The generalized Riemann hypothesis for Dirichlet L-functions

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Volume III

Trace extensions, Frobenius lifts and local cup products

Abstract

We analyze the integral extensions detected by the marked traces. Local Frobenius lattices, cup products and Taylor conditions give bounds for the scalar and higher-rank terms at spherical and ramified primes.

The guide in Volume I lists all twenty-seven volumes.

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15 Scalar trace extensions at the original level

15.1 Rank-one trace classes at places with negative filtration jumps

The local terms in (4) depend on the integral lattice and on a range of weights of the representations giving the trace relations. At some completions the target alone gives an upper bound. We first compute it at a single place. We then sum the bounds on the same fields and for the same twists as in Theorem 50.6.

Let $K/\mathbb{Q}_\ell$ be finite. Let $\mathcal{O}$ be the integer ring of a finite extension $L/\mathbb{Q}_\ell$ containing the images of all embeddings of K, and fix a uniformizer π. Put $e_L = v_\pi(\ell)$ and normalize $v_\ell(\ell) = 1$. For a G_K-stable free lattice T put $V = T[1/\ell]$, $T' = T^\vee(1)$, and $A' = T'[1/\ell]/T'$. The cyclotomic character has de Rham filtration jump -1. Write $Z_f(K,T)$ for integral cocycles whose rational class lies in $H^1_f(K,V)$. At precision $a \geq 1$, put

$$D_a(T) = Z^1(K, T/\pi^a T)/\operatorname{red}_a Z_f(K,T).$$

We use continuous cocycles throughout.

Proposition 15.1 (The full local lifting obstruction)**.** *Suppose V is de Rham, every filtration jump of V at every embedding of K is negative, and $H^0(K, T'[1/\ell]) = 0$. Then $D_a(T)$ is canonically isomorphic to $H^2(K,T)[\pi^a]$. Its normalized length is that of $H^0(K,A')/\pi^a H^0(K,A')$, divided by e_L, where length on the latter module is taken over $\mathcal{O}$. In rank one let $\beta : G_K \to \mathcal{O}^\times$ be the character of T' and $I(\beta) = (\beta(g) - 1 : g \in G_K)$. Then*

$$\frac{\operatorname{length}_{\mathcal{O}} D_a(T)}{e_L} = \min(a/e_L, v_\ell(I(\beta))). \tag{6}$$

The same right hand side bounds the excess for any submodule of raw trace cocycles, measured modulo its intersection with the integral finite cocycles reduced to this precision.

Proof. The de Rham tangent space $D_{\mathrm{dR}}(V)/\mathrm{Fil}^0$ has L-dimension $[K : \mathbb{Q}_\ell] \dim_L V$, since the filtration has no graded part in nonnegative degree. The Bloch–Kato dimension formula adds $\dim_L H^0(K,V)$ to this tangent dimension to give $\dim_L H^1_f(K,V)$. Local duality and the Euler characteristic formula give

$$\dim_L H^1(K,V) = [K : \mathbb{Q}_\ell] \dim_L V + \dim_L H^0(K,V) + \dim_L H^0(K, V^\vee(1)).$$

The last summand is zero by hypothesis. Thus every rational class is finite, and $Z_f(K,T) = Z^1(K,T)$. [3, Corollary 3.8.4 and the proof of (3.8.5)]

Every coboundary modulo π^a lifts by lifting the vector defining it. The quotient of raw cocycles is therefore the cokernel of reduction on H^1. The exact sequence for multiplication by π^a on T identifies this cokernel with $H^2(K,T)[\pi^a]$. Local Tate duality identifies the Pontryagin dual of $H^2(K,T)$ with $H^0(K,A')$ after choosing a generator of the inverse different of $L/\mathbb{Q}_\ell$. Indeed that generator and the field trace identify the $\mathcal{O}$-linear and $\mathbb{Z}_\ell$-linear dual lattices. Apply finite local duality to the reductions of the lattices and pass to the inverse limit on degree two and the direct limit on degree zero. The finite groups in the inverse systems satisfy the Mittag–Leffler condition. The dual of the kernel of multiplication by π^a is the quotient by π^a in degree zero. A finite module and its Pontryagin dual have the same length. [1, Chapter I, §2]

For completeness $H^0(K,A')$ is finite here. Over L the common kernel of $g - 1$ on $T'[1/\ell]$ is zero. Finitely many $g - 1$ have zero common kernel by finite dimensionality. The map to the direct sum given by these operators has an L-linear left inverse. Clearing its denominators

bounds the denominators modulo T' of every invariant. In rank one the invariants are precisely $I(\beta)^{-1}\mathcal{O}/\mathcal{O}$ in a basis of T'. The ideal is nonzero by the hypothesis on invariants. Its single elementary divisor gives (6). Restricting the quotient map to a submodule proves the last assertion. □

Lemma 15.2 (A test on rational local units)**.** *Suppose T has rank one and that on $\mathcal{O}_K^\times$ local reciprocity expresses its dual character in the form*

$$\beta(\mathrm{rec}_K u) = \eta(u) \prod_{\tau: K \hookrightarrow L} \tau(u)^{\epsilon b_\tau}, \qquad \epsilon \in \{1, -1\}.$$

Here η is trivial on $1 + \mathfrak{m}_K$, all b_τ are nonnegative integers, and $B = \sum_\tau b_\tau > 0$. Put $c_2 = 2$ and $c_\ell = 1$ for odd ℓ. Then

$$v_\ell(I(\beta)) \le c_\ell + v_\ell(B).$$

If η is trivial on $\mathcal{O}_K^\times$ and $\ell - 1 \nmid B$, one has $I(\beta) = \mathcal{O}$. More generally $I(\beta) = \mathcal{O}$ whenever ℓ is odd, $\eta^Q = 1$ on the units, and $\ell - 1 \nmid QB$.

Proof. Embed $\mathbb{Z}_\ell^\times$ in the local units. Every embedding sends $u = 1 + \ell^{c_\ell}$ to itself and $\eta(u) = 1$. The valuation of $u^{\epsilon B} - 1$ is $c_\ell + v_\ell(B)$. To see this, raising a principal unit to a power prime to ℓ preserves its depth by the binomial expansion. Raising it to the power ℓ increases the depth by exactly one for odd ℓ at positive integral depth and at $\ell = 2$ at depth at least two. Inversion preserves the valuation. This proves the upper bound and also proves that $\beta \neq 1$. For the residue test take a Teichmüller generator $\zeta \in \mathbb{Z}_\ell^\times$ of order $\ell - 1$. The Q-th power of its image is $\zeta^{\epsilon QB}$. When the latter has nontrivial reduction, the original image has nontrivial reduction as well. At two the divisibility $1 \mid B$ always holds. The argument uses rational units inside K for every ramification index of $K/\mathbb{Q}_\ell$. □

Theorem 15.3 (A uniform bound on the negative-jump part of the main trace)**.** *Fix q_0, d in the shorter family of Corollary 11.30. On one of its period intervals choose the designated data at each integer node x. Take all the main rank-one lattices of Theorem 50.6, their critical integers $m_x < z \le r_x$, coefficient embeddings ι, and signed powers of the prescribed twists. The estimate also applies with $1 \le z \le t_x$ to rank-one characters with the same weight windows, orientations, and tame unit factors. At each node write b_x for the large weight scale. For each such lattice T write $W^-(T, \ell)$ for the completions $w \mid \ell$ at which the filtration jumps of T are all negative and at least one jump is at most -2. Let $d_w(T)$ be the supremum over precision of the normalized length in (6) over $K = E_{x,w}$. At each of the two level primes $p_{x,i}$, $d_w(T) = 0$ for every $w \in W^-(T, p_{x,i})$. If the interval starts at V_0 sufficiently large, then*

$$\sum_\ell (\log \ell) \sup_{x,\iota,T} \frac{\sum_{w \in W^-(T,\ell)} d_w(T)}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}. \tag{7}$$

The bound applies to the raw trace classes at every auxiliary level and precision. It includes places above 2 and the discriminant. At each rational prime splitting in $\mathbb{Q}(\sqrt{-d})$, W^- contains exactly one member of every conjugate pair of places above that prime for the designated characters.

Proof. We recall explicitly the data entering the estimate. Put $n_x = [E_x : \mathbb{Q}]$, where $E_x = \mathbb{Q}(\mu_{q_{1,x}}, \sqrt{-d})$. The two base weights are b_x and $a_x/x_{1,x}$. For $1 \le j \le m_x$ the unshifted lift weight is $w_{j,\tau} = k_\tau^{(0)} + (2j+1)r_x^8$ with $k_\tau^{(0)} \in \{b_x, a_x/x_{1,x}\}$. Indeed the construction adds half the gap $2r_x^8$ to the base weight and then adds j full gaps. The defining window of a main pair

is $w_{j,\tau}+r_x-t_x \le k_\tau \le w_{j,\tau}$. Equivalently $k_\tau = w_{j,\tau}-h_\tau$, where each h_τ is an integer in $[0,4r_x]$ and is shared by the two conjugate labels of that infinite place. This is the weight condition in `CriticalPair.IsMainPair` in `New/Determinant/Twist/FitTwists.lean` of the shorter archive. The proof below allows the choices at distinct infinite places to be independent. The positivity of the base weights and $r_x^8-4r_x>5r_x$ for $r_x\ge 2$ give $k_\tau>t_x$. The bounds $k_\tau\le 4b_x$ also follow from $k_\tau^{(0)}\le b_x$, $m_x\le 2\sqrt{x}$, $r_x\le 3x^5$, and $b_x\ge x^{500}$ for sufficiently large x. The last inequality is weaker than the scale conditions $x^{10^6}\le x_{1,x}$ and $x_{1,x}^{10^6}\le b_x$.

For the pulled-back character $u(a)=\nu(a/\bar a)^{-1}$ the two exponents at conjugate embeddings have opposite signs and magnitude k_τ. This follows by applying the defining torus character to the quotient of the two coordinates. Its sign is fixed on the chosen CM type, induced from the specified embedding of $\mathbb{Q}(\sqrt{-d})$, and reversed on the conjugate type. The main lattice has character $u_\ell\kappa^{1-z}$, with κ cyclotomic. Hence its two jumps are $z-1-k_\tau$ and $z-1+k_\tau$. On a place in W^- the dual jumps are $b_\tau=k_\tau-z>0$. A prime splitting in the quadratic field separates its two embeddings at every completion, so exactly one member of each pair has only negative jumps. The finite-order auxiliary characters are trivial at the coefficient completions in the chosen twists. All these statements apply to each signed power.

On the local units the avatar of an algebraic Hecke character has its finite character multiplied by the monomial in the labelled embeddings. Indeed its definition transports the finite character through the coefficient embedding and replaces each algebraic infinity factor by the corresponding local embedding in the component above ℓ. Evaluation on the idele supported at w retains precisely those labels inducing w. Dualizing inverts both factors, and the Tate twist shifts each jump by -1. Thus on W^- the unit formula of Lemma 15.2 applies with $b_\tau=k_\tau-z$. The one overall sign depends on the reciprocity convention. The smooth unit factor is trivial off the marked places. At the marks its order divides Q, the order of the prescribed residue character. This follows from the tame parts $\omega_w^{\pm j}$ in Corollary 13.14. In particular the dual character is nontrivial by the rational principal-unit test, as required in Proposition 15.1.

For $\ell=p_{x,i}$ every completion is $\mathbb{Q}_\ell$, since $p_{x,i}\equiv 1 \bmod Q^2$ and the conductor of E_x divides Q. There is just one label, so $0<B\le 4b_x$. We have $Q\le\sqrt{p_{x,i}-1}$ and $p_{x,i}\ge b_x^{10}$ as in Section 130.1. Thus $0<QB<p_{x,i}-1$ for sufficiently large nodes. The residue test gives $I(\beta)=\mathcal{O}$ and proves the asserted vanishing.

At the remaining primes a nonzero loss requires $\ell-1\mid B$. At a fixed node the possible positive B lie in a finite set $\mathcal{B}_x$ formed as follows. Fix j,z, and take the sum of n_1 copies of the first unshifted weight minus z and n_2 copies of the second unshifted weight minus z, with $1\le n_1+n_2\le n_x$. Subtract an integer between zero and $4r_x(n_1+n_2)$, the sum of all the shifts h_τ at this completion. Retain the positive sums. Every completion and every coefficient embedding occurs in this list. Thus

$$|\mathcal{B}_x|\le m_x t_x(n_x+1)^2(1+4r_x n_x).$$

Every member is at most $4n_xb_x$. For any one positive integer B, write $\tau(B)$ for its number of positive divisors. The map $\ell\mapsto\ell-1$ sends the possible primes to distinct divisors of B. Therefore

$$\sum_{\ell-1\mid B}(c_\ell+v_\ell(B))\log\ell\le\tau(B)\log(B+1)+\log 2+\log B.$$

The extra $\log 2$ accounts for $c_2=2$. For any fixed $\delta>0$ this upper bound is $O_\delta((1+B)^\delta)$. Indeed, for $\alpha=\delta/2$, every prime p large enough satisfies $j+1\le 2^j\le p^{\alpha j}$ for $j\ge 1$. For each of the finitely many smaller primes the supremum of $(j+1)p^{-\alpha j}$ is finite. Multiplication gives $\tau(B)\ll_\alpha B^\alpha$, and the remaining logarithm is $O_\alpha((1+B)^\alpha)$.

There are at most n_x completions over a prime. Summing over the explicit list, which contains all the possible B even when the maximizing lattice varies with ℓ, gives

$$\sum_{\ell}(\log \ell)\sup_{\iota,T}\sum_{w\in W^-(T,\ell)} d_w(T) \ll_\delta n_x(n_x+1)^2 m_x t_x(1+4r_x n_x)(1+4n_x b_x)^\delta.$$

Use $\delta = 1/4$ and $n_x \ll_{q_0,d} x^{11}\log(2x) \ll_{q_0,d} x^{12}$, as proved in Corollary 11.30. Since $m_x t_x \ll x^{11/2}$, the extra factor $1+4r_x n_x$ is $O_{q_0,d}(x^{17})$. Division by $\sqrt{b_x}$ gives $O_{q_0,d}(x^{62}b_x^{-1/4})$. This is $O_{q_0,d}(x^{-63})$ by the lower bound on b_x. For each prime the supremum across nodes is at most the sum of their nonnegative bounds before the prime sum is taken. The tail $\sum_{x\ge V_0} x^{-63}$ over integer nodes is $O(V_0^{-62})$, proving (7). The final estimate for trace classes follows by restricting the quotient map of Proposition 15.1 at each place, with the very same coefficient prime and lattice. □

In (4) one may use $d_w(T)$ for the contribution at each $w \in W^-$. Indeed the maps to the full local lifting quotients bound the images of the restricted trace classes at every precision. The other completions still require their integral finite-condition estimates. The projected period in Section 6.6 also involves higher-rank lattices and the comparison of its integral and archimedean realizations.

15.2 Integral cup-product tests at all coefficient places

The obstruction to lifting a torsion cocycle to an integral cocycle and the obstruction to imposing the finite local condition on such a lift enter the trace estimate together. We bound the first at every completion of the shorter period fields, including ones where a rank-one character has jumps of both signs. We also determine how many integral cup products suffice to test the second obstruction and how their joint image measures the loss.

Proposition 15.4 (Integral tests for a finite local condition)**.** *Keep the local notation $K, L, \mathcal{O}, \pi, T, V, T'$ of Section 15.1, with V de Rham. Write $F(T)$ for the inverse image of $H^1_f(K,V)$ in $H^1(K,T)$, and $F_a(T)$ for its image in $H^1(K,T/\pi^a T)$. The finite local Tate pairing makes $F_a(T)$ and $F_a(T')$ exact annihilators. Choose lifts $y_1,\dots,y_u$ of a basis of the free quotient of $F(T')$. For every submodule $C \subset H^1(K,T/\pi^a T)$ form the joint map*

$$P_a : C \longrightarrow (\mathcal{O}/\pi^a)^u, \qquad h \longmapsto \big(\mathrm{inv}_K(h \cup \mathrm{red}_a\, y_i)\big)_i.$$

Here inv_K uses the evaluation pairing and the local invariant in degree two. If b_a is the length of the image of $F(T')_{\mathrm{tors}}$ under reduction at precision a, then

$$0 \le \mathrm{length}_{\mathcal{O}}\big(C/(C \cap F_a(T))\big) - \mathrm{length}_{\mathcal{O}}\, P_a(C) \le b_a. \tag{8}$$

The cokernel of integral reduction $H^1(K,T) \to H^1(K,T/\pi^a T)$ also has length b_a. For simultaneous restrictions to several places one may take all the displayed coordinates in a single map and sum the upper errors. These assertions apply to submodules of raw cocycles by passing to cohomology.

Proof. Write $q = |\mathcal{O}/\pi|$ and $h_i(V) = \dim_L H^i(K,V)$. Continuous cohomology of a local lattice is finitely generated over $\mathcal{O}$, and inverting π gives rational cohomology. These facts follow in degrees zero and one from the inverse limit of finite local cohomology, finiteness modulo π, and compact Nakayama. Rational cocycles have bounded image by compactness of G_K, so clearing a denominator computes the rationalization in degree one. Write $h_f(V) = \dim_L H^1_f(K,V)$. The quotient $H^1(K,T)/F(T)$ is torsion free, since it maps injectively into the L-vector space $H^1(K,V)/H^1_f(K,V)$. Thus the kernel of reduction on $F(T)$ is $\pi^a F(T)$. If $u_a(T)$ is the length of $H^1(K,T)[\pi^a]$, the structure theorem gives

$$|F_a(T)| = q^{a h_f(V) + u_a(T)}, \qquad b_a = u_a(T').$$

The second equality holds since all torsion classes belong to $F(T')$ and the torsion submodule is a direct summand of this finitely generated module.

The product of the orders of $F_a(T)$ and $F_a(T')$ is $|H^1(K,T/\pi^a T)|$. To verify this explicitly, the finite Euler characteristic and local duality give

$$|H^1(K,T/\pi^a T)| = q^{a[K:\mathbb{Q}_\ell]\dim_L V} |H^0(K,T/\pi^a T)|\,|H^0(K,T'/\pi^a T')|.$$

The first invariant group has order $q^{a h_0(V) + u_a(T)}$ by the multiplication sequence in degrees zero and one. Use the same formula for T'. The Bloch–Kato dimension formula for the two de Rham representations gives

$$h_f(V) + h_f(V^\vee(1)) = [K:\mathbb{Q}_\ell]\dim_L V + h_0(V) + h_0(V^\vee(1)).$$

Indeed at each embedding the paired jumps are j and $-1-j$, so the two tangent dimensions add to $[K:\mathbb{Q}_\ell]\dim_L V$. This proves the order identity. Integral lifts in the two finite conditions pair to zero. Their rational classes pair to zero by Bloch–Kato duality, and $H^2(K,\mathcal{O}(1)) \simeq \mathcal{O}$

is torsion free. Finite local duality is perfect, hence the order identity proves the assertion about annihilators. The $\mathcal{O}/\pi^a$-valued pairing may equivalently be tested through its trace pairing with a generator of the inverse different as in Proposition 15.1. [3, Theorem 3.8 and Corollary 3.8.4] [1, Chapter I, §2]

Every element of $\ker P_a$ pairs trivially with the free generators. Its remaining pairings define a homomorphism from the reduction of $F(T')_{\mathrm{tors}}$ to $\mathcal{O}/\pi^a$. The kernel of this test on $\ker P_a$ is $C \cap F_a(T)$ by the annihilator equality. Homomorphisms from a finite module killed by π^a to $\mathcal{O}/\pi^a$ form a module of the same length, as is seen on cyclic summands. This proves (8). The cokernel of integral reduction is $H^2(K,T)[\pi^a]$. Its dual is $H^0(K,A')/\pi^a H^0(K,A')$. The divisible image of $H^0(K,T'[1/\ell])$ in $H^0(K,A')$ has quotient $H^1(K,T')_{\mathrm{tors}}$ by the coefficient sequence. Reducing modulo π^a therefore gives length $u_a(T')$. For a family of places apply the same kernel argument to the direct sum of the torsion tests on the common subgroup C. Finally every finite coboundary lifts to an integral coboundary, which belongs to $Z_f(K,T)$. This justifies passage from raw cocycles. □

Theorem 15.5 (Torsion bounds in both CM orientations)**.** *Take the actual rank-one main lattices and the period intervals of Theorem 15.3. Write $n_x = [E_x : \mathbb{Q}]$. For a lattice character M put*

$$h_w(M) = e_L^{-1} \operatorname{length}_{\mathcal{O}} H^0(E_{x,w}, M[1/\ell]/M)$$

whenever the group is finite. For both $M = T$ and $M = T^\vee(1)$ these lengths are finite at every coefficient place. Both vanish at every place over either level prime. Moreover on any of the indicated intervals starting at V_0 one has

$$\sum_{\ell} (\log \ell) \sup_{x,\iota,T} \frac{\sum_{w|\ell} \bigl(h_w(T) + h_w(T^\vee(1))\bigr)}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}. \tag{9}$$

The twists and their signed powers are included in the supremum. Every completion in each of these sums is taken on the field of the same period node.

Proof. Put $K_0 = \mathbb{Q}(\sqrt{-d})$. Up to a common choice of reciprocity sign, the exponents on the labelled local units of each of the two characters are $c+\epsilon_\tau k_\tau$. Here $c = z-1$ or $-z$, ϵ_τ is constant on each orientation of K_0 with opposite values on the two orientations, and $t_x < k_\tau \le 4b_x$. The two base weights, the allowed labelled shifts, and the tame unit components are computed in the proof of Theorem 15.3. In particular all principal units kill the smooth factor. For any nontrivial lattice character β, taking invariants in $L/\mathcal{O}$ gives $I(\beta)^{-1}\mathcal{O}/\mathcal{O}$. Thus one local unit giving a value different from one bounds h_w by the valuation of that difference.

At a prime splitting in K_0 test on the rational units as in Lemma 15.2, allowing the common exponent to have either sign. For the set of labels at w use the positive integer

$$B = \left| \sum_{\tau|w} (c + \epsilon_\tau k_\tau) \right|.$$

It is positive since the signs agree and $k_\tau > |c|$. At each old level prime the field splits completely. There $0 < B \le 5b_x$ and $QB < p_{x,i} - 1$ for sufficiently large nodes, so raising a residue-unit value to Q proves that the full invariant group vanishes for both lattices. At the other split primes a positive contribution requires $\ell - 1 \mid B$ and is at most $c_\ell + v_\ell(B)$, in the notation of Lemma 15.2.

At a nonsplit prime let k be the quadratic completion of K_0 inside $E_{x,w}$. For a norm-one unit $u \in k$, its conjugate is u^{-1}. The two embeddings of k have equally many extensions to

$E_{x,w}$, by separability. Consequently the factors with exponent c cancel and the character on these units is, up to inversion, u^B with

$$B = \sum_{\tau|w} k_\tau > 0.$$

The tame factor is trivial because both old level primes split. If ℓ is inert, take in $\mathcal{O}_k$ an element α whose residue is outside $\mathbb{F}_\ell$. Then $u = (1+\ell^{c_\ell}\alpha)/(1+\ell^{c_\ell}\bar{\alpha})$ has norm one and $v_\ell(u-1) = c_\ell$. Its B-th power gives the same upper bound $c_\ell + v_\ell(B)$ by the binomial argument of Lemma 15.2. The Teichmüller lift of a generator of the norm-one residue group has order $\ell + 1$. Thus a positive contribution requires $\ell + 1 \mid B$. This argument also applies when $E_{x,w}/k$ is ramified.

The only prime ramifying in K_0 is d. Here d is odd, and $v_d(\sqrt{-d}) = 1/2$. The norm-one unit

$$u = \frac{1 + d\sqrt{-d}}{1 - d\sqrt{-d}}$$

satisfies $v_d(u - 1) = 3/2$. For an element of depth greater than $1/(d - 1)$, the linear binomial term in its d-th power has strictly smaller valuation than all later terms. Powers prime to d preserve depth. We obtain $h_w \le 3/2 + v_d(B)$ at this last prime. The principal-unit tests in all three cases also prove the asserted finiteness.

For each j write w_1, w_2 for the two values of the unshifted lift weight at its node. If n_i labels at the completion use w_i, their actual magnitudes sum to $S = n_1w_1 + n_2w_2 - H$, with the integer H between zero and $4r_x(n_1 + n_2)$. Thus all the positive integers B above occur among

$$S, \qquad \big|(n_1 + n_2)c \ \pm S\big|, \qquad 1 \le n_1 + n_2 \le n_x, \quad n_1, n_2 \ge 0.$$

The common list of actual sums for all w, ℓ, ι, T at this node has size $O(m_x t_x (n_x{+}1)^2(1{+}4r_x n_x))$ and maximum at most $5n_x b_x$. For a fixed B each of $\ell - 1 \mid B$ and $\ell + 1 \mid B$ gives at most $\tau(B)$ primes, all at most $B + 1$. Counting the ramified prime separately for its constant part, the weighted sum of the bounds for that B is at most

$$2\tau(B)\log(B + 1) + \log 2 + \tfrac{3}{2}\log d + \log B.$$

The valuations of B sum to at most $\log B$ even when the union of these prime sets contains d. This bound is $O_d((1 + B)^{1/4})$ by the divisor estimate in the proof of Theorem 15.3. For each prime there are at most n_x places and two characters to count. Thus after division by $\sqrt{b_x}$ the per-node prime sum is $O_{q_0,d}(x^{62}b_x^{-1/4})$, including the independent weight shifts exactly as in that proof. The scale inequality $b_x \ge x^{500}$ bounds this by $O_{q_0,d}(x^{-63})$. Summing over integer nodes $x \ge V_0$ proves (9) with the supremum inside the prime sum. □

Corollary 15.6 (The remaining integral tests in a torus lift)**.** *At every precision the local finite conditions on the rank-one main trace in Theorem 45.11 can be tested by $n_x/2$ free cup-product coordinates across all $w \mid \ell$, with an additional normalized length at most $\sum_{w|\ell} h_w(T^\vee(1))$ as in (8). This additional length vanishes at both level primes. For each scalar trace with lattice $\mathcal{O}(-e)$, $0 \le e < t_x$, there are n_x free coordinates. Summed over these scalar weights and places, the weighted prime sum of the upper errors at a fixed node is*

$$O\big((n_x t_x + 1)^2 \log(2 + n_x t_x)\big).$$

After division by $\sqrt{b_x}$, the sum of the scalar and main upper errors, with the supremum over a whole interval taken inside the prime sum, is $O_{q_0,d}(V_0^{-10})$. The same assertions hold for any common submodule of the restricted raw trace classes.

Proof. For the torus character the dual has no rational invariants by the local-unit computation. Its tangent dimension is the number of embeddings at which the original character has nonnegative jump. Exactly one of the two jumps $z - 1 \pm k_\tau$ in each conjugate pair is positive. Counting over all coefficient completions therefore gives $n_x/2$. The dimension formula used in Proposition 15.4 makes this the total rank of the free local test modules. In the coefficient sequence the torsion in $H^1(K, T')$ is $H^0(K, A')$, because $H^0(K, T'[1/\ell]) = 0$. The reduced torsion has normalized length at most $h_w(T')$, proving the claims for the main trace.

For the scalar trace the dual is $\mathcal{O}(e + 1)$. Its jumps are negative at all embeddings and it has no rational invariants. The same dimension formula gives n_x free generators over all w. On rational units in $E_{x,w}$ the cyclotomic power acts by the power of the local norm with absolute exponent $B = (e+1)[E_{x,w} : \mathbb{Q}_\ell]$. Thus rational principal and residue units bound the normalized torsion length by $\mathbf{1}_{\ell-1|B}(c_\ell + v_\ell(B))$. Put $Y = n_x t_x$. All these B's lie between 1 and Y. A positive contribution requires $\ell \le Y + 1$, and the largest possible $v_\ell(B)$ is at most $v_\ell(Y!)$. Consequently, before counting the choices of place and scalar weight, the weighted sum of simultaneous upper bounds is at most $\log((Y + 1)!) + \log 2 + \log(Y!)$. There are at most Y place and weight pairs per prime, giving the asserted per-node estimate. Using $n_x \ll_{q_0,d} x^{12}$ and $t_x \ll x^5$ bounds it by $O_{q_0,d}(x^{35})$. Divide by $\sqrt{b_x}$, use $b_x \ge x^{400}$, and sum over the nodes together with (9). The joint-map assertion in Proposition 15.4 applies even when the restricted classes at different places are dependent. □

The surviving quantity in (8) is the length of the image of the trace classes under the free cup products. Its rank bound alone allows length proportional to the precision. The period estimate requires a bound for this image independent of precision, with the growth in x prescribed in Section 6.6.

15.3 Trace extensions at the two original level primes

At a level prime the torus character has a single filtration jump at each completion. Its sign changes at the conjugate completion. We use the ends of the interval of jumps of the auxiliary representations to determine the local classes at every congruence precision. We use the filtration convention in which κ, the cyclotomic character, has jump -1.

Lemma 15.7 (Extensions at an end of a small-height interval)**.** *Let $K = \mathbb{Q}_p$, $p > 2$, and let K'/K be a totally ramified Galois extension of degree q prime to p. Fix nonnegative integers a, B, h with $0 \le h \le B$ and $q(a + B) < p - 1$. Let T be a free line over a finite coefficient ring of integers $\mathcal{O}$, with uniformizer π. Suppose its character on $G_{K'}$ is $\xi\kappa^{-h}$ for an unramified character ξ. For $h = 0$ take $T = \mathcal{O}$ with trivial action. Let $z : G_K \to T/\pi^j T$ be a continuous cocycle, $j \ge 1$. Suppose its extension of $\mathcal{O}/\pi^j$ by $T/\pi^j T$ is, on restriction to $G_{K'}$, a subquotient of a lattice in a crystalline representation with filtration jumps in $[-a, B]$. Here and below this condition concerns the underlying p-adic representation at all embeddings. If $a = 0$ or $h = B$, then the class of z is zero for $h > 0$. For $h = 0$, the cocycle is unramified and lifts to an integral unramified cocycle.*

Proof. First take $j = 1$ and write $k = \mathcal{O}/\pi$. Over K' take the $\mathbb{F}_p$-linear dual of the extension and twist by $\bar{\kappa}^a$, where the bar denotes reduction modulo p. The dual of a finite subquotient of a lattice is a subquotient of its dual lattice: for two nested full lattices take their dual lattices in reverse order using the pairing into $\mathbb{Q}_p/\mathbb{Z}_p$. Subquotients inside smaller finite groups follow by taking preimages. The twist is the reduction of the corresponding integral Tate twist. Thus the resulting module U is torsion crystalline of weights in $[0, H]$, $H = a + B$, in the Breuil–Kisin convention in which κ has weight 1. It is an extension with subobject $k^*(a)$ and quotient $(k(\bar{\xi}))^*(a + h)$, where the star means the $\mathbb{F}_p$-linear dual and k^* has trivial action.

Choose a uniformizer of K' and a compatible sequence of its p-power roots. Write G_∞ for the absolute Galois group of the resulting union of fields. We recall the integral comparison facts we use. The restriction to G_∞ of a torsion crystalline module killed by p of weights in $[0, H]$ has a contravariant Kisin model of height at most H. After inverting the parameter u, Fontaine's equivalence with étale Frobenius modules is exact. Here the residue field of K' is $\mathbb{F}_p$, so the model is a free $\mathbb{F}_p[[u]]$-lattice with semilinear Frobenius $\varphi(u) = u^p$. Its linearized Frobenius matrix F is integral and $u^{qH}F^{-1}$ is integral, since the Eisenstein polynomial reduces to u^q. In this convention an unramified module has a constant invertible Frobenius matrix. Tensoring by $\bar{\kappa}^s$, $s \ge 0$, multiplies the matrix by a scalar unit times u^{qs}. This is the reduction of the rank-one Tate model with multiplier $E(u)/c_0$, where $c_0 = E(0)/p$. The formula for an unramified module follows by extending its constant Frobenius module from $\mathbb{F}_p$ to $\mathbb{F}_p[[u]]$. [4, §§2.1, 2.4, 3.3][5, §2]

In this range the sequence of the three models is exact already over $\mathbb{F}_p[[u]]$. We give the lattice argument. A matrix A for a map after inverting u satisfies $AF_1 = F_2\varphi(A)$. If its largest pole has order $s > 0$, then $\varphi(A) = F_2^{-1}AF_1$ has largest pole at most $s + qH$, but also exactly ps. This contradicts $qH < p - 1$. In particular two height-H lattices in the same generic module agree, by applying the assertion to the identity in both directions. The image of a map of models has height at most H, by applying the map to the linearized Frobenius inclusions for its domain. Its kernel also has height at most H: the inverse linearized Frobenius of a kernel vector multiplied by u^{qH} is integral and belongs to the Frobenius pullback of the kernel. Indeed its image in the pullback of the target vanishes by injectivity of linearized Frobenius and flatness of $u \mapsto u^p$. Images and kernels are finite free over the power series discrete valuation ring. Exactness of the generic sequence and uniqueness of the lattices now give the asserted integral exactness.

Choose bases of the end models and lift the basis of the quotient of the middle model. Contravariance puts the positive additional weight h in the submodel. In these bases Frobenius has matrix

$$\begin{pmatrix} u^{q(a+h)}D & X(u) \\ 0 & u^{qa}C \end{pmatrix}, \qquad C, D \in \mathrm{GL}_{[k:\mathbb{F}_p]}(\mathbb{F}_p). \tag{11}$$

All scalar unit factors of the Tate model are included in C, D. The upper right block of the inverse matrix is $-u^{-q(2a+h)}D^{-1}X(u)C^{-1}$. Its product with u^{qH} is integral. If $h = B$, this says that u^{qa} divides every entry of X. If $a = 0$, this divisibility just says X is integral. Consequently the matrix after removing the twist $\bar{\kappa}^a$ still has integral entries. Its two diagonal blocks are $u^{qh}D_1$ and C_1, with D_1, C_1 constant and invertible. Denote its upper right block by X_1.

For $h > 0$, change the lifted quotient basis by a matrix Y in the submodel. The equation for zero upper right block is

$$Y = X_1 C_1^{-1} + u^{qh} D_1 \varphi(Y) C_1^{-1}.$$

The operator in the second term sends matrices divisible by u^m to matrices divisible by u^{qh+pm}. Iterating from zero therefore converges in $\mathbb{F}_p[[u]]$ to a solution. Retwisting gives a splitting over G_∞ of the sequence defining U. Ozeki's full faithfulness theorem for the restriction of torsion crystalline representations gives the same splitting over $G_{K'}$: the source and target of the section are torsion crystalline in $[0, H]$ and $qH < p - 1$. Dualizing back gives a splitting of the original sequence over $\mathbb{F}_p$. The invariant lift of $1 \in k$ then yields a k-linear section by scalar multiplication. Restriction followed by corestriction multiplies $H^1(K, T/\pi T)$ by q, which is invertible. Thus the original class is zero over K. [5, Theorem 1]

For $h = 0$, T is trivial. After removing the twist we may use identity matrices at both ends. Changing basis replaces the off-diagonal block by $X_1 + \varphi(Y) - Y$. Take $Y = \sum_{i \geq 0} \varphi^i(X_1 - X_1(0))$; this converges and leaves the constant block $X_1(0)$. A constant invertible Frobenius matrix gives an unramified representation of G_∞ obtained by restriction from $G_{K'}$, since the uniformizer tower is totally ramified. Its finite unramified representation lifts to a weight-zero crystalline lattice by lifting its Frobenius matrix to $\mathrm{GL}_{2[k:\mathbb{F}_p]}(\mathbb{Z}_p)$. Retwist by $\bar{\kappa}^a$. The resulting lift has weight $a \in [0, H]$, so Ozeki's theorem lifts the isomorphism over G_∞ to $G_{K'}$. Undoing the twist and the dual, z kills inertia over K'. Its values on inertia over K factor through a group of order q, which has no nonzero homomorphism to the additive group of k. An integral lift is obtained by lifting its Frobenius value and extending continuously from $\widehat{\mathbb{Z}}$ through $\mathbb{Z}_p$.

For $j > 1$, reduce modulo π. For $h > 0$, subtract a coboundary given by any lift of a splitting element. For $h = 0$, subtract the reduction of the integral unramified cocycle just constructed. In both cases the resulting extension still has the required crystalline presentation: changing a section gives an isomorphic extension, and taking a difference of extensions uses the diagonal pullback and the subtraction pushout of their direct sum. The subquotient condition is stable under these operations. The new cocycle takes values in $\pi(T/\pi^j T)$. In its extension restrict the first summand to this submodule, then divide out the fixed submodule $0 \oplus \pi^{j-1}(\mathcal{O}/\pi^j)$. The quotient is the extension for the cocycle divided by π at precision $j - 1$. Induction gives a coboundary, or an integral unramified lift, respectively. Multiplying back by π and restoring the subtracted term proves both assertions. □

Theorem 15.8 (Local trace classes at both original levels)**.** *Fix one of the shorter intervals and its nonzero designated torus lifts as in Theorem 50.11. Use the separating systems in the image of the rank-one marked idempotent, at any of the permitted auxiliary levels. At each $p = p_{x,0}, p_{x,1}$ all the main trace classes belong to the integral finite local condition at every $w \mid p$ and every precision. Put $t' = 2z - 1$. At a completion where the main jump is positive the same*

assertion holds for all the scalar weights $0 \le h < t'$. At its conjugate it holds for $h = t' - 1$ as well.

Proof. We verify the common field, tame type and jumps. Both primes split completely in $E = E_x$, so $E_w = \mathbb{Q}_p$. Put $Q = (q_0 d)^{h_x+6}$, where h_x is the level exponent of the node. We have $Q^2 \mid p - 1$ and $p \ge b_x^{10}$. Use the degree Q subextension K' of $\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p$. It is totally ramified and cyclic. Its inertia subgroup kills every character of $\mathbb{F}_p^\times$ of order dividing Q, by local reciprocity and the cyclotomic action on μ_p.

At a chosen old place the target level fixes a specified residue vector. Its line stabilizer is generated by this vector stabilizer and the diagonal matrices scaling that vector. In the marked projector the normalized action of the latter group is trivial: the source label $X-1$ is the trivial character of $\mathrm{GL}_1(\mathbb{F}_p)$. In the translation action this leaves the inverse of the prescribed residue character on that line, and trivial action on the complementary block. Its tame character is a nontrivial power $\omega_w^{\pm j_0}$ of order dividing Q, as in Theorem 50.6. The auxiliary twist splits above p and the later trace places are away from it. The characteristic-zero classification of the nontrivial line-character space used in Theorem 45.14 gives one tame line and an unramified spherical complement, with zero monodromy. The local Weil parameter thus has inertia of exponent dividing Q. The opposite place has the dual type with the unramified norm twist. The normalizing splitting character has unramified smooth part here, so it preserves this exponent and zero monodromy.

We need the compatibility at the coefficient characteristic itself. In the base change of Theorem 45.16, the normalized cuspidal factors are regular algebraic essentially conjugate self-dual, as verified in Corollary 12.16. The parameter of the isobaric sum is the direct sum of their parameters with their indicated norm twists. Every summand therefore has zero monodromy. Theorem 1.1 of Barnet-Lamb, Gee, Geraghty and Taylor identifies the semisimple Weil parameter at $w \mid p$ and bounds the monodromy Jordan blocks by those of the automorphic parameter. The upper bound has only blocks of length one, so the Galois monodromy vanishes. The restriction of the finite Weil inertia to $I_{K'}$ is trivial since its characters have order dividing Q. A de Rham representation with these properties is crystalline over K': in a larger semistable period module the vanishing of monodromy gives crystalline periods, trivial inertia descends them through the ramified part, and unramified descent preserves their full dimension by semilinear Galois descent. This reasoning applies to all the trace matrices in the projected space. [6, Theorem 1.1]

All the systems at one such level have the same algebraic coefficient representation at infinity and the same normalizing character. Their labelled Hodge multisets are identical by the weight formula in Corollary 12.16. To evaluate this common multiset use the nonzero base lift itself: its normalized representation is the sum of T and the lines $\mathcal{O}(-h)$, $0 \le h < t'$, by the parameter calculation in Theorem 50.6. Hence the jumps at w are

$$0, 1, \dots, t' - 1, \qquad J_w = z - 1 \pm k_\tau.$$

The choice of coefficient conjugate fixing $\mathbb{Q}_p$ does not change this multiset at a completely split place, since the image of the local embedding of E is in $\mathbb{Q}_p$. Restriction to K' repeats the same jumps. Set $a = \max(0, -J_w)$, $B = \max(t' - 1, J_w)$. From $t_0 < k_\tau \le 4b_x$ and $t_0 \le b_x$ we have $a + B \le 6b_x$. Therefore $Q(a + B) < p - 1$ for the indicated intervals with sufficiently large initial node.

By Lemma 12.14 each local trace extension is a subquotient of an integral lattice with rational constituents among these systems, the target and the trivial line. All are crystalline over this single K' in $[-a, B]$. If $J_w > 0$ then $a = 0$. On $G_{K'}$ the main line is an unramified character times κ^{-J_w}, since on $\mathbb{Q}_p^\times$ its unit character is the product of its smooth factor and the algebraic

character of jump J_w. Lemma 15.7 applies to this line and to every scalar line. For $h = 0$ the unramified lifts are finite classes because unramified extensions of the trivial representation are crystalline. When $J_w < 0$, the main finite-condition quotient vanishes by Theorem 15.3; its dual has no residual invariants at this level prime. Now $B = t' - 1$, so the upper endpoint in Lemma 15.7 gives the assertion for the last scalar line at the same completion. □

Thus at either old coefficient prime the main free cup products in (10) vanish for the marked separating families. The only possible scalar coordinates there occur on the negative side at $0 \le h < t' - 1$. There are at most $[E_x : \mathbb{Q}](t' - 1)/2$ of them, because the prime splits completely and one completion in each conjugate pair has negative main jump. The joint images are still taken at the original precision on the original field. The argument gives no bound for their lengths from the number of coordinates alone.

15.4 The marked character as a subrepresentation or quotient

At an original level prime p, let $K = E_{x,w} = \mathbb{Q}_p$. The normalized representations used for the torus pairing are crystalline over the totally ramified extension F/K of degree Q in Theorem 15.8. Their smooth inertial type, namely the inertia action on potentially crystalline periods, is the sum of a trivial representation and one copy of a character ξ. This character has order greater than two. Indeed its order is $Q/\gcd(Q, j)$ for $1 \le j \le m_x$, as in Theorem 50.6. We prove a full-precision local condition when the character of this type occurs on a rational line at either end of the representation. The remaining representation can have noncommutative image on every open subgroup.

Lemma 15.9 (Removing outer constituents from a finite subquotient)**.** *Let V be a characteristic-zero representation of a group on a finite-dimensional space over a local coefficient field L, with a stable lattice and a stable filtration $0 \subset V_1 \subset V_2 \subset V$. Fix a collection S of simple residual representations over its residue field k. Suppose no reduction of a stable lattice in V_1 or V/V_2 has a composition factor in S. A finite subquotient U of a lattice in V, killed by π^a and with all composition factors in S, is a subquotient of the reduction modulo π^a of a stable lattice in V_2/V_1.*

Proof. Write $U = N/N'$ with N, N' stable submodules of a lattice. By adding a common sufficiently deep multiple of that lattice to N and N' one may take them to be full lattices because $N \cap \pi^b\Lambda \subset \pi^a N \subset N'$ for large b, by elementary divisors. The new quotient is the original one. There is now a surjection $N/\pi^a N \to U$. Intersect N with V_1, V_2. The resulting filtration of N has free successive quotients, so reduces exactly modulo π^a. The outer factors have no simple in S, by filtering their reductions by powers of π. The bottom factor therefore has zero image in U. The image of the middle step is all of U, since its cokernel would be a quotient of the top factor. Thus the map from the middle step induces a surjection from the middle graded lattice modulo π^a to U. □

Proposition 15.10 (Full scalar local conditions with a marked line)**.** *Fix K as above. Suppose that each member V_y of a finite local trace family of the specified type and Hodge multiset has a G_K-stable line or a one-dimensional quotient of smooth inertial type ξ, after extending coefficients if necessary. For every $0 \le h < t'$ and every precision, all the local raw trace cocycles with target $\mathcal{O}(-h)/\pi^a$ satisfy the integral finite local condition. For $h > 0$ their classes vanish and for $h = 0$ they are unramified.*

Proof. A line of this type has inertia character $\xi\kappa^{-d}$, where d is one of the jumps of V_y, with the common identification of smooth characters by reciprocity. This is the local formula for a potentially crystalline character used in Lemma 50.12. Its reduction differs from the trivial character and from $\bar\kappa^{-h}$. To see this, evaluate at a prime-to-p lift s of a tame generator. An equality with $\bar\kappa^{-e}$, $e = 0$ or h, would give $p - 1 \mid Q(d - e)$ after raising to Q. The common jump interval $[A, B] = [\min(0, J_w), \max(t' - 1, J_w)]$ has $Q(B - A) < p - 1$. Thus $d = e$, which would make the nontrivial prime-to-p value $\xi(s)$ trivial modulo π.

After taking the quotient by a marked subline, or the kernel of a marked quotient, the remaining representation is crystalline over K. Indeed it is crystalline over F, and exactness of crystalline periods there removes the only nontrivial smooth inertia character. The extension F/K is totally ramified, so its maximal unramified subfield equals that of K. The full period space over F is now fixed by $\mathrm{Gal}(F/K)$, and supplies all required periods over K. The jumps of the remaining representation lie in $[A, B]$ by filtered exactness.

Use the graph construction of Lemma 12.14 for a raw cocycle c of weight h. The extension of $\mathcal{O}/\pi^a$ by $\mathcal{O}(-h)/\pi^a$ is a finite subquotient of a lattice whose rational representation is a

direct sum of copies of the members, the target and the trivial line. This construction works with any π^a by using that ideal in its graph and pushout. Give this direct sum a filtration of length three. Put the distinguished sublines at the bottom in members with sublines, and put the distinguished quotients at the top in members with quotients. In each case put the remaining representation in the middle, together with the target and the trivial line. Taking the direct sum of these filtrations gives one crystalline middle representation over K, with the same bound on its jumps. Both outer terms are direct sums of lines with the residual exclusion just proved. All stable lattices in those direct sums have the same residual composition factors. For example one may filter each lattice by its intersections with a flag of these lines and reduce the saturated filtration. The residual composition factors of the extension for c all belong to $\{k, k(-h)\}$, by its defining exact sequence. Apply Lemma 15.9 to see that this extension is torsion Fontaine–Laffaille over the absolutely unramified field K. Here $B - A \leq p - 2$, and torsion objects of the common Fontaine–Laffaille interval are closed under subquotients.

For precision, the lifting argument in Theorem 12.15 applies once this torsion presentation and the two ends are given. Twist the filtration interval to $[0, B - A]$, with the quotient a rank-one object of pure degree $-A$. In the filtered module choose a lift of its generator in that filtration degree, by strictness. Its divided Frobenius differs from itself by a vector in the reduced kernel. Lifting that vector to the integral kernel with the same split filtration gives an integral Fontaine–Laffaille extension reducing to the original one with both ends identified. Undoing the twist puts the class of c in the integral finite condition. This argument uses the same equations for π^a since Frobenius fixes the coefficient ring. At $K = \mathbb{Q}_p$ this condition is zero for $h > 0$ and the full unramified subgroup for $h = 0$, as in Theorem 50.16. All assertions descend through a coefficient extension by an $\mathcal{O}$-linear projection from the enlarged ring onto $\mathcal{O}$ sending one to one. □

Lemma 15.11 (A single nontrivial tame eigenvalue)**.** *Let V have the local Hodge multiset and the smooth type above. If V is potentially abelian then its marked line is a direct summand over G_K, after extending coefficients. In general, an absolutely irreducible local constituent of dimension greater than one whose smooth inertia contains ξ stays absolutely irreducible on every open subgroup of G_K.*

Proof. We first record the restriction on permutations of summands. Suppose $V = \bigoplus_i V_i$ and G_K permutes these subspaces. Choose a finite normal extension L/K containing F for which every summand is G_L-stable. All are crystalline over L. Their period spaces form a direct sum on which inertia permutes the summands in the same way, by functoriality of periods. In passing to a coefficient vector space one can use any embedding of L_0; inertia fixes L_0. On a cycle of length j for an inertia element the eigenvalues on these blocks include all j-th roots of any eigenvalue of its j-th power on the first block. This follows, for instance, by following an eigenvector round the cycle. Wild inertia acts as the identity on these periods. On a generator of tame inertia the only eigenvalues available are 1 and $\xi(s)$, and $\xi(s) \neq -1$. Thus no cycle of length $j \geq 2$ is possible. All of inertia stabilizes the blocks. In each Frobenius orbit of blocks the dimensions of their nontrivial smooth inertia parts are equal. Conjugation and semilinear transport preserve the dimension of inertia invariants. Moreover the multiplicity can be computed in any coefficient embedding of L_0, since the bijective crystalline Frobenius identifies the factors and commutes with inertia. As the total nontrivial dimension is one, the orbit which carries it consists of a single block.

If V is potentially abelian, choose L normal such that the matrices of G_L commute. Their joint generalized character spaces are permuted by G_K. Each space has all composition factors equal to its one character. At every label of L the jumps in V are pairwise distinct, by the scalar string and the separated main jump of Theorem 15.8. Filtered exactness therefore forces

each generalized character space to have dimension one. The preceding orbit argument singles out one line fixed by all of G_K. The sum of the other orbits is a stable complement.

Now take an absolutely irreducible constituent W containing ξ and restrict to an open normal subgroup H. Work over an algebraic closure of the coefficient field. The algebra spanned by H in $\operatorname{End} W$ is finite-dimensional and normalized by G_K. Its radical is nilpotent and its image on W is G_K-stable. Irreducibility and nilpotence force that image to be zero. Thus $W|_H$ is semisimple, and G_K permutes its isotypic spaces transitively. The finitely many summands and a decomposition into simples can be defined over a finite coefficient extension by choosing bases. Regularity of the jumps on restriction to H forces multiplicity one of each simple, since every repetition would repeat a jump after a further extension containing F. The permutation argument applies to this transitive collection in W, whose nontrivial smooth inertia part still has dimension one. There is therefore one block and $W|_H$ is absolutely irreducible. Every open subgroup contains an open normal subgroup, which proves the claim. □

Theorem 15.12 (The old-prime trace contribution with a marked line)**.** *At either original level prime of Theorem 50.11, the quantity $B_{\mathrm{old}}(D)$ of Theorem 50.16 is zero whenever every member at each negative completion has its marked character on a subline or on a quotient line. This includes mixtures of such members with potentially abelian members. It also suffices for a marked character to occur as a rank-one composition factor with the smallest or the largest labelled jump of the member. On this subfamily the full old-prime sum of $p(D)/\sqrt{b_x}$, with the same prime weights and suprema as in Theorem 50.16, is $O_{q_0,d}(V_0^{-10})$.*

For an arbitrary family at an old prime, all surviving joint cup coordinates are determined by local trace evaluations on integral tuples whose components vanish on every member meeting these line conditions at the place of evaluation.

Proof. If a marked line χ is a composition factor, write $V_- \subset V_1 \subset V_y$ with $V_1/V_- = \chi$, and write $V_+ = V_y/V_1$. If the jump of χ is smallest, all jumps of $V_-\chi^{-1}$ are strictly positive. Over F this crystalline representation has zero de Rham tangent space and zero invariants, hence zero H^1_f. The extension of χ by V_- is crystalline there, so splits there. Restriction on rational H^1 is injective by corestriction and $[F : K] \neq 0$. It splits over K, providing a subline of V_y. If the jump is largest, apply the same argument to $\operatorname{Hom}(V_+, \chi)$ and the extension $0 \to \chi \to V_y/V_- \to V_+ \to 0$. A splitting provides a quotient line of V_y. A zero-dimensional outer term in either argument already gives the required line.

Apply Proposition 15.10 at each negative completion. It applies simultaneously to all the indicated members, and Lemma 15.11 includes the potentially abelian ones. All remaining coordinates in Theorem 50.16 are zero. Restricting its prime sum proves the estimate for this subfamily.

Finally at one completion let $\mathcal{N}_w$ be the span of the local matrix differences for the full family, and project it to the matrix components of the members with the stated line property. The image is precisely their local difference lattice, since the same group elements generate both. Two local trace functionals of the same weight which agree on the kernel of this projection have difference factoring integrally through its image. Their cocycle difference obeys all the relations of that projected family, so Proposition 15.10 puts its class in the finite condition. For an empty projected family the assertion follows directly from the definition. Thus each surviving cup coordinate is determined on the kernel, including when the original global classes have dependent restrictions. This proves the last assertion. □

A local representation may have its marked smooth type in an irreducible constituent of rank greater than one. The existence of a one-dimensional eigenspace on its period module alone

does not supply a rational Galois subline. The integral estimates for such constituents, and for interior marked factors outside the line conditions above, remain necessary for the unrestricted prime sum.

15.5 The weight-zero trace at the original level primes

We bound the weight-zero scalar trace for the entire marked family at either original level prime. The bound is independent of the distance to the negative main jump and of the congruence precision. The proof compares two descriptions of the ramified additive character on the uniformizer tower.

Let $p \geq 5$, $K = \mathbb{Q}_p$, and let F/K be a totally ramified cyclic tame extension of degree $q \mid p-1$. Take a uniformizer of F with Eisenstein polynomial $E(u) = u^q - cp$, where $c \in \mathbb{Z}_p^\times$ is a Teichmüller unit. Here is one construction. Start with a uniformizer, and project it to its tame character on the first graded piece of the maximal ideal by averaging against the inverse Teichmüller lift of that character. The projection is unchanged modulo the square of the maximal ideal. Its q-th power is fixed by Galois, with K-valuation one. The principal-unit part of its quotient by p has a q-th root in K, which can be divided out of the uniformizer. Write F_∞ for the field obtained by adjoining compatible p-power roots of this uniformizer.

For $n \geq 1$ put $S_n = (\mathbb{Z}/p^n)[[u]]$, $D_n = S_n[1/u]$ and $c_0 = E(0)/p = -c$. Frobenius φ fixes coefficients and sends u to u^p. We use the integral comparison for G_{F_∞} in the convention of Lemma 15.7. Fontaine's equivalence identifies finite representations with torsion étale Frobenius modules over the p-adic completion of $\mathbb{Z}_p[[u]][1/u]$, whose reduction is D_n. In the contravariant convention the Tate character κ has the rank-one model with multiplier $E(u)/c_0$. A finite subquotient of a lattice in a crystalline representation of weights in $[0, H]$, in the convention giving κ weight one, has a finite-height model for its restriction to G_{F_∞}. Such a model M is a finitely generated S_n-submodule spanning the étale module after inverting u, and the image $\Phi(\varphi^* M)$ under the linearized Frobenius satisfies

$$E^H M \;\subset\; \Phi(\varphi^* M) \;\subset\; M.$$

These are the torsion Kisin comparison and rank-one Tate model used here. [7, §§2–3, especially Example 2.3.5]

Lemma 15.13 (Removing the common Tate twist at finite precision)**.** *Let U be an extension of the trivial $\mathbb{Z}/p^n$ by itself over G_F. Suppose it is a finite subquotient of a crystalline lattice in the filtration interval $[-a, B]$, in which κ has filtration jump -1. Put $H = a + B$, $D = \min(a, B)$ and assume $q(H + n - 1) < p - 1$. The étale Frobenius module on F_∞ of its dual in the contravariant convention has an extension matrix with diagonal entries one and upper right entry in $E^{-D} S_n$. The entries refer to the fixed trivializations of the two ends.*

Proof. Two height-H models for the same étale module coincide under this inequality. Indeed, if M, N are such models, let s be the smallest nonnegative integer with $u^s M \subset N$. If $s > 0$, the smallest exponent for the corresponding inclusion after Frobenius pullback is ps. To see that $ps - 1$ fails, choose $m \in M$ with $u^{s-1} m \notin N$. The coefficient of u^{p-1} in $u^{ps-1}(1 \otimes m)$, using the basis $1, u, \ldots, u^{p-1}$ over $\varphi(S_n)$, detects $u^{s-1} m$. Since $u^{q(H+n-1)} = (E + cp)^{H+n-1}$ is divisible by E^H modulo p^n, the height inclusions give

$$u^{s+q(H+n-1)} \Phi(\varphi^* M) \subset u^{q(H+n-1)} N \subset \Phi(\varphi^* N).$$

Thus $(p-1)s \leq q(H+n-1) < p-1$, a contradiction. Reversing the inclusion proves uniqueness.

Dualize U and twist by κ^a. This is a torsion crystalline subquotient of weights in $[0, H]$ with both end characters κ^a mod p^n, just as in the proof of Lemma 15.7. Take a model M of height H. Intersect it with the first end module after inverting u and take its image in the other end module. Both models have height H. For the image this follows by projecting the two inclusions. For the intersection, the preimage of $E^H m$ under linearized Frobenius in $\varphi^* M$

maps to zero in the generic quotient. Flatness of φ, and injectivity of linearized Frobenius on the generic quotient, put this preimage in the pullback of the intersection. The intersection and image span their generic modules because localization is exact. By uniqueness they are the standard free rank-one models with multiplier $t = (E/c_0)^a$. Lifting the basis of the quotient now gives a basis of M, in which

$$\Phi = \begin{pmatrix} t & X \\ 0 & t \end{pmatrix}, \qquad X \in S_n, \qquad E^H t^{-2} X \in S_n.$$

The last inclusion uses the upper right entry of $E^H \Phi^{-1}$. Undoing the Tate twist replaces X by X/t, which lies in $E^{-a} S_n \cap E^{-B} S_n = E^{-D} S_n$. All divisions take place in D_n, where E is invertible. □

Lemma 15.14 (The cyclotomic logarithm in the Laurent module)**.** *Set $\mathcal{L}(\sigma) = p^{-1} \log(\kappa(\sigma)^{p-1})$. For its additive extension modulo p^n, a Frobenius entry on F_∞, in the convention of Lemma 15.13, is, up to multiplication by a constant unit and a change $(\varphi - 1)Y$, the reduction of*

$$C(u) = \frac{p-1}{p} \log(1 - cp\, u^{-q}) = -(p-1) \sum_{i \geq 1} \frac{c^i p^{i-1}}{i}\, u^{-qi}.$$

Proof. For the calculation compose the contravariant equivalence with duality. This gives a covariant exact equivalence sending κ^{-1} to the line with multiplier E/c_0. Take $j \geq n$, $m = (p-1)p^j$, and the character $\theta = \kappa^{-m}$. It is trivial modulo p^{j+1}. The connecting class of the invariant generator modulo p^{j+1} in this line has cocycle $(\theta(\sigma) - 1)/p^{j+1}$. Modulo p^n it represents $-\mathcal{L}$, by applying the exponential to $-p^j \log(\kappa^{p-1})$ and dividing by p^{j+1}. Terms after the linear one are zero modulo p^n.

The Frobenius multiplier of the line is $(E/c_0)^m$. In its étale module change basis by u^{-qp^j}. The multiplier becomes $f = (1 - cpu^{-q})^m$, since the constant unit c_0 has order dividing $p-1$. It is one modulo p^{j+1}. Lifting the fixed generator in this basis and taking its Frobenius difference gives $(f-1)/p^{j+1}$. The connecting extension in either category is obtained by pulling back the reduction map of the line along the map from the trivial object carrying its generator to the invariant generator, identifying the kernel by division by p^{j+1}. The two choices of generator differ by a constant unit modulo p^{j+1}. Indeed Frobenius-fixed Laurent series over $\mathbb{F}_p$ are constants by comparing coefficients under $u \mapsto u^p$, and reduction and division by p prove the same fact over $\mathbb{Z}/p^{j+1}$. A change of lift adds a Frobenius coboundary. Finally the logarithm and exponential on the p-adically complete Laurent ring give $(f-1)/p^{j+1} \equiv C(u) \bmod p^n$, by the same estimate on the exponential terms. This proves the formula. □

Theorem 15.15 (A precision-independent bound for weight zero)**.** *Let an additive cocycle $z : G_K \to \mathbb{Z}/p^\nu$ have an extension which over F is a torsion crystalline subquotient in $[-a, B]$, with $a, B \geq 0$. Put $D = \min(a, B)$. If $D+1 \leq p-2$, $q(a+B+D) < p-1$, and $q \max(D+1, 2D) < p$, the order of $z(I_K)$ is at most p^D, independently of ν.*

Proof. Local reciprocity identifies the pro-p units of K with $1 + p\mathbb{Z}_p$, on which $\mathcal{L}$ is a generator of the additive character module. Subtract an unramified integral character modulo p^ν and multiply by a unit so that the ramified character is $p^{\nu-k}\mathcal{L} \bmod p^\nu$, where its image has order p^k. These operations preserve the torsion subquotient condition by the direct sum, diagonal pullback and addition pushout of extensions. The unramified extension is crystalline of weight zero. If $k > D$, the extension for $\mathcal{L} \bmod p^{D+1}$ is such a subquotient over F. For detail, first restrict the kernel of the extension with the displayed cocycle to $p^{\nu-k}(\mathbb{Z}/p^\nu)$, retaining the full

quotient coordinate. Then divide by the fixed subgroup $0 \oplus p^k(\mathbb{Z}/p^\nu)$. Identifying the kernel with $\mathbb{Z}/p^k$ gives the extension for $\mathcal{L} \bmod p^k$, which can be reduced modulo p^{D+1}.

Set $n = D+1$. Lemmas 15.13 and 15.14 give an element $X_0 \in E^{-D}S_n$ whose difference from a constant unit times $C(u)$ is a Frobenius coboundary in D_n. Both representatives have pole order strictly below p. For C only the terms $i \le n$ survive, since $i-1-v_p(i) \ge n$ for $i > n$ and $n \le p-2$. For $n < i < p$ this is immediate, and for $i \ge p$ use $i-1-\log_p i \ge p-2$. For X_0 the pole order is at most $q(D+n-1) = 2qD$ if $D > 0$, by expanding E^{-D} as a binomial series in $cp\,u^{-q}$ modulo p^n. When $D = 0$ it is integral. If $Y \in D_n$ has positive pole order s, $\varphi(Y) - Y$ has pole order exactly ps, since its leading coefficient remains nonzero. A coboundary of pole order strictly below p must consequently be integral. We obtain $E^D C(u) \in S_n$.

Its coefficient at u^{-q}, before reduction, is

$$-(p-1)c^{D+1}p^D \sum_{j=0}^{D} \frac{(-1)^{D-j}}{j+1}\binom{D}{j} = -(p-1)c^{D+1}p^D \frac{(-1)^D}{D+1}.$$

The identity follows by integrating $(x-1)^D$ from zero to one. The coefficient has valuation exactly D, hence is nonzero modulo p^{D+1}. This contradicts integrality and proves $k \le D$. □

Corollary 15.16 (Removal of the trivial scalar from the old-prime sum)**.** *In Theorem 50.16 one may replace $B_{\mathrm{old}}(D)$ by*

$$B_{\mathrm{old}}^{>0}(D) = \frac{1}{e_L} \sum_{1 \le h < t'-1} \operatorname{length}_{\mathcal{O}} \Gamma_h(D)$$

in the prime-sum inequality, for all the marked families, including those with strongly irreducible local constituents.

Proof. At a negative completion use $a = -J_w$ and $B = t' - 1$. Theorem 15.8 and the graph presentation give the required subquotient condition. Here $D = \min(a, B) \le t_0$. The hypotheses of Theorem 15.15 hold uniformly, since $a + B \le 6b_x$, $t_0 \le b_x$, $p \ge b_x^{10}$ and $Q \le \sqrt{p-1}$. At positive completions the ramified additive character already vanishes. For a coefficient ring $\mathcal{O}$ apply the theorem to the $\mathbb{Z}_p$-coordinates of the cocycle modulo p^ν. Pull back its quotient along the copy of $\mathbb{Z}/p^\nu$ generated by one, then push out the kernel along each coordinate functional. These operations give exactly the scalar extensions used in the theorem and preserve the torsion subquotient condition. Their ramified images are all killed by p^D, so the same is true over $\mathcal{O}$.

The evaluation at g of Lemma 50.14 computes the ramified coordinate. In one copy of $\mathcal{O}/p^\nu$ the subgroup killed by p^D has length at most $e_L D$. Thus the joint normalized length omitted from B_{old} is at most $n_x t_0/2$ at either prime, even when the restrictions at distinct places are dependent. At a node there are two old primes, both at most b_x^{11}. The sum of these errors with the prime weights, divided by $\sqrt{b_x}$, is $O(n_x t_0 \log b_x/\sqrt{b_x})$. Use $n_x \ll_{q_0,d} x^{12}$, $t_0 \ll x^5$ and $\log b_x \ll b_x^{1/4}$ to bound the sum over nodes $x \ge V_0$ by $O_{q_0,d}(V_0^{-107})$. Restricting the original suprema and using $(u+v)_+ \le u_+ + v$ for $v \ge 0$ gives the asserted prime sum. □

The bound removes the weight-zero ramification term on the same fields and intervals as the projected period. A uniform estimate for the positive intermediate scalar weights at the negative completions would remove the remaining old-prime term.

15.6 Positive scalar weights on the rational uniformizer tower

We pass from the varying tame extension at an original level prime to a single tower over $\mathbb{Q}_p$. For each positive scalar weight this gives a necessary local equation independent of the large negative jump. We compute its first residue class.

Fix compatible p-power roots of $-p$ and write $K_\infty = \mathbb{Q}_p((-p)^{1/p^\infty})$. For brevity put $S_n(v) = (\mathbb{Z}/p^n)[[v]]$, $\mathcal{E}_n(v) = S_n(v)[1/v]$ and $e(v) = v + p$. Let $1 \le h \le p-3$, and choose a generator of $H^1(\mathbb{Q}_p, \mathbb{Z}_p(-h))$, which is free of rank one by Lemma 50.14. At precision p^n denote by $C_{h,n}$ an upper right Frobenius entry of the corresponding extension restricted to G_{K_∞}. Use the covariant convention of Lemma 15.14, so the diagonal entries are $e(v)^h$ and one. Thus $C_{h,n}$ is well-defined up to a constant unit and addition of

$$L_h(Y) = e(v)^h \varphi(Y) - Y, \qquad \varphi(v) = v^p, \quad Y \in \mathcal{E}_n(v).$$

Membership tests below are independent of these choices.

Proposition 15.17 (Descent of the pole bound)**.** *Let F be the degree q subfield of $\mathbb{Q}_p(\mu_p)$, and let $0 \le h \le B$, $a \ge 0$. Suppose an extension of $\mathbb{Z}/p^n$ by $(\mathbb{Z}/p^n)(-h)$ over $\mathbb{Q}_p$ is a crystalline lattice subquotient over F in the filtration interval $[-a, B]$. If $q(a + B + n - 1) < p - 1$, its entry C over K_∞ satisfies*

$$C \;\in\; e(v)^{-d} S_n(v) + L_h(\mathcal{E}_n(v)), \qquad d = \min(a, B - h).$$

Proof. If ζ_p is a primitive p-th root, expansion of $(1 + (\zeta_p - 1))^p = 1$ gives $(\zeta_p - 1)^{p-1}/(-p) \equiv 1 \bmod (\zeta_p - 1)$. The map raising to $p - 1$ is an automorphism of principal units, so $\mathbb{Q}_p(\zeta_p)$ contains a uniformizer λ with $\lambda^{p-1} = -p$ and $\lambda/(\zeta_p - 1) \equiv 1 \bmod (\zeta_p - 1)$. The field F has a uniformizer $\lambda^{(p-1)/q}$ whose q-th power is $-p$. Choose its roots compatibly with the fixed tower. On its tower the Kisin parameter u satisfies $v = u^q$ and $E(u) = u^q + p = e(v)$. Indeed the parameters are Teichmüller lifts of the compatible uniformizer sequences. Restriction of étale Frobenius modules is scalar extension for the resulting inclusion of Laurent rings, by the field-of-norms equivalence. The tame group acts by $u \mapsto \eta u$ for $\eta \in \mu_q$. Its action commutes with Frobenius since $q \mid p - 1$.

As in Lemma 15.13, dualize the extension and twist by κ^a. The intersection and image models agree with the standard end models by the uniqueness proved there. In their bases the matrix has diagonal entries E^{a+h}, E^a and upper right entry $X \in S_n(u)$. The inverse height condition gives $E^{a+B-(2a+h)} X \in S_n(u)$. After undoing the twist the entry $X' = E^{-a} X$ is in both $E^{-a} S_n(u)$ and $E^{-(B-h)} S_n(u)$. This proves the pole bound E^{-d} on that tower.

The end bases can be taken to be the base changes of the two standard end bases over K_∞, since the rank-one Tate multipliers are already $e(v)^h$ and one. Therefore $C(u^q) - X' = E^h \varphi(Y) - Y$ for a Laurent series Y over the larger tower. Average this equality over μ_q, dividing by the unit q. The average of X' belongs to $e(v)^{-d} S_n(v)$, and the average of Y belongs to $\mathcal{E}_n(v)$. Indeed the average kills exactly the powers of u whose exponent is not divisible by q. This proves the assertion over the original tower. □

Lemma 15.18 (A finite system of coefficient equations)**.** *Let $h < p - 1$ and let $C \in \mathcal{E}_n(v)$ have pole order N, counting zero for an integral series. In the test of Proposition 15.17 one need only use*

$$Y = \sum_{m=1}^{M} y_m v^{-m}, \qquad M = \left\lfloor \frac{\max(N, d + n - 1) + h}{p} \right\rfloor.$$

The test is equivalent to the vanishing of all negative-power coefficients of $e(v)^d (C - L_h(Y))$ for some such Y.

Proof. Integral terms of Y give only integral terms of $L_h(Y)$. An element of $e(v)^{-d}S_n(v)$ has pole order at most $d+n-1$ by the binomial expansion, with pole order zero if $d=0$. If $m>0$ is the largest pole of Y, its coefficient occurs unchanged at exponent $-pm+h$ in $(v+p)^h\varphi(Y)$, since the leading coefficient of the polynomial is one. A different Frobenius monomial is separated by at least $p>h$, and $-Y$ cannot reach that exponent because $pm-h>m$. Thus $pm-h\le \max(N,d+n-1)$ whenever the membership holds. Conversely, any solution to the finite equations gives the required membership since $e(v)$ is a unit in $\mathcal{E}_n(v)$. Only the negative part of the input series C is needed in these equations. □

Theorem 15.19 (The first residue of a positive scalar generator)**.** *For $1\le h\le p-3$ the class of $C_{h,1}$ in $\mathbb{F}_p((v))/L_h(\mathbb{F}_p((v)))$ has a representative Av^{-1} with $A\in\mathbb{F}_p^\times$. Consequently the pole test at $n=1$ excludes a primitive trace class precisely when $d=0$.*

Proof. Put $L=\mathbb{Q}_p(\zeta_p)$ and use λ from the preceding proof. Over G_L a nonzero residual class becomes a homomorphism to $\mathbb{F}_p$. Its restriction is nonzero since $[L:\mathbb{Q}_p]$ is prime to p. Local reciprocity identifies it with a homomorphism of $L^\times$. Conjugation on its argument by $\delta\in\mathrm{Gal}(L/\mathbb{Q}_p)$ scales its value by $\bar\kappa(\delta)^{-h}$. The quotient of consecutive principal unit groups U_j/U_{j+1}, $U_j=1+\lambda^j\mathcal{O}_L$, has character $\bar\kappa^j$, since $\delta(\lambda)/\lambda$ is the Teichmüller lift of $\bar\kappa(\delta)$. The unramified direction is trivial under conjugation, and the residue unit group has order prime to p. Moreover $U_{p+1}=U_2^p$ by the logarithm on $\lambda^2\mathcal{O}_L$. Among the layers $1,\dots,p$ only $i=p-1-h$ has the required character, with $2\le i\le p-2$. Semisimplicity for the group of order $p-1$, or the successive unit quotients, shows that the homomorphism is nonzero on U_i and zero on U_{i+1}. Its cyclic degree p extension therefore has upper jump i by the ramification theorem of local reciprocity. We use the upper numbering in which inertia is the zeroth group.

Set $L_m=L(\lambda^{1/p^m})$, choosing compatible roots. Each step over L_m is cyclic of degree p with unique jump p^{m+1}. Indeed a nonidentity automorphism multiplies its new uniformizer by a nontrivial p-th root of unity, and $v_{L_{m+1}}(\zeta_p-1)=p^{m+1}$, with the uniformizer of L_{m+1} having valuation one. The degree p extension cut out by our character remains cyclic with jump i over every L_m. To check the induction step, the two cyclic extensions of L_m have different jumps, hence are linearly disjoint. In their compositum every character with nonzero component from the higher-jump extension has that higher jump, since the lower character vanishes on that last group. The upper group of the compositum is therefore the full group through i and thereafter intersects the subgroup over L_{m+1} trivially. The Herbrand function for the base change to L_{m+1} is the identity up to i, proving the assertion.

The same jump occurs in the field of norms with uniformizer z for the sequence of roots of λ, where $v=z^{p-1}$. One can see this directly in the field-of-norms construction. Take a uniformizer in the field of norms of the composite tower and its norm-compatible components in the finite layers. These components have valuation one because the extensions are totally ramified. For a generator σ of the cyclic group the differences of each component and its σ-translate have valuation $i+1$, by the preceding calculation and the equality of the two numberings for the single jump of a cyclic group of order p. The components of the difference in the field of norms are limits of the norms of these differences. Those norms have the same normalized valuation, so their limits do also. This gives the lower, hence upper, jump i in the norm field. [8]

Modulo p replace the basis of the subline by z^{-h} times its original basis. Its Frobenius multiplier becomes one, and the upper right entry becomes $z^hC_{h,1}(z^{p-1})$. It is now an Artin–Schreier class, up to the harmless choice of sign for extensions. Such a class over $\mathbb{F}_p((z))$ has a unique representative consisting of a constant and terms a_jz^{-j} with $j>0$ prime to p. Indeed remove positive powers by successive Frobenius, and replace z^{-pj} by z^{-j} modulo a coboundary. Uniqueness follows by looking at the largest pole of $\varphi(Y)-Y$, and then at the constant. The

greatest remaining pole gives the ramification jump of a ramified Artin–Schreier extension. Explicitly, for greatest pole j, a root y of $y^p - y = f$ has valuation $-j$ upstairs, where z has valuation p. Choose integers A, B with $-jA + pB = 1$. The uniformizer $y^A z^B$ is multiplied under $y \mapsto y+1$ by $(1+1/y)^A$, which differs from one by an element of valuation j, since $p \nmid A$.

The largest pole is therefore i. Under $z \mapsto \eta z$, $\eta \in \mu_{p-1}$, the class has character η^h. Uniqueness of the reduced representative forces the constant to be zero and every pole to satisfy $-j \equiv h \bmod p-1$. The only possibility is $j = i$, with nonzero coefficient A. Take an Artin–Schreier coboundary giving the difference from Az^{-i}. Average its primitive against the character η^h to keep only that eigenspace. Its primitive then has the form $z^h Y(z^{p-1})$. Dividing its Frobenius difference by z^h gives $v^h \varphi(Y) - Y$. The reduced term becomes Av^{-1}, as required. This entry belongs to $v^{-d}\mathbb{F}_p[[v]]$ for $d \geq 1$. For $d = 0$ use Lemma 15.18 with $N = 1$, which gives $M = 0$, so its nonzero pole cannot be removed. □

Corollary 15.20 (A finite-precision test for the remaining old-prime loss)**.** *At a negative completion of either old prime, take $B = t' - 1$ and $1 \leq h < B$. For any $n \geq 1$ with $Q(-J_w + B + n - 1) < p - 1$, if the test of Lemma 15.18 fails for $C_{h,n}$ and $d = B - h$, the image of the scalar cup coordinate at this completion is killed by p^{n-1}, for the full marked family at every original trace precision.*

Proof. First take $\mathbb{Z}_p$ coefficients. If the local image of a class at precision p^ν has order p^k, it is a unit times $p^{\nu-k}$ times the fixed generator in cohomology, by Lemma 50.14. Subtract a coboundary and multiply by a unit, so this identity holds on cocycles. Restricting the kernel coordinate and dividing out the redundant quotient coordinate, as in Theorem 15.15, gives the extension for the generator modulo p^k. These operations preserve the torsion subquotient condition. If $k \geq n$, reduce modulo p^n and apply Proposition 15.17. This contradicts the assumed failure. Over $\mathcal{O}/p^\nu$ apply the same argument to each $\mathbb{Z}_p$-coordinate by pullback and pushout. The evaluation isomorphism of Lemma 50.14 identifies the exponent bound with the one for the free cup coordinate. □

In this test the exponent $B - h$ and the Laurent ring on the uniformizer tower of $\mathbb{Q}_p$ do not depend on $-J_w$ or on the degree of the tame extension. The first residue calculation shows why the intermediate weights require information at higher precision. An estimate for the precision of failure, uniform over the primes and weights in the interval, is still needed.

15.7 Finite obstruction depth for the scalar traces

The Laurent test on $\mathbb{Q}_p$ in Section 15.6 eventually fails at each positive scalar weight. We prove this for its actual integral cohomology generator and remove the restriction on its precision in the tame descent argument.

Lemma 15.21 (Uniqueness of the model at every precision)**.** *Use $S_n, \mathcal{E}_n$ and the linearized Frobenius of Section 15.5, with Eisenstein polynomial of degree q and reduction u^q. If $qH < p-1$, an étale Frobenius module killed by p^n has at most one model of height at most H inside it, for every $n \geq 1$. Consequently Proposition 15.17 holds for every n under the single inequality $q(a+B) < p-1$.*

Proof. At $n = 1$ this is the pole comparison in Lemma 15.13. The same proof applies to any module killed by p inside its generic module. For general n write D for the étale module. Given a model M, intersect it with $p^i D$ and take its image in $p^i D/p^{i+1}D$, for $0 \leq i < n$. This image spans its generic module and has height at most H. In detail, intersection with the étale submodule preserves the lower Frobenius inclusion. Indeed the preimage of $E^H x$ for an intersection vector x maps to zero in the generic quotient by injectivity of the linearized Frobenius there. Flatness of the Frobenius pullback therefore puts that preimage in the pullback of the intersection. The upper inclusion follows from stability. Taking the image in a generic quotient preserves both inclusions by projecting them. Localization preserves each kernel and image used here.

For two models M, N their sum is also a model of height at most H by adding the Frobenius inclusions. In each graded piece just considered its image equals the image of M by the uniqueness at $n = 1$, since the ambient graded piece is killed by p. Now descend from $i = n$ to $i = 0$. For $x \in (M+N) \cap p^i D$ the equality of graded images gives $m \in M \cap p^i D$ such that $x - m$ belongs to the next intersection, already in M by induction. This gives $M = M + N$, and the same argument gives $N = M + N$. In Proposition 15.17 the restriction on n was used only to identify the two end models. The uniqueness just proved identifies them at every precision. □

For $1 \leq h \leq p-3$ and $d \geq 0$, let $N(p,h,d)$ be the first n at which the Laurent class $C_{h,n}$ fails the test with exponent d in Proposition 15.17. For $h = 0$ use the ramified generator $\mathcal{L}$ of Lemma 15.14 on the same $-p$ tower. Failure at one precision implies failure at every larger one, by reduction of the module and its basis. The next theorem proves existence without using any estimate for the auxiliary systems.

Theorem 15.22 (Finite depth on the rational tower)**.** *If $1 \leq h \leq p-3$ and $h + 2d < p-1$, then $N(p,h,d)$ exists and is finite. It is one for $d = 0$ and at least two for $d > 0$. For $h = 0$ and $2d < p-1$ it is exactly $d+1$.*

Proof. Let U be the integral extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p(-h)$ representing a generator. Suppose its test succeeds at every n. Put $H' = h + 2d$ and apply the contravariant étale module functor to the restrictions of $U^\vee(d)$. At precision p^n the assumed test and a change of lifted quotient basis give a Frobenius matrix

$$A_n = \begin{pmatrix} e(v)^{h+d} & e(v)^d X_n \\ 0 & e(v)^d \end{pmatrix}, \qquad X_n \in e(v)^{-d} S_n(v).$$

Indeed before twisting the dual the diagonal is $e(v)^h, 1$ in the covariant convention for U, and the extra twist multiplies the whole Frobenius by $e(v)^d$. The entries of A_n and of $e(v)^{H'} A_n^{-1}$ are integral, including the upper right entry $-e(v)^d X_n$ of the latter. The span of this basis over $S_n(v)$ is thus a free model M_n of height H' inside the fixed generic module.

These models are compatible, regardless of which successful solution was used at each level. Indeed the image of M_{n+1} at level n has height H' by the image argument above. It is M_n by Lemma 15.21 applied with $q = 1$. The induced map $M_{n+1}/p^n M_{n+1} \to M_n$ is injective as well, since a basis of M_{n+1} is a generic basis, remains one after reduction and is therefore still independent over $S_n(v)$. Lift a basis successively. The lifts are bases at every step by reduction through a nilpotent ideal. Their inverse limit identifies $M = \varprojlim M_n$ with two copies of $\mathbb{Z}_p[[v]]$. The compatible Frobenius matrices and their integral multiples $e(v)^{H'} A_n^{-1}$, expressed in these bases, give the two height inclusions on M. After localization and p-adic completion its basis gives the original integral étale module, since it gives an isomorphism modulo every p^n by construction.

This is finite $E(v)$-height on the fixed uniformizer tower for the restriction of a lattice of the full $G_{\mathbb{Q}_p}$. The finite-height theorem of Gao implies that the associated rational representation is potentially semistable. One may pass to the dual convention in that theorem, since rational potential semistability is preserved by duality and Tate twist. We conclude that $U[1/p]$ is semistable over some finite extension $L/\mathbb{Q}_p$. [9, Theorem 1.1.16]

Over L the monodromy N of this extension is zero. Indeed it kills the subline and is zero on the quotient, so could only induce a map from the trivial line to the line for $\mathbb{Q}_p(-h)$. If f is the residue degree of L, the linear operator φ^f is one on the first line and p^{hf} on the second. The relation $N\varphi^f = p^f \varphi^f N$ makes the putative map equal to $p^{(h+1)f}$ times itself. Hence it is zero. Semistability with zero monodromy gives crystallinity. But $H^1_f(L, \mathbb{Q}_p(-h)) = 0$ for $h > 0$ by the crystalline dimension formula, since all its filtration jumps are positive and its invariants vanish. The extension splits over L. Restriction followed by corestriction multiplies rational cohomology by $[L : \mathbb{Q}_p]$, so it would split over $\mathbb{Q}_p$ as well. This contradicts the choice of the integral generator in its torsion-free cohomology group. We obtain finite failure. The first-layer assertions follow from Theorem 15.19.

For $h = 0$ take $c = -1$ in Lemma 15.14. For $n \le d$, only its first n series terms survive modulo p^n since $n < p - 1$. Every negative coefficient of $(v + p)^d C(v)$ is zero there. In fact its terms involve $v^j v^{-i}$ with $j < i \le n$, and the accompanying power of p is $d - j + i - 1 \ge d$, with denominator i a unit. The test succeeds. At $n = d + 1$, the nonzero coefficient at v^{-1} of $(v + p)^d C(v)$ has valuation exactly d by the computation of Theorem 15.15. The original logarithm and every representative in $(v+p)^{-d} S_n(v)$ have pole order less than p since $2d < p-1$. A Frobenius coboundary of pole order less than p is integral at weight zero. Thus no change of basis removes that coefficient. The argument also gives failure at $n = 1$ when $d = 0$. □

Corollary 15.23 (A bound independent of the tame field and the far jump)**.** *For either old coefficient prime in the full marked family, put $B = t' - 1$ at a nonzero lift. The positive scalar term remaining in Corollary 15.16 satisfies*

$$B^{>0}_{\mathrm{old}}(D) \;\le\; \frac{[E_x : \mathbb{Q}]}{2} \sum_{h=1}^{B-1} \big(N(p, h, B - h) - 1\big).$$

Every number on the right is finite and depends only on its three displayed arguments. The estimate holds at arbitrary trace precision and for every auxiliary family at the node.

Proof. At each negative completion $-J_w > B$, and $Q(-J_w + B) < p - 1$ by Theorem 15.8. Also $h + 2(B - h) \le 2t_0 < p - 1$ by the size of the primes. We may therefore apply Corollary 15.20 with $n = N(p, h, B - h)$, using Lemma 15.21 to remove its precision restriction. This kills each cup coordinate by $p^{N(p,h,B-h)-1}$. A subgroup killed by p^s in one copy of $\mathcal{O}/p^\nu$ has normalized length at most s. Add these bounds over the negative completions and the positive intermediate

weights. The joint image is a submodule of their direct sum, so this addition also applies when the local restrictions are dependent. □

The corollary replaces the finite local defect for a long interval of jumps over the ramified field by a finite depth over $\mathbb{Q}_p$ depending only on the scalar weight and the short string at that prime. An upper bound on these positive-weight depths as p and the string vary is still required to sum the unrestricted projected period on the full interval.

15.8 Cyclotomic units detecting the positive scalar classes

For $1 \le h \le p-3$, the Laurent test in Section 15.6 uses a primitive class of $H^1(\mathbb{Q}_p, \mathbb{Z}_p(-h))$. We give its local reciprocity character at every finite precision. The unit on which the character is normalized can be chosen through one polynomial calculation over $\mathbb{F}_p$.

Choose compatible primitive roots ζ_{p^n}, with $\zeta_{p^{n+1}}^p = \zeta_{p^n}$. Put $L_n = \mathbb{Q}_p(\zeta_{p^n})$, $\lambda_n = \zeta_{p^n} - 1$, $\Gamma_n = \mathrm{Gal}(L_n/\mathbb{Q}_p) = \Delta \times P_n$, where Δ is the Teichmüller subgroup and P_n is the kernel of reduction to Γ_1. Write $\omega : \Delta \longrightarrow \mathbb{Z}_p^\times$ for the action on roots of unity, and write $[b]$ for the Teichmüller lift of $b \in \mathbb{F}_p^\times$. On a pro-$p$ abelian group with Δ-action use the projector

$$e_i = \frac{1}{p-1} \sum_{\delta \in \Delta} \omega(\delta)^{-i} \delta.$$

In a multiplicative group the coefficients of this operator act by continuous $\mathbb{Z}_p$-powers. Set $U_n = 1 + \lambda_n \mathcal{O}_{L_n}$.

Lemma 15.24 (A unit basis at every cyclotomic level)**.** *For each $2 \le i \le p-2$ there is $b \in \mathbb{F}_p^\times \setminus \{1\}$ such that, at every $n \ge 1$, the $\mathbb{Z}_p[P_n]$-module $e_i U_n$ is free of rank one with basis*

$$w_n = e_i u_n(b), \qquad u_n(b) = \frac{1 - [b]\zeta_{p^n}}{[1-b]}.$$

Both sequences are compatible under the norm.

Proof. The numerator has residue $1-b$, so the quotient is a principal unit. Taking the product over the p choices of a p-th root of ζ_{p^n} gives $1 - [b]^p \zeta_{p^n} = 1 - [b]\zeta_{p^n}$. The denominator also has p-th power equal to itself. Norms commute with the action of Δ.

We first select b at level one. In $\mathcal{O}_{L_1}/\lambda_1^{i+1}$ the characteristic is p because $i+1 \le p-1 = v_{L_1}(p)$. The truncated exponential and logarithm through degree i are mutually inverse. In the coordinate $X = \log(1+\lambda_1)$, an element $\delta \in \Delta$ acts by $X \mapsto \bar{\omega}(\delta) X$. This follows by expanding $(1+\lambda_1)^a$, with any integer a representing δ modulo p. Thus in the logarithm of a unit the projector e_i retains exactly the coefficient of X^i through this degree. For $u_1(b)$ that coefficient is

$$A_i(\beta) = -[X^i] \sum_{j=1}^{i} \frac{\beta^j}{j} (\exp X - 1)^j, \qquad \beta = \frac{b}{1-b}.$$

Indeed the constant factor $(1-[b])/[1-b]$ is 1 in the truncated ring. The polynomial A_i has degree i, leading coefficient $-1/i$ and constant term zero. The admissible β's are all elements of $\mathbb{F}_p$ except $0, -1$. Vanishing at every admissible value would give at least $p-1$ roots, including 0. Choose a value with $A_i(\beta) \neq 0$. It follows that $w_1 - 1$ has valuation exactly i.

The only torsion in U_n is μ_{p^n}, on which Δ acts by ω. After tensoring with $\mathbb{Q}_p$, the logarithm identifies $U_n \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with L_n, since it is an isomorphism on sufficiently deep units. The normal basis theorem applied to $L_n/\mathbb{Q}_p$ now gives $\mathrm{rank}_{\mathbb{Z}_p} e_i U_n = |P_n|$. This module is finite free over $\mathbb{Z}_p$, since its projection kills the torsion. Every p-th power in U_1 is 1 mod λ_1^p, by the binomial formula. Hence w_1 is a basis of the rank-one module $e_i U_1$.

For $n > 1$ let σ generate P_n. Norm induces an isomorphism from the coinvariants $(e_i U_n)_{P_n}$ to $e_i U_1$. Surjectivity follows from the norm formula for w_n. For injectivity, a unit of norm one is $\sigma(y)/y$ for some $y \in L_n^\times$ by Hilbert's Theorem 90. In the pro-p completion of $L_n^\times$, the valuation quotient is $\mathbb{Z}_p$ with trivial Δ-action and the kernel is U_n. Thus $e_i y$ belongs to $e_i U_n$. Applying e_i to the identity proves injectivity. The ring $\mathbb{Z}_p[P_n]$ is local with maximal ideal $(p, \sigma - 1)$, since modulo p its group ring is $\mathbb{F}_p[T]/T^{|P_n|}$ with $T = \sigma - 1$ and the ring is finite over $\mathbb{Z}_p$. Nakayama's lemma now shows that w_n generates $e_i U_n$ over this ring. Both are free of the same finite rank over $\mathbb{Z}_p$, so the resulting surjection is an isomorphism. □

Theorem 15.25 (An integral reciprocity coordinate for each scalar weight). *Fix $1 \le h \le p-3$, take $i = p-1-h$, and choose b as in Lemma 15.24. Write $R_n = \mathbb{Z}/p^n\mathbb{Z}$, fixing the usual scalar basis of $R_n(-h)$. Restrict a cohomology class to G_{L_n} and use local reciprocity there. Evaluation of the resulting character on $u_n(b)$ is an isomorphism*

$$H^1(\mathbb{Q}_p, R_n(-h)) \longrightarrow R_n.$$

These isomorphisms commute with reduction. In particular the inverse images of 1 are the reductions of a single generator of $H^1(\mathbb{Q}_p, \mathbb{Z}_p(-h))$.

Proof. The restriction of $R_n(-h)$ to G_{L_n} is trivial. On Δ it has nontrivial character ω^i, so its Δ-invariants are zero. All higher cohomology of Δ with these coefficients vanishes by averaging. The extension of P_n by Δ therefore gives $H^j(\Gamma_n, R_n(-h)) = 0$ for every j. By inflation and restriction, followed by local reciprocity, the group on the left is precisely

$$\mathrm{Hom}^{\mathrm{cont}}_{\Gamma_n}\left((L_n^{\times})^{\wedge p}, R_n(-h)\right).$$

Here the superscript on the source denotes pro-p completion. We use one fixed convention for reciprocity at every level. Its compatibility with norms identifies the map of characters under field restriction with precomposition by the norm [2].

Only $e_i(L_n^{\times})^{\wedge p} = e_i U_n$ contributes. A homomorphism on this component is determined by any assigned value $t \in R_n$ on w_n. In fact the value on σw_n must be $\kappa(\sigma)^{-h}t$ for $\sigma \in P_n$, where κ in this formula is reduced modulo p^n. These assignments extend uniquely on the group-ring basis of the lemma and give an equivariant continuous homomorphism. It takes the same value on $u_n(b)$ as on w_n. The norm formula for $u_{n+1}(b)$ proves compatibility with reduction, even though the evaluating field varies with n. Finally continuous cohomology with coefficients in the inverse limit agrees here with the inverse limit of the H^1's: the finite coefficient transition maps are surjective and the groups of invariants are zero. This proves the integral statement. □

Corollary 15.26 (Finite unit calculation for a trace class). *For $e_n = (p-1)p^{n-1}$, put $m_n = p^{n-1}+1+ne_n$. The finite reciprocity character giving the primitive class above can be calculated on units in*

$$\mathbb{Z}_p[T]/(\Phi_{p^n}(1+T), T^{m_n}).$$

If a local scalar trace class at a split place over an old prime has coefficient ring $\mathcal{O}/p^n\mathcal{O}$, its normalized length in the free scalar coordinate equals the normalized length of the ideal generated by its value on $u_n(b)$.

Proof. The indicated cyclotomic polynomial is Eisenstein and T maps to λ_n. For $s = 1 + p^{n-1} > e_n/(p-1)$, logarithm and exponential identify $1 + \lambda_n^s \mathcal{O}_{L_n}$ with $\lambda_n^s \mathcal{O}_{L_n}$. Raising to p^n carries it onto $1 + \lambda_n^{m_n}\mathcal{O}_{L_n}$. Every R_n-valued character kills this last subgroup. One can therefore decompose the projected unit in the finite group modulo p^n-th powers using the basis $\{\sigma w_n : \sigma \in P_n\}$ and evaluate each basis vector by the formula in the theorem.

The coefficient ring $\mathcal{O}$ is finite free over $\mathbb{Z}_p$. Choosing a basis and applying the theorem to each component identifies the scalar cohomology with $\mathcal{O}/p^n\mathcal{O}$ by evaluation on this one unit. Any two free scalar coordinates differ by a unit, so they give the same ideal length. This applies also to the ideal of values of a family of classes. □

This determines the primitive character and its normalization at all precisions. To bound the period defect by this calculation one still needs to bound the orders of the values attained by the actual trace classes, uniformly over the primes and weights of Corollary 15.23.

15.9 Crystalline monodromy at the original level primes

The Frobenius condition of Section 15.6 allows a simple pole modulo p at the intermediate weights. The crystalline lattice also gives a monodromy operator on its residual Breuil module. We use this operator to exclude that pole for the actual trace classes.

Let $p \geq 5$, let $q \mid p-1$, and use the totally ramified field $F = \mathbb{Q}_p((-p)^{1/q})$ and uniformizer from Proposition 15.17. The Eisenstein polynomial is $u^q + p$. We use the covariant Frobenius-module functor: the representation on the uniformizer tower consists of Frobenius-fixed vectors after separable scalar extension. The filtration jump of κ is -1. The residual module of κ^{-j} for $j \geq 0$ has Frobenius multiplier u^{qj} in this convention. Equivalently one may first dualize the representation and then apply the contravariant functor of Lemma 15.7.

Lemma 15.27 (The lowest coefficient allowed by crystalline monodromy)**.** *Let a crystalline Breuil module in residue characteristic p be an extension of rank-one objects. Suppose its underlying Breuil–Kisin module has Frobenius*

$$\varphi(e) = u^t e, \qquad \varphi(f) = u^s f + y(u)e, \qquad 0 \leq s \leq t < p-1,$$

where e spans the submodule, and $y \in k'[[u]]$ for a finite coefficient field k' of characteristic p. The residue field of F is $\mathbb{F}_p$, so Frobenius acts trivially on the coefficient field. Then

$$\operatorname{ord}_u(y) \;\geq\; \max(0, s-q+1),$$

with the order of zero taken to be infinite.

Proof. We give the one-residue-embedding calculation underlying [10, Lemma 2.2.7]. Put $d = q(p-2)$ and $R = k'[u]/u^{qp}$. Write e_0, f_0 for the two basis vectors of the Frobenius pullback modulo u^{qp}. Its filtered part in degree $p-2$ contains

$$E = u^{d-t}e_0, \qquad G = u^{d-s}f_0 - u^{d-s-t}y\,e_0.$$

Indeed linearized Frobenius sends these to $u^d e, u^d f$. The indicated coefficient of y is integral by the height condition, or by the upper right entry of u^d times the inverse Frobenius matrix. The intersection of the filtered part with Re_0 is RE, because a lift along this line has linearized image divisible by u^d exactly when its coefficient is divisible by u^{d-t}. The divided Frobenius φ_{p-2} takes E, G to e_0, f_0, respectively.

The monodromy on scalars is $-u\,d/du$. On the rank-one ends it kills the displayed Frobenius-pullback generators by [10, Lemma 2.2.3]. Thus $N(e_0) = 0$ and $N(f_0) = \mu e_0$ with $\mu \in uR$, the last inclusion being the crystalline condition. For every filtered vector z, $u^q N(z)$ is filtered and $\varphi_{p-2}(u^q N(z)) = N(\varphi_{p-2}(z))$. In particular the vector

$$U = u^q N(G) + (d-s)u^q G$$

is in RE. Applying divided Frobenius to it gives $\mu e_0 \bmod u\mathcal{M}$, hence zero modulo u, since $\varphi(u^q) = u^{pq} = 0$ in R. Here $\mathcal{M}$ is the Breuil module. If $U = zE$, it follows that z has zero constant coefficient.

The coefficient of e_0 in U, expanded with the Leibniz rule, is

$$u^{q+d-s}\mu + u^{q+d-s-t}(uy' - ty).$$

If y had least degree $l \leq s-q$, the term of degree $d-t+q-s+l \leq d-t$ in the second summand would be nonzero: $0 \leq l < s \leq t < p-1$, so $l - t \neq 0$ in k'. This degree is nonnegative since $u^{d-s-t}y$ is integral, and $d - t < qp$, so the truncation preserves it. The first summand has strictly greater degree since $t \geq s$ and $\mu \in uR$. This contradicts $U \in uRE$. When $s < q$ the asserted bound is just integrality. □

Theorem 15.28 (Integral scalar trace conditions throughout the tame range)**.** *Let a, B be nonnegative integers with $q(a+B) < p-1$. Suppose an extension of $\mathbb{Z}/p^\nu$ by $(\mathbb{Z}/p^\nu)(-h)$ over $\mathbb{Q}_p$, $0 \le h \le \min(B, p-3)$, restricts to a finite subquotient of a lattice crystalline over F with filtration jumps in $[-a, B]$. For $h > 0$ its class is zero. For $h = 0$ its class is unramified. The same conclusions hold with coefficients in $\mathcal{O}/p^\nu\mathcal{O}$ for every finite extension of coefficient rings $\mathcal{O}/\mathbb{Z}_p$.*

Proof. Since $q \mid p-1$ and $qh \le q(a+B) < p-1$, $q(h+1) \le p-1$. First suppose $h > 0$. A nonzero class gives as a subquotient the primitive nonsplit scalar extension modulo p. Indeed $H^1(\mathbb{Q}_p, (\mathbb{Z}/p^\nu)(-h))$ is free of rank one by Theorem 15.25. After changing the section and a unit in the kernel, the cocycle is $p^{\nu-k}$ times a primitive integral cocycle, where $p^k > 1$ is its order. Take the submodule with kernel $p^{\nu-k}(\mathbb{Z}/p^\nu)(-h)$, still containing the chosen quotient generator. In it the subgroup generated by p^k times that quotient generator is fixed. Divide out this subgroup and then reduce modulo p. In the result the kernel and quotient each have dimension one and the cocycle is the primitive reduction.

If $h = 0$ and the class is ramified, first subtract its unramified part. This is a Baer sum with the reduction of an integral unramified extension, crystalline with jumps zero. The diagonal pullback and addition pushout are subquotients of the direct sum of the two extensions. The ramified character of $\mathbb{Q}_p$ is now a multiple of $\mathcal{L} = p^{-1}\log(\kappa^{p-1})$ by local reciprocity. The same kernel and quotient construction yields its nonzero primitive reduction.

Denote the resulting two-dimensional residual extension by V. The twist $V(-a)|_{G_F}$ is a subquotient of a crystalline lattice with jumps in $[0, H]$, $H = a + B \le p-2$. A killed-p subquotient can be obtained from the reduction of a lattice: in the inverse images of its numerator and denominator take the numerator as the new lattice, whose p-multiple lies in the denominator. We can extend coefficients to contain all embeddings of F and take the residue field there, since extending this lattice and then reducing gives scalar extension of its reduction. By the crystalline Breuil comparison and the correspondence for subobjects and quotients, $V(-a)|_{G_F}$ has a crystalline Breuil module, with a crystalline subline and quotient line. The underlying Kisin modules form an exact sequence. Their heights are at most H: the parent has this height and the linearized Frobenius inclusions pass to saturated intersections and images as in Lemma 15.7. [10, Theorems 2.1.4 and 2.1.5 and §2.1]

Over $k'[[u]]$ the two end lattices are the standard ones for the characters $\bar\kappa^{-(a+h)}$ and $\bar\kappa^{-a}$. Indeed any two height-H lattices in the same generic module agree when $qH < p-1$, by Lemma 15.21. Thus in the order subline, quotient the exponents in Lemma 15.27 are

$$t = q(a+h), \qquad s = qa.$$

Lift a basis of the quotient. After undoing the twist, its upper right entry is $u^{-s}y$. By the lemma this has pole order at most $q-1$.

On the other hand the primitive class on the rational uniformizer tower has upper right entry Av^{-1}, where $A \neq 0$, by Theorem 15.19. At $h = 0$ this follows from the first coefficient of the logarithm in Lemma 15.14. On the tower over F the substitution is $v = u^q$, so the entry is Au^{-q}. Two entries for the same extension with the same end bases differ by $u^{qh}\varphi(Y) - Y$ in $k'((u))$. If Y has a pole of order $m \ge 1$, the pole of this difference has order $pm - qh$, since $pm - qh > m$. In fact $pm - qh \ge p - qh > q$. If Y is integral its difference is integral. Neither case can account for the difference between a pole of order q and a series with pole order at most $q-1$. This contradicts the existence of V.

Finally for coefficients $\mathcal{O}/p^\nu\mathcal{O}$, pull back the quotient along the rational scalar copy and push out the kernel along each vector coordinate of a $\mathbb{Z}_p$-basis of $\mathcal{O}$. Each resulting extension

satisfies the same subquotient condition, by adding copies of the endpoint characters if needed. Its class is zero, or unramified for $h = 0$, by what we have proved. The coordinate maps identify the original cohomology group with their direct sum over $\mathbb{Z}/p^{\nu}$, giving the conclusion. □

Corollary 15.29 (The projected period at the two original level primes)**.** *On the fields E_x, nodes x, coefficient embeddings and shorter intervals of Theorem 50.16, all the local scalar trace classes at the two original coefficient primes belong to the integral finite condition at every precision. For every marked family there one has $r(D) = 0$. With $p(D)$ the normalized integral projected period valuation of Theorem 50.11, one consequently has*

$$\sum_{p} (\log p) \sup_{\substack{D \text{ at } p \\ p \in \{p_{x,0}, p_{x,1}\}}} \frac{p(D)}{\sqrt{b_x}} \ll_{q_0,d} V_0^{-10}.$$

An empty supremum contributes zero.

Proof. At a negative completion the local field is $\mathbb{Q}_p$. The graph lattice of Lemma 12.14 is crystalline over the common degree-Q tame subfield of $\mathbb{Q}_p(\mu_p)$. Theorem 15.8 gives the common window with $a = -J_w$, $B = t' - 1$, and $Q(a + B) < p - 1$. This is the field F above with $q = Q$. Theorem 15.28 therefore applies to every scalar restriction, including the trivial character. Positive completions and the main summand at both signs were covered in Theorem 15.8. Zero lifts have zero defect by definition. All the groups $\Gamma_h(D)$ in Theorem 50.16 thus vanish, including the ramified coordinate at $h = 0$. Substitute in its prime sum. This uses the same set of families at every node, including those with a locally irreducible constituent of dimension greater than one. □

The corollary settles the contribution at the two original level primes of each node. The other coefficient primes in the full projected period and the remaining comparisons in the global argument still require estimates.

15.10 Tame inertia characters in a scalar trace extension

At a primitive label of rank f, the characters of tame inertia can have order p^f-1. The small-ramification inequality of Theorem 15.28 then fails even for a short weight interval. We give a necessary condition on the inertial type of a lattice that can contribute a scalar trace class. The condition involves the two rank-one ends of the extension at every residue embedding.

Let $U/\mathbb{Q}_p$ be unramified of degree f, let $e=p^f-1$, and put $F=U((-p)^{1/e})$. Fix the compatible uniformizer towers, with parameters v on U and u on F, so that $v=u^e$. Let $\Delta=\mathrm{Gal}(F/U)$. On the i-th residue embedding let $\theta_i(\delta)=\delta(u)/u$ in that component, for $\delta\in\Delta$. The θ_i have order e and are the Frobenius conjugates of one another. Index components cyclically so that Frobenius goes from $i-1$ to i. We extend the coefficient field to split the residue embeddings.

Theorem 15.30 (Necessary tame characters for a nonfinite scalar class)**.** *Let $H=a+B$ with $a,B\ge 0$ and $3H+1<p$, $p\ge 5$. Let $0\le h\le \min(B,p-3)$. Suppose an extension of $\mathbb{Z}/p^\nu$ by $(\mathbb{Z}/p^\nu)(-h)$ over $\mathbb{Q}_p$ is a Galois-stable subquotient of a lattice over $\mathbb{Q}_p$ which is crystalline over F, with jumps in $[-a,B]$. Write $\mathcal{T}$ for the set of its tame inertial characters together with the trivial character, closed under inverses and residue Frobenius. If the class is nonzero for $h>0$, or ramified for $h=0$, there exist cyclic lists of integers b_i,c_i satisfying, at every i,*

$$|b_i|,\ |c_i|\le\frac{eH}{p-1},\qquad b_i>c_i,\qquad \theta_i^{b_i},\theta_i^{c_i}\in\mathcal{T},$$
$$0\le ea+pb_{i-1}-b_i\le eH,\qquad 0\le e(a+h)+pc_{i-1}-c_i\le eH.$$

The same test applies at all precisions over finite extensions of the coefficient ring, with the tame types of the parent lattices and their local coefficient conjugates.

Proof. The extraction in Theorem 15.28 gives the primitive residual extension. All the submodules and quotients in that construction are stable already over $\mathbb{Q}_p$. Any added unramified or Tate representation has trivial smooth inertial character. Twist by $\bar\kappa^{-a}$ and take the crystalline Breuil module over F of the extracted subquotient; its underlying Kisin module has height at most H. Here we need its tame descent as well. For the new numerator lattice in the characteristic-zero parent, the integral Kisin module carries the descent action, and its reduction modulo u has in each component the tame type of the parent. This is the integral comparison with descent data of [11, Definition 3.6 and Propositions 5.16–5.17]. It uses the canonical integral Kisin module of the lattice over F, whose reduction also underlies the crystalline Breuil module by [10, Theorem 2.1.5]. Either choice of dual convention only reverses characters in the type test.

By the subobject correspondence of [10, Theorem 2.1.4], the subobjects defining our quotient and its two ends have saturated underlying Kisin submodules. They are preserved by tame descent: they are the intersections of the parent model with their generic subspaces, and those subspaces descend. Thus the underlying Kisin sequences remain exact after reduction modulo u, since their quotients are free over the component power series rings. Every tame character of an end modulo u occurs in the parent. Taking subobjects and quotients preserves the bound on height, by the linearized Frobenius argument of Lemma 15.21; that argument for intersections and images does not require uniqueness of the models.

In the canonical generic bases pulled up from U, the quotient and subline lattices in component i are generated by u^{b_i} and u^{c_i} times those bases. Indeed a lattice in a Laurent line is a power of u times the standard lattice. These generators have tame characters $\theta_i^{b_i}$ and $\theta_i^{c_i}$. Their Frobenius exponents are, respectively,

$$s_i=ea+pb_{i-1}-b_i,\qquad t_i=e(a+h)+pc_{i-1}-c_i.$$

The height condition puts both in $[0, eH]$. In particular $|pb_{i-1} - b_i| \le eH$. At an index with $|b_{i-1}|$ maximal, this implies $(p-1)\max_i |b_i| \le eH$. Apply the same argument to c_i. It remains to prove the strict ordering.

Take a generic basis for the primitive extension over the rational tower in which its untwisted off-diagonal term is Av^{-1}, $A \neq 0$, as in Theorem 15.19 and Lemma 15.14. After the unramified extension this term occurs in every component. Lift the chosen quotient generator in the middle lattice, and project the lift to its tame eigencharacter by averaging over Δ, of order prime to p. Dividing the lift by u^{b_i} gives a generic tame-invariant lift of the original quotient generator. Its difference from the original lift is $Y_i(v)$ times the canonical subline generator, with $Y_i(v) \in k'((v))$, since only powers of u^e are invariant. In the resulting lattice basis the off-diagonal term is

$$y_i = u^{ea + pb_{i-1} - c_i}\bigl(Av^{-1} + v^h \varphi(Y_{i-1}) - Y_i\bigr).$$

This follows by applying Frobenius to the lifted quotient generator and subtracting u^{s_i} times its next lift. Frobenius raises the parameter to its p-th power.

If some Y_j has a pole, let m be the largest pole order and take $i - 1 = j$ attaining it. In component i the parenthesis then has pole order $pm - h > \max(m, 1)$. Integrality of y_i bounds this order above by

$$\frac{s_i + b_i - c_i}{e} \le H + \frac{2H}{p-1} < p - h.$$

For the strict inequality use $h \le H$ and $3H + 1 < p$. This contradicts $pm - h \ge p - h$. Consequently all Y_i are integral. In particular the least degree of y_i and its residue relative to t_i are

$$l_i = ea + pb_{i-1} - c_i - e = s_i - e + b_i - c_i, \qquad l_i - t_i \equiv -e(h+1) \not\equiv 0 \pmod{p}.$$

We spell out the part of the monodromy calculation needed when s_i, t_i can exceed p. Use $d = e(p-2)$ in the proof of Lemma 15.27. In component i the filtered vectors E_i, G_i in that proof generate the filtered submodule: the condition for a pair of power series is exactly that its linearized image be divisible by u^d, which gives those two generators by solving the triangular system. Their divided Frobenius images are the next basis vectors. For $N(f_{0,i}) = \mu_i e_{0,i}$, the commutation relation therefore gives $\mu_i \in k'[u^p]/(u^{ep})$ in the sense that only degrees divisible by p occur. Indeed it expresses this vector as divided Frobenius of a combination of the two filtered generators in the preceding component. Crystallinity gives $\mu_i \in uR$ as well.

Exactly as in that lemma, the following coefficient of the subline lies in $u^{d-t_i+1}R$, $R = k'[u]/(u^{ep})$:

$$u^{e+d-s_i}\mu_i + u^{e+d-s_i-t_i}(uy_i' - t_i y_i).$$

If $l_i \le s_i - e$, the lowest term of y_i contributes nontrivially in degree $d - t_i + e - s_i + l_i \le d - t_i < ep$. Its degree is nonnegative by the inverse height condition. Cancellation by the first summand would require a term of degree $l_i - t_i$ in μ_i, impossible by the congruence above. This contradiction proves $l_i > s_i - e$, hence $b_i > c_i$ at every index. Coordinate pullbacks and pushouts as in Theorem 15.28 give the coefficient-ring assertion. On forgetting the coefficient action the additional tame characters are local conjugates, included in the test. □

Corollary 15.31 (A count for a primitive marked orbit)**.** *If, for some i, the only character in $\mathcal{T}$ of the form θ_i^j, $|j| \le eH/(p-1)$, is the trivial one, all the scalar classes in the theorem satisfy the integral finite condition. Suppose the nontrivial types consist of a single primitive orbit of order e, its inverses, and characters of orders dividing Q with $QH < p-1$. Among all*

embeddings of the field of values of the primitive character into $\overline{\mathbb{Q}}_p$, the proportion for which this vanishing test can fail is at most

$$\frac{2f\lfloor eH/(p-1)\rfloor}{\varphi(e)} \le \frac{2fH(1+\log_2 e)}{p-1}.$$

Proof. The absolute value bound for either shift is strictly less than e. A trivial character therefore forces that shift to be zero, and both shifts cannot be zero at the same index. An element of order dividing Q other than the identity has least absolute exponent at least e/Q; this exceeds $eH/(p-1)$.

Fix one component. The image of a primitive character, as the coefficient embedding varies, has exponent running over the units modulo e, each once for its field of values. The orbit gives the exponents $p^j k$ for $0 \le j < f$. For each j, multiplication by p^j permutes the unit classes. At most $2\lfloor eH/(p-1)\rfloor$ of them have a nonzero representative in the indicated interval. The interval is symmetric, so inversion gives no further exceptions. Take the union over j. Finally if e has m distinct prime divisors, its j-th smallest prime divisor is at least $j+1$. Thus $e/\varphi(e) \le \prod_{j=1}^{m}(1+1/j) = m+1 \le 1+\log_2 e$. □

At a primitive marked place of the shorter construction the orbit in Lemma 47.35 has precisely this order with $f = r$, even after the prescribed scalar twist. For families with the indicated types the bound therefore remains effective when e is as large as $p^r - 1$. The count is over coefficient embeddings. A bound for the supremum in the period inequality must also control the exceptional embeddings and the main representation of higher rank.

16 Cup products along Frobenius orbits

16.1 Frobenius and the integral cup product

We give a cochain proof of the local pairing argument of `New/Congruences/Lifting/CupProductBound.lean`. It applies to matrices of arbitrary rank and retains the precision of the original torsion cocycle. The geometric input to this argument consists of lifts satisfying explicit inequalities. Keeping track of the choices will let us apply the result to the same dual lattice that defines the projected period.

Let $K/\mathbb{Q}_p$ be finite, $G = G_K$, $F = \mathbb{C}_p^\flat$, and $n \geq 1$. The tilt F is an algebraically closed complete valued field of characteristic p, with continuous action of G. These are the usual tilting and finite-extension theorems for perfectoid fields applied to $\mathbb{C}_p$. [12] Write $A_n = W_n(F)$ for the length-n Witt vectors, with Witt Frobenius ϕ. Let $S_n \subset A_n$ be the additive subgroup of Witt vectors all of whose coordinates lie in the maximal ideal $\mathfrak{m}_F$. It is the kernel of residue reduction on $W_n(\mathcal{O}_F)$. We use A_n/S_n as an additive group. For a Tate twist (1), G acts also on the twist and ϕ acts on the Witt vector alone.

Lemma 16.1 (Constants modulo the maximal ideal of Witt vectors)**.** *The map $\phi - 1$ is onto A_n and bijective on S_n. The fixed subgroup of A_n/S_n under ϕ is exactly the image of $\mathbb{Z}/p^n$, which embeds there. These statements also hold after twisting by one. The quotient has the discrete topology for the coordinate topology on A_n.*

Proof. On F, surjectivity is the solvability of $X^p - X = t$. On successive quotients of the filtration of A_n by powers of p the same assertion holds, since F is perfect. Lifting a solution one quotient at a time proves surjectivity. For $z \in S_n$ a solution in S_n is $-\sum_{r\geq 0} \phi^r z$. In Witt coordinates ϕ is the p-power map. All these coordinate valuations therefore tend to infinity. For each positive lower bound on the valuations the corresponding coordinate neighborhood of zero is an additive subgroup, by the integral Witt addition and inverse polynomials. Completeness gives the stated sum, still in S_n. The fixed vectors in A_n have coordinates in $\mathbb{F}_p$, so are $W_n(\mathbb{F}_p) = \mathbb{Z}/p^n$. None except zero belongs to S_n. This proves injectivity on S_n and on the constants. If $(\phi - 1)t \in S_n$, subtract its unique preimage in S_n from t to obtain a fixed vector. Finally S_n is open, since $\mathfrak{m}_F$ is open and there are only finitely many coordinates. □

Let Y, Y' be finite free $\mathbb{Z}_p$-lattices with continuous G-action, a, a' continuous integral 1-cocycles on these lattices, and

$$P : Y \times Y' \longrightarrow (\mathbb{Z}/p^n)(1)$$

a G-equivariant bilinear map. For nonnegative integers i, j, s, put $b = p^i a$, $b' = p^j a'$ and extend $B = p^s P$ bilinearly over A_n. Thus B takes values in $A_n(1)$. For $u \in A_n \otimes_{\mathbb{Z}_p} Y$ and $u' \in A_n \otimes_{\mathbb{Z}_p} Y'$ use the displacements

$$d_u(g) = b(g) + gu - u, \qquad d'_{u'}(g) = b'(g) + gu' - u'.$$

They are 1-cocycles for the ordinary linear action. Frobenius fixes the integral terms in these formulas.

Theorem 16.2 (A Frobenius path that annihilates a cup product)**.** *Suppose $H \subseteq G$ is open of index prime to p, and there exist x, x' as above with*

$$B(d_x(g), g d'_{x'}(h)) \in S_n(1) \qquad (g, h \in H).$$

Suppose further that a finite sequence of pairs joins (x, x') to $(\phi x, \phi x')$, changing one component at a time. Require the following condition on each step, for every $g \in H$.

1. *In a step from* (u,u') *to* $(u+v,u')$ *require* $B(v,d'_{u'}(g))\in S_n(1)$.

2. *In a step from* (u,u') *to* $(u,u'+v')$ *require* $B(d_u(g),gv')\in S_n(1)$.

Then $p^{i+j+s}[P(a\cup a')]=0$ *in* $H^2(K,(\mathbb{Z}/p^n)(1))$.

Proof. Use the left inhomogeneous cochain differential $\partial t(g)=gt-t$ and $\partial f(g,h)=gf(h)-f(gh)+f(g)$. Define the following continuous cochain on H, initially with values in $A_n(1)$,

$$f_{u,u'}(g)=B(b(g),gu')-B(u,d'_{u'}(g)).$$

Its differential is

$$\partial f_{u,u'}(g,h)=B(b(g),gb'(h))-B(d_u(g),gd'_{u'}(h)).$$

Indeed the first term in f has differential $-B(b(g),g(hu'-u'))$, by the cocycle identity for b. The differential of $B(u,d'_{u'}(g))$ is $B(gu-u,gd'_{u'}(h))$, by the cocycle identity for $d'_{u'}$. Combining gives the formula. At $u=x,u'=x'$ its second term vanishes modulo $S_n(1)$.

In a first-component step the change of f is $-B(v,d'_{u'}(g))$. In a second-component step it is

$$B(d_u(g),gv')-\partial B(u,v')(g).$$

The second identity follows by expanding $d_u(g)$ and using $B(gu,gv')=gB(u,v')$. Consequently, modulo $S_n(1)$, the first steps contribute zero and the second steps contribute coboundaries. If t is the sum of $B(u,v')$ over the second steps, we obtain, for $f=f_{x,x'}$,

$$(\phi-1)f\equiv-\partial t\pmod{S_n(1)}.$$

Here $f_{\phi x,\phi x'}=\phi f_{x,x'}$ because ϕ commutes with H, fixes b,b', and respects the pairing.

Choose $y\in A_n(1)$ with $(\phi-1)y=t$ by Lemma 16.1. The image of $f+\partial y$ in the quotient is now fixed by ϕ pointwise. Hence it is the image of a unique cochain $\kappa:H\to(\mathbb{Z}/p^n)(1)$. It is continuous because $f+\partial y$ is continuous and the quotient is discrete. Taking its differential leaves the differential of f unchanged in the quotient. Both that differential and $B(b(g),gb'(h))$ have the same reduction there. The latter is constant in the Witt direction. Injectivity on constants therefore gives the equality in finite coefficients

$$\partial\kappa(g,h)=p^{i+j+s}P(a(g),ga'(h))\qquad(g,h\in H).$$

Thus the indicated class restricts to zero on H. Corestriction composed with restriction multiplies the class by $[G:H]$. Multiplication by this index is invertible on the p^n-torsion group, proving the result over K. □

Corollary 16.3 (A bound on the joint tests at the original precision)**.** *Fix an actual datum D of Theorem 50.11, a spherical coefficient place $w\mid p$, and one target lattice T. Write $K=E_{x,w}$. Let L_w be the free sublattice spanned by the chosen lifts of the free finite dual tests in Proposition 15.4, with $T'=T^\vee(1)$. For a test $y\in L_w$, use an integral cocycle representing it. For each restricted global trace c at precision π^a, write $c=t\circ a_Y$ as in Lemma 12.14, with a_Y an integral cocycle on Y and $t:Y\to T/\pi^aT$ equivariant.*

Form an $\mathcal{O}$-sublattice $L'_w\subset L_w$ from the elements p^ky for which every such trace admits a path of Theorem 16.2 with $i+j+s\le k$, for the pairing

$$(v,\lambda)\longmapsto\lambda(t(v))\quad\text{in }(\mathcal{O}/\pi^a)(1)$$

and the two cocycles on Y, T'. Use one path for each cyclic $\mathbb{Z}_p$-coordinate of the output if necessary. Any collection of these generators may be used. The normalized length of the local image in the free coordinates of this target is at most

$$\frac{1}{e_{\mathcal{O}}}\ \mathrm{length}_{\mathcal{O}}\left(L_w/(L'_w+\pi^a L_w)\right), \qquad e_{\mathcal{O}}=v_\pi(p).$$

This bound is valid for simultaneous tests and for arbitrary local trace representations.

Proof. The graph construction realizes c at precisely the stated modulus by a map of lattices, with no division of a local length. The displayed pairing is the pairing computing $c\cup \mathrm{red}\, y$, by functoriality of cup products. As a constant $\mathbb{Z}_p$-module $\mathcal{O}/\pi^a$ is a finite direct sum of cyclic p-groups. After twisting by one, each coordinate projection is G_K-equivariant. Apply the theorem to each output coordinate, choosing its order p^n as modulus. Since their joint map is an isomorphism on coefficients, their vanishing gives vanishing of the full cup with $p^k y$. The same holds on the entire $\mathcal{O}$-span of these tests. Hence every local evaluation functional factors through $L_w/(L'_w+\pi^a L_w)$. For a cyclic module $\mathcal{O}/\pi^d$ with $d\le a$, the maps into $\mathcal{O}/\pi^a$ form the ideal $\pi^{a-d}\mathcal{O}/\pi^a$ of length d. The elementary divisor theorem gives the bound after summation and division by $e_{\mathcal{O}}$.

In the period estimate the source of the local maps is the actual joint global image. If any set of local targets is discarded, the kernel of the projection to those retained injects into the product of the discarded coordinate images. Thus $r(D)$ in Theorem 50.11 is bounded by the length on the retained targets plus the sum of the displayed bounds on the others. No field, test precision or interval in that theorem changes. Obtaining the full period bound this way requires quantitative control of these indices for the unrestricted crystalline lattices. □

16.2 Integral lifts of quotient generators at every precision

The two cocycles in Theorem 16.2 define extensions of a trivial lattice. To construct its Frobenius path one needs control of lifts of the trivial quotient inside the integral Breuil–Kisin modules. We bound the power of the uniformizer parameter needed for each such lift. This is the estimate underlying the lift ideals in `New/Congruences/Lifting/CupProductBound.lean`. It applies to both cocycles on the actual period fields, including when those fields are ramified at the coefficient prime.

Let $K/\mathbb{Q}_p$ have residue field k and absolute ramification index e_K. Use the ring $\mathfrak{S} = W(k)[[u]]$ with Witt Frobenius on coefficients and $\phi(u) = u^p$. The Eisenstein polynomial $E(u)$ of a chosen uniformizer is monic of degree e_K and reduces to u^{e_K} modulo p. In particular $u^{e_K} = E(u) - pq(u)$ for some $q \in \mathfrak{S}$.

Lemma 16.4 (A Frobenius bound for an ideal modulo p^ν)**.** *Suppose $I \subseteq \mathfrak{S}$ is an ideal with nonzero reduction in $k[[u]]$ and*

$$E(u)^h I \;\subseteq\; \mathfrak{S}\,\phi(I) \qquad (h \geq 0).$$

For $\nu \geq 1$, let a_ν be the least $a \geq 0$ for which $u^a \in I + p^\nu\mathfrak{S}$. Then a_ν exists and

$$(p-1)a_\nu \leq e_K(h + \nu - 1).$$

If $e_K h < p - 1$, then $I = \mathfrak{S}$ and $a_\nu = 0$ for every ν.

Proof. The reduction of I is (u^m) for some $m \geq 0$. In $\mathfrak{S}/I$ the element u^m is divisible by p. Raising it to the ν-th power proves existence. Work henceforth in $\mathfrak{S}_\nu = \mathfrak{S}/p^\nu$ and write I_ν for the image of I. In this ring $u^{e_K(h+\nu-1)}$ is divisible by E^h. Indeed, in the expansion of $(E - pq)^{h+\nu-1}$ every term with fewer than h factors of E has at least ν factors of p. Consequently

$$u^{a_\nu + e_K(h+\nu-1)} \in E^h I_\nu \subseteq \mathfrak{S}_\nu \phi(I_\nu).$$

Since k is perfect, Witt Frobenius is an automorphism on the coefficient ring. Every series in $\mathfrak{S}_\nu$ therefore has a unique expression $\sum_{i=0}^{p-1} u^i \phi(f_i)$. In that expression the extended ideal $\mathfrak{S}_\nu\phi(I_\nu)$ consists exactly of series with $f_i \in I_\nu$ for all i. Applying this to the displayed monomial shows that $u^{\lfloor (a_\nu + e_K(h+\nu-1))/p \rfloor} \in I_\nu$. Minimality of a_ν gives the asserted inequality. If $e_K h < p - 1$, take $\nu = 1$ to get $a_1 = 0$. Thus I contains an element congruent to one modulo p. It is a unit in the local ring $\mathfrak{S}$. □

Theorem 16.5 (Lifts in a crystalline extension with a trivial quotient)**.** *Let $0 \to Y \to V \to \mathbb{Z}_p \to 0$ be an exact sequence of G_K-lattices with crystalline rational middle term. Suppose the jumps of V, in the convention where the cyclotomic character has jump -1, lie in $[-r, h]$, $r, h \geq 0$. Take the covariant Breuil–Kisin modules after twisting the sequence by $-r$, and choose a generator $\mathbf{t}_r$ of the rank-one module of $\mathbb{Z}_p(-r)$. For the resulting map*

$$\beta : \mathfrak{M}(V(-r)) \longrightarrow \mathfrak{S}\mathbf{t}_r$$

at precision p^ν, the quotient generator multiplied by u^{a_ν} has an integral lift, with $(p-1)a_\nu \leq e_K(h+\nu-1)$. If $e_K h < p-1$, the generator itself has a lift in $\mathfrak{M}(V(-r))$ before any reduction.

Proof. The covariant module of $V(-r)$ has height at most $r+h$. This is the integral Breuil–Kisin correspondence for crystalline lattices: in the convention giving $\mathbb{Z}_p(1)$ Hodge–Tate weight 1, the twist has weights in $[-(r+h), 0]$. The height assertion means that $E^{r+h}\mathfrak{M}(V(-r))$ is contained in the $\mathfrak{S}$-span of its Frobenius image. The quotient line has Frobenius $\phi(\mathbf{t}_r) = (E/c_0)^r \mathbf{t}_r$, where

$c_0 = E(0)/p \in W(k)^\times$. We also use the integral comparison after passage to the étale Frobenius module. It identifies the scalar extension of β to $W(\mathbb{C}_p^\flat)$ with the given quotient map after the same scalar extension. In particular it is surjective there, also modulo p. These standard properties of the covariant functor, including this normalization of the Tate line, are recorded in Sections 2–3 of the cited paper. [13]

Write the image as $I\mathbf{t}_r$. Its reduction modulo p is nonzero by the étale comparison: otherwise the scalar extension modulo p of the map to the line would vanish. Applying β to the height inclusion gives

$$E^{r+h} I \subseteq (E/c_0)^r \mathfrak{S}\phi(I).$$

Cancel E^r in the integral domain $\mathfrak{S}$ and apply Lemma 16.4. The resulting membership $u^{a_\nu} \in I + p^\nu \mathfrak{S}$ is exactly the assertion of a lift at that precision. In the case $h = 0$ the integral surjectivity is also the conclusion about the quotient in the exactness theorem of Čoupek, Gazaki and Marmora. The argument here gives the bound at positive h as well. □

Corollary 16.6 (The two lifts for the period data)**.** *Let $K = E_{x,w}$ be any spherical completion of the marked period data and apply Corollary 16.3. For every scalar target $\mathcal{O}(-d)$, $0 \le d \le B$, the integral crystalline extension attached to a finite test of the dual $\mathcal{O}(d+1)$ has a lift of its quotient generator with $a_\nu = 0$ at every precision, over every K. The same holds for the main dual when $J_\tau \ge -1$ at every embedding of K.*

For the trace side use the extension from Lemma 12.14. At every spherical place it admits the bound of Theorem 16.5 with

$$h = \max\left(\{0, B\} \cup \{J_\tau : \tau \mid w\}\right).$$

In particular, when all $J_\tau < 0$ and $e_K B < p - 1$, the quotient generator on the trace side also lifts integrally for arbitrary ranks, even with arbitrarily large negative jumps.

Proof. For a single integral cocycle on an $\mathcal{O}$-lattice, forget coefficients to $\mathbb{Z}_p$ and take the extension of $\mathbb{Z}_p$ defined by that cocycle. This is the pullback of its $\mathcal{O}$-extension along the inclusion of $\mathbb{Z}_p$ in the trivial quotient. It is crystalline for a finite dual test, by the defining rational finite condition. The trace side extension is rationally split with crystalline constituents. These constructions keep the original cocycles on the two kernel lattices. Under restriction of coefficients the jumps are taken over all labelled embeddings. The jumps on each extension are the union of those on the kernel with zero. For the scalar dual the kernel jump is $-d - 1$; for the main dual its jumps are $-1 - J_\tau$. The respective sign conditions thus give $h = 0$. For the trace side the kernel has weights from the scalar string and the main jump by the graph description. This proves all the claims by the theorem, including the assertion at ramified places.

To relate these lifts precisely to the ambient modules of Theorem 16.2, send u to the Teichmüller lift $[\underline{\varpi}]$ of a compatible tower of roots of the uniformizer in $F = \mathbb{C}_p^\flat$. Choose a basis of $\mathbb{Z}_p(-r)$ in the comparison and write ξ_r for the coefficient of the image of $\mathbf{t}_r$ there. It is a unit in $W(F)$, since it generates the quotient line after this base change. Frobenius gives $\phi(\xi_r) = (E([\underline{\varpi}])/c_0)^r \xi_r$. With $v_F(\underline{\varpi}) = 1/e_K$, reduction modulo p shows $v_F(\bar{\xi}_r) = r/(p-1)$, because $\bar{E}(\underline{\varpi}) = \underline{\varpi}^{e_K}$. Thus division by $[\underline{\varpi}]^{a_\nu}\xi_r$ in $W_\nu(F)$, and removal of the chosen basis of the twist, turn the integral lift into an affine lift over one. The valuation of the reduced divisor is exactly $a_\nu/e_K + r/(p-1)$. This constructs starting vectors for the original precision without any triangularity requirement. The estimates on their pairings with Galois displacements, which occur in the path hypotheses of Theorem 16.2, are additional requirements. □

16.3 Uniform integral lifts from Galois displacements

Retain the notation of Theorem 16.5. We convert a power of u in the image ideal at finite precision to a power of p in the integral image. The result is uniform in the torsion precision and in the rank of the crystalline representation. The conversion uses the Galois action on the Breuil–Kisin module after scalar extension.

Put $F = \mathbb{C}_p^{\flat}$ and $A = W(\mathcal{O}_F)$, and embed $\mathfrak{S}$ in A using $u = [\underline{\varpi}]$. Choose compatible primitive p^n-th roots of unity and write $\epsilon \in \mathcal{O}_F$ for the resulting element. Set $\mu = [\epsilon] - 1$ and $\tau = \phi^{-1}(\mu)$. For a crystalline lattice of effective height let M be its covariant Breuil–Kisin module and $M_A = A \otimes_{\mathfrak{S}} M$. The crystalline comparison supplies the semilinear G_K-action on M_A and gives the divisibility

$$(g-1)M \subseteq u\tau M_A \qquad (g \in G_K).$$

This is the necessity direction of the crystalline descent theorem of Gee and Liu, in its covariant convention, for the chosen tower of uniformizer roots. It also applies to the trivial module and to the rank-one quotient of a twisted extension. [14, Appendix F, (F.10) and Theorem F.11]

Lemma 16.7 (Trading one parameter for a power of the prime)**.** *Let $I\mathbf{t}_r$ be the image of the map β in Theorem 16.5. If $p^c u^m \in I + p^\nu \mathfrak{S}$, where $m \geq 1$, $c \geq 0$ and $\nu \geq 1$, then*

$$p^{c+1+v_p(m)} u^{m-1} \in I + p^\nu \mathfrak{S}.$$

Proof. For $f \in I$ lift $f\mathbf{t}_r$ to M. Put $g(\mathbf{t}_r) = b_g \mathbf{t}_r$. The factor b_g is a unit of A, congruent to one modulo $u\tau A$ by crystalline descent. Equivariance of β and the same divisibility for M give

$$b_g g(f) - f \in u\tau I A.$$

In particular $g(f) \in IA$, so subtraction gives $(g(f) - f)/(u\tau) \in IA$. For $t \in \mathfrak{S}$, descent for the trivial module gives $(g(t) - t)/(u\tau) \in A$. Applying the two statements to $f + p^\nu t$ proves that the divided displacement of every element of $I + p^\nu \mathfrak{S}$ belongs to $(I + p^\nu \mathfrak{S})A$.

Choose $g \in G_K$ with $g(u) = [\epsilon]u$. Indeed $X^{p^n} - \varpi$ is Eisenstein for every n, and the embeddings sending the chosen root to its product with ζ_{p^n} are compatible. Their union extends to the algebraic closure over K. Applying the divided displacement to the monomial in the hypothesis gives

$$p^c u^{m-1} \frac{[\epsilon]^m - 1}{\tau} \in (I + p^\nu \mathfrak{S})A.$$

To compute the factor modulo u, write $m = p^v m_0$ with $p \nmid m_0$. The geometric sum for m_0 evaluated at $[\epsilon]^{p^v}$ is a unit of A, since its residue in the residue field of A is m_0. The remaining factor $([\epsilon]^{p^v} - 1)/\tau$ equals $\prod_{j=0}^{v} \phi^j(\mu/\tau)$. Both μ/τ and $E(u)$ are generators of the kernel of Fontaine's map $\theta : A \to \mathcal{O}_{\mathbb{C}_p}$. Thus each factor is a unit times $\phi^j(E(u))$, which is congruent modulo u to $p\phi^j(c_0)$ for $c_0 = E(0)/p$. It follows that the entire factor is p^{v+1} times a unit, modulo uA. The error after multiplication by $p^c u^{m-1}$ is in $p^c u^m A$, already in the extended ideal. This gives the required membership over A.

It descends modulo p^ν to $\mathfrak{S}$. Indeed A/p^ν is faithfully flat over $\mathfrak{S}/p^\nu$. Modulo p this is the local injection of the discrete valuation ring $k[[u]]$ into the valuation ring $\mathcal{O}_F$ taking u to an element of positive valuation. It is flat by torsion freeness and faithful since the maximal ideal stays proper. For higher ν use the flatness criterion for a nilpotent ideal with the ideal generated by p. Its required first Tor vanishes by the resolution with alternating differentials p and $p^{\nu-1}$, since A is p-torsion free. Faithfulness follows from the same reduction modulo p. □

Theorem 16.8 (A lift independent of the precision)**.** *Write $e = e_K$ and keep the upper jump bound h of Theorem 16.5. If $eh < p-1$ take $C = 0$. Otherwise suppose $e \le p-2$ and take any nonnegative integer C satisfying*

$$C\big((p-1)^2 - pe\big) \;\ge\; peh.$$

Then $(p,u)^C \subseteq I$. In particular the constant $p^C\mathbf{t}_r$ has an integral lift, and so does $p^s E(u)^t\mathbf{t}_r$ whenever $s+t \ge C$. At an unramified place with $p \ge 5$ one may always take $C = \lceil h/(p-3)\rceil$.

Proof. The $C = 0$ case is Lemma 16.4. In the second case the coefficient of C is positive. Work modulo p^{C+1} and apply that lemma to find u^a in the reduced image with $(p-1)a \le e(h+C)$. Legendre's formula gives $v_p(a!) \le a/(p-1)$. Hence

$$a + v_p(a!) \le \frac{pe(h+C)}{(p-1)^2} \le C.$$

For $m < a$, apply Lemma 16.7 at $a, a-1, \dots, m+1$. The total exponent paid is $a-m+v_p(a!)-v_p(m!) \le C-m$. For $a \le m \le C$, simply multiply u^a by u^{m-a}. Thus $p^{C-m}u^m \in I + p^{C+1}\mathfrak{S}$ for every $0 \le m \le C$.

At $m = 0$ this means $p^C(1-pb) \in I$ for some $b \in \mathfrak{S}$. The factor in parentheses is a unit. Thus $p^C \in I$, which absorbs the modulus in all the other memberships as well. These monomials generate $(p,u)^C$. Since $E \in (p,u)$, the stated products lie in the image. For $e = 1$ one has $(p-1)^2 - p = p(p-3)+1$, proving the last assertion. □

Corollary 16.9 (Uniform lifts on the marked period fields)**.** *At each spherical place of the data in Theorem 50.11 to which the preceding theorem applies, the graph extension of every restricted trace admits an integral lift of $p^C\mathbf{t}_r$ with $h = \max(0, B, \max_{\tau|w} J_\tau)$. For a scalar dual test take $C = 0$ at every spherical place, including the places of larger ramification. For the main dual use $h = \max(0, \max_{\tau|w}(-1 - J_\tau))$ in the theorem. These choices are independent of the test precision and the number of trace matrices.*

Proof. These are the two crystalline integral extensions of Corollary 16.6; their quotient is the trivial lattice before twisting. It suffices to apply the theorem to each one, using the labelled jumps computed in that corollary. The resulting lift can be reduced to every precision in the same period datum. □

16.4 Uniform integral lifts over ramified fields

For large absolute ramification index, passing repeatedly through Frobenius decreases the parameter exponent before we convert it to a power of p. The cost of a passage depends on the different of the local field and the upper Hodge jump. We keep the same period field, uniformizer tower and coefficient prime as in Theorem 16.5.

Write $e = e_K$, $d_K = \lceil v_p(E'(\varpi)) \rceil$, and $I_\nu = I + p^\nu \mathfrak{S}$, where I is the ideal of that theorem. In particular $d_K = 0$ if $e = 1$, and $d_K \le 1 + v_p(e)$ otherwise. Indeed the nonzero terms in the derivative of an Eisenstein polynomial, evaluated at ϖ, have valuations with distinct fractional parts $(i-1)/e$, $1 \le i \le e$. Their minimum is at most the valuation $v_p(e) + (e-1)/e$ of the leading term.

Lemma 16.10 (One passage through Frobenius)**.** *Let h be the upper jump bound from Theorem 16.5. For $\nu \ge 1$ one has*

$$p^c u^a \in I_\nu \quad \Longrightarrow \quad p^{c + d_K h + v_p(h!)} u^{\lfloor (a+h)/p \rfloor} \in I_\nu.$$

Proof. Let $J = \mathfrak{S}\phi(I)$ and $J_\nu = J + p^\nu \mathfrak{S}$. We first prove an operation in J_ν. Choose $g(u) = qu$ with $q = [\epsilon]$ as in Lemma 16.7, and retain A, μ, τ from that lemma. Then for $f \in J_\nu$ one has

$$\frac{g(f) - f}{u\mu} \in J_\nu A.$$

Indeed J is the image ideal on the $\mathfrak{S}$-span of the Frobenius transforms of the integral modules, using $\phi(\mathbf{t}_r)$ as the generator on the quotient. For a generator $\phi(m)$, the displacement is $\phi((g-1)m)$, divisible by $u^p\mu$ in the A-span of these Frobenius transforms. For a coefficient in $\mathfrak{S}$ the displacement is divisible by $u\mu$, since g fixes $W(k)$ and $g(u) - u = u\mu$. For a power series this follows term by term, with its divided series converging in the Witt coordinate topology. The induced action on the quotient generator differs from one by a multiple of $u\mu$. The lift and subtraction used in Lemma 16.7 now prove the inclusion, including the summand $p^\nu \mathfrak{S}$.

The element $g(E) = \omega E$ has $\omega \in A^\times$, because both elements generate the kernel of θ. Write $E = \sum_i c_i u^i$ and $[i]_q = 1 + \cdots + q^{i-1}$ for $i \ge 1$. If $\eta = \mu/(E\tau) \in A^\times$, then

$$D = \frac{\omega - 1}{u\tau} = \eta \sum_{i \ge 1} c_i [i]_q u^{i-1}, \qquad \theta(D) = \theta(\eta) E'(\varpi).$$

The last identity uses $\theta(q) = 1$. Since $\theta(\tau) = \zeta_p - 1$, we have

$$v_p(\theta(\omega) - 1) = 1/e + 1/(p-1) + v_p(E'(\varpi)) > 1/(p-1).$$

For any $n \ge 1$ set $D_n = (\omega^n - 1)/(u\tau)$. At that depth the p-th power adds one to the valuation of a principal unit minus one, by the binomial formula. Raising to an exponent prime to p preserves the valuation. Applying these facts to $\theta(\omega)$ gives

$$v_p(\theta(D_n)) = v_p(E'(\varpi)) + v_p(n).$$

Thus some multiple of D_n in A is congruent to $p^{d_K + v_p(n)}$ modulo E: the required multiplier modulo E is integral in $\mathcal{O}_{\mathbb{C}_p}$, and $\theta : A \to \mathcal{O}_{\mathbb{C}_p}$ is onto.

Suppose $f = p^c u^a E^n \in J_\nu$, with $n \ge 1$. Multiplying its divided displacement by u gives the identity

$$\frac{g(f) - f}{\mu} = p^c u^{a+1} E^{n-1} \frac{E\tau}{\mu} D_n + f\, \omega^n \frac{q^a - 1}{\mu}.$$

The second term belongs to $J_\nu A$, as does the left side. Multiply the first term by a unit and use the congruence for D_n. Its error is in ufA. Hence $p^{c+d_K+v_p(n)}u^{a+1}E^{n-1} \in J_\nu A$. Faithful flatness modulo p^ν, proved in Lemma 16.7, returns this membership to $\mathfrak{S}$.

Now $p^c u^a E^h \in J_\nu$ by the height inclusion $E^h I \subset J$. Apply the operation h times. It gives $p^{c+d_K h+v_p(h!)}u^{a+h} \in J_\nu$. Decomposing the series modulo p^ν according to the exponent of u modulo p, as in Lemma 16.4, returns the assertion in I_ν. □

Theorem 16.11 (Integral quotient lifts at every ramification index)**.** *Keep the crystalline extension and the ideal I of Theorem 16.5, with arbitrary $K/\mathbb{Q}_p$. For $J \geq 0$ put*

$$\alpha_J = \frac{pe}{(p-1)^2 p^J}, \qquad b_J = J\big(d_K h + v_p(h!)\big).$$

Choose any J with $\alpha_J < 1$. The ideal I contains p^C, independently of the torsion precision, for every integer C satisfying

$$(1-\alpha_J)C \;\geq\; b_J + \left(\alpha_J + \frac{p(1-p^{-J})}{(p-1)^2}\right) h.$$

More generally $p^{b_J}(p,u)^{C-b_J} \subset I$. Denote the least nonnegative C allowed by these choices by $C_K(h)$, using $C_K(h) = 0$ when $eh < p-1$.

Proof. Work at precision $\nu = C+1$. Lemma 16.4 starts with $u^{a_0} \in I_\nu$ and $(p-1)a_0 \leq e(h+C)$. Successive applications of Lemma 16.10 give $p^{b_J}u^{a_J} \in I_\nu$, where $a_{j+1} = \lfloor (a_j+h)/p \rfloor$. Summing the geometric progression yields

$$a_J \leq p^{-J}\frac{e(h+C)}{p-1} + (1-p^{-J})\frac{h}{p-1}.$$

Multiplying by $p/(p-1)$ bounds $a_J + v_p(a_J!)$. The assumed inequality on C is precisely sufficient for $b_J + a_J + v_p(a_J!) \leq C$. Applying Lemma 16.7 to remove each remaining parameter factor now gives $p^C \in I_\nu$. As in Theorem 16.8 this gives $p^C \in I$. For $m < a_J$ one may stop the removals at m, obtaining $p^{C-m}u^m \in I_\nu$. For $a_J \leq m \leq C - b_J$ this membership follows by multiplying $p^{b_J}u^{a_J}$ itself. The modulus is absorbed by $p^C \in I$, proving the ideal inclusion. Such J exist for every e, including when $p = 2$. □

Corollary 16.12 (The three integral extensions used by a cup test)**.** *At any spherical coefficient place of the marked period data, let Y, Y' be the graph and dual-test kernel lattices of Corollary 16.3, considered over $\mathbb{Z}_p$. Write X, X' for the extensions of $\mathbb{Z}_p$ defined by their two integral cocycles. For a cyclic coordinate of the finite output of order p^n, write $P : Y \otimes Y' \to (\mathbb{Z}/p^n)(1)$ for its equivariant pairing. There is a third crystalline extension X'' of $\mathbb{Z}_p$ with kernel $Z = (Y \otimes Y')^\vee(1)$ and an integral equivariant map $X'' \to Z$ whose reduction modulo p^n is P times projection onto the trivial quotient.*

For each of X, X', X'', take h to be the maximum of zero and its kernel jumps, over the coefficient embeddings at that place. The corresponding power $p^{C_K(h)}$ has an integral quotient lift in its normalized Kisin module, at every spherical place and for every value of n.

Proof. Choose a $\mathbb{Z}_p$-linear lift $\psi \in Z$ of P. The cocycle $b(g) = (g\psi - \psi)/p^n$ is continuous and integral because P is equivariant. Its class is rationally zero. On $Z \oplus \mathbb{Z}_p$ define $g(z,t) = (gz + tb(g), t)$. The map $(z,t) \mapsto p^n z + t\psi$ has the asserted equivariance and reduction. The rational kernel is crystalline by tensor products and Tate duality, and the extension is rationally split. Its kernel jumps are the sums $-1-j-j'$ for the jumps j, j' in the tensor product, at

the same embedding. Adding zero gives all the jumps on each extension. Choose the twist by $-r$ in Theorem 16.5 separately on the three extensions, then apply Theorem 16.11. Dividing by the common rank-one Tate generator restores the untwisted normalization and projection to the stated power of p. All three constructions use the original K and the original cyclic output. □

16.5 A local cup bound from two adjacent Frobenius lattices

Let $K/\mathbb{Q}_p$ be finite. We use the rings $F, A, \mathfrak{S}$ and elements u, E, μ, τ of Lemma 16.7. For a crystalline $\mathbb{Z}_p$-lattice V write $N(V)$ for its covariant Breuil–Kisin module in $W(F) \otimes_{\mathbb{Z}_p} V$. Concretely, twist by $-r$ until the jumps are nonnegative, then tensor the effective module with the inverse of the rank-one module of $\mathbb{Z}_p(-r)$. In the comparison this divides by the generator of the same Tate line on both sides. Put

$$L(V) = \mathfrak{S}\,\phi N(V), \qquad N_A(V) = AN(V), \qquad L_A(V) = AL(V).$$

Here ϕ acts on the Witt vector and fixes V. These are two lattices in a common ambient module. Changing r preserves the normalized module by compatibility with Tate twists. More generally $V \mapsto N(V)$ respects tensor products, the unit and integral equivariant maps. One description is to take the inverse of the tensor equivalence between crystalline lattices and prismatic F-crystals, then evaluate on the Breuil–Kisin prism. Both operations respect tensor products. The covariant comparison identifies the result with the modules just described. [15, Theorems 5.6 and 7.9] [13, Sections 2–3]

Lemma 16.13 (The two lattices, their displacements and the Tate line)**.** *If the jumps of V lie in $[-r, h]$, then*

$$E^h N(V) \;\subseteq\; L(V) \;\subseteq\; E^{-r} N(V).$$

Both A-lattices are stable under G_K. For every $g \in G_K$ the displacements of the indicated $\mathfrak{S}$-lattices satisfy

$$(g-1)N(V) \subseteq u\tau N_A(V), \qquad (g-1)L(V) \subseteq u\mu L_A(V).$$

In the Witt comparison the two A-lattices of $\mathbb{Z}_p(1)$ are $\tau^{-1}A(1)$ and $\mu^{-1}A(1)$, respectively. For an exact sequence $0 \to Y \to X \to \mathbb{Z}_p \to 0$ of crystalline lattices, the kernel of projection on each A-lattice of X is the corresponding A-lattice of Y.

Proof. Before untwisting, the span of Frobenius is contained in the effective module and contains its multiple by E^{r+h}. Untwisting divides Frobenius on the line by $(E/c_0)^r$; this gives both inclusions. For an effective module the first displacement assertion is the crystalline Galois descent formula used in Lemma 16.7. It also gives stability after extending scalars to A. On the Tate generator the ratio of its translate to itself is a unit congruent to one modulo $u\tau$. Its inverse has the same property, so the assertion passes to the untwisted module. [14, Appendix F, (F.10) and Theorem F.11]

For an element $\phi(m)$ the first displacement bound gives $\phi(u\tau) = u^p\mu$ at the second lattice. For a coefficient $a \in \mathfrak{S}$, one has $(g-1)a \in u\mu A$. Indeed g fixes $W(k)$ and $g(u) = [\epsilon^c]u$ for some $c \in \mathbb{Z}_p$. The difference $[\epsilon^c] - 1$ is divisible by μ. To see this, expand $(1+\mu)^c$ by the binomial series, first for nonnegative integers and then by continuity for $c \in \mathbb{Z}_p$. The series converges in the Witt coordinate topology since μ reduces to zero in the residue Witt vectors. Subtracting the two power series $a(g(u)), a(u)$ now gives the claim, with convergence in the same topology. Applying the product rule to sums of $a\phi(m)$ gives the second displacement bound. Stability also follows by commuting the Galois action with ϕ.

For the Tate line let ξ be the coefficient of a generator of $N(\mathbb{Z}_p(-1))$ in the basis of the comparison lattice. Then $\phi(\xi)/\xi = E/c_0$ and $\phi(\tau)/\tau = \mu/\tau$. The ratio v of these two elements belongs to $A^\times$, since E and μ/τ generate $\ker\theta$. There is $b \in A^\times$ with $\phi(b)/b = v$. Modulo p choose a $(p-1)$-st root of $\bar{v}$ in $\mathcal{O}_F^\times$. Given a solution modulo p^m, correcting it by $1 + p^m t$ requires solving $\bar{t}^p - \bar{t} = a$ with $a \in \mathcal{O}_F$. A root exists in F by algebraic closure and has nonnegative valuation by the equation. Completeness gives the required unit. Now $\xi/(b\tau)$ is

fixed by ϕ and is a unit in $W(F)$, so it belongs to $\mathbb{Z}_p^\times$. Taking the inverse Tate line gives $N_A(\mathbb{Z}_p(1)) = \tau^{-1}A(1)$; applying ϕ gives the formula for L_A.

The covariant Kisin functor preserves kernels on these sequences. This is Proposition 3.4 of the exactness paper, applied after a common effective twist. The map $\mathfrak{S} \to A$ is faithfully flat. Indeed modulo p it is the faithfully flat map from $k[[u]]$ to the valuation ring $\mathcal{O}_F$, and the p-adic flatness criterion applies to these p-complete, p-torsion-free rings with noetherian source. Thus the kernel identity holds for N_A. Applying the automorphism ϕ of A gives the identity for L_A. [13, Sections 2.1 and 3, Proposition 3.4] □

Proposition 16.14 (A quotient lift in the intersection)**.** *Let $\beta : X \to \mathbb{Z}_p$ be a surjection of crystalline lattices, h the largest of zero and the jumps of X, and $d_K, C_K(h)$ as in Theorem 16.11. Set*

$$Q_K(X) = \begin{cases} C_K(h), & \text{if all jumps are nonnegative or all are nonpositive,} \\ C_K(h) + d_K h + v_p(h!), & \text{otherwise.} \end{cases}$$

The intersection $N(X) \cap L(X)$ contains a vector over $p^{Q_K(X)}$.

Proof. By Theorem 16.11 choose $x_0 \in N(X)$ over p^C, with $C = C_K(h)$. If all jumps are nonpositive, $N(X) \subseteq L(X)$ by the lemma; if all are nonnegative, $L(X) \subseteq N(X)$ and $\phi(x_0)$ is in both lattices over the same constant.

For arbitrary jumps put $M = N(X)$, $L = L(X)$ and $H_n = E^n M \cap L$ for $0 \le n \le h$. Flatness identifies AH_n with $E^n M_A \cap L_A$, by taking the kernel of subtraction in $E^{-r}M$. We claim that whenever $n \ge 1$ and $p^c E^n \in \beta(H_n)$, one has

$$p^{c+d_K+v_p(n)} E^{n-1} \in \beta(H_{n-1}).$$

Choose $y = E^n m \in H_n$ lifting $p^c E^n$. Use the element g of Lemma 16.10, with $\omega = g(E)/E$ and $D_n = (\omega^n - 1)/(u\tau)$. The vector $(gy - y)/(u\mu)$ belongs to L_A by the displacement assertion for L. It also belongs to $E^{n-1}M_A$, since it equals

$$E^{n-1}\frac{E\tau}{\mu}\left(\omega^n \frac{gm - m}{u\tau} + D_n m\right).$$

Its projection is $p^c E^{n-1}(E\tau/\mu)D_n$. The factor $E\tau/\mu$ is a unit. By the calculation of $\theta(D_n)$ in Lemma 16.10, some multiple of D_n equals $p^{d_K+v_p(n)}$ modulo E. Consequently $p^{c+d_K+v_p(n)}E^{n-1}$ belongs to the extended image ideal $\beta(H_{n-1})A$. Indeed the error is a multiple of $p^c E^n = \beta(y)$, already in that ideal. Faithful flatness contracts the ideal to $\beta(H_{n-1})$. This proves the claim.

The height inclusion puts $E^h x_0$ in H_h over $p^C E^h$. Apply the claim for $n = h, h-1, \dots, 1$. The resulting exponent is $C + d_K h + v_p(h!)$, and $H_0 = M \cap L$. □

Theorem 16.15 (A precision-independent bound for crystalline cup tests)**.** *Let a, a' be integral cocycles on $\mathbb{Z}_p$-lattices Y, Y', whose extensions X, X' of $\mathbb{Z}_p$ are crystalline. Let $P : Y \otimes Y' \to (\mathbb{Z}/p^n)(1)$ be any equivariant pairing. Put $Z = (Y \otimes Y')^\vee(1)$, and let h_Z be the maximum of zero and its jumps. Then*

$$p^{Q_K(X)+Q_K(X')+C_K(h_Z)}[P(a \cup a')] = 0 \quad \text{in } H^2(K, (\mathbb{Z}/p^n)(1)).$$

The bound holds for every $K/\mathbb{Q}_p$, every $n \ge 1$, and arbitrary ranks.

Proof. Write $i = Q_K(X),\ j = Q_K(X'),\ s = C_K(h_Z)$. Construct the third extension X'' and its map to Z as in Corollary 16.12. That construction uses just the lattices and P, so it applies to the data here. Evaluation gives an integral equivariant map

$$X'' \otimes Y \otimes Y' \longrightarrow \mathbb{Z}_p(1)$$

whose reduction modulo p^n is projection on the first quotient followed by multiplication by P. Take $z \in N(X'')$ over p^s. Tensor compatibility sends $z, N_A(Y), N_A(Y')$ to $N_A(\mathbb{Z}_p(1))$. Applying ϕ, it sends $\phi z, L_A(Y), L_A(Y')$ to $L_A(\mathbb{Z}_p(1))$. In both cases reduce the integral identity in $W_n(F)$. The scaled pairing $B = p^s P$ on the reductions of the two kernel lattices takes values in the image of $\tau^{-1}A(1)$ at the first level and of $\mu^{-1}A(1)$ at the second. These conclusions use integral representatives of vectors in each level before reducing the pairing.

Choose vectors

$$y \in N(X) \cap L(X), \quad y' \in N(X') \cap L(X')$$

over p^i, p^j by Proposition 16.14. Set $x = \phi^{-1}y,\ x' = y'$ in the Witt comparison. Both belong to their respective first A-lattices, because $\phi^{-1}L_A(X) = N_A(X)$. Under fixed integral splittings of the two extensions they specify the affine points in Theorem 16.2 after reduction. For example $gx - x$ in the extension has kernel component $p^i a(g) + gx_{\mathrm{ker}} - x_{\mathrm{ker}}$. We denote these displacements by d_x and $d'_{x'}$.

Since $y \in L(X)$, its displacement lies in $u\mu L_A(Y)$. Indeed it belongs to the indicated multiple of $L_A(X)$ by Lemma 16.13; its quotient coordinate vanishes, and division by $u\mu$ still gives zero on the quotient since A is a domain. Applying ϕ^{-1} gives $d_x(g) \in \phi^{-1}(u)\tau N_A(Y)$. Likewise $d'_{x'}(h) \in u\tau N_A(Y')$. Both assertions hold for all $g, h \in G_K$. Stability of $N_A(Y')$ then implies that $B(d_x(g), g d'_{x'}(h))$ lies in the reduction of $\phi^{-1}(u)A(1)$. This is contained in $S_n(1)$: multiplication by the Teichmüller vector of any element of positive valuation sends integral Witt vectors to zero in the residue Witt vectors.

Use the path

$$(x, x') \ \longrightarrow\ (\phi x, x') \ \longrightarrow\ (\phi x, \phi x').$$

In the first move $\phi x - x = y - x \in N_A(Y)$, since both endpoints belong to $N_A(X)$ over the same quotient. Pairing this difference with $d'_{x'}(g)$ thus gives an element of the reduction of $uA(1)$. In the second move $\phi x' - x' \in L_A(Y')$, since $x' = y' \in L(X')$ and $x' \in N(X')$. Its g-translate also lies in $L_A(Y')$. The displacement $d_{\phi x}(g)$ lies in $u\mu L_A(Y)$ as shown above. Their pairing is again in the reduction of $uA(1)$. These two subgroups lie in $S_n(1)$. The cochain argument of Theorem 16.2, with $H = G_K$, gives the stated annihilator over the original field. □

Corollary 16.16 (A local contribution to the original joint image)**.** *For the data D of Theorem 50.11, choose any set $\mathcal{W}$ of pairs consisting of a spherical coefficient place $w \mid p$ and a target T. Let $r_{\mathrm{rest}}(D)$ be the normalized length on the retained coordinates, with the original global source. Then*

$$r(D) \ \le\ r_{\mathrm{rest}}(D) + \sum_{(w,T)\in\mathcal{W}} [E_{x,w} : \mathbb{Q}_p]\,\big(Q_K(X) + Q_K(X') + C_K(h_Z)\big)_{w,T}, \qquad K = E_{x,w}.$$

In each summand the exponent may be bounded using the labelled weights of the local trace family and the dual target. The same bound works at every trace precision.

Proof. At a fixed w, T, use the graph of Lemma 12.14 for each restricted trace and the integral extension of its finite dual test. The jumps of their extensions are the jumps of their kernels together with zero. For a nonempty fixed trace family the rational graph kernel is the same direct sum for all the cocycles, and for a fixed target the rational dual kernel is fixed as well.

Thus the exponents just proved work for every test and trace simultaneously. For an empty kernel the cup vanishes. The conclusion applies to every cyclic $\mathbb{Z}_p$-coordinate of the output with its original modulus. Hence it applies to the full $(\mathcal{O}/\pi^a)(1)$-valued evaluation.

In a single free output coordinate the subgroup annihilated by p^k has $\mathcal{O}$-length at most $kv_\pi(p)$. Each target uses at most $[K:\mathbb{Q}_p]$ such coordinates. As in Corollary 16.3, the kernel of projection of the joint image injects into the sum of the discarded coordinate images. Dividing their lengths by $v_\pi(p)$ proves the estimate. To sum it over primes and nodes requires control of the sizes of the exponents relative to $\sqrt{b_x}$. The local annihilator alone imposes no such restriction on the largest Hodge jumps. □

16.6 Integral lifts along an unramified Frobenius orbit

Suppose that $K/\mathbb{Q}_p$ is unramified of degree f, and that $\mathcal{O}$ is the ring of integers in a coefficient field containing all images of K. Use the uniformizer p of K, so $E(u) = u - p$. Put

$$R = \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}, \qquad R_0 = W(k_K) \otimes_{\mathbb{Z}_p} \mathcal{O}.$$

The factors of $R_0 = \prod_i \mathcal{O}$ are indexed by the embeddings $i : K \hookrightarrow \mathrm{Frac}(\mathcal{O})$; write $\mathbf{e}_i$ for the idempotents. Thus $R_i = \mathbf{e}_i R = \mathcal{O}[[u]]$. Let Fi be the index of $\phi(\mathbf{e}_i)$. This permutation is a single cycle of length f; on the power series coefficients in $\mathcal{O}$, Frobenius acts as the identity.

Let X be a crystalline $\mathcal{O}$-lattice with an equivariant surjection $\beta : X \to \mathcal{O}$ to the trivial representation. Use the normalized module of Lemma 16.13 with the action of $\mathcal{O}$ on it induced by its action on X. Put $M_i = \mathbf{e}_i N(X)$ and $L_i = R_i \phi(M_{F^{-1}i})$. At i, let h_i and r_i be the least nonnegative integers such that all labelled jumps belong to $[-r_i, h_i]$.

Lemma 16.17 (The height at one embedding)**.** *Both M_i, L_i are free over R_i, and*

$$E^{h_i} M_i \;\subseteq\; L_i \;\subseteq\; E^{-r_i} M_i.$$

Proof. Write π for a uniformizer of $\mathcal{O}$, with $p\mathcal{O} = \pi^{e_{\mathcal{O}}}\mathcal{O}$. The underlying $N(X)$ is free over $\mathfrak{S}$. Multiplication by π on it is injective since a unit times its $e_{\mathcal{O}}$-th power acts as p. The module $M_i/\pi M_i$ embeds into $N(X)/pN(X)$ by multiplication by $\pi^{e_{\mathcal{O}}-1}$. It has no u-torsion and is finite over the discrete valuation ring $(\mathcal{O}/\pi)[[u]]$. Lift a basis of this free module to M_i. Nakayama's lemma gives generators, and any relation has all coefficients divisible by π. Division by π and iteration show that every coefficient is zero. This proves freeness. Frobenius base change of a basis gives a basis for L_i, by linear independence in the Witt comparison.

Choose a single twist making all jumps nonnegative. For the resulting effective module, the filtration on the linearized Frobenius image modulo E, after inverting p, is the de Rham filtration. Its n-th piece is the image of the intersection of that Frobenius image with E^n times the original module. This is the Hodge comparison for crystalline Breuil–Kisin modules with coefficients, in the convention giving the cyclotomic character jump -1. [16, Section 10.3, Theorem 10.5 and Remark 10.6] Over $\mathrm{Frac}(\mathcal{O})[[u-p]]$ the elementary divisor E^j contributes a line in degree j to this filtration. Thus on the i-component these exponents are exactly its labelled jumps. Removing the common twist subtracts its jump from every exponent.

The unlabelled height bounds in Lemma 16.13 put both transition matrices between the two bases in $R_i[1/E]$. If $a \in R_i$ is divisible by E^m in $\mathrm{Frac}(\mathcal{O})[[u-p]]$, it is already divisible in R_i. Indeed the substitution $u = (u-p)+p$ identifies R_i with $\mathcal{O}[[u-p]]$, and the first m coefficients vanish in $\mathcal{O}$. Apply this to the entries of the transition matrices after clearing denominators. The two labelled inclusions over the discrete valuation ring therefore hold over R_i. □

Theorem 16.18 (Frobenius-weighted upper jumps)**.** *For $p \geq 3$ put*

$$H_i = \sum_{j=1}^{\infty} \frac{h_{F^j i}}{p^j}, \qquad c_i = \begin{cases} 0, & H_i < 1, \\ \left\lceil \dfrac{p(p-1)}{(p-1)^2 - p} H_i \right\rceil, & H_i \geq 1. \end{cases}$$

The image of M_i contains $p^{c_i}\mathbf{e}_i$. The image of $M_i \cap L_i$ contains $p^{q_i}\mathbf{e}_i$, where one may take $q_i = c_i + v_p(h_i!)$. If $r_i = 0$ one may lower q_i to the minimum of this integer and $c_{F^{-1}i}$.

Proof. Omit the factor $\mathbf{e}_i$ from the notation for scalars in component i. Write $I_i = \beta(M_i) \subset R_i$. Its image modulo π is nonzero. In fact β is surjective after passing to the Witt comparison, and stays so on each component after reduction modulo π. The corresponding component

of the quotient there is nonzero since the two copies of the residue field of K split over the algebraically closed tilt. In the discrete valuation ring R_i/π, the image ideal therefore contains a power of u. Raising an expression $u^a \in I_i + (\pi)$ to the $e_{\mathcal{O}}\nu$-th power gives a pure power of u in $I_i + (p^\nu)$ for every $\nu \geq 1$.

Let a_i be the least exponent of such a power at precision p^ν. The height inclusion on component Fi, after projection, is

$$E^{h_{Fi}} I_{Fi} \subseteq R_{Fi}\phi(I_i).$$

Modulo p^ν, $u^{h_{Fi}+\nu-1}$ is divisible by $E^{h_{Fi}}$, by expanding $u = E + p$. Thus the ideal on the right contains $u^{a_{Fi}+h_{Fi}+\nu-1}$ at this precision. Decompose its power series by their exponents modulo p, as in Lemma 16.4. The coefficient map between these two copies of $\mathcal{O}$ is the identity. It follows that

$$p a_i \;\leq\; a_{Fi} + h_{Fi} + \nu - 1.$$

Iterating the inequality and using periodicity gives $a_i \leq H_i + (\nu - 1)/(p - 1)$.

The operation of Lemma 16.7 works in each $I_i + (p^\nu)$. To verify this, G_K fixes the idempotents and the coefficients $\mathcal{O}$, so a lift in M_i has displacement in $u\tau A M_i$. For the chosen $g(u) = [\epsilon]u$ the divided displacement of $p^c u^m$ therefore gives the same scalar $([\epsilon]^m - 1)/\tau$ as in that lemma. Its congruence modulo u trades one u for $p^{1+v_p(m)}$. The extension $\mathfrak{S} \to A$ stays faithfully flat after tensoring with $\mathcal{O}$ and on each idempotent, so the ideal membership descends to $R_i/(p^\nu)$.

If $H_i < 1$, take $\nu = 1$. Then $a_i = 0$ and $I_i = R_i$. Otherwise take $\nu = c_i + 1$. Our choice of c_i ensures that

$$a_i + v_p(a_i!) \leq \frac{p}{p-1}\left(H_i + \frac{c_i}{p-1}\right) \leq c_i.$$

Remove the a_i parameter factors. This gives $p^{c_i} \in I_i + (p^{c_i+1})$, hence $p^{c_i} \in I_i$ by division by a unit congruent to one modulo p.

For the intersection, choose $x_i \in M_i$ over p^{c_i}. Then $E^{h_i} x_i \in L_i$. Apply the divided-displacement argument of Proposition 16.14 to $E^n M_i \cap L_i$, for $n = h_i, \ldots, 1$. All its operators preserve the label, since the chosen Galois element fixes K. In the scalar congruence for D_n one now has $d_K = 0$ because $E' = 1$. Thus the successive costs are $v_p(n)$. This proves the bound for q_i. When $r_i = 0$, $L_i \subset M_i$, and $\phi(x_{F^{-1}i})$ is a common lift over $p^{c_{F^{-1}i}}$, proving the improvement. □

Corollary 16.19 (Length of the missing constants)**.** *Let $U = R_0 \cap \beta(N(X) \cap L(X))$. Then*

$$\frac{\operatorname{length}_{\mathcal{O}}(R_0/U)}{v_\pi(p)} \leq \sum_i q_i \leq \frac{8}{p-1}\sum_i h_i.$$

For the first lattice alone, $R_0 \cap \beta(N(X))$, one may replace 8 by 7. If the next m labels after i have upper jumps at most B, and every upper jump is at most H, the entire i-component of the constant lattice lifts into M_i whenever

$$B(1 - p^{-m}) + H p^{-m} < p - 1.$$

Proof. Each component of U contains $p^{q_i}\mathcal{O}$; the corresponding normalized length is at most q_i. For $p \geq 3$ one has $p(p-1)/((p-1)^2 - p) \leq 6$. On $H_i \geq 1$ rounding up thus gives $c_i \leq 7H_i$; on the other components $c_i = 0$. Since F permutes the indices, $\sum_i H_i = (p-1)^{-1}\sum_i h_i$. The estimate $v_p(h_i!) \leq h_i/(p-1)$ proves the intersection bound. Splitting the geometric series for H_i after m terms gives the asserted sufficient condition for $H_i < 1$. □

Corollary 16.20 (The labelled lifts of a period pairing)**.** *At an unramified spherical completion $K = E_{x,w}$ with $p \geq 3$, use the original $\mathcal{O}$-linear graph map $t : Y \to T/\pi^a T$ and the test on $Y' = T^\vee(1)$, with the dual taken over $\mathcal{O}$. After a coefficient extension containing K if necessary, the preceding theorem and corollary apply separately to the two extensions of $\mathcal{O}$ and to a third extension with kernel $Z = \mathrm{Hom}_{\mathcal{O}}(Y, T)$. At each label i, the jumps of this last kernel are $j_T(i) - j$ for the jumps j of Y at that same label.*

Proof. Use the original graph cocycle before restricting scalars. The integral dual-test cocycle likewise defines an extension of $\mathcal{O}$, crystalline by the finite condition. Lift t to an $\mathcal{O}$-linear map $\psi : Y \to T$, which is possible since Y is free. The integral cocycle $(g\psi - \psi)/\pi^a$ defines the required rationally split extension of $\mathcal{O}$ by Z. The map from this extension to Z sending (z, b) to $\pi^a z + b\psi$ is equivariant. Evaluating it on Y and then on Y' recovers the original pairing times b modulo π^a. The labelled jumps follow from taking the dual and tensor product over $\mathcal{O}$. The same constructions commute with finite free extension of the coefficient ring, using the extended ideal $\pi^a \mathcal{O}'$. The local field, cocycles and test precision in the original ring have all been kept fixed. The length in Corollary 16.19 refers to constants in a Kisin-module quotient. Passing from this length to the joint image of the local cup tests requires compatibility of the points on successive Frobenius components. □

16.7 Comparing the cup cochains at successive embeddings

Keep the field K, the coefficient ring $\mathcal{O}$, the idempotents $\mathbf{e}_i$ and the cyclic permutation F of Theorem 16.18. The exponents supplied at individual embeddings may be different. To apply Frobenius descent to one cup class at its original precision, we match the exponents on each move of the path. We write $i^- = F^{-1}i$ and $i^+ = Fi$.

Theorem 16.21 (An annihilator using adjacent labelled exponents)**.** *Let a, a' be $\mathcal{O}$-valued lattice cocycles with crystalline extensions X, X' of $\mathcal{O}$ by Y, Y', respectively. Let $P : Y \otimes_{\mathcal{O}} Y' \to (\mathcal{O}/\pi^m)(1)$ be an equivariant pairing at precision $m \geq 1$. Write X'' for the third extension with kernel $\mathrm{Hom}_{\mathcal{O}}(Y \otimes_{\mathcal{O}} Y', \mathcal{O}(1))$ constructed by lifting P and dividing its displacement by π^m. Let $q_{X,i}, q_{X',i}$ be the common-lift exponents of Theorem 16.18 for the first two extensions, and $s_i = c_{X'',i}$ its single-lattice exponent for the third. Put*

$$A_i = q_{X,i^+}, \qquad B_i = q_{X',i}.$$

Then p^k annihilates the class of $P(a \cup a')$, where

$$k = \max_i \max \left\{ \max(A_i, A_{i^-}) + B_i + s_i, \quad A_{i^-} + \max(B_i, B_{i^-}) + s_{i^-} \right\}.$$

Proof. Write $D = (W(\mathbb{C}_p^\flat) \otimes_{\mathbb{Z}_p} \mathcal{O})/\pi^m$, and let D_{sm} be the image of $W(\mathfrak{m}_{\mathbb{C}_p^\flat}) \otimes_{\mathbb{Z}_p} \mathcal{O}$ there. The idempotents act on the small subgroup because they are integral. The assertions of Lemma 16.1 extend to D: the map $\phi - 1$ is onto, and the fixed points in D/D_{sm} are precisely the embedded copy of $\mathcal{O}/\pi^m$. Indeed, as a $\mathbb{Z}_p$-module $\mathcal{O}/\pi^m$ is a direct sum of finite cyclic p-groups, and on each cyclic coordinate the two groups in question are $W_n(F_0)$ and its quotient by $W_n(\mathfrak{m}_{F_0})$, with $F_0 = \mathbb{C}_p^\flat$. The assertion commutes with twisting by one. Since G_K fixes $W(k_K)$, it fixes every $\mathbf{e}_i$.

Choose y_i, y_i' in the common lattices for X, X' over $p^{q_{X,i}}\mathbf{e}_i, p^{q_{X',i}}\mathbf{e}_i$. Take z_i in the first lattice for X'' over $p^{s_i}\mathbf{e}_i$. Contraction of z_i with the two first kernel lattices at i shows that $p^{s_i}P$ on their reductions takes values in the reduction of $\tau^{-1}\mathbf{e}_i(A \otimes \mathcal{O})(1)$. Here and below the unmarked tensor with A is over $\mathbb{Z}_p$. In fact the trilinear contraction is an integral $\mathbb{Z}_p$-multilinear equivariant map. It respects the three $\mathcal{O}$-actions by balancing. Hence the tensor and Tate-line assertions of Lemma 16.13 apply component by component. At the second lattices at i use ϕz_{i^-} instead. The bound there is $\mu^{-1}\mathbf{e}_i(A \otimes \mathcal{O})(1)$ for $p^{s_{i^-}}P$.

At i start with the points $x_i = \phi^{-1}y_{i^+}$ and $x_i' = y_i'$. Their constant coordinates are $p^{A_i}\mathbf{e}_i, p^{B_i}\mathbf{e}_i$. Fix $\mathcal{O}$-linear splittings of the two extensions and write u_i, v_i for the kernel coordinates. Put

$$b_i(g) = p^{A_i}\mathbf{e}_i a(g), \qquad b_i'(g) = p^{B_i}\mathbf{e}_i a'(g),$$
$$d_i(g) = b_i(g) + gu_i - u_i, \quad d_i'(g) = b_i'(g) + gv_i - v_i.$$

Use the continuous $D(1)$-valued cochain

$$f_i(g) = p^{k-A_i-B_i}\big(P(b_i(g), gv_i) - P(u_i, d_i'(g))\big).$$

The two terms are bilinear in the first affine point and the second affine point, including their constant coordinates. Consequently we may first multiply either point by an integral power of p and subtract the same exponent from the prefactor whenever that prefactor remains integral.

The cochain calculation of Theorem 16.2 gives, modulo $D_{\mathrm{sm}}(1)$,

$$\partial f_i(g, h) \equiv p^k \mathbf{e}_i P(a(g), ga'(h)).$$

Indeed $d_i(g)$ lies in the first kernel lattice times $\phi^{-1}(u)\tau$, by applying ϕ^{-1} to the displacement of y_{i^+} in its second lattice. The translate $gd_i'(h)$ belongs to the first kernel lattice by stability.

Since $k - A_i - B_i \geq s_i$, their scaled pairing lies in the reduction of $\phi^{-1}(u)(A \otimes \mathcal{O})(1)$, which is small.

We compare f_i with ϕf_{i^-} in two moves. For the first move fix the second point x'_i. Put $I = \max(A_i, A_{i^-})$. Move the first point from $p^{I-A_i}x_i$ to $p^{I-A_{i^-}}\phi x_{i^-}$, using the prefactor p^{k-I-B_i}. The initial cochain is f_i by bilinearity. Both first points are in the first lattice at i over $p^I\mathbf{e}_i$, since $\phi x_{i^-} = y_i$. Their difference is in the kernel. The displacement of x'_i is divisible by $u\tau$ there. Our prefactor contains p^{s_i}; the change in cochain is therefore small, by the first-move identity of Theorem 16.2.

For the second move use the point ϕx_{i^-} with its original constant $p^{A_{i^-}}\mathbf{e}_i$, and put $J = \max(B_i, B_{i^-})$. Move the second point from $p^{J-B_i}x'_i$ to $p^{J-B_{i^-}}\phi x'_{i^-}$, using $p^{k-A_{i^-}-J}$. By bilinearity the initial cochain here is exactly the terminal cochain of the first move: both equal the cochain on $(\phi x_{i^-}, x'_i)$ with prefactor $p^{k-A_{i^-}-B_i}$. Now both second points are in the second lattice at i with the same constant coordinate. Their difference and its translates are in the second kernel lattice. The displacement of $\phi x_{i^-} = y_i$ is divisible by $u\mu$ there. The prefactor contains $p^{s_{i^-}}$. Thus the second-move identity says that the change in cochain is a coboundary modulo the small subgroup. The terminal cochain is ϕf_{i^-}, since ϕ carries the idempotent at i^- to the one at i and fixes the original coefficients and cocycles.

Sum over i. For $f_* = \sum_i f_i$, we have $(\phi - 1)f_* \equiv -\partial t$ modulo $D_{\mathrm{sm}}(1)$ for a zero-cochain $t \in D(1)$, while $\partial f_* \equiv p^k P(a \cup a')$ since $\sum_i \mathbf{e}_i = 1$. Choose z with $(\phi - 1)z = t$. The cochain $f_* + \partial z$ now reduces to a continuous cochain in the fixed subgroup $(\mathcal{O}/\pi^m)(1)$; the quotient topology is discrete by Lemma 16.1. Injectivity on these constants identifies its differential with $p^k P(a \cup a')$ in the original finite coefficients. This proves the claim. □

Corollary 16.22 (The labelled bound on a spherical period coordinate)**.** *At an unramified spherical coefficient place $w \mid p$, $p \geq 3$, the contribution of a discarded target to the drop in the normalized lengths of the joint images in Corollary 16.16 is at most $[E_{x,w} : \mathbb{Q}_p]\,k$, with k as above using the three $\mathcal{O}$-linear lattices of Corollary 16.20. In particular*

$$k \leq \frac{8}{p-1}\sum_i (h_{X,i} + h_{X',i} + h_{X'',i}).$$

The maximum formula in the theorem retains the positions of the large jumps on the orbit.

Proof. The graph and the dual test give the two cocycles at the original precision. For the third extension evaluation of $\mathrm{Hom}_{\mathcal{O}}(Y, T)$ on Y followed by the dual test is the required integral contraction. The kernels have fixed rational jumps for the given trace family and target. The resulting k works simultaneously for the corresponding cocycles. If coefficients were enlarged, 1 is primitive in $\mathcal{O}'$ over $\mathcal{O}$, so an $\mathcal{O}$-linear retraction $\mathcal{O}' \to \mathcal{O}$ sending 1 to 1 descends a coboundary at the extended modulus $\pi^m\mathcal{O}'$. Counting the free coordinates and intersecting with the joint global image now gives the bound just as in Corollary 16.16. Finally each expression in the maximum uses at most $\sum_i q_{X,i} + \sum_i q_{X',i} + \sum_i s_i$. Apply Corollary 16.19 to the three sums. A prime sum for arbitrary large jumps would require additional savings beyond this last inequality. □

16.8 Removing nonconstant terms at the Eisenstein prime

Keep the unramified field K, coefficient ring $\mathcal{O}$, and component i of Section 16.6. Write $C = \mathrm{Frac}(\mathcal{O})$, $E = u - p$, $\delta = u\tau$, $R_i = \mathcal{O}[[u]]$, and $A_i = \mathbf{e}_i(A \otimes_{\mathbb{Z}_p} \mathcal{O})$, with $A = W(\mathcal{O}_{\mathbb{C}_p^\flat})$. We identify the unit of R_i with $\mathbf{e}_i$. For a crystalline lattice X write $M = M_i(X)$ and $L = L_i(X)$ for the two lattices in Lemma 16.17. A subscript A denotes extension of scalars to A_i. The construction below raises the order of $gx - x$ for all $g \in G_K$ at once. It uses Taylor coefficients at $u = p$.

Lemma 16.23 (Horizontal coordinates at the uniformizer)**.** *In the E-adic comparison over $C((E))$ there is a basis $d_1, \dots, d_m$ fixed by G_K, with labelled jumps $j_1, \dots, j_m$, such that*

$$L \otimes_{R_i} C[[E]] = \bigoplus_k C[[E]]d_k, \qquad M \otimes_{R_i} C[[E]] = \bigoplus_k E^{-j_k} C[[E]]d_k.$$

Here $R_i \to C[[E]]$ is expansion at $u = p$. In the de Rham period ring, g acts on a coordinate $f(E)d_k$ by $f(gE)d_k$, with the coefficient field fixed. This assertion applies to a finite Laurent principal part and to every finite Taylor expansion.

Proof. First twist into effective height. In Kisin's rational comparison the module $D_{\mathrm{cris}}(X[1/p])$, tensored with the ring of analytic functions on the open unit disc, maps to the analytic span of the linearized Frobenius image. This becomes an isomorphism upon inverting $\phi(\lambda)$, where $\lambda = \prod_{m \geq 0} \phi^m(E/E(0))$. In the period comparison the constant vectors in this description are the G_K-invariant crystalline periods. At $u = p$ the factors of $\phi(\lambda)$ have values $1 - p^{p^m - 1}$, $m \geq 1$. Their product converges to a nonzero element of K. Taylor expansion of the analytic comparison at this point therefore gives the first lattice in any basis of D_{cris}. Because K is unramified, its scalar field is already K and $D_{\mathrm{cris}} = D_{\mathrm{dR}}$ under comparison. Taking the component selects a C-basis of the corresponding filtered space. Untwisting is compatible with all these comparisons by the tensor construction. [17, Lemma 2.1.8, (2.1.9), and Proposition 2.1.10]

To identify the first Frobenius source lattice M, let $\mathcal{B}_i^+ = \mathbf{e}_i(B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} C)$. The integral comparison for the Frobenius image identifies its A-span after inverting μ with the representation lattice over the same ring. Applying ϕ^{-1} gives the identification for M after inverting τ. Since $\theta(\tau) = \zeta_p - 1 \neq 0$, τ is a unit in B_{dR}^+. Thus M extends over $\mathcal{B}_i^+$ to the representation tensored with B_{dR}^+, on this component. Choose a basis d_k adapted to the de Rham filtration, with d_k of degree j_k. The filtered de Rham comparison makes this last lattice the degree-zero part of $B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}$. In the adapted basis it is $\bigoplus_k E^{-j_k}\mathcal{B}_i^+ d_k$, since E and Fontaine's t generate the same maximal ideal of B_{dR}^+. The two rational lattices over $C[[E]]$ can be compared there by the height inclusions of Lemma 16.17. The extension to $\mathcal{B}_i^+$ is faithfully flat: it is a nonempty product of extensions of discrete valuation rings sending the parameter to a uniformizer. Hence the equality descends. The basis vectors are horizontal, and G_K fixes the scalar coefficient factor. Substitution in finite Taylor expansions, using $g(E)B_{\mathrm{dR}}^+ = EB_{\mathrm{dR}}^+$, gives the asserted action. □

Lemma 16.24 (Retracting finite Taylor conditions)**.** *There is an R_i-linear retraction $\rho : A_i \to R_i$. Let $\mathcal{M}$ denote M or L, and apply ρ coefficientwise to $E^{-a}\mathcal{M}_A$, for $a \geq 0$. This map fixes $E^{-a}\mathcal{M}$ and commutes with integral maps of these lattices. If a horizontal coordinate of $v \in E^{-a}\mathcal{M}_A$ agrees with a prescribed Laurent polynomial in $C[E, E^{-1}]$ modulo $E^n\mathcal{B}_i^+$, the same congruence holds for that coordinate of $\rho(v)$. It holds simultaneously for any finite collection of these conditions, with arbitrary integer orders n.*

Proof. The ring A_i is faithfully flat over R_i by base change from $\mathfrak{S} \to A$ and passage to the idempotent. Prescribe unknowns in R_i for the images of all elements of A_i, with the equations

of linearity and the requirement that the images of elements of R_i equal themselves. Each finite collection is a system of linear equations over R_i with a solution in A_i, given by the elements themselves. Its right-hand side is therefore zero in the cokernel of its coefficient matrix after faithfully flat extension, so is already zero in the original cokernel. Each finite collection is solvable in R_i. This ring is compact in the (π, u)-adic topology, with finite residue field. Compactness of the product of all the copies of R_i gives a solution to every equation, since each equation is closed. This gives ρ. Tensoring with the indicated free lattice gives the coefficientwise retraction independently of its basis.

In a basis of $E^{-a}\mathcal{M}$ each horizontal coordinate is a linear form with coefficients in $C((E))$ by the previous lemma. Multiply the required congruence by a power of E so all the coefficients and its prescribed value are power series and the new precision N is positive. Truncate those series below exponent N and clear their finitely many coefficient denominators by p^b. The result is a linear congruence over R_i in the entries of v, with known right-hand side. It holds modulo $E^N A_i$: indeed A/E^N is p-torsion-free, by the successive quotients isomorphic to $A/E = \mathcal{O}_{\mathbb{C}_p}$. After inverting p, its injection into B_{dR}^+/E^N follows from the definition of B_{dR}^+ as the E-adic completion of $A[1/p]$. Tensoring with the finite free coefficients and taking the idempotent preserves these injections. Apply ρ to the integral congruence. In the period ring divide back by p^b and the initial power of E. This proves the assertion. The same injection shows, entrywise, that a vector already in $E^{-a}\mathcal{M}_A$ and lying in $E^D\mathcal{M}$ after extension to $\mathcal{B}_i^+$ lies in $E^D\mathcal{M}_A$ for $D \geq -a$. □

Theorem 16.25 (An integral lift with deep Galois displacements)**.** *Let $\beta : X \to \mathcal{O}$ be a crystalline lattice surjection with trivial quotient. Suppose $x_0 \in E^{-a}M$ has constant coordinate $\beta(x_0) = \mathbf{e}_i$, where $a \geq 0$. For any integer $D \geq 0$, form the finite set*

$$\mathcal{I} = \bigcup_k \{n \in \mathbb{Z} \setminus \{0\} : -a - j_k \leq n < D - j_k\}, \qquad s = \sum_{n \in \mathcal{I}} v_p(n).$$

Here $v_p(n) = v_p(|n|)$. There exists $x \in E^{-a}M$ such that

$$\beta(x) = p^s \mathbf{e}_i, \qquad (g-1)x \in \delta E^D M_A \quad (g \in G_K).$$

If $x_0 \in M$, one has $x \in M$. The same conclusion holds for an initial point in $E^{-a}M$ if X has no labelled jump in $[-a, 0)$. Each integer in the union is counted just once, irrespective of the multiplicities.

Proof. Every $g \in G_K$ sends E to $\omega_g E$ for a unit $\omega_g \in A$ with $\omega_g - 1 \in \delta A$. In fact $g(u)/u = [\epsilon^c]$ for some $c \in \mathbb{Z}_p$, $[\epsilon^c] - 1 \in \mu A$, and $\mu/(E\tau)$ is a unit. Apply this identity and Lemma 16.13 to an element of $E^{-a}M$. Its divided displacement by δ belongs to $E^{-a}M_A$, including for $a = 0$.

Choose g_0 with $g_0(u) = [\epsilon]u$ and put $\omega = \omega_{g_0}$. For $n \neq 0$ write $D_n = (\omega^n - 1)/\delta$. The calculation in Lemma 16.10 gives $v_p(\theta(D_n)) = v_p(n)$ since $E' = 1$. For negative n use $D_{-n} = -\omega^{-n} D_n$. Surjectivity of θ supplies $\alpha_n \in A$ with $\alpha_n D_n \equiv p^{v_p(n)} \pmod{E}$. On $E^{-a}M$ use the additive operator

$$T_n v = p^{v_p(n)} v - \rho\left(\alpha_n \frac{g_0 v - v}{\delta}\right).$$

If $\beta(v)$ is constant, the correction has zero projection, by the preceding lemma applied to the projection map. Thus T_n multiplies this coordinate by $p^{v_p(n)}$. It preserves $E^{-a}M$; on inputs in M it also preserves M.

Consider a horizontal coordinate $f(E) \in C((E))$ of v. Suppose $f - c \in E^b C[[E]]$ for some $c \in C$. For every g, $f(gE) - f(E) \in E^b \mathcal{B}_i^+$. Division by δ preserves this order in $\mathcal{B}_i^+$, since $\theta(\delta) = p(\zeta_p - 1) \neq 0$. The correction after applying ρ still has order at least b by Lemma 16.24.

Hence the new coordinate equals $p^{v_p(n)}c$ modulo E^b. If $b = n \neq 0$, write $f - c = a_n E^n + O(E^{n+1})$, where $a_n \in C$ is fixed, possibly zero. Then

$$\alpha_n \frac{f(g_0 E) - f(E)}{\delta} \equiv p^{v_p(n)} a_n E^n \pmod{E^{n+1}\mathcal{B}_i^+}.$$

The prescribed term on the right is rational in these coordinates, so the same congruence holds after the retraction. Subtraction raises the order of the difference from $p^{v_p(n)}c$ to at least $n+1$. At $n = 0$ use the identity operator: a power series of order at least zero is its constant coefficient modulo E.

Initially each coordinate has order at least $-a - j_k$. Process the union of all integer intervals $[-a - j_k, D - j_k)$ in increasing order, inserting the identity at zero. Whenever a coordinate reaches an integer in its interval, its agreement with a constant gains one order by the preceding calculation. All other coordinates retain their already attained orders of agreement. In the resulting point each coordinate agrees with a constant modulo E^{D-j_k}. Thus for all g the horizontal coordinates of $(gx - x)/\delta$ have orders at least $D - j_k$. The description of M in Lemma 16.23 puts this vector in $E^D M$ over $\mathcal{B}_i^+$. It already belongs to $E^{-a}M_A$, so Lemma 16.24 gives membership in $E^D M_A$.

It remains to verify the assertion about a pole when no jump belongs to $[-a, 0)$. For $j_k < -a$, the initial order is strictly positive. Its constant stays zero throughout: before or after any raising step a correction respecting a positive order also has zero constant term. The final order on this coordinate is at least $D - j_k \geq -j_k$. For $j_k \geq 0$, agreement with a constant modulo E^{D-j_k} gives order at least $\min(0, D - j_k) \geq -j_k$. These are all the coordinates under the gap hypothesis, so x belongs to M over $C[[E]]$. Its known membership in $E^{-a}M$ then gives $x \in M$ over R_i, by the divisibility test in Lemma 16.17. □

Corollary 16.26 (Deepening at one prime while retaining the other)**.** *Put $P = \phi(E) = u^p - p$. Let $y \in L$ have a constant quotient coordinate and suppose $(g-1)y \in \delta P^a L_A$ for all $g \in G_K$. For $B \geq 0$ let $s_0 = v_p(B!)$. There exists $q \in L$ with*

$$q - p^{s_0} y \in P^a L \cap \ker \beta, \qquad (g-1)q \in u\mu\, E^B P^a L_A \quad (g \in G_K).$$

Proof. Use the same operators on L, whose horizontal coordinates are power series with no shifts by Lemma 16.23. The initial Taylor step at zero is free. Process degrees $1, \ldots, B$. At every step the correction lies in $P^a L$, because ρ is R_i-linear and the divided displacement belongs to $P^a L_A$; the quotient of the correction is zero. This last divisibility persists after the correction. In fact $(g-1)L \subseteq u\mu L_A \subseteq \delta L_A$, and $g(P)/P = \phi(\omega_g) \equiv 1 \pmod{\phi(\delta)}$, where $\phi(\delta) = u^p \mu \in \delta A$. The product rule therefore gives $(g-1)(P^a L) \subseteq \delta P^a L_A$. The final divided displacement is also in $E^{B+1} L_A$, by the Taylor calculation and integral divisibility. These two divisibilities give their product: modulo E, P has nonzero scalar value $p^p - p$, a nonzero divisor on A_i/EA_i. Successively cancelling E shows $P^a L_A \cap E^{B+1} L_A = P^a E^{B+1} L_A$. Finally δE is a unit times $u\mu$. This gives both asserted conclusions. □

16.9 The local cup product near the five weight intervals

We use the two lattices of Section 16.8 to compare the cup cochains for the original graph and test. The order in which we change their two arguments depends on the signs of the labelled jumps at the embedding where the comparison takes place.

Theorem 16.27 (A cup bound from two adjacent lattices)**.** *Let $K/\mathbb{Q}_p$ be unramified, $p \geq 5$, and take coefficients containing K. Use the crystalline extensions X, X' of $\mathcal{O}$ by Y, Y', their cocycles c, c', and an equivariant pairing*

$$Q : Y \otimes_{\mathcal{O}} Y' \longrightarrow (\mathcal{O}/\pi^m)(1).$$

Assume a third crystalline extension X'' of $\mathcal{O}$ admits an integral equivariant contraction with Y, Y' to $\mathcal{O}(1)$ which reduces to its quotient coordinate times Q. Let $a, B \geq 0$ be integers with $B < p$, put $\eta = B + 1$ and $D = a + \eta$. Assume at every label i that $E^a \mathbf{e}_i$ lifts in each of the three lattices M_i. For each of the first two extensions at each label, assume that either $\mathbf{e}_i$ itself lifts in M_i or no jump of that extension belongs to $[-a, 0)$. At each label assume at least one of the following conditions.

1. *All jumps of X are nonnegative and all jumps of X' are nonpositive.*
2. *All jumps of X are at most B and all jumps of X' are at least $-\eta$.*

For $U = X, X'$, let s_U be the maximum over i of

$$\sum_{\substack{n \neq 0:\ \exists\, j \text{ a jump of } U \text{ at } i \\ -a-j \leq n < D-j}} v_p(|n|).$$

Then $p^{s_X + s_{X'}}[Q(c \cup c')] = 0$ in $H^2(K, (\mathcal{O}/\pi^m)(1))$.

Proof. Set $P = \phi(E)$, $\delta = u\tau$ as before. At each label divide a lift of $E^a \mathbf{e}_i$ by E^a. When a unit lift is available, start with that lift instead. Apply Theorem 16.25 to the first two extensions and multiply by the additional power of p needed to attain the respective maximum. This gives $x_i \in M_i(X)$, $x_i' \in M_i(X')$ over $p^{s_X}\mathbf{e}_i, p^{s_{X'}}\mathbf{e}_i$. Write $d_v(g) = gv - v$. Their displacements satisfy

$$d_{x_i}(g) \in \delta E^D M_{i,A}(Y), \qquad d_{x_i'}(g) \in \delta E^D M_{i,A}(Y').$$

Indeed the quotient coordinate vanishes, and the factors δ, E are nonzero divisors on the quotient ring, also after the finite free coefficient extension. Use the kernel assertion of Lemma 16.13. For the third extension choose $z_i \in E^{-a} M_i(X'')$ over $\mathbf{e}_i$. The contraction shows that pairing the two first kernel lattices over A_i by Q takes values in the image of $E^{-a}\tau^{-1}A_i(1)$; for the two second lattices it takes values in the image of $P^{-a}\mu^{-1}A_i(1)$, using ϕz_{i^-}. These are statements in the finite Witt comparison of Theorem 16.21. For example evaluate the integral contraction on $E^a z_i$ before reducing, then divide by E^a there. The elements E, P, τ, μ are units in $W(\mathbb{C}_p^\flat)$, since their reductions are nonzero in $\mathbb{C}_p^\flat$. Thus both pairing bounds concern the given modulus.

We verify the cochain comparisons of Theorem 16.2. Throughout, any value in the reduction of $uA_i(1)$ is small in the sense of Lemma 16.1. At the initial pair (x_i, x_i') the value $Q(d_{x_i}(g), g d_{x_i'}(h))$ is small. The first input has factor δE^D; the second belongs to the stable first kernel lattice. The first pairing bound leaves the factor uE^{D-a}. The quotient coordinates match those of $\phi x_{i^-}, \phi x'_{i^-}$, respectively. The latter points belong to the second lattices, with displacements in $\phi(\delta)P^D$ times their second kernel lattices; here $\phi(\delta) = u^p\mu$.

Under condition (1), move the first argument from x_i to ϕx_{i^-} and then move the second from x'_i to $\phi x'_{i^-}$. The first difference lies in the first kernel lattice of Y, since $L_i(X) \subseteq M_i(X)$ by Lemma 16.17. Pair it with $d_{x'_i}(g)$ and use δE^D and the first pairing bound. The second difference and all its translates lie in the second kernel lattice of Y', since $M_i(X') \subseteq L_i(X')$. Pair a translate with $d_{\phi x_{i^-}}(g)$ and use $u^p \mu P^D$ and the second pairing bound. Both results are small.

Under condition (2), put $y = \phi x'_{i^-}$. Its displacement belongs to $\delta P^a L_{i,A}(Y')$ because $\phi(\delta)/\delta \in A$ and $D \geq a$. Apply Corollary 16.26 to y, with $v_p(B!) = 0$. We obtain $q'_i \in L_i(X')$ over the same constant as x'_i and with displacement in $u\mu E^B P^a L_{i,A}(Y')$. Take the path

$$\begin{aligned}(x_i, x'_i) &\longrightarrow (x_i, q'_i) \longrightarrow (\phi x_{i^-}, q'_i) \\ &\longrightarrow (\phi x_{i^-}, \phi x'_{i^-}).\end{aligned}$$

For the first move, the difference of the second points and its translates lie in $E^{-\eta} M_{i,A}(Y')$ by the lower jump bound. Every translate of E is E times a unit. The fixed first displacement supplies $\delta E^{a+\eta}$, so the first pairing bound leaves $uA_i(1)$. For the middle move, the difference of the first points lies in $E^{-B} L_{i,A}(Y)$ by the upper jump bound. The fixed second displacement supplies $u\mu E^B P^a$, so the second pairing bound again leaves $uA_i(1)$. For the last move, both second points are in $L_i(X')$; pair a translate of the kernel difference with the displacement of the fixed first point, using $u^p \mu P^D$. This proves smallness for every move also under condition (2).

For completeness, the cochain of Theorem 16.2 at (x_i, x'_i) consequently has differential $p^{s_X + s_{X'}} \mathbf{e}_i Q(c \cup c')$ modulo the small subgroup. A first-argument move changes it by a small cochain; a second-argument move changes it by a coboundary modulo that subgroup. The terminal cochain is the Frobenius transform of the initial cochain at i^-. Summing over the cycle, $\sum_i \mathbf{e}_i = 1$, and the Frobenius difference of the sum is a coboundary modulo the small subgroup. Surjectivity of $\phi - 1$ on the finite Witt comparison corrects this difference by a coboundary, exactly as in Theorem 16.21. The corrected cochain takes values modulo the small subgroup in its Frobenius constants $(\mathcal{O}/\pi^m)(1)$, which inject into that quotient. Its differential is the required multiple of the original cup. □

Corollary 16.28 (The five intervals for the marked torus data)**.** *For the data of Theorem 50.11 at a sufficiently large node x, write $b = b_x$ and $b_2 = a_x/x_{1,x} = xb/x_{1,x}$. There is a constant $C_0 \geq 1$, depending only on q_0, d, with the following property. For any rational prime $p \geq x_{1,x}$ put*

$$\mathcal{N}_{x,p} = \{n \in \mathbb{Z} \setminus \{0\} : \min_{c \in \{0, b, -b, b_2, -b_2\}} |n - c| \leq C_0(x^{42} + b/p)\},$$
$$S_{x,p} = \sum_{n \in \mathcal{N}_{x,p}} v_p(|n|).$$

At every spherical place $w \mid p$, each original local cup between a trace and a free dual test is annihilated by $p^{2S_{x,p}}$. The exponent is simultaneous in the trace ranks, targets, powers, precision and coefficient embeddings. If the five intervals contain no nonzero multiple of p, each of these cups vanishes at its original precision.

Proof. The bound $x_{1,x} \geq x^{10^6}$ puts p above the ramified primes of E_x for large x, by Lemma 51.1. Work over $K = E_{x,w}$. Enlarge coefficients to contain K as in Corollary 16.20. The graph, the dual test and its integral contraction there provide the three extensions of the theorem. All their upper jumps are at most $12b$, as calculated in Lemma 51.4. Put

$$a = \left\lceil \frac{12b}{p-3} \right\rceil, \qquad B = t' - 1.$$

Theorem 16.8 supplies the lift of $E^a\mathbf{e}_i$ on all three extensions. To see this with the coefficient action, pull the quotient back along $\mathbb{Z}_p \to \mathcal{O}$, $1 \mapsto 1$. The rational pullback is a subrepresentation of the sum of the crystalline extension and a trivial line, and has the same upper bound. Apply the theorem to its quotient $\mathbb{Z}_p$, map the resulting lift to the original module, and multiply by $\mathbf{e}_i$.

At label i, the graph jumps belong to $\{0, \dots, B, J_i\}$, with $J_i = z - 1 + \epsilon_i k_i$. The dual-test extension adds zero to the jump $-1 - j_T$, where $j_T = J_i$ for the main target and $0 \le j_T \le B$ for a scalar target. The weight formulas in Theorem 15.3 and the shifts of Lemma 51.4 put J_i within $O_{q_0,d}(x^{41})$ of one of $\pm b, \pm b_2$. Indeed $r_x \ll x^5$, $m_x \le 2\sqrt{x}$, the shift from a base angular magnitude is $(2j+1)r_x^8 - h_i$ with $0 \le h_i \le 4r_x$, and $|z-1| \le t_0 \ll x^5$. Here $0 < b_2 \le b$, while $b_2 = xb/x_{1,x} \ge xb/p$ and $b_2 \ge b^{1-10^{-6}}$ by the scale conditions. Consequently $|J_i| > a + B + 2$ for large x, simultaneously for all $p \ge x_{1,x}$. Both the graph and the main dual therefore have the required gap $[-a, 0)$. For a scalar dual every jump is nonpositive at every embedding, so a unit already lifts by Theorem 16.5, using the same pullback for coefficients.

If $J_i > 0$, the graph has only nonnegative jumps and the dual test only nonpositive jumps. If $J_i < 0$, the graph has upper bound B, and the dual has lower bound $-B - 1$ for either target. These give conditions (1) and (2), respectively, with $B < p$. Every nonzero integer in either sum of the theorem lies within $a + B + 2$ of the negative of one of the graph or dual jumps. Using $a \ll 1 + b/p$ and the displayed bounds on the jumps, all such integers belong to $\mathcal{N}_{x,p}$ for one constant C_0. Thus each exponent is at most $S_{x,p}$. The theorem applies over K. An $\mathcal{O}$-linear retraction of the extended coefficients fixing 1, applied to the cobounding cochain, descends the assertion to the original coefficient ring and modulus. □

16.10 Scalar tests when the graph lift has a pole

At primes below $x_{1,x}$, a negative main jump on the graph can lie inside the interval $[-a, 0)$ of Theorem 16.25. The graph point then supplied can have an E-pole. A scalar dual has bounded jumps at both ends and admits an integral point at each label. We use this point to absorb both the pole of the graph and the pole at $\phi(E)$ of its translate.

Theorem 16.29 (A cup annihilator for a bounded nonpositive test)**.** *Keep the local field, coefficients, contraction and cocycles of Theorem 16.27, with $p \geq 5$. Let $a, B \geq 0$, $\eta = B + 1$, and $d = 2a + 2\eta + 1$. Assume $E^a \mathbf{e}_i$ lifts in the first lattices for X, X'' and $\mathbf{e}_i$ lifts in the first lattice for X', at every label. All jumps of X' are to belong to $[-\eta, 0]$. At each label the jumps of X are either all nonnegative or all at most B. For $U = X, X'$, put $a_X = a$, $a_{X'} = 0$, and set*

$$s_U = \max_i \sum_{\substack{n \neq 0:\ \exists\, j \text{ a jump of } U \text{ at } i \\ -a_U - j \leq n < d - j}} v_p(|n|).$$

Then $p^{s_X + s_{X'} + v_p(d!)}$ annihilates $[Q(c \cup c')]$ at its original modulus.

Proof. Write $P = \phi(E)$, $e_0 = v_p(d!)$. Take the graph point over 1 on component i by dividing its given lift by E^a; take the integral unit lift for the test. Theorem 16.25 gives

$$x_i \in E^{-a} M_i(X), \qquad x_i' \in M_i(X')$$

over $p^{s_X} \mathbf{e}_i, p^{s_{X'}} \mathbf{e}_i$, after rescaling to the maximum for each extension. For x_i and x_i' their respective displacements lie in $\delta E^d M_{i,A}(Y)$ and $\delta E^d M_{i,A}(Y')$. In particular the former displacement has no pole. Write $v_i = p^{e_0} x_i'$. The first and second lattice pairing bounds of Theorem 16.27 hold with factors $E^{-a}\tau^{-1}$ and $P^{-a}\mu^{-1}$, using the given lift in X''. We use those bounds also on scalar multiples of their inputs inside the Witt comparison, where E and P are invertible.

The point $y_i = \phi x_{i^-}'$ belongs to $L_i(X')$, and $d_{y_i}(g) \in u^p \mu P^d L_{i,A}(Y')$. Apply Corollary 16.26 with both of its exponents equal to d: indeed $u^p \mu / \delta \in A$. We obtain $q_i \in L_i(X')$ with the same constant as v_i and with

$$d_{q_i}(g) \in u\mu E^d P^d L_{i,A}(Y') \quad (g \in G_K).$$

The power p^{e_0} used here is the same at all labels. The graph translate ϕx_{i^-} is in $P^{-a} L_i(X)$, with displacement in $u^p \mu P^d L_{i,A}(Y)$. Thus at every label the two pairs (x_i, v_i) and $(\phi x_{i^-}, \phi v_{i^-})$ have the same two constants.

At the first pair the two displacements give a small cup: the graph supplies δE^d, and the translate of the other displacement belongs to the stable first test lattice. Move first from v_i to q_i with the graph fixed. The difference and its translates belong to $E^{-\eta} M_{i,A}(Y')$ by the lower jump bound on X'. The first pairing bound and the graph displacement leave $uE^{d-a-\eta} A_i(1)$, which is small. Next move from x_i to ϕx_{i^-} with q_i fixed. We check both possible bounds for the graph at this label.

If its jumps are nonnegative, $L_i(X) \subseteq M_i(X)$. The difference of the two graph points belongs to $E^{-a} P^{-a} M_{i,A}(Y)$. Indeed after multiplication by $E^a P^a$ both endpoints are in the first lattice and have equal quotient. The displacement of q_i belongs to $u\mu E^{d-\eta} P^d M_{i,A}(Y')$. Under the first pairing bound the product is in

$$u(\mu/\tau) E^{d-\eta-2a} P^{d-a} A_i(1).$$

Both exponents are nonnegative and $\mu/\tau \in A$, so this is small.

If the graph jumps are at most B, use $M_i(X) \subseteq E^{-B}L_i(X)$. The difference of the graph points now belongs to $E^{-a-B}P^{-a}L_{i,A}(Y)$ by the same kernel argument. Use the displacement of q_i in its second lattice and the second pairing bound. The product is in

$$uE^{d-a-B}P^{d-2a}A_i(1),$$

again small. Finally move the second point from q_i to ϕv_{i^-}. Their difference and its translates belong to $L_{i,A}(Y')$. Pair a translate with the graph displacement $u^p\mu P^d L_{i,A}(Y)$, using the second pairing bound. This is small as well. Applying the two cochain identities and summing over Frobenius as in Theorem 16.27 now gives the coboundary at the product of the two constants, which is the asserted exponent. □

Corollary 16.30 (All scalar targets at unramified spherical places)**.** *In the marked torus data, for any sufficiently large node, let $w \mid p$ be spherical with $E_{x,w}$ unramified and $p \geq 5$. Use the same five centers as in Corollary 16.28. With a fixed enlargement of C_0, define $S_{x,p}$ by its formula at this prime, also when $p < x_{1,x}$. Then $p^{3S_{x,p}}$ annihilates every free cup evaluation of every scalar target, in arbitrary graph rank and at the given precision.*

Proof. Use $B = t' - 1$, $a = \lceil 12b_x/(p-3) \rceil$. The height and pullback argument of Corollary 16.28 gives the E^a-lifts over this unramified field for every $p \geq 5$. The scalar target of jump $j_T \in [0, B]$ has test extension of jumps $0, -1 - j_T$. Its unit lift follows from its nonpositive jumps by Theorem 16.5. The graph satisfies the first alternative of the theorem where $J_i > 0$ and the second where $J_i < 0$. Since $d \ll_{q_0,d_{\mathrm{quad}}} 1 + x^5 + b_x/p$, with d_{quad} the fixed discriminant parameter, every integer in the two valuation sums and every positive integer up to d belongs, when nonzero, to the five intervals with the enlarged constant. This follows from the shift $O_{q_0,d_{\mathrm{quad}}}(x^{41})$ about each base weight as in Corollary 16.28. The three costs are each at most $S_{x,p}$. Descent of the cobounding cochain from the enlarged coefficients preserves the original modulus. □

16.11 Taylor expansion at the next Frobenius prime

Keep the unramified field K, the coefficients $\mathcal{O}$, and $p \geq 5$ of Theorem 16.27. Set $P = \phi(E) = u^p - p$. At $E = 0$ the parameter u specializes to p; at $P = 0$ it specializes instead to a root π_1 of $T^p - p$. Fix the root in the compatible system defining u, and write $H = G_{K(\pi_1)}$. We first construct points on which the displacements for H have prescribed orders at both primes.

Lemma 16.31 (Increasing the order at P with the first order fixed)**.** *Let $M = M_i(U)$ for a crystalline extension with trivial quotient map β. Suppose $v \in E^{-e}M$, $e \geq 0$, has constant quotient, and $d_v(g) = (g-1)v$ belongs to $\delta E^D M_A$ for all $g \in H$, with $D \geq 0$. For $t \geq 0$, let $\mathcal{J}$ be the union of the integer intervals $[-j, t-j)$, over the jumps j of U at i^-, with zero removed. Put*

$$c = \sum_{n \in \mathcal{J}} (2 + v_p(|n|)).$$

There is a point $q \in E^{-e}M$ such that

$$q - p^c v \in E^D M \cap \ker \beta, \qquad (g-1)q \in \delta E^D P^t M_A \quad (g \in H).$$

Proof. Use the period ring obtained by applying ϕ^{-1} before the de Rham comparison, with residue map $\theta_1 = \theta\phi^{-1}$ on A. Its uniformizer is P. We describe the rational Taylor expansions. For $C = \mathrm{Frac}(\mathcal{O})$ put $C_1 = C[T]/(T^p - p)$. This is a product of fields. At every finite order N, the expansion

$$u = T(1 + P/p)^{1/p} \pmod{P^N}$$

identifies $R_i[1/p]/P^N$ with $C_1[P]/P^N$. Indeed division by the distinguished monic polynomial $(u^p - p)^N$ gives the left side a basis of pN monomials over C. In its quotient $u^p = p + P$, so $u(1 + P/p)^{-1/p}$ gives the inverse image of T by the truncated binomial expansion. In the period ring, this element at infinite order is the unique root of $T^p - p$ reducing to π_1 in the internal period factor. Uniqueness follows by Hensel's lemma after inverting p. Hence H fixes these Taylor constants as well as the external coefficients.

The element E is a unit in the P-adic rational completion: no root of $T^p - p$ equals p. Thus M and $L_i(U)$ agree there by the height inclusions. The latter is spanned by $\phi M_{i^-}(U)$. Transporting Lemma 16.23 from i^- by ϕ shows that its horizontal coordinates are $P^{-j}C_1[[P]]$ in directions indexed by precisely the jumps j at i^-. This description also applies to $E^{-e}M$. The transported horizontal basis vectors are H-invariant since G_K commutes with Frobenius. In these coordinates $g \in H$ acts on $f(P)$ by substitution of gP for P.

The retraction ρ in Lemma 16.24 preserves finite rational Taylor congruences in these coordinates. To check this, multiply by a power of P to remove poles and truncate the coordinate matrices and the prescribed terms modulo P^N. The isomorphism above represents the resulting coefficients by polynomials in u over C; clear their denominators by a power of p. The integral congruence holds over A_i/P^N because ϕ^{-1} takes P to E, and the injection into the rational period ring modulo E^N was proved in Lemma 16.24. Apply ρ in the free lattice $E^{-e}M_A$ and divide back in the rational comparison. This also proves that divisibility by P^t for an element of M_A can be tested in this rational comparison.

There is an element $g_0 \in H$ fixing all p-power roots of unity and taking u to $[\epsilon]^p u$. In fact $K(\pi_1)/K$ is Eisenstein of degree p and is not Galois: a normal extension containing π_1 would contain the p-th roots of unity, whose degree over the unramified K is $p-1$. The abelian extension $K(\mu_{p^\infty})$ cannot contain π_1. The Kummer image over this extension acting on the compatible roots of p is therefore a closed subgroup of $\mathbb{Z}_p$ nonzero modulo p, hence all of $\mathbb{Z}_p$. The element corresponding to p is as required. Write $\omega = g_0 E/E$. Using $(g_0 u - u) = u\mu \sum_{h=0}^{p-1} [\epsilon]^h$

and the unit $\mu/(E\tau)$, we get

$$v_p\big(\theta(\omega-1)\big)=2+\frac{1}{p-1}.$$

Also $g_0P/P=\phi(\omega)$, and $\theta_1(\delta)=\pi_1(\zeta_{p^2}-1)$ has valuation $1/(p-1)$. For $n\neq 0$ it follows that

$$Z_n=\frac{(g_0P/P)^n-1}{\delta}\in A,\qquad v_p(\theta_1 Z_n)=2+v_p(|n|).$$

Integrality follows from $\omega-1\in\delta A$ and $\phi(\delta)\in\delta A$. For the valuation, a unit $1+y$ with $v_p(y)>1/(p-1)$ gains exactly $v_p(|n|)$ in $v_p((1+y)^n-1)$ by the binomial formula, first for n prime to p, then for powers of p; inversion handles negative n.

Lift $p^{2+v_p(|n|)}/\theta_1 Z_n$ through the surjection θ_1 to $\alpha_n\in A$. At degree $n\neq 0$ apply

$$v_1\longmapsto p^{2+v_p(|n|)}v_1-\rho\big(\alpha_n(g_0v_1-v_1)/\delta\big).$$

At every step the correction belongs to E^DM and has zero quotient. This remains true at the next step, since $(g-1)(E^DM)\subseteq\delta E^DM_A$. In a horizontal coordinate agreeing with a constant through degree $n-1$, the divided difference multiplies the term of degree n by Z_n modulo the next degree. Multiplication by α_n and the Taylor retraction therefore remove this term after the indicated scaling. Previously attained orders persist, since δ is a unit in the rational P-adic comparison. At degree zero take the actual Taylor constant; it is fixed by H. Processing the union of the intervals in increasing order gives agreement with a constant modulo P^{t-j} in each direction, just as in Theorem 16.25. Thus $(g-1)q/\delta\in P^tM_A$ for $g\in H$, by the integral divisibility test. This vector is also in E^DM_A by construction. The image of E modulo P is the nonzero divisor π_1-p on the internal integral period ring, and remains a nonzero divisor after finite free scalar extension and projection to i. Cancelling powers of P gives $E^DM_A\cap P^tM_A=E^DP^tM_A$, as required. □

Lemma 16.32 (The pole remaining after the first expansion)**.** *For the point produced in Theorem 16.25 let e be the largest $-j$ for jumps $j\in[-a,0)$ at its label, or zero for the empty set. Then the point belongs to $E^{-e}M$. If every jump at that label is at most D, it also belongs to L.*

Proof. For a jump $j<-a$ the horizontal coordinate has zero constant and final order at least $D-j$, as proved in that theorem. For any other jump its order is at least $\min(0,D-j)$. In every case this gives order at least $-j-e$, which is the bound for $E^{-e}M$. When $j\le D$ all these orders are nonnegative, the condition for L. After clearing an E-power these divisibilities descend from $C[[E]]$ to R_i by Lemma 16.17. □

Theorem 16.33 (The main cup with both signs and both base weights)**.** *At any sufficiently large node let $p\ge x^{50}$ and let $w\mid p$ be spherical. Put $b=b_x$ and $b_2=xb/x_{1,x}$. For some constant C depending only on q_0,d, every original cup for the main target is killed by p^h with an integer h satisfying*

$$h\le 2S_{x,p}+C\left(b_2+x^{42}+1+\frac{\log(2b)}{\log p}\right).$$

Here the radius constant in $S_{x,p}$ is enlarged once if necessary. Both signs and base weights may vary at a single place; the graph rank is arbitrary.

Proof. Work first over $K=E_{x,w}$ with coefficients containing K. This field is unramified by our bound on p. Set $a=\lceil 12b/(p-3)\rceil$, $B=t'-1$, $D=2a+B+3$. The three extensions of Corollary 16.20 have E^a-lifts as in Corollary 16.28. Deepen the first two at E by Theorem 16.25.

The valuation sums for depth D are at most $S_{x,p}$ since their radius is $O_{q_0,d}(x^{42} + b/p)$. By scaling, take every initial point on both sides over $p^{S_{x,p}}\mathbf{e}_i$. All their displacements are in δE^D times the respective first kernel lattices.

If the base weight at a label is b, the far jump and its dual lie at distance greater than $a + B + 2$ from zero, since $p \geq x^{50}$ and the shifts are $O_{q_0,d}(x^{41})$. At a label with the other base weight, their absolute values are at most $b_2 + O_{q_0,d}(x^{41})$. By Lemma 16.32 all of the initial points belong to $E^{-e}M_i$ for one integer

$$0 \leq e \leq a, \qquad e \ll_{q_0,d} b_2 + x^{42}.$$

One may take the largest remaining pole bound from that lemma over both sides and all labels.

At label i name the two sides U, V temporarily. Choose U to have nonnegative jumps: take the graph when $J_i > 0$, the main dual when $J_i < 0$. The other side V has upper jump at most B. Write u_i, v_i for their initial points, and use the same side names for their antecedents at i^- in this comparison. We have $L_i(U) \subseteq M_i(U)$, whereas $v_i \in L_i(V)$ by Lemma 16.32 since $D \geq B$. Apply Lemma 16.31 to v_i with $t = e$. Let k be the maximum of its costs over all labels. Multiplying its output at each label by the remaining power of p gives q_i with

$$q_i - p^k v_i \in E^D M_i(V) \cap \ker \beta, \qquad q_i \in L_i(V),$$
$$d_{q_i}(g) \in \delta E^D P^e M_{i,A}(\ker \beta) \quad (g \in H).$$

The membership in $L_i(V)$ follows from $E^D M_i(V) \subseteq L_i(V)$. The kernel equality in the displacement follows by dividing out the nonzero factors in the trivial quotient.

Use the cochains on H, scaling *both* original sides by p^k at every label. The initial cup of displacements is small by the first pairing bound $E^{-a}\tau^{-1}A_i(1)$ from Theorem 16.27. First change $p^k v_i$ to q_i; their difference is in the first kernel lattice, while the other displacement supplies δE^D. Next move $p^k u_i$ to $p^k \phi u_{i^-}$. Their difference belongs to $E^{-e}P^{-e}M_{i,A}(\ker \beta_U)$, since the translate lies in $P^{-e}L_i(U)$ and the two constants agree. The displacement of q_i just obtained clears both poles in the first pairing bound and leaves $uE^{D-e-a}A_i(1)$. Finally move q_i to $p^k \phi v_{i^-}$. Both belong to $P^{-e}L_{i,A}(V)$. The fixed point $p^k \phi u_{i^-}$ has displacement in $u^p \mu P^D$ times its second kernel lattice. Under the second pairing bound $P^{-a}\mu^{-1}A_i(1)$ the result lies in $u^p P^{D-e-a}A_i(1)$.

Every displayed bound is small and persists for the translates of the differences required when the second argument moves: H preserves the lattices over A_i and the principal ideals of E, P. Thus the same computations apply with either assignment of the graph to U or V. The terminal pair is Frobenius of the initial pair at i^- with the original argument order and the same common scaling. The cochain descent in Theorem 16.27, now for H, shows that $p^{2S_{x,p}+2k}$ kills the restriction of the cup. Corestriction composed with restriction multiplies group cohomology by $[G_K : H] = p$, so $h = 2S_{x,p} + 2k + 1$ works over the original local field. Retracting the coefficient extension descends the coboundary at the original precision.

It remains to estimate k. At the preceding label the graph jumps are contained in $[0, B]$ and one far jump. For the dual they consist of zero and one far jump. Thus each set of integers in Lemma 16.31 is covered by two intervals with a total of $O(e + B + 1)$ integers of absolute value at most $16b$. In an interval of m successive integers there are at most $m/p^l + 1$ nonzero multiples of p^l. Sum over $1 \leq l \leq \log(16b)/\log p$ to bound their total valuation by $m/(p-1) + \log(16b)/\log p$. It follows that $k \ll e + B + 1 + \log(2b)/\log p$. The bound on e proves the theorem. □

16.12 Horizontal Taylor coefficients after Frobenius

Keep the unramified field K, $p \geq 5$, $E = u - p$, A_i, $R_i = \mathcal{O}[[u]]$, and $C = \mathrm{Frac}(\mathcal{O})$ of Section 16.8. For a crystalline lattice U put $M_{0,i} = M_i(U)$ and, for $l \geq 1$,

$$M_{l,i} = R_i \, \phi^l M_{0,F^{-\iota}i}.$$

In particular $M_{1,i} = L_i(U)$. To construct a point in $M_{2,i}$ deeply fixed at E, we need a bound for many Taylor coefficients at once. The bound comes from the series at $u = 0$ before substituting u^{p^l} for u.

Lemma 16.34 (The horizontal matrix at the origin)**.** *Choose an R_i-basis of $M_{0,i}$. There is a horizontal basis of the rational crystalline periods in which the coordinate matrix of the chosen basis is*

$$H(u) = I + \sum_{n \geq 1} H_n u^n.$$

This is a matrix of meromorphic functions regular at the origin. For every $n \geq 1$, both $p^{n+v_p(n!)}H_n$ and the coefficient of u^n in $p^{n+v_p(n!)}H(u)^{-1}$ are integral over $\mathcal{O}$, entry by entry (that is, their entries lie in $\mathcal{O}$).

Proof. The analytic crystalline comparison used in Lemma 16.23 identifies the first Frobenius image rationally with the span of the constant crystalline periods on the open unit disc after inverting $\phi(\lambda)$, and is an isomorphism at $u = 0$. The change between $M_{0,i}$ and $M_{1,i}$ is over $R_i[1/E]$, with $E(0) = -p \neq 0$. Hence its horizontal matrix and inverse are also regular at the origin. Choose the constant basis to make its value there equal to I. These assertions commute with a common effective twist and its inverse. The comparison identifies the constant vectors with G_K-invariants inside the crystalline period ring. [17, (2.1.7)–(2.1.9) and Proposition 2.1.10]

Take $g(u) = qu$, $q = [\epsilon]$, as in Lemma 16.7. By crystalline descent on the chosen basis its matrix is $G = I + u\tau D$ with $D \in \mathrm{Mat}(A_i)$. In horizontal coordinates this means

$$H(qu) = H(u)G.$$

We explain how to compare finite powers of u in this identity despite the period coefficients in G. Put $\mathcal{R}_i = \varprojlim_m (A_i/u^m A_i)[1/p]$. The map from A into the corresponding rational completion extends to $B^+_{\max}$: the latter is the p-adic completion of $A[u/p]$ with p inverted, for our unramified parameter. Modulo u^m, terms of degree at least m in u/p vanish and the remaining denominators are bounded by p^{m-1}. The ring A/u^m is p-adically complete and has no p-torsion. Indeed multiplication by the Teichmüller element u^m multiplies each coefficient in a p-adic Teichmüller expansion by $\underline{p}^m$, where $u = [\underline{p}]$; the resulting ideal is closed and p-saturated. Finite free coefficients and the idempotent preserve these facts. In particular if an element of A_i vanishes at order m in this rational completion, it lies in $u^m A_i$. Here vanishing at order m means that its image modulo u^m is zero.

An analytic function on the open disc evaluates in $B^+_{\max}$ and its image in this completion is its formal series in u. The same holds with qu in place of u. Denominators with nonzero constant term are units in the rational completion. Thus identities of such meromorphic expressions in the period ring may be transferred by first clearing the analytic denominators in $B^+_{\max}$. This also justifies the divided differences used below. For any one of these functions W, $(W(X) - W(Y))/(X - Y)$ is a meromorphic function regular at the origin of the bidisc: multiply by the two denominators and use the divided difference of each analytic numerator or denominator. After substitution $X = qu$, $Y = u$, it follows that $(W(qu) - W(u))/\tau$ is $u(\mu/\tau)$ times a regular

formal series in the rational completion. The cancellation by τ here takes place in the original period ring before mapping to this completion.

Induct on $m \geq 1$. Set $c = m - 1 + v_p((m-1)!)$. By induction write

$$p^c H(u) = T(u) + u^m W(u),$$

where T is the polynomial of degree less than m with the indicated coefficients and is integral. The matrix $J = (T(u)G - T(qu))/\tau$ belongs to $\mathrm{Mat}(A_i)$, since $G - I$ is divisible by τ, and $(q^h - 1)/\tau \in A$ for all $h \geq 0$. The horizontal identity gives

$$J = u^m \left(\frac{q^m - 1}{\tau} W(u) + q^m \frac{W(qu) - W(u)}{\tau} - uW(u)D \right).$$

The expression in parentheses is in $\mathrm{Mat}(\mathcal{R}_i)$. Therefore J/u^m is integral over A_i in the sense that its entries lie in A_i. Its residue modulo u, after inverting p, is $((q^m - 1)/\tau)W(0)$. Cancellation of u^m in the completion follows by reducing modulo u^{m+n} for every n, since u is a nonzero divisor before completion.

By the factorization in Lemma 16.7, $(q^m - 1)/\tau$ modulo uA is $p^{1+v_p(m)}$ times a unit. In fact its factors up to units are $\phi^j(\mu/\tau)$ for $0 \leq j \leq v_p(m)$, each associated to $\phi^j(E) \equiv -p \bmod u$. Multiplying by the inverse unit shows $p^{1+v_p(m)}W(0)$ is in the image of A_i/uA_i. These entries are rational constants in C. The R_i-linear retraction of Lemma 16.24, reduced modulo u, maps integral entries to $\mathcal{O}$ and fixes the constants after inverting p. Hence $p^{1+v_p(m)}W(0)$ has entries in $\mathcal{O}$. Since $W(0) = p^c H_m$, this completes the induction. The series with these coefficient bounds are closed under multiplication: $j!(n-j)!$ divides $n!$. The recursion for the inverse of a matrix series with constant I then gives the same assertion for H^{-1}. □

Theorem 16.35 (Integral approximation of a horizontal coefficient)**.** *Let $Q = p^l \geq 3$, $A_0 \geq 1$, $d_0 = 2\lceil A_0/Q \rceil$. In the horizontal basis transported by ϕ^l from the origin-normalized basis at $F^{-l}i$, the following hold for $M_{l,i}$ at $u = p$.*

1. *For $y \in M_{l,i}$, every coefficient of E^j, $0 \leq j < A_0$, in its horizontal expansion belongs to $p^{-d_0}\mathcal{O}^{\operatorname{rank} U}$.*

2. *For an integral constant vector v in that basis, there is $z \in M_{l,i}$ with horizontal expansion $p^{d_0}v + O(E^{A_0})$. In particular, for $y \in M_{l,i}$ and $N < A_0$, $N \geq 0$, one can realize $p^{2d_0}[E^N]y$ up to $O(E^{A_0})$ in $M_{l,i}$, where brackets denote the horizontal Taylor coefficient.*

Proof. The basis obtained by applying ϕ^l to the original lattice basis has horizontal matrix $H_l(u) = H(u^Q)$, with the corresponding Frobenius transport of the constants. It is a basis of $M_{l,i}$ by its definition. Its inverse has the same substitution. The lemma and $v_p(n!) \leq n$ bound the coefficients of u^{Qn} in both matrices below by valuation $-2n$. In particular these series converge near $u = p$. The coefficient of E^j from the term of index n, for $Qn \geq j$, has valuation at least

$$Qn - j - 2n = (1 - 2/Q)(Qn - j) - 2j/Q \;\geq\; -2j/Q.$$

The terms tend to zero p-adically as n grows. Thus the Taylor polynomials of both $p^{d_0}H_l$ and $p^{d_0}H_l^{-1}$ below degree A_0 have integral coefficients. Multiplying an integral lattice coordinate vector by H_l proves (1), since the expansion of $R_i = \mathcal{O}[[E]]$ is integral. For (2), use as lattice coordinates the Taylor polynomial of $p^{d_0}H_l^{-1}v$ below that degree. Multiplying back by H_l gives the asserted congruence. Apply (1) to y and then (2) to $v = p^{d_0}[E^N]y$ for the last assertion. □

16.13 A common point in two Frobenius lattices

Let $p \geq 5$ and $K/\mathbb{Q}_p$ be unramified. Use the lattices $M_{l,i}(U)$ of Section 16.12 for a crystalline extension $\beta : U \to \mathcal{O}$ with trivial quotient. We suppress $\mathbf{e}_i$ on its scalar coordinate. Put $i^- = F^{-1}i$, $i^{--} = F^{-2}i$, $P = u^p - p$ and $H = G_{K(p^{1/p})}$, with the choice of root in Section 16.11. We will use an integral point in $M_{2,i}$ in a comparison that passes over the intervening label i^-.

Lemma 16.36 (Two divisibilities for the second Frobenius lattice)**.** *Let $z \in M_{2,i}$ have constant quotient, and suppose $(g-1)z \in \delta E^A M_{2,i,A}$ for $g \in H$, $A \geq 0$. For $t \geq 1$ set $c_t = 2(t-1) + v_p((t-1)!)$. There exists $q \in M_{2,i}$ such that*

$$q - p^{c_t} z \in E^A M_{2,i} \cap \ker \beta, \qquad (g-1)q \in \delta E^A P^t M_{2,i,A} \quad (g \in H).$$

Proof. The A_i-span of $M_{2,i}$ is stable. Its generators $\phi^2 m$ have displacement divisible by $\phi^2(\delta) \in \delta A_i$, since g commutes with ϕ. For coefficients $r \in R_i$, $gr - r \in \delta A_i$ by crystalline descent on the trivial line. As gE/E belongs to $1 + \delta A_i$, the same argument gives $(g-1)(E^A M_{2,i}) \subseteq \delta E^A M_{2,i,A}$.

At $P = 0$ all the horizontal coordinates of $M_{2,i}$ are regular and form the full power series lattice. Indeed the span of $\phi M_{0,i^{--}}$ at i^- is M_{1,i^-}, whose E-adic rational lattice is the span of a constant basis by Lemma 16.23. Apply ϕ and extend scalars to the P-adic rational completion at i. A basis and its inverse remain regular, since $\phi(E) = P$. The Taylor constants are in $C[T]/(T^p - p)$ in these coordinates, and are fixed by H as in Lemma 16.31.

Apply the coefficientwise R_i-linear retraction in a basis of $M_{2,i}$. It respects congruences in horizontal coordinates at P: truncate the regular change of basis modulo the required P^n, represent the coefficients by polynomials over C using the finite Taylor isomorphism of Lemma 16.31, then clear a common power of p and retract. The injection of A_i/P^n in the rational period quotient justifying this step is the one in that lemma. Starting with the constant term in each coordinate, process $n = 1, \dots, t-1$. With its g_0, α_n replace the current point v by

$$p^{2+v_p(n)} v - \rho(\alpha_n (g_0 - 1) v/\delta).$$

The correction lies in $E^A M_{2,i}$ and has zero quotient. The established stability preserves the hypothesis about the divided displacement for the next step. Since $\alpha_n((g_0 P/P)^n - 1)/\delta \equiv p^{2+v_p(n)} \bmod P$, the term of degree n beyond the constant is removed, and all lower orders are preserved. The point obtained agrees with an H-fixed constant modulo P^t. For every $g \in H$, δ is a unit in the rational P-adic comparison. Thus its divided displacement lies in $P^t M_{2,i,A}$ by the same integral divisibility test. It also belongs to $E^A M_{2,i,A}$. The intersection is the product, since E is a nonzero divisor modulo P, as in Lemma 16.31. The sum of the costs is c_t. □

Theorem 16.37 (A lift fixed to high order across an intervening label)**.** *Let s, κ, h_0, h_1 be nonnegative integers. Assume E^s lifts to $M_{0,i}$ and p^κ lifts to $M_{0,i^{--}}$. Suppose all jumps at i are at most h_0, and those at i^- are at most h_1. Put $N = s + h_0 + p h_1$. For integers $A > N$, $t \geq 1$ there is $q \in M_{2,i}$ with*

$$\beta(q) = p^{\kappa + 4\lceil A/p^2 \rceil + c_t},$$
$$(g-1)q \in \delta E^A P^t M_{2,i,A} \qquad (g \in H).$$

In a horizontal basis adapted to the jumps at i, the coordinate of q at every jump j with $h_0 - j > N$ has order at least A at $E = 0$. In particular $q \in M_{0,i} \cap M_{2,i}$ if all jumps at i^- are nonnegative and every negative jump j at i satisfies $h_0 - j > N$ and $-j \leq A$.

Proof. Take $v \in E^{-s} M_{0,i}$ over 1. With $E = u - p$, the polynomial

$$f(E) = E^{s+h_0}\big((p+E)^p - p\big)^{h_1}$$

is monic of degree N. The successive upper-height inclusions put $y = f(E)v$ in $M_{2,i}$: first $E^{s+h_0}v \in M_{1,i}$, then $P^{h_1}M_{1,i} \subseteq M_{2,i}$. At $E = 0$, $M_{2,i}$ has the same rational lattice as $M_{1,i}$, since P is a unit there and their two transition matrices have only powers of P as denominators. These are the regular horizontal coordinates of Lemma 16.23. In the direction of jump j, the coordinate of y has order at least $h_0 - j$, using the description of $M_{0,i}$ in the same lemma. Thus $[E^N]y$ is zero in all the directions specified in the theorem.

With $d = 4\lceil A/p^2 \rceil$, Theorem 16.35 supplies $\xi \in M_{2,i}$ equal to the constant $p^d[E^N]y$ modulo order A. This congruence is unchanged by any constant change of horizontal basis, including the change to one adapted to the filtration. The quotient map in horizontal bases is constant by functoriality of crystalline periods. Since $\beta(y) = f(E)$ and its coefficient of E^N equals one, $\beta(\xi) = p^d + E^A w$ with $w \in R_i$; integral divisibility follows from $R_i = \mathcal{O}[[E]]$. Furthermore $(g-1)\xi/\delta \in M_{2,i,A}$ before any Taylor comparison. In that comparison g fixes the indicated constant, gE/E is a unit, and δ specializes to $p(\zeta_p - 1) \neq 0$. Thus this divided displacement lies in $E^A M_{2,i,A}$ by the integral test from Lemma 16.24, applied entrywise in a basis of $M_{2,i}$.

Apply ϕ^2 to a lift at i^{--} to get $y_0 \in M_{2,i}$ over p^κ. Set

$$z = p^\kappa \xi - E^A w y_0.$$

Its quotient is $p^{\kappa+d}$. The second term has displacement in $\delta E^A M_{2,i,A}$ by the stability proved above. All its horizontal coordinates have order at least A at $E = 0$. Lemma 16.36 applied to z gives the stated q, preserving the orders in the specified directions.

For the last assertion, nonnegative previous jumps give $M_{2,i} \subseteq M_{1,i}$ by Frobenius transport. Consequently q is in $E^{-r_i}M_{0,i}$, where $r_i = \max(0, \max_j(-j))$. In every nonnegative direction its coordinate is regular, hence has order at least $-j$. In every negative direction the hypothesis gives order at least $A \geq -j$. The horizontal description now puts q in $M_{0,i}$ rationally at E. Divisibility after clearing E^{r_i} descends in $\mathcal{O}[[E]]$. □

16.14 The main cup on an orbit with both base weights

Write $b = b_x,\ G = xb/x_{1,x}$ for the two integral base weights of the shorter torus family. At each labelled embedding there is one of these two weights. For a fixed datum let $B = t' - 1$. Choose an integer $W = \lceil C_0 x^{42} \rceil$, with C_0 depending only on q_0, d, large enough for the following bounds at every label. If its base weight is $\beta \in \{b, G\}$, the far jump of the graph and the far jump of the main dual, when present, have absolute values in $[\beta - W, \beta + W]$. The other jumps on these two extensions are in $[0, B]$. Every upper jump on all three extensions, including the extension defining the integral contraction, is at most $\beta + W$. Indeed the graph jumps belong to $\{0, \dots, B, J_i\}$, and the main dual jumps belong to $\{0, -1 - J_i\}$. The third kernel has jumps $J_i - j$ at the same label by Corollary 16.20; these are zero or have absolute value at most $|J_i| + B$. The shifts of J_i are $O_{q_0,d}(x^{41})$ and $B \ll x^5$. We take $W \geq B + 10$.

Theorem 16.38 (A uniform bound for the main cup at large primes)**.** *At sufficiently large nodes, for every prime $p \geq x^{50}$ and every spherical place $w \mid p$, each original cup for the main target is killed by p^h, where one can take a nonnegative integer h satisfying*

$$h \ll_{q_0,d} W + 1 + \frac{b}{p^2} + \frac{G}{p} + \frac{\log(2b)}{\log p}.$$

The estimate is uniform in all twists in the designated family, trace ranks, tests and precisions.

Proof. Let $K = E_{x,w}$, unramified in this range. Extend coefficients to contain K, using the original ideal $\pi^m \mathcal{O}'$. Denote the graph and dual extensions by X, X', their finite pairing by Q, and the extension defining the contraction by X''. Put

$$a = \left\lceil \frac{b + W}{p - 3} \right\rceil, \quad H_p = W + 1 + \left\lceil b/p^2 \right\rceil + \lceil G/p \rceil, \qquad D = 2a + B + 4.$$

Since $p \geq 7$, $a \leq b/p + 8H_p$ and $a/p \leq 4H_p$; these follow by subtracting b/p before estimating the ceiling. At every label the image of each of the three lattices M_0 contains $p^k E^s$ for nonnegative $k + s \geq a$. This is Theorem 16.8 with upper jump bound $b + W$, pulling back along the trivial $\mathbb{Z}_p$-line, then projecting by the labelled idempotent as in Corollary 16.28.

If the sign or the base weight is constant on the orbit, Lemma 51.9 gives $2S_{x,p}$. Each of its five intervals has $O(W + 1 + b/p)$ integers with absolute value at most $16b$. In an interval of m consecutive integers, the number of nonzero multiples of p^r is at most $m/p^r + 1$. Summing this inequality for $1 \leq r \leq \log(16b)/\log p$ bounds the sum of valuations by $m/(p-1) + \log(16b)/\log p$. This proves the assertion in this case.

It remains to consider a place with both signs and both weights. The rational prime is inert in $\mathbb{Q}(\sqrt{-d})$: it is unramified, and a split prime has constant sign. Consecutive labels restrict to conjugate embeddings on the unramified quadratic completion, since F acts by one Frobenius step. The sign is determined by this restriction by Theorem 15.3. In particular the signs alternate. At a positive label the graph is nonnegative and the dual has upper jump zero. At a negative label the dual is nonnegative and the graph has upper jump at most B.

We choose a set of labels at which to start integral points. Write s_i for a permitted pole order at a chosen label and κ for a common exponent. The following two choices will be used.

1. If $b/p - G \leq 40H_p$, choose every label, put $s_i = s = \min(a, G - W - 2)$ and $\kappa = a - s$. We have $G \geq b^{1-10^{-6}}$, so $s \geq 0$ for large x. The inequality for a gives $\kappa \leq 50H_p$. The images contain $p^\kappa E^s$. A negative far jump has size at least $G - W > s$, so neither of the first two extensions has a jump in $[-s, 0)$.

2. Suppose $b/p - G > 40H_p$. Choose every label of weight b and every label whose successor has weight G. At a chosen label with successor of weight G put $s_i = 0$. At each such label a pure power p^κ lifts on all three extensions for one $\kappa \ll H_p$. Indeed the sum of the successive upper jumps divided by the respective powers of p is bounded by

$$(G + W)/p + (b + W)/(p(p - 1)).$$

The estimate for c_i in Theorem 16.18 is at most a constant times 1 plus this sum. Take the largest of those bounds, rounded up, for κ. At every remaining chosen label put $s_i = a$. Its weight is b, so the negative far jumps have size at least $b - W > a$. Also E^a lifts there on all three extensions. Use the same κ by scaling. A label not chosen has weight G and successor of weight b; its predecessor has a successor of weight G, hence was chosen. Thus successive chosen labels are one or two steps apart.

At a chosen label take, on X and X', lifts of $p^\kappa E^{s_i}$ and divide by E^{s_i}. Apply Theorem 16.25 at depth D. Its operators multiply a constant quotient by the same powers for any initial constant, hence apply also to p^κ. The resulting points are in $M_{0,i}$: for $s_i = 0$ they began there, and otherwise the required gap was just checked. Their displacements lie in δE^D times their first kernel lattices. The cost in powers of p beyond κ is

$$O(H_p + \log(2b)/\log p).$$

Indeed the union of the intervals $[-s_i - j, D - j)$ is covered by the short cluster and one far interval with a total of $O(a + B + 1)$ consecutive indices, counted within each interval. Their nonzero absolute values are at most $16b$. Apply the preceding valuation count and use $a/p \le 4H_p$. On X'' choose $z_i \in E^{-s_i} M_{0,i}(X'')$ over p^κ. Contraction with z_i gives the bound $E^{-s_i}\tau^{-1}A_i(1)$ for $p^\kappa Q$ on the first kernel lattices, taken at the original precision. Transport by ϕ^l gives

$$\phi^l(E)^{-s_i}\phi^l(\tau)^{-1}A_{F^l i}(1)$$

on their l-th Frobenius lattices at $F^l i$. These statements take place after reduction in the finite Witt comparison, where the displayed denominators are units before imposing the indicated lattice bounds.

For each chosen endpoint i two steps from its preceding chosen label, name the nonnegative side at i by U and the other by V. Here i has weight b, its predecessor i^- has weight G, and $s_{i^{--}} = 0$. The extension V has upper jump B as a bound at i, is nonnegative at i^-, and has upper bound $G + W$ there. Each of its negative jumps j at i satisfies $b - W \le -j \le b + W$. Apply Theorem 16.37 to V with

$$s = a, \quad h_0 = B, \quad h_1 = G + W, \quad A = 2(b + W) + B + 4, \quad t = B + 1.$$

The pure lifts required are E^a at i and p^κ at i^{--}. To verify the degree condition, $p(G + W) < b - 39pH_p$, and $a + W < 4pH_p$ by the bound for a and $b/p \le pH_p$. Hence

$$a + p(G + W) < b - W, \qquad N = a + B + p(G + W) < A.$$

In each negative direction $B - j > N$ and $A \ge -j$. The theorem supplies $q_i \in M_{0,i}(V) \cap M_{2,i}(V)$ with displacement in $\delta E^A P^{B+1} M_{2,i,A}$ on $H = G_{K(p^{1/p})}$. The exponent of its quotient is $\kappa + 4\lceil A/p^2 \rceil + 2B + v_p(B!) \ll H_p$; here $B < p$.

Choose e_* to be the maximum of the quotient exponents of the points on the first two extensions at all chosen labels and of all the q_i just constructed. Multiply each point by the additional power of p needed for quotient p^{e_*}. This preserves all membership and displacement

bounds. Thus $e_* \ll H_p + \log(2b)/\log p$. Use the pairing $\widetilde{Q} = p^\kappa Q$ throughout. At each unchosen label use, for both initial points, the Frobenius of the initial points at its predecessor. For the following comparisons a value in the image of $uA_i(1)$ at the given precision will be called small, as in Theorem 16.27.

At a chosen label the initial cup of displacements is small since one supplies δE^D and the first pairing bound costs $E^{s_i}\tau$, with $D \geq s_i$. At an unchosen label the initial cup is the Frobenius of the previous one and is still small since $\phi(u) = u^p$; its comparison with the preceding Frobenius is the identity.

First suppose that the preceding label i^- and i are both chosen. Name their two sides by reference to i, taking U nonnegative there and V with upper jumps bounded by B. Denote the points at i by u_i, v_i. Move u_i to ϕu_{i^-} at fixed v_i. The difference is in the first kernel lattice since $M_{1,i}(U) \subseteq M_{0,i}(U)$. Pairing with d_{v_i} leaves $uE^{D-s_i}A_i(1)$. Next move v_i to ϕv_{i^-}. Both are in $M_{1,i}(V)$, the current point by Lemma 16.32 since $D \geq B$. Pair the kernel difference with the displacement of ϕu_{i^-}; the surviving factor is $u^p P^{D-s_{i^-}}$. Both moves give small values.

Next suppose that i^- is unchosen and i is chosen. The preceding initial pair at i^- is the Frobenius of the pair at i^{--}, so the required terminal points are $\phi^2 u_{i^{--}}, \phi^2 v_{i^{--}}$. Use the side names for i in all three moves.

1. Move v_i to q_i at fixed u_i. The difference lies in the first kernel lattice. The factor δE^D of the other displacement and the first pairing bound give smallness as above.

2. Move u_i to $\phi^2 u_{i^{--}}$ at fixed q_i. The difference belongs to $E^{-(b+W)}P^{-B}M_{2,i,A}(\ker \beta_U)$. Indeed the upper bounds on U are $b+W$ at i and B at i^-, because the sign alternates. Successive height inclusions give the assertion in the extension lattice. The quotient coordinates agree, so it holds in the kernel. Here and in the other moves the kernel assertion is Lemma 16.13, transported by Frobenius. Base change from $\phi^l(R_{F^{-l}i})$ to R_i is flat, with basis $1, u, \ldots, u^{p^l-1}$, and extension to A_i is flat. The second Frobenius pairing at i costs only $\phi^2(\tau)$, since $s_{i^{--}} = 0$. This element is a unit times $EP\tau$, by applying Frobenius to μ/τ as in Lemma 16.7. Pair with $d_{q_i} \in \delta E^A P^{B+1} M_{2,i,A}$; what remains is in $uE^{A-(b+W)-1}A_i(1)$.

3. Move q_i to $\phi^2 v_{i^{--}}$. Both points are in $M_{2,i}(V)$. The fixed first point has displacement divisible in that lattice for U by $\phi^2(\delta E^D) = u^{p^2}\phi^2(\tau)\phi^2(E)^D$. The same pairing bound leaves a multiple of u^{p^2}.

All bounds apply also to the translates of kernel differences in second-argument moves: H preserves the A_i-lattices and the ideals of E, P. Thus the argument uses the given order of the graph and dual in the cup, irrespective of their temporary names.

The terminal pair at every label is the single Frobenius of the actual initial pair at i^-, also when two moves around the orbit return to the original label. The cochain argument of Theorem 16.21 therefore applies on H at precision π^m. Explicitly, the initial displacements give differential $p^{2e_*}\mathbf{e}_i\widetilde{Q}(c \cup c')$ modulo small cochains, and the moves identify the initial and terminal cochains there up to a coboundary. Summing over all labels, correcting the resulting Frobenius difference by a coboundary, and taking Frobenius constants gives a coboundary over the original finite module, as in Theorem 16.27. Thus restriction is killed by $p^{2e_*+\kappa}$. The extension $K(p^{1/p})/K$ has degree p by Eisenstein's criterion. Corestriction shows that $h = 2e_*+\kappa+1$ works over K. An integral coefficient retraction fixing 1 descends the coboundary to the original π^m. The bounds for e_* and κ give the estimate. □

16.15 Taylor conditions at several Frobenius primes

Keep $K/\mathbb{Q}_p$ unramified, $p \geq 5$, and use the coefficient ring and Frobenius lattices of Section 16.12. Put $P_j = \phi^j(E) = u^{p^j} - p$ for $j \geq 0$. Choose the compatible roots $\pi_j = p^{1/p^j}$ used to define u. For $L \geq 1$ let $H_L = G_{K(\pi_L)}$. The rational Taylor expansion at P_j, for $0 \leq j \leq L$, is computed with the map $\theta_j = \theta\phi^{-j}$. Its coefficient algebra for the variable u is $C[T]/(T^{p^j} - p)$, a product of fields.

Proposition 16.39 (Simultaneous orders on a Frobenius lattice)**.** *Let $\beta : U \to \mathcal{O}$ be a crystalline lattice surjection with trivial quotient. Fix a label i and $s \geq 0$, and write $M = M_{s,i}(U)$. Let F_0 be a product of nonnegative powers of the P_v, $0 \leq v \leq L$, excluding indices in a set $I \subseteq \{0, \dots, L\}$. Suppose $z \in M$ has constant quotient and $(g-1)z \in \delta F_0 M_A$ for $g \in H_L$. Prescribe an order $T_j \geq 1$ for every $j \in I$. For $j < s$ put $\mathcal{D}_j = \{0\}$; for $j \geq s$ take for $\mathcal{D}_j$ the jumps of U at $F^{-j}i$. Set*

$$I_j = \bigcup_{a \in \mathcal{D}_j} ([-a, T_j - a) \cap \mathbb{Z}) \ \setminus \{0\}, \qquad c = \sum_{j \in I} \sum_{n \in I_j} (L + 2 + v_p(|n|)).$$

There is $z' \in M$ such that

$$z' - p^c z \in F_0 M \cap \ker \beta, \qquad (g-1)z' \in \delta F_0 \prod_{j \in I} P_j^{T_j} M_A \quad (g \in H_L).$$

Proof. For each j, at finite rational precision in P_j one has $u = T(1 + P_j/p)^{1/p^j}$, by the truncated binomial formula, as in Lemma 16.31. The inverse takes T to $u(1+P_j/p)^{-1/p^j}$. The two algebras agree by these inverse maps, or by division by the distinguished monic polynomial P_j^n at precision n. In the internal period factor the lift of π_j satisfying $T^{p^j} = p$ is unique after inverting p, since the derivative at π_j is nonzero. Thus the Taylor constants are fixed by H_L for $j \leq L$, including after extension of coefficients.

At P_j, the transition from $M_{a,i}$ to $M_{a+1,i}$ has denominators only at P_a. Thus if $s > j$ the rational lattice equals that of $M_{j+1,i}$, and if $s \leq j$ it equals that of $M_{j,i}$. Transport Lemma 16.23 from $F^{-j}i$ by ϕ^j. It gives a constant horizontal basis in which the lattice at P_j is a direct sum of the power series modules shifted by $-a$, for precisely the a stated in the proposition, with multiplicities. The higher Frobenius lattice is regular in all directions. The retraction of Lemma 16.24 in a basis of M preserves finite Taylor congruences here. Indeed one clears poles, represents the truncated coefficients by polynomials over C using the preceding isomorphism, clears powers of p, and applies the R_i-linear retraction. The required injection of A_i/P_j^n into the rational period quotient follows from the E-adic injection by ϕ^j, also on finite free coefficients. This likewise detects divisibility of a vector of M_A by P_j^n.

There exists $g_0 \in H_L$ acting by $g_0 u = [\epsilon]^{p^L} u$ and fixing all p-power roots of unity. The Kummer image over $K(\mu_{p^\infty})$ is all of $\mathbb{Z}_p$, as proved in Lemma 16.31; take the element with parameter p^L. Write $\omega = g_0 E/E$. Since $\mu/(E\tau)$ is a unit and $\theta(\sum_{h=0}^{p^L-1} [\epsilon]^h) = p^L$,

$$v_p(\theta(\omega - 1)) = L + 1 + \frac{1}{p-1}, \qquad v_p(\theta_j(\delta)) = \frac{1 + 1/(p-1)}{p^j}.$$

The second formula uses $\theta_j(\delta) = \pi_j(\zeta_{p^{j+1}} - 1)$. Now $g_0 P_j/P_j = \phi^j(\omega)$, and $\phi^j(\delta) \in \delta A$. Consequently, for $n \neq 0$ the integral element

$$Z_{j,n} = ((g_0 P_j/P_j)^n - 1)/\delta$$

has nonzero θ_j-value of valuation

$$L + 1 + \frac{1}{p-1} + v_p(|n|) - \frac{1 + 1/(p-1)}{p^j}.$$

Here one uses the binomial formula for a unit differing from one by valuation greater than $1/(p-1)$, including inversion for $n<0$. This is at most $c_n = L+2+v_p(|n|)$. Lift $p^{c_n}/\theta_j(Z_{j,n})$ via θ_j to $\alpha \in A$.

When the already imposed product is F, process the integers for the next j in increasing order using

$$v \longmapsto p^{c_n} v - \rho\big(\alpha(g_0-1)v/\delta\big).$$

The correction is in FM and has zero quotient. Indeed every generator $\phi^s m$ has displacement in $\phi^s(\delta)M_A$, the ring coefficients have displacement in δA_i, and for all $g \in H_L$ the ratios gP_v/P_v belong to $1+\delta A_i$. Thus $(g-1)(FM) \subseteq \delta F M_A$, preserving the hypothesis for the next operation. In a horizontal coordinate agreeing with a constant up to degree n (terms below degree n agree), the degree-n term of the divided displacement is $Z_{j,n}$ times its degree-n coefficient. Multiplication by α and retraction therefore cancel that term after scaling by p^{c_n}. Other coordinates retain their attained orders, and at degree zero one takes the current constant coefficient. This is the induction of Theorem 16.25 for the shifted intervals $[-a, T_j - a)$. It gives agreement with a constant modulo order $T_j - a$ in each direction. Since δ has nonzero specialization, the final divided displacement belongs to $P_j^{T_j} M_A$.

For $v \neq j$, the specialization of P_v at θ_j is $\pi_j^{p^v} - p \neq 0$. Hence any product of the other primes is a nonzero divisor modulo P_j, also after finite free extension and on the labelled factor. Cancelling P_j repeatedly gives $FM_A \cap P_j^{T_j} M_A = F P_j^{T_j} M_A$. Use this product for the subsequent indices. The corrections at every stage belong to $F_0 M \cap \ker \beta$, so their scaled sum gives the first assertion as well. □

Theorem 16.40 (A common point across a string of bounded jumps)**.** *Let $l \geq 1$, $p^l \geq 3$, and suppose E^a lifts at i in $M_{0,i}$ and p^κ lifts at $F^{-l}i$ in $M_{0,F^{-l}i}$. Let $h_j \geq 0$ bound the upper jumps at $F^{-j}i$ for $0 \leq j < l$. Suppose the intermediate jumps ($1 \leq j < l$) are all at least $-\eta$, with $\eta \geq 0$. Put*

$$N = a + \sum_{j=0}^{l-1} p^j h_j.$$

Let $A > N$, and suppose every jump $t < -\eta$ at the current label satisfies $h_0 - t > N$ and $-t \leq A$. There exists $v \in M_{l,i}$ such that

$$\beta(v) = p^{\kappa + 4\lceil A/p^l \rceil}, \qquad (g-1)v \in \delta E^A M_{l,i,A} \quad (g \in G_K),$$
$$\left(\prod_{j=0}^{l-1} P_j^{\eta} \right) v \in M_{0,i}.$$

Proof. Divide a lift of E^a by E^a to get x over 1. Take $f = E^a \prod_{j<l} P_j^{h_j}$, a monic polynomial of degree N in $E = u - p$. The successive upper-height inclusions give $y = fx \in M_{l,i}$. At $E = 0$ all factors P_j, $j > 0$, are rational units. The coordinates of y in a basis adapted to the jumps t have orders at least $h_0 - t$. The coefficient $[E^N]y$ consequently vanishes in all directions $t < -\eta$, and its quotient is 1. Realize $p^{4\lceil A/p^l \rceil}[E^N]y$ modulo E^A by an integral $\xi \in M_{l,i}$ using Theorem 16.35. As in Theorem 16.37, its quotient differs from the prescribed constant by $E^A w$, $w \in R_i$. For $y_0 = \phi^l$ of the indicated lift of p^κ, set $v = p^\kappa \xi - E^A w y_0$. The adapted coordinates of v are all regular and have order at least A where $t < -\eta$. Its quotient is as asserted. Its divided displacement is integral in $M_{l,i,A}$, is of order A in rational horizontal coordinates, and is therefore divisible by E^A integrally, by the entrywise divisibility argument of that theorem.

Multiplying by $\prod_{1 \leq j < l} P_j^{\eta}$ sends $M_{l,i}$ into $M_{1,i}$ by the intermediate lower bounds. In the resulting vector, the directions at $E = 0$ still have the orders just obtained. After the further

factor E^{η}, the order in every direction is at least $-t$: for $t \geq -\eta$ this uses regularity, and otherwise $A \geq -t$. After clearing its possible E-denominator in $M_{0,i}$, the horizontal divisibility descends over $\mathcal{O}[[E]]$ by Lemma 16.17. $\square$

16.16 Cup products when the larger weight is sparse

At an unramified spherical place use b, G, W as in Section 16.14, increasing W by a fixed factor for the scalar targets if necessary. At a label of weight G, every jump of the first two extensions has absolute value at most $\eta = 2(G+W+1)$. At a label of weight b, each side has short jumps in $[-\eta, \eta]$ and at most one further jump of absolute value in $[b-W, b+W]$. The two possible far jumps have opposite signs. In a scalar test the dual has no far jump. We assign its side the sign opposite to the graph's far window even if that window is empty on the dual. Thus at every such label the side assigned positive sign has all jumps at least $-\eta$, and the other has all jumps at most η. The third extension has upper jumps at most $\beta + W$ at a label of weight β, since its kernel has differences $j_T - j$ at that label.

Theorem 16.41 (Sparse occurrences of the larger weight on an unramified orbit)**.** *Let $p \geq 5$, $J \geq 1$. In the periodically repeated Frobenius orbit of labels, suppose no three occurrences of weight b have both successive distances at most J. The distances must be positive and this condition includes repetitions around the orbit. Assume*

$$16(J+1)p^J\eta \leq b.$$

At sufficiently large nodes, for either the main or a scalar target, the original local cup is killed by p^h with

$$h \ll_{q_0,d} 1 + \frac{b}{p^{J+1}} + (J+1)^2\eta + (J+1)\frac{\log(2b)}{\log p}.$$

There is no restriction on changes of sign among the larger weights. No change in the finite precision of the cup is required.

Proof. Work first with coefficients containing the local field K. Call a label a cut if the next J labels all have weight G. If there is any occurrence of weight b, either it is a cut or its next such occurrence within J steps is a cut, by the hypothesis on triples. If there are no occurrences, every label is a cut. Let c, i be consecutive cuts in positive cyclic order, and write $i = F^l c$, $l > 0$. If $l > 1$, the label immediately after c is not a cut. Hence the first larger weight following c is exactly $J+1$ steps away. No intervening label can be a cut, since it sees this larger weight within its next J steps. If the first larger weight is not itself a cut, there is a second one within J steps, and it is a cut for the same reason as above. Thus $l \leq L = 2J+1$. The indices after c through i contain at most two larger weights. When there are two, the first one, denoted by v, is $F^{J+1}c = F^{-r}i$ with $1 \leq r \leq J$, and the second is i.

At every cut a common p^κ lifts integrally on all three extensions. In fact the weighted upper-jump sum of Theorem 16.18 is at most

$$(G+W)/(p-1) + (b+W)/(p^J(p-1)).$$

Consequently one may take $\kappa \ll 1 + \eta + b/p^{J+1}$. Choose such points on all three extensions. The third one shows that $\widetilde{Q} = p^\kappa Q$ on the level zero kernel lattices takes values in the image of $\tau^{-1}A_i(1)$. Its l-fold Frobenius from c gives the bound $\tau_l^{-1}A_i(1)$ on the level l kernels at i, with $\tau_l = \phi^l(\tau)$. One has

$$\tau_l \;\in\; A_i^\times \tau \prod_{j=0}^{l-1} P_j,$$

by iterating $\mu/\tau \in A^\times E$.

Put $D = \eta + 2$ and restrict temporarily to $H_L = G_{K(\pi_L)}$. At each cut use Proposition 16.39 on both points in M_0 with $T_j = D$ for $0 \leq j < L$. Their displacements now have factor

$\delta \prod_{j<L} P_j^D$ in the corresponding kernel. The increase in the quotient exponent is bounded by

$$C\left((L+1)^2\eta + (L+1)\log(2b)/\log p\right) =: C_1.$$

Indeed at each prime the integer indices come from at most two intervals, the short cluster and the far window, of total length $O(\eta)$, with absolute values $\le 8b$. For a light label the whole cluster already lies in $[-\eta,\eta]$. Summing $v_p(|n|)$ on each interval costs its length divided by $p-1$ plus at most $\log(8b)/\log p$, as in Theorem 16.38. The same bound C_1, with an absolute enlargement of C, will suffice for all subsequent applications at $j<l$ on M_l, where the shifted index is simply zero.

We construct one auxiliary point for each pair of cuts. All choices below are made before scaling the points to a common quotient. Write $S = \prod_{0\le j<l} P_j^\eta$ and omit the endpoint index i on its lattices.

Suppose first that all larger weights after c through i assign the same positive side U (choose either if there are none). Denote the other side by V. The lower bound on U and upper bound on V, respectively, give by successive height inclusions

$$SM_l(U) \subseteq M_0(U), \qquad SM_0(V) \subseteq M_l(V).$$

Take the l-fold Frobenius of an original integral lift of p^κ on U at c. In $M_l(U)$ impose orders D at $0 \le j < l$ by the proposition. This gives $y \in M_l(U)$, over cost at most $\kappa + C_1$, whose displacement belongs to $\delta \prod_{j<l} P_j^D M_{l,A}(U)$.

In the other case the two larger weights have opposite signs. Name V the side assigned negative at their first occurrence v, hence positive at i. Let U be the other side. At v, E^a lifts on V for $a = \lceil (b+W)/(p-3) \rceil$, by Theorem 16.8 and the coefficient pullback of Corollary 16.28. At c we have the pure lift of p^κ. Apply Theorem 16.40 at v, with $l' = J+1$, $h_0 = \cdots = h_J = \eta$ and $A = 2(b+W) + 2\eta + 4$. The preceding J labels are light and all their jumps lie in $[-\eta,\eta]$. At v, each jump $t < -\eta$ satisfies $-t \ge b - W$. The required strict inequality follows from

$$a + \sum_{j=1}^{J} p^j \eta \le (b+W)/2 + 1 + Jp^J\eta < b - W.$$

The last inequality uses the stated size assumption and $W = o(b)$. In particular the degree is less than A. We get a point of $M_{l',v}(V)$ over $p^{\kappa + 4\lceil A/p^{J+1}\rceil}$, with displacement divisible there by δE^A; after multiplication by $\prod_{0\le j<l'} P_j^\eta$ it lies in $M_{0,v}(V)$.

Transport this point by ϕ^r to a point q_0 of $M_l(V)$ at i. The product $\prod_{r\le j<l} P_j^\eta$ puts it in $M_r(V)$. All the lower jumps of V between v and i, including i but excluding v, are at least $-\eta$. Thus $Sq_0 \in M_0(V)$, and its displacement in $M_l(V)$ before multiplying by S belongs to $\delta P_r^A M_{l,A}(V)$. Apply Proposition 16.39 with $F_0 = P_r^A$ and impose D at all $j<l$ except r. Its correction in $P_r^A M_l(V)$ still belongs to $S^{-1}M_0(V)$. Indeed at r the lower jump is at least $-(b+W)$; at all other indices it is at least $-\eta$. The lower-height inclusions prove the membership since $A \ge b+W$. The resulting point $q \in M_l(V) \cap S^{-1}M_0(V)$ has displacement in

$$\delta P_r^A \prod_{j<l,\ j\ne r} P_j^D M_{l,A}(V).$$

Its quotient exponent is at most $\kappa + 4\lceil A/p^{J+1}\rceil + C_1$.

Scale the initial points at every cut and all auxiliary points to quotient p^{e_*}, taking e_* to be the largest exponent obtained. At a noncut define both initial points by applying Frobenius the appropriate number of times to the pair at the preceding cut. On the following paths a value is small if it belongs to the image of $uA_i(1)$ at the given precision. Initial cups at cuts

are small by the level zero pairing and the factor $\delta = u\tau$; this persists under Frobenius at all other labels. Write u_i, v_i for the cut points on U, V.

In the case of common sign move u_i to y, then v_i to $\phi^l v_c$, then y to $\phi^l u_c$. For the first move the difference belongs to $S^{-1}M_0(\ker \beta_U)$. Its poles are cleared by the displacement of v_i in the level zero pairing. For the second the difference is in $S^{-1}M_l(\ker \beta_V)$, while d_y supplies $\delta \prod_{j<l} P_j^D$. The formula for τ_l and $D \geq \eta + 1$ leave at least a factor u. The third difference is in the level l kernel, and $d_{\phi^l v_c}$ has factor $u^{p^l}\tau_l$.

In the case of opposite signs move v_i to q, then u_i to $\phi^l u_c$, then q to $\phi^l v_c$. The first difference lies in $S^{-1}M_0(\ker \beta_V)$ and is handled by d_{u_i}. For the middle move the upper jumps of U are at most η except at the index r, where they are at most $b + W$. Therefore multiplication by $P_r^{b+W} \prod_{j<l,\ j\neq r} P_j^{\eta}$ puts its difference in $M_l(\ker \beta_U)$. The displacement of q supplies each of these factors and, in addition, the full product in τ_l/τ, since $A \geq b + W + 1$. This gives smallness. Both points in the final move are in $M_l(V)$, and $d_{\phi^l u_c}$ supplies $u^{p^l}\tau_l$. In both cases the differences are in the kernels because the quotient constants agree. All bounds hold for translates on H_L.

The final pair at a cut is the single Frobenius of the actual initial pair at its preceding label. At a noncut this comparison is the identity. Summing the cochains over the labels and taking Frobenius constants as in Theorems 16.21 and 16.38 shows that the restricted cup is killed by $p^{2e_* + \kappa}$. Corestriction from $K(\pi_L)$, of degree p^L by Eisenstein's criterion, adds L to the exponent. The coefficient retraction descends the original finite modulus as before. Since $A \ll b$, the bounds for κ, C_1, e_*, L prove the result. □

16.17 Pure lifts and Taylor orders at ramified places

Let $K/\mathbb{Q}_p$ be finite Galois, with maximal unramified subfield K_0, ramification index e, and uniformizer ϖ. Coefficients contain all images of K. Use $u = [\underline{\varpi}]$, its Eisenstein polynomial $E(u)$ over K_0, and the normalized crystalline lattices of Lemma 16.13. Now the idempotents i correspond to embeddings of K_0. At a fixed i, its ring is $R_i = \mathcal{O}[[u]]$; different extensions of i to K can carry different Hodge jumps. Let h_i, r_i be nonnegative integers bounding these jumps above by h_i and below by $-r_i$ at *all* those embeddings. With components understood, write $P_j = \phi^j(E)$, $\delta = u\tau$, and set $d_K = \lceil v_p(E'(\varpi)) \rceil \le 1 + v_p(e)$. We use all primes p, including two.

Lemma 16.42 (Height and horizontal coordinates over the residue-field labels)**.** *At each label i,*

$$E^{h_i} M_{0,i} \subseteq M_{1,i} \subseteq E^{-r_i} M_{0,i}.$$

At P_j the rational horizontal coordinates of $M_{s,i}$ are regular if $s > j$. If $s \le j$, on each rational factor the shifts are the negatives of the jumps at the corresponding embedding of K above $F^{-j}i$. The Taylor constants in these coordinates are fixed by $G_{K(\varpi^{1/p^L})}$ for $j \le L$.

Proof. First work at E. On the indicated coefficient component E is separable over $C = \mathrm{Frac}(\mathcal{O})$; its roots are the e extensions of the chosen embedding applied to ϖ. The rational E-adic completion is the product of the power series rings for those roots, each with parameter E. In each factor the analytic comparison of Lemma 16.23 identifies M_1 with the regular horizontal lattice. In fact the zeros of $\phi^t(E)$, $t > 0$, have valuation $1/(ep^t)$, different from $1/e$; hence $\phi(\lambda) = \prod_{t \ge 1} \phi^t(E/E(0))$ is a unit at these roots. For convergence there, each normalized factor tends to one geometrically in the exponent p^t, since $E(0)$ has valuation one. The constant space is the crystalline period space after extension from K_0 to K. This space identifies with the filtered de Rham periods since the representation is crystalline. After choosing a basis adapted to the filtration at the given embedding, the level zero B^+_{dR}-lattice has shifts $-t$ for its Hodge jumps t, by filtered comparison. As in Lemma 16.23, the level zero comparison uses that τ specializes to $\zeta_p - 1 \neq 0$. The equality of rational lattices descends from the period ring factorwise by faithful flatness over a discrete valuation ring, since E is a uniformizer there. These comparisons, including a common twist and untwist, are those of the rational Breuil–Kisin comparison. [17, Section 2.1]

The two transition matrices are in $R_i[1/E]$ by Lemma 16.13. If an element of R_i vanishes modulo E^n at all rational factors, it already vanishes in R_i/E^n: Weierstrass division by the distinguished monic polynomial E^n makes this quotient finite free over $\mathcal{O}$. The rational shifts thus give the two integral inclusions. Freeness of $M_{s,i}$ is proved as in Lemma 16.17, decomposing the residue-field coefficient ring and then applying Frobenius. At P_j all transitions except the one from level j to $j+1$ are regular isomorphisms rationally; the root valuations $1/(ep^v)$ are different for different v. Transport the description at level zero and one by ϕ^j.

More explicitly, $P_j(u)$ is separable over C. At each finite order, division by P_j^n and Hensel lifting the simple roots identify the rational Taylor algebra with $(C[T]/P_j(T))[P_j]/P_j^n$: T lifts the residue of u as an exact root of $P_j(T) = 0$, and $u = T + O(P_j)$ is recovered successively using $P_j'(T)$ invertible. In the internal period factor the root reducing to ϖ^{1/p^j} is unique. The indicated subgroup fixes that root by uniqueness, and fixes K and the external coefficient field. Thus it fixes the Taylor constants, including those used to adapt the de Rham filtration separately in each factor. It acts on the parameter by substitution of gP_j. This gives the last assertion. □

Theorem 16.43 (A pure quotient lift after J bounded labels)**.** *For a crystalline surjection $U \to \mathcal{O}$ with trivial quotient suppose $h_i \le H$ everywhere. Suppose $h_{F^t i} \le \eta$ for $1 \le t \le J$, with*

$J \geq 1$ and $p^{J+1} \geq 8e$. There is a lift of p^κ in $M_{0,i}$ with

$$\kappa \ll 1 + J(d_K+1)\eta + \frac{eH}{p^{J+1}}.$$

The implied constant is absolute.

Proof. Set $I_j = \beta(M_{0,j}) \subset R_j$. At precision p^ν let a_j be the least exponent of a pure power of u in $I_j + (p^\nu)$. It exists by the residue-field and Witt comparison argument in Theorem 16.18. That argument uses only freeness, nonzero images on each residue-field component after reduction, and R_j/π a discrete valuation ring, all of which remain valid here. By the height lemma $E^{h_{Fj}} I_{Fj} \subseteq R_{Fj}\phi(I_j)$. Write $u^e = E + pB(u)$ with integral B. Modulo p^ν, $u^{e(h+\nu-1)}$ is a multiple of E^h by binomial expansion. Decomposing a power series into its p residue classes of exponents gives

$$pa_j \leq a_{Fj} + e(h_{Fj} + \nu - 1).$$

Taking the maximum around the finite cycle yields $\max a_j \leq e(H+\nu-1)/(p-1)$.

Starting at $F^J i$, make J passages back to i by Lemma 16.10. To verify the labelled version, on a step from Fj to j the image in the Frobenius lattice is $R_{Fj}\phi(I_j)$. Multiply the current power by $E^{h_{Fj}}$ using the height lemma. The divided displacements and descent modulo p^ν in that proof act on this labelled ideal, because G_K fixes the residue-field constants and the coefficient factors. For each successive exponent t of E, they remove one E, add one u, and cost $d_K + v_p(t)$. The residue-class decomposition on extraction of ϕ then replaces the parameter exponent a by $\lfloor (a + h_{Fj})/p \rfloor$. Therefore at label i we have $p^c u^{a_*}$ in $I_i + (p^\nu)$, with

$$c \leq J(d_K+1)\eta, \qquad a_* \leq \frac{e(H+\nu-1)}{p^J(p-1)} + \frac{\eta}{p-1}.$$

The remaining parameter powers cost $a_* + v_p(a_*!) \leq pa_*/(p-1)$ by Lemma 16.7, likewise on this component. Let $\nu = C+1$. The coefficient of $H + C$ in the last cost is $pe/(p^J(p-1)^2) \leq 4e/p^{J+1} \leq 1/2$. Taking $C = \lceil 2(J(d_K+1)\eta + 2\eta + 4eH/p^{J+1} + 1) \rceil$ makes the total cost at most C, including for $p = 2$. Thus $p^C \in I_i + (p^{C+1})$, and division by an element of $1 + pR_i$ gives $p^C \in I_i$. □

Proposition 16.44 (Taylor orders with the different included)**.** *The assertion of Proposition 16.39 holds over the present K at every prime, with $H_L = G_{K(\varpi^{1/p^L})}$. For $j \geq s$ use for $\mathcal{D}_j$ the union of the jumps above $F^{-j}i$; for $j < s$ still use $\{0\}$. In its cost replace $L + 2 + v_p(|n|)$ by $L + 3 + d_K + v_p(|n|)$.*

Proof. There exists $g_0 \in H_L$ with $g_0 u = [\epsilon]^{p^L} u$. Indeed $X^{p^a} - \varpi$ is Eisenstein over K. The embeddings on its root fields sending ϖ^{1/p^a} to $\zeta_{p^a}^{p^L}\varpi^{1/p^a}$ are compatible under the p-th power maps. Extend their union to the algebraic closure; the resulting automorphism fixes the chosen root at level L. For $\omega = g_0 E/E$, the polynomial divided-difference formula in Lemma 16.10 gives

$$v_p(\theta(\omega - 1)) = L + 1/e + 1/(p-1) + v_p(E'(\varpi)).$$

To see the extra L, write the displacement of u as $u\mu\sum_{h=0}^{p^L-1}[\epsilon]^h$. The sum has specialization p^L, and the divided difference of E specializes to $E'(\varpi)$. As $\mu/(E\tau)$ is a unit, $gE/E \in 1 + \delta A$ for every $g \in G_K$. Also $v_p(\theta_j(\delta)) = (1/e + 1/(p-1))/p^j$. It follows that $((g_0 P_j/P_j)^n - 1)/\delta$ belongs to A and its nonzero θ_j-value has valuation at most $L + 3 + d_K + v_p(|n|)$. The binomial valuation rule applies at two as well: the displayed input valuation strictly exceeds $1/(p-1)$. Surjectivity of θ_j gives the integral multiplier of the divided difference with the prescribed power of p as specialization.

Use the retraction in an R_i-basis, whose construction in Lemma 16.24 uses faithful flatness and compactness, valid also for this Eisenstein parameter. To check its Taylor property on several rational factors, clear poles by a common P_j-power. The finitely many component congruences are encoded in $R_i[1/p]/P_j^n$ by the finite Taylor isomorphism in the height lemma, using the product idempotents there. Clear one power of p to get integral linear congruences. The map A_i/P_j^n into the corresponding rational period quotient is injective as before: internally $A/E = \mathcal{O}_{\mathbb{C}_p}$, the successive quotients for E^n are p-torsion free, and the rational map is the de Rham completion. Transport by ϕ^j and pass to the finite free coefficients. Retraction thus preserves these congruences, with no precision loss on the original lattice.

Now use exactly the degree-raising operators in Proposition 16.39, with the multiplier just constructed. The horizontal comparison in the height lemma gives the intervals to process, and the same operator at an integer n works on all factors simultaneously. Every earlier product of P_v's is preserved, since $gP_v/P_v \in 1 + \delta A$. The image of another P_v modulo P_j is a nonzero divisor: its internal specialization is nonzero by the different root valuations, and the assertion survives coefficient extension and taking an idempotent. Therefore intersecting the divisibilities multiplies their ideals. This proves both conclusions of the proposition at the stated cost. □

17 Higher-rank local tests at spherical and ramified primes

17.1 The local conditions for the higher-rank summand

Let R_ξ be the rank $s = r_x$ summand of a true system supplied by Theorem 52.10. Use $t = 5s,\ k = s+1-m_x,\ h = s^8,\ L = (t-1)/2$. At an infinite place τ, write $B_\tau \in \{b_x, a_x/x_{1,x}\}$ for the base weight and $w_j = B_\tau + (2j+1)h$. The notation σ_p, $0 \le p < s$, is the old row at this place in Lemma 52.11. All Hodge–Tate weights here use weight -1 for κ.

Proposition 17.1 (Labelled weights and exact widths)**.** *At the two conjugate labels over τ, the Hodge–Tate multisets of R_ξ, in one order, are*

$$\{L + (\sigma_p + w_j) : 0 \le p < s\}, \qquad \{L - (\sigma_p + w_j) : 0 \le p < s\}.$$

For $0 \le a < t$, the multiset for $u_\xi \otimes \kappa^a$ contains zero. At either label its exact width (largest minus smallest weight) is

$$W_{j,\tau} = \begin{cases} B_\tau + 2m_x h + k/2 - 1, & j < m_x, \\ B_\tau + (2m_x+1)h + k/2 - 1 + L, & j = m_x. \end{cases}$$

All normalized target systems at the same weight, including those with additional finite auxiliary level, have the same labelled multiset as u_ξ before applying the common Tate twist.

Proof. We explain the local compatibility needed at the coefficient prime. Apply the construction of Theorem 45.16 block by block to the cohomological base change of the definite target. If (ρ, b) has cuspidal degree d, then for a constituent with norm exponent $z = (b-1)/2 - i$ one uses $\rho|\det|^{z+(d-n)/2}$ in the rank d Galois normalization. Its holomorphic infinitesimal entries, before the shift by $(d-n)/2$, form a sublist of the regular target entries of parity $(n-1)/2 \bmod \mathbb{Z}$. At the conjugate labels the same is true. Thus this twist is regular algebraic of the rank d parity. Its cuspidal factor ρ is conjugate self-dual by the cohomological base change. After the norm twist it is polarized by a norm power on E_x^+: the required exponent is $2z + d - n$. Multiplying the polarizing character by the quadratic character of E_x/E_x^+, if needed, sets its prescribed common real sign without changing its base change. The Galois theorem for regular algebraic cuspidal polarized pairs now gives de Rham representations at every place over ℓ, with the labelled weights of these algebraic parameters. It gives crystalline representations at such places where the local cuspidal representation is unramified. In degree one this is the corresponding statement of local class field theory for algebraic Hecke characters. The good split Frobenius polynomials identify these semisimple representations with the ones used in Theorem 45.16. [18, Theorem 2.1.1]

For clarity about signs and translations, the labelled weight formula takes the infinitesimal entries at a fixed label to their negatives or to themselves according to the dual convention, with a common shift by the half sum normalization. In fact for decreasing highest weight (a_i) the two lists in the usual dual convention are formed from $a_i + (d+1)/2 - i$ and $a_i + d - i$: the difference is constant. Dualizing reverses and negates the list. The shift $(d-n)/2$ above makes the translation common to the whole rank n list, and the normalizing character adds a single labelled weight. The character block of length t has the known weights $0, \dots, t-1$, so its center after all shifts is L. The old entries in Lemma 52.11 are at displacements $\sigma_p + w_j$ from that center before the possible reversal of sign. Removing the character block gives one of the two displayed multisets. At the conjugate label the sign reverses. Indeed $u_\xi^c \simeq u_\xi^\vee \otimes \kappa^{1-t}$ by Corollary 45.17; the character block is stable under this operation, and semisimplicity permits cancellation. The paired weights are therefore d' and $t-1-d'$. The target calculation used only

its infinitesimal character and the common normalizer, so it applies also to the other systems of its weight.

It remains to take the width of the union of $[-L, L] \cap \mathbb{Z}$ and the displacements $\sigma_p + w_j$. Reflection and translation preserve this width. The largest displacement is $w_j + k/2 - 1 > L$. The smallest old displacement is $h + w_j - w_{m_x} = (1 + 2(j - m_x))h$. For $j < m_x$ it lies below $-L$; for $j = m_x$ the smallest entry of the union is $-L$. Subtraction gives the two formulas. Translating the full multiset by $-a$ retains zero because it initially contains all of $0, \dots, t-1$. □

Theorem 17.2 (The corrected finite local lengths at every coefficient prime)**.** *Fix $0 \le a < t$ and any stable lattice T_a in $R_\xi \otimes \kappa^a$, over a sufficiently large finite coefficient field with valuation ring $\mathcal{O}$ and uniformizer π. For $v \mid \ell$ and $N \ge 1$ let $\mathcal{F}_{v,N}$ be the reduction in $H^1(E_{x,v}, T_a/\pi^N)$ of the integral classes whose rationalizations lie in the Bloch–Kato finite local condition. Then*

$$\sum_{v|\ell} (\mathrm{length}_{\mathcal{O}} \mathcal{F}_{v,N} - \mathrm{length}_{\mathcal{O}} H^0(E_{x,v}, T_a/\pi^N)) = N f_x s.$$

This holds at every node and every permitted twist on the period interval.

Proof. All old displacements have absolute value at least $h > L$, as in the proof of Lemma 52.11. Consequently $L - a + (\sigma_p + w_j)$ and $L - a - (\sigma_p + w_j)$ have opposite strict signs: $|L - a| \le L$. Each conjugate pair of embeddings therefore contributes exactly s negative weights. For a de Rham representation the sum of the dimensions of $D_{\mathrm{dR}}/\mathrm{Fil}^0$ over $v \mid \ell$ counts these negative weights over all labels, with multiplicity. Each embedding induces one place over ℓ, and there are f_x conjugate pairs. By Lemma 12.27, at each place the difference of the two integral lengths on the left is N times its tangent dimension. This proves the identity, including at primes of ramification of the field or of the tame level. □

Theorem 17.3 (Vanishing of the integral lifting defect on short split orbits)**.** *Take a place $v \mid \ell > 2$ of E_x unramified over $\mathbb{Q}_\ell$, split over E_x^+, and spherical for the chosen level and splitting character. The auxiliary levels in a trace relation are allowed to vary away from v and its conjugate. Suppose*

$$W_{j,\tau} \le \ell - 1 \quad \text{for all labels inducing } v, \qquad W_{j,\tau_0} \le \ell - 2 \quad \text{for at least one.}$$

For the unshifted trace relations take as target either a lattice in R_ξ or one in a prescribed cyclotomic summand of u_ξ. More generally one can twist all the representations in the local relation by κ^a, $0 \le a < t$, with the target chosen among the corresponding summands. At every finite precision every cocycle of Lemma 12.14 for these data lifts to an integral cocycle with crystalline rational extension. Its class in the quotient by $\mathcal{F}_{v,N}$ for that target is zero.

Proof. At a split place the cohomological base change matches the local GL_n representation. A spherical parameter has trivial inertia and zero monodromy, and the local parameter of the isobaric sum is the direct sum of its twisted cuspidal parameters. Hence every one of those cuspidal factors is unramified at v. The crystalline clause used in Proposition 17.1 applies, including at the additional levels which avoid this place. The avatar of the locally unramified algebraic normalizing character is crystalline there, as are Tate twists. Taking a direct summand preserves crystallinity.

At each label inducing v, take the interval from the smallest to the largest weight of the full normalized and commonly twisted target representation. It contains zero and all weights of every lattice in the relation, since by Proposition 17.1 the target summand is a subrepresentation and the other full target systems have the same weights. Lemma 12.14 realizes the cocycle extension as a subquotient at the given precision of lattices with these rational constituents

and the trivial line. The hypotheses on the widths are precisely those of the labelled integral Fontaine–Laffaille lifting theorem, Theorem 13.15. Applying it to this extension gives a lift with the stated end identifications and the stated cohomology class. □

For example, away from the old level and field discriminant, a prime split in $\mathbb{Q}(\sqrt{-d})$ has all its places split over E_x^+. A decomposition group there fixes that quadratic subfield and so cannot contain complex conjugation. The levels chosen away from ℓ introduce no further exception above it. Thus the width bound with B_τ replaced by the larger of the two base weights suffices simultaneously at all these places. On an orbit containing only the smaller base weight one may use that smaller value. The exact finite-condition length above and this integral lifting result have different ranges. The congruence-class loss for long or ramified orbits still requires an integral estimate.

17.2 The higher-rank trace defects at small unramified primes

Use the sparse choice of infinite places of Lemma 51.19 on the original period field E_x. Put $s = r_x,\ t = 5s,\ Q_x = k_x t,\ b = b_x$ and $G = xb/x_{1,x}$. At a coefficient prime p use the selected twist and any of the auxiliary levels away from p in the trace relations. The targets T_i in this section consist of a stable lattice in R_ξ and the t scalar lattices $\mathcal{O}(-a)$, $0 \le a < t$, from Theorem 52.10.

Lemma 17.4 (Crystallinity at the spherical coefficient places)**.** *At a place $v \mid p$ where the level and the smooth part of the normalizing character are unramified, the normalized definite target systems are crystalline. This includes the spherical places inert over E_x^+. In particular Theorem 17.3 also holds at these places when $E_{x,v}/\mathbb{Q}_p$ is unramified and its stated width conditions hold.*

Proof. The identity Hermitian space is quasi-split at the nonsplit finite places on these fields, as checked in Theorem 45.16. The extension at such a place is unramified quadratic and the level here is hyperspecial. Labesse's Corollary 5.3 identifies the base change locally at unramified as well as at split places, for the definite cohomological space used there. Thus a hyperspecial constituent has unramified standard base change at v. [19, Corollaire 5.3] Its local parameter is the direct sum of those of the norm twists of the cuspidal factors. Trivial inertia and zero monodromy on a direct sum hold on each summand. The local cuspidal representations are therefore unramified, and the crystalline clause of Proposition 17.1 applies at p. The locally unramified algebraic normalizer has crystalline avatar. This supplies exactly the crystallinity used in Theorem 17.3; its remaining hypotheses and proof use only the local field and the labelled widths. □

For a spherical $v \mid p$, write $K = E_{x,v}$. For a precision π^N in a coefficient valuation ring $\mathcal{O}$, let $C_i \subset H^1(K, T_i/\pi^N)$ be the image of the cocycles satisfying the matrix relations of Lemma 12.14 for the given automorphic family. Let F_i be the reduction of the integral Bloch–Kato finite condition for T_i. Define the nonnegative length

$$d_{v,i} = \frac{1}{v_\pi(p)} \operatorname{length}_{\mathcal{O}}(C_i/(C_i \cap F_i)) .$$

Proposition 17.5 (Cup products with several large jumps at one label)**.** *For $5 \le p \le Q_x$, $p \nmid q_1 d$, and spherical $v \mid p$, every cup of a class in C_i with the reduction of an integral finite-condition class for $T_i^\vee(1)$ is annihilated by p^H with*

$$H \ll_{q_0,d} bx^{-120}.$$

This bound is uniform in i, ξ, j, in the coefficient field and embedding, and in the auxiliary levels and precision.

Proof. The graph construction of Lemma 12.14 gives a lattice extension of the trivial line with kernel Y and an equivariant map $Y \to T_i/\pi^N$ carrying its cocycle to the specified one. The graph extension splits rationally, and its rational kernel is a direct sum of copies of the given systems. The entire extension is therefore crystalline by Lemma 17.4. Use the integral dual-test extension for the second side. For the contraction, lift the map on Y to a linear map $\psi : Y \to T_i$. The cocycle $(g\psi - \psi)/\pi^N$ defines a rationally split extension by $\operatorname{Hom}_{\mathcal{O}}(Y, T_i)$. Evaluation after multiplication by π^N and addition of the quotient coordinate times ψ recovers the original pairing modulo π^N. This is the construction of Corollary 16.20 with the full target lattice in the Hom space.

We verify the estimates on jumps that enter the Frobenius argument of Theorem 16.41. At a label of base weight $\beta \in \{G, b\}$, Proposition 17.1 puts all the graph jumps into two intervals. One is $[-W, W]$. The other is $\epsilon\beta + [-W, W]$ for a single sign ϵ at that label. Here $W = \lceil Cx^{42} \rceil$ works on the whole interval since the shifts $(2j+1)s^8$, $(2m_x+1)s^8$, s, and t are all $O(x^{41})$. The dual jumps are in the short interval and in the window of the opposite sign, by duality followed by the shift -1. The far window is empty on the dual for a scalar target. In the Hom space every jump is a difference of two jumps at the same label. Two far terms there have the same sign before taking their difference. Thus increasing C once puts all upper jumps of the third extension below $\beta + W$. These assertions allow every one of the $k_x - 1$ old entries in the far cluster.

Here are the details that permit these intervals in the proof of Theorem 16.41, which was stated for the torus data. Put $\eta = 2(G + W + 1)$ and $J = \lfloor 120 \log x / \log p \rfloor$. Use the cuts followed by J light labels. By Lemma 51.19 no three heavy labels have two successive positive gaps at most J, since $2J < l_0$. Thus consecutive cuts are at distance at most $2J + 1$. At most two heavy labels occur after the initial cut and through the terminal cut. The upper jumps of each of the three extensions give a common integral lift of p^{κ} at each cut with $\kappa \ll 1 + \eta + b/p^{J+1}$, by Theorem 16.18. On a light label all jumps on the first two sides are in $[-\eta, \eta]$. On a heavy label the side whose far window is positive has lower jumps at least $-\eta$, and the opposite side has upper jumps at most η.

The simultaneous order corrections of Proposition 16.39 use the *union* of the shifted integer intervals of length $\eta + 2$ over the jumps. Each of our two clusters gives a single containing interval of length $O(\eta)$, regardless of the number of repeated or distinct jumps in it. Their endpoints have absolute value at most $8b$. The sum of $v_p(|u|)$ over the nonzero integers in either interval of length M is $O(M/(p-1) + \log(8b)/\log p)$, by counting multiples of p^a. Thus the correction over a path still costs

$$O\big((J+1)^2 \eta + (J+1)\log(2b)/\log p\big).$$

For paths with a common heavy sign the two one-sided height inclusions in Theorem 16.41 use only the lower and upper bounds just verified. For opposite signs the negative jumps $u < -\eta$ at the first heavy label all satisfy $-u \geq b - W$. With $a' = \lceil (b + W)/(p-3) \rceil$ one has

$$a' + \sum_{i=1}^{J} p^i \eta < b - W \quad \text{if } 16(J+1)p^J \eta \leq b$$

at large x. The degree test in Theorem 16.40 therefore holds for every negative direction, taking Taylor order $2(b + W) + 2\eta + 4$. That theorem incurs the cost $O(1 + b/p^{J+1})$ once for the whole vector. These are all uses of the jumps in constructing the paths. The three changes of points and the contraction estimates in Theorem 16.41 now apply to the same lattices and precision. Their kernel inclusions depend only on the two one-sided bounds and the bound for the third extension. Taking Frobenius constants and corestricting from an extension of degree at most p^{2J+1} gives

$$H \ll 1 + b/p^{J+1} + (J+1)^2 \eta + (J+1)\log(2b)/\log p.$$

All operations can take place after extending the coefficient ring to contain K, since an integral coefficient retraction fixing one descends the same annihilation at the original precision.

Finally $p^J \leq x^{120} < p^{J+1}$, $J \ll \log(2x)$, $G/b \leq x^{1-10^6}$ and $b \geq x^{400}$. These inequalities imply the degree condition and bound the last display by $O_{q_0,d}(bx^{-120})$, including the logarithmic term. They hold for the same choices that enter Theorem 51.20. □

Theorem 17.6 (A small-prime sum for the actual higher-rank local quotients)**.** *On the same fields and interval, at each sufficiently large node one has*

$$\sum_{\substack{p\le Q_x\\ p\nmid 6q_0d}} (\log p)\sup \sum_{\substack{v\mid p\\ v \text{ spherical}}} \sum_i d_{v,i} \ll_{q_0,d} f_x(s+t)b_x x^{-110}.$$

The supremum includes the higher-rank true system, its simultaneous twist, the auxiliary systems at that level, the embeddings and all precisions. After division by $sB_x^{\det}$ the right hand side is $O_{q_0,d}(x^{-105})$.

Proof. The integral finite conditions for T_i and $T_i^\vee(1)$, reduced at precision π^N, are exact annihilators under finite local Tate duality. This is the order and pairing calculation in Proposition 15.4, including the integral torsion classes. Thus $C_i/(C_i\cap F_i)$ injects by cup products into a product of copies of $\mathcal{O}/\pi^N$, using generators of the dual finite condition. One can take the images of a basis of its integral finite-condition raw cocycles. Their rank is the target rank d_i plus the de Rham tangent dimension of the dual, by Lemma 12.27, and is at most $(1+[K:\mathbb{Q}_p])d_i$. Applying Proposition 17.5 to these integral representatives handles every generator, including those defining torsion cohomology classes. The p^H-torsion in one copy of $\mathcal{O}/\pi^N$ has normalized length at most H. Summing the coordinate bounds gives

$$\sum_{v,i} d_{v,i} \ll_{q_0,d} 4f_x(s+t)b_x x^{-120},$$

since $\sum_i d_i = s+t$ and the local degrees over p sum to $2f_x$. All prime divisors of q_1 divide $6q_0$. The hypotheses of the proposition therefore hold at the spherical places retained here. The elementary estimate $\sum_{p\le Q_x}\log p \ll Q_x \ll x^{10}$ proves the prime sum, also after the indicated supremum. Finally $s+t=6s$ and $B_x^{\det} = f_x x b_x(\log b_x)^3/(m_x k_x)$, with $m_x k_x \ll x^{11/2}$. Substitution proves the normalized estimate. □

This controls the loss of the finite local condition for all the listed summands and primes. The determinant comparison still requires the other primes and the passage from these local quotients to the full tested Hecke lattice.

17.3 The higher-rank local quotients at the fixed conductor

Put $\mathcal{R} = \{p : p \mid 6q_0 d\}$. We use the fields, the sparse set of infinite places, and the local conditions $C_i, F_i, d_{v,i}$ of Section 17.2. The ramification index of a completion above a fixed p can grow along this family.

Lemma 17.7 (The sign of the large weights above a conductor prime)**.** *Fix $p \in \mathcal{R}$, a place $v \mid p$, a coefficient embedding, and a lift index j. Put $K = E_{x,v}$ and let K_0 be its maximal unramified subfield. At every full embedding of K having base weight $b = b_x$, the large weights of all the graph representations in the local trace relation lie in $\epsilon b + [-W, W]$ for a common sign ϵ and a common integer $W \ll x^{42}$. At these embeddings the large weights of the dual test for R_ξ lie in the opposite window. All other weights of either representation at these embeddings lie in $[-W, W]$, and a scalar dual has only these latter weights. The statement holds after a common twist by κ^a, $0 \le a < t$, and replacement of the dual by the dual of the twisted target with its Tate twist.*

Proof. Identify the complex labels using the specified coefficient embedding. The field has CM type Φ induced by a fixed embedding of $K_d = \mathbb{Q}(\sqrt{-d})$. The labelling in Proposition 17.1 can be tracked simultaneously at all infinite places. Indeed, at each real place τ the complexified unitary group is identified with GL_n by the coordinate indexed by $\Phi(\tau)$. The two coordinates of the complexified unitary group are related by inverse transpose, up to the change of basis of the Hermitian form. Consequently passage to the other coordinate negates the infinitesimal entries and reverses their order. The prescribed positive columns act on exterior powers of the $\Phi(\tau)$ coordinate, and the negative columns act on their duals. Taking the successive differences in this order is exactly the calculation of Lemma 52.11 at every $\Phi(\tau)$.

When passing to the labelled Galois weights the common translation places the center of the string at $L = (t-1)/2$. Taking a dual, if it is used in the Galois normalization, negates the displacements from its center at every label. It follows that for one fixed sign δ the old displacements at σ are

$$\delta\, \epsilon_\Phi(\sigma)(\sigma_h + w_j), \qquad 0 \le h < s,$$

where ϵ_Φ is 1 on Φ and -1 on its conjugate, and σ_h is the row entry at the underlying infinite place. The normalized scalar entries are $0, \dots, t-1$. For $h < k_x - 1$, $\sigma_h + w_j$ equals the base weight plus an error $O(x^{41})$. The other displacements are $(1 + 2(j - (h + 2 - k_x)))s^8 = O(x^{41})$, since $s \ll x^5$ and $m_x \ll \sqrt{x}$. The same highest weights occur at each auxiliary level in the local graph. A common Tate twist changes each entry by at most t. Duality followed by the Tate twist changes a weight u into $-1 - u$.

The field calculation in Lemma 51.21 shows that all the full embeddings of this completion carrying b induce the same CM orientation. For primes dividing q_0 this follows from splitting in K_d. At $p = d$ the ratios of images of a q_1-th root of unity belong to $\langle d \rangle \bmod q_1$, and its image modulo q_0 omits -1. These statements also hold for the translated set of large weights. Multiplying the common orientation by δ gives the asserted sign. Enlarging W absorbs the center, the twists and the bounded displacements. □

Proposition 17.8 (A conductor bound uniform in rank and precision)**.** *At every spherical $v \mid p$, $p \in \mathcal{R}$, a cup of a class in C_i with a reduced integral finite-condition class for $T_i^\vee(1)$ is annihilated by p^H, with*

$$H \ll_{q_0,d} b_x x^{-108}.$$

The constant is uniform in the level choices away from p, the coefficient embedding, the test, and the precision.

Proof. Use the graph, the integral finite dual extension, and the extension by $\mathrm{Hom}(Y, T_i)$ constructed in Proposition 17.5. The first and third extensions are rationally split, the rational systems at v are crystalline by Lemma 17.4, and the dual extension is crystalline by its finite local condition. Write $G = xb_x/x_{1,x}$ and set $\eta = 2(G+W+1)$, with W enlarged by an absolute factor. Mark a residue-field embedding when at least one full embedding over it has weight b_x. At unmarked embeddings all upper and lower jumps on the three extensions have absolute value at most η. At each full embedding, the jumps of the Hom kernel are differences of target and graph jumps. In any such difference two large weights have equal sign before subtraction. All upper jumps on the three extensions are therefore at most $b_x + W$ everywhere. Lemma 17.7 gives a choice of side U, among graph and dual, with lower jumps at least $-\eta$ at every full embedding. The other side V has upper jumps at most η at all of them. This also holds for a scalar dual, whose weights are all short.

We give the dependence of the ramified comparison on these data. Take $J = \lfloor 120 \log x / \log p \rfloor$, $A = 2J + 1$ and mark cuts followed by J consecutive unmarked residue-field embeddings. Lemma 51.21 gives the spacing condition $2J < l_0$. The gap argument of Theorem 16.41 then bounds the number of steps between consecutive cuts by A. For $e = [K : K_0]$ and for d_K as in Section 16.17, one has $e \le 2f_x \ll_{q_0,d} x^{12}$ and $d_K \le 1 + v_p(e)$. In particular $p^{J+1} \ge 8e$ at large nodes. Theorem 16.43 gives integral points over p^κ at each cut for the three extensions, with

$$\kappa \ll 1 + J(d_K + 1)\eta + e(b_x + W)/p^{J+1}.$$

Impose Taylor orders $D = \eta + 2$ at each of the first A primes $P_j = \phi^j(E)$ on the two cut points. For a single residue-field embedding collect the jumps over all its full embeddings. The short jumps give one interval of indices of length $O(\eta)$. On each side all far jumps, if present, give one further interval of length $O(\eta)$, by the sign lemma. This uses the actual multiplicities only to specify the jumps. The order correction in Proposition 16.44 uses their union, taking each integer index once. For an interval of length M inside $[-8b_x, 8b_x]$ the sum of the valuations of its nonzero integers is $O(M/(p-1) + \log(2b_x)/\log p)$, by counting multiples of each p^a. The cost over these primes is therefore bounded by

$$C_1 \ll_{q_0,d} (A+1)(A + d_K + 4)\eta + (A+1)\log(2b_x)/\log p.$$

For completeness, let c, i be consecutive cuts at distance a, and $S = \prod_{j<a} P_j^\eta$. In the notation of Lemma 16.42, the one-sided bounds give $SM_{a,i}(U) \subseteq M_{0,i}(U)$ and $SM_{0,i}(V) \subseteq M_{a,i}(V)$. The a-fold Frobenius of an initial uncorrected U-point at c is integral in $M_{a,i}(U)$. Imposing orders D there costs at most C_1, since that lattice is regular at P_j for $j < a$. Scale all points to their maximum quotient exponent $e_* \le \kappa + C_1$, up to rounding. At i, replace the U-point by this corrected Frobenius point, replace the V-point by Frobenius of the corrected one at c, and then do the same on U. The three kernel differences belong respectively to $S^{-1}M_{0,i}(U)$, $S^{-1}M_{a,i}(V)$ and $M_{a,i}(U)$. In the first two moves the unchanged point has displacement divisible by $u\tau \prod_{j<a} P_j^D$. In the last move its displacement contains $u^{p^a}\phi^a(\tau)$. The contraction from the third extension, multiplied by p^κ, has denominator τ at level zero and $\phi^a(\tau)$ at level a. Their ratio is a unit times $\prod_{j<a} P_j$. In each move a factor u survives after clearing these denominators and S. The same divisibilities hold on translates in $G_{K(\varpi^{1/p^A})}$, as used in Proposition 51.22. The Frobenius cochain descent of that proposition thus kills the cup by $p^{2e_*+\kappa}$ on this subgroup. Eisenstein's criterion gives index p^A, so corestriction adds A to the exponent. Extending coefficients for the residue-field components and then applying an integral coefficient projection fixing 1 gives the same annihilator in the original coefficient ring.

Finally $A + d_K = O_{q_0,d}(\log(2x))$, $p^{J+1} > x^{120}$, $G/b_x \le x^{1-10^6}$ and $b_x \ge x^{400}$. They bound $e(b_x + W)/p^{J+1}$ by $O_{q_0,d}(b_x x^{-108})$, and bound all the other terms in $2e_* + \kappa + A$ by the same quantity. □

Theorem 17.9 (The integral local comparison through the product of the ranks)**.** *At every sufficiently large node in the original interval,*

$$\sum_{p\le Q_x}(\log p)\sup\sum_{\substack{v\mid p\\ v\ \text{spherical}}}\sum_{i} d_{v,i}\ll_{q_0,d} f_x(s+t)b_x x^{-108},\qquad Q_x=k_x t.$$

In particular the sum over primes of the supremum over integer nodes $x\ge V_0$ in the interval of these local terms, divided at each node by $sB_x^{\det}$, is $O_{q_0,d}(V_0^{-102})$. Only nodes with $p\le Q_x$ are counted in this last assertion.

Proof. For $p\in\mathcal{R}$, the integral annihilator and the generator count in Theorem 17.6 give $d_{v,i}\ll_{q_0,d}(1+[E_{x,v}:\mathbb{Q}_p])\operatorname{rank}(T_i)b_x x^{-108}$ by the proposition. The sum over all v of their local degrees is $2f_x$, and the sum over i of the ranks is $s+t$. The finite sum of $\log p$ over $\mathcal{R}$ depends only on q_0,d. The complementary sum for $p\le Q_x$ is bounded in Theorem 17.6 by the same expression with x^{-110}. This proves the first assertion. Since $s+t=6s$ and $m_xk_x\ll x^{11/2}$, division by $sB_x^{\det}=sf_xxb_x(\log b_x)^3/(m_xk_x)$ gives $O_{q_0,d}(x^{-103})$ at a node. For each prime the supremum of nonnegative normalized terms is bounded by their sum over nodes. Interchange the two nonnegative sums and use $\sum_{x\ge V_0}x^{-103}\ll V_0^{-102}$. □

17.4 Repeated large weights and the middle range of coefficient primes

Write $Z = x_{1,x}$, $q = q_{1,x}$, $b = b_x$ and $G = xb/Z$. For each coefficient prime define

$$D_{x,p} = \sup \sum_{\substack{v \mid p \\ v \text{ spherical}}} \sum_{i} d_{v,i},$$

with the data and the supremum of Theorem 17.6. The modulus q, which grows along the interval, controls the primes at which the sparse argument can encounter three successive large weights. This continues to hold above $Q_x = k_x t$.

Lemma 17.10 (Three successive large weights)**.** *On the sparse sets chosen in Lemma 51.19, at any prime $p \nmid qd$, three successive large weights in the periodic Frobenius orbit imply $p \equiv 1$ or $-1 \bmod q$. The implication is uniform under translation of the set.*

Proof. Let $M = 2^s = m_x^{22}$ in the notation of the spacing construction, using s here only for the exponent of two. The selected place classes have the form $-ug^e \bmod q$, where u is a fixed unit, $g = 1 + 3l_0q_0$ has order M, and the selected exponents belong to $[0, M/(8L))$, $L = 2l_0$. They contain no nonconstant three-term arithmetic progression. Indeed their construction in Lemma 6.25 gives this property with both positive coefficients equal to one. Thinning the set preserves the property.

Take the three embeddings in the order of arithmetic Frobenius, reversing them if necessary. Its action on a q-th root of unity raises the root to the power p. Reorienting a label to the CM type changes its exponent on the root by a sign. The ratios of the second and third place classes to the first are thus $\pm p$ and $\pm p^2 \bmod q$, respectively. If their three selected exponents are e_0, e_1, e_2, raising the first ratio to the fourth power and the second to the second power gives

$$2(2e_1 - e_0 - e_2) \equiv 0 \pmod{M}.$$

The integer on the left has absolute value less than M by the range of the exponents. Thus $2e_1 = e_0 + e_2$ and the progression property gives $e_0 = e_1 = e_2$. In particular the first ratio is one modulo the full q, proving the claim. The argument allows repeated visits to the same label. □

Proposition 17.11 (The total loss in the two exceptional classes)**.** *At a sufficiently large node,*

$$\sum_{p \equiv \pm 1 \bmod q} (\log p) D_{x,p} \ll_{q_0,d} f_x(s+t) b \, \frac{(\log(2b))^2}{q}.$$

Here and below $s = r_x$ denotes the rank. The sum includes all primes in the two residue classes.

Proof. For x sufficiently large, a prime congruent to $\pm 1 \bmod q$ is at least $q - 1$, hence greater than any prime divisor of $6q_0d$. The completion is unramified and $p \geq 5$. By Proposition 17.5 and its calculation of the labelled weights, all three crystalline extensions used in the pairing have upper jumps $h_i \leq Cb$. Only this last weight bound is used here. In Theorem 16.18 the forward sum is at most $Cb/(p-1)$. The single-lattice exponent c_i is zero when that sum is less than one and otherwise at most seven times the sum. The common-lattice exponent is at most $c_i + v_p(h_i!) \leq c_i + h_i/(p-1)$. The two-point cochain comparison in Theorem 16.21 therefore gives a cup annihilator p^a with $a \ll b/p$. The three exponents are all zero if $p > Cb + 2$, on increasing C. The contraction here can be formed from $\mathrm{Hom}(Y, T_i)$, as in Corollary 16.20. The generator count of Theorem 17.6 now gives $D_{x,p} \ll f_x(s+t)b/p$, with vanishing beyond $Cb+2$. These bounds allow torsion tests at the given finite precision.

Write the positive integers in the two classes, apart from one, as $kq\pm1,\ k\ge1$. For $p\le Cb+2$ we have $\log p\ll\log(2b)$. Since $kq-1\ge kq/2$,

$$\sum_{\substack{p\le Cb+2\\ p\equiv\pm1 \bmod q}}\frac{\log p}{p}\ll\frac{\log(2b)}{q}\sum_{1\le k\le(Cb+3)/q}\frac{1}{k}\ll\frac{(\log(2b))^2}{q}.$$

This proves the estimate for the actual prime sum. □

Theorem 17.12 (The higher-rank comparison up to $x_{1,x}/x^{12}$)**.** *For a node x put $P_x=x_{1,x}/x^{12}$ and*

$$\mathcal{P}_x=\{p:p\le P_x\}\ \cup\ \{p:p\equiv\pm1 \bmod q_{1,x}\}.$$

On the original fields and interval, one has

$$\sum_p(\log p)\sup_{\substack{x \text{ in the interval}\\ p\in\mathcal{P}_x}}\frac{D_{x,p}}{sB_x^{\det}}\ll_{q_0,d}V_0^{-9/2}.$$

An empty supremum contributes zero. The level and the coefficient choices in $D_{x,p}$ are taken at that node.

Proof. Fix x large. Theorem 17.9 covers $p\le Q_x$. Consider $Q_x<p\le P_x$ outside the two classes of the proposition. These primes are unramified in E_x. Lemma 17.10 gives the no-triple condition for the Frobenius argument with $J=1$. The part of the proof of Proposition 17.5 through its bound for general J uses the prime cutoff only to obtain that condition and the size inequality $16(J+1)p^J\eta\le b$, where $\eta=2(G+W+1)$ and $W\ll x^{42}$. Here $x^{10^6}\le Z\le b^{10^{-6}}$, so $\eta\ll G$ and $p\eta/b\ll x^{-11}$. Thus the size inequality holds. More explicitly, between cuts there are at most three steps. For a pair of large weights of opposite signs, the critical degree inequality is

$$\lceil(b+W)/(p-3)\rceil+p\eta<b-W.$$

It follows from $p>Q_x$, $W=o(b)$, and the preceding bound on $p\eta/b$. The order corrections for the short and far windows and the three changes of points consequently give the uniform exponent

$$H_p\ll_{q_0,d}1+b/p^2+\eta+\log(2b)/\log p.$$

All pairings here are at a spherical place of the original field and use the original precision. The same local generator count as in the proposition bounds $D_{x,p}$ by $O_{q_0,d}(f_x(s+t)H_p)$.

Summing over these primes after removing the factor $f_x(s+t)$ costs at most

$$C_{q_0,d}(b/Q_x+P_x(1+\eta+\log(2b))).$$

Indeed $\sum_{p\le y}\log p\ll y$ and dyadic summation gives $\sum_{p>Q_x}(\log p)/p^2\ll1/Q_x$. We have $G\ge b^{1-10^{-6}}$, so $1+\eta+\log(2b)\ll G$. Now $P_xG=bx^{-11}$ and $Q_x\asymp x^{10}$. The displayed sum is $O_{q_0,d}(bx^{-10})$. For the exceptional classes at every prime the proposition gives $O_{q_0,d}(b(\log(2b))^2/q)$ after the same factor is removed. Since $q\ge m_x^{22}\ge x^{11}$, adding these terms and the small-prime bound gives

$$\frac{1}{sB_x^{\det}}\sum_{p\in\mathcal{P}_x}(\log p)D_{x,p}\ll_{q_0,d}\frac{x^{9/2}}{(\log b)^3}\left(x^{-10}+x^{-11}(\log(2b))^2\right)\ll_{q_0,d}x^{-11/2}.$$

Here $s+t=6s$ and $m_xk_x/x\ll x^{9/2}$. For each prime bound the interval supremum of the nonnegative terms by their sum over its eligible nodes. Interchanging the sums and summing $x^{-11/2}$ over integers $x\ge V_0$ proves the assertion. □

17.5 Integral lifting at primes above the inverse weight gap

Keep the trace relations and $D_{x,p}$ of Sections 17.2 and 17.4. The normalized graph systems at the chosen automorphic twist have scalar weights $0, \dots, t-1$. We use this normalization throughout. Let $h = s^8$. The old weights have absolute value at least $h - (t-1)/2$, which permits integral comparison of both sides of a cup product near the upper end of the range of coefficient primes.

Lemma 17.13 (Choosing the order of the two moves at each label)**.** *Use the crystalline extensions, integral contraction, unramified field, lifts of E^a and integer $B < p$ in Theorem 16.27. At each embedding suppose that each of the two initial extensions has a unit lift or has no jump in $[-a, 0)$. Suppose the two sides at this embedding can be denoted by U, V so that all jumps of U are at most B and all jumps of V are at least $-(B+1)$. The choice of U may depend on the embedding. Then the exponent $s_X + s_{X'}$ of that theorem still annihilates the cup at its given precision.*

Proof. Take $D = a + B + 1$ and $P = \phi(E)$. On the two original sides X, X', independently, construct the integral points of Theorem 16.25 and scale them to the respective maxima $s_X, s_{X'}$ over all labels. Their displacements have factors $u\tau E^D$ in the first kernel lattices. The initial cup is small, in the sense of Theorem 16.2, since the pairing on the first lattices has denominator $E^a\tau$. On the adjacent Frobenius lattices it has denominator $P^a\mu$, with $\mu = \phi(\tau)$ in this notation.

Fix a label i and name the two points u_i, v_i using the choice U, V at i. A subscript i^- in the following comparison denotes the point on that *same* original side at the preceding label, regardless of the names chosen there. For ϕv_{i^-} the displacement has factor $u^p\mu P^D$ in its adjacent lattice. Corollary 16.26 supplies a point q_i in that lattice of the same quotient, with displacement in $u\mu E^B P^a$ times its kernel. The cost $v_p(B!)$ is zero. First change v_i to q_i. The difference and its translates lie in $E^{-(B+1)}M_{0,i}(V)$ on the kernel, by the lower jump bound. The fixed U-displacement clears this power, the pairing denominator, and leaves u. Next change u_i to ϕu_{i^-}. The difference now lies in $E^{-B}M_{1,i}(U)$ by the upper jump bound, and the displacement of q_i clears it and the adjacent denominator. Finally change q_i to ϕv_{i^-}. Both are in the adjacent lattice, and the displacement of ϕu_{i^-} leaves a factor $u^p P^{D-a}$.

All these inclusions apply to the translates needed for either order of the two arguments, since G_K preserves the lattices and the displayed principal ideals. Thus a first-argument change is small and a second-argument change is a coboundary modulo the small subgroup, by the two cochain identities of Theorem 16.2. In the original X, X' coordinates the terminal pair is exactly the Frobenius transform of the pair at i^- on every label. Summing over the labels and taking Frobenius constants as in Theorem 16.27 gives the original cup multiplied by $p^{s_X + s_{X'}}$. □

Theorem 17.14 (An explicit integer supporting the very large prime defects)**.** *There is a constant $C = C(q_0, d)$ such that, at every sufficiently large node, the positive integer*

$$A_x = \prod_{\substack{n \in \mathbb{Z} \\ |n - b_x| \le C x^{42}}} n$$

has the following property. At every prime $p \ge 8b_x/h$ and spherical place $v \mid p$, the cohomology quotient $C_i/(C_i \cap F_i)$ vanishes when $p \nmid A_x$. At the remaining primes in this range it is killed by p^4. In particular every such prime contributing nonzero length divides just one integer in the displayed product and divides it to the first power.

Proof. Let $b = b_x$, $G = xb_x/x_{1,x}$. Choose an integer $W = O_{q_0,d}(x^{42})$ bounding the deviations from the centers $0, \pm G, \pm b$ in Proposition 17.5, enlarged to cover the upper jumps of the

contraction, which are at most $b + W$. At a spherical place above $p \geq 8b/h$, the field is unramified and all three representations are crystalline by Lemma 17.4. Put

$$a = \lceil (b + W)/(p - 3) \rceil, \qquad B = W.$$

For large x, $a < h/4$ and $B < p$, since $h \ll x^{40}$, $W = o(b)$, and $b \geq x^{400}$. By Theorem 16.8 the quotient E^a lifts on each of the three first lattices. For general coefficients one applies that theorem to the pullback of the trivial quotient along $\mathbb{Z}_p \to \mathcal{O}$ and projects the lift to each residue-field component, as in Corollary 16.28.

We check the two conditions on the initial points. The graph weights consist of the nonnegative scalar entries and the old entries $L \pm (\sigma_u + w_j)$ at the indicated labels, where $L = (t-1)/2$. In the far window the displacement has magnitude at least $G - O(x^{41})$; in the other old entries it is the absolute value of a nonzero odd multiple of h. Hence every negative old graph weight has absolute value at least $h - L$. For the dual of the old target a nonzero weight is of the form $-1 - L \mp (\sigma_u + w_j)$ and has absolute value at least $h - L - 1$. In particular the graph and this dual extension have no jump in $[-a, 0)$. For a scalar dual the jumps of the test extension are $0, -1 - u$ for $0 \leq u < t$; all are nonpositive. Its unit quotient thus lifts by Theorem 16.18, since all upper jump bounds are zero. These facts include every integral dual test.

At each label the graph has a single sign for all its far jumps and the old dual has the opposite sign. A scalar dual has no far jump. Thus the side with negative far jumps has upper jumps bounded by B, and the opposite side has lower jumps at least $-(B + 1)$, increasing the original W once to absorb the shift of the dual. If the negative window is empty, its side still has the required upper bound. Lemma 17.13 applies. Put $D = a + B + 1$. Each valuation index occurring in the exponent on either side is a nonzero integer in some $[-a - u, D - u)$, where u is a jump at the label in question. All these indices lie within Cx^{42} of a center in $\{0, b, -b, G, -G\}$, with C fixed independently of the datum, since $a < h/4$. We may take the same constant in the definition of A_x.

For large x, $G + Cx^{42} < 8b/h$. Indeed $Gh/b = xh/x_{1,x} \to 0$ and $x^{42}h/b \to 0$. Thus the three intervals about $0, \pm G$ contain no nonzero multiple of p. The remaining two intervals are negatives of each other, and the positive one has length $2Cx^{42} < p$. It contains at most one multiple of p, which cannot be a multiple of p^2, since $p^2 \geq 64b^2/h^2 > b + Cx^{42}$. The union therefore contributes at most two to each valuation sum, and zero if $p \nmid A_x$. The lemma gives the claimed exponent for every cup. The annihilator equality in Proposition 15.4 gives the same exponent for the quotient, and gives its vanishing when all cups are zero. □

Corollary 17.15 (The upper tail on the original period interval)**.** *With the data and denominator of Theorem 17.12,*

$$\sum_p (\log p) \sup_{\substack{x \text{ in the interval} \\ p \geq 8b_x/s^8}} \frac{D_{x,p}}{sB_x^{\det}} \ll_{q_0,d} V_0^{-349}.$$

Proof. At an eligible prime the generator count of Theorem 17.6 gives $D_{x,p} \leq 16 f_x(s + t)$. Theorem 17.14 restricts its support to primes dividing A_x. There are $O_{q_0,d}(x^{42})$ positive factors and each is at most $2b_x$. Unique factorization gives $\sum_{p | A_x} \log p \leq \log A_x \ll_{q_0,d} x^{42} \log(2b_x)$. The normalized sum at the node is therefore

$$\ll_{q_0,d} \frac{x^{42+9/2}}{b_x (\log b_x)^2} \ll_{q_0,d} x^{-350},$$

using $b_x \geq x^{400}$. Bounding the primewise suprema by their sums over nodes proves the corollary. □

Together with Theorem 17.12, this leaves the spherical coefficient primes satisfying

$$\frac{x_{1,x}}{x^{12}} < p < \frac{8b_x}{s^8}, \qquad p \not\equiv \pm 1 \pmod{q_{1,x}},$$

for the sum of the local quotients considered here. The integral index of the full tested Hecke lattice in Corollary 52.3 still requires a global comparison.

17.6 All scalar tests in the higher-rank graph

Consider the full higher-rank graph, including its old weights of both signs. For $0 \le c < t$, the scalar summand $\mathcal{O}(-c)$ of the true lift has dual test lattice $\mathcal{O}(c+1)$. Its test extension has only the weights $0, -1-c$ at every label. This permits an integral comparison for the scalar targets throughout the interval of primes left by Theorem 17.14.

Lemma 17.16 (A scalar test against a graph with short weights of both signs)**.** *Let $K/\mathbb{Q}_p$ be unramified, $p \ge 5$. Use the two crystalline extensions X, X' and the integral contraction X'' of Theorem 16.27. Suppose E^a lifts in the first lattice for X, X'' at every label, and a unit lifts there for X'. Let $B \ge 0$, $\eta = B+1$ be integers. Assume all the jumps of X' belong to $[-\eta, 0]$. At each label assume either that all jumps of X are at least $-B$, or that all its jumps are at most B. Put $d = 2a + 2\eta + 1$, $a_X = a$, $a_{X'} = 0$. For $U = X, X'$ define*

$$s_U = \max_i \sum_{\substack{n \ne 0 \\ -a_U - j \le n < d - j \\ \text{for some jump } j \text{ at } i}} v_p(|n|).$$

Then $p^{s_X + s_{X'} + v_p(d!)}$ kills the cup of the two cocycles at its original modulus.

Proof. We give the changes of points, including the effect of negative short graph weights. Write M_i, L_i for the first and adjacent Frobenius lattices at i and $P = \phi(E)$. By Theorem 16.25 there are graph points $x_i \in E^{-a} M_i(X)$ over p^{s_X} and test points $x'_i \in M_i(X')$ over $p^{s_{X'}}$, with displacements in $u\tau E^d$ times the respective first kernel lattices. We suppress the label idempotent on scalar quotients. A point over E^a in X'' gives the denominator $E^a\tau$ for the pairing of first lattices. Frobenius gives $P^a\mu$ for the pairing of adjacent lattices. As in Theorem 16.27, these are inclusions in the integral lattices inside the finite Witt comparison, where the denominators are invertible.

Put $e_0 = v_p(d!)$, $v_i = p^{e_0} x'_i$. Apply Corollary 16.26 to $\phi x'_{i^-} \in L_i(X')$. This point has displacement in $u^p \mu P^d$ times the adjacent kernel, which is contained in $u\tau P^d$ times that kernel. The corollary with both exponents d gives $q_i \in L_i(X')$ over the same quotient as v_i, with displacement in $u\mu E^d P^d$ times the adjacent kernel. On the graph, $\phi x_{i^-} \in P^{-a} L_i(X)$ has displacement in $u^p \mu P^d$ times its adjacent kernel.

The initial cup of displacements at (x_i, v_i) is small since one input supplies $u\tau E^d$ and the first denominator is $E^a\tau$. First change v_i to q_i. The test lower bound gives $L_i(X') \subset E^{-\eta} M_i(X')$, so the kernel difference pairs with the graph displacement into $uE^{d-a-\eta} A_i(1)$, in the notation of Theorem 16.29. Next change x_i to ϕx_{i^-} at fixed q_i. If the graph jumps are at least $-B$, then $L_i(X) \subset E^{-B} M_i(X)$. Its kernel difference belongs to $E^{-a-B} P^{-a} M_i(Y)$, where Y is its kernel. Indeed multiplication by $E^{a+B} P^a$ puts both points in the first lattice with the same quotient. The displacement of q_i belongs to $u\mu E^{d-\eta} P^d$ times the first test kernel. The first denominator leaves the factor

$$u\,(\mu/\tau)\, E^{d-\eta-2a-B} P^{d-a}.$$

Both exponents are nonnegative. If instead the graph jumps are at most B, we have $M_i(X) \subset E^{-B} L_i(X)$. The same multiplication puts the graph difference in the adjacent kernel. Using q_i in its adjacent lattice and the adjacent denominator leaves $uE^{d-a-B} P^{d-2a}$, again small. Finally change q_i to ϕv_{i^-}. Both are in the adjacent test lattice and the fixed graph displacement contributes $u^p \mu P^d$. Every translate satisfies the same inclusions.

The terminal pair is the Frobenius of the actual initial pair at the preceding label. The first and second cochain identities and descent to the Frobenius constants, as in Theorems 16.2 and 16.21, therefore give a coboundary for the stated multiple. The argument over coefficients containing all labels descends by an integral coefficient projection fixing 1. □

Theorem 17.17 (The full spherical sum for the scalar summands of a true lift)**.** *With the original field, twists and interval, write*

$$D^{\mathrm{sc}}_{x,p} = \sup \sum_{\substack{v\mid p \\ v \text{ spherical}}} \sum_{0\le c<t} d_{v,\mathcal{O}(-c)}.$$

Then for sufficiently large V_0

$$\sum_p (\log p) \sup_{x \text{ in the interval}} \frac{D^{\mathrm{sc}}_{x,p}}{sB^{\det}_x} \ll_{q_0,d} V_0^{-9/2}.$$

Here the trace cocycles can come from the entire higher-rank graph at the specified auxiliary levels. All finite precisions and all integral dual classes are included.

Proof. At $p \le P_x = x_{1,x}/x^{12}$, this is included in Theorem 17.12 by nonnegativity. Fix a node and write $b = b_x$, $G = xb/x_{1,x}$. For $p > P_x$ the field is unramified. At each spherical place its graph systems and their normalized summands are crystalline by Lemma 17.4. Form the two extensions plus the contraction for a single test as in Proposition 17.5. Choose $W = O_{q_0,d}(x^{42})$, enlarged to exceed $t+2$, so that all short weights have absolute value at most W and at a label of base weight $\beta \in \{b, G\}$ all the graph's far weights lie in one signed window $\epsilon\beta + [-W, W]$. At such a label either the lower bound $-W$ or the upper bound W applies to all graph weights. Both the graph and the contraction have upper jumps at most $b + W$, by increasing W to cover weight differences. The scalar test has only $0, -1 - c$ as jumps. Hence its unit lifts at every label, by Theorem 16.18. For X, X'' use Theorem 16.8 to lift E^a at all labels, where $a = \lceil (b+W)/(p-3) \rceil$. It applies to these coefficient lattices by the pullback and projection in Corollary 16.28.

Apply the lemma with $B = W$. Each integer in either of its two Taylor sums, and each positive integer through d in the factorial, lies within $C(W + 1 + b/p)$ of a member of $\{0, b, -b, G, -G\}$. Indeed the index interval about the negative of a jump has endpoints $-a_U - j, d - j$, and $a, d = O(W + 1 + b/p)$. The exponent for any scalar test is at most

$$3 \sum_{n\in\mathcal{I}_x} v_p(|n|), \qquad \mathcal{I}_x = \{n \in \mathbb{Z} \setminus \{0\} : \min_{\beta\in\{0,b,-b,G,-G\}} |n - \beta| \le C(W + 1 + b/P_x)\}.$$

This single set works for all $p > P_x$ and all the data at the node. Each integer in one Taylor union is counted only once. A unit of normalized length in a dual coordinate is measured using $v_\pi(p)$, so the local generator count and exact annihilator in Theorem 17.6 give

$$D^{\mathrm{sc}}_{x,p} \ll_{q_0,d} f_x t \sum_{n\in\mathcal{I}_x} v_p(|n|).$$

Here we summed the ranks of the scalar targets, whose sum is t.

There are $O_{q_0,d}(W + 1 + b/P_x)$ nonzero integers in $\mathcal{I}_x$, all of absolute value at most $2b$ for large x. By unique factorization,

$$\sum_{p>P_x} (\log p) D^{\mathrm{sc}}_{x,p} \ll_{q_0,d} f_x t \sum_{n\in\mathcal{I}_x} \log|n| \ll_{q_0,d} f_x t (W + 1 + b/P_x) \log(2b).$$

Since $P_x \ge x^{10^6-12}$, $W \ll x^{42}$, and $b \ge x^{400}$, the factor $W + 1 + b/P_x$ is $O_{q_0,d}(bx^{-120})$. Division by $sB^{\det}_x$ costs at most $O(x^{9/2}/(f_x t b(\log b)^3))$ as a multiplicative factor, by $t = 5s$ and $m_x k_x/x \ll x^{9/2}$. The normalized contribution is $O_{q_0,d}(x^{-115})$. Sum over the integer nodes to bound the primewise suprema for $p > P_x$, and add the bound already proved for $p \le P_x$. □

For the local sum over spherical places, only the target R_ξ of rank s remains in the intermediate range of Section 17.5. Its dual contains large weights, so the bounded test hypothesis in Lemma 17.16 holds here specifically for the scalar targets.

17.7 Removing the poles of a family of integral dual tests

Let $K/\mathbb{Q}_p$ be unramified of degree f, with $p \geq 5$. Let $\mathcal{O}$ be the valuation ring of a finite coefficient field L containing the images of K, and π a uniformizer. We measure finite length by $\ell_p(A) = \mathrm{length}_{\mathcal{O}}(A)/v_\pi(p)$. The f labels are the embeddings of K into L. At a label i use $R_i = \mathcal{O}[[u]]$, $E = u - p$ and the first Breuil–Kisin lattice M_i of Lemma 16.23. We first keep track of all dual cocycles together.

Proposition 17.18 (The cost of removing a pole at one labelled weight)**.** *Let $X \to B$ be a surjection of crystalline $\mathcal{O}$-lattices with trivial free quotient B and kernel Y. Take integers $a, D \geq 0$. Suppose that for each basis vector of B and each label its multiples by E^a and by p^a have lifts in $M_i(X)$. Let S be an integer at least the maximum over labels of*

$$\sum_{\substack{n \neq 0 \\ -a-j \leq n < D-j \quad \text{for some labelled weight } j \text{ of } X}} v_p(|n|).$$

There is a sublattice $B_* \subseteq B$ with

$$\ell_p(B/B_*) \leq S \operatorname{rank} B + (a+S) \sum_i \sum_{-a \leq t < 0} \mathrm{mult}_{i,Y}(t).$$

Every $c \in B_*$ has, at every label i, an integral point in $M_i(X)$ over $\mathbf{e}_i c$ with displacement in $u\tau E^D M_{i,A}(Y)$ for every $g \in G_K$. Here $\mathbf{e}_i$ denotes the label idempotent.

Proof. Fix a horizontal basis d_j adapted to the filtration at i, allowing repeated indices for repeated weights. An element of $E^t M_i(X)$ has, in the direction of weight j, a coordinate in $E^{t-j}L[[E]]$. For $-a \leq t \leq 0$ let $I_{i,t} \subseteq B$ be the constants with a lift in $E^t M_i(X)$ satisfying the following two conditions. Its coordinate in every direction of weight j agrees with an element of L modulo $E^{D-j}L[[E]]$, and its displacement belongs to $u\tau E^D M_{i,A}(Y)$. These are $\mathcal{O}$-linear conditions on the space of points over constants. The kernel statement of Lemma 16.13 applies to a trivial free quotient of any finite rank, by preservation of kernels or by taking the kernels of its coordinate maps successively.

The Taylor construction in Theorem 16.25 applies to a point over any constant vector. At a nonzero integer order the divided displacement has quotient zero, so the correction multiplies the quotient vector by $p^{v_p(|n|)}$. At order zero one takes the actual Taylor constant in each horizontal coordinate. Thus the proof gives both conditions defining $I_{i,t}$. Starting with an E^a-lift divided by E^a yields $p^S B \subset I_{i,-a}$. Starting with an integral p^a-lift yields $p^{a+S}B \subset I_{i,0}$. We use the same bound S at both starts, multiplying up to the indicated exponent.

Consider a step $t \to t+1$, with $t < 0$. For $j < t$ the initial order $t - j$ is positive, so the correcting Taylor constant in that direction must be zero, since $D - j > t - j$. The order is therefore at least $t + 1 - j$. For $j > t$, agreement with a constant gives order at least $\min(0, D - j) \geq t + 1 - j$. For $j = t$, exactly one coefficient in each such direction can obstruct the next step, namely its constant term. Vanishing of all these coefficients puts the point in $E^{t+1}M_i(X)$. Indeed after division by E^t the horizontal calculation shows divisibility by E in the rational Taylor lattice. It holds in $M_i(X)$ since this module is free over R_i and $R_i/E = \mathcal{O}$ injects into L.

The map taking these leading coefficients on the space of allowed points takes values in L^h, where $h = \mathrm{mult}_{i,Y}(t)$. The quotient B adds only weight zero. The image is a finitely generated $\mathcal{O}$-module of rank at most h. In fact the map on $E^t M_i(X)$ in just the directions $j = t$ factors through its finite free $\mathcal{O}$-fiber modulo E, and $\mathcal{O}$ is noetherian. The image on the allowed points is free and surjects onto $I_{i,t}/I_{i,t+1}$ by projection to B. Thus that latter group needs at most h

generators. It is killed by p^{a+S} by the integral start above. Its normalized length is at most $h(a+S)$.

Set $B_* = \bigcap_i I_{i,0}$. The quotient of $p^S B$ by its intersection with B_* embeds in $\bigoplus_i I_{i,-a}/I_{i,0}$. Adding the length of $B/p^S B$ and summing the successive pole steps gives the stated bound. This counts the initial Taylor cost $S \operatorname{rank} B$ only once. □

Theorem 17.19 (The main summand at the split places with negative far graph weights)**.** *Use the original fields and twists and $P_x = x_{1,x}/x^{12}$. For each datum let $\mathcal{V}^-(x,p)$ consist of the spherical places over p at which all labelled far weights of the normalized graph are negative. For the main target, a lattice in R_ξ, one has*

$$\sum_p (\log p) \sup_{\substack{x \text{ in the interval} \\ p > P_x}} \sup_{\text{data}} \frac{\sum_{v \in \mathcal{V}^-(x,p)} d_{v,R_\xi}}{s B_x^{\det}} \ll_{q_0,d} (\log(2V_0))^{-2}.$$

At every prime splitting in $K_d = \mathbb{Q}(\sqrt{-d})$, the negative orientation selects one place in each conjugate pair before restriction to the spherical level.

Proof. Fix a node, prime and place in the assertion. Write $b = b_x$, $G = xb/x_{1,x}$, and choose $W = O_{q_0,d}(x^{42})$ large enough for all the deviations in Proposition 17.5. Enlarge it to bound the short jumps on graph and dual and to make $b+W$ an upper jump bound on their integral contraction. Here $p > P_x$ is unramified in E_x and $W+2 < p$. All the representations in question are crystalline at this spherical place. Put $a = \lceil (b+W)/(p-3) \rceil$, $D = a + W + 1$. Choose one integer S_p, independent of the place and the datum at the node, as the sum of $v_p(|n|)$ over all nonzero integers within $C(W + 1 + b/P_x)$ of one of $0, b, -b, G, -G$. As in Theorem 17.17, this includes every index in the Taylor sums for these choices of a, D.

Let T be the main target lattice and $T' = T^\vee(1)$. For B take the lattice of all integral finite-condition raw cocycles with values in T'. Its rank is at most $(1+f)s$ by Lemma 12.27. The lattice $X' = T' \oplus B$ has action

$$g(y,c) = (gy + c(g), c).$$

It is crystalline after inverting p. Indeed for an $\mathcal{O}$-basis of B, take the direct sum of the crystalline extensions defined by its basis cocycles and push out the T' kernels by addition. The resulting extension has the displayed action. For every vector $c \in B$, apply Theorem 16.8 to the pullback along $1 \mapsto c$. It gives the two lifts required in Proposition 17.18; mapping to X' and projecting to the label gives the indicated quotient vector. This uses only the common upper bound $b+W$, including the trivial quotient.

The far weights of T' at this place are positive. At most m_x weights of T' per label can lie in $[-a, 0)$. Indeed the old row has $k_x - 1$ far entries out of s, leaving $s + 1 - k_x = m_x$ entries, by Proposition 17.1. Moreover every negative old dual weight has absolute value at least $h - (t-1)/2 - 1$, $h = s^8$. This exceeds a whenever $p \geq 8b/h$, as checked in Theorem 17.14. Proposition 17.18 therefore produces integral deep points for all c in a sublattice B_* whose normalized colength is at most

$$S_p(1+f)s + f m_x (a + S_p) \mathbf{1}_{\{p < 8b/h\}}.$$

On the graph all upper jumps are at most $W < p-1$, so Theorem 16.18 gives an integral unit point at every label. Correct it to depth D with Theorem 16.25 and scale to quotient p^{S_p} simultaneously. The contraction lattice has its E^a-lifts by Theorem 16.8. For each $c \in B_*$

both initial points are now integral at all labels, with the displacements of Lemma 17.13. Its three moves apply with the graph as upper-bounded side and the dual as lower-bounded side. The dual point has constant vector c, and the graph point has constant p^{S_p}. In the cochain identity the displacement on the dual is $c(g) + gy - y$ for a kernel coordinate y. Thus the identity gives $p^{S_p}[z \cup \operatorname{red} c] = 0$ for every local graph class z, also at the original modulus. The cochain argument uses kernel differences at equal quotient vectors, exactly as for a trivial line. In particular $p^{S_p} B_*$ annihilates all the graph classes. This sublattice is the same for all z at the given target and local datum, because its construction used only T'.

By the annihilator equality of Proposition 15.4, pairing injects the quotient measured by d_{v,R_ξ} into $\operatorname{Hom}_{\mathcal{O}}(B/p^{S_p}B_*, \mathcal{O}/\pi^N)$ at precision π^N. On a cyclic summand the length of this Hom is bounded by that of its domain. Since $\sum_{v|p}[E_{x,v} : \mathbb{Q}_p] = 2f_x$, the preceding estimates give, uniformly at the node,

$$\sum_{v \in \mathcal{V}^-(x,p)} d_{v,R_\xi} \ll_{q_0,d} f_x\big(sS_p + m_x(b/p)\mathbf{1}_{\{p<8b/h\}}\big).$$

We used $a \ll b/p$ on the indicated range and $m_x \le s$. The proof may use coefficients containing the images of the local field. Finite-condition raw cocycles commute with finite free scalar extension, by expansion in a coefficient basis and the rational finite condition. Local cohomology with the extended finite module is likewise the corresponding scalar extension. The original graph classes after extension satisfy the same relations, by flatness. Normalized length of a finite module is unchanged under this extension, so the inequality applies over the original coefficient ring.

The total of $(\log p)S_p$ for $p > P_x$ is at most $C(W+1+b/P_x)\log(2b)$, by factoring the integers in the fixed windows. As in Theorem 17.17, summing the corresponding normalized costs over nodes is $O_{q_0,d}(V_0^{-110})$. For the second term divide by $sB_x^{\det} = sf_x xb(\log b)^3/(m_x k_x)$. Since $m_x^2 k_x/(sx) \ll 1$ and $p < 8b/h \le b$, the result at this prime is $O(1/(p(\log p)^3))$, uniformly over eligible nodes. Each such prime exceeds $V_0^{10^6-12}$. On a dyadic interval $[y, 2y)$, $\sum_{p<2y} \log p \ll y$ bounds the weighted sum by $O((\log y)^{-3})$. Summing those intervals gives the claimed squared logarithm. Finally the orientation of the far weights is fixed by the restriction of the label to K_d, as in the first part of Lemma 17.7. At a split prime all labels of one completion induce the same embedding of K_d; conjugation reverses it. □

17.8 A two-prime comparison for the main summand on arbitrary orbits

The pole bound in Lemma 16.32 applies to the main summand as soon as the far weights miss the possible pole interval. The short weights may still include negative entries on both sides. We give the comparison for these signs, allowing the sign of the far weights to change at each step.

Proposition 17.20 (Comparison with bounded residual poles on both sides)**.** *Take the unramified local setting, three crystalline extensions, and integral contraction of Theorem 16.27. Assume $p \geq 5$ and E^a lifts on every label for all three extensions. Choose nonnegative integers e, B. For the first two extensions at each label, suppose the negative jumps in $[-a, 0)$ have absolute value at most e. At each label suppose the two sides can be denoted U, V so that U has lower jumps at least $-B$ and V has upper jumps at most B. Put $D = a + e + B + 2$. Let S be the maximum over both sides and all labels of $\sum v_p(|n|)$ over nonzero integers in the union of the intervals $[-a - j, D - j)$ for the jumps j. Let k bound the maximum of $\sum(2 + v_p(|n|))$ over nonzero integers in the union of $[-j, e - j)$ at any label on either side. The cup is annihilated by $p^{2S+2k+1}$.*

Proof. Use $E = u - p$, $P = \phi(E)$, $\delta = u\tau$, $\mu = \phi(\tau)$. At each label and on each side, Theorem 16.25 constructs a point from the E^a-lift divided by E^a. Scale to constant p^S and call the points u_i, v_i using the assignment at i. Their displacements are in δE^D times the first kernel lattices. By Lemma 16.32 these points belong to $E^{-e}M_i$, and v_i also belongs to L_i because its jumps are at most $B \leq D$. Here M_i, L_i are the first and adjacent lattices. The contraction gives denominators $E^a\tau$ and $P^a\mu$ in the first and adjacent pairings, respectively.

Write $H = G_{K(p^{1/p})}$. Apply Lemma 16.31 at P to v_i with order e. Its cost uses the jumps of the same side at the preceding label and is at most k. After scaling the result at i to quotient p^{S+k}, write it as q_i. Then

$$q_i - p^k v_i \in E^D M_i(V) \cap \ker \beta, \qquad (g-1)q_i \in \delta E^D P^e M_{i,A}(\ker \beta_V) \quad (g \in H).$$

Here β denotes projection onto the trivial quotient. We have $q_i \in L_i(V)$ since $E^D M_i(V) \subseteq L_i(V)$. In the next comparison the subscripts i^- refer to the same original side as at i, before assigning names to the two points at i^-.

Start with the two initial points multiplied by p^k. Their displacements pair into $uA_i(1)$ because $D \geq a$. First change $p^k v_i$ to q_i. Its difference is in the first kernel lattice, where the unchanged displacement and the first pairing denominator leave a factor u. Next change $p^k u_i$ to $p^k \phi u_{i^-}$ at fixed q_i. The first point has at most an E^e denominator and the second lies in $P^{-e}L_i(U)$. The lower jump bound gives $L_i(U) \subset E^{-B}M_i(U)$. Multiplying both points by $E^{e+B}P^e$ puts them in the first lattice, where their quotient coordinates agree. The displacement of q_i and the first pairing bound consequently give $uE^{D-e-B-a}A_i(1)$ for the move. Finally change q_i to $p^k \phi v_{i^-}$. Both are in $P^{-e}L_i(V)$. Pair the difference with the displacement of the Frobenius point on U in $u^p \mu P^D L_{i,A}$. The second denominator leaves $u^p P^{D-e-a}$. All the remaining exponents are nonnegative. The same inclusions hold for translates by H.

The three comparisons leave elements of the small subgroup at every label. The final ordered pair is Frobenius of the preceding initial pair in the original two coordinates. The cochain identities in Theorem 16.2 and descent on H, as in Theorem 16.33, kill the restricted cup by p^{2S+2k}. Since $X^p - p$ is Eisenstein over the unramified field K, corestriction adds one to the exponent. Retraction of any coefficient extension gives the result at the original precision. □

Theorem 17.21 (The interior prime range for all spherical places)**.** *With $D_{x,p}$ as in Section 17.4, put*

$$R_x = 8x_{1,x}/x, \qquad Y_x = b_x/x^{45}.$$

On the original interval and for the original twists one has

$$\sum_{p}(\log p)\sup_{\substack{x \text{ in the interval}\\ R_x\le p\le Y_x}}\frac{D_{x,p}}{sB_x^{\det}}\ll_{q_0,d}(\log(2V_0))^{-2}.$$

This includes the main target and both orientations of the far weights, including alternating orientations on a single Frobenius orbit.

Proof. Write $b=b_x,\ G=xb/x_{1,x},\ h=s^8$. We record the actual size of the short window. In the row of Proposition 17.1 each non-far old displacement is an odd multiple of h with multiplier of absolute value at most $2m_x+1$. The offsets from either far center have size at most $(2m_x+1)h+O(s+t)$. The scalar entries, translations and dual shifts cost at most $O(s+t+1)$. Thus a single integer

$$W\ll_{q_0,d}(m_x+1)h+s+t+1\ll_{q_0,d}x^{81/2}$$

bounds the short jumps and the deviations from the centers $\pm b,\pm G$; take it large enough also to bound the upper jumps of the contraction by $b+W$, using same-label differences. In particular $W=o(G)$. The coefficient prime is unramified in E_x and exceeds $W+2$ for large x. Put $a=\lceil(b+W)/(p-3)\rceil$. Since $p\ge 8b/G$ and $b/G\to\infty$, we have $a<G/4$. The far jumps of either side have absolute value at least $G-W>a$. All possible negative jumps in $[-a,0)$ therefore have absolute value at most W.

At every spherical place above p, use the graph, dual and contraction of Proposition 17.5. Their upper jump bounds give the E^a-lifts, by Theorem 16.8. At any one label the two far windows on the first two sides have opposite signs when both occur. Hence one side has all jumps at least $-W$ and the other has all jumps at most W, with the same conclusion for a scalar dual. Apply the proposition with $B=e=W$. For the first Taylor cost take S_p to be the sum of $v_p(|n|)$ over nonzero integers within $C(W+1+b/P_x)$ of $0,\pm b,\pm G$, where $P_x=x_{1,x}/x^{12}$. Indeed $R_x>P_x$ and $D+a=O(W+1+b/p)$. For the second cost at any one label, the union of $[-j,W-j)$ is covered by a short interval and a single far interval of total length $O(W+1)$, with absolute endpoints at most $3b$. Counting multiples of powers of p as in Proposition 17.5 gives

$$k\ll_{q_0,d}W+1+\log(2b)/\log p.$$

The actual sums and their multiplicities have been taken before any supremum over levels or embeddings. Uniformly over those choices, exact local duality and the generator count in Theorem 17.6 now give

$$D_{x,p}\ll_{q_0,d}f_x(s+t)\big(S_p+W+1+\log(2b)/\log p\big).$$

As in Theorem 17.17, unique factorization bounds the weighted sum of the S_p at a node by $O_{q_0,d}((W+1+b/P_x)\log(2b))$. After division by the determinant normalization the sum of the corresponding node costs over $x\ge V_0$ is $O_{q_0,d}(V_0^{-110})$. For the other terms use $m_xk_x/x\ll x^{9/2}$, $s+t=6s$, and $p\le b/x^{45}$. In particular

$$\frac{x^{9/2}(W+1)}{b}\ll_{q_0,d}\frac{1}{p}\qquad\text{since } 9/2+81/2=45.$$

The term involving $\log(2b)/\log p$ after normalization is also $O_{q_0,d}(1/(p(\log p)^3))$, using $x^{9/2}/b\le 1/p$ and $\log p\le\log b$. The resulting primewise majorant holds before taking the interval supremum. Since $p\ge R_x\ge x^{10^6-1}$, summing with weight $\log p$ by the dyadic calculation of Theorem 17.19 proves the bound. □

17.9 The transition between the two base weights

Let $Z = x_{1,x}$, $b = b_x$, $G = xb/Z$ and $P_x = Z/x^{12}$. We use the integral base weights at the node in this notation, as in Section 16.14. When p is close to b/G, the possible pole order of a crystalline lift can reach the smaller weight G. Moreover some short jumps can be negative on *both* sides of the pairing. The next proposition allows these poles on the actual higher-rank graph and its dual.

Proposition 17.22 (A local cup bound across the smaller weight)**.** *On the higher-rank family of Section 17.1, choose an integral width $W \ll_{q_0,d} x^{81/2}$ as in Theorem 17.21. For every sufficiently large node, every prime $p > P_x$, every spherical place $v \mid p$ and every target, the pairing of a graph class with an integral finite-condition test at the original precision is killed by p^h for an integer*

$$h \ll_{q_0,d} W + 1 + \frac{b}{p^2} + \frac{G}{p} + \frac{\log(2b)}{\log p}.$$

All implied constants are independent of the orbit, the trace rank, the level, the coefficient field and the precision.

Proof. Fix a place and extend coefficients to contain the unramified field $K = E_{x,v}$. Write $M_{\ell,i}$ for the ℓ-th Frobenius lattice at embedding i, $i^- = F^{-1}i$, $E = u - p$, $P = \phi E$, $\delta = u\tau$, and $\tau_\ell = \phi^\ell \tau$. We use the three integral extensions of Proposition 17.5, associated to the graph, the test and the contraction Q. At every embedding with base weight $\beta_i = b$ or G, all short jumps of the first two extensions lie in $[-W, W]$. Their other jumps are within W of $+\beta_i$ on one side and $-\beta_i$ on the other. For a scalar test the latter assignment on the side without far jumps is arbitrary once the graph sign is fixed. All upper jumps of the third extension are at most $\beta_i + W$. These are the same-label bounds of Proposition 17.5, with the width calculated in Theorem 17.21.

Put

$$a = \left\lceil \frac{b+W}{p-3} \right\rceil, \quad J = W + 1 + \lceil b/p^2 \rceil + \lceil G/p \rceil, \quad D = 2a + 3W + 10.$$

Our scale assumptions $x^{10^6} \le Z \le b^{10^{-6}}$ give $p > 4(W+2)$, $G > 2(W+2)$, and $b - W > a$ for large x. In particular $a \le b/p + 8J$ and $a/p \le 4J$. Theorem 16.8, applied to the trivial-line pullback and then projected to a label, gives integral lifts of $p^k E^s$ on all three extensions whenever $k + s \ge a$.

We select labels on the cyclic orbit and at each selected label specify an integer s_i, together with a common κ, so that $p^\kappa E^{s_i}$ lifts.

1. If $b/p - G \le 40J$, select all labels and take $s_i = \min(a, \lfloor G - W - 2 \rfloor)$, $\kappa = a - s_i$. Then $0 \le \kappa \le 51J$. A negative far jump has absolute value at least $G - W > s_i$.

2. If $b/p - G > 40J$, select each label of weight b and each label followed by one of weight G. At labels of the second kind put $s_i = 0$. For each of the three extensions the weighted sum of future upper bounds there is at most

$$(G+W)/p + (b+W)/(p(p-1)).$$

   Theorem 16.18 thus supplies integral lifts of a common p^κ with $\kappa \ll J$; take the largest required exponent, or zero for the empty set. At every other selected label put $s_i = a$. Its weight is b, so $b - W > a$ excludes every negative far jump from $[-s_i, 0)$. The unselected labels have weight G and are followed by weight b. They are isolated. A gap of two steps starts at a selected label with $s_i = 0$.

At each selected label, divide a lift on each of the first two sides by E^{s_i}. Raise its displacement to order δE^D in $M_{0,i,A}$ by Theorem 16.25. Its Taylor operators apply to any constant quotient since their corrections have zero quotient. The exponent paid is the sum of $v_p(|n|)$ over the nonzero integers in the union of $[-s_i-j, D-j)$ for the jumps j. This union lies in two intervals of total length $O(a+W+1)$ and absolute endpoints at most $5b$. In an interval of length N, the nonzero multiples of p^r number at most $1+N/p^r$. Summing over $1 \le r \le \log(5b)/\log p$ bounds the Taylor cost by $O(J+\log(2b)/\log p)$.

Lemma 16.32 puts the resulting point in $E^{-W}M_{0,i}$, since every possible negative jump of absolute value at most s_i is short. At a label with $s_i = 0$ the point is integral. On a side with upper jumps at most W the point is also in $M_{1,i}$. On each side use Lemma 16.31 to impose order W at P over $H = G_{K(p^{1/p})}$. Its cost is the sum of $2+v_p(|n|)$ over the nonzero integers in the union of $[-j, W-j)$ at the preceding label. The two containing intervals now have total length $O(W+1)$, so the additional exponent is $O(W+1+\log(2b)/\log p)$. The correction is in $E^D M_{0,i}$, hence preserves all three membership properties just stated, using $E^D M_{0,i} \subset M_{1,i}$ when the upper jumps are at most W. Write $d_z(g) = (g-1)z$. The selected points now have

$$d_z(g) \in \delta E^D P^W M_{0,i,A}(\ker \beta) \qquad (g \in H),$$

where β is projection to the trivial line.

On the third extension divide a lift of $p^\kappa E^{s_c}$ at a selected label c by E^{s_c}. By the integral contraction of Corollary 16.20 and Frobenius transport, $p^\kappa Q$ on the two ℓ-th kernel lattices at $F^\ell c$ takes values in

$$\phi^\ell(E)^{-s_c}\tau_\ell^{-1} A_{F^\ell c}(1)$$

after reduction at the given precision. We shall use τ_1/τ equal to E times a unit and τ_2/τ equal to EP times a unit. Call a value small when it belongs to the image of $uA_i(1)$.

Consider any two consecutive selected labels c, i, allowing wraparound. Name the positive far side at i by U and the other by V, using these names for the same original systems at the preceding labels. Thus at i the lower jumps on U are at least $-W$ and the upper jumps on V are at most W. Denote the selected points by u_i, v_i. We describe comparisons between points with the same quotient. All auxiliary points below are constructed first. We then scale each selected or auxiliary point to the largest quotient exponent e_* that occurs on the orbit. This does not affect membership or displacement divisibility.

For a gap of one step, first change u_i to ϕu_c at fixed v_i. Both points lie in $E^{-W}P^{-W}M_{0,i}(U)$: the second lies in $P^{-W}M_{1,i}(U)$, and $M_{1,i}(U) \subseteq E^{-W}M_{0,i}(U)$ by the lower bound. Their difference belongs to the corresponding kernel lattice. Pairing with d_{v_i} in the first lattice leaves a multiple of uE^{D-W-s_i}. Next change v_i to ϕv_c. Both are in $P^{-W}M_{1,i}(V)$. Pairing their kernel difference with ϕd_{u_c} in the adjacent lattice leaves a multiple of $u^p P^{D-W-s_c}$.

For a gap of two steps, the intervening label i^- has weight G, i has weight b, and $s_c = 0$. Hence $\phi^2 u_c, \phi^2 v_c$ lie in their second lattices. Put $S = E^W P^W$. There are two possible sign patterns.

1. Suppose the signs at i^-, i agree. Successive one-sided height inclusions give $SM_{2,i}(U) \subset M_{0,i}(U)$ and $SM_{0,i}(V) \subset M_{2,i}(V)$. For U start with the twice-transported pure lift of p^κ from c. Apply Proposition 16.39 in $M_{2,i}(U)$ with $L = 1$, raising the orders at both E, P to $T = 2W+2$. The rational Taylor lattices at both primes are regular on $M_{2,i}$, and $T < p$, so the exponent added is at most $6(T-1)$. The resulting $q \in M_{2,i}(U)$ has $Sq \in M_{0,i}(U)$ and $d_q \in \delta E^T P^T M_{2,i,A}$ on H.

   First change u_i to q at fixed v_i. The two points are in $S^{-1}M_{0,i}(U)$, and the first pairing leaves a multiple of uE^{D-W-s_i}. At fixed q change v_i to $\phi^2 v_c$. The difference is in

$E^{-2W}P^{-W}M_{2,i}(V)$ with zero quotient, by $v_i \in E^{-W}M_{0,i}(V)$ and the second height inclusion. The pairing with d_q costs τ_2 and leaves a multiple of $uE^{T-2W-1}P^{T-W-1}$. Finally change q to $\phi^2 u_c$ at fixed $\phi^2 v_c$. Both moving points are integral in $M_{2,i}(U)$, and the fixed displacement supplies $u^{p^2}\tau_2$.

2. Suppose the signs at i^-, i differ. At i^- the upper and lower bounds on V are $G+W$ and $-W$, and U has upper jump at most W. At i the upper bounds on V, U are $W, b+W$ respectively. Use Theorem 16.40 on V with $\ell = 2$, $\eta = W$, $h_0 = W$, $h_1 = G+W$ and $A = 2(b+W)+3W+6$. The required E^a-lift at i and pure p^κ-lift at c were proved above. In this two-step case $pG < b - 40pJ$. Since $W \le J$, $b/p \le pJ$, and $a \le b/p + 8J$, it follows that

$$a + p(G+W) < b - W, \qquad N = a + W + p(G+W) < A.$$

Every jump $j < -W$ at i has $b - W \le -j \le b + W$; in particular $W - j > N$ and $-j \le A$. The theorem gives $q_0 \in M_{2,i}(V)$ with $Sq_0 \in M_{0,i}(V)$, quotient $p^{\kappa+4\lceil A/p^2\rceil}$, and $d_{q_0} \in \delta E^A M_{2,i,A}$. Apply Lemma 16.36 with order $W+1$. It gives $q \in M_{2,i}(V)$ with displacement in $\delta E^A P^{W+1} M_{2,i,A}$ on H, at cost $2W + v_p(W!) = 2W$. The correction is in $E^A M_{2,i}(V)$. Here $M_{2,i}(V) \subseteq P^{-W}M_{1,i}(V) \subseteq P^{-W}E^{-(b+W)}M_{0,i}(V)$, by the two lower bounds. Consequently $Sq \in M_{0,i}(V)$ as well.

Change v_i to q at fixed u_i. Their difference is in the first kernel lattice after multiplication by S, and d_{u_i} supplies that factor as well as $uE^{s_i}\tau$. Change u_i to $\phi^2 u_c$ at fixed q. The successive upper bounds on U and the pole of u_i put the difference in $E^{-(b+2W)}P^{-W}M_{2,i}(U)$ with zero quotient. The displacement of q and the second Frobenius pairing leave a multiple of $uE^{A-b-2W-1}$. Finally change q to $\phi^2 v_c$. Both are integral in $M_{2,i}(V)$ and the fixed displacement supplies $u^{p^2}\tau_2$.

All the displayed exponents after cancellation are nonnegative. The kernel inclusions follow from Lemma 16.13 after flat Frobenius base change; H preserves each indicated A_i-lattice and principal ideal, so translates of differences have the same bounds. The estimates for the selected and auxiliary points give $e_* \ll J + \log(2b)/\log p$. At an unselected label take both initial points to be the Frobenius of the points at its predecessor. At a selected label the initial cup of displacements is small since $D \ge s_i$. Its Frobenius at an unselected label is again small. Thus the initial cup and every change of a point lie in the small subgroup in the cochain comparison of Theorem 16.2. The terminal pair at each label is exactly the single Frobenius of the preceding initial pair, including across a gap of two steps. The descent of Theorem 16.21 on H therefore kills the restricted cup by $p^{2e_*+\kappa}$: the two affine points contribute p^{e_*} each, and the integral pairing used throughout was $p^\kappa Q$. Restriction followed by corestriction costs one further power of p since $X^p - p$ is Eisenstein over K. An integral coefficient retraction fixing one descends the coboundary at the original modulus. This proves the stated bound for $h = 2e_* + \kappa + 1$. □

Theorem 17.23 (All spherical places through b_x/x^{45})**.** *With $D_{x,p}$ of Section 17.4, on the original fields and interval one has*

$$\sum_p (\log p) \sup_{\substack{x \text{ in the interval} \\ p \le b_x/x^{45}}} \frac{D_{x,p}}{sB_x^{\det}} \ll_{q_0,d} (\log(2V_0))^{-2}.$$

Both scalar and main targets are included. On the transition range $P_x < p < 8Z/x$ alone, the left side is $O_{q_0,d}(V_0^{-100})$.

Proof. Theorem 17.12 treats $p \le P_x$; Theorem 17.21 treats $8Z/x \le p \le b_x/x^{45}$. It suffices to check the transition at the same nodes. By exact local duality and the count of generators in Theorem 17.6, the proposition bounds $D_{x,p}$ by $Cf_x(s+t)h$. Set $R = 8Z/x$. Chebyshev's inequality for the sum of $\log p$ and a dyadic decomposition imply $\sum_{p>P_x}(\log p)/p^2 \ll 1/P_x$ and $\sum_{p<R}(\log p)/p \ll 1 + \log R$. Also each prime contributes $\log(2b)$ to the weighted sum of $\log(2b)/\log p$. Thus at a node the total, after removing $Cf_x(s+t)$, is bounded by

$$R(W + 1 + \log(2b)) + b/P_x + G(1 + \log R).$$

Here $R \le 8b^{10^{-6}}$, $W \ll x^{81/2}$, $b \ge x^{400}$, $P_x \ge x^{10^6-12}$, and $G/b = x/Z \le x^{1-10^6}$. Since $f_x(s+t)/(sB_x^{\mathrm{det}}) \ll x^{9/2}/(b(\log b)^3)$, the normalized total is $O_{q_0,d}(x^{-110})$; for example $x^{9/2}R(W+1)/b \ll x^{45}b^{-1+10^{-6}}$. Bound the supremum at each prime by the sum over eligible integer nodes and sum x^{-110} for $x \ge V_0$. Combining the three ranges gives the assertion. □

17.10 The length of one horizontal Taylor condition

Fix an unramified $K/\mathbb{Q}_p$ of degree f, $p \geq 5$. Write C for a finite coefficient field containing the images of K, and $\mathcal{O}$ for its valuation ring. As in Proposition 17.18, let $X \to B$ be a surjection of crystalline lattices with kernel Y and trivial free quotient. Fix a label i, put $R = \mathcal{O}[[u]]$, $E = u - p$, $P = u^p - p$, and write $M = M_{0,i}$. At E use the horizontal coordinates of Lemma 16.23; at P use their Frobenius transports from i^- as in Lemma 16.31. At each prime choose a basis adapted to the filtration on Y and extend it by horizontal lifts of a basis of B of weight zero. Such lifts exist by strict exactness of the filtered de Rham comparison on crystalline representations. The two choices may be made separately. In these coordinates the quotient coordinates of any point over a constant vector are constant. Only kernel coordinates will impose conditions.

At E the rational Taylor coefficients lie in C. At P they lie in $C_1 = C[T]/(T^p - p)$, with $u = T(1 + P/p)^{1/p}$. Call a coordinate of depth n if it agrees with some member of C modulo order n at the prime in question. In particular the approximating constant at P belongs to C even when C_1 has more than one factor. This condition implies order at least n for its Galois displacement under the full group G_K. The horizontal vectors and the external coefficients are fixed and the prime is taken to a unit multiple of itself.

Prescribe integers for the individual coordinate depths at both primes. Write X_c for the integral points of $M(X)$ over constant vectors that meet the prescriptions c. Write $N_c \subset M(Y)$ for integral kernel points whose coordinates themselves vanish to the prescribed orders. Thus N_c is an R-module contained in X_c, whereas X_c is an $\mathcal{O}$-module. Denote by $I_c \subset B$ the image of X_c. For a finite $\mathcal{O}$-module let ℓ_p be its $\mathcal{O}$-length divided by $v_\pi(p)$. Equivalently, it is its $\mathbb{Z}_p$-length divided by $[C : \mathbb{Q}_p]$.

Lemma 17.24 (Raising one coordinate order over G_K)**.** *Raise a set of coordinates of common prescribed depth n by one at one of the two primes, obtaining d from c. There is an additive map $A : X_c \to X_d$ with $Az - p^k z \in N_c$. At E take $k = v_p(|n|)$ for $n \neq 0$ and $k = 0$ for $n = 0$. At P one may take $k = 2 + v_p(|n|)$ for $n \neq 0$, and $k = 1$ for $n = 0$. For a nonzero step the map also eliminates the leading term of any kernel coordinate vanishing through all orders below n. On kernel points, every other previously prescribed vanishing order persists.*

Proof. Use the R-linear retraction ρ of Lemma 16.24 on the lattice. It respects finite rational Taylor congruences at E and P; the latter assertion, including right sides in $C_1[P, P^{-1}]$, is proved in Lemma 16.31 by Weierstrass division by P^N, clearing coefficient denominators, then retracting. In particular homogeneous orders in several coordinates at both primes are preserved simultaneously.

For $n \neq 0$ the operator is

$$z \longmapsto p^k z - \rho\big(\alpha(gz - z)/\delta\big), \qquad \delta = u\tau.$$

At E use g, α from Theorem 16.25, and at P those of Lemma 16.31. In the latter case g fixes the Taylor constants in C_1 because it fixes $p^{1/p}$. At degree n the factor $\alpha((gE/E)^n - 1)/\delta$, or its companion with P in place of E, specializes to p^k. Thus the indicated coefficient is cancelled, also when $n < 0$. Every divided displacement used here is integral in the kernel lattice over A_i, by crystalline descent. It has the prescribed homogeneous orders at both primes: δ is a rational unit at both, and all approximating constants belong to C. Retraction therefore puts the correction in N_c. These observations also prove the assertions on kernel points. At E, degree zero requires only taking the actual constant Taylor coefficient; the identity acts on X_c.

At P, degree zero requires a separate argument. Choose $g_* \in G_K$ fixing the p-power roots of unity and sending the chosen system of roots of p to its product with the compatible roots

of unity of Kummer parameter one. Existence was proved in Lemma 16.31. Use

$$Az = pz + \rho\left(\sum_{j=0}^{p-1}(g_*^j z - z)\right).$$

The correction is integral, has zero quotient and has all prescribed homogeneous orders. At a coordinate now regular at P, write its constant term as $\sum_{m=0}^{p-1} a_m T^m$ with $a_m \in C$. In the internal period factor the lifted root T is taken to $\zeta_p^j T$, and $g_*^j P/P$ is a unit. Hence the sum of the p translates has constant term pa_0, by the geometric sum of roots of unity. This is the same rational prescription on all external coefficient factors, so it persists under ρ. This gives agreement to order one with an element of C. □

Theorem 17.25 (The length of a single Taylor step)**.** *Suppose a step raises r kernel coordinates of common prescribed depth n by one, keeping all other prescriptions. At E one has $\ell_p(I_c/I_d) \le rk$, with k as in Lemma 17.24. The same bound holds at P provided all the kernel coordinates at E have a common prescribed depth M_0, where*

$$M_0 > 0, \qquad p \mid M_0, \qquad M_0 \ge -j \quad \textit{for each weight } j \textit{ of } Y \textit{ at } i.$$

Proof. At E a nonzero step imposes the vanishing of one Taylor coefficient in each of r directions. The resulting map on X_c has image in C^r with bounded denominators. Indeed the points belong to a fixed integral free lattice and each finite horizontal coefficient uses finitely many Taylor coefficients of its basis and of the integral coordinates. The image is consequently a finitely generated $\mathcal{O}$-module of rank at most r. Its quotient I_c/I_d is killed by p^k, by the lemma. This proves the bound. At degree zero there is no loss at E.

At P, put $Q = X_c/(X_d + N_c)$. This group surjects onto I_c/I_d, and the lemma gives $p^k Q = 0$. We first compute a different quotient, using only kernel points. For $w \in N_c$ record the coefficient of P^n in the r directions in question. Its kernel is N_d. The image $L \subset C_1^r$ is stable under $\mathcal{O}_K[[u]]$, acting on it through specialization at P. Write $K_1 = K(\pi_1)$, $\pi_1^p = p$. The specialized ring is $\mathcal{O}_{K_1} = \mathcal{O}_K[\pi_1]$. Indeed $T^p - p$ is Eisenstein, and the valuations of $a_j \pi_1^j$, $0 \le j < p$, for $a_j \in K^\times$, are distinct modulo integral valuations of $K^\times$; hence an integral sum has integral coefficients. View $C_1 = C \otimes_K K_1$ as a K_1-space of dimension $[C : K]$. The image L has bounded denominators: the integral coefficients of a vector reduced modulo P^m have representatives of degree $< pm$ with coefficients in $\mathcal{O}$, and the fixed rational coordinate matrices and the truncated substitution for u introduce only a fixed denominator for any specified coefficient. Thus L is a submodule of a finite free fractional lattice over the discrete valuation ring $\mathcal{O}_{K_1}$, and is free of rank at most $r[C : K]$. The image of E in this ring is $\pi_1 - p$, a uniformizer. We obtain

$$\dim_{\mathbb{F}_p} N_c/(N_d + EN_c) = \dim_{\mathbb{F}_p} L/(\pi_1 - p)L \le fr[C : K] = r[C : \mathbb{Q}_p].$$

This computation uses the unramified base K, so it applies also to ramified external coefficients.

Set $d_0 = r[C : \mathbb{Q}_p]$, and take any $d_0 + 1$ elements of $Q[p]$, represented by $x_j \in X_c$. Write $px_j = x_j' + w_j$ with $x_j' \in X_d$, $w_j \in N_c$. The preceding dimension bound supplies a relation among the w_j modulo $N_d + EN_c$, with coefficients in $\mathbb{Z}_p$ not all divisible by p. For the same combination x of the x_j we therefore have

$$px = z + Ew, \qquad z \in X_d, \quad w \in N_c.$$

Apply in succession the E-operators of the lemma for the degrees $M_0 + 1, \dots, M_0 + p - 1$, using the displayed nonzero-step formula on all coordinates. Denote the composite by A. All these operators cost zero since $p \mid M_0$. They preserve X_d, and $Ax - x \in N_c$: whenever a

coordinate agrees with a constant to its prescribed order, the correction has that homogeneous order, whether or not the degree being cancelled is its next degree. Starting with Ew, whose kernel coordinates vanish to order M_0+1 at E, they give $A(Ew) = E^p w'$ with $w' \in N_c$. In fact division by E^p leaves order M_0 at E, at least the required integral order $-j$ in each direction. Integrality descends from the horizontal comparison after clearing the E-denominator, as in Lemma 16.17. At P this division is by a rational unit and preserves the homogeneous orders of the kernel point.

Now $E^p = P + pr_0$ for some $r_0 \in \mathcal{O}_K[u]$ by the binomial formula. Additivity gives

$$p(Ax - r_0 w') = Az + Pw' \in X_d.$$

Indeed $Pw' \in N_d$. The element $Ax - r_0 w'$ belongs to $x + N_c$; since X_d is given there by rational Taylor conditions, the factor p cancels. This proves a nonzero $\mathbb{F}_p$-relation among the chosen elements of $Q[p]$. Thus $\dim_{\mathbb{F}_p} Q[p] \le d_0$. Multiplication by p^{j-1} embeds $Q[p^j]/Q[p^{j-1}]$ into $Q[p]$. The exponent bound now gives $\operatorname{length}_{\mathbb{Z}_p} Q \le k d_0$. Pass to its quotient I_c/I_d and divide by $[C : \mathbb{Q}_p]$. □

Corollary 17.26 (Integral tests with prescribed orders at two primes)**.** *Suppose the weights of Y at label i lie in $[-\eta, \eta]$ and a single signed interval $\pm[\gamma_i, \gamma_i + w]$. Let m_i count those in the first interval and let $s = \operatorname{rank} Y$. Take positive integers a, η, w, put $J = a+\eta+w+1$, $D = 3a + 2\eta + 2w + 4$, and suppose*

$$32J < \min_i(\gamma_i, p), \qquad \max_i(\gamma_i + w + \eta + 1) \le (p-3)a.$$

Assume that p divides no nonzero integer within $32J$ of 0 or any $\pm\gamma_i$. For every basis vector of B assume its E^a- and p^a-multiples lift at every label of $M(X)$. There is a sublattice $B' \subset B$ with

$$\ell_p(B/B') \ll (a+1)\sum_i m_i + fs(w+1)$$

whose vectors have integral points at all labels, of displacement in $\delta E^D M_{i,A}(Y)$ over G_K. At a label with negative far interval one may also require that the divided displacement have horizontal orders

$$\begin{aligned} E:&\quad M_i^* = p\,\lceil(\gamma_i + w + D + 1)/p\rceil \quad \text{in every direction,}\\ P:&\quad w + 1 - v + 2a\,\mathbf{1}_{\{|v|\le\eta\}} \quad \text{in each direction of weight } v \text{ at } i^-. \end{aligned}$$

The point itself is regular in horizontal coordinates at E on those labels.

Proof. Proposition 17.18 applies with its $S = 0$ by the excluded multiples, since the indices in the intervals $[-a-j, D-j)$ are within $32J$ of the indicated centers. The only weights in $[-a, 0)$ are short. Its proof thus gives, at total normalized cost at most $a\sum_i m_i$, integral lifts whose kernel coordinates agree with constants through depth $D-j$ at E. At P, integrality already implies depth $-v$.

At a negative label next impose the uniform E-depth M_i^*, coordinate by coordinate. Directions which already meet that requirement incur no cost. On a short line the sum of $v_p(|n|)$ over the steps is at most $v_p(M_i^*!) \le M_i^*/(p-1) \ll a+1$. Here $D-j > 0$ and $\gamma_i + w + \eta + 1 \le (p-3)a$. On a far line $-j \in [\gamma_i, \gamma_i + w]$. A multiple of p in $[D-j, M_i^*)$ is at most $M_i^* - p < \gamma_i + w + D + 1$ and at least $\gamma_i + D$, contrary to the exclusion. The first part of Theorem 17.25 shows that imposing the uniform E-depth costs $O((a+1)m_i)$. Now $M_i^* \ge D - j$ on every line. In particular the new depth retains the original divisibility and satisfies all hypotheses on M_0 in the second part of that theorem.

Raise the P-orders from $-v$ to the stated values, using that second part at each step. The nonzero integers visited are within $32J$ of 0 or $\pm\gamma_{i^-}$, so the cost per direction and step is at most two, including degree zero. This costs at most $2s(w+1)+4am_{i^-}$ at this label. The quotient coordinates throughout are the given constants of B. Intersect the resulting sublattices over the labels; the quotient by an intersection embeds in the direct sum of the quotients, giving the claimed total. Agreement with external constants gives the asserted displacement orders even after division by δ, which is a unit in both rational comparisons. The original integral δE^D-divisibility persists by the integral test of Lemma 16.24. Since $M_i^* > 0$, the points themselves are regular at E on the specified labels. □

17.11 The spherical local sum for all tested summands

At a large coefficient prime a graph may have many coordinates of nearly equal large weight. The graph itself can have arbitrarily large rank in the trace construction. We impose the extra Taylor conditions of Corollary 17.26 on the dual tests, whose kernel is the specified target dual. The proof below uses the width of the large weights at a *single* label.

Proposition 17.27 (Pairing the graph with the restricted dual tests)**.** *Fix an unramified spherical place K of degree f over $\mathbb{Q}_p$, a main target T of rank s, and the graph, dual and integral contraction of Proposition 17.5. Use $\eta, w, \gamma_i, a, J, D$ satisfying the numerical and excluded-multiple hypotheses of Corollary 17.26. At label i suppose that both kernels have short jumps in $[-\eta, \eta]$ and that their other jumps belong to opposite signed intervals $\pm[\gamma_i, \gamma_i + w]$. Suppose $w \le \eta$, and that the third kernel is $\mathrm{Hom}(Y, T)$, where Y is the graph kernel and the second kernel is $T' = T^\vee(1)$. If m_i counts the short jumps of T' at i, then*

$$d_{v,T} \ll (a+1) \sum_i m_i + fs(w+1).$$

The same bound works for every trace graph of the indicated weights, at the original test precision.

Proof. Work first over coefficients containing K. Take B to be the integral finite-condition raw cocycles of T', and use the crystalline extension $T' \oplus B$ with $g(y, c) = (gy + c(g), c)$, constructed and justified in Theorem 17.19. The third extension is the integral lift of the pairing modulo π^N: if $\psi : Y \to T$ is a linear lift, its map to $\mathrm{Hom}(Y, T)$ takes a kernel coordinate z and a quotient coordinate e to $\pi^N z + e\psi$. This map is equivariant by the defining cocycle. Evaluation and pairing with T' give an integral equivariant trilinear map to the Tate line, equal to the required pairing Q modulo π^N when the third point has quotient one.

All three extensions have upper jumps at most $(p-3)a$. For the third, a jump in the kernel is $-u - v - 1$ for a graph jump u and a test jump v at the same label. Two far jumps have opposite signs and $|u + v| \le w$. A short and a far jump give magnitude at most $\gamma_i + w + \eta + 1$, and two short ones give magnitude at most $2\eta + 1$. This also shows that the third jumps are within $2\eta + w + 1$ of $0, \pm\gamma_i$. All quotients add only the zero weight. By Theorem 16.8 every basis vector of a trivial quotient admits lifts of its multiples by E^a and p^a, first pulling back along the vector and then projecting to the label.

Choose $c \in B'$ from Corollary 17.26 and use the points y_i there over c. On the graph divide an E^a-lift by E^a and correct it to order δE^D by Theorem 16.25. All the relevant nonzero Taylor indices are within $32J$ of the excluded centers, so the quotient remains one. Call this point $x_i \in E^{-a} M_{0,i}(X)$. In a horizontal coordinate of weight u its order is at least $d_D(u)$, where

$$d_t(l) = \begin{cases} t - l, & l < -a, \\ \min(0, t - l), & l \ge -a. \end{cases}$$

Indeed the initial order is $-a - l$; if this is positive the Taylor constant must vanish, and in all cases there is agreement with a constant to order $t - l$, as in Lemma 16.32. On the third extension likewise choose an E^{-a}-point z_i over one corrected at E with $t = a$. Its corrected orders are $d_a(l)$. Its nonzero Taylor indices also miss the multiples of p, by the bounds for the third jumps. The function $d_a(l)$ is nonincreasing in l. All constructions so far use the full group G_K.

Use $\tau_1 = \phi(\tau)$, $P = \phi E$, and say that a pairing is small if its reduction lies in $uA_i(1)$ at the original precision. Integral contraction with z_i gives the first-lattice denominator $E^a\tau$, and

with ϕz_{i^-} the adjacent denominator $P^a\tau_1$. The initial displacements pair to a small value since both have factor δE^D, where $D \geq a$. We compare the pair (x_i, y_i) to $(\phi x_{i^-}, \phi y_{i^-})$ separately at each label.

If the test has positive far sign, the graph has upper jump at most η and the test has lower jump at least $-\eta$ at i. The point ϕy_{i^-} is integral in $M_{1,i}$; its displacement contains $u^p\tau_1 P^D$ in that lattice. By Corollary 16.26 there is a point $q \in M_{1,i}$ of the same quotient whose displacement has factor $u\tau_1 E^{a+\eta+1}P^{2a}$. Here $D \geq 2a$ and $v_p((a+\eta+1)!) = 0$. First change y_i to q. Their difference lies in $E^{-\eta}M_{0,i}(T')$ since y_i is integral and the lower bound gives this inclusion for the adjacent lattice. The fixed graph displacement supplies $u\tau E^D$. Next change x_i to ϕx_{i^-}. The upper bound on the graph puts their difference in $E^{-(a+\eta)}P^{-a}M_{1,i}(Y)$, including the possible pole of each point. The displacement of q cancels these factors and the adjacent denominator, leaving a multiple of u. Finally change q to ϕy_{i^-}; both are in the adjacent lattice and the displacement of ϕx_{i^-} supplies $u^p\tau_1 P^D$.

Suppose instead that the test has negative far sign at i. First change x_i to ϕx_{i^-} at fixed y_i. Write $v_i(g) = (g-1)y_i/\delta$ and $\Delta = \phi x_{i^-} - x_i$. These are kernel vectors. Before reduction, contract $\phi z_{i^-}, \Delta, v_i(g)$ by the integral trilinear map. The result belongs to the first Tate lattice with at most E, P denominators. Indeed $v_i(g)$ belongs to the first integral kernel lattice, x_i has at most an E^a denominator and the transported points have at most P^a denominators in the adjacent lattices. Passing between the first and adjacent lattices introduces only powers of E.

We verify that the contraction actually belongs to the integral first Tate lattice. At E, the Frobenius of the third point is regular in horizontal coordinates, since the adjacent lattice is regular there and P is a unit in this rational comparison. A graph direction of weight u in Δ has order at least $\min(0, d_D(u))$, which is at least $-(\gamma_i + w)$. Every direction in $v_i(g)$ has absolute order $M_i^* \geq \gamma_i + w + D + 1$. Thus the product in the horizontal Tate coordinate has order at least one.

At P take the horizontal bases transported from i^-. In a nonzero component of the trilinear map let u, v, l be the weights of the graph kernel, test kernel and third extension, respectively, at i^-. The filtered de Rham map vanishes on that component if $u+v+l > -1$, since the Tate line has filtration jump -1. Its coefficient in horizontal bases is constant. Hence $l \leq -u-v-1$ on every component that contributes. For ϕz_{i^-} the order is at least $d_a(-u-v-1)$, by monotonicity. The graph difference has order at least $\min(-u, d_D(u))$: the current graph point has only a possible E-pole and the transported point has its corrected orders from i^-. Corollary 17.26 gives the third term in the following lower bound for the product order, with $l_0 = -u-v-1$,

$$d_a(l_0) + \min(-u, d_D(u)) + w + 1 - v + 2a\,\mathbf{1}_{\{|v|\leq\eta\}}.$$

If v is short, use $d_a(l_0) \geq -a - l_0$ and $\min(-u, d_D(u)) \geq -a - u$. The sum is at least $w+2$. If v is far and u is short, then $|l_0| \geq \gamma_{i^-} - \eta - 1 > a$ and $d_a(l_0) = a - l_0$, giving the same lower bound. If both are far, $|u+v| \leq w$ and $|u| \geq \gamma_{i^-} > D + a$. Then $\min(-u, d_D(u)) = -u$ and $d_a(l_0) \geq -(w+1) - l_0$. This gives order at least one. These inequalities apply to every component and hence to their sum. Horizontal kernel bases can be chosen as part of bases of the extensions, as in Section 17.10; the equal quotient coordinates cancel in each difference.

In the first Tate lattice the required horizontal order is one at E and one at P, since its single weight is -1 at both labels. The first assertion is Lemma 16.23, and the second follows by transport since the height transition between the first and adjacent lattices is invertible at P. For clarity, these two rational tests remove the integral denominators: after multiplying the contraction by $E^bP^{b'}$ to put it in the first Tate lattice, the tests give divisibility there by E^b and $P^{b'}$ individually, using the integral injections of Lemmas 16.24 and 16.31. The image

of P modulo E is $p^p - p$, a nonzero divisor on A_i/EA_i; cancellation gives divisibility by their product. Thus the trilinear contraction is in $\tau^{-1}A_i(1)$, the first Tate lattice. Multiplying back by $\delta = u\tau$ and reducing gives the small pairing for this change of the graph point.

Now change y_i to ϕy_{i^-}. The latter is integral in the adjacent lattice, and the former lies there because it is integral in the first lattice and regular at E in the horizontal basis, by the corollary. The height inclusion and integral divisibility test give this membership in $M_{1,i}$. The fixed graph displacement supplies $u^p\tau_1 P^D$, so the adjacent pairing is small.

The estimates hold under translates in the cochain identities: the group preserves the integral lattices, the two prime ideals and the horizontal order bounds. At every label the terminal pair is exactly Frobenius of the previous initial pair. Summing the cochain identities of Theorem 16.2 and taking Frobenius constants as in Theorem 16.21 gives $[z \cup \operatorname{red} c] = 0$ for the original graph class z and every $c \in B'$. For fixed T the dual sublattice works for every graph and every precision. By Proposition 15.4 the local quotient in the assertion now injects into $\operatorname{Hom}_{\mathcal{O}}(B/B', \mathcal{O}/\pi^N)$. Its length is at most that of B/B', by cyclic decomposition. Corollary 17.26 gives the assertion. Descent to the original coefficients follows by flat scalar extension and invariance of normalized length as in Theorem 17.19. □

Theorem 17.28 (The full spherical local sum on the period interval)**.** *Use the fields, simultaneous twists and full range of levels and precisions of Section 17.4. For $D_{x,p}$, the supremum of the sum of normalized cohomology lengths $d_{v,T}$ over the spherical places above p and all tested summands, one has*

$$\sum_{p} (\log p) \sup_{x \text{ in the interval}} \frac{D_{x,p}}{sB_x^{\det}} \ll_{q_0,d} (\log(2V_0))^{-2}.$$

Proof. Theorem 17.23 covers the primes up to $Y = b/x^{45}$, where $b = b_x$. Write $h = s^8$. Corollary 17.15 covers $p \geq 8b/h$ and Theorem 17.17 covers the scalar targets. We need the main target on $Y < p < 8b/h$. Write $G = xb/x_{1,x}$, and $\beta_i \in \{b, G\}$ for the base weight. At a given twist index j use the explicit row of Lemma 52.11. Its $k_x - 1$ far entries have displacements $\beta_i + (2j+1)h + k_x/2 - 1 - r$, $0 \leq r < k_x - 1$, before reflection and translation. The remaining $m_x = s + 1 - k_x$ old entries have displacements which are odd multiples of h of magnitude at most $(2m_x + 1)h$. The scalar graph entries lie in $[0, t-1]$; the translation is $(t-1)/2$ and passage to the target dual negates and subtracts one. In particular the far entries on either side have magnitudes within $s + t + 2$ of $\beta_i + (2j+1)h$, with opposite signs on the graph and dual. This also holds on every constituent of the rational graph kernel by Proposition 17.5. Put

$$c_0 = 2(s+t+3), \qquad \gamma_i = \lfloor \beta_i + (2j+1)h \rfloor - c_0, \qquad w = 2c_0 + 3,$$
$$\eta = \lceil 4((m_x+2)h + s + t + w + 2) \rceil, \quad a = \lceil (b + 3\eta + w + 2)/(p-3) \rceil.$$

These intervals contain the weights just calculated; the number of short weights on the dual is m_x per label. Also $\gamma_i + w + \eta + 1 \leq (p-3)a$. Uniformly in the node, $\eta \ll x^{81/2}$, $w \ll x^5$, $a \ll x^{45}$, $p > b/x^{45} \geq x^{355}$ and $\gamma_i \gg G \geq b^{1-10^{-6}}$. Hence $32J < \min(p, \gamma_i)$ for sufficiently large x.

Take a fixed $C = C(q_0, d)$ large enough that the intervals of radius $32J$ about all $0, \pm\gamma_i$, for every twist at the node, are contained within Cx^{45} of $\{0, \pm b, \pm G\}$. Call a prime in the remaining range exceptional if it divides a nonzero integer in these last five intervals. The sum of $\log p$ over them is at most $O_{q_0,d}(x^{45} \log(2b))$, by factoring the product of the absolute values of those integers. At *every* such prime the bound using only the largest upper jump in the proof of Proposition 17.11 gives $D_{x,p} \ll f_x(s+t)b/p$. Indeed the future sums in Theorem 16.18

are $O(b/(p-1))$, as are the factorial valuations, so the cup costs $O(b/p)$ per generator. Here $b/p \le x^{45}$. The normalized exceptional contribution is consequently

$$\ll_{q_0,d} \frac{x^{90+9/2}}{b(\log b)^2} \ll_{q_0,d} x^{-300}$$

at the node. The normalizing identity is $B_x^{\mathrm{det}} = f_x x b(\log b)^3/(m_x k_x)$, with $m_x k_x/x \ll x^{9/2}$ and $t = 5s$.

At every remaining prime the proposition and $\sum_{v|p}[E_{x,v} : \mathbb{Q}_p] = 2f_x$ give, for the main target,

$$\sum_{v|p,\ v \text{ spherical}} d_{v,T} \ll f_x(m_x b/p + s(w+1)),$$

using $a + 1 \ll b/p$ on this range. The first term after normalization is at most $C/(p(\log p)^3)$, since $m_x^2 k_x/(sx) \ll 1$ and $p \le b$. This bound precedes the interval supremum. All eligible p exceed V_0^{355}. On a dyadic interval starting at y, Chebyshev's inequality bounds the sum of $\log p/(p(\log p)^3)$ by $O((\log y)^{-3})$; summation gives $O((\log(2V_0))^{-2})$. For the second term sum over primes *at the node* first: $\sum_{p<8b/h} \log p \ll b/h$. After normalization this contributes

$$\ll \frac{x^{9/2}(w+1)}{h(\log b)^3} \ll_{q_0,d} x^{-30},$$

since $s \asymp x^5$. For this term and the exceptional term, the supremum at each prime is bounded by the sum over its eligible integer nodes. Summing for $x \ge V_0$ and adding the previously covered ranges proves the theorem. □

The sum in this theorem is the spherical cohomology term. The tested projection in Corollary 52.3 uses an index in a global Hecke lattice. Bounding that index also requires a comparison with the integral theta periods.

17.12 Tame inertia and integral lifts at a level prime

At either original level prime the local field is $\mathbb{Q}_p$. Trivializing a tame type can produce a ramification index much larger than p. The number of different inertia characters, however, controls the number of parameter powers for which lowering the exponent costs a power of p.

Lemma 17.29 (Lifting a constant with tame descent data)**.** *Let $U/\mathbb{Q}_p$ be unramified with $\mu_e \subset U$, where $p \nmid e$, and put $F = U(p^{1/e})$. Write $E(u) = u^e - p$. Let $X \to \mathcal{O}$ be a surjection of $G_{\mathbb{Q}_p}$-lattices with trivial quotient, crystalline over F, with upper Hodge jump at most $h \geq 0$. At each residue label of F suppose the tame generator on the rational crystalline periods is annihilated by a monic polynomial of degree at most d with integral coefficients, having 1 as a root. We allow a different polynomial at each label. If $e \leq p^m$, $m \geq 1$, $\kappa = md$, and $h + \kappa < p - 1$, then p^κ has an integral lift in each first Breuil–Kisin lattice of X over F.*

Proof. Enlarge the coefficient ring to contain the images of F. On each component let $R_i = \mathcal{O}[[u]]$, and let I_i be the image ideal in the module of the trivial quotient for the normalized first lattice. A generator σ of $\mathrm{Gal}(F/U)$ acts on u by $\zeta_i u$, for a primitive e-th root ζ_i. Indeed one may take compatible roots of $p^{1/e}$ and send their p^n-th roots to their multiples by $\zeta^{p^{-n} \bmod e}$. These assignments give the tame action on the prism, fixing its residue-field coefficients. The associated integral lattices carry this semilinear descent. To see the descent on a lattice, apply functoriality of the crystalline lattice functor to the automorphism of the field and its uniformizer tower. The action on the given representation identifies the transported lattice with the original one, compatibly with projection. On M_i/uM_i after inverting p this is the action on the crystalline periods under specialization of the Kisin module at zero. In an effective twist this is the equivariant specialization for a tame descent datum; untwisting by a Tate line preserves the inertia type. Equivalently one may make this identification after extending the tame group to all roots of order $|k_U^\times|$ and restrict to the given group, using functoriality of the rational period comparison. See the tame descent action and the equivariant specialization in the proof of Proposition 5.17 of Caraiani and Levin. [20]

Set $\nu = \kappa + 1$. At each label let a_i be the least a with $u^a \in I_i + p^\nu R_i$. It exists by the nonzero reduction of the quotient ideal: modulo a coefficient uniformizer π it contains a power of u, as follows by base change to the Witt comparison of the surjection. Raising this membership to $\nu v_\pi(p)$ in R_i/I_i gives one modulo p^ν. The height inclusion yields $E^h I_i \subset R_i \phi(I_{i^-})$. Since $u^{e(h+\nu-1)} = (E+p)^{h+\nu-1}$ is divisible by E^h modulo p^ν, extracting the component of a pure power according to its exponent modulo p gives

$$p a_{i^-} \leq a_i + e(h + \nu - 1).$$

This is the finite free decomposition of R_i over the Frobenius image with basis $1, u, \ldots, u^{p-1}$, used in Lemma 16.4. Taking the maximum over the orbit gives $a_i \leq e(h+\kappa)/(p-1) < e$.

Fix i; write $M = M_i(X)$, q for its indicated annihilating polynomial. Then $q(\sigma)M \subset uM$. Indeed M/uM is free over $\mathcal{O}$, and σ is linear on this quotient. The rational annihilation just identified therefore holds integrally on the quotient. Suppose $p^t u^k \in I_i + p^\nu R_i$, $1 \leq k \leq a_i$. Choose $x \in M$ whose projection is $p^t u^k + p^\nu f(u)$. The integral vector $q(\sigma)x/u$ projects to

$$q(\zeta_i^k) p^t u^{k-1} + p^\nu f_1(u) \quad (f_1 \in R_i).$$

For the error term, evaluating $q(\sigma)f$ at zero gives $q(1)f(0) = 0$, so division by u is integral. Thus the exponent drops by one at no cost whenever $q(\zeta_i^k)$ is a unit. At another step Lemma 16.7 gives $p^{t+1+v_p(k)} u^{k-1} \in I_i + p^\nu R_i$. Its divided-displacement proof applies in a residue component since G_F fixes its idempotent and the external coefficients. Here $1 + v_p(k) \leq m$ as $k < e \leq p^m$.

The residues of ζ_i^k, $1 \le k \le a_i < e$, are distinct because reduction is injective on roots of unity of order prime to p. The reduction of q is monic, so at most d steps require the latter operation. Starting with u^{a_i}, we obtain $p^\kappa \in I_i + p^{\kappa+1}R_i$, multiplying by a power of p if the actual cost is smaller. This gives $p^\kappa(1-pg) \in I_i$ with $g \in R_i$; the second factor is a unit. This proves the claim before reduction at any test precision. □

Theorem 17.30 (The local cup for a common tame spectrum)**.** *Over $\mathbb{Q}_p$, $p \ge 5$, take a trace graph, a target T, the dual test and their integral contraction as in Proposition 17.5. Suppose the rational systems in the graph and T are de Rham, their Weil–Deligne monodromy is zero, and tame inertia on them has a common eigenvalue set Σ at a generator (an enclosing set suffices). Let $p \nmid e$ be an integer killing these finite tame characters. Assume $e \mid p^f - 1$ with $p \nmid f$, and $e \le p^m$, $m \ge 1$. Put $d = (|\Sigma|+2)^2$. For two positive integers η, w and a positive base $\gamma > 2\eta + w$, suppose all jumps on graph and dual are in $[-\eta, \eta]$ and opposite signed intervals of magnitudes $[\gamma, \gamma+w]$. One of the two far intervals may be empty. If $p > \gamma+6(\eta+w+1)+md+2$, then every such finite cup has an annihilator p^b with*

$$b \ll md + \eta + w + 1.$$

Proof. Take U unramified of degree f, and $F = U(p^{1/e})$. The Weil–Deligne restrictions have trivial inertia over F and zero monodromy. They are crystalline over F by the semistable comparison and descent of crystalline periods, as in Theorem 15.8. The graph and contraction extensions split rationally and hence are crystalline there. The test extension also becomes crystalline: its class is finite over $\mathbb{Q}_p$, and restriction preserves the kernel defined by B_{cris}; the restricted kernel representation is crystalline. All the embeddings of F restrict to the original embedding of $\mathbb{Q}_p$. Thus each residue label has the same enclosing short and far intervals, including after a coefficient extension.

Enlarge coefficients to split the desired finite type. A tame generator on the periods has finite order prime to p, so is semisimple. The kernel of the graph has spectrum contained in $\Sigma \cup \{1\}$; this allows arbitrary repetitions of the rational trace systems. The test kernel $T^\vee(1)$ has inverse spectrum, since the Tate line has trivial smooth inertia. The contraction kernel $\mathrm{Hom}(Y, T)$ has spectrum in the set of all quotients of two members of $\Sigma \cup \{1\}$. The spectra on the three extensions add at most the eigenvalue one. The products of $X - \lambda$ over the respective distinct eigenvalues, including one, are integral monic annihilators of degree at most d, uniformly over all labels. The upper jumps on all three extensions are at most $\gamma + w + 2\eta + 1$ by the same-label differences in the contraction. Hence the lemma yields pure integral lifts of p^κ on all three extensions with $\kappa = md$.

Let $D = 2\eta + 3$ and $H = G_{F(\varpi^{1/p})}$ with $\varpi = p^{1/e}$. On the first two extensions take these pure lifts and impose order D at $E = u^e - p$ on their divided displacements over H, using Proposition 16.44 with $L = 1$, the first lattice, and the single prime E. The cost uses the union of the intervals $[-j, D-j)$ for the jumps j. They lie in two intervals of total length $O(\eta + w + 1)$, independently of the ranks and of the number of embeddings of F. Every nonzero index has absolute value less than p, and the different exponent in that proposition is $\lceil v_p(e\varpi^{e-1}) \rceil \le 1$. Thus the exponent cost is $O(\eta + w + 1)$. The points are integral in $M_{0,i}$, their quotient coordinates can all be scaled to p^{e_*} for $e_* \le \kappa + O(\eta + w + 1)$, and their displacements on H belong to $\delta E^D M_{0,i,A}(\ker \beta)$. On a side of upper weight at most η, the points also lie in $M_{1,i}$. Indeed their horizontal coordinates at every root of E agree with constants modulo order $D - j > 0$; regularity at all roots gives integral membership by Lemma 16.42.

Use the third pure point and its Frobenius transport. On the first kernel lattices $p^\kappa Q$ costs only τ in the denominator, on the adjacent ones only $\phi\tau$. At a label i, call the side of lower jumps at least $-\eta$ the first moving side. Change its point to the Frobenius of its preceding point.

Their difference is in $E^{-\eta}M_{0,i}$ on the kernel, because the transported point is in $M_{1,i}$ there. The opposite displacement supplies $u\tau E^{D}$, so the pairing is small. Now change the upper-bounded point: both it and the transported point are in $M_{1,i}$. The fixed Frobenius displacement supplies $u^{p}(\phi\tau)(\phi E)^{D}$ and again the pairing is small. These bounds apply to translates, and the initial displacements give a small cup. The labelled cochain descent of Theorem 16.21 therefore kills the restriction with exponent $2e_{*}+\kappa$. The final restriction index is pef, whose p-part is p. Corestriction adds only one to the exponent, and an integral coefficient retraction returns the coboundary at the original precision. □

Corollary 17.31 (The size of the tame term on the marked fields)**.** *At each of the two original level primes and each place above it, restrict to trace families and targets having zero monodromy and a common tame spectrum of at most $(s+t)s+1$ roots of unity of orders dividing $e=\operatorname{lcm}_{1\le j\le s}(p^{j}-1)$. The supremum can be taken separately over such spectra, embeddings, twists and precisions. For these families the sum over primes of the interval suprema of the cohomology loss, with weight $\log p/(sB_{x}^{\det})$ as in Theorem 17.28, is $O_{q_0,d}(V_0^{-350})$ at the two level primes. This statement concerns each fixed spectrum before taking the supremum.*

Proof. Here $p_0,p_1\in[b_x^{10},b_x^{11}]$ and the places split completely by Section 130.1. Take $f=\operatorname{lcm}(1,\dots,s)$; then $e\mid p^{f}-1$ and $p\nmid f$. Also $e\le\prod_{j=1}^{s}(p^{j}-1)<p^{s(s+1)/2}$, so $m=s^{2}$ works. The row computation in Theorem 17.28 gives a single far sign on each side at a fixed split place, with $w=O(s+t+1)$, $\eta=O_{q_0,d}(x^{81/2})$, and γ within $O_{q_0,d}(x^{81/2})$ of b_x or $xb_x/x_{1,x}$. For a scalar target put its dual weights in the short interval. In particular $md\ll x^{30}$, $p>\gamma+6(\eta+w+1)+md+2$, and every cup costs $O_{q_0,d}(x^{41})$. Exact local duality and the integral raw-cocycle generator count of Theorem 17.6 bound the sum of normalized lengths at each of these primes by $O_{q_0,d}(f_x(s+t)x^{41})$. The count takes place over the original split completions, after the preceding descent. Multiplication by $\log(p_0p_1)\ll\log b_x$ and division by $sB_{x}^{\det}$ give

$$O_{q_0,d}\left(\frac{x^{41+9/2}}{b_x(\log b_x)^{2}}\right).$$

Finally $b_x\ge x^{400}$. Replace each nonnegative primewise supremum by its sum over eligible nodes and sum the displayed bound for integers $x\ge V_0$. □

The finite character mark splits the tested index as in Theorem 52.13. Applying the last corollary to an entire marked type requires identifying the Weil–Deligne spectra of the target systems occurring in that type, including its competitors at auxiliary levels.

17.13 Inertia of a marked factor at the original level

At an original level place the marking specifies a representation of the group acting on the residue frame. We first determine what this requires of the local parameter of a form containing that marking. The action on the complementary block will enter the argument.

Lemma 17.32 (A squarefree block with spherical complement)**.** *Let K be a nonarchimedean local field with residue field $k = \mathbb{F}_q$. For pairwise distinct Frobenius orbits of nontrivial regular characters θ_ν of $\mathbb{F}_{q^{d_\nu}}^\times$, let ρ_ν denote the corresponding finite cuspidal representations of $\mathrm{GL}_{d_\nu}(k)$. Here regular means that the orbit has size d_ν. Put $s = \sum_\nu d_\nu$, and let $t \geq 1$. Write $\odot$ for finite parabolic induction, inflating the tensor product on a Levi subgroup to its parabolic. The representation*

$$\sigma = (\bigodot_\nu \rho_\nu) \odot \mathbf{1}_{\mathrm{GL}_t(k)}$$

is irreducible. If an irreducible smooth representation π of $\mathrm{GL}_{s+t}(K)$ contains its inflation on the maximal compact subgroup, then the monodromy of its Weil–Deligne parameter is zero. The smooth inertia characters consist of t trivial characters and the d_ν conjugates belonging to each θ_ν, with the characters read by local reciprocity over the unramified degree-d_ν extension of K.

Proof. In the finite Green parametrization, induction of irreducibles with disjoint cuspidal supports joins their partition data and gives one irreducible. The support of the trivial representation of $\mathrm{GL}_t(k)$ consists of t copies of the trivial character in degree one. All the other supports are distinct from it and from one another, proving the first assertion. [21, §2.3]

For $l \geq 1$ let K_l^1 denote the kernel of reduction on $\mathrm{GL}_l(\mathcal{O}_K)$. Taking K_l^1-invariants is exact on smooth representations in characteristic zero by averaging. For normalized parabolic induction its result is the finite parabolic induction of the principal congruence invariants on the Levi blocks. Indeed, by Iwasawa decomposition an inducing function is determined by its restriction to the maximal compact subgroup. Right invariance under the kernel of reduction forces the values there into the invariants on the blocks. The modulus character on the integral Levi is one, giving the finite function model in both directions.

Use the supercuspidal support of π and realize π as a subquotient of induction from that support. Each supercuspidal factor has nonzero principal congruence invariants by the preceding calculation. Such a factor has depth zero. By the depth-zero classification for the general linear group it is compactly induced from the extension to the center times a maximal compact subgroup of a finite cuspidal representation. Its principal congruence invariants give just that finite cuspidal representation. To see the last point use the Cartan representatives for the restriction of the induction to the maximal compact. If the diagonal exponents have a strict gap, the intersection with the conjugated principal congruence subgroup reduces to a subgroup containing the unipotent radical across that gap. This follows entry by entry from conjugation by the diagonal powers of a uniformizer. The cuspidal inducing representation has zero invariants under this radical. Equal diagonal exponents give the single central double coset. [22, Prop. 6.8]

Uniqueness of finite cuspidal support now shows that the supercuspidal factors of π have finite types ρ_ν, once each, and trivial degree-one type, t times. A segment of length a in the Langlands classification uses a unramified twists of one supercuspidal. Hence every segment using a ρ_ν has length one. A longer segment could only occur among the unramified characters. We exclude this using the standard module surjecting onto π. An unramified segment of length a gives an unramified determinant twist of the Steinberg representation. Its principal congruence invariants are the finite Steinberg representation $\mathrm{St}_a(k)$. This follows by taking invariants in the presentation by functions on full flags modulo the sum of the functions pulled back from proper partial flags. [24]

Thus the invariant space of the standard module is the finite induction of the ρ_ν with the product $V_t = \odot_j \mathrm{St}_{a_j}(k)$, where the a_j's are the segment lengths on the trivial line. Each constituent of V_t has support on that line. The disjoint-support factorization distinguishes these constituents after induction with the ρ_ν. Exactness and the occurrence of σ in π force $\mathbf{1}_{\mathrm{GL}_t(k)}$ to occur in V_t. By finite Frobenius reciprocity this requires a trivial vector in each $\mathrm{St}_{a_j}(k)$. For $a_j > 1$ the finite Steinberg representation is irreducible of dimension $q^{a_j(a_j-1)/2} > 1$. Consequently all segments have length one. Local Langlands for the Langlands quotient gives the direct sum of their parameters. The depth-zero cuspidal formula gives the stated inertia characters and zero monodromy on each nontrivial orbit. The other summands are unramified characters by local class field theory. [23, Prop. 2.4.1(i)] □

Theorem 17.33 (Common inertia at each marked level place)**.** *Use the higher-rank fields, weights and simultaneous twists of Section 17.2. Fix a node, a lift index, an embedding of the coefficient field and a tuple ω in Theorem 52.13. At $p = p_{x,0}$ or $p_{x,1}$ fix a place $v \mid p$ of E_x. Every normalized target system occurring in this tuple, at any of the auxiliary trace levels away from p, has zero Weil–Deligne monodromy at v. There is one set Σ_v of at most $s+1$ eigenvalues on a tame generator that contains the spectra of all these systems, their true summands, the scalar targets and their common Tate twists. Every eigenvalue has order dividing* $\mathrm{lcm}_{1\le d\le s}(p^d-1)$.

Proof. At a chosen place write c for the degree-s member of the tuple. The completion is $\mathbb{Q}_p$. The level fixes the ordered frame formed by the first s residue columns. Its stabilizer in the full finite group is the subgroup of the parabolic preserving their span on which the first diagonal block is the identity. In particular it contains both the unipotent radical and the whole complementary GL_t. They act trivially at this place. On the quotient of the parabolic, acting through the first diagonal block g, the marked action is translation multiplied by $\psi_v(\det g)$. This is the determinant factor in the frame equivariance of Proposition 52.4. Thus a copy of the marked finite representation χ_c in the frame action is $\chi_c \otimes (\psi_v^{-1} \circ \det)$ in the ordinary translation action. Frobenius reciprocity supplies a nonzero map from the finite induction of this representation with $\mathbf{1}_{\mathrm{GL}_t}$ into the local principal congruence invariants. The same argument works after projection to any automorphic constituent with a nonzero component in the tuple, since translation and projection to its isotypic space commute.

The prescription of Section 130.1 labels χ_c by distinct Frobenius orbits. Tensoring by the determinant character multiplies the degree-d character on each torus by $\psi_v^{-1} \circ N_{\mathbb{F}_{p^d}/\mathbb{F}_p}$, since the determinant there is the norm. In the character group of tame inertia this is multiplication by one Frobenius-fixed character, and hence preserves the orbit sizes and disjointness. The residue character ψ_v has order dividing Q. Indeed its old part is a power of the prescribed order-dividing-Q level character, and the auxiliary twist is split at the coefficient prime as in Theorem 15.8. An orbit of size one that became trivial would have had order dividing Q, excluded by the condition on the roots of c. Thus the lemma applies. The resulting set of smooth eigenvalues, including one, is fixed by the tuple, the local character, and the coefficient embedding. A degree-d orbit contributes d elements of order dividing p^d-1. At the opposite place complex conjugation replaces the type by its dual with an unramified norm twist. There we take the inverses of the same eigenvalues.

It remains to pass to the coefficient-p Galois systems. At this split place the cohomological base change in Theorem 45.16 has the parameter of the local automorphic representation. Its expression as an isobaric sum is a direct sum of the local parameters of its cuspidal factors with norm twists. Hence each factor has zero monodromy and inertia contained in the set just found. The regular algebraic polarized cuspidal twists on which the Galois construction acts were identified in Proposition 17.1. Theorem 1.1 of Barnet-Lamb, Gee, Geraghty and Taylor gives equality of the semisimplified Weil representations there, and bounds the Galois monodromy

partitions above by the automorphic partitions. Every block on the automorphic side has size one, forcing the same on the Galois side. Semisimplification preserves the representation of finite inertia in characteristic zero. The normalizing character has unramified smooth part at the original levels, and Tate and norm twists have trivial smooth inertia. Direct summands restrict to subsets of the same spectrum. This also covers the Tate lines used for the scalar tests. Raising the auxiliary level at other places leaves the frame condition at v in force on each new constituent. [6, Thm. 1.1] □

Corollary 17.34 (Local cohomology on the marked components)**.** *For the data of the theorem let $D_{x,p}^{\mathrm{mark}}$ be the supremum, with the tuple fixed inside each test, of the sum of the defects $d_{v,T}$ of Section 17.2 over the target summands and places above p which are spherical or lie at the two original levels. The supremum includes the tuple itself. On the original period fields and interval,*

$$\sum_{p}(\log p)\sup_{x}\frac{D_{x,p}^{\mathrm{mark}}}{sB_{x}^{\det}}\ll_{q_0,d}(\log(2V_0))^{-2}.$$

At the two original level primes alone the sum of interval suprema is $O_{q_0,d}(V_0^{-350})$.

Proof. At each original level place the theorem supplies a common spectrum for the entire individual trace family and its target. The graph of Lemma 12.14 takes direct sums of the corresponding rational systems. This adds no new eigenvalues. The bound $s+1\le(s+t)s+1$ permits the use of Corollary 17.31, at the chosen place and independently at its conjugate. Its bound for the sum over places and targets is uniform in the tuple before the primewise supremum. This proves the assertion about the two levels. For the spherical part use Theorem 17.28 on the same data. The supremum of a sum of nonnegative terms is at most the sum of their suprema. Adding the two estimates gives the displayed inequality. The global index λ_ω of the tested theta projection within a tuple, defined in Theorem 52.13, still requires a comparison with these local quantities. □

18 References

[1] J. S. Milne, *Arithmetic Duality Theorems*, 2nd ed., 2006, Chapters I and II. Book.

[2] J. S. Milne, *Class Field Theory*. Local and global Artin reciprocity and the correspondence with idele class quotients.

[3] S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progress in Mathematics **86**, Birkhäuser, 1990, 333–400, Corollary 3.8.4 and Example 3.9. Published chapter.

[4] X. Caruso and T. Liu, *Quasi-semi-stable representations*, Bull. Soc. Math. France **137** (2009), 185–223. Article.

[5] Y. Ozeki, *Full faithfulness theorem for torsion crystalline representations*, New York J. Math. **20** (2014), 1043–1061. Author's text.

[6] T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, *Local-global compatibility for $l = p$, II*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), 165–179. Theorem 1.1.

[7] T. Liu, *Torsion p-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. École Norm. Sup. (4) **40** (2007), 633–674, §§2–3. Original paper.

[8] J.-P. Wintenberger, *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. (4) **16** (1983), 59–89. Original paper.

[9] H. Gao, *Breuil–Kisin modules and integral p-adic Hodge theory*, Theorem 1.1.16, with appendices by Y. Ozeki and with T. Liu. Author's text.

[10] G. Boxer, F. Calegari, T. Gee, J. Newton and J. A. Thorne, *The Ramanujan and Sato–Tate conjectures for Bianchi modular forms*, Forum Math. Pi **13** (2025), e10, §§2.1–2.2. Journal article.

[11] A. Caraiani and B. Levin, *Kisin modules with descent data and parahoric local models*, Definition 3.6 and Propositions 5.16–5.17. Authors' text.

[12] P. Scholze, *Perfectoid spaces*, Publ. Math. IHÉS 116 (2012), 245–313. The author's text, Section 3 and Theorem 3.7.

[13] P. Čoupek, E. Gazaki and A. Marmora, *Exactness Property of Breuil–Kisin Functors and Bloch–Kato Selmer Groups*, Sections 2 and 3, especially Proposition 3.4, Theorem 3.6 and Remark 3.7. Authors' manuscript.

[14] M. Emerton and T. Gee, *Moduli Stacks of Étale (φ, Γ)-Modules and the Existence of Crystalline Lifts*, Appendix F by T. Gee and T. Liu, Definition F.9, (F.10), and Theorem F.11. Authors' text.

[15] B. Bhatt and P. Scholze, *Prismatic F-crystals and crystalline Galois representations*, Theorems 5.6 and 7.9. Authors' text.

[16] R. Bartlett, *Degenerating products of flag varieties and applications to the Breuil–Mézard conjecture*, Selecta Mathematica 30 (2024), article 17, Section 10.3 and Theorem 10.5. Journal article.

[17] R. Bartlett, *On the irreducible components of some crystalline deformation rings*, Forum of Mathematics, Sigma 8 (2020), e22, Section 2.1, especially Lemma 2.1.8, (2.1.9), and Proposition 2.1.10. Journal article.

[18] T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), 501–609, Theorem 2.1.1(3)–(4). Published paper.

[19] J.-P. Labesse, *Changement de base CM et séries discrètes*, in *On the Stabilization of the Trace Formula*, International Press, 2011, 429–470, Corollaire 5.3. Author's text.

[20] A. Caraiani and B. Levin, *Kisin modules with descent data and parahoric local models*, Def. 3.1, Prop. 3.8 and Prop. 5.17. Authors' paper.

[21] A.-M. Aubert, *On the Macdonald correspondence*, §2.3, especially the factorization over distinct cuspidal supports. Author's paper.

[22] A. Moy and G. Prasad, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), 98–121, Prop. 6.8. Publication record.

[23] M. Emerton, T. Gee and F. Herzig, *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), Prop. 2.4.1(i). Authors' paper.

[24] A. V. Zelevinsky, *Induced representations of reductive p-adic groups II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. Éc. Norm. Supér. **13** (1980), 165–210. Published paper.