are $0,0,1,-1,2,-2$. The other mark gives the two entries at $(0,1),(1,0)$ of $M(g_*) - M(cg_*^{-1}c)$, of weights $1,-1$. Lift these eight equations homogeneously from B to P and use Theorem 32.11 for the twelve base variables. This gives the table. □

The global count is the one sought in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/IsLocPres.lean`, line 34; the eight marking equations are listed in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/MopBase.lean`, line 105. Theorem 32.13 gives a presentation on the actual coefficient rings. The coefficient sizes of its relations are still needed for the length in (32.2). Applying the intersection argument of Section 31.5 also requires control of its weight-zero algebra, characteristic-zero points and traces. The variable and relation counts alone do not provide the required bound on the common intervals.

32.7 The component at the first cyclotomic twist

We treat the case $r = 1$ left open by the vanishing hypothesis in Theorem 32.11. We keep the same E_x, the same auxiliary field $K' = \mathbb{Q}(i)$, and the same range $\ell > \exp(x^{\varepsilon}/12) > 2\lfloor x^{401} \rfloor + 3$. In particular both finite characters α, η are unramified at ℓ. Denote the cyclotomic character by κ.

Lemma 32.14 (Kummer classes with a variable unramified twist)**.** *Let $K/\mathbb{Q}_\ell$ be unramified of degree f, let I be its inertia group, and let ϕ be arithmetic Frobenius on the maximal unramified extension K^{nr}. Let C be a finite local Artin $\mathcal{O}$-algebra, N a finite C-module, and $\beta : G_K \to C^\times$ unramified with Frobenius value u. There is a decomposition natural in these coefficients*

$$H^1(G_K, N(\kappa\beta)) \simeq N[u-1] \oplus U_K(N,u), \tag{32.4}$$
$$U_K(N,u) = \ker\bigl(1 - u\phi : \mathcal{O}_{K^{\mathrm{nr}}} \otimes_{\mathbb{Z}_\ell} N \longrightarrow \mathcal{O}_{K^{\mathrm{nr}}} \otimes_{\mathbb{Z}_\ell} N\bigr).$$

Here $N[u-1] = \{a : (u-1)a = 0\}$, and ϕ acts on the first tensor factor. For a small extension $C' \to C$ and a lift $u' \in C'^\times$ of u, every element of $U_K(\mathfrak{m}_C, u)$ lifts to $U_K(\mathfrak{m}_{C'}, u')$. On a one-dimensional residual coefficient module $N = k$, the dimension of $U_K(k, \bar{u})$ over k is f.

Proof. For the untwisted coefficients $N(1)$ and a finite unramified extension K_n/K, the Kummer map identifies

$$H^1(G_{K_n}, N(1)) = (K_n^\times)^\wedge_\ell \otimes_{\mathbb{Z}_\ell} N.$$

For $N = \mathbb{Z}/\ell^b$ this is the Kummer sequence and Hilbert's Theorem 90. For general finite N, send $a \otimes n$ to the image of the compatible Kummer classes of a under the coefficient map $1 \mapsto n$. The cyclic case and a decomposition of the underlying finite abelian group prove this natural map is an isomorphism. The uniformizer ℓ splits the valuation. The residue units have order prime to ℓ. The logarithm divided by ℓ identifies the principal units of K_n with its additive ring of integers. Indeed the logarithm and exponential series are inverse on $1 + \ell\mathcal{O}_{K_n}$ and $\ell\mathcal{O}_{K_n}$ because ℓ is odd and K_n is absolutely unramified. These isomorphisms commute with Frobenius and with inclusions of unramified fields. Taking the union over K_n gives

$$H^1(I, N(1)) = N \oplus (\mathcal{O}_{K^{\mathrm{nr}}} \otimes_{\mathbb{Z}_\ell} N).$$

Here we use continuity of Galois cohomology under a union of fields with finite discrete coefficients. Equivalently, a Kummer class over the union is represented by an element of a finite stage, and an equality of Kummer classes also occurs at a finite stage.

The twist β is trivial on I and multiplies its Frobenius action on this decomposition by u. An inertia invariant in $N(\kappa\beta)$ is zero, since an inertial cyclotomic value differs from one by a unit. Inflation–restriction therefore identifies $H^1(G_K, N(\kappa\beta))$ with the Frobenius invariants of the displayed direct sum. This proves (32.4).

For the lifting assertion write $J = \ker(C' \to C)$, killed by $\mathfrak{m}_{C'}$. Tensoring the surjection of maximal ideals with $\mathcal{O}_{K^{\mathrm{nr}}}$ stays exact, since this ring is torsion-free over $\mathbb{Z}_\ell$. Lift the proposed unit term as an element of the larger tensor product. Its error under $1 - u'\phi$ belongs to $\overline{\mathbf{F}}_\ell \otimes_{\mathbf{F}_\ell} J$. On this space $1 - \bar{u}\phi$ is surjective. Indeed if m is the finite order of $\bar{u}$, then $(1-\bar{u}\phi)(1+\bar{u}\phi+\cdots+(\bar{u}\phi)^{m-1}) = 1-\phi^m$. The last operator is surjective because the equation $X - X^{\ell^{fm}} = a$ has a solution in $\overline{\mathbf{F}}_\ell$ for every a. Correct the lift by a preimage of its error. Finally local Tate duality gives $\dim_k H^2(G_K, k(\bar{\kappa}\bar{\beta})) = \mathbf{1}_{\bar{u}=1}$. The local Euler characteristic and $H^0 = 0$ give first cohomology of dimension $f + \mathbf{1}_{\bar{u}=1}$. Subtract the dimension of the first summand of (32.4). This proves the dimension assertion. □

Theorem 32.15 (The integral ordinary chart at every first twist)**.** *For $r = 1$, including $\bar{\alpha} = 1$ or $\bar{\eta}\bar{\alpha} = 1$ on $G_{\mathbb{Q}_\ell}$, the ring U/I_t is formally smooth over $\mathcal{O}$. The counts of its integral coordinates in weights $-2, -1, 0, 1, 2$ are $1, 2, 3, 4, 2$. In combination with Theorem 32.11 this gives the same chart for every $1 \le r < \lfloor x^{401} \rfloor$ on the indicated large-prime range.*

Proof. Write $G = G_{\mathbb{Q}_\ell}$ and choose an arithmetic Frobenius lift F. By Lemma 32.10 a framed ordinary deformation is a lower unipotent flag coordinate in three variables together with an upper triangular deformation. Its diagonal has parameter ring $A = \mathcal{O}[[t_0, t_1, t_2]]$. The ratio of its diagonal at F to the base diagonal has entries $1 + t_0, 1 + t_1, 1 + t_2$. This is an unramified diagonal cocycle with the sign action, as in the proof of Theorem 32.11. Denote the diagonal matrix at a positive-sign element g by $D(g)$.

Quotient the upper unipotent group by its central entry 02. The two entries $01, 12$ give an additive group in this quotient. With the diagonal prescribed, their values are continuous cocycles for its conjugation action, by writing a matrix as unipotent times diagonal. For a finite coefficient algebra C the entries of the cocycles belong to $N = \mathfrak{m}_C$. If $\eta(F) = 1$, they are separate lines. On line $a, a + 1$, $a = 0, 1$, the character is $\kappa\beta_a$ with β_a unramified and value

$$u_a = \alpha(F)(1 + t_a)/(1 + t_{a+1})$$

at F. In this case take two copies of $K = \mathbb{Q}_\ell$. If $\eta(F) = -1$, the sign exchanges the two lines with a minus sign. Let K be the unramified quadratic extension. Over G_K the first line has character $\kappa\beta$; its unramified factor has value

$$u = \alpha(F^2)(1 + t_0)/(1 + t_2)$$

at F^2. Indeed the square of the diagonal at F in the semidirect product has entries $D_{00}(F)/D_{22}(F), 1, D_{22}(F)/D_{00}(F)$, where here $D_{ii}(F)$ denotes the ordinary diagonal coordinate of the negative-sign matrix. Both adjacent ratios at F^2 coincide. The two-dimensional module is induced from the first line over G_K. Its standard vector and its image under F form a basis since the second vector is a unit multiple of the other standard vector. This construction commutes with coefficient changes. Shapiro's lemma identifies its first cohomology with that of this single line over G_K. Use one index ν for each of the two lines in the first case, and one index for the induced module in the second. The sum of the associated degrees f_ν is always two.

Passing to cohomology removes two framed coordinates. In fact g_* acts on the adjacent module by the scalar $\kappa(g_*)$ and $\kappa(g_*) - 1$ is a unit. Subtracting a unique coboundary makes any adjacent cocycle vanish at g_*. Thus the class and two freely chosen coboundary coordinates recover the cocycle functorially over every C. Applying Lemma 32.14 to these classes gives valuation coordinates $v_\nu \in \mathfrak{m}_C$ with $(u_\nu - 1)v_\nu = 0$, and unit terms in its second summand. Consequently the ordinary ring maps, on formal spectra, to the base with coordinate ring

$$A_{\mathrm{val}} = A[[v_\nu : \nu]]/((u_\nu - 1)v_\nu : \nu).$$

The valuation assignment gives a ring homomorphism in the opposite direction by passing through the Artinian quotients of the universal ring. When $u_\nu - 1$ is a unit, the corresponding variable simply vanishes.

This map of deformation functors is formally smooth. Given a small extension with a prescribed lift of the diagonal and the valuation coordinates, lift the unit terms by Lemma 32.14 and the two coboundary coordinates freely. This lifts the entire upper representation modulo its central entry. Choose any lift of the old central entry as a continuous function, which is possible by finiteness of the coefficient rings. Its defect of multiplication takes values in the residual central line tensored with the kernel of the small extension. This is a two-cocycle by associativity. On inertia the central line has character $\bar{\kappa}^2$; hence its dual twisted by 1 has no invariants. Local Tate duality gives $H^2 = 0$, so a central one-cochain corrects the defect. The flag coordinates lift freely. Over the residual point of A_{val}, the relative tangent space has three flag coordinates, two adjacent coboundaries, $\sum f_\nu = 2$ unit directions, and two central cocycles, giving dimension nine. For the last number the local Euler formula over $\mathbb{Q}_\ell$ gives $\dim Z^1 = 2 \dim W + \dim H^2 = 2$ on the one-dimensional central line W. A formally smooth map between these complete Noetherian local rings with the same residue field and relative tangent dimension nine identifies the target with a formal power series ring in nine variables over the source. The power series lifting argument in Theorem 32.11 applies over A_{val} as well. Thus, up to a choice of coordinates,

$$U/I_{\mathrm{ord}} = A_{\mathrm{val}}[[y_1, \dots, y_9]].$$

We now compute the saturation in the source. Work in the upper flag basis. The elements used to define q_a have positive sign, so their traces are unchanged by this basis choice. Products of upper triangular matrices have diagonal equal to the product of the diagonals. The definitions in Section 32.4 therefore give exactly

$$q_a = c_a c_{a+1}(D_{aa}(\gamma) - \kappa(\gamma) D_{a+1,a+1}(\gamma)).$$

The positive-sign element γ induces a positive integer power n_ν of arithmetic Frobenius over the associated unramified field. Its adjacent diagonal ratio divided by $\kappa(\gamma)$ equals $u_\nu^{n_\nu}$. In the induced case the two ratios are equal. It follows that $u_\nu - 1$ divides the corresponding q_a. Thus $\Delta v_\nu = 0$ for every ν, where $\Delta = \ell q_0 q_1$. Modulo all the v_ν, the ring is a power series domain over $\mathcal{O}$. The image of Δ is nonzero there because each $c_a \neq 0$ and the displayed formulas for the u_ν give $u_\nu^{n_\nu} - 1 \neq 0$ in the power series domain A. Hence the elements annihilated by powers of Δ in U/I_{ord} form precisely the ideal generated by the v_ν. Its quotient is reduced and has one minimal prime. The basepoint factors through it because the diagonal representation has zero adjacent classes. This identifies the quotient with U/I_t.

The residual tangent directions after saturation have weights $-2, -1, -1$ from the flag, three zeros from the diagonal, weight one for all four adjacent directions, and weight two for the two central directions. The split torus acts on the adjacent quotient by a common scalar and on its central line by its square, so the decomposition of tangent dimensions is compatible with the action, including in the induced case. The saturated chart is a power series ring, and homogeneous entry differences lift a basis of its relative cotangent space as in Theorem 32.11. Choose those differences as the integral chart coordinates. This proves the stated counts. □

Corollary 32.16 (Global relations for the first twist)**.** *For $r = 1$, the variable counts of Theorem 32.13 hold with the same definition of a_j. So do its upper bounds on relations, except that the weight-one bound is*

$$a_1 + 3 + \mathbf{1}_{\alpha=\eta}.$$

The equality of characters here is global. For the other twists the table of Theorem 32.13 applies without change.

Proof. The relative cone of Lemma 32.12 still computes the framed tangent with a fixed local lift. Global duality identifies its third cohomology with the dual of the global invariants of $W_j^*(1)$. Only $j = 1$ can contribute, by inspection on inertia at ℓ. At weight one the two characters in the dual twist are $\bar{\alpha}^{-1}$ and $\bar{\eta}\bar{\alpha}^{-1}$, since the sign involution is minus the swap. Now $\alpha(c) = -1$. Since the finite characters have order prime to ℓ, reduction is injective on their values. Thus the dimension of the global invariants is one precisely when $\alpha = \eta$, and zero otherwise. The Euler characteristic computation in Lemma 32.12 adds this dimension to $\dim H^2$. The obstruction argument, the eight marking equations, and the local parameter counts of Theorem 32.15 give the result. □

The Frobenius factors and saturation are defined in `Development/AutomorphicArithmetic/AutomorphicVarieties/PadicHodgeTheory/IsInertiaInfiniteCyclotomic.lean`, line 99 and lines 104–135. The outstanding global length and period estimates are stated for these fields in Section 6.6.

32.8 The fixed point and its nonpositive cotangent spaces

Use the ordinary marked deformation ring of Sections 32.4 through 32.7. We take its input character α from the actual period field $E = \widetilde{E}$ of Section 6.6. Write $n = [E : \mathbb{Q}]$, let N be its cyclotomic modulus, and retain $t = \lfloor x^{401} \rfloor$, $1 \leq r < t$ and $\ell > \exp(x^{\varepsilon}/12)$. On the intervals in question $\ell > 2t + 3$, $\ell \nmid nN$, and $\alpha(c) = (-1)^r$. These are the field and character hypotheses used in the local chart and relative presentation. Write $H = G_{\mathbb{Q}(i)} = \ker \eta$, $\Psi = \kappa_\ell^r \alpha$, and $d = \mathrm{diag}(\Psi, 1, \Psi^{-1})$.

We denote the completed marked ring by B. Its matrix M satisfies $M(c) = 1$. The remaining two marks set the adjacent entries $01, 10$ of $M(g_*) - M(cg_*^{-1}c)$ to zero. Here $g_* \in I_\ell$ gives three

distinct residual diagonal eigenvalues. The action by $\mathrm{diag}(z, 1, z^{-1})$ gives the integer weight $b - a$ to the entry ab.

Lemma 32.17 (Passing from the completed presentation to a graded ring)**.** *Take the presentation $B = P/(f_s)$ of Theorem 32.13 or Corollary 32.16, where $P = \mathcal{O}[[X_1, \dots, X_m]]$ and each variable is chosen to vanish at the split point. Let P_j consist of series all of whose monomials have weight j. There is a Noetherian graded ring $A = \bigoplus_j P_j \subset P$. Its invariant part P_0 is complete local and every P_j is finite over P_0. The ring $A/(f_s)$ embeds in B, and its degree-zero part is the full ring of invariants of B. The cotangent module of A at the split point is free over $\mathcal{O}$ on the X_i. In weight j, the cotangent module after the equations is the quotient of this free module by the linear terms of the f_s of weight j, in both presentations.*

Proof. The possible weights are $0, \pm 1, \pm 2$. Weight-zero monomials are generated by variables of weight zero, opposite pairs, and triples of weights $1, 1, -2$ or $-1, -1, 2$. Remove opposite pairs first to see this. Choose one such factorization of every invariant monomial. A monomial of ordinary degree d uses at least $d/3$ factors. Grouping a series by the chosen products therefore expresses it as a convergent power series in finitely many invariant monomials. Thus P_0 is a quotient of a complete Noetherian local power series ring over $\mathcal{O}$.

For fixed weight j, remove invariant factors until none remain. If all remaining weights have the same sign there are at most $|j|$ factors. With both signs present there can be only weights of absolute sizes one and two with opposite signs, and the weight of absolute size one occurs at most once. This bounds the number of remaining factors by $|j| + 2$. Grouping by the finitely many possible remainders proves that P_j is finite over P_0, and that A is generated as an algebra by the original variables over P_0.

Taking the weight-j part of any expression $\sum_s f_s b_s$ in P replaces b_s by its component of weight $j - \mathrm{wt}(f_s)$. This proves the asserted injection. On each Artinian truncation torus invariance is coefficientwise equality for an indeterminate invertible z. Project a lift of an invariant in B to weight zero. It continues to lift that invariant on every truncation and hence in B by separatedness.

Every homogeneous series with zero constant and linear terms is in the square of the augmentation ideal of A. In each monomial factor off its first two variables in a fixed ordering and group by the resulting pair. Each remaining series still has one weight. This proves the assertion about the free cotangent module. Taking the linear term of an arbitrary multiple of f_s multiplies its linear term by the constant of the multiplier. This proves the last assertion. The inclusion into $J[[X_1, \dots, X_m]]$, where $J = \mathrm{Frac}\,\mathcal{O}$, gives the formal chart of Section 31.5 on A. □

Proposition 32.18 (The fixed subscheme is the split point)**.** *In the graded solution ring $A/(f_s)$, the quotient by all elements of nonzero weight is $\mathcal{O}$, via the split augmentation. In particular the weight-zero cotangent module of the marked solution at this point is zero.*

Proof. First work over any finite Artinian local coefficient algebra with the prescribed residue field. A fixed lift has diagonal matrices on H, since nonzero-weight entries vanish. The matrix on hc equals that on h, so it is enough to determine the three coordinate characters on H. On I_ℓ the ordered ordinary flag of Lemma 32.10 has graded characters $\Psi, 1, \Psi^{-1}$. The coordinates are already diagonal, and the residual eigenvalues of g_* distinguish their order. Thus each coordinate agrees on this inertia group with the corresponding entry of d. The same is true at the other primes over ℓ by conjugation, including the inverse transpose and reversal for c.

On inertia away from ℓ the quotient of a coordinate by its base character has image of order prime to ℓ, by the prescribed inertia kernel. It reduces to one and lies in the finite ℓ-group

of principal units of the Artinian algebra. It is therefore trivial. Each of the three quotient characters on H is now everywhere unramified. Artin reciprocity and the trivial ideal class group of $\mathbb{Q}(i)$ make it trivial globally. The latter class number follows from Euclidean division in $\mathbb{Z}[i]$. Thus the only fixed lift is the split one, over every such algebra.

Killing the nonzero-weight variables in P gives the complete ring on the weight-zero variables, with the specialized equations. It gives exactly the same quotient in A. Indeed, in a homogeneous series group the monomials that contain a nonzero-weight variable by their first such variable, leaving coefficients of a single weight in each group. Fixed lifts are represented by this quotient because the homogeneous variables are entry differences or parameters homogeneous in those differences, and fixedness is equivalent to killing every nonzero weight. Apply uniqueness to its own maximal-ideal truncations. All its augmented variables vanish in each, so separatedness gives the isomorphism with $\mathcal{O}$. In the cotangent module the removed variables have nonzero weight, proving the assertion in weight zero. □

Let C_j be the weight-j cotangent module of the marked solution at this augmentation, and use $D = J/\mathcal{O}$. For a character ξ write $S(\xi) \subset H^1(\mathbb{Q}, D(\xi))$ for the classes unramified at finite primes away from ℓ and zero on $G_{\mathbb{Q}_\ell}$.

Theorem 32.19 (The two negative cotangent spaces)**.** *There are identifications*

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}}(C_{-1}, D) &\simeq S(\Psi^{-1}) \oplus S(\eta\Psi^{-1}), \\ \operatorname{Hom}_{\mathcal{O}}(C_{-2}, D) &\simeq S(\eta\Psi^{-2}). \end{aligned} \tag{32.5}$$

Both C_{-1} and C_{-2} have finite length. Hence the negative equations span both rational negative cotangent spaces of the ambient algebra. The weight-zero equations span its integral weight-zero cotangent space.

Proof. Homogeneous derivations at the split point are determined by the weighted variables in Lemma 32.17. Their values on matrix entries have the same weight, and those entries generate the completed deformation. The weight--1 functor is therefore exactly the one in Corollary 32.9.

For weight -2, let $W = \mathcal{O}/\varpi^a$ and write a lift to $\mathcal{O} \oplus W$, with square-zero ideal W, as $(1 + z(g)E_{20})d(g)$ in its matrix part. The differential of the sign involution sends E_{20} to $-E_{20}$, and the split diagonal acts by Ψ^{-2}. Multiplication is precisely the cocycle law for $\xi = \eta\Psi^{-2}$. Since $\xi(c) = -1$, subtracting a unique coboundary makes $z(c) = 0$. The other two marks are in adjacent entries and are satisfied automatically.

At ℓ, the unit $\xi(g_*) - 1$ allows us to use a local coboundary to set $z(g_*) = 0$. The ordinary flag must now be the standard one, by the three distinct eigenvalues and Lemma 32.10. The negative entry vanishes on inertia. For $g \in G_{\mathbb{Q}_\ell}$, evaluate the cocycle at $g_* g = g(g^{-1} g_* g)$. This gives $(\xi(g_*) - 1)z(g) = 0$, hence $z(g) = 0$. Conversely a local coboundary gives a conjugate of the split point and belongs to the chosen component. One may conjugate the split lift by $1 + YE_{20}$ over $\mathcal{O}[[Y]]$ and then specialize Y to the square-zero value. The component is preserved by change of frame, or directly by its defining matrix words, traces and saturation as in Lemma 32.7. Away from ℓ the inertia kernel condition gives the unramified class by averaging on the prime-to-ℓ inertia quotient.

Globally and on $G_{\mathbb{Q}_\ell}$ the divisible invariants for ξ vanish by evaluation at g_*. On each other inertia group they are zero or divisible. Thus the multiplication sequence identifies the finite strict group just computed with $S(\xi)[\varpi^a]$. Take the union over a. This proves (32.5).

All three groups on the right are character summands of strict groups on $E\mathbb{Q}(i)$, by prime-to-ℓ descent. Their twists are $r, r, 2r$. They are finite by Theorem 25.1. A free summand of a finitely

generated $\mathcal{O}$-module would give an infinite summand D in its dual. Hence both cotangent modules are torsion. Lemma 32.17 now proves the rational spanning, and Proposition 32.18 gives the integral assertion at weight zero. □

Corollary 32.20 (The numerical presentation for every contributing character)**.** *For all sufficiently distant intervals, every large-prime component with $S(\Psi^{-1}) \neq 0$ has the variable and equation counts of Section 31.5 and satisfies its formal chart and rational cotangent spanning requirements. For those counts set, in the order used in that section,*

$$R = a_{-1} + 2, \quad S = a_{-2} + 1, \quad R' = a_1 + 3, \quad S' = a_2 + 1, \quad m_0 = a_0 + 3.$$

In particular the quotient of the ambient weight--1 lattice by the negative weight-one equations has length $\operatorname{length}_{\mathcal{O}} S(\Psi^{-1}) + \operatorname{length}_{\mathcal{O}} S(\eta\Psi^{-1})$.

Proof. If $r = 1, \alpha = \eta$, the required $S(\Psi^{-1})$ is a summand at ℓ of the first-twist strict group of the fixed field $\mathbb{Q}(i)$. Its dual injects into the ℓ-part of the finite group $K_2(\mathbb{Z}[i])$, by Theorem 25.1. All sufficiently large ℓ therefore contribute zero, uniformly in E. This removes the only case where Corollary 32.16 permits an extra equation. In every other case the table in Theorem 32.13 gives $R, S, R'+1, S'+1, m_0$ variables and at most $R+1, S, R', S', m_0$ equations in the five indicated weights. Pad lists by zero equations if necessary. The free cotangent module of the ambient algebra embeds into its rationalization, so saturation of its augmentation square adds no linear terms. The preceding theorem identifies its quotient in weight -1, including its full length by duality over the DVR. □

The integral intersection inequality of Theorem 31.8 also requires finiteness of the weight-zero solution ring, rigidity at the mixed points, and a nonvanishing trace difference there. The results here establish the chart, the equations and the distinguished-point conditions for the original ordinary component. The comparison in the source occurs in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/ModelUpper.lean`, line 103 and the strict-class injection in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/VlenLe.lean`, line 52.

32.9 Steinberg conditions at split primes and strict classes in weight one

Retain the fields, characters, coefficient ring $\mathcal{O}$, and marking of Theorem 32.1. A Steinberg condition at a prime of residue cardinality Q requires a single Jordan block for the logarithm of tame inertia at a characteristic-zero point. Its integral closure also contains points with smaller monodromy. We first compute its intersection with zero monodromy. This determines which of the marked lifts can be tested at such a prime.

Proposition 32.21 (The unramified intersection)**.** *Let $Q > 1$ be a power of a prime different from ℓ, with $\ell > 3$. In the affine space of pairs of three by three matrices (Φ, N) over $\mathcal{O}$, take the schematic closure $\mathcal{Z}_Q$ of the reduced characteristic-zero locus on which N is regular nilpotent, $\Phi N = QN\Phi$, and*

$$\det(X - \Phi) = P_Q(X) = (X - Q)(X - 1)(X - Q^{-1}).$$

Its intersection with $N = 0$, as a scheme over $\mathcal{O}$, is defined by the three coefficients of $\det(X - \Phi) - P_Q(X)$.

Proof. Write $A = \mathcal{O}[\Phi_{ij}]$ and $C \subset A$ for the ideal of these three coefficients. Over an algebraic closure of the residue field the fiber of A/C is a finite union of Jordan conjugacy classes. The

centralizer of a matrix has dimension at least three. Indeed, blocks of sizes s_i, s_j with the same eigenvalue contribute $\min(s_i, s_j)$ to the dimension of the commuting algebra, and the diagonal contributions already sum to three. Each class therefore has dimension at most six. Krull's height theorem gives the opposite bound for the dimension of every component of this three-equation fiber in nine-space. Consequently (C, ℓ) has height four in A at each closed point of the special fiber. In the regular local ring there the three coefficients followed by ℓ form part of a system of parameters and hence a regular sequence. It follows that A/C has no ℓ-torsion. This assertion can be checked at those closed points because any nonzero ℓ-torsion has support in the special fiber.

The generic fiber is reduced. In an eigenbasis the differential of the three coefficient functions on diagonal perturbations has Vandermonde determinant up to sign. The eigenvalues $Q, 1, Q^{-1}$ are distinct, so this differential has rank three at each geometric point. For any such Φ, let N_0 carry its Q^{-1}-line isomorphically to its 1-line, its 1-line to its Q-line, and its Q-line to zero. The curve (Φ, tN_0) specializes a regular block to $(\Phi, 0)$. If f vanishes on the defining locus of $\mathcal{Z}_Q$, then $f(\Phi, 0)$ vanishes on the reduced generic fiber of A/C. Torsion-freeness gives $f(\Phi, 0) \in C$. Conversely the three coefficient differences already vanish on $\mathcal{Z}_Q$. These two ideal inclusions give the asserted scheme equality, including its values on coefficient rings with nilpotents. □

Proposition 32.22 (Contact with the marked weights)**.** *Let q be a rational prime different from ℓ splitting completely in $E_x\mathbb{Q}(i)$, and use arithmetic Frobenius. On the unramified lower triangular lifts of the marked diagonal with diagonal characters $\Psi, 1, \Psi^{-1}$, the normalized condition $\mathcal{Z}_q$ cuts out the principal scalar ideal generated by*

$$a_{q,r} = (q^r - q)(1 - q^{-r-1}).$$

It vanishes identically in weight $r = 1$. If $r > 1$ and $d_q = v_\ell(q-1) \geq 1$, its valuation is

$$2d_q + v_\ell(r-1) + v_\ell(r+1).$$

In particular this is the largest m for which those triangular lifts over $(\mathcal{O}/\ell^m)[\epsilon]/(\epsilon^2)$ lie on $\mathcal{Z}_q$.

Proof. The characteristic polynomial of Frobenius is determined by $q^r, 1, q^{-r}$. Its leading and constant coefficients agree with those of P_q, and the other two differences are, up to sign,

$$q^r + q^{-r} - q - q^{-1} = (q^r - q)(1 - q^{-r-1}).$$

Apply Proposition 32.21. For odd ℓ and $d_q \geq 1$, expanding successive ℓ-th powers by the binomial theorem gives $v_\ell(q^j - 1) = d_q + v_\ell(j)$ for $j \geq 1$. Apply this to $r-1$ and $r+1$. Over $\mathcal{O} \oplus \epsilon(\mathcal{O}/\ell^m)$ the scalar component still takes values in $\mathcal{O}$, so the same equation at an unramified point forces $r = 1$ among the positive weights. □

There is an integral version with a variable weight when $d_q \geq 1$. Put $u = 1 + \ell$, $a = \log_\ell(q)/\log_\ell(u)$, and assign $Y = (1+T)^a$ to the first Frobenius eigenvalue. Evaluating at $T = u^r - 1$ recovers $Y = q^r$. The same computation cuts out $(Y - q)(Y - q^{-1})$ up to a unit. Write $a = \ell^{d_q - 1} b$, where $b \in \mathbb{Z}_\ell^\times$, and $d = \ell^{d_q - 1}$. In $\mathcal{O}[[T]]$ this ideal is generated by the monic polynomial

$$\big((1+T)^d - u^d\big)\big((1+T)^d - u^{-d}\big).$$

Indeed substitute $V = ((1+T)/u^{\pm 1})^d$ in the binomial series $(V^b - 1)/(V - 1)$. It is a unit since its residue at the closed point is $\bar{b}$. The polynomial reduces to T^{2d} in the residue field. Its geometric generic roots are the $2d$ distinct values $u^{\pm 1}\zeta - 1$ with $\zeta^d = 1$, since u^2 has infinite order.

Theorem 32.23 (Strict tangents at ramified weight-one lifts)**.** *On an interval with common cyclotomic modulus N_* divisible by the conductors of the period fields, take distinct rational primes $q_1, q_2 \equiv 1 \bmod 4N_*$, different from ℓ. Use the range of ℓ in Theorem 32.1. For every E_x on that interval and every odd α of $\mathrm{Gal}(E_x/\mathbb{Q})$, there is a marked integral family ρ_b, $b \in \ell\mathcal{O}$, with the original local conditions away from q_1, q_2. At both primes its nonzero members have regular tame monodromy and conductor exponent two. For every $m \geq 1$, the group*

$$e_\alpha(\mathcal{S}_{\ell,1}(E_x) \otimes \mathcal{O})[\ell^m]$$

injects $\mathcal{O}$-linearly into the tangent module over $\mathcal{O} \oplus \epsilon(\mathcal{O}/\ell^m)$ at the integral point ρ_{ℓ^m} satisfying both closed Steinberg conditions. The injection persists modulo infinitesimal changes of basis.

Proof. Let D be the field cut out by α, let $\Delta = \mathrm{Gal}(D/\mathbb{Q})$, and put $\Psi = \alpha\kappa_\ell$. Consider first one of the two primes, q_i. If h is the class number of D, a prime $\mathfrak{q}_i \mid q_i$ satisfies $\mathfrak{q}_i^h = (a_i)$. The divisors of the elements σa_i, $\sigma \in \Delta$, are supported on distinct primes. Their classes are multiplicatively independent modulo roots of unity. Project a_i in $D^\times \otimes_{\mathbb{Z}} J$ by the idempotent

$$e_{\alpha^{-1}} = \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \alpha(\sigma)\sigma.$$

Kummer theory, followed by twisting by α and descent along $D/\mathbb{Q}$, gives a class $w_i \in H^1(\mathbb{Q}, J(\Psi))$. Here descent takes invariants, since positive-degree cohomology of the finite group Δ with J-coefficients is zero by averaging. At a prime above q_i the projected valuation is nonzero. At primes away from q_i, ℓ it is zero. For a local field of residue characteristic different from ℓ, the unit group has finite pro-ℓ completion, while the valuation gives a copy of $\mathbb{Z}_\ell$ in the completion of the multiplicative group. The Kummer class of a uniformizer has nonzero restriction to tame inertia, as is also seen by adjoining its successive ℓ-power roots over the maximal unramified extension. Thus w_i is ramified at q_i and is zero on inertia over D away from q_i, ℓ.

Each a_i is a unit at every place over ℓ. Localization of these unit Kummer classes, followed by the ℓ-adic logarithm, takes values in the α^{-1}-line of $D \otimes_{\mathbb{Q}} J$. Indeed the logarithm identifies the completed local units tensored with J with the product of local fields tensored with J over $\mathbb{Q}_\ell$. The normal basis theorem identifies $D \otimes_{\mathbb{Q}} J$ with $J[\Delta]$. Each of the two projected logarithms is nonzero. To see this, fix an embedding in $\overline{\mathbb{Q}}_\ell$. The logarithms of all σa_i are linearly independent over $\mathbb{Q}$, since a rational relation, after clearing denominators, would give a product which is a local unit of logarithm zero. Such a unit is a root of unity, and the divisor calculation forces all exponents to vanish. Brumer's theorem on logarithms of algebraic numbers upgrades the independence to algebraic coefficients. It applies to these units after taking a common positive power in the logarithm neighborhood of one. The coefficients of $e_{\alpha^{-1}}$ are algebraic, so its sum of logarithms is nonzero. [11]

There are consequently $c_1, c_2 \in J^\times$ such that $c_1 w_1 + c_2 w_2$ restricts to zero on G_ℓ. The two distinct ramified primes prevent cancellation on inertia at either of them. Subtracting a coboundary makes a representing cocycle identically zero on the specified G_ℓ. Its image is compact, so a common scalar multiple gives a continuous cocycle $z : G_{\mathbb{Q}} \to \mathcal{O}(\Psi)$. On the old inertia kernels in Theorem 32.1 it is zero. Indeed these kernels lie in G_D because κ_ℓ is unramified away from ℓ, and coboundaries vanish on inertia acting trivially on the coefficients. Also $z(c) = 0$, since $\Psi(c) = 1$ and $c^2 = 1$.

In one-based matrix indices set $U = E_{12} - E_{23}$ and $L = E_{21} - E_{32}$. The involution in Theorem 32.1 fixes $\exp(tU)$, and its differential fixes L. Conjugation by $d(g)$ scales U by $\Psi(g)$ and L by $\Psi(g)^{-1}$. The formula

$$\rho_b(g) = \big(\exp(bz(g)U)d(g),\, \eta(g)\big)$$

therefore gives a homomorphism by the cocycle rule. Here the exponential stops in degree two. It keeps the specified local matrices at ℓ and at c. Since $b \in \ell\mathcal{O}$, it reduces to the marked residual point. At q_i, tame inertia has logarithm a nonzero scalar multiple of U when $b \neq 0$, and Frobenius is upper triangular with diagonal $q_i, 1, q_i^{-1}$. Thus the pair belongs to the regular locus in characteristic zero and to its closure over $\mathcal{O}$. Wild inertia is trivial and the inertia-fixed space has dimension one, giving conductor exponent $3 - 1 = 2$.

Now take $b = \ell^m$, $M = \mathcal{O}/\ell^m$. Represent a strict class by the normalized cocycle $y : G_{\mathbb{Q}} \to M(\Psi^{-1})$ of Theorem 32.1. Multiply the matrix part of $\rho_b(g)$ on the left by $1 + \epsilon y(g)L$. On ϵM the action of ρ_b agrees with the split action, so this is again a homomorphism. The original marking is preserved. At q_i, y is unramified, so the logarithm of tame inertia is unchanged. Choose a tame generator and an arithmetic Frobenius. The integral pair at $b = \ell^m$ reduces modulo ℓ^m to $(d(\mathrm{Frob}_{q_i}), 0)$. For every polynomial equation f of $\mathcal{Z}_{q_i}$, its linear term on the square-zero ideal is its differential at this reduction. The lower triangular change of Frobenius there leaves the characteristic polynomial fixed. Proposition 32.21 therefore makes this derivative zero. The constant term already vanishes at ρ_b. This proves both Steinberg conditions for the same tangent.

Finally $X \mapsto (X_{21} - X_{32})/2$ retracts the inclusion $y \mapsto yL$ on the split adjoint module. The diagonal action scales both entries by Ψ^{-1}, and the differential of the involution interchanges them with a minus sign. Hence the retraction is equivariant also for the full semidirect action. It sends any infinitesimal change of basis to a scalar coboundary, which proves injectivity on classes and the asserted linearity. □

The primes can be chosen once on each sufficiently distant power interval, for all its large coefficient primes. Linnik's theorem applied first modulo $4N_*$ and then modulo $4N_*q_1$ supplies distinct q_1, q_2 bounded by a fixed power of $4N_*$. The interval bounds on N_* place both below $\exp(x^{\varepsilon}/12)$ for all sufficiently large x in the interval. Their conductor contribution on the rational base is $q_1^2 q_2^2$. All nonzero ρ_b are conjugate over J by the matrices $\mathrm{diag}(b, 1, b^{-1})$, which commute with the involution in the semidirect product. They preserve the upper triangular flag. Moreover the strict tangent has zero first variation of every matrix trace, because ρ_{ℓ^m} reduces to the diagonal on M. Converting this realization into an upper bound on strict length would require control of integral directions over this reducible point. Counting its rational isomorphism class gives no such bound.

33 Rigidity and orthogonal monodromy

33.1 Ordinary deformations with a separated middle line

Use the field, primes, twists and ordinary marked ring of Section 32.8. Write $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $C = \mathrm{diag}(1, -1)$. Let B^{out} be the completed quotient obtained by setting the four matrix entries between the middle line and the two outer coordinates to zero, for every $h \in H = G_{\mathbb{Q}(i)}$. The group law and $M(c) = 1$ impose the same zeros on the other coset. Denote the invariant ring of this quotient by D^{out}. The tangent directions of the full rank-three ring may include deformations that leave this quotient.

Lemma 33.1 (An odd two-dimensional representation over the integral ring)**.** *There is a continuous representation $W : G_{\mathbb{Q}} \to \mathrm{GL}_2(B^{\mathrm{out}})$ of determinant η. For $h \in H$, $M(h)$ is $W(h)$ on the outer coordinates and 1 on the middle line. One has $W(c) = C$, and the residual representation is $\overline{\Psi} \oplus \overline{\eta}\,\overline{\Psi}^{-1}$. On $G_{\mathbb{Q}_\ell}$ there is a stable direct-summand line whose inertia character is Ψ, with quotient inertia character Ψ^{-1}. The line splits as a module over the coefficient ring.*

Proof. Work first in a finite Artinian quotient. The determinant of $M|_H$ is one on inertia at ℓ, by the three prescribed ordinary characters. On that inertia the middle character is also one. In fact its line is the eigenline of $M(g_*)$ for the middle eigenvalue, by its distinct residue. The first ordinary flag line belongs to the outer coordinates, and the middle line maps isomorphically to the second graded piece. These statements follow from the three spectral projectors in Lemma 32.10 over the Artinian ring. The ordered characters give the assertion on all inertia. At the other primes above ℓ, conjugating the flag or its reflected dual preserves triviality of the middle and determinant characters. On the remaining inertia groups both characters reduce to one and factor through a finite prime-to-ℓ group. The class group argument of Proposition 32.18 now gives determinant and middle character one on all of H.

Let $U(h)$ be the outer block. Thus $U(h) \in \mathrm{SL}_2$, and the global sign involution gives

$$U(chc) = J_2 U(h)^{-\mathsf{T}} J_2 = C U(h) C^{-1}.$$

The second identity follows by multiplying a two-by-two matrix of determinant one. Since $G_{\mathbb{Q}} = H \rtimes \langle c \rangle$, these identities define W with the indicated value at c. Reduction at the split point gives the stated two characters. On inertia at ℓ the first ordinary line belongs to the outer block and carries Ψ. The $\Psi(g_*)$-eigenline is unique. For any $g \in G_{\mathbb{Q}_\ell}$, conjugating this line by $W(g)$ gives another inertia line with the same character, because Ψ is global. It must coincide with the first line. Its quotient has the inverse character on inertia. All the constructions are compatible on Artinian quotients. The spectral projector at g_* exhibits the limit line as a direct summand, proving the statements over B^{out}. □

Theorem 33.2 (Finiteness of the integral trace algebra on the outer block)**.** *The ring D^{out} is finite over $\mathcal{O}$ for every large coefficient prime and every twist in the prescribed range. In particular, on this quotient every continuous local specialization of the invariant ring into $k'[[s]]$ over a finite residue extension k'/k, in characteristic ℓ, sends all traces to constants.*

Proof. Twist W by Ψ^{-1}. Its residual trace is $1+\bar{\chi}$, where its determinant is the fixed character $\chi = \eta\Psi^{-2}$. This determinant is odd. It is de Rham at ℓ as a cyclotomic power times an unramified finite character. Its residual restriction at ℓ is nontrivial because on inertia it is $\bar{\kappa}_\ell^{-2r}$ and $0 < 2r < \ell - 1$. Choose the finite set of primes containing ℓ and all divisors of $4N$. The representation is unramified outside this set. Its trace on $G_{\mathbb{Q}_\ell}$ is the sum of the two ordinary characters, and the one lifting the trivial residual character is unramified there, by Lemma 33.1.

Apply Pan's Theorem 5.1.2(1) for the totally real abelian base field $\mathbb{Q}$. In the setup of his Section 5.1.1 this is the ordinary pseudodeformation ring of the trace $1 + \bar{\chi}$ with fixed determinant χ and reducible local trace at ℓ. It is finite over $\Lambda = \mathcal{O}[[1 + \ell\mathbb{Z}_\ell]]$, the completed group algebra acting by local reciprocity through the restriction to inertia of the local character lifting 1. The local residual characters are distinct, so that character is unambiguously labelled. These hypotheses hold here, including that ℓ is unramified in the base field. [8]

The trace of $W\Psi^{-1}$ induces a map from this universal ring on all Artinian quotients, hence on their limit. Its weight character is trivial on $1 + \ell\mathbb{Z}_\ell$. Thus the map factors through the tensor product of the universal ring with $\mathcal{O}$ along the augmentation of Λ. The image is a finite $\mathcal{O}$-module. It contains the traces of W, obtained by multiplying the twisted traces by $\Psi(g)$. A finite $\mathcal{O}$-module in B^{out} is closed as a continuous image of a finite product of compact copies of $\mathcal{O}$.

To identify the invariant ring, the proof of Theorem 34.2 applies in this quotient. In detail, $Z = M(g_*) - M(cg_*^{-1}c)$ is $\mathrm{diag}(z, 0, -z)$ for a unit z, and $z^2 = \frac{1}{2}\operatorname{tr} Z^2$ is a polynomial in traces on H. The square root with the prescribed residue and the inverse of z^2 lie in the closed trace

algebra by expansion at their unit residues. The spectral projectors of Z are polynomials in $Z, 1/z$. Taking traces of products with those projectors inserted gives each diagonal entry and every opposite pair of outer entries, also when the entries come from different matrices. These generate the weight-zero entry monomials in this two-coordinate block. The entry differences topologically generate the deformation ring, so projection on finite truncations puts every invariant in the closed trace algebra. Conversely the traces are invariant. On H they are $1 + \operatorname{tr} W(h)$. Hence D^{out} is a submodule of the finite module above and is finite over the Noetherian DVR $\mathcal{O}$. Reduction of a finite local $\mathcal{O}$-algebra modulo ϖ is Artinian. Its image in the domain $k'[[s]]$ is its residue field, proving the last assertion. □

Theorem 33.3 (The modular form and its conductor)**.** *Let a characteristic-zero integral point of B^{out} have some nonzero entries in both outer off-diagonal directions. Here an integral point is a continuous local specialization into the integers of a finite extension of J. Then $W \otimes \kappa_\ell^r \alpha^{-1}$ is the Galois representation of a normalized primitive cuspidal eigenform f of weight $2r+1$, nebentypus $\eta\alpha^{-2}$, and level exactly the conductor of $\eta\alpha^{-2}$. In particular its level divides $4N$. If v is a rational prime split in $\mathbb{Q}(i)$, prime to $N\ell$, then with arithmetic Frobenius*

$$\operatorname{tr} M(\mathrm{Frob}_v) = 1 + v^{-r}\alpha(v)a_v(f). \tag{33.1}$$

For fixed ℓ, r, α the isomorphism class of this form with its coefficient embedding determines at most one scaling orbit of such points.

Proof. If $W|_H$ had an invariant line over the algebraic closure, the line would be stable under $U(g_*) - U(cg_*^{-1}c) = \operatorname{diag}(z, -z)$. Its distinct eigenvalues force the line to be a coordinate line. The two outer entry directions exclude both choices. Thus W is absolutely irreducible and it is odd since $W(c) = C$.

We check crystallinity. The stable line on $G_{\mathbb{Q}_\ell}$ and its quotient are an unramified character times κ_ℓ^r and an unramified character times κ_ℓ^{-r}, respectively. Their ratio V is crystalline of the form $\kappa_\ell^{2r}\nu$ with ν unramified. Over the coefficient field both $H^0(V)$ and $H^0(V^*(1))$ vanish by inertia, since $2r \geq 2$. Local duality and the local Euler characteristic give $\dim H^1(\mathbb{Q}_\ell, V) = 1$. The Bloch–Kato fundamental exact sequence for a crystalline representation gives $\dim H^1_f = \dim H^0 + \dim(D_{\mathrm{dR}}/\mathrm{Fil}^0) = 1$ in this positive cyclotomic twist. Hence every extension in the indicated order is crystalline. This applies to W and gives the two distinct Hodge–Tate weights. [12]

The representation W is continuous, odd, absolutely irreducible, finitely ramified, and de Rham with distinct weights at $\ell > 5$. Pan's regular Fontaine–Mazur theorem over $\mathbb{Q}$ therefore gives a cuspidal eigenform up to twist. The difference of the weights is $2r$. The twist displayed in the statement normalizes the two cyclotomic exponents to $0, 2r$. Any remaining de Rham character of weight zero over $\mathbb{Q}$, unramified almost everywhere, has finite order. Indeed its restriction to local units at ℓ has finite image by the classification of de Rham characters. At each other ramified prime the image of inertia is finite by local reciprocity. Global reciprocity for $\mathbb{Q}$ then gives a finite ray class character. Absorb such a character into the eigenform and take its primitive representative. The determinant equals $\kappa_\ell^{2r}\eta\alpha^{-2}$, giving the nebentypus. We use arithmetic Frobenius for the usual Deligne representation of a newform. [8]

At a finite $q \neq \ell$, inertia on W factors through the image of (η, α) on I_q, since it kills their common kernel. This is a finite abelian group of order prime to ℓ. Its two characters lift the distinct or repeated residual characters with multiplicity, by the integral idempotents after unramified coefficient extension. Prime-to-ℓ characters are determined by reduction. The two inertia characters are therefore α and $\eta\alpha^{-1}$. After the indicated twist these become 1 and $\eta\alpha^{-2}$. Monodromy is zero since inertia has finite image. Their Artin conductor is exactly that

of $\eta\alpha^{-2}$, including at wildly ramified primes. At ℓ the crystalline representation has unramified Weil–Deligne parameter and zero monodromy. Local–global compatibility and the newform conductor theorem identify these exponents with the primitive level. This gives the stated divisibility and (33.1). [13, 9]

An isomorphism of two such representations W commutes at c with C, hence is diagonal, say with entries a, b. On the outer block its conjugation is realized by $\mathrm{diag}(z, 1, z^{-1})$ with $z^2 = a/b$ over the algebraic closure. The middle line is fixed and $M(hc) = M(h)$. Thus the two marked systems are related by this scaling. Their invariant-ring values agree by the closed trace description and continuity. This gives the assertion about orbits. □

The character $\alpha(v)$ in (33.1) records the finite twist that gives the level $4N$. It will enter the integer bounding the trace differences. The ordinary two-dimensional input in the source is specified in `Development/AutomorphicArithmetic/AutomorphicVarieties/AutomorphicCases/AutomorphyUnramifiedCoefficients.lean`, line 19.

33.2 The cyclotomic restriction and the tangent space at an irreducible point

Keep the ordinary ring B of Section 32.8 and the same x, E, ℓ, r, α. A characteristic-zero integral point P takes its values in the integers of a finite extension of the coefficient field. Write $U = M_P|_H$ for its representation of $H = G_{K'}$, where $K' = \mathbb{Q}(i)$. An absolutely irreducible U means that it has no proper nonzero invariant subspace over the algebraic closure of its coefficient field. We also consider the integral realizations of nonconstant trace curves in the special fiber supplied by Theorem 34.6.

Lemma 33.4 (Strong irreducibility in characteristic zero)**.** *The determinant of $M|_H$ in the entire ordinary ring is one. For every sufficiently distant interval, an absolutely irreducible U at a characteristic-zero integral point remains absolutely irreducible on every open subgroup of H.*

Proof. Work first in a finite Artin quotient of B. On inertia at ℓ the determinant on H is the product $\Psi \cdot 1 \cdot \Psi^{-1}$, also at a conjugate place. Elsewhere it is a principal unit of order prime to ℓ by the inertia kernel condition, and thus equals one. The everywhere unramified determinant factors through the ideal class group of $\mathbb{Q}(i)$, which is trivial. Passing to the inverse limit proves the assertion over B.

Suppose the restriction of U became reducible over some finite extension of K'. It is then reducible on an open normal subgroup H_0 of H. Over the algebraic closure choose a simple H_0-submodule. Its translates under H span the three-dimensional space. Their sum is semisimple and all the simple constituents have the same dimension, which divides three. Reducibility forces this dimension to be one. The distinct character spaces are permuted transitively by H and have equal dimensions. Thus H_0 either acts by scalars or has three distinct character lines.

As ℓ is unramified in K', inertia at ℓ lies in H and its cyclotomic character surjects onto $\mathbb{Z}_\ell^\times$. The latter fact follows from total ramification of $\mathbb{Q}_\ell(\mu_{\ell^a})$ for every a. An open subgroup of inertia therefore contains an element s with $\kappa_\ell(s)$ of infinite order. The ordinary eigenvalues on s are $\kappa_\ell(s)^r, 1, \kappa_\ell(s)^{-r}$ since α is unramified there. Taking $s \in H_0$ excludes the scalar case.

In the other case the action on the three character lines gives a transitive permutation quotient of the compact image of U. It is continuous, since the stabilizer of a line is closed and there are only three lines. The kernel of reduction of this image is pro-ℓ, as is seen using the successive congruence subgroups of the integral matrix group. Its image in S_3 is trivial since $\ell > 3$. The residual matrix image is the cyclic group of the matrices $\mathrm{diag}(\bar{\Psi}(h), 1, \bar{\Psi}(h)^{-1})$. A generator must act by a three-cycle. Any lift $h \in H$ of that generator has trace zero in the

basis of the three permuted lines. Reduction then gives $a+1+a^{-1}=0$ for a generator a of $\bar{\Psi}(H)$, so $a^3=1$. On inertia the order of the image of $\bar{\Psi}$ is $(\ell-1)/\gcd(\ell-1,r)>3$, since $\exp(x^{\varepsilon}/12)>3t+1$ eventually. This is a contradiction. □

Theorem 33.5 (The image over the cyclotomic extension)**.** *Let U be as in Lemma 33.4 with absolutely irreducible image, or let U realize a nonconstant trace curve in the special fiber. Over the algebraic closure of the coefficient field let G denote the Zariski closure of $U(H)$. Its identity component G^0 is semisimple and acts irreducibly. For every finite extension L/K' the Zariski closure of $U(G_{L(\mu_{\ell^\infty})})$ contains G^0. In characteristic zero each of these cyclotomic images is enormous in the sense of Newton and Thorne.*

Proof. The inverse image of G^0 is closed and of finite index, so it is open. Lemma 33.4, or Theorem 34.6 in characteristic ℓ, shows that G^0 acts irreducibly. We take reduced algebraic subgroups over the algebraic closure throughout. The unipotent radical of G^0 has a nonzero fixed vector by triangularization. Its fixed space is G^0-stable by normality and hence is the whole space. The matrix action is faithful, so this radical is trivial. The connected reductive group G^0 has finite center by Schur's lemma and $\det U=1$. It is therefore semisimple.

The closure of $U(G_L)$ contains G^0 since finitely many of its cosets cover G. In the closure of the image of $[G_L,G_L]$ one finds all commutators of elements of G^0. Indeed the commutator is a morphism on the product and $U(G_L)\times U(G_L)$ is dense in the product of its closures. A connected semisimple group equals its algebraic derived group. This last statement follows, for example, from its generation by root groups and the formula $[t,x_\beta(a)]=x_\beta((\beta(t)-1)a)$ for a root β and a torus element t with $\beta(t)\neq 1$. Thus the indicated closure contains G^0. The cyclotomic character has abelian image, so it kills $[G_L,G_L]$. This proves the assertion for the full infinite cyclotomic extension.

In characteristic zero G^0 contains an element with three distinct eigenvalues, by applying the inertia argument of Lemma 33.4 to its open inverse image. It is also the identity component of the closure of each cyclotomic image above. Here are the details of the enormous-image test. The elements with distinct eigenvalues form a nonempty Zariski open subset of this connected component. The intersection of the cyclotomic image with that subset is dense in G^0. Since G^0 acts absolutely irreducibly, its linear span is the whole endomorphism algebra by Burnside's theorem. For any nonzero endomorphism X, the trace pairing therefore gives such an image element h with $\operatorname{tr}(hX)\neq 0$. Writing $h=\sum_\lambda \lambda e_\lambda$ in terms of its eigenprojectors, at least one $\operatorname{tr}(e_\lambda X)$ is nonzero. Applying this to an element of each simple submodule of the endomorphism algebra gives the required test. One may enlarge the coefficient field finitely so all these eigenvalues lie in it. Indeed an eigenvalue has degree at most three over the initial local field, which has only finitely many extensions of bounded degree in its algebraic closure. This is the criterion in Newton and Thorne, Section 2.5. [14] □

Corollary 33.6 (Tangent classes and potentially automorphic points)**.** *Suppose $2\le r<t$ and P is a characteristic-zero integral point for which U is absolutely irreducible. On $\mathcal{V}=\operatorname{End}(U)$ let H act by conjugation and let c act by $X\mapsto -J_3X^{\mathsf T}J_3$. The tangent space of the graded solution at P, modulo the diagonal scaling direction, injects into $H^1_f(\mathbb{Q},\mathcal{V})$, with scalars extended to the algebraic closure. Suppose that for some CM field $F\supset K'$ the representation $(U\otimes\kappa_\ell^{-1})|_{G_F}$ is attached to a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ in the normalization of Newton and Thorne. Then this Selmer group vanishes and the tangent space has dimension one.*

Proof. An algebraic tangent vector gives continuous infinitesimal matrices at P. To verify continuity without any finiteness assumption on the solution's invariant ring, use the ambient

graded ring in Lemma 32.17. Its invariant part is a quotient of a power series ring $\mathcal{O}[[Y_1,\dots,Y_d]]$ on finitely many invariant monomials. Enlarge the coefficient field to contain their values b_j, which are in its maximal ideal. Extending the integer coefficients finitely commutes with forming this power series ring by finite freeness. In the extended ring translation by $b=(b_j)$ is a continuous change of variables. At b a series modulo $(Y_1-b_1,\dots,Y_d-b_d)^2$ is its constant plus its ordinary linear term at b. Every algebraic derivation of this ring at P therefore acts by that linear term. The weight-zero series and the finitely many nonzero-weight variables generate the graded ring. After a common scalar multiple, all their generator derivatives are integral in one finite extension. Evaluating the variables plus their infinitesimal changes now converges in the dual numbers over that integer ring, and agrees with the original derivation on every homogeneous equation. Undo the scalar multiple. This produces continuous infinitesimal matrices over a finite characteristic-zero field.

Writing a lift as $(1+\epsilon a(g))M_P(g)$ makes a a one-cocycle for the displayed action, by the semidirect product law. At primes different from ℓ its restriction to inertia factors through the prescribed finite quotient. Averaging shows that this restriction is a coboundary.

Let K be a completion of K' at ℓ. On G_K both the ordinary representation and its infinitesimal lift have a flag with inertia exponents $r,0,-r$, in that order. The flag follows from the integral spectral projectors at g_* and Lemma 32.10, applied on successive Artin quotients. A change to the conjugate place takes a reflected dual flag and gives the same exponents. Regard a line over the dual numbers as a two-dimensional representation over the coefficient field. Each graded piece is then an unramified representation times its stated cyclotomic power, and is crystalline. The same is true of each graded line of the original point.

Build the representation down its flag. If T is the crystalline part already built and Q is the next graded piece, $V=\operatorname{Hom}(Q,T)$ is crystalline. All its cyclotomic exponents are at least two because $r\ge 2$. Consequently $H^0(K,V^*(1))=0$ and $\dim(D_{\mathrm{dR}}(V)/\mathrm{Fil}^0)=[K:\mathbb{Q}_\ell]\dim V$, with dimensions over the coefficient field. The first vanishing also follows directly from the flag since all graded exponents of $V^*(1)$ are strictly negative. Local duality, the Euler characteristic and the Bloch–Kato dimension formula now give

$$\dim H^1(K,V)=[K:\mathbb{Q}_\ell]\dim V+\dim H^0(K,V)=\dim H^1_f(K,V).$$

Thus every such extension is crystalline. Induction proves crystallinity of the specialization and of the underlying self-extension given by its infinitesimal lift. This places the local adjoint class in $H^1_f(K,\mathcal{V})$, by the crystalline extension criterion. [12] It also lies in $H^1_f(\mathbb{Q}_\ell,\mathcal{V})$. Indeed its image in cohomology with B_{cris}-coefficients vanishes upon restriction to K, and restriction followed by corestriction multiplies by the nonzero scalar $[K:\mathbb{Q}_\ell]$.

If the global cohomology class vanishes, write $a(g)=X-gX$. The mark at c gives $X=-J_3X^{\mathsf{T}}J_3$. On H, every infinitesimal matrix is the commutator of X with its original matrix. The marked difference $Z=M(g_*)-M(cg_*^{-1}c)$ stays diagonal, so $[X,Z]$ is diagonal. Since Z has distinct eigenvalues in the marked coordinates, X is diagonal. The equation at c then makes X a multiple of $\operatorname{diag}(1,0,-1)$. These multiples give precisely the infinitesimal scaling directions on entries, hence on the graded ring by the continuity argument. At an irreducible point some off-diagonal entry is nonzero, so the scaling direction has dimension one. This proves the injection.

Now assume the stated automorphy over F, and write F^+ for its maximal totally real field. Then $F=F^+K'$. The matrix J_3 is symmetric and the marked inverse-transpose identity identifies the c-conjugate of $U\kappa_\ell^{-1}$ with its dual times κ_ℓ^{-2}. This gives the polarization used in the adjoint Selmer theorem. Its pairing is unique up to scalar by absolute irreducibility on G_F. Therefore its adjoint module on G_{F^+} is the restriction of $\mathcal{V}$, including the indicated

negative adjoint action at c. The cyclotomic twist is trivial on $G_{F(\mu_{\ell^\infty})}$. Theorem 33.5 verifies the enormous-image hypothesis. Newton and Thorne's Theorem A thus gives $H^1_f(F^+, \mathcal{V}) = 0$. Restriction preserves the local Selmer conditions, and the restriction from $\mathbb{Q}$ is injective by corestriction in characteristic zero. This proves the vanishing over $\mathbb{Q}$ and the assertion about the tangent dimension. [14] □

For the full cohomology bound on the fields of Section 6.6, automorphy of all these three-dimensional points still requires a proof. So does exclusion of the characteristic-ℓ curves in Theorem 33.5. In the source the cyclotomic irreducibility argument occurs in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/TypeIIrrCyc.lean`, line 77 and the adjoint application in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/LiftingTheory.lean`, line 55.

33.3 Rigidity in the full space at the two-dimensional points

Let P be a characteristic-zero integral mixed point whose four adjacent entries vanish, in the ordinary deformation for the fields and intervals of Section 32.8. Theorem 33.3 gives a modular representation W of $G_{\mathbb{Q}}$ on the outer two coordinates with determinant η. Write J for a finite coefficient extension large enough for the point and its normalized primitive eigenform f. At a good prime we use arithmetic Frobenius for V_f, so that

$$k = 2r + 1, \qquad V_f = W \otimes \kappa_\ell^r \alpha^{-1}, \qquad \operatorname{level}(f) = f(\eta\alpha^{-2}) \mid 4N.$$

Here $f(\xi)$ on the right denotes the conductor of a character. We consider the full nine-dimensional adjoint module $\mathcal{V}$ from Corollary 33.6, including the entries between the outer coordinates and the middle coordinate. In particular this calculation tests infinitesimal deformations in the original three-dimensional ring.

Lemma 33.7 (The modular form at a mixed point has no complex multiplication)**.** *The representation $W|_H$ remains absolutely irreducible on every open subgroup of $H = G_{\mathbb{Q}(i)}$ on all sufficiently distant intervals. In particular f is a non-CM newform.*

Proof. Irreducibility on H follows from the two opposite outer entries and the marked matrix with two distinct outer eigenvalues, as in Theorem 33.3. If an open restriction is reducible, choose an open normal subgroup H_0 on which it is reducible. The translates of a simple submodule span the space. In dimension two the restriction is thus a sum of characters. If the two characters are equal, an open subgroup of inertia at ℓ acts by scalars. This is impossible since its eigenvalues on an element with cyclotomic value of infinite order are $\kappa_\ell(s)^r$ and $\kappa_\ell(s)^{-r}$.

If the characters differ, $W(H)$ permutes their lines transitively. The resulting continuous quotient of order two kills the pro-ℓ kernel of reduction. The residual image of H consists of $\operatorname{diag}(a, a^{-1})$ for $a \in \bar{\Psi}(H)$ and is cyclic. A lift of a generator must swap the lines, so its trace is zero. Thus a generator a satisfies $a + a^{-1} = 0$, giving order four. But on inertia the order of the image is at least $(\ell - 1)/r > 4$ throughout all sufficiently distant intervals. This proves the assertion about restriction. A CM newform comes by induction of a Hecke character of an imaginary quadratic field. Its Galois representation is a sum of characters on that field, as follows either from the induced representation or by comparing the Frobenius polynomials of the theta series. Restriction to its compositum with $\mathbb{Q}(i)$ would contradict what we have just proved. A character twist does not affect irreducibility on a subgroup. □

Theorem 33.8 (Vanishing for the full adjoint module at a mixed outer block)**.** *At every such P one has $H^1_f(\mathbb{Q}, \mathcal{V}) = 0$ for all $1 \le r < t$, including the first twist.*

Proof. For $h \in H$ the action is conjugation by the block matrix with outer block $W(h)$ and middle entry one. Complex conjugation acts on endomorphisms by the negative reflected transpose. The two scalar matrices on the outer and middle blocks therefore give two copies of η. On a traceless outer matrix X, the action at c is $-J_2 X^{\mathsf{T}} J_2 = CXC^{-1}$, with $C = W(c)$ as in Lemma 33.1. This gives the usual trace-zero adjoint of W. The space of maps from the middle line to the outer plane is $W|_H$ under H. The action of c exchanges it with the space of maps in the reverse direction. Their direct sum is consequently $\operatorname{Ind}_H^{G_{\mathbb{Q}}}(W|_H)$. Since W is already defined on $G_{\mathbb{Q}}$, this induction is $W \otimes (1 \oplus \eta)$. Altogether we obtain

$$\mathcal{V} \simeq \operatorname{ad}^0 W \;\oplus\; \eta^{\oplus 2} \;\oplus\; W \;\oplus\; (W \otimes \eta). \tag{33.2}$$

Bloch–Kato local conditions commute with this direct sum.

An $H^1_f(\mathbb{Q}, \eta)$ class restricts on $\mathbb{Q}(i)$ to an additive character unramified away from ℓ. At a place above ℓ, the unramified classes for the trivial coefficient field J are crystalline and have dimension one. The Bloch–Kato dimension formula for J gives $\dim H^1_f = 1$ there, so the character is unramified at this place as well. It factors through the finite ideal class group, which has no nonzero image in the additive group of J. Restriction is injective by corestriction. This disposes of both scalar summands. For $\operatorname{ad}^0 W$ the scalar twist relating W and V_f disappears under conjugation. Theorem B(1) of Newton and Thorne gives the vanishing for the adjoint of this non-CM regular algebraic cuspidal modular representation over $\mathbb{Q}$. Its trace-zero direct summand therefore has zero Selmer group. [14]

For the other two summands take $L = E\mathbb{Q}(i)$, which is finite abelian over $\mathbb{Q}$. Both α and η are trivial on G_L. Kato's representation $V_\lambda(f)$, where $\lambda \mid \ell$ is the coefficient place, uses the inverse of arithmetic Frobenius in its Euler polynomial. Thus $V_\lambda(f) \simeq V_f^\vee$ after extending coefficients. Indeed the Euler polynomial of its inverse Frobenius agrees with that of arithmetic Frobenius on V_f at every good prime, by Kato's formula (14.10.3). Chebotarev and equality of semisimplifications give the isomorphism since V_f is absolutely irreducible. Consequently

$$V_\lambda(f)(r) \simeq W^\vee \otimes \alpha \simeq W \otimes \eta\alpha,$$

using the exterior pairing in dimension two. Over L this is $W|_{G_L}$ and also $(W \otimes \eta)|_{G_L}$. Since $1 \le r \le k-1$ and $2r \neq k$, Theorem 14.2(1) of Kato gives finiteness of the Selmer group over L with divisible torsion coefficients obtained from a stable lattice in $V_\lambda(f)(r)$. This is the off-center case of that theorem. [15]

We explain the passage to field coefficients. Write V for Kato's twisted representation and T for the chosen lattice. Fix a finite set S of places of L containing the ramification, the archimedean places and those above ℓ. Classes in $H^1_f(L, V)$ are unramified outside S, so they lie in $H^1(G_{L,S}, V)$. Their images in $H^1(G_{L,S}, V/T)$ lie in Kato's finite group by his local definition, using the injection by inflation into global cohomology. A vector space is divisible as an additive group, and thus has zero image in a finite group. The coefficient sequence now puts this entire vector space in the image of $H^1(G_{L,S}, T)$.

This integral cohomology is a finitely generated module over the coefficient DVR. In fact the degree-one groups of all finite quotients of T are finite by finiteness of extensions of bounded degree unramified outside S. The degree-zero groups in the same inverse system are finite as well. Thus the integral group is their compact degree-one inverse limit. Reduction modulo the uniformizer embeds its quotient by that uniformizer into the first cohomology of the finite residual module. The compact form of Nakayama's lemma gives finite generation. The image of a finitely generated integral module in a vector space contains no nonzero vector subspace over the coefficient field. Hence $H^1_f(L, V) = 0$. The conclusion persists under every finite extension of coefficient fields by exactness of scalar extension and the definition of the local conditions.

Finally restrict either of the last two summands in (33.2) from $\mathbb{Q}$ to L. Their restrictions are zero in Selmer cohomology by the preceding calculation, and corestriction multiplies by $[L : \mathbb{Q}]$. This proves both vanishings over $\mathbb{Q}$. The use of L is an injective restriction for the point associated with the original period field E. □

Corollary 33.9 (Only scaling survives in the full ordinary tangent space)**.** *At every two-dimensional mixed point in Theorem 34.7, the geometric tangent space of the full graded solution has dimension one. It is generated by diagonal scaling. This holds uniformly over the indicated fields and intervals for all* $1 \le r < t$.

Proof. By Lemma 34.5 and Theorem 33.2, such a point has a continuous integral representative after scaling. The construction of the global adjoint cocycle from an algebraic tangent vector in Corollary 33.6 used only integrality of the representative and the finite-weight presentation. It applies in the full algebra B, so the cocycle may have arbitrary adjacent entries. Its inertia classes away from ℓ vanish by the same averaging.

At a place of K' over ℓ write K for the local field, of residue degree d. The ordinary flag of P has diagonal characters $\xi_0, 1, \xi_2$, of cyclotomic exponents $r, 0, -r$. They are crystalline, as is the representation W by Theorem 33.3. The primitive form f has level prime to ℓ. The two roots of its Hecke polynomial at ℓ have complex absolute value ℓ^r at every embedding by Deligne. Crystalline local–global compatibility identifies them with the arithmetic-normalized Weil–Deligne eigenvalues of V_f there. Undoing the twist $\kappa_\ell^r \alpha^{-1}$ makes the two eigenvalues for W have complex absolute value one. Passing to the unramified extension K raises them to the power d. The middle character has eigenvalue one as well. The eigenvalues of the three diagonal characters are exactly these eigenvalues, by exactness on crystalline representations. [10, 9]

It follows that no ratio of two of these diagonal characters equals κ_ℓ on G_K. Such a ratio would have a Weil–Deligne Frobenius eigenvalue of absolute value ℓ^d in the same normalization. In an infinitesimal ordinary deformation, each graded line, regarded over the coefficient field, has two copies of its original character as composition factors. Its unramified factor may acquire a Jordan block, which is crystalline. Now repeat the flag extension argument of Corollary 33.6. For an already constructed crystalline part T' and next graded piece Q', the graded exponents of $V' = \mathrm{Hom}(Q', T')$ are positive. No composition factor of V' is κ_ℓ, by the ratio calculation. Hence $H^0(K, (V')^*(1)) = 0$, also when $r = 1$. The local Euler characteristic and the Bloch–Kato dimension formula give $H^1(K, V') = H^1_f(K, V')$. This proves crystallinity of the whole self-extension and places the local adjoint class in H^1_f, descending to $\mathbb{Q}_\ell$ by corestriction.

The global class vanishes by Theorem 33.8. For a coboundary the mark at c makes its matrix X satisfy $X = -J_3 X^{\mathsf{T}} J_3$. The diagonal mark Z with distinct eigenvalues forces that matrix to be diagonal, exactly as in Corollary 33.6. It is a multiple of $\mathrm{diag}(1, 0, -1)$. The two opposite outer entries at a mixed point make the corresponding scaling tangent nonzero. This proves the assertion in the full solution. □

Thus on this part of the solution the integral trace algebra, the characteristic-zero tangent dimension and the uniform trace sum are all controlled. The intersection estimate of Theorem 31.8 still requires the corresponding arithmetic statements on the irreducible three-dimensional part. The matrix decomposition behind the Selmer calculation appears in `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/ReducibleThreeDim.lean`, line 263.

33.4 Integral representatives and the crystalline first twist

Let B be the ordinary marked deformation ring on the fields and intervals in Section 32.8. Denote its invariant ring by D and its graded algebra of finite sums of homogeneous elements by B^{gr}. The coefficient DVR is $\mathcal{O}$, with fraction field J. We take geometric characteristic-zero points with values in $\overline{J}$.

Lemma 33.10 (Integral representatives of all algebraic points)**.** *Every $\mathcal{O}$-algebra homomorphism $D \to \overline{J}$ is a continuous local map to the integers of a finite extension of J. Every geometric point of B^{gr} can consequently be scaled to a continuous integral point of B.*

Proof. The invariant ring D is a complete local quotient of a formal power series ring in finitely many variables over $\mathcal{O}$, with the same residue field, by Lemma 32.17. Let $\xi : D \to \overline{J}$, $d \in \mathfrak{m}_D$, and write $y = \xi(d)$. If $y \neq 0$ and $v(y) \leq 0$, every conjugate of $1/y$ over J is integral since the valuation on the complete local field has a unique extension. Thus

$$f(T) = \prod_{\sigma : J(y) \hookrightarrow \overline{J}} (1 - T/\sigma(y)) \in \mathcal{O}[T], \qquad f(0) = 1.$$

Then $f(d)$ is a unit in D with zero image under ξ, a contradiction. Every element of $\mathfrak{m}_D$ therefore has zero image or strictly positive valuation. Subtracting an integral lift of the residue shows that $\xi(D)$ is integral.

Let $d_1, \dots, d_b$ be the images of formal parameters. Together with a uniformizer they generate $\mathfrak{m}_D$. All their images belong to one finite extension J'. The minimum valuation of their nonzero images and that of the uniformizer is a positive number a. Every element of $\mathfrak{m}_D^n$ has image of valuation at least na, since all coefficients in a finite expression in the ideal generators have integral image. It follows that ξ is continuous and agrees, on any power series, with the limit of the images of the polynomial truncations. The complete field J' contains each such limit. This proves the first assertion. For a point of B^{gr} apply Lemma 34.5 to its invariant values and the prescribed point of the fiber. The resulting scaling gives the integral representative. □

Theorem 33.11 (Crystallinity on the entire ordinary component)**.** *Let K be a completion of $K' = \mathbb{Q}(i)$ at ℓ. Restrict the local ordinary component of Sections 32.4–32.7 to G_K. Every characteristic-zero specialization in a finite local J-algebra, obtained from its framed deformation ring, is crystalline with the labeled Hodge type of the three cyclotomic exponents $r, 0, -r$. This holds for the whole range $1 \leq r < t$, including specializations to dual numbers at points with an adjacent character ratio equal to κ_ℓ.*

Proof. Write $A = U_{\mathrm{loc}}/I_t$ for the ordinary local component. It is a formal power series ring over $\mathcal{O}$ by Theorems 32.11 and 32.15. Use first the place where its ordered flag is defined. The restricted representation of G_K is an ordinary matrix representation since the sign character is trivial there. At a field-valued integral specialization it has diagonal graded characters equal to unramified characters times $\kappa_\ell^r, 1, \kappa_\ell^{-r}$ in that order.

Set $a = \Delta$ as in Section 32.7 when $r = 1$, and set $a = 1$ when $r \geq 2$. In the first case $\Delta \neq 0$ in A, by the formula for its image in the diagonal parameters in Theorem 32.15. At a field point where $a \neq 0$, no upper graded character divided by a lower one equals κ_ℓ on G_K. For $r \geq 2$ their exponents already differ by at least two. For $r = 1$, the outer pair has difference two. In an adjacent pair the factor q_j in Δ is a nonzero constant times $\xi_j(\gamma) - \kappa_\ell(\gamma)\xi_{j+1}(\gamma)$ for the positive-sign element $\gamma \in G_K$. Its nonzero value excludes the asserted equality of characters.

At such points all three graded characters are crystalline. Given a crystalline part T already constructed from the top of the flag and the next graded character Q, let $V = \mathrm{Hom}(Q, T)$. Its

graded cyclotomic exponents are positive and it has no composition factor equal to κ_ℓ. Thus $H^0(K, V^*(1)) = 0$ and the dimension of $D_{\mathrm{dR}}(V)/\mathrm{Fil}^0$ is $[K : \mathbb{Q}_\ell] \dim V$ over the coefficient field. The local Euler characteristic and the Bloch–Kato dimension formula give $H^1(K, V) = H^1_f(K, V)$. Every extension at this step is crystalline, proving the assertion at all these field points with their indicated Hodge type. [12]

Now apply the framed potentially crystalline deformation theorem of Kisin to G_K, the trivial inertial type, and the indicated Hodge type at every embedding of K. We may first enlarge the finite coefficient field to contain the fields of definition of the embeddings. Formation of our power series ring commutes with this finite flat coefficient extension. Kisin gives a quotient of the framed ring of the residual three-dimensional representation that tests the prescribed type on every finite local coefficient algebra in characteristic zero. Potential crystallinity of trivial inertial type here is crystallinity. Indeed over a finite Galois extension where a representation is crystalline, inertia acts trivially on its crystalline period module when the inertial type is trivial. Semilinear descent on the maximal unramified coefficient field then gives the full dimension of crystalline periods downstairs. For finite local coefficients one can apply this dimension test to the underlying vector space. A finite inertial action with trivial type acts trivially there by semisimplicity in characteristic zero. [16, Theorem 1.2.2]

The restricted framed family gives a homomorphism from this universal framed ring to A. Every element in the image of the defining ideal of the crystalline quotient vanishes at the integral field points where $a \neq 0$. These points detect every nonzero element of A. In fact, if $u \in A$ is nonzero, consider the nonzero series ua. A nonzero one-variable integral series takes a nonzero value at some sufficiently small nonzero power of the uniformizer. Its lowest nonzero term then has strictly smaller valuation than its tail. In several variables first make one nonzero coefficient series nonzero by induction, then apply the one-variable assertion to the remaining variable. All chosen coordinates have positive valuation. Thus ua is nonzero at an integral field point. This proves that the whole crystalline ideal maps to zero in A. Every specialization now factors through Kisin's quotient, including a specialization with nonzero nilpotents. The simple rank and period conditions descend by a finite extension of the coefficient field. At a conjugate place the family is conjugate to the original family or its reflected dual. Its Hodge type is the same after reflection and duality, and crystallinity is preserved. This proves the theorem at every place above ℓ. □

Corollary 33.12 (The first twist at arbitrary irreducible mixed points)**.** *For every geometric irreducible mixed point of B^{gr} the injection into $H^1_f(\mathbb{Q}, \mathcal{V})$ in Corollary 33.6 holds throughout $1 \leq r < t$. At any such point which is potentially automorphic as specified in that corollary the geometric tangent dimension is one.*

Proof. Lemma 33.10 allows an integral representative. An algebraic tangent at it yields continuous infinitesimal matrices after a common scalar multiple, by the power series argument of Corollary 33.6. Theorem 33.11 puts their restriction at every place of K' over ℓ in the crystalline self-extensions. Therefore the class satisfies the Bloch–Kato local condition at ℓ, using restriction and corestriction as in that corollary. Its inertia classes away from ℓ vanish by finite averaging. The two marks identify the kernel with diagonal scaling. Finally Theorem 33.5 supplies the cyclotomic image hypothesis over the CM field of potential automorphy, and the adjoint Selmer vanishing descends to $\mathbb{Q}$ by corestriction, as in Corollary 33.6. All these arguments now apply at $r = 1$ as well. □

33.5 Orthogonal monodromy on the ordinary component

Take U from Theorem 33.5. Thus U is either an absolutely irreducible integral characteristic-zero point or an integral realization of a nonconstant trace curve in residue characteristic ℓ. Write F for its local coefficient field, R for the valuation ring, and k_F for its finite residue field. We may extend F finitely. In equal characteristic the symbols Ψ, α, η denote the images of the original characters under the coefficient map. On $H = G_{\mathbb{Q}(i)}$ the residual matrix in the marked lattice is $\operatorname{diag}(\bar{\Psi}, 1, \bar{\Psi}^{-1})$. On the intervals under consideration the order of $\bar{\Psi}(I_\ell)$ exceeds four for every sufficiently large x, uniformly for $\ell > \exp(x^{\varepsilon}/12)$ and $1 \le r < t$.

Theorem 33.13 (The two possible monodromy groups)**.** *Over $\overline{F}$ the Zariski closure of $U(H)$ is either SL_3 or the special orthogonal group of a nondegenerate symmetric bilinear form S. The full image preserves S in the latter case, and its projective class of forms is unique.*

Proof. By Theorem 33.5 the identity component G^0 is semisimple and irreducible in its faithful three-dimensional representation, and the full group G is in SL_3. The rank of G^0 is one or two. Rank zero would give the trivial group. We use the root subgroups and Weyl group of a connected reductive group over an algebraically closed field. [17]

In rank two its maximal torus is conjugate to the full diagonal torus of SL_3. A root of G^0 occurs in its Lie algebra and hence is one of the characters t_i/t_j on the ambient off-diagonal matrix entries. Its root subgroup is $I + aE_{ij}$, $a \in \overline{F}$. Indeed a parametrization by the additive group is equivariant for scaling the parameter by t_i/t_j. A term a^m in an entry must have that entry's torus character equal to $(t_i/t_j)^m$. Among the ambient characters only t_i/t_j itself is a positive multiple of this root in additive weight notation. Hence the parametrization has only the indicated linear term, which is nonzero by the inclusion of its tangent space. Roots occur in opposite pairs. The graph of these pairs on the three coordinates is connected, since otherwise the torus and all root groups preserve a proper coordinate subspace and together generate G^0. A connected graph gives all three pairs using commutators of elementary matrices. The groups for these pairs generate SL_3.

In rank one the nonidentity Weyl element inverts a maximal torus. The trace on the representation therefore equals the trace on its dual on all conjugates of that torus, a dense subset of G^0. It agrees everywhere. These traces identify the two irreducible representations in either coefficient characteristic. Indeed, if two absolutely irreducible modules are inequivalent, the image of the group algebra on their direct sum is a product of two full matrix algebras. Its radical annihilates both simple modules and is zero in this faithful image, so this follows from the semisimple algebra decomposition. A rank-one projector on just the first factor would distinguish the traces already on a linear combination of group elements. Thus there is a nondegenerate invariant bilinear form, unique up to scalar by Schur's lemma. Its transpose is a multiple by a scalar of square one. Since the dimension is odd and the characteristic differs from two, the form is symmetric. A connected semisimple group of rank one has dimension three by its root decomposition. Its closed image therefore equals $\mathrm{SO}(S)$, also connected of dimension three.

The group G normalizes G^0 and hence preserves the line of S. Write $U(h)^{\mathsf{T}} S U(h) = \nu(h) S$. The determinant gives $\nu^3 = 1$. From $U \simeq U^{\vee} \otimes \nu$, reduction in stable lattices gives the same isomorphism on residual semisimplifications. Here and below lattice independence follows from equality of the characteristic polynomials after reduction and the Brauer–Nesbitt theorem on the common finite residual quotient. In particular $\bar{\nu}$ is one of $1, \bar{\Psi}, \bar{\Psi}^{-1}$ on H, since the residual dual has a trivial constituent. The order on inertia excludes the last two choices. Reduction is injective on cube roots of unity at $\ell > 3$, so $\nu = 1$. We have $G^0 \subset G \subset \mathrm{SO}(S) = G^0$, as asserted. □

Lemma 33.14 (An integral lift of the projective representation)**.** *In the orthogonal case, after a finite coefficient extension there are an integer $e \in \{0,1\}$ and a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ satisfying*

$$\mathrm{Ad}^0 \rho|_H \simeq U, \qquad \phi := \det \rho = \eta^e \Psi, \qquad \bar{\rho} = \mathrm{diag}(\bar{\phi}, 1). \tag{33.3}$$

Here Ad^0 is conjugation on trace-zero two-by-two matrices. The parity e is determined by the marked representation U. For $e = 0$ one has $\rho(c) = I$. For $e = 1$ its eigenvalues at c are $1, -1$.

Proof. The symmetric form S can be defined over a finite extension. With the marked involution J_3 the relation on H becomes

$$U(chc) = J_3 U(h)^{-\mathsf{T}} J_3 = A U(h) A^{-1}, \qquad A = J_3 S.$$

Since $c^2 = 1$, irreducibility gives $A^2 = bI$ for a nonzero scalar b. Symmetry of S and J_3 gives $A^{\mathsf{T}} S A = S A^2 = bS$. Put $C = (b/\det A)A$. Taking determinants shows $(\det A)^2 = b^3$, so $C \in \mathrm{SO}(S)$ and $C^2 = I$. The semidirect product $G_{\mathbb{Q}} = H \rtimes \langle c \rangle$ thus extends U into this orthogonal group, with c acting by C. Multiplying S by a scalar has no effect on C. It is the unique element of $\mathrm{SO}(S)$ implementing this conjugation, since the centralizer of $U(H)$ consists of scalars and a scalar in SO_3 is one.

Over a finite extension identify this split orthogonal group with PGL_2 acting on $\mathfrak{sl}_2$ by conjugation. For this identification the trace pairing on $\mathfrak{sl}_2$ is nondegenerate since the characteristic differs from two. Conjugation is faithful on PGL_2 and identifies it with the full special orthogonal group, as one can also see by recovering the matrix algebra from the pairing and bracket in Theorem 32.6. Write w for the resulting continuous projective representation of $G_{\mathbb{Q}}$.

We give the lattice argument since S in the original marked basis need not be unimodular. The local field F has finitely many square classes by the valuation, finiteness of its residue field and Hensel's lemma. Adjoin square roots of their representatives. Every element of the original $\mathrm{PGL}_2(F)$ now has a determinant-one lift. The preimage of the compact group $w(G_{\mathbb{Q}})$ in SL_2 over the larger local field is compact. Indeed the determinant-one covering is finite and proper. Equivalently one may cover the compact image by finitely many compact pieces in matrix charts, normalizing one entry to one. On each piece its determinant and the inverse determinant are bounded. Its determinant-one representatives are therefore bounded as well, and their union is closed. The span over the integer ring of the orbit of a lattice under this compact group is a stable lattice, since it contains the original lattice and is contained in a fixed scalar multiple of it. Choosing a basis makes w integral projectively.

Reduce in that basis. On H, the adjoint of $\bar{w}$ has semisimple constituents $\bar{\Psi}, 1, \bar{\Psi}^{-1}$ by lattice independence. In a basis adapted to a composition series, the image of $\ker(\bar{\Psi}|_H)$ therefore consists of unipotent upper triangular matrices. Its projective image is an ℓ-group because the adjoint is faithful and the residue field is finite. This subgroup is normal in the whole residual projective image. If nontrivial, it fixes a unique line of the two-dimensional projective representation. Indeed a central element of order ℓ is nontrivial projectively unipotent and has a unique eigenline, which its centralizer preserves. The unique common fixed line is preserved by every normalizer. If the subgroup is trivial, $\bar{w}(H)$ is cyclic of order equal to that of $\bar{\Psi}(H)$, as the adjoint diagonal characters show. Conjugation by c acts trivially on this cyclic quotient because $\bar{\Psi}$ is global. The centralizer of a projective semisimple generator of order greater than two preserves both of its eigenlines individually. Thus in both cases $\bar{w}(G_{\mathbb{Q}})$ has an invariant line over a finite residue extension.

Use this line to make the projective reduction upper triangular. Integral invertible representatives then have their lower-left entries in the maximal ideal. Conjugate by $\mathrm{diag}(t, 1)$ with

$0 < v(t) < v(\varpi_F)$ after a finite ramified extension. Both off-diagonal entries of every representative now belong to the maximal ideal, and determinants stay units. The residual projective image is diagonal and its ratio character on H is $\bar{\Psi}$ or $\bar{\Psi}^{-1}$ by the same adjoint constituents. Exchange the axes if necessary. On $G_{\mathbb{Q}}$ this ratio must equal $\bar{\eta}^e\bar{\Psi}$ for exactly one $e \in \{0,1\}$, since $G_{\mathbb{Q}}/H$ is quadratic. The nontrivial projective involution at c, if present, stays nontrivial after reduction. The projective congruence kernel is pro-ℓ, as follows from successive matrix congruences modulo scalars. Hence $e = 0$ exactly when $C = I$.

For each g scale an integral projective representative to have reduction exactly $\operatorname{diag}(\bar{\phi}(g), 1)$. The quotient of $\phi(g)$ by its determinant is a principal unit, with a unique principal square root. Scaling by that root gives the unique matrix with the prescribed determinant and residual matrix. Uniqueness makes the result multiplicative. Continuity follows either from square roots on principal units or from the continuous inverse of $D \mapsto ([D], \det D)$ on the compact set of integral matrices with diagonal reduction and lower-right residue one. This map is injective there because a scalar of square and residue one equals one. This constructs ρ. Since $\Psi(c) = 1$, its determinant at c is $(-1)^e$. The eigenvalues of an involution and their reductions now give the stated action. This construction applies over a Laurent series coefficient field of characteristic ℓ as well. □

Theorem 33.15 (Local characters and crystallinity of the lift)**.** *The lift ρ is absolutely irreducible on H. On $G_{\mathbb{Q}_\ell}$ it has a stable line of inertia character Ψ, and the quotient is unramified. Away from ℓ it kills $I_p \cap \ker\eta \cap \ker\alpha$. In characteristic zero it is crystalline at ℓ with cyclotomic exponents $r, 0$, and its prime-to-ℓ Artin conductor is exactly the conductor of $\eta^e\alpha$.*

Proof. If $\rho|_H$ preserved a line L, the endomorphisms with image in L and vanishing on L would form an invariant line of $\mathfrak{sl}_2$. This contradicts the irreducibility of U. On I_ℓ, the first line of the ordinary flag in U has character Ψ and is distinguished by the simple eigenvalues at g_*. Transfer this line into $\mathfrak{sl}_2$. A generator X satisfies $\operatorname{tr}(X^2) = 0$, because conjugation preserves that trace and Ψ^2 is nontrivial on inertia. Hence X is nilpotent of rank one, since $\operatorname{tr} X = 0$ and two is invertible. Its image equals its kernel. The line of such matrices is stable under $G_{\mathbb{Q}_\ell}$, since conjugating the inertia group preserves its Ψ-character and the $\Psi(g_*)$-eigenspace is one-dimensional. Its kernel therefore gives a stable line for ρ on the whole decomposition group.

Write a, b for the characters on this line and its quotient. Conjugation on the nilpotent direction has character a/b. Thus $a/b = \Psi$ on I_ℓ and $ab = \phi = \Psi$ there, so $b^2 = 1$ on inertia. The two residual characters there are $\bar{\Psi}, 1$. Reduction of the line and quotient gives these characters in some order, since the intersection of a field-valued line with a lattice is saturated. The order of $\bar{\Psi}(I_\ell)$ exceeds two, so $\bar{b} = 1$ there. It follows that $b = 1$ on inertia, and $a = \Psi$ there.

In characteristic zero restrict to G_K, with $K = \mathbb{Q}_\ell(i)$ in the chosen completion. By Theorem 33.11 the trace-zero adjoint is crystalline there for every r. In a flag basis write $\rho(g) = \left(\begin{smallmatrix} a(g) & b(g)z(g) \\ 0 & b(g) \end{smallmatrix}\right)$. Its action on E_{12} is by a/b and it sends $\operatorname{diag}(1,-1)$ to $\operatorname{diag}(1,-1) - 2z(g)E_{12}$. The span of these two matrices is consequently isomorphic to $\rho \otimes b^{-1}$ by rescaling the off-diagonal basis vector. It is a subrepresentation of a crystalline representation, and b is unramified. This proves crystallinity of $\rho|_{G_K}$. As $K/\mathbb{Q}_\ell$ is unramified of degree one or two, semilinear descent of crystalline periods gives crystallinity over $\mathbb{Q}_\ell$. The two exponents follow from the diagonal characters.

If $g \in I_p \cap \ker\eta \cap \ker\alpha$ with $p \neq \ell$, then $U(g) = 1$ and $\phi(g) = 1$. Thus $\rho(g)$ is a scalar of square one reducing to 1, so is the identity. The residual inertia characters at such a prime are $\overline{\eta^e\alpha}, 1$. The full inertia factors through the finite abelian image of (η, α), of order prime

to ℓ. Reduction identifies its characters with their prime-to-ℓ lifts. In characteristic zero the two characters on inertia are therefore $\eta^{e}\alpha, 1$, with zero monodromy. This gives the asserted conductor, including the possible wild conductor at p. □

33.6 Frobenius primes for the full invariant ring

Retain the period field E, $x \in [V, V^{3/2}]$, $\ell > \exp(x^{\varepsilon}/12)$, $1 \le r < t$, and α of Section 32.8. Write $H = G_{\mathbb{Q}(i)}$, $\Psi = \kappa_{\ell}^{r}\alpha$, and $M|_{H}$ for the universal matrix on the ordinary component. The invariant subring $D \subset B$ has the augmentation $\epsilon : D \to \mathcal{O}$ given by the diagonal representation. Set $I = \ker \epsilon$, write $\mathfrak{m}_D, \mathfrak{m}_B$ for the two maximal ideals, and let $q = \#k$ be the cardinality of their common residue field. For $h \in H$ put

$$t_h = \operatorname{tr} M(h) - \left(\Psi(h) + 1 + \Psi(h)^{-1}\right) \in I.$$

An oriented split prime is a rational prime split in $\mathbb{Q}(i)$ together with a choice of one place of $\mathbb{Q}(i)$ over it. We take arithmetic Frobenius at that place. Outside the ramification set its H-conjugacy class determines t_h.

Theorem 33.16 (Frobenius generators before specialization)**.** *Put $d = \dim_k I/\mathfrak{m}_D I$. Given any finite set of excluded primes, there exist d oriented split primes outside it and outside $2N\ell$ for which*

$$I = (t_{\mathrm{Frob}_{v_1}}, \dots, t_{\mathrm{Frob}_{v_d}}). \tag{33.4}$$

They can be selected from conjugacy classes in a single finite Galois extension determined by the universal matrices modulo $\mathfrak{m}_B^4$. If s is the number of parameters in the presentation of Lemma 32.17, this extension has degree at most $2q^{9\binom{s+4}{3}}$ over $\mathbb{Q}$. If s_j parameters have weight j, the length d of the list is bounded by

$$s_0 + s_1 s_{-1} + s_2 s_{-2} + \binom{s_1+1}{2} s_{-2} + \binom{s_{-1}+1}{2} s_2.$$

Proof. The quotient of D by the ideal generated by all t_h is topologically generated by the scalars. Indeed D is topologically generated by the traces, by Theorem 34.2. Every ideal of a complete Noetherian local ring is closed, by applying Krull intersection to its quotient. The image of $\mathcal{O}$ in this quotient is closed by compactness. Thus the scalars fill the quotient. Augmentation splits the scalar inclusion, proving that the ideal generated by all t_h is I. Choose $h_1, \dots, h_d$ whose trace differences give a basis of $I/\mathfrak{m}_D I$.

We verify what is detected by the truncated matrices. In the ambient presentation $P = \mathcal{O}[[X_1, \dots, X_s]]$, take each parameter to vanish at the diagonal point as in Lemma 32.17. The weights belong to $\{0, 1, -1, 2, -2\}$, and the uniformizer has weight zero. A weight-zero monomial of total degree at least four, counting the uniformizer among its factors, splits into two nonempty weight-zero monomials. A zero weight or a pair of opposite weights can be taken out first. If neither occurs, the zero sum consists of triples of weights $1, 1, -2$ or $-1, -1, 2$. This also proves that the invariant monomials of degrees one, two and three generate the invariant power series ring topologically. If $y \in D \cap \mathfrak{m}_B^4$, lift it into the fourth power of the ambient maximal ideal. Taking the weight-zero part still maps to y, since the relations are homogeneous and extraction of each weight commutes with passage to B, by Lemma 32.17. The monomial factorization and closedness of $\mathfrak{m}_D^2$ give

$$D \cap \mathfrak{m}_B^4 \subset \mathfrak{m}_D^2.$$

The single variables of weight zero, the opposite pairs, and the two types of triples in the parameter variables all vanish at the augmentation. Together with $\mathcal{O}$ they generate D topologically. Their classes span $I/\mathfrak{m}_D I$, which gives the stated count. The augmentation extends

to B at the split point and maps $\mathfrak{m}_B^4$ into $\varpi^4\mathcal{O}$, where ϖ is a uniformizer. Also $I \cap \mathfrak{m}_D^2 = \mathfrak{m}_D I$. Indeed $\mathfrak{m}_D = \varpi\mathcal{O} \oplus I$ as an $\mathcal{O}$-module and its square is $\varpi^2\mathcal{O} \oplus (\varpi I + I^2)$. Hence $M(h)$ modulo $\mathfrak{m}_B^4$ determines t_h modulo $\mathfrak{m}_D I$.

The kernel in H of this finite matrix representation is normal in $G_{\mathbb{Q}}$, since conjugation by c acts on the matrices by the marked inverse transpose. Its fixed field L contains $\mathbb{Q}(i)$ and is unramified outside $2N\ell$. The maximal ideal of P has $s+1$ generators including ϖ. There are $\binom{s+4}{3}$ monomials of degree at most three in them. Their graded pieces bound $\#(B/\mathfrak{m}_B^4)$ by $q^{\binom{s+4}{3}}$. Counting the nine entries of a matrix and the two cosets of H gives the claimed degree bound.

For each h_i the Chebotarev theorem supplies infinitely many rational primes with its conjugacy class in $\mathrm{Gal}(L/\mathbb{Q})$. They split in $\mathbb{Q}(i)$. Choosing a prime of L above each one realizes the specified element of the class, fixing the orientation. Its trace difference has the required residue in $I/\mathfrak{m}_D I$. Nakayama's lemma now gives (33.4). For $d = 0$ the same lemma gives $I = 0$. □

Corollary 33.17 (Size and specialization of the prime list)**.** *Let S contain $2, 3, 5$, the divisors of $N\ell$, and any further primes to be excluded, and put $b_S = \prod_{p\in S} p$, $n_* = 4q^{9\binom{s+4}{3}}$. The oriented primes in* (33.4) *can all be taken with*

$$\log v_i \;\le\; C n_* \log(b_S n_*) \tag{33.5}$$

for an effective absolute constant C. Two orientations may use the same underlying rational prime in this bound. For any integral specialization into a valuation ring R with valuation v_R, the same list computes the full trace ideal. More precisely,

$$\inf_{h\in H} v_R(t_h) = \min_{1\le i\le d} v_R(t_{\mathrm{Frob}_{v_i}}),$$

where the elements on both sides are evaluated in R; the value for the zero ideal or the empty list is $+\infty$.

Proof. Adjoin $\sqrt{b_S}$ to the field L of the theorem. The resulting Galois field L' has degree $n \le n_*$. Every prime of S ramifies in the quadratic subfield since b_S is squarefree and even. The entire extension is unramified outside S. For $p \in S$, write e for the ramification index in L'. The local different exponent is at most $e - 1 + e\,v_p(e)$. To see this, pass to the maximal unramified subextension of the completion. A uniformizer of the remaining totally ramified extension generates the integer ring, and its Eisenstein polynomial has degree e. The nonzero terms of its derivative at the uniformizer have valuations in distinct residue classes modulo e. Its valuation is therefore at most that of the leading derivative term, $e-1+e\,v_p(e)$. The different is generated by this derivative. Summing over the primes over p gives $v_p(|\operatorname{disc} L'|) \le n(1+v_p(n))$. It follows that $\log|\operatorname{disc} L'| \le n(\log b_S + \log n)$.

Lift each required Frobenius element of $\mathrm{Gal}(L/\mathbb{Q})$ to $\mathrm{Gal}(L'/\mathbb{Q})$. The unconditional least-prime theorem of Lagarias, Montgomery and Odlyzko applied to its conjugacy class gives an unramified rational prime $\ll |\operatorname{disc} L'|^{C_0}$ with an effective absolute C_0. Such a prime automatically avoids S. Orient it as in the theorem. This proves (33.5). For the last assertion (33.4) gives equality of the generated ideals after every integral specialization. The valuation of an ideal generated by finitely many elements in a valuation ring is the least valuation of any generator. [18] □

Theorem 33.18 (Simultaneous separation of irreducible points)**.** *Fix finitely many irreducible characteristic-zero points or realizations of nonconstant trace curves from Theorem 33.5, allowing different coefficient primes, twists and period fields from the prescribed family. Among*

rational primes split in $\mathbb{Q}(i)$, a set of relative natural density one has nonzero trace difference at both places for every member of the list. The same holds within any nonempty finite Frobenius condition compatible with splitting, interpreted on its split part. In (33.4) *each prime can therefore also be chosen to separate this entire finite list, without the size assertion* (33.5)*.*

Proof. Fix one representation U over its local field. Consider the compact image Γ of (U, Ψ) on H. Over the algebraic closure of the coefficient field let G denote the algebraic monodromy of U and C_Ψ the Zariski closure of $\Psi(H)$ in the multiplicative group. Theorem 33.13 identifies G with SL_3 or SO_3. Restrict to $G_{K'E(\mu_{\ell^\infty})}$. Here $\Psi = 1$ and, by Theorem 33.5, the closure of the matrix image contains G. Thus the closure of Γ contains $G \times 1$. Multiplying by this subgroup shows that its closure is the full $G \times C_\Psi$. On each connected component the regular function

$$(g, z) \longmapsto \operatorname{tr}(g) - z - 1 - z^{-1}$$

is nonconstant: in each of the two possible groups G a one-parameter torus has eigenvalues $a, 1, a^{-1}$. Consequently its zero set contains no component.

Here is a measure argument valid in both coefficient characteristics. If a compact linear group is Zariski dense in an algebraic group, its intersection with a closed subvariety containing no component has Haar measure zero. Pass first to the identity component and translate each of the finitely many cosets back to it. In the connected case every open subgroup of the compact group is Zariski dense: its finitely many translates cover a dense subgroup of an irreducible variety. The same is true of every nonempty open subset, since here the compact groups are profinite. If a proper closed subvariety has positive measure, choose an irreducible one Z of least dimension with that property. Its intersection A with the compact group is measurable. Continuity of translations in L^1 implies $\mu(A \cap uA) > 0$ for all u in a neighborhood of the identity. For instance approximate 1_A in L^1 by functions on finite quotients. Some u in that neighborhood satisfies $uZ \neq Z$. Otherwise the orbit of any point of Z under the neighborhood is dense in the whole connected group and is contained in Z. The closed set $Z \cap uZ$ has smaller dimension and positive measure, contrary to the choice of Z. This proves the claim.

Pulling back the zero set from Γ gives a closed Haar null set in H. Take the union for the finitely many representations and for their conjugates by c, to treat both choices of place. The union is conjugacy invariant in $G_{\mathbb{Q}}$, closed and null. Its images in finite Galois quotients have inverse images of arbitrarily small measure: the clopen neighborhoods of the closed set decrease to it and Haar probability is regular. On each finite quotient the Chebotarev theorem identifies the density of the corresponding unramified primes with this measure. It follows that the exceptional primes have upper natural density zero. Any fixed nonempty compatible finite Frobenius condition on split primes has positive density. Removing the exceptional primes preserves its density, which proves all the assertions. □

Equations (33.4) and (33.5) concern the original invariant ring even when it has positive-dimensional special fiber. They give a finite test for congruence with the diagonal. The size bound depends on the coefficient prime, and it bounds the sampling primes. A uniform bound for the sum of trace valuations on the remaining irreducible loci is still required in the intersection estimate of Theorem 31.8.

34 Trace congruences on the common interval

34.1 The traces in the marked deformation ring

Fix x on one of the common intervals of Theorem 11.9, put $t = \lfloor x^{401} \rfloor$, and take a coefficient prime $\ell > \exp(x^{\varepsilon}/12)$ and a twist $1 \le r < t$. Enlarge the lower endpoint so that $\ell > 2t + 3$.

Use the character summands of the strict group for the field E_x and the auxiliary quadratic character η of $K' = \mathbb{Q}(i)$ from Sections 32.4–32.6. Write O for the coefficient discrete valuation ring, ϖ for a uniformizer and k for its finite residue field. In particular $\Psi = \kappa_\ell^r \alpha$, with α of order prime to ℓ and unramified at ℓ. For a complex conjugation c, we have $\eta(c) = -1$ and $\Psi(c) = 1$.

Let B be the completed marked global deformation ring. Its base matrices are $d(g) = \operatorname{diag}(\Psi(g), 1, \Psi(g)^{-1})$. It represents lifts with the prescribed sign η, the inertia kernels away from ℓ, and the ordinary component through the basepoint at ℓ described in Section 32.4. Write $M(g)$ for the matrix coordinate of the universal lift. On $H = G_{K'} = \ker \eta$ it is a homomorphism. We use absolute Galois groups and pull back matrices from their specified unramified-outside-finitely-many-places quotients. The element $g_* \in I_\ell \subset H$ is chosen so that the residual entries of $d(g_*)$ are distinct. Let J reverse the three coordinates. The sign acts by $A \mapsto JA^{-\mathsf{T}}J$. For rows numbered $0, 1, 2$, the marking at c sets the entries $00, 11, 01, 10, 02, 20$ of $M(c)$ equal to those of the identity. Since $c^2 = 1$, the group law gives $M(c) = JM(c)^{\mathsf{T}}J$. Hence $M(c) = 1$. The other two marking equations set the entries $01, 10$ of

$$Z = M(g_*) - M(cg_*^{-1}c)$$

equal to zero. The split torus acts by conjugation with $\operatorname{diag}(a, 1, a^{-1})$. Let D be the invariant subring of B. Invariance means equality of the two Laurent expressions for an indeterminate a on every Artinian truncation. Let $\mathcal{T}$ denote the O-algebra in B generated by $T(h) = \operatorname{tr} M(h)$, $h \in H$. All closures below use the maximal-ideal topology of B.

Lemma 34.1 (Matrix entries around a cycle)**.** *Put $z = M(g_*)_{00} - M(g_*)_{22}$. Then z is a unit, $z^2 \in \mathcal{T}$, and every product of matrix entries around a directed cycle of indices belongs to $\mathcal{T}[1/z^2, z]$. For this statement each edge may use the matrix of a different group element of H.*

Proof. The group law and the marking at c give

$$M(ch^{-1}c) = JM(h)^{\mathsf{T}}J \qquad (h \in H).$$

Consequently $Z = -JZ^{\mathsf{T}}J$. The two zero entries and this symmetry give $Z = \operatorname{diag}(z, 0, -z)$ since 2 is a unit. Its residue has $z \neq 0$, by the choice of g_*. Expanding the trace of the square of the defining difference shows that $z^2 = \operatorname{tr}(Z^2)/2 \in \mathcal{T}$. The diagonal matrix units are

$$P_0 = (Z^2 + zZ)/(2z^2), \qquad P_1 = 1 - Z^2/z^2, \qquad P_2 = (Z^2 - zZ)/(2z^2).$$

For indices $i_0, \dots, i_n = i_0$, the entry product is

$$\prod_{j=0}^{n-1} M(h_j)_{i_j i_{j+1}} = \operatorname{tr}\left(P_{i_0} M(h_0) P_{i_1} M(h_1) \cdots P_{i_{n-1}} M(h_{n-1})\right).$$

Insert the three polynomial formulas. After expansion, the trace of every matrix word is a trace of a single group element of H, or the trace of the identity, since g_* and $cg_*^{-1}c$ both belong to H. This proves the assertion. □

Theorem 34.2 (The invariant ring is generated topologically by traces)**.** *Let A be the O-subalgebra of B generated by the entries $M(h)_{ij}$, $h \in H$. Write $(A[1/z^2])_0$ for its weight-zero part after the indicated localization. Then*

$$(A[1/z^2])_0 = \mathcal{T}[1/z^2, z], \qquad D = \overline{\mathcal{T}}.$$

The algebra in the first equality is generated as a module by $1, z$ over $\mathcal{T}[1/z^2]$. The closed algebra D is complete Noetherian local, with residue field k.

Proof. Write $\bar{i} = 2 - i$. An entry $M(h)_{ij}$ equals $M(ch^{-1}c)_{\bar{j},\bar{i}}$ and has weight $j - i$. An entry of weight 1 may thus be used in either of the positions $01, 12$, and a weight -1 entry in either of the positions $10, 21$. Weight 2 has position 02 and weight -2 has position 20.

Any list of weights in $\{0, \pm 1, \pm 2\}$ with sum zero splits into singles (0), pairs $(1, -1), (2, -2)$, and triples $(1, 1, -2), (-1, -1, 2)$. Remove singles and opposite pairs. If anything remains, its positive and negative weights must have different absolute sizes, one on each side. The sum condition now partitions the remainder into triples. The reflection identity in the first paragraph puts each pair or triple around a directed cycle. Multiplication in the coefficient ring allows us to reorder the entries. Hence Lemma 34.1 applies to each weight-zero monomial. Every invariant polynomial is the image of its weight-zero part, by uniqueness of Laurent coefficients for the torus. Localizing at the invariant z^2 preserves this description. For the opposite inclusion, traces, z and $1/z^2$ are invariant. The module assertion follows from $z^2 \in \mathcal{T}$.

Choose $z_0 \in O^\times$ lifting the residue of z. Then $y = z^2/z_0^2 - 1$ is topologically nilpotent. In B, the geometric series inverts $1 + y$, and $z = z_0 \sum_{n \geq 0} \binom{1/2}{n} y^n$. Indeed the binomial coefficients belong to $\mathbb{Z}_\ell$, by continuity from nonnegative integral exponents. The two sides have the same square and their sum has unit residue $2\bar{z}_0$. Thus z and $1/z^2$ belong to $\overline{\mathcal{T}}$.

The framed presentation uses finitely many differences of entries from their base values as topological generators. To recall why, on the kernel of the finite residual image the congruence subgroup over every finite Artin coefficient ring is an ℓ-group. Its universal source is the maximal pro-ℓ quotient of that kernel with the allowed ramification. This quotient is finitely generated by the pro-ℓ generator criterion and the finiteness of the first cohomology with finite trivial coefficients over a number field unramified outside a fixed finite set. The latter follows from class field theory, as in the proof of the framed presentation in Section 32.1. Add finitely many lifts of the residual image. The entries of this list determine all the coordinates by multiplication, inversion and limits. Inverses of the unit determinants expand about their base values. Elements of negative sign give the same matrices as their products on the right with c, so all entries may be taken on H. The entry differences are homogeneous, with base constants subtracted only in weight zero.

This yields a surjection from a power series ring with finitely many weighted variables onto B. On any truncation an invariant element of the target lifts to a polynomial, and taking the weight-zero part still lifts that element. The weight-zero monomials are generated by the finitely many singles, pairs and triples of the variables just described. The power series ring over O on those invariant monomials maps continuously into B because each monomial vanishes at the residual point. Its image is compact, since its coefficients have finite residue field and its Artinian quotients are finite. It is therefore closed in B. The polynomial truncations prove that its image is all of D. This also identifies the induced topology on D with the quotient topology of this compact power series ring. In particular D has the asserted local properties. The algebraic equality already proved puts each of its polynomial generators in $\overline{\mathcal{T}}$. Traces are invariant, and invariance is closed on every truncation. This proves the second equality. □

Proposition 34.3 (A test on curves over finite fields)**.** *The O-module D is finite if and only if for every finite extension k'/k and continuous local k-algebra map*

$$u : D/\varpi D \longrightarrow k'[[s]]$$

all $u(T(h))$ are constant. In these expressions a trace in D is reduced modulo ϖ before applying u.

Proof. Put $D_k = D/\varpi D$. If D is finite, D_k is an Artinian local k-algebra and its maximal ideal is nilpotent. Every such map into a domain factors through k. Conversely, if D_k is Artinian,

a power of the maximal ideal of D is in ϖD. Thus the maximal-ideal and ϖ-adic topologies on D agree. Lift a finite k-basis of D_k, expand successively modulo powers of ϖ, and use completeness of O and D. The lifts generate D over O.

If D_k has positive dimension, choose a prime quotient C of dimension one along a maximal chain of primes. It is a complete Noetherian local domain. Its normalization is finite because complete Noetherian local rings are Nagata. It is local: a finite algebra over a complete local ring splits as a product of local algebras, and the normalization is a domain. Being one-dimensional and normal, it is a discrete valuation ring, complete with finite residue extension k'/k. The roots of $X^{|k'|} - X$ give its field of constants by Hensel's lemma, and expansion in a uniformizer identifies it with $k'[[s]]$. The finite-normalization theorem used here is proved in the Stacks Project on Nagata rings. The composition $D_k \twoheadrightarrow C \hookrightarrow k'[[s]]$ is local, continuous and has a nonconstant image, since every nonzero element of the maximal ideal of C has strictly positive valuation. If it sent all traces to constants, Theorem 34.2 and closedness of the finite constant field would make its entire image constant. This proves the criterion. □

Theorem 34.4 (Rigidity after restriction to one-dimensional characters)**.** *For a map u as above, suppose there are a number field $L \supset K'$ finite over K', a finite extension F of $k'((s))$, and three continuous characters $\lambda_i : G_L \to F^\times$ with*

$$u(T(h)) = \sum_{i=1}^{3} \lambda_i(h) \qquad (h \in G_L).$$

Then $u(T(h))$ is constant for every $h \in H$, and u factors through the residue field k.

Proof. Let $P_h(X)$ be the characteristic polynomial of $M(h)$. Its coefficients are in D. Since $\ell > 3$, Newton's identities express these coefficients in terms of $T(h)$, $T(h^2)$, and $T(h^3)$. Thus for $h \in G_L$, the roots of $u(P_h)$ are exactly the $\lambda_i(h)$, with multiplicities.

At a finite place of L over ℓ, the ordinary equations give the three eigenvalues on inertia after reduction: they are $\bar\Psi(h), 1, \bar\Psi(h)^{-1}$. Moving the place by conjugation either preserves this multiset or takes its inverse, which is the same multiset. At all other finite places the imposed inertia kernel has finite index in inertia, since the specified finite characters have finite image there. Hence for every such inertia element h, some $M(h^n)$ is the identity. Newton's identities at h^n give $\lambda_i(h)^n = 1$ for all three characters. In particular each λ_i takes roots of unity on inertia at every finite place of L.

Every λ_i has compact image, so its valuation has finite image in $\mathbb{Z}$, necessarily zero. Let k_F be the finite residue field of F. Divide λ_i by its reduction lifted through the constant field $k_F \subset F$. The quotient is a continuous character into the principal units of F. This group is torsion-free: if j is prime to ℓ, then $(1+b)^j - 1$ has valuation $v(b)$ for $v(b) > 0$, and raising to an ℓ-power sends $1+b$ to $1+b^\ell$. Hence the quotient is unramified at all finite places. The field L is totally imaginary. By the Hilbert class field theorem its unramified abelian Galois group is the finite ideal class group. Thus the quotient character is trivial, and every λ_i takes values in $k_F^\times$. The class field theorem used here is recalled in Milne, *Class Field Theory*.

For arbitrary $h \in H$, choose $m > 0$ with $h^m \in G_L$; this follows from the action on the finitely many cosets. The roots of $u(P_{h^m})$ are the m-th powers of the roots of $u(P_h)$. Indeed the coefficients of the polynomial of the powered roots are symmetric integral polynomials in the original roots. The resulting identity for matrices holds first for the generic matrix with distinct eigenvalues and hence identically over $\mathbb{Z}$; it commutes with specialization through D. The roots of $u(P_{h^m})$ lie in a finite field, so the roots of $u(P_h)$ are algebraic over a finite field. Their sum in $k'[[s]]$ is constant: any element of that ring algebraic over a finite field satisfies $a^q = a$ for some prime power $q > 1$, forcing its positive-order part to vanish. All specialized

traces are therefore constant. Theorem 34.2 makes the image of u constant, and locality makes it the image of k. □

The local strict classes detected in Corollary 32.9 and the theta projections of Corollary 66.19 thus refer to the same E_x and the same interval in x. The preceding theorem excludes every nonconstant trace curve whose restriction becomes a sum of characters over a finite extension. Finiteness still requires the other trace curves to be excluded. Even finiteness alone would give no uniform bound on the integral length of the strict groups. The required comparison of the nonzero full Bessel period also remains open. For the original marking and the specialization argument compare `Development/AutomorphicArithmetic/AutomorphicVarieties/UnitaryThreeGroups/MopBase.lean`, line 105 and `Development/AutomorphicArithmetic/AutomorphicVarieties/CurveDVRSpecialization/FinCrit.lean`, line 45.

34.2 Trace curves and the contribution from the two-dimensional block

Work on the actual field $E = \widetilde{E}$ of Proposition 9.9. For each $\ell > P = \exp(x^{\varepsilon}/12)$ and $\alpha(c) = (-1)^r$, take the ordinary marked deformation of $\operatorname{diag}(\Psi, 1, \Psi^{-1})$ with $\Psi = \kappa_\ell^r \alpha$, $1 \le r < t$. As in Theorem 34.2, the auxiliary group is $H = G_{\mathbb{Q}(i)}$, and the trace algebra D is the invariant part of the completed marked deformation ring B. The sign involution is inverse transpose with the coordinates reversed. The field E supplies the character α, and the auxiliary quadratic group H is fixed for all x.

Lemma 34.5 (Realizing a trace specialization by integral matrices)**.** *Let V be a complete discrete valuation ring with finite residue field and fraction field L. Given a continuous local coefficient-algebra map $D \to V$, there is a finite extension L'/L and a continuous local map $B \to \mathcal{O}_{L'}$ extending the given map. This holds also when V has characteristic ℓ. For a prescribed point of the graded fiber over $\overline{L}$, its orbit under the diagonal scaling contains a point giving such a map.*

Proof. Use the finite-weight algebra $A/(f_i)$ from Lemma 32.17. Write it B^{gr}. Its degree-zero part is precisely D, a direct summand as a D-module. Thus the base change to L is nonzero. It is a finitely generated L-algebra, generated by the finitely many nonzero-weight variables. A closed point over $\overline{L}$ exists, and the values of the variables belong to a finite extension of L, also for any prescribed $\overline{L}$-point. Write the nonzero values of positive weight as a_i with weight $d_i > 0$ and those of negative weight as b_j with weight $-e_j < 0$. Normalize the valuation by a uniformizer of L.

Each $a_i^{e_j} b_j^{d_i}$ is the image of an invariant in the maximal ideal of D, so it has positive valuation. Hence $-v(a_i)/d_i < v(b_j)/e_j$ for every such pair. Choose a rational number u strictly between all lower and all upper bounds, ignoring either list when empty. After a finite extension there exists a scalar of valuation u. Scaling by it puts every nonzero-weight variable in the maximal ideal, including those with value zero. The original weight-zero variables are there by locality. Thus evaluation converges on the ambient complete power series ring.

The two evaluations on ambient weight-zero series are continuous and coincide on the finitely many invariant monomials of degree at most three. They therefore agree on all such series. For a homogeneous series of any fixed weight, grouping by the finitely many residual monomials in the proof of Lemma 32.17 expresses it as a polynomial in the variables with weight-zero series coefficients. The two evaluations agree on all such series and in particular on every defining relation. This proves the claimed extension. The universal matrices give continuous matrices over $\mathcal{O}_{L'}$. □

Theorem 34.6 (A nonconstant trace curve is strongly irreducible)**.** *Let $u : D/\varpi D \to k'[[s]]$ be continuous and local, with k'/k finite. If its image contains a nonconstant series, each realization of Lemma 34.5 has absolutely irreducible three-dimensional restriction to G_L for every finite extension $L/\mathbb{Q}(i)$. Consequently the failure of D to be finite over the coefficient ring would give a curve with precisely this property on the large-prime ordinary component.*

Proof. Let M be the specialized matrix on H, now regarded over the algebraic closure of the Laurent series field. The element

$$M(g_*) - M(cg_*^{-1}c) = \operatorname{diag}(z, 0, -z)$$

has distinct eigenvalues by Lemma 34.1. Every H-invariant subspace is therefore spanned by a subset of the three coordinate vectors. Draw an arrow $j \to i$ when some $M(h)_{ij} \neq 0$, $h \in H$, $i \neq j$. A coordinate subspace is invariant precisely when no arrow leaves its index set. Since $M(ch^{-1}c)$ is the reflected transpose of $M(h)$, each adjacent arrow occurs with its reflection of the same direction along the chain $0, 1, 2$.

If both adjacent directions occur, the graph is strongly connected. If just one adjacent direction occurs, an outer arrow in the opposite direction makes a directed cycle through all three vertices. In each of these cases M is irreducible. If all arrows have the same direction the representation is triangular. Its traces are sums of three continuous characters already on H, hence constant by Theorem 34.4. The only other reducible possibility has all adjacent entries zero and both outer directions present. It factors through the quotient studied in Theorem 33.2. That integral invariant algebra is finite over the coefficient ring, by ordinary pseudodeformation finiteness. Its image in the characteristic-ℓ valuation ring is constant. This last assertion still holds after the finite extension in Lemma 34.5: its residue field is finite, and the elements algebraic over a finite field are constants. Both reducible possibilities contradict the hypothesis.

Suppose the restriction became reducible on some finite extension. Pass to a finite normal extension of $\mathbb{Q}(i)$ inside the chosen algebraic closure which contains it, and let H_0 be its open normal subgroup of H. Choose a simple H_0-submodule of the representation over the algebraically closed coefficient field. Its finitely many H-translates are simple H_0-modules of the same dimension. Their sum is the whole space by H-irreducibility. A sum of simple submodules is semisimple, and one can select a direct-sum subcollection by successively discarding a summand contained in the previous sum. Their common dimension divides three. Since restriction was reducible, this dimension is one. Thus over H_0 the traces are sums of three characters. A basis for these lines is defined over a finite extension of the coefficient field, and the characters are continuous there. Theorem 34.4 again makes all the original traces constant. This proves strong irreducibility. The final assertion follows by Proposition 34.3, which produces a nonconstant one-dimensional specialization if D is not finite. □

Next choose a single rational prime $v \geq 7$ split in $\mathbb{Q}(i)$ and prime to $2F$, where F is the conductor of ψ. On sufficiently distant selected intervals the growing auxiliary primes q, d_0 avoid v, so $v \nmid N$ and $v < \ell$. Such a fixed prime exists by Dirichlet's theorem. Use arithmetic Frobenius at v throughout, and write

$$d_{\alpha,r} = 1 + v^r \alpha(v) + v^{-r} \alpha(v)^{-1}.$$

At fixed x, ℓ, r, α, denote by $\mathcal{P}_{\ell,r,\alpha}$ the scaling orbits of geometric mixed points on the graded solution whose four adjacent entries are zero on H. Thus a representative has some nonzero entries in each outer direction. This definition does not assume finiteness of the invariant algebra of the full rank-three space. Write $b(P)$ for the trace of its matrix at Frob_v.

Theorem 34.7 (A uniform sum for the two-dimensional trace differences)**.** *With $v_\ell(\ell) = 1$, on the original period fields and intervals one has*

$$\sum_{\ell>P}(\log \ell) \sum_{1\le r<t} \sum_{\alpha(c)=(-1)^r} \sum_{P'\in\mathcal{P}_{\ell,r,\alpha}} v_\ell(b(P') - d_{\alpha,r}) \;\le\; \exp(x^{\varepsilon/3}). \tag{34.1}$$

Here ℓ runs over primes and α over the indicated characters of $\mathrm{Gal}(E/\mathbb{Q})$.

Proof. The invariant algebra on this quotient is local and finite over the coefficient DVR by Theorem 33.2. At a geometric point its values are integral and its maximal ideal maps into the maximal ideal of a finite extension. Indeed the contraction of that residue maximal ideal lies over the closed point of the DVR and is maximal by finiteness. The point can thus be scaled to a continuous integral one by Lemma 34.5 applied on the quotient. Its weight grading, homogeneous equations and degree-zero lifting follow from the same proof by imposing the additional homogeneous entry equations. By Theorem 33.3, for fixed ℓ, r, α these orbits inject into the coefficient-embedded newforms of weight $2r+1$, character $\eta\alpha^{-2}$, and primitive level $f(\eta\alpha^{-2}) \mid 4N$. Their traces are precisely of the form $1 + v^{-r}\alpha(v)a_v(f)$.

For each r, take all complex characters α in this sum and all normalized primitive eigenforms f of the stated weight, character and level, including their conjugate systems of coefficients. Form the single number

$$I_r = \prod_{\alpha,f} \left(\alpha(v)a_v(f) - v^{2r}\alpha(v) - \alpha(v)^{-1}\right).$$

Each factor is an algebraic integer, since the Hecke coefficient of a normalized newform is integral. The collection is stable under every automorphism of the algebraic numbers, so I_r is a rational integer. Deligne's bound gives $|a_v(f)| \le 2v^r$ at every embedding. Thus the normalized trace has absolute value at most three, while $|d_{\alpha,r}| \ge v^r - 1 - v^{-r} > 3$. Thus $I_r \neq 0$, with the empty product equal to one. Each factor has absolute value at most $(v^r+1)^2$. The dimension bound for cusp forms of weight $2r+1$ and level dividing $4N$, proved in Section 30.4, bounds the number of factors by a fixed polynomial in $N(r+2)$, since the number of possible α's is at most $n \le N$.

For each ℓ choose a common coefficient embedding into $\overline{\mathbb{Q}}_\ell$. All admissible conjugates are included. Every orbit contributes a distinct labelled factor for fixed α, even when traces agree between different forms. Its valuation is exactly the displayed difference valuation since v^r is an ℓ-adic unit. All unselected factors have nonnegative valuation. Summing first at each ℓ and then over primes bounds the left side by $\sum_{r<t} \log|I_r|$. This is at most $[N(t+2)]^A$ for a fixed A and sufficiently large x. Since $\log N \ll_{\psi,\varepsilon} x^{\varepsilon/4}$ and $\log(t+2) = O(\log x)$, this proves (34.1). □

In particular all the two-dimensional orbit terms of the intersection inequality of Theorem 31.8, would together be $o(\exp(x^\varepsilon/2))$, even with its factor two. Applying that inequality globally still requires finiteness on the strongly irreducible rank-three locus, control of the tangent spaces in the full solution and of the rank-three trace differences. The small-prime tail in Corollary 19.20 and the nonzero comparison on the integral theta image also remain open.

34.3 The odd orthogonal part of the three-dimensional calculation

Keep the fields E, the intervals $[V, V^{3/2}]$, the threshold $P = \exp(x^\varepsilon/12)$, and the split prime v of Theorem 34.7. Put $K' = \mathbb{Q}(i)$, $H = G_{K'}$, and let η be the quadratic character of $K'/\mathbb{Q}$. At a prime $\ell > P$, let κ_ℓ be the cyclotomic character and put $\Psi = \kappa_\ell^r\alpha$ for a character α of $\mathrm{Gal}(E/\mathbb{Q})$ with $\alpha(c) = (-1)^r$, $1 \le r < t$. In particular $\Psi(c) = 1$. All assertions in this

paragraph hold on sufficiently distant intervals with the thresholds of Section 6. The order of $\bar{\Psi}$ on inertia at ℓ then exceeds four.

Let U denote the three-dimensional matrix representation on H of an absolutely irreducible point, with the marking fixed as in Lemma 34.5. The same notation applies to a realization of a nonconstant trace curve in Theorem 34.6. By Theorem 33.13 and Lemma 33.14, if the algebraic monodromy of U preserves a symmetric form, U is the trace-zero adjoint of a two-dimensional representation ρ on restriction to H. The marking determines $e \in \{0,1\}$ with $\det \rho = \eta^e \Psi$ and residual representation $\operatorname{diag}(\overline{\eta^e \Psi}, 1)$ on $G_{\mathbb{Q}}$. We call this case *odd orthogonal* if $e = 1$, that is, if $\rho(c)$ has determinant -1. The term refers to this two-dimensional lift. The determinant of U is always one.

Proposition 34.8 (Rigidity of odd orthogonal traces in characteristic ℓ)**.** *There is no nonconstant trace curve of odd orthogonal type.*

Proof. Suppose such a curve existed, and take the integral realization over a complete discrete valuation ring R of characteristic ℓ and finite residue field provided by Lemma 34.5 and Lemma 33.14. Enlarge the mixed-characteristic coefficient ring $\mathcal{O}$ by a finite unramified extension to accommodate the residue field of R. The ring R then receives the coefficient map through the finite field of constants in it. The original coefficient map has this property since its image is a finite field. Write $\phi = \eta\Psi$ for the fixed mixed-characteristic character and use its reduction for the corresponding determinant in R. It is totally odd and de Rham. Its reduction on inertia at ℓ is nontrivial.

The residual trace of ρ is $1 + \bar{\phi}$. The representation is unramified outside the primes dividing $4N\ell$. At ℓ it has two ordinary diagonal characters. The quotient character b is unramified by Theorem 33.15. Its reduction on the decomposition group is the trivial character. Indeed the distinct residual inertia characters distinguish the trivial summand from $\bar{\phi}$. Pan's ordinary pseudodeformation theorem over $\mathbb{Q}$ applies to this trace with determinant ϕ, this set of primes and the local reducibility condition at $\ell > 5$. The universal ordinary trace algebra is finite over $\Lambda = \mathcal{O}[[1 + \ell\mathbb{Z}_\ell]]$. The weight map acts through the local character lifting 1 on principal units under local reciprocity. [19, Theorem 5.1.2(1)]

The pseudotraces modulo the powers of the maximal ideal of R give compatible maps from this algebra, hence one map to R. Since b is unramified, the map factors through specialization at the augmentation $\Lambda \to \mathcal{O}$. Its image in R is a domain finite over a finite field, hence a finite field. All traces of ρ and all values of its determinant are constant. For $h \in H$,

$$\operatorname{tr} U(h) = \frac{(\operatorname{tr} \rho(h))^2}{\det \rho(h)} - 1.$$

This follows by writing the conjugation action on all endomorphisms as $\rho \otimes \rho^\vee$, and removing its scalar summand. Every trace of U is consequently constant. The invariant algebra D is topologically generated by these traces by Theorem 34.2. Its continuous image lies in the same finite, hence closed, field of constants. The finite extension used to realize the curve was injective on its coefficient field, so the original image was constant as well. □

Let $\mathcal{Q}_{\ell,r,\alpha}$ be the scaling orbits of geometric characteristic-zero points of odd orthogonal type with irreducible U. Let $b(Q)$ be the trace of U at arithmetic Frobenius at v. The prime v splits in K', so this Frobenius lies in H.

Theorem 34.9 (Odd orthogonal points on the period fields)**.** *For each such orbit there is a normalized primitive cuspidal newform g of weight $r+1$, character $\eta\alpha$ and level $f(\eta\alpha) \mid 4N$*

such that

$$b(Q) = \frac{a_v(g)^2}{v^r \alpha(v)} - 1.$$

The geometric tangent space of the full graded solution at Q is generated by scaling and has dimension one. Moreover,

$$\sum_{\ell > P} (\log \ell) \sum_{1 \le r < t} \sum_{\alpha(c) = (-1)^r} \sum_{Q \in \mathcal{Q}_{\ell,r,\alpha}} v_\ell(b(Q) - d_{\alpha,r}) \le \exp(x^{\varepsilon}/3). \tag{34.2}$$

Proof. By Lemma 33.10 and Theorem 33.15, the point has a continuous integral representative. The lift ρ is odd, absolutely irreducible, finitely ramified, and crystalline at ℓ, with cyclotomic exponents $0, r$. It has split residual semisimplification $1 \oplus \bar{\phi}$ with distinct inertia characters at ℓ. Pan's Fontaine–Mazur theorem over $\mathbb{Q}$ applies at $\ell > 5$ and identifies ρ, up to twist, with the representation of a cuspidal eigenform. The weight difference gives weight $r + 1$; normalizing the smaller cyclotomic exponent to zero leaves a de Rham character of weight zero. Such a finitely ramified character of $G_{\mathbb{Q}}$ is of finite order. Its inertia at ℓ has finite image by the local classification of de Rham characters, and at each other ramified prime the local units have finite image in the ℓ-adic coefficient units. Global class field theory for $\mathbb{Q}$ then makes it a ray class character. Absorb this character into the eigenform and take its primitive newform. With arithmetic Frobenius in the normalization of Deligne, the resulting representation V_g equals ρ up to isomorphism. Its determinant prescribes the character $\eta\alpha$. Theorem 33.15 computes its prime-to-ℓ conductor as $f(\eta\alpha)$. Its Weil–Deligne representation at ℓ is unramified with zero monodromy by crystallinity. The local–global compatibility and newform conductor theorem used in Theorem 33.3 give the asserted primitive level. The adjoint trace identity of Proposition 34.8 now gives the displayed formula. [19, 20]

For the tangent assertion, take the unitary automorphic normalization π_g of g, with central character ω_g. It has no nontrivial self-twist by a Hecke character. The central character forces any such twist to be quadratic. At unramified primes the self-twist would also yield $V_g \simeq V_g \otimes \xi$ for the corresponding quadratic Galois character ξ, by equality of Frobenius polynomials and Chebotarev density. The intertwiner could not be scalar, since $\xi \neq 1$. Yet V_g remains absolutely irreducible on $H \cap \ker \xi$. An invariant line there would give an invariant nilpotent line in its trace-zero adjoint, contradicting strong irreducibility of U in Lemma 33.4. Schur's lemma gives a contradiction.

The adjoint lift of Gelbart and Jacquet, $\Pi = \operatorname{Sym}^2 \pi_g \otimes \omega_g^{-1}$, is therefore cuspidal. It is self-dual. Its cyclic base change $\Pi_{K'}$ remains cuspidal by the cuspidality criterion of Arthur and Clozel. Indeed a failure of cuspidality would give a self-twist by η; in degree three this changes the central character by the nontrivial character η^3. The base change is conjugate self-dual. At the complex place its three characters are $(z/\bar{z})^r, 1, (z/\bar{z})^{-r}$, so it is regular algebraic for $r \ge 1$. [21, 22]

To match the twist in Corollary 33.12, let w be an unramified finite place of K' away from ℓ, outside the finitely many primes where g ramifies. For eigenvalues a, b of arithmetic Frobenius of V_g at w, the normalized Satake multiset of $\Pi_{K',w}$ is $\{a/b, 1, b/a\}$ under the fixed coefficient embedding. This multiset is invariant under inversion. In the Tate normalization of Newton and Thorne the local parameter for rank three uses $\Pi_{K',w} \otimes |\det|^{-1}$, so at geometric Frobenius these parameters acquire the factor $\#k(w)$. The same factor is the value of κ_ℓ^{-1} at that Frobenius. Thus $U \otimes \kappa_\ell^{-1}$ and the representation attached to $\Pi_{K'}$ have the same Frobenius polynomials. Their semisimplifications are isomorphic by Chebotarev density, and U is irreducible. This supplies the automorphic representation required in Corollary 33.12 with $F = K'$. That corollary gives the tangent assertion for every $1 \le r < t$. [23]

We next check the multiplicity in the orbit sum. Fix ℓ, r, α and an embedding of the algebraic coefficients into $\overline{\mathbb{Q}}_\ell$. If the same coefficient-embedded newform serves two orbits, their representations U, U' on H are isomorphic. After scaling to integral representatives, an intertwiner D_0 conjugates the two matrices

$$U(g_*) - U(cg_*^{-1}c) = \operatorname{diag}(z_*, 0, -z_*), \qquad U'(g_*) - U'(cg_*^{-1}c) = \operatorname{diag}(z'_*, 0, -z'_*).$$

Distinct eigenvalues give $z'_* = \pm z_*$. The prescribed common nonzero residue at the mark distinguishes the signs at $\ell > 2$. Thus $z'_* = z_*$ and D_0 is diagonal. The marked conjugate-dual relation $U(chc) = J_3 U(h)^{-\mathsf{T}} J_3$ for both representations makes $D_0^{-1} J_3 D_0^{-\mathsf{T}} J_3$ centralize $U(H)$. It is scalar by absolute irreducibility. If the diagonal of D_0 is (d_0, d_1, d_2), this gives $d_0 d_2 = d_1^2$. After dividing by d_1, its diagonal is $(u, 1, u^{-1})$. The coset matrices are determined by the matrices on H and the fixed value at c. On D, the evaluations agree by the trace description in Theorem 34.2 and continuity for the two integral representatives. The homogeneous parts are generated over the invariant series by finitely many monomials in the marked entry coordinates as in Lemma 34.5. The scalar u therefore relates the evaluations on the entire graded algebra. The orbits coincide. Thus at this ℓ, r, α a labelled newform accounts for at most one orbit.

Put $z = v^r \alpha(v)$. Multiplication of $b(Q) - d_{\alpha,r}$ by z, an ℓ-adic unit, gives $a_v(g)^2 - (z+1)^2$. For each r form the product

$$J_r = \prod_{\alpha, g} (a_v(g)^2 - (v^r \alpha(v) + 1)^2)$$

over every permitted complex character and all normalized primitive newforms of the weight, character and primitive level above, including all conjugate systems. The factors are algebraic integers and the set of labels is Galois stable, so $J_r \in \mathbb{Z}$. At every complex embedding the two Satake roots of g at v have absolute value $v^{r/2}$ by Deligne's theorem. Their adjoint trace has absolute value at most three. The comparison trace has absolute value at least $v^r - 1 - v^{-r} > 3$, so every factor is nonzero. Its absolute value is at most $4v^r + (v^r + 1)^2$. An empty product has value one.

Every selected orbit at a fixed ℓ thus contributes the valuation of a distinct labelled factor, with valuations of all factors nonnegative. The sum in (34.2) is bounded by $\sum_{r<t} \log |J_r|$. The dimension estimate of Section 30.4 and the bound of N for the number of characters bound the number of factors per r by a polynomial in $N(r+2)$. Since v is fixed, the logarithm of the bound for one factor is $O_v(r+1)$. For some fixed A the sum is at most $[N(t+2)]^A$. The estimates $\log N \ll_{\psi,\varepsilon} x^{\varepsilon/4}$ and $\log(t+2) = O(\log x)$ give (34.2). □

Corollary 34.10 (The remaining large-prime cases)**.** *If the full weight-zero algebra fails to be finite, there is a nonconstant residual trace curve whose algebraic monodromy is* SL_3 *or whose monodromy is orthogonal with an even two-dimensional lift. In characteristic zero the two-dimensional mixed orbits and the odd orthogonal irreducible orbits together have one-dimensional tangent spaces, and the sum of their trace valuations with the weights in* (34.2) *is* $o(\exp(x^\varepsilon/2))$.

Proof. Combine Theorem 34.6, Theorem 33.13, and Proposition 34.8 for the first assertion. Corollary 33.9 gives the tangent statement in the two-dimensional case and Theorem 34.9 gives it in the odd orthogonal case. Summing (34.1) and (34.2) gives the bound $2\exp(x^\varepsilon/3) = o(\exp(x^\varepsilon/2))$. □

For the interval estimate (W), the remaining three-dimensional cases require a proof both in the special fiber and at characteristic-zero points. The smaller-prime strict cohomology in Corollary 19.20 and the comparison for the nonzero projected integral theta period on the same fields and intervals also remain open here.

34.4 The squared characters in the rank-three argument

Write $\mathrm{Gal}(E/\mathbb{Q}) = P \times H$, with P the p-Sylow subgroup. View a character α of H as a Dirichlet character by extending it trivially on P, and let 2^s be the two-part of its primitive conductor. Choose $\alpha(c) = (-1)^r$. Let η_4 be the quadratic character of $\mathbb{Q}(i)$, and set $\Psi = \kappa_p^r \alpha$ as a character of $G_{\mathbb{Q}}$, where κ_p is the p-adic cyclotomic character. Write

$$D = 2\varphi(o), \qquad e_p = v_2(p^D - 1), \qquad H_p = 4(F+2)(p+1)4^{e_p}, \qquad \varphi(1) = 1.$$

Proposition 34.11 (Vanishing for the two squared characters)**.** *Suppose $2^{s-1} \geq H_p$. Let $\vartheta = \eta_4 \Psi^2$ or $\vartheta = (\eta_4 \Psi^2)^{-1}$. Take a finite unramified coefficient ring $O/\mathbb{Z}_p$ containing the values of α, with fraction field J_p and residue field k. Put $\delta = \bar{\vartheta}$. Every continuous cocycle with values in $k(\delta)$ which vanishes on a subgroup of index prime to p in $I_v \cap \ker \delta$ for each finite inertia group I_v is a coboundary. Moreover,*

$$\ker\left(H^1(\mathbb{Q}, (J_p/O)(\vartheta)) \longrightarrow H^1(\mathbb{Q}_p, (J_p/O)(\vartheta)) \times \prod_{v \nmid p\infty} H^1(I_v, (J_p/O)(\vartheta)) \right) = 0.$$

Proof. Extend a character of $\mathrm{Gal}(B_J/\mathbb{Q})$ to the product of the Galois groups of K_ψ and C_J as in Theorem 25.17. The independence of the two quadratic fields gives a factorization

$$\alpha = \lambda\, \zeta\, \chi_q^u \chi_{d_0}^v, \qquad u, v \in \{0, 1\},$$

where λ comes from K_ψ with order dividing the prime-to-p part of o, ζ comes from C_J, and the final factors are quadratic. Squaring eliminates both of those factors, also when $p = q$ or $p = d_0$. Let ω be the Teichmüller lift of $\bar{\kappa}_p$, and factor the primitive Dirichlet character inducing $\eta_4 \omega^{2r} \alpha^2$ as $\phi\xi$, where ϕ has odd conductor f and ξ has conductor a power of two. This product is odd. One has $f \mid pF$, and the order of ϕ divides $\mathrm{lcm}(p-1, o/p^{v_p(o)})$, hence divides $p^D - 1$. The same divisor works for the inverse.

Our size hypothesis gives $s \geq 5$. The two-part of α takes at 5 a primitive 2^{s-2}-th root of unity, because 5 generates the units congruent to 1 mod 4 at that modulus and the character is primitive there. Squaring reduces the order to 2^{s-3}. Multiplication by η_4 preserves that value at 5. On -1 the value has order at most two. The conductor of ξ is consequently exactly 2^{s-1}, also after reduction since p is odd. The inequality $2^{s-1} \geq H_p \geq f 4^{e_p}$ now gives the finite-cocycle assertion by Theorem 25.14 and Corollary 25.15.

The local residual character at p is nontrivial. If $p \mid f$, its primitive inertia character there is nontrivial and has prime-to-p order, so survives reduction. Otherwise suppose it is unramified and trivial on Frobenius at p. Then the reduction of $\xi(p)$ lies in $\mathbf{F}_{p^D}^\times$, because it is the inverse of the reduction of $\phi(p)$. The unit-group computation in the proof of Theorem 25.14 gives $p \equiv \pm 1 \bmod 2^{s-1-e_p}$. But $2^{s-1-e_p} \geq H_p/2^{e_p} > p+1$, which rules this out. The same reasoning applies to the inverse character.

Write $A = (J_p/O)(\vartheta)$. Nontrivial residual action locally at p gives an element acting by a scalar whose difference from one is a unit. In particular $A^{G_{\mathbb{Q}_p}} = A^{G_{\mathbb{Q}}} = 0$. The sequence $0 \to k(\delta) \to A \xrightarrow{p} A \to 0$ now identifies $H^1(\mathbb{Q}, k(\delta))$ with $H^1(\mathbb{Q}, A)[p]$; it gives the same identification over $\mathbb{Q}_p$. A strict class killed by p thus has a finite lift which is a local coboundary at p; its cocycle vanishes on $I_p \cap \ker \delta$. At $v \nmid p\infty$, κ_p is unramified and every value of the remaining inertia action has order prime to p. On $I_v \cap \ker \delta$ the entire action on A is consequently trivial. Local triviality in $H^1(I_v, A)$ forces the finite cocycle to be zero on that subgroup, since a coboundary restricts there to zero. The finite-cocycle assertion just proved annihilates the lift. Finally a continuous cocycle from a profinite group to this discrete p-primary module has finite image. Each individual class is killed by a power of p. A nonzero element in the

displayed kernel would therefore have a nonzero multiple killed by p, contrary to what we have just proved. □

Corollary 34.12 (Highest two-power conductor on the period field)**.** *On the intervals of Theorem 20.11, take $s = J + 2$. For every sufficiently large x, every α of this conductor at two, and every indicated twist, Proposition 34.11 applies at every odd $p \le x^{2/3}$. For $p \equiv 3, 5 \bmod 8$ it applies through $p \le x^{2+4\varepsilon}$.*

Proof. By Proposition 9.9, $2^J \ge x^{2+8\varepsilon}/o$. Thus $2^{s-1} \ge 2x^{2+8\varepsilon}/o$. equation 25.13 and the valuation formula following it give $H_p \ll_\psi p^3$, uniformly in p. For $p \le x^{2/3}$ this is $O_\psi(x^2)$. For the indicated classes modulo eight one has $\max(v_2(p-1), v_2(p+1)) = 2$, hence $e_p = 2 + v_2(D)$ and $H_p \ll_\psi p$. Up to $x^{2+4\varepsilon}$ this is again smaller than the displayed lower bound for 2^{s-1} eventually. These inequalities are independent of r, q, d_0. □

The source identifies the squared character in `Development/AutomorphicArithmetic/AutomorphicVarieties/LevelCohomology/DirGPowOne.lean`, line 23 and its inverse on the $(3,1)$ matrix entry in `Development/AutomorphicArithmetic/AutomorphicVarieties/LevelCohomology/HcXppOf.lean`, line 36. The other matrix entries used for the strict groups in (19.15) include unsquared characters. Their remaining lengths and the contribution from $H \ge \sqrt{g_1 g_2}$ to the prescribed, arithmetically normalized theta periods are still required for (W).

34.5 Trace congruences on the special fiber

Fix a large coefficient prime and a twist on the common interval as in Theorem 34.6. Let $H = G_{K'}$, where $K' = \mathbb{Q}(i)$, and let D be the complete local ring of invariant functions in the marked ordinary deformation. For $h \in H$, put $t_h = T(h) - \Psi(h) - 1 - \Psi(h)^{-1}$. These elements vanish at the diagonal point. On a characteristic-ℓ curve through that point, the last three terms specialize to the lifts of their residues in the finite constant field. In particular the specialization of t_h is the difference between a trace and its constant term.

Lemma 34.13 (Integral matrix differences in the congruence subgroups)**.** *Let U be an integral realization of a nonconstant trace curve, and let $V \subset H$ be any open subgroup. After a fixed finite extension of its equal-characteristic coefficient field, write R for the valuation ring and z for a uniformizer. For every integer $m \ge 1$, put $V_m = \{v \in V : U(v) \equiv 1 \pmod{z^m}\}$. There is one integer $C \ge 1$, depending on U, V, for which*

$$z^{Cm} M_3(R) \;\subseteq\; \sum_{v \in V_m} R\,(U(v) - 1) \;\subseteq\; z^m M_3(R) \qquad (m \ge 1).$$

Proof. The Zariski closure of $U(H)$ over the algebraic closure is connected and is either SL_3 or SO_3 in its irreducible representation of degree three, by Theorem 33.13. The image of each open subgroup is dense there, since finitely many translates of its closure cover this irreducible group. In both groups the set of matrices with distinct eigenvalues is nonempty: a diagonalizable torus element has eigenvalues $b, 1, b^{-1}$ for a choice $b^2 \ne 1$. It is an open set, defined by nonvanishing of the discriminant. Choose $h \in V$ in the kernel of reduction with $A = U(h)$ in that set. Adjoin its eigenvalues to the coefficient field. In the resulting ring R the matrix $A - 1$ still has entries in zR. Choose an eigenvalue $\lambda \ne 1$ of A, and let P be the rank-one projection onto its eigenline along the other two eigenlines.

The conjugates $P_w = U(w) P U(w)^{-1}$, $w \in V$, generate the full matrix algebra. To see this over the algebraic closure, let $\mathcal{B}$ be the unital algebra they generate. Its Jacobson radical is preserved by $U(V)$. The radical is nilpotent, so it has a nonzero common kernel on the natural module. That kernel is invariant under $U(V)$, which acts irreducibly by Theorem 34.6.

Consequently the radical acts as zero and is zero. The summands of the module corresponding to the simple factors of the semisimple algebra $\mathcal{B}$ are permuted by $U(V)$. Their set is finite and is preserved by its Zariski closure. A connected group acts trivially on this finite set. Irreducibility gives a single summand, so in a suitable basis the algebra is a full matrix algebra acting on several copies of its natural module. The rank of every member is a multiple of the number of copies. The element P has rank one, forcing one copy. This proves the assertion, which descends to the coefficient field by linear independence. The span of the nonempty words in the P_w is a nonzero two-sided ideal of this algebra, hence fills it. Words of length at most nine suffice: starting at length one, the span of words of length at most j grows strictly until it stabilizes under multiplication by the generators, and the ambient dimension is nine. Choose nine such words forming a basis. For some $d_0 \geq 0$ their R-span contains $z^{d_0} M_3(R)$, by clearing the denominators of the coordinates of the nine matrix units in this fixed basis.

Let S consist of 1 and the eigenvalues of A other than λ, counting each value once, and put $d_1 = \sum_{\mu \in S} v_z(\lambda - \mu)$. The eigenvalues are integral. For $b = \ell^a$ the numbers λ^b, μ^b remain distinct, and

$$p_b(X) = \prod_{\mu \in S} \frac{X - \mu^b}{\lambda^b - \mu^b}, \qquad p_b(A^b) = P, \qquad p_b(1) = 0.$$

The equality $\lambda^b - \mu^b = (\lambda - \mu)^b$ bounds the denominators of the coefficients by z^{bd_1}. Since $p_b(1) = 0$, it is a combination of $X^j - 1$, $1 \leq j \leq 3$, with those same denominators. Choose b least among the powers of ℓ with $b \geq m$. Then $A^b - 1 = (A - 1)^b \in z^m M_3(R)$. Thus each $z^{bd_1} P_w$ lies in the indicated integral span of differences from V_m, using the elements $w h^{bj} w^{-1}$. This span is closed under multiplication, since for group matrices $(X - 1)(Y - 1) = (XY - 1) - (X - 1) - (Y - 1)$. The nine chosen words now place $z^{d_0 + 9bd_1} M_3(R)$ inside the span. As $b \leq \ell m$, take $C = \max(1, d_0 + 9\ell d_1)$. The other inclusion is the definition of V_m. □

Theorem 34.14 (Density of congruence to the diagonal at finite precision)**.** *Fix a nonconstant curve as above and a nonempty finite Galois Frobenius condition on primes of K'. There is a constant $\delta > 0$ such that for every integer $n \geq 1$ the relative natural density within that condition of the unramified primes with $u(t_{\mathrm{Frob}_v}) \equiv 0 \pmod{z^n}$ is at most $(10/9) n^{-\delta}$. The same uniform exponent works on every conjugacy class of the chosen finite quotient. The constant and the valuation refer to the curve and that quotient.*

Proof. Take V to be the kernel of the finite quotient and use Lemma 34.13. In a coset $g V_m$ the residual trace is a single constant a. The functions $f(v) = \mathrm{tr}(U(g) U(v)) - a$ on V_m have the following property, for each choice of g. Some value has valuation at most Cm. Indeed $U(g)$ has a unit entry, so its trace pairing with a matrix unit is a unit. By the lemma, the differences $f(v) - f(1)$ cannot all be divisible by z^{Cm+1}.

Let d be the minimum valuation of f and set $F = z^{-d} f$. This is integral with nonzero reduction. The linear span over the local field of its right translates has dimension at most ten: each translate is a linear combination of the nine entries of $U(v)$ and the constant function. The integral-valued functions in this span form a free R-module of rank at most ten. In fact, finitely many evaluations embed the span in a coordinate space of its own dimension, by choosing successive evaluations that reduce the kernel. Integral functions map into the standard integral lattice and every submodule of a finite free module over R is finite free. Reduction therefore puts the span of the translates of F modulo z in dimension at most ten. Choose a basis of that span from the translates. At every point of V_m at least one basis function is nonzero, since a translate can move any point to a point where F is a unit. The supports of the basis translates all have the same Haar measure. Their union covers V_m, so f has valuation

exactly d on at least one tenth of the coset. These elements fail the congruence at precision $Cm+1$.

Set $m_0 = 1$ and $m_{j+1} = Cm_j + 1$. Matching the constant term at precision m_j is a union of cosets of V_{m_j}. Conditional on each matching coset at least one tenth fails at precision m_{j+1}. Consequently at most $(9/10)^j$ of any initial union of V-cosets matches at precision m_j. We have $m_j \le (C+1)^j$. Taking $j = \lfloor \log n / \log(C+1) \rfloor$ gives the claimed bound with $\delta = \log(10/9)/\log(C+1)$. For each finite precision the condition on traces is conjugacy invariant and factors through a finite quotient, by continuity and finiteness of R/z^nR. Chebotarev's theorem identifies this Haar probability with relative natural density in the prescribed union of conjugacy classes, as in Theorem 33.18. □

Theorem 34.15 (Frobenius parameters from a set of positive upper density)**.** *Put $d = \dim(D/\varpi D)$. Let $\mathcal{P}$ be any set of rational primes of positive upper natural density that split in K', outside the ramification set. There are d distinct primes in $\mathcal{P}$ and places v_j above them such that*

$$O[[X_1, \dots, X_d]] \longrightarrow D, \qquad X_j \longmapsto t_{\mathrm{Frob}_{v_j}}$$

makes D a finite module. Fewer traces never suffice for such a finite map. The places can be chosen with either orientation at each step. For any finite list of these rings, one list of rational primes works simultaneously, using the first d primes for a ring of special-fiber dimension d.

Proof. Work initially in $A = D/\varpi D$. At any step consider the quotient by the traces already chosen. For every minimal prime with a positive-dimensional quotient, choose a further prime quotient of dimension one on that component. Its normalization gives a continuous map to a power series ring over a finite residue extension, nonconstant on D, by the argument of Proposition 34.3. There are only finitely many such curves, including those from the other rings being treated at this step. By Theorem 33.18, outside a set of rational split primes of density zero the trace difference is nonzero on every chosen curve, at either place. Theorem 34.14 also bounds the density of agreement on each curve at finite precision. A set of positive upper density still contains infinitely many choices after the excluded primes and the previous choices are removed. The next trace avoids every positive-dimensional minimal prime that was considered, because it has nonzero image on the chosen curve. On a positive-dimensional domain of finite Krull dimension, quotient by a nonzero element lowers dimension by at least one: prepend the zero prime to any lifted chain of primes containing that element. Zero-dimensional components stay zero-dimensional. This proves that a ring of dimension d reaches dimension zero in at most d steps. If fewer than d elements cut it to dimension zero, their ideal would be primary to its maximal ideal, contrary to Krull's height theorem. This also proves the lower bound for a finite map, since its quotient by the maximal ideal of the source is a finite algebra over the residue field.

The quotient of D by $(\varpi, t_{\mathrm{Frob}_{v_1}}, \dots, t_{\mathrm{Frob}_{v_d}})$ is now Artinian local with finite residue field. This ideal defines the original topology since it contains a power of the maximal ideal. Lift a residue-field basis of the quotient to D. To express an arbitrary element, take its expansion on that basis modulo the ideal, lift the coefficients from the residue field to O, and write the error as a sum of the ideal generators times new elements. Repeat on these elements. At step j all unexpanded terms lie in the j-th ideal power. The coefficient polynomials of the chosen lifts converge in $O[[X_1, \dots, X_d]]$ since their successive differences have total $(\varpi, X_1, \dots, X_d)$-order tending to infinity. Their evaluations converge in the complete ring D, giving the required finite generating set. This argument includes $d = 0$. □

34.6 Ordinary cuspidal adjoints with the prescribed residual characters

Use a large coefficient prime ℓ, a twist $1 \le r < t(x)$ and one of the complementary characters α for E_x in Theorem 34.6. Put $K' = \mathbb{Q}(i)$, write η for its quadratic character, and set $\Psi = \kappa_\ell^r \alpha$, where κ_ℓ is the cyclotomic character. Finite characters are viewed either as Dirichlet characters or as Galois characters by reciprocity with arithmetic Frobenius.

Theorem 34.16 (An automorphic lift of the residual diagonal)**.** *Use the primitive character χ inducing $\eta\alpha$, its conductor N, and $k = r+1$ in Theorem 21.20. On every sufficiently distant common interval its hypotheses on k, N, ℓ hold. For the auxiliary characters there put $\xi = \xi_1\xi_2$. There is a cuspidal self-dual automorphic representation of $\mathrm{GL}_3/\mathbb{Q}$ whose associated Galois representation U has determinant one, is absolutely irreducible on every open subgroup, and is ordinary crystalline at ℓ with cyclotomic exponents $r, 0, -r$. It has a lattice whose reduction on $G_{K'}$ is $\mathrm{diag}(\bar\Psi, 1, \bar\Psi^{-1})$. At every finite place away from ℓ its monodromy is zero and its inertia characters are $\chi\xi, 1, (\chi\xi)^{-1}$. Its base change stays cuspidal along every finite solvable Galois extension. In particular it gives a cuspidal system on each original E_x.*

Proof. We have $\chi(-1) = (-1)^{r+1}$, and N divides four times the modulus of E_x. On these intervals the logarithm of the latter is $o(x^\varepsilon)$. Since $\ell > \exp(x^\varepsilon/12)$ and $r < x^{401}$, we have $\ell > \max(k+1, (8N)^3)$ eventually. The inertia order of $\bar\Psi$ is $(\ell-1)/\gcd(\ell-1, r) > 4$ throughout this range, as α is unramified at ℓ.

Let f be the newform just constructed and ρ_f its two-dimensional ℓ-adic representation. We use the classical theorems on Galois representations of newforms of weight at least two. The determinant is $\kappa_\ell^r \chi\xi$, and the trace at any good arithmetic Frobenius is $a_p(f)$. The representation is absolutely irreducible. At ℓ the level is prime to ℓ. Good reduction and ordinarity give a crystalline representation with an unramified quotient line and with inertia character κ_ℓ^r on the subline. At a different prime its Artin conductor exponent equals the local newform conductor exponent. We use the normalization in which the cyclotomic character takes arithmetic Frobenius at $p \neq \ell$ to p. [13, 20] Skinner and Wiles, *Ordinary representations and modular forms*.

Chebotarev and the Brauer–Nesbitt theorem give $\bar\rho_f^{\mathrm{ss}} = 1 \oplus \bar\phi$, where $\bar\phi = \bar\eta\,\bar\Psi$, by the determinant and the congruences of Theorem 21.20. One can make the reduction split. Start with a stable lattice and an upper triangular residual basis. If π is a uniformizer of its coefficient field, pass to a field containing u with $u^2 = \pi$ and conjugate the matrices by $\mathrm{diag}(u, 1)$. The new upper-right entries are divisible by u, and the old lower-left entries were divisible by π, so the new lower-left entries are also divisible by u. Thus the residual diagonal is realized in a lattice.

The projective image of ρ_f is Zariski dense in PGL_2. It is infinite, since a finite projective image becomes scalar on a finite extension, incompatible with the two distinct Hodge–Tate exponents there. A proper connected positive-dimensional subgroup of PGL_2 is solvable. If it has a nontrivial unipotent radical, the unique point fixed by that radical is preserved by its normalizer, contradicting irreducibility. The other possibility for the identity component is a torus, in which case the full image permutes two lines. It would then admit a nontrivial quadratic self-twist. Indeed the operator which is $1, -1$ on the two lines intertwines the representation with its twist by their permutation sign. After reduction such a twist would preserve the two characters $1, \bar\phi$. The quadratic character reduces nontrivially at odd ℓ, so it would equal $\bar\phi$, contrary to the inertia order greater than four. This proves density. Every open subgroup has the same projective Zariski closure, since finitely many translates of a proper closed subgroup cannot cover the connected algebraic group.

Take $U = \mathrm{Ad}^0 \rho_f$, the conjugation action on the trace-zero matrices. The adjoint action of PGL_2 is irreducible, orthogonal for the trace pairing and of determinant one. Its automorphic realization is the adjoint lift of f by Gelbart and Jacquet. It is cuspidal because the quadratic self-twist characterizing the dihedral case was just excluded. The unramified Hecke parameters match those of U by symmetric square and determinant twist. For a cyclic extension of prime degree, failure of cuspidality under base change would give a nontrivial finite character self-twist of the cuspidal source. The matching good Frobenius polynomials and Chebotarev would then give this same self-twist of the restricted Galois representation. On the kernel of the character an intertwiner must be scalar by absolute irreducibility there, which forces the character to be trivial. Induction along a cyclic series proves the base-change assertion. [21, 22]

The adjoint of the split residual lattice has characters $\bar{\phi}, 1, \bar{\phi}^{-1}$, giving the required diagonal on $G_{K'}$. The ordinary flag on ρ_f gives the adjoint flag consisting successively of the maps from quotient to subline, the trace-zero maps preserving the subline, and all trace-zero maps. Its three exponents are as stated, and crystallinity is preserved by tensors and direct summands.

At $p \mid N s_1 s_2$, $p \neq \ell$, the conductor exponents of ρ_f and its determinant coincide, since the nebentype is primitive of the full newform level. For the finite-inertia part of the Weil–Deligne parameter, the codimension of invariants for every ramification group is at least the codimension for the determinant. The monodromy correction to the Artin conductor is nonnegative. Equality therefore forces a one-dimensional inertia-fixed space when the determinant is ramified. The finite inertia action splits as $1 \oplus \chi\xi$. Monodromy commutes with inertia and is nilpotent, so vanishes on both distinct character lines. Away from $N s_1 s_2 \ell$ the representation is unramified. Passing to the adjoint proves the claimed local description. □

For the full deformation ring a further issue arises in using automorphy lifting. Theorem 1.1(iv) of Allen, Newton and Thorne requires the distinct residual constituents to be individually polarized by the same multiplier. On any CM extension of K', complex conjugation fixes the restriction of $\bar{\Psi}$, since it comes from an abelian character of $G_{\mathbb{Q}}$. If a common scalar twist λ satisfied the polarization condition, division of the equations for $\lambda\bar{\Psi}$ and λ would give $\bar{\Psi}^{c} = \bar{\Psi}^{-1}$. Then $\lambda\bar{\Psi}$ and $\lambda\bar{\Psi}^{-1}$ would be equal, violating distinctness. Thus the existence of the adjoint above leaves the other components of the original ring to be controlled. Allen, Newton and Thorne, *Automorphy lifting for residually reducible ℓ-adic Galois representations, II*.

34.7 Two extension classes from an even orthogonal point

Use the large coefficient primes and marked representations of Corollary 34.10. Suppose a characteristic-zero irreducible point or a nonconstant residue-characteristic trace curve is even orthogonal. By Lemma 33.14 it has a two-dimensional lift ρ over a complete discrete valuation ring R with finite residue field, after a finite coefficient extension. Put $H = G_{\mathbb{Q}(i)}$. Its determinant is $\Psi = \alpha\kappa^{r}$, its residual matrix is $\mathrm{diag}(\bar{\Psi}, 1)$, and $\rho(c) = 1$. In residue characteristic the character is understood through the coefficient map. Let $U = \mathrm{Ad}^0 \rho|_H$. Choose $g_* \in I_\ell \subset H$ for which $u = \Psi(g_*)$ has residual order greater than four. Such a value exists on the given interval. On inertia the line and quotient characters are $\Psi, 1$, by Theorem 33.15. The projectors formed from $\rho(g_*)$ and the unit $u - 1$ give an integral eigenbasis with this ordering. For $g \in G_{\mathbb{Q}}$ denote the matrix entries in this basis by a_g, b_g, c_g, d_g in row order. Set

$$B = (b_g : g \in G_{\mathbb{Q}}), \qquad C' = (c_g : g \in G_{\mathbb{Q}}), \qquad I = (\operatorname{tr} U(h) - \Psi(h) - 1 - \Psi(h)^{-1} : h \in H).$$

Proposition 34.17 (The simultaneous trace congruence and the two extensions)**.** *One has $I = BC'$, a nonzero proper ideal. In the coefficients R/I, there is a class of zero annihilator*

for each of Ψ and Ψ^{-1}. Both classes are unramified away from ℓ, and the class for Ψ^{-1} is zero on the whole decomposition group at ℓ. For the adjoint trace ideal it suffices to use Frobenius elements at good rational primes splitting in $\mathbb{Q}(i)$.

Proof. Both off-diagonal ideals are nonzero by irreducibility and lie in the maximal ideal by the residual splitting. Modulo BC' matrix multiplication makes a_g, d_g characters with product $\Psi(g)$. The character d_g is one on inertia at ℓ, since the chosen first eigenline is the ordinary line. Away from ℓ the image of inertia in ρ has order prime to ℓ, by Theorem 33.15. There d_g reduces to one and takes values in a group of principal units of ℓ-power order. Thus it is unramified at every finite prime. The class number and the narrow class number of $\mathbb{Q}$ are one, so $d_g = 1$ and $a_g = \Psi(g)$ in this quotient. Since $1 + \operatorname{tr} U(h) = (\operatorname{tr}\rho(h))^2/\Psi(h)$, this proves $I \subseteq BC'$.

Conversely reduce modulo I. For any $h \in H$ use $g_*^j h$, for $j = 0, 1, 2$. At least two of these choices have $1 + \bar{u}^j\bar{\Psi}(h) \neq 0$. For each such choice the adjoint trace identity gives

$$(u^j a_h + d_h)^2 = (u^j \Psi(h) + 1)^2 \pmod{I}.$$

The sum of the two expressions being squared is a unit by residual splitting and odd residue characteristic. The two linear congruences therefore hold with the plus sign. The difference of the two chosen powers of u is a unit. It follows that $a_h = \Psi(h)$ and $d_h = 1$ modulo I. For $g \notin H$ apply this to gc, since both $\rho(c)$ and $\Psi(c)$ are one. Now $b_g c_h = a_{gh} - a_g a_h \in I$ for every pair of group elements. This proves the equality of ideals.

Scale the first basis vector so that the upper-right ideal becomes R and the lower-left ideal becomes I. This preserves integrality and the diagonal and gives an upper extension modulo I. Its normalized upper entry b_g/d_g is a cocycle for Ψ. In a second lattice scale in the opposite direction to make the lower-left ideal R and the upper-right ideal I. Its normalized lower entry c_g/a_g is a cocycle for Ψ^{-1}. Both cocycles vanish at g_* and each takes a unit value somewhere. A multiple which is a coboundary must vanish identically, since evaluation at g_* forces the cobounding vector to be zero. This proves the assertion on annihilators. Inertia away from ℓ acts through the same finite prime-to-ℓ quotient in each lattice, where averaging kills degree-one cohomology. On the local group at ℓ the lower-left entry is identically zero.

Write $I = (\pi^m)$. An element of H realizes the minimum valuation m, since the discrete valuation is integral. Reduce the lift modulo π^{m+1}, including Ψ and the quadratic character defining H. The conjugacy class of this element in the resulting finite Galois quotient occurs at good rational primes by Chebotarev. Those primes split in $\mathbb{Q}(i)$, and either orientation has the same trace difference since $\rho(c) = 1$. They realize the same minimum. □

34.8 Simultaneous upper and lower extension classes

For the finite family and the ring of Lemma 34.20 put $\overline{B} = B/IB$ and $\overline{L} = L/IL$. These are finite-type O-modules. Write $D^* = \operatorname{Hom}_O(D, J/O)$ for the O-linear dual of such a module. Denote the actual complementary strict group over $\mathbb{Q}$ with coefficient $(J/O)(\Psi^{-1})$ by S^-. Character descent identifies it with the prescribed component on E_x.

Proposition 34.18 (An integral scalar for every member of the family)**.** *There are injections $\overline{B}^* \hookrightarrow P$ and $\overline{L}^* \hookrightarrow S^-$. If $h = \operatorname{length}_O \overline{B}$, then*

$$h \le v_\ell(L(1-r, \alpha^*)), \qquad (\ell^h, 0, \dots, 0) \in T, \qquad \ell^h \overline{L} = 0.$$

These statements include coefficient rings of characteristic ℓ and arbitrary congruences among the different traces.

Proof. Modulo I the diagonal entries are $\Psi, 1$. Thus $g \mapsto b_g \bmod IB$ is a cocycle for Ψ, and $g \mapsto c_g/\Psi(g) \bmod IL$ is a cocycle for Ψ^{-1}. Compose them with the corresponding O-linear functionals. These are continuous, since the modules are finitely generated over O with their compact quotient topology. Their values generate their respective modules and both cocycles vanish at g_*. If the resulting class is zero, evaluation there forces its cobounding vector to vanish since $u - 1$ is a unit. The functional therefore vanishes. Away from ℓ the joint inertia representation is finite of order prime to ℓ, so its degree-one classes vanish on inertia. The lower entry vanishes identically on the full local group at ℓ, by ordinarity.

We check the upper local condition when $r = 1$ before quotienting by I. On inertia at ℓ the actual diagonals in every tuple are $\Psi, 1$ and the lower entries vanish. Thus b_g itself is a cocycle there with values in $B(\kappa)$. For any compact pro-ℓ coefficient module the Kummer valuation on inertia is defined by taking the inverse limit of the valuations on finite abelian quotients as in Lemma 34.19. The module B is compact and a closed additive subgroup of the product of coefficient rings. In each coordinate its Kummer valuation is zero by the proof of that lemma, which applies in the original integral eigenbasis as well, since the two adjacent cocycles differ by a nonzero scalar in the fraction field. The base coordinate is identically zero. By injectivity into the product, the valuation in B is zero. Its image under every functional on B/IB is zero too. This proves both injections. The first one and Theorem 28.2 give finiteness and the claimed bound on h, since a free O-summand would have infinite dual.

Choose lifts $b_1, \dots, b_s$ in B of elementary-divisor generators of $\overline{B}$, of orders $\ell^{a_1}, \dots, \ell^{a_s}$. They generate B over the local ring T by Nakayama's lemma. There is a matrix $Z = (z_{ij})$ with entries in I such that $\ell^{a_i} b_i = \sum_j z_{ij} b_j$. Set

$$y = \det\bigl(\operatorname{diag}(\ell^{a_1}, \dots, \ell^{a_s}) - Z\bigr).$$

Multiplying the relations by the adjugate shows that $yB = 0$. Its base value is $\ell^{\sum_i a_i} = \ell^h$. By Lemma 34.20, y vanishes in every irreducible coordinate and annihilates L as well. Since $\ell^h - y \in I$, it follows that $\ell^h L \subset IL$. When $s = 0$ use the empty determinant, equal to one. □

34.9 The upper extension at the first twist

At $r = 1$, an unrestricted local Kummer class at ℓ can have nonzero valuation. Such a class would contribute an extra local term precisely when $\alpha(\ell) = 1$. We determine the valuation of the upper class in Proposition 34.17, including on a curve with coefficient field of characteristic ℓ.

Lemma 34.19 (The upper spin extension has unit Kummer class)**.** *On the ordinary component through the marked point, the upper class in $H^1(\mathbb{Q}, (R/I)(\alpha\kappa))$ of Proposition 34.17 satisfies the Kummer unit condition at ℓ.*

Proof. Let K be the completion of $\mathbb{Q}(i)$ at the chosen place over ℓ. It is unramified over $\mathbb{Q}_\ell$, so the two inertia groups agree. In the scaled upper lattice from the proposition the local matrices on G_K have diagonal characters a, d, with d unramified and $a/d = \kappa\beta$ for an unramified β. The upper entry divided by d is an integral cocycle z for this character before reduction modulo I.

There is an R-linear valuation map on these cocycles after restriction to inertia. For each finite coefficient module R/π^j, apply the Kummer valuation on the maximal unramified extension, with κ-twisted coefficients. Explicitly a Kummer class represented at a finite unramified stage by y maps to its normalized valuation, extended additively in the coefficient module. The valuations agree under passage up the unramified tower. The cyclic case is the Kummer sequence and the assertion for finite abelian coefficient modules follows by additivity. These

maps are natural in the coefficients, kill coboundaries, and commute with reduction modulo π^j. Taking their inverse limit defines $v_R(z) \in R$. This is the first coordinate in the local decomposition of Lemma 32.14. This argument uses finite ℓ-groups at every stage for either characteristic of R.

Use the original integral rank-three lattice at this place. The eigenvalues $u, 1, u^{-1}$ of the marked inertia element are residually distinct, so their eigenvectors can be chosen as an integral basis by the spectral projectors. They are ordered along the ordinary flag. In the first two steps of the flag the normalized adjacent cocycle has zero Kummer valuation by Theorem 32.15. Indeed its valuation is the coordinate killed by saturation in the ordinary chart. At a place with negative Frobenius sign the same test is taken on G_K via the induced pair of adjacent coordinates. Changing the adapted basis by diagonal units rescales the class by a unit and an upper unipotent change adds a coboundary on the adjacent quotient. Hence the vanishing holds in the integral eigenbasis just chosen, at all coefficient levels.

Over the fraction field of R, identify the rank-three representation with $\mathrm{Ad}^0 \rho$. In the ordered eigenbasis $E_{12}, \mathrm{diag}(1,-1), E_{21}$ of this adjoint, conjugation sends the middle vector to itself minus $2z(g)E_{12}$. The distinct eigenvalues of g_* show that the two ordered eigenbases correspond up to three nonzero scalars. The first adjacent cocycle in the original integral basis is therefore a fixed nonzero fraction-field multiple sz. It is integral and its valuation vanishes. Choose $h \neq 0$ in R with $hs \in R$. Linearity gives $(hs)v_R(z) = hv_R(sz) = 0$. The ring R is a domain, so $v_R(z) = 0$. Reducing modulo I proves the unit condition for the upper extension, including at split values of $\alpha(\ell)$. □

34.10 The integral trace ring of several even orthogonal lifts

Fix $\ell, r, \alpha, O, \Psi, H$ as in Proposition 34.17. Consider any finite family of its even orthogonal lifts ρ_i, over complete discrete valuation rings R_i of either coefficient characteristic. Add the diagonal representation $\rho_0 = \mathrm{diag}(\Psi, 1)$ over O. In each original residually split lattice diagonalize the same g_* with $u = \Psi(g_*)$ in the first position. Denote the tuples of entries of $\rho_i(g)$, including position zero, by a_g, b_g, c_g, d_g. All tuples lie in $A = O \times \prod_i R_i$, with its product topology. Let T be the closed O-algebra generated by the traces of the adjoints on H, and set

$$B = \sum_{g \in G_{\mathbb{Q}}} T b_g, \qquad L = \sum_{g \in G_{\mathbb{Q}}} T c_g, \qquad I = BL.$$

The last product denotes the module of finite sums of coordinatewise products.

Lemma 34.20 (The augmentation and the two off-diagonal modules)**.** *The ring T is also the closed algebra generated by the two-dimensional traces on $G_{\mathbb{Q}}$. It contains every a_g, d_g. The modules B, L are finite over T, and I is the kernel of the projection $T \to O$ at the diagonal. The annihilators of B, L in T coincide and consist of the tuples zero at every irreducible member of the family.*

Proof. Every trace polynomial has a common residue in the naturally embedded copies of the residue field of O. The same holds in its closure. An element of T with nonzero common residue is a unit in T. To see this, lift its residual inverse from O and use the convergent geometric series for the error in A. Hence T is compact and local. For $h \in H$, the square of $s_j = \mathrm{tr}\, \rho(g_*^j h)$ is in T by the adjoint identity, since the determinant tuple is the scalar $u^j \Psi(h)$. If the common residue $1 + \bar{u}^j \bar{\Psi}(h)$ is nonzero, lift it from O and apply Newton iteration to $X^2 - s_j^2$. The derivative is a unit and the residual error is squared at every step up to a unit. In every coordinate the iteration converges to the unique square root with that residue. Closedness gives $s_j \in T$. Among $j = 0, 1, 2$ two satisfy the residue test. Solve $s_j = u^j a_h + d_h$

using their unit determinant to get $a_h, d_h \in T$. The equalities $\rho(c) = 1$ and $\Psi(c) = 1$ give this for all g. Conversely all adjoint traces are polynomials in the two-dimensional traces divided by the prescribed unit determinants. This proves equality of the two closed trace rings.

The joint image of all the two-dimensional representations is topologically finitely generated. To see this, pass to the kernel of the common residual character in the absolute Galois group unramified outside the finite set of allowed primes. Its joint image lies in a product of principal congruence kernels and is pro-ℓ. Its maximal pro-ℓ quotient before taking this image is finitely generated. Indeed the classes of degree ℓ characters unramified outside that finite set form a finite group. For example, adjoining ℓ-th roots of unity reduces this finiteness to the finiteness of the class group and finite generation of the group of units with the specified primes inverted, by Kummer theory and prime-to-degree restriction. The generator criterion for pro-ℓ groups applies. Reinstating the finite residual quotient gives finitely many topological generators g_s of the joint image. Matrix multiplication gives $b_{gh} = a_g b_h + b_g d_h$ and its lower analogue. Inversion divides the negative off-diagonal entries by the known unit determinant. The sums of Tb_{g_s} and of Tc_{g_s} are compact, hence closed, and contain the entries of all words. They give B, L.

The identities $b_g c_h = a_{gh} - a_g a_h$ put I in T. It is a closed finitely generated ideal, with zero base coordinate. Modulo I both diagonal entries are characters. The lower one is one on inertia at ℓ, and unramified away from ℓ by the finite prime-to-ℓ inertial types and its trivial reduction. For this last assertion T/I is a profinite local ring whose principal units are pro-ℓ, by reduction at finite precision in the product and passage to the quotient by the closed ideal. The ordinary class field theorem over $\mathbb{Q}$ gives $d_g = 1$ modulo I. The determinant then gives $a_g = \Psi(g)$ there. The trace generators and compactness of O now give $T/I = O$ under base evaluation. Finally, at each irreducible coordinate some upper entry and some lower entry are nonzero. Those coefficient rings are domains. At position zero both modules vanish. Coordinatewise multiplication proves the annihilator assertion. □

34.11 The even orthogonal case on a common interval

Take an original interval $[V, V^{3/2}]$ from Theorem 19.15. Let E_* be its largest period field, N_* its cyclotomic modulus and g_* its degree. Write $t_* = \lfloor V^{(3/2)401} \rfloor$. For $2 \le r < t_*$ let $D_r(E_*)$ be the nonzero integer of Lemma 11.3, at $n = r$, on this very field. Define the positive integer

$$Z_* = |\mathrm{Cl}(E_*)| \prod_{2 \le r < t_*} |D_r(E_*)|.$$

The class number here is the ordinary ideal class number. In particular every factor is defined independently of any deformation ring.

Theorem 34.21 (Prime support and congruence depth for even orthogonal points)**.** *At every x of a sufficiently distant such interval use the original period field E_x, a prime $\ell > \exp(x^{\varepsilon}/12)$, and a complementary character α at $1 \le r < \lfloor x^{401} \rfloor$. An even orthogonal irreducible point in characteristic zero, or a nonconstant even orthogonal trace curve in characteristic ℓ, requires $\ell \mid Z_*$. Outside these primes, every nonconstant trace curve on the marked component has monodromy* SL_3.

For each triple $(\ell, r, [\alpha])$ let $d_{\ell,r,\alpha}$ be the O-length of C in Theorem 28.1, with one representative per residue Frobenius orbit. The following inequality sums each distinct triple on the whole interval once:

$$\sum_{\ell, r, [\alpha]} b_\alpha d_{\ell,r,\alpha} \log \ell \le \log Z_* \ll_{\psi,\varepsilon} V^{1220}.$$

For a characteristic-zero even orthogonal point with coefficient ring R, the greatest integer m for which all its adjoint traces agree with the diagonal traces modulo π_R^m satisfies

$$m/e(R/O) \le \min\{j \ge 0 : \ell^j C = 0\} \le d_{\ell,r,\alpha}.$$

Thus the same sum bounds the weighted sum of the suprema of these normalized depths, one supremum per triple. At a single x, the construction with $E_x, t(x)$ gives the bound $O_{\psi,\varepsilon}(x^{812})$.

Proof. The fields on the interval are nested. The two auxiliary quadratic fields and K_ψ are fixed there and only the index of the real cyclotomic layer varies. Uniformly on this interval, $\exp(V^\varepsilon/12) > 2t_* + 3$ and exceeds g_* and every prime dividing N_* for large V, by the degree and conductor bounds of Theorem 19.15. Inflate each α to the largest group. The groups P, C are defined over $\mathbb{Q}$ with the same local conditions, so this leaves them unchanged. At $r = 1$ their dual-order description uses character components of the class group. The sum of their lengths at ℓ, weighted by b_α, is at most $v_\ell(|\mathrm{Cl}(E_*)|)$. Indeed $\mathbb{Z}_\ell[\mathrm{Gal}(E_*/\mathbb{Q})]$ is a product of the unramified integer rings for the different Frobenius orbits. An O-module of length one in the selected factor has $\mathbb{Z}_\ell$-length b_α.

At $r \ge 2$, Theorem 28.1 gives components in the critical sign at twist $r-1$. On each of these unramified factors, the zeroth Fitting ideal of a finite module is generated by ℓ to its length. The injection and Fitting inclusion of Lemma 11.2 therefore bound the weighted sum at this ℓ, r by $v_\ell(D_r(E_*))$, as in Theorem 11.1. Taking products proves the first inequality. At a single field of degree g and modulus N, the conductor–discriminant formula gives $|\operatorname{disc} E| \le N^g$. Minkowski represents each ideal class by an ideal of norm at most $2^g|\operatorname{disc} E|^{1/2}$. Counting such ideals by $\zeta_E(2) \le \zeta(2)^g$ gives $\log|\mathrm{Cl}(E)| \ll g\log(2N)$. Lemma 11.3 bounds the sum of the remaining logarithms by $O_\psi(gt^2(g + \log(2Nt)))$. Use $g \ll_\psi x^{2+8\varepsilon}$, $\log N \ll_{\psi,\varepsilon} x^{\varepsilon/4}$, $t \le x^{401}$ and $0 < \varepsilon < 1/52$. This gives the claimed pointwise bound and, with $x = V^{3/2}$, the stated interval bound.

For an even orthogonal point let (π_R^m) be the trace ideal from Proposition 34.17. Its upper extension class meets the conditions of P, including the unit restriction at $r = 1$ by Lemma 34.19. After reduction modulo π_R the normalized cocycle still takes a unit value and vanishes at the marked inertia element. It gives a nonzero class in P_1 after residue extension. Indeed finite coefficient scalar extension commutes with cohomology and these local conditions by using a basis of the extension. Hence $d_{\ell,r,\alpha} > 0$ in both coefficient characteristics by Theorem 28.1. This proves the prime exclusion. Corollary 34.10 supplies the monodromy conclusion.

In mixed characteristic embed R/π_R^m as torsion in $\mathrm{Frac}(R)/R$. Global degree-one cohomology injects under this map since $\Psi(g_*) - 1$ is a unit, and all the imposed local conditions persist. The divisible positive group over these coefficients is $P \otimes_O R$. Indeed R is finite free over O, so the assertion follows componentwise in a basis, also for the Kummer valuation. The upper class has annihilator exactly (π_R^m). By Theorem 28.1, every element of $P \otimes_O R$ is killed by ℓ^j for the least j which kills C. This gives the claimed depth. □

The estimate measures simultaneous agreement of all traces at one point. The number of points over an exceptional prime, the full strict group for Ψ^{-1}, and the dense three-dimensional case still require control. The interval estimate (W) also requires the arithmetic normalization of the theta pairing.

34.12 One trace function on the even orthogonal locus

Use the largest period field E_*, its degree g_*, modulus N_*, and last twist cutoff t_* from Theorem 34.21. For each primitive character associated with a character of $\mathrm{Gal}(E_*/\mathbb{Q})$, let f_α be

its conductor. Define

$$M_* = \left| \prod_{r=1}^{t_*-1} \prod_{\alpha(c)=(-1)^r} r f_\alpha (r+1)!\, L(1-r,\alpha^*) \right|.$$

The inner product takes every complex character of the specified parity once.

Fix an actual marked ring on the interval at a large prime ℓ and write D for its complete invariant trace ring with coefficient ring O, as in Theorem 34.2. Its diagonal evaluation is denoted by x_0. In the complementary strict group S^-, take the sum S^{ev} of the images of $(L/IL)^*$ in Proposition 34.18, ranging over all finite families of even orthogonal points and nonconstant even orthogonal trace curves of this ring. This definition permits all their integral eigenbases. It includes the lower extension classes obtained from individual points followed by any coefficient functional. Indeed at one point the lower-entry lattice generates the ideal C' of Proposition 34.17. Dividing the lower entries by a generator of that ideal, then reducing modulo the trace ideal at the point, factors through L/IL.

Theorem 34.22 (A common scalar and the exponents of the strict classes)**.** *The integer M_* is positive and satisfies*

$$\log M_* \ll g_* t_*^2 \log(2N_* t_*).$$

For each $\ell > \exp(V^\varepsilon/12)$ on a sufficiently distant original interval its valuation is exactly

$$v_\ell(M_*) = \sum_{r<t_*,\ r\geq 1} \sum_{[\alpha]} b_\alpha\, v_\ell(L(1-r,\alpha^*)),$$

where the inner sum uses the residue Frobenius orbits on the same parity of characters of E_*.

For every marked ring as above, put $d = v_\ell(L(1-r,\alpha^*))$. There is an element $y \in D$ with $x_0(y) = \ell^d$ whose image is zero on every even orthogonal irreducible characteristic-zero point and every nonconstant even orthogonal trace curve. One may instead prescribe the same diagonal value M_* in all these rings. The actual strict subgroup S^{ev} is killed by ℓ^d. In particular, writing $a_{\ell,r,\alpha}$ for the smallest exponent killing it, meaning $\ell^{a_{\ell,r,\alpha}} S^{\mathrm{ev}} = 0$, one has on the whole interval

$$\sum_{\ell,r,[\alpha]} b_\alpha a_{\ell,r,\alpha} \log \ell \leq \log M_*.$$

Count a repeated triple only once, taking the sum of its detected subgroups inside the same strict group over $\mathbb{Q}$.

Proof. The Bernoulli polynomial formula in Lemma 25.4 makes each factor defining M_* an algebraic integer. Its denominator computation holds also for $B_1(X) = X - 1/2$. In that weight all the characters are odd and nonprincipal, and the functional equation and nonvanishing at one give nonvanishing at zero. The higher weights use the nonvanishing in the same lemma. Galois conjugation permutes the factors, giving a nonzero rational integer before the absolute value. The Bernoulli estimate there bounds the logarithm of the absolute value of each cleared factor from above by $O(r \log(2N_* r))$, also at weight one. Summing over at most g_* characters at each weight proves the size estimate. The conductor, factorial and weight in each clearing factor are ℓ-adic units by the inequalities in Theorem 34.21. The local orbit of α has b_α elements with equal valuations, since its coefficient ring is unramified. Taking valuations gives the identity. Theorem 28.2 identifies each summand before weighting with the positive Selmer length.

Consider first any finite list of the indicated specializations of D and include its base evaluation. Its image in the product of the coefficient rings is exactly T of Lemma 34.20. Indeed the

invariant ring is topologically generated by the traces on H by Theorem 34.2, and it is compact. The continuous product image therefore contains the entire closed algebra of these trace tuples and is contained in that algebra. Proposition 34.18 puts $(\ell^h, 0, \dots, 0)$ in that image, for some $h \le d$. Multiply by ℓ^{d-h} and lift the resulting tuple to D. In the compact fiber $x_0^{-1}(\ell^d)$ each specialization imposes a closed vanishing condition. The finite-list result gives the finite intersection property, so there is a single y satisfying all of them. Since $M_*/\ell^d \in \mathbb{Z}$, multiplying y by this integer gives the second specified value.

For every finite family, ℓ^d kills L/IL, hence its dual and its image in the actual strict group. This annihilation persists on their sum and under inflation of the character from a smaller period field on the interval. Thus $a_{\ell,r,\alpha} \le d$, and the valuation identity and prime factorization of M_* give the final inequality. □

To obtain the required bound for the order of the full strict group one must also control the number of its generators and its classes outside S^{ev}. The dense three-dimensional deformations and the comparison with the arithmetically normalized theta pairing remain open in the argument for (W).

34.13 Trace congruences and three strict character groups

Fix $x \in [V, V^{3/2}]$, $1 \le r < \lfloor x^{401} \rfloor$, a prime $\ell > \exp(x^{\varepsilon}/12)$, and a character α of the period field with $\alpha(c) = (-1)^r$. Put $K' = \mathbb{Q}(i)$, $H = G_{K'}$, $\eta = \chi_{-4}$, $\Psi = \alpha\kappa_\ell^r$, and $(\chi_1, \chi_2, \chi_3) = (\Psi, 1, \Psi^{-1})$. In particular $\chi_j(c) = 1$ for each j. Let O be the minimal unramified coefficient ring of α. Consider an integral specialization of the original marked ordinary representation with restriction $U : H \to \mathrm{GL}_3(R)$ irreducible over the fraction field of R. Here R is the integer ring of a finite extension of $\mathrm{Frac}(O)$, or a complete discrete valuation ring of characteristic ℓ with finite residue field extending that of O. Write π for its uniformizer and $v(\pi) = 1$. The split residual basis, the marked element g_* in inertia and the polarization are those of Lemmas 32.10 and 34.1. These hypotheses apply to the marked integral realizations of irreducible points and nonconstant trace curves considered above.

Proposition 34.23 (The two lower extension lengths)**.** *Let I_{tr} be the ideal of R generated by $\operatorname{tr} U(h) - \sum_j \chi_j(h)$ for $h \in H$. For $i \ne j$, let A_{ij} be the ideal generated by $U(h)_{ij}$ as h runs over H. Put*

$$B = A_{12} = A_{23}, \qquad C = A_{21} = A_{32}, \qquad U_0 = A_{13}, \qquad V_0 = A_{31}.$$

All four ideals are nonzero and proper. Write b, c_0, u, v_0 for their valuations in the displayed order. Then

$$I_{\mathrm{tr}} = BC + U_0V_0, \qquad m = v(I_{\mathrm{tr}}) = \min(b + c_0, u + v_0).$$

The two lower adjacent entries give strict classes over K' with character Ψ^{-1}, and the lower corner gives a strict class there with character Ψ^{-2}. Their cyclic coefficient modules have respective lengths

$$d_{21} = d_{32} = \min(m, v_0 + b - c_0), \qquad d_{31} = \min(m, 2c_0 - v_0), \qquad d_{21} + d_{31} \ge m.$$

Each class has zero annihilator over its cyclic coefficient ring.

Proof. Write $\bar{j} = 4 - j$ for index reversal and let J be the reversal matrix. The marking gives $U(ch^{-1}c) = JU(h)^{\mathsf{T}}J$, which identifies the reflected entry ideals. The orbit of each basis vector spans the fraction-field representation by irreducibility, so each entry ideal is nonzero. The residual matrices are diagonal.

Let $\mathcal{A}$ be the R-span of $U(H)$. This is an algebra by the group law. The difference $Z = U(g_*) - U(cg_*^{-1}c)$ is $\mathrm{diag}(z, 0, -z)$ with z a unit. The same difference for the split diagonal representation has first entry $z_0 = \Psi(g_*) - \Psi(g_*)^{-1}$, congruent to z modulo π. The three residues $z_0, 0, -z_0$ are distinct. Polynomial interpolation in Z puts each coordinate projector P_j in $\mathcal{A}$. Inserting P_j between two matrices shows $A_{ij}A_{jk} \subset A_{ik}$ for distinct indices.

Set $I = BC + U_0V_0$. Taking the j-th diagonal entry modulo I defines an algebra homomorphism $\mathcal{A} \to R/I$, since every off-diagonal term in the diagonal of a product belongs to I. We identify its restriction to H. The element $U(g_*)$ has the three eigenvalues $\chi_j(g_*)$ with pairwise unit differences by ordinarity. Its spectral projectors Π_j are polynomials in $U(g_*)$, hence belong to $\mathcal{A}$. The matrix $Q = \sum_j \Pi_j P_j$ reduces to the identity. Its inverse belongs to $\mathcal{A}$ by Cayley–Hamilton and its unit determinant. We may scale its columns by diagonal units to give Q diagonal entries one. The columns are ordered eigenvectors, so their successive spans give the unique ordinary flag. In this basis the local matrices on $H \cap G_{\mathbb{Q}_\ell}$ are upper triangular and have the prescribed diagonal on inertia by Lemma 32.10. Passage through the diagonal algebra homomorphism is invariant under conjugation by Q. Thus $U(h)_{jj} \bmod I$ equals $\chi_j(h)$ on the chosen inertia. At a conjugate place the same assertion follows by inner conjugation on H or by the polarized formula, which reverses the indices and inverts the characters. Away from ℓ the prescribed inertia kernels give a finite quotient of order prime to ℓ. The ratio of the two characters reduces to one and takes values in the principal units of the finite local ring R/I, an ℓ-group. It is therefore trivial on this inertia as well. The ratio is now everywhere unramified on H. The ring $\mathbb{Z}[i]$ is Euclidean, so class field theory makes the ratio trivial globally. This proves $I_{\mathrm{tr}} \subset I$.

For the other inclusion, traces of all words in $U(H)$ agree with their split diagonal evaluations modulo I_{tr}, by the group law. In particular $z^2 = \mathrm{tr}(Z^2)/2$ agrees with z_0^2, and $z + z_0$ is a unit. Thus $z \equiv z_0 \bmod I_{\mathrm{tr}}$. Lagrange interpolation for Z expresses P_i using the three nodes $z, 0, -z$, all their differences units. The same expressions for the split evaluation reduce to the same word coefficients modulo I_{tr}. For $i \neq j$ the trace $\mathrm{tr}(P_iU(h)P_jU(h'))$ therefore agrees modulo that ideal with its split value zero. It equals $U(h)_{ij}U(h')_{ji}$. These products generate I, proving equality.

For distinct i, j, k form the quotient

$$L_{ij} = A_{ij}/(IA_{ij} + A_{ik}A_{kj}).$$

Choose a generator of A_{ij}, the same one for each reflected pair. Division by this generator identifies L_{ij} with $R/\pi^{d_{ij}}$ for some $d_{ij} \geq 0$. In this quotient the entry $U(h)_{ij}/\chi_j(h)$, expressed in the chosen generator, is a cocycle for χ_i/χ_j. Indeed the term through k in matrix multiplication vanishes, and both diagonal factors can be replaced by their characters modulo I. If a scalar a times the cocycle is a coboundary, then

$$a\,U(h)_{ij} = (\chi_i(h) - \chi_j(h))w \quad \text{in } L_{ij}$$

for some $w \in L_{ij}$, with the entry here in its original ideal. Subtract the equations at g_* and $cg_*^{-1}c$. The left side vanishes because Z is diagonal. On the right the multiplier of w is a difference of distinct elements of $\{z_0, 0, -z_0\}$, hence a unit. Thus $w = 0$. The entries generate L_{ij}, so $aL_{ij} = 0$.

For $i > j$ take a local $h \in H \cap G_{\mathbb{Q}_\ell}$ and put $U' = Q^{-1}U(h)Q$. In $U(h)Q = QU'$ the term through k on each side belongs to $A_{ik}A_{kj}$, since $Q, U' \in \mathcal{A}$. We have $U'_{ij} = 0$, $Q_{ii} = Q_{jj} = 1$, and the diagonal entries of both representations are $\chi_j(h)$ in coordinate j modulo I. Consequently $U(h)_{ij} = (\chi_j(h) - \chi_i(h))Q_{ij}$ in L_{ij}. This gives a coboundary on the full decomposition group at the chosen place. At the conjugate place, the polarized identity relates the cocycle to the

negative of its reflected lower cocycle, using the cocycle formula for the inverse. Thus both lower adjacent classes and the lower corner vanish locally there too. At every prime away from ℓ the inertia cocycle factors through the prescribed finite group of order prime to ℓ, whose positive-degree cohomology vanishes by averaging. These are exactly the strict conditions on K'.

Finally $C^2 \subset V_0$ and $V_0 B \subset C$ by multiplication of entries. The two lower quotients therefore have the asserted nonnegative lengths. Before truncation at m, the two arguments $v_0 + b - c_0$ and $2c_0 - v_0$ sum to $b + c_0 \geq m$. If either is at least m, that term alone suffices. Otherwise truncation changes neither. This proves their summed lower bound. □

For a character ξ over O, let $a(\xi)$ be the least $a \geq 0$ such that ℓ^a kills the strict group over $\mathbb{Q}$ for $(\mathrm{Frac}(O)/O)(\xi)$. The conditions are vanishing on the decomposition group at ℓ and vanishing on inertia at all other finite places. The groups below are finite by descent to the period field or its compositum with K', and the finiteness in Lemma 19.2. Write

$$a_0 = a(\Psi^{-1}), \qquad a_1 = a(\eta\Psi^{-1}), \qquad a_2 = a(\eta\Psi^{-2}).$$

Theorem 34.24 (Depth at an irreducible ordinary point)**.** *For each mixed-characteristic specialization above with ramification index $e = v(\ell)$,*

$$\frac{m}{e} \leq \max(a_0, a_1) + a_2.$$

If $a_1 = a_2 = 0$, one lower adjacent class descends in the invariant part to the complementary character Ψ^{-1} at full length m. In equal characteristic, under the same vanishing, this class has zero annihilator over R/π^m and gives a nonzero strict class over the residue coefficient field for $\overline{\Psi}^{-1}$.

Proof. Conjugation by c sends each normalized lower cocycle to the negative of its reflected cocycle. Indeed $U(chc)_{ij} = U(h^{-1})_{\bar{\jmath},\bar{\imath}}$, and division by $\chi_j(h)$, followed by the inverse cocycle identity, yields the sign minus. The generators for a reflected pair agree. All $\chi_i(c)$ are one. Thus the two projections of the 21-class under $(1 + c)/2$ and $(1 - c)/2$ are strict classes for Ψ^{-1} and $\eta\Psi^{-1}$ over $\mathbb{Q}$, respectively. Here restriction identifies $H^1(\mathbb{Q}, -)$ with the invariants on H, since the quotient has order two. Twisting by η identifies the anti-invariants in the same way. On local inertia and decomposition groups, the quotients also have order dividing two, so restriction detects the required local vanishing. The corner is anti-invariant and descends with character $\eta\Psi^{-2}$.

The two projected adjacent classes sum on H to a class of full length d_{21}. The annihilators are ideals in a discrete valuation ring. At least one projection has full length, since otherwise $\pi^{d_{21}-1}$ would kill both when $d_{21} > 0$. The corner has full length d_{31} by Proposition 34.23.

In mixed characteristic embed R/π^d into $\mathrm{Frac}(R)/R$ by multiplication by π^{-d}. The induced global degree-one map for each of these characters is injective. In the coefficient exact sequence its possible kernel comes from invariants of the divisible group, which vanish because g_* acts by a scalar distinct from one modulo π. This also holds for the corner since $\bar{\Psi}(g_*)$ has order greater than two. The strict local restrictions persist. Extending scalars from O to the finite free O-module R carries the original divisible strict group to the one with coefficients in $\mathrm{Frac}(R)/R$, as is seen by using an O-basis in the global cochains and their local restrictions. Consequently $d_{21} \leq e \max(a_0, a_1)$ and $d_{31} \leq e a_2$. Their sum bounds m from below by the proposition.

For equal characteristic the character values in R/π^d come from the residue field of O. A basis of this finite coefficient space over that field separates its strict classes by coordinate projections. A nonzero coordinate gives a strict class with finite residue coefficients, which

injects into the divisible strict group for the corresponding characteristic-zero character by the same invariants argument. Thus when $a_1 = a_2 = 0$ the corner and the anti-invariant adjacent projection vanish, in either characteristic. Then $d_{31} = 0$ and $d_{21} = m$, since $d_{21} \le m$. The invariant projection has the full annihilator length m. As $m > 0$, at least one residue coefficient projection in equal characteristic is nonzero. This proves all the assertions. □

Corollary 34.25 (A single exceptional integer for the two critical groups)**.** *Let E_* be the largest period field on the original interval. Apply Lemma 11.3 to $E'_* = E_*(i)$ with cyclotomic modulus $4N_*$, and write*

$$B_* = \prod_{n=2}^{2t_*+1} |D_n(E'_*)|, \qquad t_* = \lfloor V^{(3/2)401} \rfloor.$$

Write S_i for the strict character group defining a_i, and $\lambda_i = \operatorname{length}_O S_i$. For the distinct triples $(\ell, r, [\alpha])$ occurring anywhere on the interval above their respective large-prime cutoffs,

$$\sum_{\ell,r,[\alpha]} b_\alpha \, (\lambda_1 + \lambda_2) \log \ell \le 17 \log B_* \ll_{\psi,\varepsilon} V^{1220}, \qquad b_\alpha = [O : \mathbb{Z}_\ell].$$

Consequently the same bound holds with $\lambda_1 + \lambda_2$ replaced by $\max(0, \sup m/e - a_0)$, where the supremum is taken over all the irreducible mixed-characteristic specializations of the triple, with value zero for an empty family. At every prime in this range with $\ell \nmid B_*$, the full-length conclusion of Theorem 34.24 applies.

Proof. All the fields E_x are subfields of E_*. Inflate their characters to this largest field and then to E'_*. This last field realizes the fixed character η, including when $i \in E_*$ already. On a sufficiently distant interval the primes considered exceed $2t_* + 3$, $2g_*$, and every prime dividing $4N_*$, by Theorem 34.21. Character descent therefore identifies the two groups with components on E'_* at weights $s = r, 2r$, respectively. Their finite characters in the convention $\beta^{-1}\kappa_\ell^{-s}$ are $\beta = \eta\alpha, \eta\alpha^2$. Both satisfy $\beta(c) = (-1)^{s+1}$. They lie in the critical eigenspace of Lemma 11.2.

At a fixed ℓ, s, the integral group ring splits into unramified valuation rings. On each factor, the zeroth Fitting valuation is the module length. The Fitting inclusion and the strict dual injection in Lemma 11.2 bound the sum of the strict lengths, weighted by the residue degrees, by $v_\ell(D_{s+1}(E'_*))$, as in the proof of Theorem 11.1. The proof uses only abelianity of the CM field, its cyclotomic modulus and the two fixed good primes, which also apply to E'_*. Each exponent is at most the corresponding length. To count multiplicities one may count each geometric character once, since its residue orbit has size b_α. Unramified extension to a larger coefficient ring preserves the exponent. The map $\alpha \mapsto \eta\alpha$ is injective. The fibers of $\alpha \mapsto \eta\alpha^2$ have at most sixteen elements. Indeed $\mathrm{Gal}(E_*/\mathbb{Q})$ embeds by restriction in the product of the cyclic group of K_ψ, the cyclic real cyclotomic group, and two groups of order two. Its subgroup killed by two has at most sixteen elements. The same holds in the character group by the elementary divisor theorem. For a given s the adjacent term can come only from $r = s$, and the corner only from $r = s/2$. Thus the two lists together use each critical factor at most seventeen times. Summing valuations times $\log \ell$ proves the first bound.

Writing $g'_* = [E'_* : \mathbb{Q}] \le 2g_*$, Lemma 11.3 gives $\log B_* \ll g'_* t_*^2 (g'_* + \log(8N_* t_*))$. Theorem 19.15 bounds $g_* \ll V^{3+12\varepsilon}$ and $\log(2N_*) \ll V^{3\varepsilon/8}$, while $t_*^2 \le V^{1203}$. Together these give the displayed power of V. The assertion about suprema follows by subtracting a_0 in Theorem 34.24. If $\ell \nmid B_*$, all the corresponding nonnegative critical exponents must vanish. □

Corollary 34.26 (Full negative cotangent lengths on the period fields)**.** *At each triple above, let $C = C_{-1} \oplus C_{-2}$ be the two negative cotangent modules at the split augmentation of the actual ordinary ring, as in Theorem 32.19. Then*

$$\operatorname{length}_O C - \lambda_0 = \lambda_1 + \lambda_2, \qquad \operatorname{length}_O(B_* C) = \operatorname{length}_O(B_* S_0).$$

In particular, the weighted sum of the first difference, over the distinct triples on the whole interval, is at most $17 \log B_*$.

At a given x put $P = \exp(x^{\varepsilon}/12)$, $Y = x^{-1240} \exp(x^{\varepsilon}/2)$, $B = M_* B_*$. For every prime $\ell > P$ and complementary triple on E_x set

$$C'_{\ell,r,\alpha} = \begin{cases} BC, & P < \ell \le Y, \\ C, & \ell > Y. \end{cases}$$

Let $\mathcal{L}_{>}(x)$ be the total logarithmic order of the complementary strict groups for these primes and all $1 \le r < t$ on E_x. With one summand per residue character orbit, there is a constant $K_{\psi,\varepsilon}$ uniform on the intervals such that

$$-17 \log B_* \;\le\; \mathcal{L}_{>}(x) - \sum_{\ell,r,[\alpha]} b_\alpha \log \ell \, \operatorname{length}_O C'_{\ell,r,\alpha} \;\le\; K_{\psi,\varepsilon}\, x^{-15} \exp(x^{\varepsilon}/2).$$

Proof. Theorem 32.19 identifies the O-linear dual of C with $S_0 \oplus S_1 \oplus S_2$, with dualizing module $\operatorname{Frac}(O)/O$. A cyclic O-module of length j has dual of the same length and the same elementary divisor. This proves the first equality. At each fixed prime each critical component has length at most the valuation of its own determinant $D_{s+1}(E'_*)$ by the Fitting calculation in Corollary 34.25. Since this determinant divides B_* up to sign, B_* kills S_1 and S_2 separately. Taking images under multiplication by B_* in each cyclic factor gives the second equality, which holds for every multiple of B_* as well.

By descent the contribution to $\mathcal{L}_{>}(x)$ from a single triple is $b_\alpha \log \ell \, \lambda_0$. At $P < \ell \le Y$, the difference of the two length terms is exactly $\operatorname{length}_O S_0[B]$, by the identity for the images and the exact sequence of multiplication by B. At $\ell > Y$ the difference is $-(\lambda_1 + \lambda_2)$. The positive sum is bounded by Theorem 22.9 on E_x with this very B. Corollary 34.25 bounds the absolute value of the negative sum. Both sums have finite support at each x by Proposition 11.4 and the nonzero determinant product for the critical terms. □

The depth inequality has the direction $m/e \le a_0$ away from B_*. Obtaining the upper bound in Section 6.6 still requires control of the entire complementary group, including its classes outside those detected by the trace congruences.

35 References

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