The generalized Riemann hypothesis for Dirichlet L-functions

Volume XX · Heights and divisors of projected theta periods

26 November 2026 · Edition 700

Abstract

We compare the complex norms and finite period ideals of the same signed columns. Their marked determinants, terminal Fourier components and common divisors enter the height of the projected period.

The guide in Volume I lists the sections and page counts for the series.

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108 Complex norms and finite period ideals

108.1 Testing a signed period on rational unitary frames

Fix x on one of the intervals of Section 6.6, and put $E = E_x$, $F = E^+$. Use the matched small endpoint, finite characters and finite class measure of Section 97.6. A small class a has an underlying unimodular Hermitian lattice M_a, stabilizer Γ_a, and a polynomial coefficient space at each member τ of the CM type. Choose an E-basis of the rational space giving a basis of M_a at every place above g_1, g_2, T. Such a basis exists. In any starting basis the condition at a finite place is an open condition on an invertible matrix, and weak approximation on the entries meets these finitely many conditions. Use the dual basis on the opposite coefficient line. These choices write an algebraic output at a as a flag polynomial Q_a times the fixed basis of its determinant line. We evaluate its polynomial part. Restoring a determinant character on a unitary frame multiplies the value by a unit at every place where the frame and its inverse are integral.

Lemma 108.1 (Simultaneous rational frames for the small classes)**.** *At each a there is a finite set $\mathcal{F}_a$ in $U(M_a \otimes E)(F)$, integral on M_a at all three marked rational primes, with the following property. At each of these primes p, and at every coefficient place $w \mid p$, its reductions in the CM coordinates range over all of $\prod_\tau \mathrm{GL}_n(\mathbb{F}_p)$. One set works for all polynomial weights on this Hermitian lattice.*

Proof. The primes split completely in E. For each $v \mid p$ of F the two factors of $E \otimes_F F_v$ are paired by the Hermitian form. An isometry is determined by its matrix on the first factor. On the other it acts by the inverse transpose in paired bases. Unimodularity makes the second lattice the full dual of the first, so every invertible integral matrix on the first preserves both. Choose one of these two factors at each v, and lift any list of invertible residue matrices to integral isometries there.

To meet a prescribed list of residue classes by a single rational isometry, let H be the Hermitian matrix over E in our basis and let $X^* = H^{-1}\bar{X}^{\mathsf{t}}H$. The transformation $X \mapsto (I - X)(I + X)^{-1}$ identifies open subsets of the F-vector space $X^* = -X$ and the unitary group. Its inverse is given by the same formula, wherever $I + g$ is invertible. Any of the chosen local isometries can be moved into this open subset within its residue class. Multiply it on the two paired factors by t, t^{-1} with $t \in 1 + p\mathbb{Z}_p$. Only finitely many t give eigenvalue -1. The inverse transforms of the resulting isometries have open neighborhoods which the transform carries into the prescribed classes. Weak approximation in an F-basis of the skew-Hermitian space meets all these neighborhoods at once. Its rational transform is defined, since its denominator is nonzero at the chosen completions. Taking one such lift of each list gives a finite set. Changing w permutes the factors and may change a chosen factor to its paired one. In integral bases inverse transpose and changes of basis induce bijections on the invertible residue matrices. This proves the assertion for every w. □

For the small weight μ_τ put $m_\tau = \mu_\tau - \mu_{\tau n}\mathbf{1}$ and $h = \max_\tau m_{\tau 1}$. Use the bilinear highest-vector normalization $B = \bigotimes_\tau B_{m_\tau}$. For $g \in \mathcal{F}_a$ let $l_{a,g}$ be the vector in the opposite coefficient fiber such that

$$B(Q, l_{a,g}) = Q((\tau g)_\tau)$$

in the chosen determinant frames. Concretely its polynomial at each embedding is $p_{m_\tau}((\tau g)^{\mathsf{t}} X_\tau)$, where p_m is the product of the leading minors. This is the polynomial identity of Lemma 98.4. Form a test $t_{a,g}$ supported on a by taking the sum of the Γ_a-translates

of $l_{a,g}$, with the dual coefficient action and the inverse finite character. At the other representatives set the value to zero and extend by equivariance. Only the coefficient argument of the test varies. The finite representative, its residue flags and its level character remain those of a.

Proposition 108.2 (Integral generation of the terminal tests)**.** *Let $w \mid p$ be a place of the coefficient field above one of the three marked primes. Suppose $p > h + n$ and $p \nmid |\Gamma_a|$ for each small class. Then over the valuation ring O_w the tests $t_{a,g}$ generate the entire integral small test lattice S_w. For every invariant output table Y, the period with $t_{a,g}$ is exactly $Y(a)((\tau g)_\tau)$ in these frames.*

Proof. In a single factor the dual Weyl lattice ∇_m consists of the flag polynomials with integral coefficients. The Weyl lattice Δ_m lies in it, by its construction from integral row substitutions. The form B_m on this polynomial lattice has monomial weights $\alpha!/H_m$, with H_m the hook product. Each hook is at most m_1+n-1. Thus B_m is integral on $\nabla_m \times \nabla_m$ at w. Lemma 98.4 identifies the dual of ∇_m with Δ_m, giving equality of the two lattices and a perfect pairing. The same holds on their tensor product with the paired determinant lines. In its polynomial realization this tensor product is saturated in the ambient coefficient lattice. Indeed each individual integral flag space is the intersection of a vector subspace with the monomial lattice, hence a direct summand over a discrete valuation ring. Tensoring the splittings proves the assertion.

Each $l_{a,g}$ is integral by its displayed polynomial and our choice of bases. Suppose its evaluations failed to detect some nonzero reduction of a vector in the paired lattice. By saturation the reduced polynomial Q of that vector is nonzero. Each matrix entry at τ has degree at most $m_{\tau 1}$, since its entire matrix column has degree at most $m_{\tau 1}$. The polynomial

$$Q(Y) \prod_{\tau} \det Y_\tau$$

is nonzero over the residue field of O_w, of degree at most $h + 1 < p$ in each variable. A polynomial of degrees strictly less than p in the separate variables and zero on the full grid $\mathbb{F}_p^d$ is zero, even with coefficients in an extension field. Successive use of the one-variable root bound proves this. Thus Q is nonzero on a tuple of invertible residue matrices. Lemma 108.1 supplies a frame giving that tuple. By perfect duality the reductions of the $l_{a,g}$ therefore span the opposite fiber, and Nakayama's lemma lifts this spanning statement to O_w.

The stabilizer trace maps that fiber onto its invariant submodule, since on an invariant vector it is multiplication by the unit $|\Gamma_a|$. Tracing the generators, then summing over the disjoint classes, proves generation of S_w. Contracting an invariant output with each term in the trace gives the same value $B(Y(a), l_{a,g})$ by duality of the character actions. There are $|\Gamma_a|$ terms and the class has measure $1/|\Gamma_a|$, proving the period identity. □

Theorem 108.3 (Determinantal ideals of the actual signed columns at the marked primes)**.** *At every sufficiently large x on the stated intervals, take any finite list of the prescribed signed theta columns with a common small testing space. For each column j let $Y_j = R_1 P_j$ be its actual lowered small table, including all earlier operations. Fix the finite central projector e of Theorem 101.15, allowing also $e = 1$. Write $Z_j = e^{\mathrm{t}} Y_j$, where the transpose is for the finite class pairing. At every $w \mid g_1 g_2 T$, and for every choice of s columns, the fractional ideal of the s-rowed period determinants tested on eS_w is generated by*

$$\det \left[Z_j(a_i)((\tau g_i)_\tau) \right]_{i,j=1}^{s}, \qquad a_i \text{ small classes}, \quad g_i \in \mathcal{F}_{a_i}.$$

It is also generated by the coefficient minors $\det[(Z_j(a_i))_{\alpha_i}]_{i,j=1}^{s}$, in the same polynomial frames, with one joint monomial α_i for each row. Both identities include zero ideals. There is no loss in valuation at any exterior degree in passing between these tests and coefficients.

Proof. The cutoff and stabilizer assertion in the proof of Theorem 105.1 give the hypotheses of Proposition 108.2 for all three primes on these fields and intervals. Indeed the small weight width is bounded by the large weight width by interlacing. The character values and the original column entries lie in the common coefficient field. The projector can have a denominator there. Its transpose exists over the field because the pairings on the opposite finite table spaces are perfect in characteristic zero, by the classwise nondegenerate pairing and stabilizer averaging. Since eS_w is generated by $et_{a,g}$, multilinearity of the determinant proves the first equality of ideals from $\langle Y_j, et_{a,g}\rangle = \langle Z_j, t_{a,g}\rangle$.

For fixed classes use independent matrix variables X_i in each of the s rows of $\det[Z_j(a_i)(X_i)]$. Its coefficients are exactly the asserted determinants, by expansion in those independent variables. Each evaluation is an integral combination of them. For a nonzero determinant polynomial divide it by a coefficient of least valuation. Its reduction is nonzero and its degree in any matrix entry is at most h, since any determinant term uses just one polynomial in each row. Multiplication by one determinant for every row and embedding allows the grid argument in the proposition to find a tuple of invertible residues giving a nonzero value. Choose the rational lifts separately in each row. The resulting determinant attains the least coefficient valuation. If the polynomial is zero, every evaluation and coefficient is zero. Taking all class lists proves the second equality. □

For the saturated projected tests $S_{0,w} = S_w \cap e(S_w \otimes \operatorname{Frac} O_w)$ the distinction between the two integral domains can likewise be kept exact. Let L send the standard generators of a free O_w-module to the tests $t_{a,g}$. Compute $K = \ker((1-e)L)$ over O_w, clearing any scalar denominator of $1-e$ before computing the kernel. Proposition 108.2 gives $L(K) = S_{0,w}$: lift any element of $S_{0,w}$ through L, and the lift belongs to K. Consequently a basis of K, applied as row combinations to the unprojected frame evaluations of Y_j, gives the period minors for $S_{0,w}$ by multilinearity. This computes both choices of integral domain on the given columns. Their minimum valuations still depend on the coefficients of the signed lifts. An estimate for those valuations and an identification with the analytic incoming row are required in Section 6.6.

108.2 A common set of tests for compact norms and marked period ideals

We work on the small definite space at a class a of Section 108.1. Put $\mathcal{S} = \{g_1, g_2, T\}$ and $G = \operatorname{Res}_{F/\mathbb{Q}} U(M_a \otimes E)$. Its real points form a compact group. Give this group Haar measure of total mass one. Each rational point of G gives a frame in the same Hermitian space over F. We shall use one finite set of such frames both for the integral periods and for integration on the compact group. For positive finite integration formulas over the real numbers, see Bayer and Teichmann [1].

Write V for the underlying rational vector space of M_a, and ρ for the action of G on V. In a rational basis let W_d be the space of restrictions to G of polynomials of total degree at most d in the entries of ρ. It is a finite-dimensional rational space of regular functions, unchanged by a rational change of basis. In this section $d \ge 1$.

Lemma 108.4 (The local denominator of compact integration)**.** *Normalized integration on W_d is a rational linear functional I. At a prime p split completely in E, let $K_p = \prod_{v\mid p} \mathrm{GL}_n(\mathbb{Z}_p)$ be the stabilizer of the underlying unimodular lattice. If $p > \max(2d+1, d+n)$, the extension of I to $\mathbb{Q}_p$ is integral on every function in $W_d \otimes \mathbb{Q}_p$ taking integral values on K_p.*

Proof. The rational points of G are dense in $G(\mathbb{R})$. Indeed the Cayley argument of Lemma 108.1 also works at every archimedean place of F. Multiplication by a central unit complex number sufficiently near one avoids eigenvalue -1 there. Weak approximation on the skew-Hermitian

coordinates can include these neighborhoods and any finite list of the open conditions in that lemma. The same Cayley chart shows that $G(\mathbb{R})$ is Zariski dense, since a polynomial vanishing on an open real coordinate box is identically zero.

The translates $R_g f(x) = f(xg)$ preserve the real space of functions $W_d \otimes \mathbb{R}$. With its Haar L^2 inner product, the orthogonal complement of the span of $R_g f - f$, for rational g, consists of the invariant functions. By continuity and density these are the constants. Thus I is the unique functional annihilating the translation differences with $I(1) = 1$. A rational basis and the rational translations give rational linear equations for it. Finitely many of their rows have the full rank, so the solution is rational. Uniqueness for this finite system persists over $\mathbb{Q}_p$.

In paired local bases write Y_v for the matrix on one of the two factors above v. The action on the other factor is its inverse transpose. After any fixed change of basis each entry of ρ is a linear combination of entries of these matrices and their inverse transposes. Write $\Delta(Y) = \prod_{v|p} \det Y_v$. Every $f \in W_d \otimes \mathbb{Q}_p$ has a unique polynomial numerator P_f in $f = \Delta^{-d} P_f$. Its degree in any entry is at most $2d$. Indeed an inverse entry contributes a cofactor divided by a determinant. In one factor, b inverse entries and c forward entries give at most $b + c + (d - b) = d + c$ in each variable after clearing the denominator. A function integral on K_p has P_f integral. Otherwise scale the numerator to a primitive integral polynomial by a positive power of p. Its reduction times Δ is nonzero, with every individual degree less than p. The grid argument of Proposition 108.2 yields an invertible residue tuple on which it is nonzero, contradicting integrality of the original function. The converse follows since Δ is a unit on K_p. The argument uses just one representative of each invertible residue tuple.

For a square variable Y put $\partial_Y = (\partial/\partial Y_{ij})_{ij}$ and $H = \prod_{j=0}^{n-1} (d + j)!/j!$. Then

$$I(f) = \left. \frac{\prod_{v|p} \det(\partial_{Y_v})^d P_f(Y)}{H^{[F:\mathbb{Q}]}} \right|_{Y=0}.$$

To verify the formula, first take $f = 1$. The Fischer squared norm of $(\det Y)^d$ is the hook product of the rectangle (d^n), namely H, by Lemma 70.4. For a right translation $Y_v \mapsto Y_v A_v$ the numerator polynomial is replaced by $P_f((Y_v A_v)_v)/\Delta(A)^d$. The chain rule multiplies its indicated determinant of derivatives by $\prod_v \det(A_v)^d$ at zero, cancelling the denominator. The formula therefore satisfies the same rational translation equations and normalization as I. Uniqueness proves it. The factorial ratios defining H are products of integers at most $d+n-1$. The differential operators have integral coefficients, so an integral numerator gives an integral answer. □

Theorem 108.5 (Positive rational integration weights at the three marked primes)**.** *Suppose each $p \in \mathcal{S}$ exceeds $\max(2d + 1, d + n)$. There is a finite set A_a of rational unitary frames integral at every $p \in \mathcal{S}$, containing the complete residue tests $\mathcal{F}_a$ of Lemma 108.1, and positive rational numbers $\lambda_{a,g}$ such that*

$$\sum_{g \in A_a} \lambda_{a,g} = 1, \qquad I(f) = \sum_{g \in A_a} \lambda_{a,g} f(g) \quad (f \in W_d).$$

Every $\lambda_{a,g}$ is a unit at every prime in $\mathcal{S}$. The same formula integrates W_d after complex scalar extension.

Proof. In $W_d^\vee \otimes \mathbb{R}$ write u_g for evaluation at g. The compact convex hull of the evaluations on $G(\mathbb{R})$ has affine span $u(1) = 1$. Indeed any further affine relation would give a nonconstant element of W_d equal to a constant on the group. Its barycenter I is in the relative interior.

Otherwise a supporting affine functional nonconstant on this hull would attain its maximum at its integral. Continuity and positive Haar mass on every nonempty open set exclude this.

There is a finite set of rational frames integral at the chosen primes whose evaluations have I in the interior of their convex hull relative to the same hyperplane. To see this, take points in the hull on both sides of I along each coordinate axis of that hyperplane, all at a common small distance. Each is a finite convex combination of evaluations. The resulting polytope contains a ball about I. Perturb the finitely many frames to rational frames integral at the chosen primes by the simultaneous approximation in the preceding proof. If all the evaluations move by less than half the radius, every supporting linear functional of norm one still has maximum strictly above its value at I. Separation proves the assertion. The zero-dimensional case requires just one frame. Adjoin $\mathcal{F}_a$ to this set and call the result B. Adjoin also a distinct frame g' for each $g \in B$ congruent to g at all the marked primes. Such frames exist by the same approximation with varying archimedean neighborhoods, since the unitary center has positive real dimension. Call the enlarged set A_a. There are strictly positive real coefficients on all of A_a representing I. In fact assign a sufficiently small positive mass at every point. The barycenter required of the remaining mass is still in the original interior neighborhood.

At each marked p, the functions integral on K_p form a full $\mathbb{Z}_p$-lattice Λ_p in $W_d \otimes \mathbb{Q}_p$. This follows by intersecting its numerator space with the finite monomial lattice. The evaluations from $\mathcal{F}_a$ span $\Lambda_p^\vee$ over $\mathbb{Z}_p$. Indeed their dual lattice is exactly Λ_p by the grid argument in Lemma 108.4, and double duality over a discrete valuation ring gives the assertion. That lemma puts I in their integral span. Write it as $\sum_{g\in B} c_g u_g$ with $c_g \in \mathbb{Z}_p$. Since $p > 2$, write each residue $\bar{c}_g$ as a sum of two nonzero residues: choose the first different from both zero and $\bar{c}_g$. Lift the two to units at g, g'. Their weighted evaluations reproduce I modulo $p\Lambda_p^\vee$, because integral numerators divided by Δ^d take the same residue at g and g'. The old evaluations span $\Lambda_p^\vee$, so changing their coefficients by elements of $p\mathbb{Z}_p$ removes the error. This gives an exact local solution with all coefficients units.

The equations on the coefficients are rational linear equations. Gaussian elimination makes their solution space a rational affine space, since they admit a real solution. Approximate the strictly positive solution at infinity and the constructed unit solutions at the three primes in its free rational coordinates. Positivity and being a local unit are open conditions. The resulting single rational solution gives the asserted weights. □

We now apply the integration formula to the signed columns $Y_j = R_1 P_j$ and $Z_j = e^{\mathrm{t}} Y_j$ of Theorem 108.3. For a small weight set

$$d = \max(1, 2\sum_{\tau} |m_\tau|), \qquad |m_\tau| = \sum_{i=1}^{n} m_{\tau i}.$$

One may take the largest such d for all the common small types in use at x. This is $O(x^C b)$ for a fixed C. Indeed the field degrees and ranks have polynomial bounds and each width is $O(xb + x^{C'})$ by Theorem 105.1. The bounds $g_i > (kb)^8$ and $\log T \gg m x^{\varepsilon}$, with $m \longrightarrow \infty$, put these same three primes above $\max(2d+1, d+n)$ for all sufficiently distant x.

For a complex embedding σ use the transported positive Hermitian form on the small fiber in the highest-vector Fischer normalization. Write r_μ for its complex dimension. We may and do choose the rational basis at each class to be orthonormal and integral at all three marks. Indeed the small definite group in Theorem 101.15 is the isometry group of the standard Hermitian space E^n, as in `Formalism/LocalToGlobal/Columns/StdLatUnimodular.lean`, line 29. Start in the standard rational orthonormal basis. At each split v choose a basis of the first factor of M_a and its paired dual basis in the second. The resulting basis in the product algebra is orthonormal, by unimodularity. A local unitary matrix takes the starting basis to this one.

Generating that same local lattice is an open condition on the matrix. The rational Cayley approximation used in Lemma 108.1 meets all these conditions at once. Its rational matrix is exactly unitary, so the resulting basis is orthonormal at every infinite place. In these frames the representing vector of evaluation at the identity is the tensor of the p_{m_τ}, of norm one by Lemma 98.4, with the unit determinant vector for the separate line. All the tests in what follows use these frames.

Theorem 108.6 (The definite Gram determinant on the same rational tests)**.** *Let G_σ be the Hermitian Gram matrix of s such projected signed columns in the small class metric with masses $1/|\Gamma_a|$. Put*

$$z_{a,g;j} = \langle Y_j, et_{a,g} \rangle, \qquad q_{a,g} = \frac{r_\mu \, \lambda_{a,g}}{|\Gamma_a|}.$$

For $g \in A_a$ use the stabilizer trace defining $t_{a,g}$ in Section 108.1. Then

$$\det G_\sigma = \sum_{\substack{B \subseteq \{(a,g) : g \in A_a\} \\ |B| = s}} \left(\prod_{(a,g) \in B} q_{a,g} \right) \left| \det [\sigma z_{a,g;j}]_{(a,g) \in B, j} \right|^2 .$$

All the weights are positive rational numbers, independent of σ and units at the three marks. At every marked coefficient place the minors in the sum generate the exact determinantal ideal on eS_w.

Proof. Fix a class and an embedding. The product of the small flag modules, meaning their external tensor product over the real places, is an irreducible unitary module of the product of the compact unitary groups. This follows from the irreducibility of each highest-weight representation and the tensor product rule for a direct product. Let l be its functional evaluating at the identity. The average over the group of the positive operators $(l \circ g)^*(l \circ g)$ commutes with the group. Its trace is $\|l\|^2 = 1$, so Schur's lemma makes it $1/r_\mu$ times the identity. Row substitution translates the functional at the identity into matrix evaluation. In the orthonormal bases the transpose and inverse conventions both preserve normalized Haar measure on $U(n)$. Removing the determinant character multiplies each translated functional by a scalar of absolute value one, which cancels in these products.

For vectors P, Q in this fiber, the product of the polynomial value $P(g)$ and the conjugate of $Q(g)$ belongs to $W_d \otimes \mathbb{C}$. Each entry of the complex matrix at a CM embedding is a complex linear combination of entries of $\rho(g)$, by extension of scalars in the restriction of scalars representation. Those real matrix entries are real on $G(\mathbb{R})$, so the same assertion holds for their complex conjugates. The product degree is at most $2 \sum_\tau |m_\tau|$. This also applies at every σ, which permutes the CM embeddings or selects their conjugates. Theorem 108.5 integrates each such product exactly with the same rational weights. Applying the preceding operator identity to the small values of the Z_j, and inserting the class masses, gives the Gram matrix of the array $(\sqrt{q_{a,g}}\, \sigma z_{a,g;j})$. Here the trace identity in Proposition 108.2 identifies each entry before taking absolute values. Cauchy–Binet gives the formula. Finally $\mathcal{F}_a \subseteq A_a$, and every additional frame is integral on the same lattices at the marked primes. Its trace test belongs to S_w, so applying e gives a test in eS_w. The two inclusions of the determinantal ideals follow from Theorem 108.3. The stabilizer orders are units at the marks by Theorem 105.1. The factors in the Weyl dimension formula for r_μ are $(m_{\tau i} - m_{\tau j} + j - i)/(j - i)$, $1 \le i < j \le n$. Both integers in each ratio lie strictly between zero and p at all three primes. The weights $\lambda_{a,g}$ are also units there by construction, proving the assertion about $q_{a,g}$. □

Corollary 108.7 (A simultaneous minor at three prescribed coefficient places)**.** *For each $p \in \mathcal{S}$ choose one coefficient place $w_p \mid p$. For fixed s columns of full rank on the projected tests, there*

are s semilocally integral tests before projection for which the same period determinant z, after projection, attains the least valuation at each of the three w_p. Set $P_{\mathcal{S}} = g_1 g_2 T$, and let M bound all $|\Gamma_a|$ as in Theorem 97.10. For every complex embedding one has

$$|\sigma z|^2 \le (9MP_{\mathcal{S}}^2)^s \det G_\sigma.$$

The logarithm of this comparison factor per exterior degree is $o(b/x^A)$ for every fixed $A > 0$ on the same intervals.

Proof. For w_p, Theorem 108.3 gives rows $t_{i,p} = t_{a,g}$ achieving its minimum. Choose rational integers $0 \le c_p < P_{\mathcal{S}}$ equal to one modulo p and zero modulo the other two primes, by the Chinese remainder theorem. Use $t_i = \sum_p c_p t_{i,p}$ before projection. At w_p the determinant expansion has one term of minimum valuation multiplied by c_p^s. Every other term has one coefficient divisible by p, and its period determinant has valuation at least the same minimum since its rows too are permissible. This proves simultaneous attainment.

At infinity the norm of the linear functional on a small class table given by $t_{a,g}$ is at most $\sqrt{|\Gamma_a|}$. In fact the trace and mass cancel on invariant values, evaluation on that fiber has norm one by unitarity, and taking its class value costs at most $\sqrt{|\Gamma_a|}$ in the mass norm. Thus each combined row functional has norm at most $3P_{\mathcal{S}}\sqrt{M}$. Apply their exterior product to $Z_1 \wedge \cdots \wedge Z_s$. Its norm is at most the product of their norms by the Gram determinant inequality, and the squared norm of the column wedge is $\det G_\sigma$. The upper bounds on the prime sizes in Theorems 7.11 and 9.13 make $\log P_{\mathcal{S}}$ polynomial in x, and Theorem 97.10 gives the same for $\log M$. The asserted comparison follows from $b = \exp(x^{\varepsilon})$. □

The primes in the corollary can be those of Theorem 15.12. Thus on these same E_x the critical-sign summands $e_{r+1}\mathcal{S}_{p,r}(E_x)$ vanish through that theorem's weight range, while the comparison above uses the same three primes for the same signed determinant. The rational frames can introduce denominators away from these primes. The formula determines the Gram matrix of the definite class tables. Comparison with the indefinite cuspidal period and a uniform estimate of the signed coefficients are still needed for Section 6.6.

108.3 All finite period ideals and the definite Gram determinant

Fix the fields, weights and class measures of Section 108.2. Work in a common finite Galois coefficient field $J \subset \mathbb{C}$ and write c for complex conjugation in this embedding. At $w \mid p$ use the valuation $v_w(p) = 1$. Let $Y_1, \dots, Y_s$ be the actual lowered signed columns for a common small type, and set $Z_i = e^{\mathrm{t}} Y_i$. The finite central projector e, which may be the identity, acts on the test side. On each class the bilinear period pairs opposite coefficient spaces by the highest-vector Fischer form with weight $1/|\Gamma_a|$. Denote this sum by $[Y, F]$.

For $D = eS$ or $D = S_0 = S \cap e(S \otimes J)$, define $\mathfrak{a}(D)$ to be the fractional ideal generated by $\det([Y_i, F_j])_{i,j=1}^s$ for $F_1, \dots, F_s \in D$. Use the abbreviations $\mathfrak{a}_{\mathrm{im}} = \mathfrak{a}(eS)$ and $\mathfrak{a}_0 = \mathfrak{a}(S_0)$. We assume the columns are independent on the projected tests, so these ideals are nonzero. They can be computed after localization on local generators of the corresponding lattices, by multilinearity and flatness. Write $b_{\mathrm{im},w} = v_w(\mathfrak{a}_{\mathrm{im}})$ and $b_{0,w} = v_w(\mathfrak{a}_0)$.

There is a conjugate-linear identification C from the algebraic column space to the opposite test space. In standard rational orthonormal coordinates conjugate the coefficients of the flag polynomials and use the dual unit vector of each determinant line. The bilinear Fischer contraction of P with CQ is the Hermitian form of the small fiber, linear in P. In particular

$$\mathcal{D} = \det\left([Z_i, CZ_j]\right)_{i,j=1}^s$$

is an element of J, strictly positive at the chosen complex embedding by the independence hypothesis. Here all finite characters have modulus one, so conjugation changes a column's stabilizer character to the inverse character of a test.

Lemma 108.8 (Conjugating a primitive exterior column into the test lattice)**.** *Fix w. In the direct sum of the class coefficient Weyl lattices at w, every exterior coordinate of $\xi = Z_1 \wedge \cdots \wedge Z_s$ has valuation at least $b_{\mathrm{im},w}$. At the conjugate place cw, conjugation carries the corresponding local Weyl lattice integrally into the opposite dual Weyl lattice at w. Consequently, if $t \in J_w^{\times}$ has $v_w(t) = b_{\mathrm{im},cw}$, then*

$$t^{-1} C\xi \ \in \ \bigwedge^{s} S_{0,w}.$$

Here C on an exterior power acts on every factor.

Proof. Choose a basis at w of the direct sum of the full coefficient lattices. Their integral duals for the classwise contraction are the opposite dual Weyl lattices by Lemma 98.4. They include the dual basis functionals. For any one such functional, take the stabilizer trace on each class, including its inverse level character. This gives an element of S_w. The class-mass pairing of this test with Z_i equals the value of the original functional on Z_i. Indeed each term of the trace has the same pairing on an invariant output, and the number of terms cancels the class mass. This calculation holds also when $p \mid |\Gamma_a|$, since only the unnormalized trace was used in the integral test. Pairing Z_i with a test in S_w equals pairing Y_i with its image under e. Thus every coordinate minor for ξ belongs to $\mathfrak{a}_{\mathrm{im}} O_w$.

For the integral effect of C, start with the Hermitian lattice M_a itself. In standard orthonormal coordinates its conjugate is its bilinear dual: integrality of $\sum_i x_i \overline{y}_i$ for all $x \in M_a$ is equivalent to $y \in M_a$, by unimodularity. After extension of scalars the same equality pairs the factor at cw with the opposite one at w. Thus conjugating an integral basis of a factor at cw gives an integral basis of the opposite lattice at w. The flag construction respects changes of basis in $\mathrm{GL}_n(O_w)$. In particular the leading-minor polynomial and its iterated integral row coefficients defining Δ_m have integral coefficients in the paired frames. Hence conjugation carries Δ_m into the opposite integral polynomial lattice ∇_m there. Exterior determinant lines of any integer power on the two sides pair integrally by the same unimodularity, using the dual line for a negative power. Tensoring and taking the direct sum gives the required integral map in every residue characteristic.

The central idempotent is self-adjoint for the Hermitian class metric. On a common full residue marking the group acts unitarily, and its central coefficients $\sum_{\chi} \chi(1)\chi(g^{-1})$ at g^{-1} are the conjugates of those at g. Restriction to the original level gives Theorem 101.15. In particular $Ce^{\mathsf{t}} = eC$, by the defining bilinear and Hermitian adjoint identities. It follows that $C\xi$ lies rationally in the exterior power of the projected test space. By the first part at cw and the integral map just proved, $t^{-1}C\xi$ belongs to the exterior power of the full ambient test lattice. Inside this lattice the submodule $S_{0,w}$ is the intersection with the rational projected test space. Indeed the invariant table is obtained by the rational stabilizer equations and intersection with the ambient lattice, and S_0 imposes the additional rational equation $eF = F$. This intersection is saturated and hence a direct summand over the valuation ring. A basis extending a basis of the summand shows that the same intersection equality holds in exterior degree s. This proves the assertion. □

Theorem 108.9 (An all-prime inequality for the actual projected definite periods)**.** *For the independent list above one has the fractional ideal memberships*

$$\mathcal{D} \ \in \ \mathfrak{a}_0 \, c(\mathfrak{a}_{\mathrm{im}}) \ \cap \ \mathfrak{a}_{\mathrm{im}} \, c(\mathfrak{a}_0).$$

For every embedding $\sigma : J \hookrightarrow \mathbb{C}$ let G_σ be the positive Gram matrix of $\sigma Z_1, \dots, \sigma Z_s$ in the class metric as in Theorem 108.6. With the absolute norm on fractional ideals, this gives

$$\log N\mathfrak{a}_0 + \log N\mathfrak{a}_{\mathrm{im}} \;\le\; \sum_{\sigma : J \hookrightarrow \mathbb{C}} \log\det G_\sigma .$$

In particular this comparison between finite valuations and complex Gram determinants includes every coefficient prime of the original definite columns, with the specified signs, level and class masses.

Proof. At w consider the alternating functional on s tests given by the determinant of the periods with $Y_1, \dots, Y_s$. On $\bigwedge^s S_{0,w}$ its image is $\mathfrak{a}_0 O_w$ by definition. The action of e on these tests is the identity, so using Z_i gives the same functional. At $C\xi$ its value is $\mathcal{D}$. Lemma 108.8 therefore gives

$$v_w(\mathcal{D}) \ge b_{0,w} + b_{\mathrm{im},cw}.$$

Since $c(\mathcal{D}) = \mathcal{D}$, applying this at the conjugate place gives the second membership. No choice of a single simultaneous generator of the local ideals is needed.

For an embedding σ write $\sigma' = \mathrm{conj}_{\mathbb{C}} \circ \sigma \circ c$. In standard rational orthonormal polynomial coordinates use the direct sum of the complex coefficient spaces over the class representatives, with the rational positive masses and normalized Fischer forms. In this ambient Hilbert space $\sigma\mathcal{D}$ is the Hermitian product of the wedges with coordinate arrays $\sigma\xi$ and $\sigma'\xi$. Indeed conjugating the latter coordinates gives exactly $\sigma(c\xi)$, while the monomial weights in the Fischer pairing and the masses are rational. In particular this computation does not require c to commute with each embedding. Cauchy–Schwarz gives

$$|\sigma\mathcal{D}| \le (\det G_\sigma \, \det G_{\sigma'})^{1/2}.$$

Both determinants are positive, since linear independence persists under the embeddings. The map $\sigma \mapsto \sigma'$ permutes the embeddings. Consequently the sum of their log determinant bounds is $\sum_\sigma \log\det G_\sigma$. At the finite places the first membership gives $\log N\mathfrak{a}_0 + \log N\mathfrak{a}_{\mathrm{im}} \le \log|N_{J/\mathbb{Q}}\mathcal{D}|$ by factorization of fractional ideals. Summing the archimedean inequality gives the result. □

Corollary 108.10 (Uniform cost of using saturated projected tests)**.** *Define $B(D) = (s[J : \mathbb{Q}])^{-1} \log N\mathfrak{a}(D)$ and*

$$H = \frac{1}{s[J : \mathbb{Q}]} \sum_{\sigma : J \hookrightarrow \mathbb{C}} \log\det G_\sigma .$$

On the original fields and intervals,

$$2B(eS) \le H, \qquad 2B(S_0) \le H + \log m_G,$$

where m_G is the order of the two marked finite block groups taken together. For $e = 1$ the latter cost can also be taken to be zero. On the prescribed intervals $\log m_G = o(b/x^A)$ for every fixed $A > 0$.

Proof. The inclusions $m_G eS \subseteq S_0 \subseteq eS$ from Theorem 101.15 give

$$m_G^s \mathfrak{a}_{\mathrm{im}} \subseteq \mathfrak{a}_0 \subseteq \mathfrak{a}_{\mathrm{im}}$$

by evaluating the determinant. Thus $0 \le B(S_0) - B(eS) \le \log m_G$, while the theorem gives $B(S_0) + B(eS) \le H$. These prove both inequalities. For $e = 1$ the two lattices agree. The logarithm of the finite group order is polynomially bounded in x by Theorem 101.15, which

proves the final estimate. In the local normalization above, $\log N\mathfrak{a}(D) = \sum_p \log p \sum_{w|p}[J_w : \mathbb{Q}_p]v_w(\mathfrak{a}(D))$. All three normalized quantities are unchanged on extending the coefficient field: localization and flat base change preserve the period ideals, the rational invariant subspaces and their integral intersections, while each complex embedding contributes with the same multiplicity. The period field is still E_x. □

The right side is the actual definite Gram quantity computed in Theorem 108.6. To use this result in Section 6.6 one still needs a lower valuation estimate encoding the prescribed arithmetic counts, and an upper estimate for these Gram determinants in the specified analytic row, on a family of sufficient rank. The complementary strict cohomology summands in Proposition 6.11 also enter the interval estimate.

108.4 The signal prime count on the signed definite columns

Fix a translate and the ordinary words of Theorem 9.13 on its field E_x. We work with the algebraic class pairing of Theorem 105.1, in the marked elliptic differential. The high set is $T_{\mathrm{hi}} \subset \Phi_0$. Write $n = |\Phi_1|$, $a = \lceil xb \rceil$, $b = \exp(x^{\varepsilon})$, $X' = \exp(x^{\varepsilon/3})$, $b' = xb/X'$. For $1 \le i \le m$, the count uses the weights $e_{i,\tau}$ following (9.1). At an unramified prime inert in K_d and split in K_q put

$$U_p = \sum_{i=1}^{m} \sum_{t \in T_{\mathrm{hi}}} \sum_{\substack{j \ge 1 \\ j \text{ odd}}} [a/p^j - e_{i,\pi^j t}]_+, \qquad \pi = \sigma_p c_E.$$

Its contribution to the uncentered count in this sign is $2U_p \log p$ by (54.1).

Lemma 108.11 (The first row at the actual word weights)**.** *On every sufficiently distant common interval, uniformly in the word, translate and edge, the partition ν_i of the i-th definite kernel satisfies*

$$G_p(\nu_i) \;\ge\; \sum_{t \in T_{\mathrm{hi}}} \sum_{j \text{ odd}} [a/p^j - e_{i,\pi^j t}]_+.$$

Here G_p is the integer in Theorem 98.12.

Proof. Write $d = t_{\mathrm{r}} + 1$, $r = k + (i-1)d$, $s = k + id$. At stage $i-1$ the first highest weight is $u_1 - (k-1)/2 - (i-1)d/2$. Indeed the initial algebraic weight subtracts the half-sum of positive roots, with first coordinate $(k-1)/2$, from u_1. At each step the input determinant shift is $e_{i,\tau} - s/2$ and the output subtracts $e_{i,\tau} - r/2$ from the zero-padded partition. Thus previous entries decrease by $d/2$ per step, and a new entry equals $-e_{i,\tau} + r/2$. These are the shifts in the prescribed polynomial substitution, given in `Formalism/LocalRingCardinality/Params.lean`, line 71. In particular the partition obtained by adding the input shift has first row

$$\nu_{i,\tau,1} = u_1(\tau) + e_{i,\tau} - (r + s - 1)/2.$$

In the polynomial model one has first subtracted the last incoming weight c to trivialize its determinant line. The determinant power in (54.22) is consequently $c + e_{i,\tau} - s/2$. At an edge after the first this is $2H_0 - d$, by the formula for a new entry. At the first edge it is nonnegative by the choices of B_τ, H_0 in (9.1), also at a high place with $c = -\lceil b' \rceil$. This identifies the partition in the operator.

At $t \in T_{\mathrm{hi}}$, $u_1 = a + (k-1)/2$ and $e_{i,t} \ge b'$. At $\mu \in \Phi_1$, u_1 is bounded by a fixed power of x. The ranks and n also have polynomial bounds. Thus $\nu_{i,t,1} \ge a + b'/2$ and $\nu_{i,\mu,1} \le e_{i,\mu} + Cx^B$ for fixed B, C. All $e_{i,\mu}$ on Φ_1 are at least $b/(nX')$. If $M = p^j$ contributes positively at t, then $M < a/e_{i,\pi^j t} \le 2nxX'$. Hence

$$\frac{b'}{2M} \;\ge\; \frac{b}{4n(X')^2} > s + Cx^B$$

eventually. The last inequality follows on taking logarithms, since $x^{\varepsilon} - 2x^{\varepsilon/3}$ dominates every constant multiple of $\log x$. As $d_M(t', s) \geq t'/M - s$, it follows for each active M that

$$d_M(\nu_{i,t,1}, s) - \nu_{i,\pi^j t,1} \;\geq\; a/M - e_{i,\pi^j t}.$$

The positive parts also give the claim for inactive M. Sum the $l = 1$ terms of G_p. Its other terms are nonnegative. □

Let $Y_1, \dots, Y_q$ be columns with this translate and a common small testing space, arising from the full integral initial Weyl tables, with the prescribed signs and coupled last edges. Use S_0, eS, m_G as in Theorem 97.10; here S_0 denotes the saturated projected test lattice. Let $\mathcal{S}$ be any finite set of the unramified primes of the indicated sign, $p \geq 5$, at which the finite class cosets in the path formula have prime-to-p shifts and the good local vacuum of Theorem 54.24. These requirements concern the supplied tests before applying any derivatives. For instance, in the presentation of Theorem 100.4 the primes prime to all its original shift orders Q_i^0 meet the shift condition. One can use representatives equal to the identity at these primes. The proof of that theorem chooses representatives supported at completely split primes outside any prescribed finite set. Thus at p the standard frames are integral frames on the common lattices of consecutive class maps.

We record separately the scalar coefficients that may occur if the finite tests have denominators. In a given column expand the finite coset sections in translates of the good common vacuum, keeping the matrix coset restrictions and the shared coupling label on the last two edges. For a path γ, including one summand of each such expansion, let a_γ be the product of its external scalar weights, its coefficients of translation, and its determinant-character scalars. Include any algebraic change of the marked scalar normalization in this product. Use integral determinant frames at p. Exclude the stabilizer masses and the terminal test projector from a_γ. Zero terms can be omitted. Define

$$L_{i,w} = \max\left(\{0\} \cup \{-v_w(a_\gamma) : \gamma \text{ in column } i\}\right), \qquad w \mid p.$$

These numbers use only the coefficients before applying the theta derivatives. In particular they vanish for integral coefficients. The expansion by normalized finite lattice intertwiners, including the reciprocal indices and the splitting characters, is given in the proof of Theorem 100.13. Scaling an additive differential by c_τ scales a kernel of block degree $|\nu_\tau|$ by $c_\tau^{|\nu_\tau|}$. This explains how to include an algebraic differential rescaling if a different normalization of that line is used.

For each of the m source vertices of an edge, and once for the small endpoint, take $M_v = |\mathrm{GL}_{n_v[E_x:\mathbb{Q}]}(\mathbb{F}_3)|$, where n_v is its rank. Set $D = \prod_v M_v$. In the definitions of $\mathfrak{a}_0, \mathfrak{a}_{\mathrm{im}}$ in Section 108.3 use these q columns. Their valuations are denoted by $b_{0,w}, b_{\mathrm{im},w}$, allowing $+\infty$ for a zero ideal.

Theorem 108.12 (The good signal primes on both projected test lattices)**.** *At each $p \in \mathcal{S}$ and each $w \mid p$, one has*

$$b_{0,w} \geq qU_p - \sum_{i=1}^{q} L_{i,w} - qv_p(D),$$

$$b_{\mathrm{im},w} \geq qU_p - \sum_{i=1}^{q} L_{i,w} - qv_p(Dm_G).$$

For the identity projection one takes $m_G = 1$. The logarithm of Dm_G is $o(b/x^A)$ for every fixed $A > 0$ on the same intervals.

Proof. At an unrefined initial class, any nonzero terminal covector tuple on a path gives an allowed refined input on its Weyl lattice, as in Theorem 60.34. The lattice modification made at T localizes to the original lattice at p since T splits in K_d. Starting at this input, apply the coefficient map on each matrix coset. In the marked differential the rule is to twist the source polynomial by the determinant power, substitute $D^{\mathrm{t}}Y_{[r]}$, and replace D^{α} by the undivided derivative of the corresponding CM section. For a finite coset indicator this is the same rule on its finite sum of translated vacua, by the linearity of differentiation. Its scalar factors are exactly the external factors recorded above. This describes (54.22) at consecutive vertices and the coefficient maps in Theorem 60.32. In the archive the substitution and evaluation are written in `Formalism/LocalToGlobal/KernelLattices/ThDistC.lean`, line 27.

At w, each prime-to-p translated product with its coefficient removed meets the local germ hypothesis. Its ratio to the reference vacuum is an integral elliptic germ by Corollary 53.28 and its proof; this also follows directly by multiplying the individual translation ratios. Theorem 98.12 puts the output in the Weyl lattice times $p^{G_p(\nu_j)}$ at edge j. Changes between two integral frames preserve this lattice. Indeed the Fischer adjoint is transpose substitution on the integral polynomial dual, which is preserved by every integral invertible change of variables. Determinant lines are changed by units. Induction on the edges and Lemma 108.11 give a valuation of at least U_p in every terminal Weyl coordinate before inserting the scalar weights and masses. The inequality involving a real U_p means that each coordinate has at least that valuation. It follows from the integer exponent $\sum_j G_p(\nu_j)$.

Now use the path identity of Theorem 60.32. It sums only paths with the specified initial covector, and keeps each coupling label common to both incident maps. On each path the original scalar product costs at most $L_{i,w}$. Its stabilizer order at each vertex divides M_v, by the faithful integral action and torsion-free level-three kernel used in Theorem 97.10. Clearing D thus clears every path mass, including the endpoint. Summing with the original signs preserves the common bound. The terminal restriction acts integrally on Weyl lattices in matched frames by Theorem 98.6 and Proposition 82.34. The contraction with an integral small test is integral by Fischer duality. This proves the bound for a single entry with tests in S_0. For tests in eS, multiplication by m_G puts every test in the integral small lattice, by the finite character sum for the projector. In each term of a q-row determinant every column occurs exactly once. Adding the entrywise bounds and then taking the ideal of all such determinants proves both inequalities. Finally $\log|\mathrm{GL}_d(\mathbb{F}_3)| \le d^2\log 3$; all vertex ranks, their number and the field degree are polynomial in x. The same polynomial bound for $\log m_G$ is given by Theorem 97.10. □

Corollary 108.13 (The good-prime count and the Gram determinant of the same columns)**.** *Suppose the chosen columns have rank q on the projected tests. Write $\omega_w = [J_w : \mathbb{Q}_p]/[J : \mathbb{Q}]$ and set*

$$L_{\mathcal{S}} = \sum_{p\in\mathcal{S}} \log p \sum_{w\mid p} \omega_w \sum_{i=1}^{q} L_{i,w},$$
$$B_{\mathrm{out}} = \sum_{p\notin\mathcal{S}} \log p \sum_{w\mid p} \omega_w \bigl([-b_{0,w}]_+ + [-b_{\mathrm{im},w}]_+\bigr).$$

Then

$$2\sum_{p\in\mathcal{S}} U_p \log p \;\le\; \frac{\sum_{\sigma} \log\det G_{\sigma}}{q[J:\mathbb{Q}]} + \frac{2L_{\mathcal{S}} + B_{\mathrm{out}}}{q} + 2\log D + \log m_G.$$

Proof. For a nonzero fractional ideal $\mathfrak{a}$, its normalized log norm is $\sum_p \log p \sum_{w\mid p} \omega_w v_w(\mathfrak{a})$. This follows from the residue degrees in the ideal norm and $v_w(p) = 1$. Sum both inequalities of Theorem 108.12 with these weights on $\mathcal{S}$; their sum on the other primes is bounded below by

$-B_{\mathrm{out}}$. Theorem 108.9 bounds the sum of the two normalized ideal log norms above by the mean of the log Gram determinants. Divide by q and use $\sum_{w|p}\omega_w = 1$. □

This compares the signal-prime contribution and the complex Gram determinant for one set of actual definite columns. The rank assumption, the costs $L_{\mathcal{S}}, B_{\mathrm{out}}$, and the size of that Gram determinant enter any application to (6.3).

108.5 All active signal primes for the prescribed finite tests

Work with one K_d-oriented field and translate of Theorem 9.22. Let $\mathcal{P}$ consist of the unramified $p \geq 5$ split in K_q, inert in K_d, for which $U_p > 0$. This is a finite set since $p \leq 2nxX'$. Use the marked elliptic differential and the elementary finite tests of the definite column in Theorem 60.32. These tests include the row and flag conditions, the terminal covector condition, and the common vector coset on the last two edges. Stabilizer masses and the small test projector have the normalizations of Theorem 108.12.

Lemma 108.14 (Support and coefficients of the finite tests)**.** *At each $p \in \mathcal{P}$ the elementary finite tests have the standard local lattice vector. On every edge there is a single choice of product basis and class representatives for which their expansions into elliptic products have prime-to-p shifts and coefficients integral above p, simultaneously for $p \in \mathcal{P}$. This holds for each allowed auxiliary character, each matrix coset, and each value of the common vector coset. The coefficients here precede insertion of the stabilizer masses, the small test projector, and any additional scalar linear combination of elementary columns.*

Proof. Write $F = E^+$. At a place v of F an allowed auxiliary character is trivial on all local units unless v splits in E/F, its norm is a rational prime ℓ, and $\ell \notin v^2$. These are the defining conductor restrictions in `Formalism/LocalToGlobal/Divisibility/Tuples.lean`, line 59–66 and `Formalism/LocalToGlobal/Divisibility/Tuples.lean`, line 110–118. The norm condition gives residue degree one and the valuation condition gives ramification index one. At either place of E over a split v the completion is therefore $\mathbb{Q}_\ell$. Since $E/\mathbb{Q}$ is Galois, this forces ℓ to split completely in E. The two flag primes g_1, g_2 and the word's terminal prime T also split completely by construction. Let S_{row} be the selected places at the conductor of the auxiliary character and at g_1, g_2. Every prime below S_{row} or T is split in K_d.

To check that this list includes every modification of the finite vector, consider the matrix test in its position coordinates. At the row places it is a function on lattice cosets. The two integer bounds on denominators and moduli are

$$c_A = \Big(\prod_{P \in S_{\mathrm{row}}} NP\Big)^A, \qquad c_B = T\Big(\prod_{P \in S_{\mathrm{row}}} NP\Big)^B.$$

Away from their support the position lattice and its congruence sublattice agree. At T the extra test is the indicator of the linear congruence on the terminal covectors. On the last two edges the shared variable Y is a vector coset with conditions only at S_{row}. In its Fourier transform the normal covector is multiplied by π_T^{-1}, where $\pi_T\overline{\pi_T} = T$. Hence it too is integral at every $p \in \mathcal{P}$. These are the tests written in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1110–1139 and `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1150–1170. Keeping Y fixed throughout this calculation preserves the coupling. At $p \in \mathcal{P}$ the matrix lattice is consequently the original unramified Hom lattice, with the reference vacuum.

We check the scalar weights as well. At a modified place of residue characteristic ℓ, the flag and block tests take the value zero or a product of residue-field characters, their inverses, and

an additive character of a finite quotient. Every nonzero such character value is a root of unity. The free entries in the unipotent integral are summed over lattice cosets, each with measure a power of ℓ. The Fourier transform of Y has the same form, with a power of ℓ from the coset volume. The determinant changes of variables contribute powers of $(NP)^{1/2}$ and values of the splitting character at ideles supported over ℓ. For the latter, raise the idele to a ray class power to write it as a principal idele times a unit idele on which the finite character is trivial. This can be done since the quotient of the ideal class group with the finite unit conditions is finite. The principal generator is a unit at every prime over p. The algebraic infinity type, after squaring to clear the half weights, is a monomial in its conjugates with integral exponents. Every factor is a p-unit because $E/\mathbb{Q}$ permutes the primes above p. This proves the same unit assertion for the original character value.

There is also a ratio of two compact open volumes on the unipotent radical in the row integral. In a basis adapted to its flag this group acts by upper unitriangular matrices. Conjugation by a diagonal matrix of sufficiently separated uniformizer powers puts both compact subgroups in the group of integral upper unitriangular matrices. Every congruence quotient of this ambient compact group has order a power of ℓ, by counting its strict upper entries. A sufficiently deep congruence kernel intersects each subgroup inside their common intersection, by openness in the radical. The two indices over that intersection are therefore powers of ℓ. This proves the assertion also for the volume ratio and its inverse. The change of variables, finite additive sum, reciprocal radical volume, and terminal Fourier transform occur respectively in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 552–586, `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 941–950 and `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 1150–1155. All these factors are integral above p.

It remains to pass from cosets to the vacuum expansion. At a split place let L be the position lattice of the vacuum and refine a coset to $z + L'$ with $L' \subset L$ an open sublattice. On $z + L$ its indicator is

$$[L : L']^{-1} \sum_{\eta \in (L/L')^\vee} \eta(y - z).$$

This follows from the character sum on the finite additive group. The index is a power of ℓ. Hence every term is a position and frequency translate of the vacuum with p-integral coefficient. Its shifts are supported over ℓ. Clearing a common denominator and applying the Chinese remainder theorem identifies these finite local shift classes with a rational class in the period space modulo its lattice. Its order is prime to every member of $\mathcal{P}$.

Apply Theorem 53.11 at each vertex with the whole finite set $\mathcal{P}$ excluded. The representatives move only at other completely split primes. Their lattice intertwiners add shifts there, with root-of-unity factors and inverse square roots of indices supported there. Their character factors are units by the ray class argument above. At p the class lattices at a shared vertex still agree on both incident edges, as in Theorem 100.4. Finally p is unramified and inert in the elliptic CM field K_d. Lemma 53.14 gives one orthonormal product basis per edge generating the trace lattice at all $p \in \mathcal{P}$. The intertwiner to this product lattice again has index prime to every p in the set. In the discriminant-one elliptic model of Section 53.5 prime-to-p translations have integral germs relative to the marked vacuum, including when the translated section vanishes at the mark, by Corollary 53.28. This proves the assertions. □

Theorem 108.15 (The complete signal-prime contribution to the period ideals)**.** *Let column Y_i use an integral initial Weyl table and the prescribed elementary tests. One may use a scalar linear combination of such columns on the same directed tower, with the same weights, for Y_i. Let C_i be a positive integer clearing these additional scalar coefficients. For an elementary*

column or an integral combination take $C_i = 1$. With $\omega_w = [J_w : \mathbb{Q}_p]/[J : \mathbb{Q}]$, one has for every $p \in \mathcal{P}$

$$\sum_{w|p} \omega_w b_{0,w} \geq qU_p - \sum_{i=1}^{q} v_p(C_i) - qv_p(D),$$

$$\sum_{w|p} \omega_w b_{\mathrm{im},w} \geq qU_p - \sum_{i=1}^{q} v_p(C_i) - qv_p(Dm_G).$$

The bounds use the same period ideals in any marked product bases, with the differential and physical coordinates fixed.

Proof. In the bases and representatives of Lemma 108.14, clearing C_i makes every premass path coefficient integral at $w \mid p$, including the character factors. At a common vertex use the integral frames on its Weyl lattice; a unit change of determinant frame preserves it. Theorem 108.12 thus bounds the valuations of the minors in these bases at each individual w by the right sides. This argument applies termwise to the integral combination C_iY_i and then divides by C_i once per column in the minor.

Changing to any other fixed marked product basis multiplies all entries of column i by the same scalar u_i, by Theorem 100.3. Every term of a combination on that tower contains one copy of each directed edge, so it has that same scalar. Every minor is multiplied by $\prod_i u_i$. Enlarge J to contain these scalars if necessary. Since $|\sigma u_i| = 1$ for all embeddings, $|N_{J/\mathbb{Q}} u_i| = 1$, and therefore

$$\sum_{w|p} \omega_w v_w(u_i) = [J : \mathbb{Q}]^{-1} v_p(N_{J/\mathbb{Q}} u_i) = 0$$

at each rational p. Both weighted ideal valuations are therefore preserved. The local degree identity in a tower preserves the averages after extending coefficients. □

Corollary 108.16 (The centered bound for the prescribed column tests)**.** *If these q columns are independent on the projected small tests, the centered sum on the same field and interval satisfies*

$$\Delta_u \leq \frac{\sum_\sigma \log\det G_\sigma}{q[J : \mathbb{Q}]} - A_{\mathrm{sig}}(x) + \frac{B_{\mathrm{data}}}{q} + \frac{2}{q}\sum_{i=1}^{q} \log C_i + 2\log D + \log m_G + o_{\psi,\varepsilon,A}(b/x^A)$$

for every fixed $A > 0$. In B_{data} use $\mathcal{S} = \mathcal{P}$ and the fixed normalization used to evaluate the Gram determinants.

Proof. Theorem 108.15 supplies both lower bounds at every active prime in Theorem 9.22. At the other primes use Proposition 9.23 in the original bases. Sum and apply the product formula inequality of Theorem 108.9 in those bases. The restricted sum of $v_p(C_i)\log p$ is at most $\log C_i$, and the identical assertion applies to D, m_G. Substitution of the centered identity proves the claim. Basis change preserves the determinant of the Gram matrix at each embedding, since it multiplies the columns by scalars of absolute value one. □

108.6 The different and the split tests in the centered bound

Use the q columns of Theorem 108.15. Denote the total derivative degree at embedding τ on edge j of column i by $d_{ij,\tau} = |\nu_{ij,\tau}|$, and let $d_{ij} = \sum_\tau d_{ij,\tau}$. In a linear combination take the common directed tower and weights of that theorem. Let $\mathcal{M}$ be the rational primes below the

row places, the two flag primes, and all terminal primes of the selected columns. Write $\mathcal{T} \subset \mathcal{M}$ for those terminal primes. All of $\mathcal{M}$ splits completely in E by Lemma 108.14. We use the prescribed integral Weyl tables at the incoming endpoint.

Lemma 108.17 (The loss of an undivided derivative on the Weyl lattices)**.** *Work over a coefficient valuation ring, using integral coefficient frames at the two ends of an edge. Suppose each undivided derivative of its class section of the given block degrees has valuation at least t, for a real t. Every coordinate of the resulting edge operator between the source and target Weyl lattices has valuation at least t.*

Proof. Take a generator of the source as a coefficient of the highest polynomial on a product of elementary row matrices with independent parameters. After the determinant twist, its polynomial is the corresponding coefficient of $p_\nu(AX)$, since $\det A = 1$. Pair with an integral dual polynomial P_τ at each target embedding. As in the proof of Theorem 98.12, the differential symbol in that block is $P_\tau([DA_\tau^{\mathsf{t}}, 0])$. Its coefficients in the indeterminates and in the entries of D are integral. The product over embeddings is homogeneous of degrees $d_{ij,\tau}$. After parameter extraction its evaluation is an integral linear combination of precisely the derivatives in the hypothesis. Such generators span the source, and the tensor products of these tests span the integral dual of the target. Testing on a dual basis proves the assertion. A determinant line trivialized in an integral frame contributes a unit under an integral change of frame. □

Here is finite data for a prime $p \in \mathcal{M} \setminus \mathcal{T}$. At each $v \mid p$ in F, let L_v be the standard split position lattice of the given edge. Its $\mathbb{Z}_p$-rank is $D_{jv} = r_j s_j$ for edge ranks r_j, s_j, since $F_v = \mathbb{Q}_p$. For all elementary functions of this edge in this column, divided by the amplitude of the reference position vacuum, choose nonnegative integers a_v, b_v so that their support lies in $p^{-a_v} L_v$ and they are constant on cosets of $p^{b_v} L_v$. Use the matrix-coset indicators appearing in the path sum, with the value of the coupled coset held fixed. Such bounds exist by the denominator and period lattices in Lemma 108.14. For a spherical factor take $a_v = b_v = 0$. Let $c_{ij,p}$ be the maximum of zero and the negatives of the valuations of the nonzero products of these original local values at $v \mid p$, over all the cosets in the finite list and all coefficient places above p. The sum of the corresponding nonnegative maxima place by place also gives an admissible upper bound. This includes the row integral scalars, prior to inserting the class masses.

Label the two places of $K = K_d$ above p by $+, -$. Let ϵ_v be the label of the position factor at v. For $\epsilon \in \{+, -\}$, put

$$h_\epsilon = \max\big(\{0\} \cup \{a_v : \epsilon_v = \epsilon\} \cup \{b_v : \epsilon_v \neq \epsilon\}\big), \qquad J_{ij,p}^{\mathrm{row}} = c_{ij,p} + \sum_{v \mid p} D_{jv} b_v + \frac{d_{ij}}{2}(h_+ + h_-).$$

The lattices, bounds and exponents here refer to the specified column and edge; they can be read off before any differentiation.

At every rational prime put $\rho_p = v_w(\mathfrak{D}_{E/K})$, with $v_w(p) = 1$, independent of $w \mid p$ by normality. Let κ_p be the bound for the marked elliptic differential in Theorem 55.12. For $p \notin \mathcal{M}$ set

$$J_{ij,p}^{\mathrm{vac}} = \tfrac{1}{2} d_{ij} \rho_p + \kappa_p \lfloor d_{ij}/2 \rfloor.$$

Theorem 108.18 (Outside-prime valuations of the two period ideals)**.** *For $p \notin \mathcal{T} \cup \mathcal{P}$ use $J_{ij,p} = J_{ij,p}^{\mathrm{vac}}$ off $\mathcal{M}$ and $J_{ij,p} = J_{ij,p}^{\mathrm{row}}$ on $\mathcal{M}$. In the original normalization, writing $b_{S_0,w} = b_{0,w}$ and $b_{eS,w} = b_{\mathrm{im},w}$, take $m' = 1$ in the saturated case and $m' = m_G$ in the image case. One has*

$$\sum_{w \mid p} \omega_w b_{D',w} \;\geq\; -\sum_{i,j} J_{ij,p} - \sum_i v_p(C_i) - q v_p(D) - q v_p(m')$$

for the saturated and image test lattices, respectively. In particular an unmarked, unramified prime with $\kappa_p = 0$ costs only the explicitly displayed scalar, mass, and projector valuations.

Proof. Fix p and choose class representatives equal to the identity over p, by Theorem 100.4. All class movements occur at other split primes and contribute p-unit multipliers and shifts with zero p-primary component. Also the finite tests and changes of variables at the other marked primes have integral coefficients above p, by the local character, volume and finite Fourier calculations in Lemma 108.14. Choose on each edge the product basis of Theorem 100.14 generating the local Hermitian trace lattice. Its vacuum intertwiner adds shifts of order prime to p and has unit normalization.

Suppose first $p \notin \mathcal{M}$. Every translated elliptic factor now has order prime to p in its translation parameter. Theorem 54.35 bounds the loss at derivative order s by $\kappa_p \lfloor s/2 \rfloor$. In an undivided derivative of degree d_{ij} the orders sum to d_{ij}, so these losses sum to at most $\kappa_p \lfloor d_{ij}/2 \rfloor$. In physical block τ the change of coordinates has entries of valuation at least $-v_w(\bar{\tau}\mathfrak{b})$, since product basis vectors belong locally to $\mathfrak{b}^{-1}\operatorname{Mat}(\mathcal{O}_E)$ and the inverse embedding matrix is the conjugate transpose. Here $\mathfrak{b}\overline{\mathfrak{b}} = \mathfrak{D}_{E/K}$. There are exactly $d_{ij,\tau}$ differentiations in this block. All multinomial coefficients in the chain rule are integral. Thus a bound for the loss above p is

$$H_{ij,w} = \kappa_p \lfloor d_{ij}/2 \rfloor + \sum_{\tau} d_{ij,\tau} v_w(\bar{\tau}\mathfrak{b}).$$

Use τ in place of $\bar{\tau}$ on a conjugated edge. In a common normal coefficient field, conjugation permutes the places above p and preserves their local degrees. For each individual τ the two conjugate ideals consequently have the same weighted average valuation. Their product has valuation ρ_p at every place above p. It follows that $\sum_{w|p} \omega_w H_{ij,w} = J^{\mathrm{vac}}_{ij,p}$.

Suppose next $p \in \mathcal{M} \setminus \mathcal{T}$. For one position coset $z + p^{b_v} L_v$, Fourier expansion on L_v costs exactly the reciprocal index $p^{-D_{j v} b_v}$ before cancellation. Its position translate has order dividing p^{a_v}, and its frequency character has order dividing p^{b_v} modulo the dual lattice. This is the coset character identity in Lemma 108.14. On taking products, the p-parts of the two idempotent components of any resulting shift have orders at most p^{h_+}, p^{h_-}, respectively. This follows by taking the maximum of the exponents on the factors of the direct product. The local basis change is $\mathcal{O}_K \otimes \mathbb{Z}_p$-linear, so these exponents also work for each elliptic coordinate. The conjugate annihilator ideal in Theorem 54.35 exchanges the two components. Since $\kappa_p = 0$, at a place over ϵ the loss of all d_{ij} differentiations is at most $d_{ij} h_{-\epsilon}$. The physical coordinate conversion costs zero here. Indeed $\mathfrak{b}$ in Theorem 52.10 is integral, and its product with its conjugate is a unit at every prime over this unramified p. The two places of K each have weight $1/2$ on averaging over $w \mid p$, by the local degree formula in a tower. Including the original scalar and the Fourier indices gives the bound $J^{\mathrm{row}}_{ij,p}$ on the weighted average.

Apply Lemma 108.17 to every edge at each w, before averaging. Since $p \notin \mathcal{T}$, the terminal refinements localize to the unrefined lattice at each vertex. Thus the identity representatives give matching integral frames at both ends of every composition. The terminal restriction is integral by Theorem 98.6. Each term of a column contains exactly one derivative operator per edge, hence its coordinate loss is at most the sum of the individual bounds plus $v_p(C_i)$. The finite signed sums preserve this lower valuation bound. Clearing D clears the path masses and $m_G e S \subset S_0$ clears the projector. Expand the minors, add over the columns, and take the minimum over the tests. Changing from these product bases to the original ones preserves each weighted ideal valuation by the norm calculation in Theorem 108.15. This proves both inequalities. □

Corollary 108.19 (A finite cost supported on the terminal and row primes)**.** *For independent projected columns let $B_{\mathcal{T}}$ be the sum in Proposition 9.23 restricted to $p \in \mathcal{T}$, in the original*

normalization. On the same intervals let $d_0 = |\operatorname{disc} K|$*. Then*

$$\begin{aligned}\Delta_u \le{}& \frac{\sum_\sigma \log\det G_\sigma}{q[J:\mathbb{Q}]} - A_{\mathrm{sig}}(x)\\ &+ \frac{1}{q}\sum_{i,j}\Big(d_{ij}\log \mathrm{rd}(E) - \tfrac{1}{2}\mathbf{1}_{d_{ij}\ \mathrm{odd}}\log d_0\Big)\\ &+ \frac{2}{q}\sum_{p\in\mathcal{M}\setminus\mathcal{T}}\sum_{i,j} J^{\mathrm{row}}_{ij,p}\log p + \frac{B_{\mathcal{T}}}{q}\\ &+ \frac{2}{q}\sum_i \log C_i + 2\log D + \log m_G + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

Proof. The different tower formula gives $\sum_p \rho_p \log p = \log \mathrm{rd}(E) - \frac{1}{2}\log d_0$. Here 3 splits in K on the selected intervals. Consequently the only nonzero κ_p is $\kappa_{d_0} = 1/2$. Every prime of $\mathcal{M}\cup\mathcal{P}$ is unramified in E and has $\kappa_p = 0$. Twice the sum of the vacuum costs of edge ij is therefore

$$d_{ij}\big(\log \mathrm{rd}(E) - \tfrac{1}{2}\log d_0\big) + \lfloor d_{ij}/2\rfloor \log d_0.$$

This is the displayed discriminant term. At the terminal primes use the finite symbolic bound, which includes the actual input and terminal frames there. At active signal primes use Theorem 108.15; at all others use Theorem 108.18. Sum the two normalized ideal log norms before taking the archimedean bound of Theorem 108.9. For each C_i, D, m_G the sum off the terminal primes is at most its full logarithm. Inserting the centered identity of Theorem 9.22 proves the inequality. □

The loss at unmarked primes in this estimate depends on the field and the degrees alone. The quantitative comparison still involves the size and rank of the signed projected periods and the finite tests at their specified levels.

108.7 The terminal covector congruence in the centered bound

Fix a translate and the common terminal generator π of Corollary 9.15. Thus $T = \pi\bar{\pi}$ splits completely in E, $S_T = \{P : P \mid \pi\}$ contains one place over each place $v \mid T$ of F, and $f = |S_T| = [F:\mathbb{Q}]$. The auxiliary characters are unramified there and T is distinct from the flag primes. We use the elementary column tests of Theorem 108.15. Let r_j, s_j be the ranks at the two ends of edge j. At a fixed pair of unrefined classes write $L_{j,P}$ for the lattice of integral position matrices at P; its reduction consists of all s_j-by-r_j matrices over $\mathbb{F}_T$.

Lemma 108.20 (Fourier expansion with the intermediate covectors fixed)**.** *In the signed path sum one can sum over nonzero terminal covectors at the vertices and use on each edge the indicator of the single affine linear condition*

$$\xi' \bar{X} = \xi$$

on $L_{j,P}$*, for each* $P \in S_T$*. Relative to the unit vacuum, this indicator is a sum of pure frequency translates with common denominator* T^{r_j} *at* P*. Their* T*-primary shift classes are killed by* T *and lie in the component at the conjugate place* $\bar{P}$*.*

Proof. Keep the original classes, their masses, all nonterminal local data and the common vector coset on the last two edges. For an integral matrix path and the specified final covector, successive multiplication backwards gives a unique tuple of intermediate covectors. If any intermediate covector is zero, the initial one is zero and the path is discarded. Conversely every

surviving path occurs once on summing over the nonzero tuples and imposing the displayed equations. This is the path identity of Theorem 60.32. With the tuples fixed the conditions on the separate edges are independent. The matrix-coset terms can be summed into their union because their indicator functions add, the theta section is linear in the finite test, and the iterated map is linear in the section on each edge separately. Chinese remaindering makes this regrouping independent of the conditions at the row places. The finite test at T in `Formalism/LocalToGlobal/Columns/FiniteSumNonzeroLargeDepth.lean`, line 646 is exactly this covector indicator.

Fix a nontrivial additive character ψ_T of $\mathbb{F}_T$. On $L_{j,P}$ the indicator is

$$T^{-r_j} \sum_{u \in \mathbb{F}_T^{r_j}} \psi_T(\xi' \bar{X} u - \xi u).$$

The sum is one after division if $\xi' \bar{X} = \xi$, and zero otherwise by character orthogonality. Multiplication by $\mathbf{1}_{L_{j,P}}$ supplies the support condition. The coefficient phase is a root of unity. The linear functional $X \mapsto \xi' \bar{X} u$ is a character of $L_{j,P}/TL_{j,P}$, so its frequency is a class in $T^{-1}L_{j,P}^{\vee}/L_{j,P}^{\vee}$. Under the split Hermitian pairing the dual of the position component at P is the component at $\bar{P}$. This proves all the assertions, including the support of each individual shift. □

For column i, let $\lambda_{i,\tau,k}^{\mathrm{in}}$ be the smallest entry of its initial highest weight at embedding τ, and let $\delta_{i,\tau}$ be its terminal normal degree in Corollary 105.2. Define

$$z_i = \prod_{\tau} \tau(\bar{\pi})^{\lambda_{i,\tau,k}^{\mathrm{in}}}, \qquad Z_i^{\mathrm{norm}} = \prod_{\tau} \tau(\bar{\pi})^{-\delta_{i,\tau}}, \qquad \widetilde{Y}_i = (z_i Z_i^{\mathrm{norm}})^{-1} Y_i.$$

The two scalars are units away from T. Use tildes on the valuation of an ideal or on a Gram matrix to indicate that these rescaled columns are used, with the original small test lattices.

Theorem 108.21 (The terminal valuation on both projected period ideals)**.** *For $D' = S_0, eS$, take $m' = 1, m_G$, respectively. One has*

$$\sum_{w \mid T} \omega_w \widetilde{b}_{D',w} \;\geq\; -\sum_{i,j} \left(f r_j + \tfrac{1}{2} d_{ij} \right) - \sum_i v_T(C_i) - q v_T(Dm').$$

Proof. Choose unrefined class representatives equal to the identity above T, by Theorem 100.4. The small class and its lift then have the standard unrefined frames at T. For each initial residue covector ξ its refined Weyl lattice satisfies $D_{a,\xi} \subset z_i D_a$, where D_a is the unrefined lattice. Indeed the prescribed kernel at P changes the column lattice at $\bar{P}$ by replacing one basis vector by T times that vector, by the two dualizations in the proof of Theorem 77.20. In the polynomial representation with its last weight subtracted, all degrees of that vector are nonnegative. Reinstating the determinant power gives the factor $T^{\lambda_{i,\tau,k}^{\mathrm{in}}}$ at the affected embedding. Their product differs from z_i by a local unit. Thus $z_i^{-1} Y_i$ may be computed with initial values integral in the unrefined frames. At the last class, in those frames the actual restriction is Z_i^{norm} times the integral standard restriction, by Theorem 105.1 and Theorem 98.6. Removing this second scalar allows us to use unrefined large frames all the way through the path sum. There is no choice of a test or a new projection at an intermediate vertex.

We make a single choice of product coordinates at T for each directed edge. The split trace lattice is the orthogonal sum over $v \mid T$ of the position lattice and its dual, paired with each other. For a position basis and its exact dual, the sums of corresponding vectors give a local orthonormal $K \otimes \mathbb{Q}_T$-basis, as in Theorem 100.14. Label its coordinates by v and a matrix-entry index. Approximate this local basis by a rational orthonormal basis agreeing modulo T

at both places of K, using the Cayley approximation in that theorem. The congruence is an open condition. Both the change of basis and its inverse then preserve the individual v-blocks modulo T. In particular an order-T shift has in each new local coordinate the same component modulo the period lattice as under the exact local basis, after identification by these bases. The vacuum intertwiner for this rational basis has index prime to T. Its shifts and the shifts from the other finite primes add no T-primary components.

Fix $w \mid T$ and an embedding τ in the CM type. Let $P_{\tau,w}$ be the prime of E obtained by pulling w back under τ, and put $v = P_{\tau,w} \cap F$. For a coordinate in the v-block, the conjugate annihilator bound of Theorem 54.35 uses the part of the shift at the place of K opposite the one induced by τ, w. If $P_{\tau,w} \notin S_T$, this opposite component of the shift at v is zero modulo the lattice. In fact the position prime there is $\overline{P_{\tau,w}}$, and the lemma puts the entire frequency shift at $P_{\tau,w}$. Hence the undivided derivatives on that coordinate are integral in every order. If $P_{\tau,w} \in S_T$, the same argument permits a loss of one per differentiation, since T kills the primary shift. Here $\kappa_T = 0$.

The physical derivative in a τ-matrix entry is a sum of product derivatives with integral coefficients. For coordinates belonging to $v' \neq v$ these coefficients lie in $T\mathcal{O}_w$. Indeed the coefficient is the conjugate embedded basis entry, which vanishes in those blocks for the exact local basis and agrees with it modulo T for the chosen basis. Conjugation exchanges $P_{\tau,w}$ and its partner over the same v. Each such off-block factor therefore pays the loss of at most one from a differentiation there. To see this in all orders, expand the product of the linear differential operators. In a product coordinate whose primary shift can cost one per derivative, its derivative order counts exactly the individual differentiations landing there. Each arriving from an uncharged block has a factor of T. Terms from charged blocks have integral coefficients, and integral multinomial coefficients preserve the resulting lower bound.

Consequently the loss in a derivative of the edge’s block degrees, apart from the Fourier indices, is at most

$$\sum_{\tau} d_{ij,\tau}\, \mathbf{1}_{P_{\tau,w} \in S_T}.$$

For a conjugated edge evaluate this formula at the conjugate place. There are f Fourier indices from the lemma, costing fr_j in either orientation. Other finite inputs have coefficients integral at T by the split-prime character and volume calculation in Lemma 108.14. At the fixed w, use Lemma 108.17 on the unrefined Weyl lattices, then compose the maps at the common vertices. Taking determinants adds the costs for the columns. The path masses, projector, and coefficient denominators contribute precisely the allowances displayed in the statement.

For each fixed τ, the total local degree weight of $P_{\tau,w} \in S_T$ is $f/[E : \mathbb{Q}] = 1/2$, because T splits completely and local degrees add in the tower $J/\tau(E)/\mathbb{Q}$. The same holds at the conjugate places. Average the preceding bound. A return to the original product bases multiplies the minors by a common product of column phases of rational norm of absolute value one, as proved in Theorem 108.15. This preserves the average and proves the theorem. □

Corollary 108.22 (The centered inequality with explicit finite terms)**.** *At the common prime*

T of the original family, the q independent projected columns satisfy, for every fixed $A>0$,

$$\begin{aligned}\Delta_u \le{}& \frac{\sum_\sigma \log\det \widetilde{G}_\sigma}{q[J:\mathbb{Q}]} - A_{\mathrm{sig}}(x)\\ &+\frac{1}{q}\sum_{i,j}\Big(d_{ij}\log \mathrm{rd}(E) - \tfrac{1}{2}\mathbf{1}_{d_{ij}\ \mathrm{odd}}\log d_0\Big)\\ &+\frac{2}{q}\sum_{p\in\mathcal{M}\setminus\{T\}}\sum_{i,j} J^{\mathrm{row}}_{ij,p}\log p + \frac{1}{q}\sum_{i,j}(2fr_j+d_{ij})\log T\\ &+\frac{2}{q}\sum_i \log C_i + 2\log D + \log m_G + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

In the first term one may equivalently use the original $\log\det G_\sigma$ *corrected by* $-\sum_{i,\tau}(\lambda^{\mathrm{in}}_{i,\tau,k}-\delta_{i,\tau})\log T$ *at every embedding.*

Proof. Use Theorem 108.21 at T in the proof of Corollary 108.19 and use the rescaled columns throughout. The two scalars removed from each column are units at every other prime, so the previous estimates for both ideals hold there with the same valuations. The T-terms from the two test lattices sum to the displayed terminal cost. The original and rescaled Gram determinants differ by $\prod_i|\sigma(z_iZ_i^{\mathrm{norm}})|^{-2}$, since the class metric and projector are fixed. Now $|\sigma\tau(\bar\pi)|=\sqrt{T}$ for every σ,τ. This gives the asserted correction. □

All terms of this inequality are evaluated on the field and translate of the prime count. Its use in (6.3) requires a uniform bound for their sum and a supply of nonzero projected columns on those intervals.

108.8 The contribution of primes inert in both quadratic fields

Work on the polynomial-conductor fields and intervals of Theorem 9.26. Put $Y=X'=\exp(x^{\varepsilon/3})$. Write T_{hi} for the high set of size h. The extra divisor $\mathcal{D}_u$ of Corollary 56.3 measures divisibility of the signed word periods beyond the primes used in U_p. We estimate a part of it for which the dyadic Frobenius moves every high place to a place of small weight.

Let $G_j=\mathrm{Gal}(C_j/\mathbb{Q})$ with $j=j_x$. For a power of two $M\mid 2^j$ use the cyclic quotient G_j/MG_j of order M. For a prime unramified in E_x let $\bar p$ denote its Frobenius in this quotient. It is also the image of $\pi_p=\sigma_p c_E$, since C_j is totally real. For a function on a finite abelian group normalize its Fourier coefficients so that it equals their sum times the corresponding characters, and write $\widehat{w}(1)$ for the average of w.

Lemma 108.23 (A weighted prime sum on the dyadic quotient)**.** *Let w be real valued on G_j/MG_j with $\sum_\chi|\widehat{w}(\chi)|\le K$. Uniformly in M,w,x,*

$$\sum_{\substack{p\le Y,\ p\ge 5,\ p\ \text{unramified in } E_x\\ p\ \text{inert in } K_q,\ K_d}} w(\bar p)\frac{\log p}{p} = \frac{\widehat{w}(1)}{4}\log Y + O_{\psi,\varepsilon}(K\log^2(2x)).$$

Proof. On unramified primes multiply w by $(1-\xi_q)(1-\xi_d)/4$. The two quadratic fields and the dyadic quotient have pairwise coprime conductors, so their compositum has the direct product Galois group. This follows also by independent variation of the units at the three moduli. The product has constant Fourier coefficient $\widehat{w}(1)/4$, and its sum of absolute Fourier coefficients is at most K. Every character in the expansion belongs to E_x.

For the primitive representative of each character, use $R_\chi(Y)/Y$ from (1.1). Replacing the sum over unramified primes by these expressions costs $O_\psi(K)$. Indeed removing prime powers

of exponent at least two costs at most $\sum_p (\log p)/(p(p-1)) < \infty$ per character. The linear smoothing term costs at most $Y^{-1}\sum_{p\le Y}\log p = O(1)$ by Chebyshev's bound. At $2, 3$ and the primes dividing N_x, the remaining difference costs at most a constant times $\sum_{p\mid 6Fqd_0}\log(p)/(p-1) = O_\psi(1)$. This accounts also for primitive representatives with smaller conductors.

The large-argument estimate in the proof of Theorem 9.26 gives $G_\chi(Y)/Y = o(1)$ uniformly. For the principal character the term remaining in $R_\chi(Y)/Y$ is $\log Y - 1 - \gamma$. For a nonprincipal character it is $-L'(1,\chi)/L(1,\chi)$, bounded in absolute value by $O_\psi(\log^2(2N_x))$ by the zero-free estimate and the logarithmic derivative calculation of Lemma 7.1. Since $N_x \le x^{D_0}$, Fourier summation gives the assertion. □

Theorem 108.24 (A lower bound for the extra divisor of the projected periods)**.** *For any of the high coset choices in Theorem 6.29, the additional term in Corollary 56.3 satisfies, uniformly over translates and the admissible lists of independent signed columns,*

$$\mathcal{D}_u \;\ge\; (\tfrac14 - o_{\psi,\varepsilon}(1))\, hma \log X'.$$

Proof. Let D be the dyadic index used to choose the high set, so $2x^{\varepsilon/8} \le D < 4x^{\varepsilon/8}$. In G_j/DG_j the image of T_{hi} contains at most $3o+1$ points, $o = [K_\psi : \mathbb{Q}]$, since T_{hi} is a union of that many cosets of DA_0. Write S for their difference set. It contains zero and has size at most $(3o+1)^2$. Take $w = 1 - \mathbf{1}_S$. Each point indicator has Fourier coefficient sum in absolute value one, so the lemma applies with $K \le 1 + |S|$ and $\widehat{w}(1) = 1 - |S|/D$.

For a prime counted with this weight, π_p has nonzero image in the dyadic quotient and hence even order. Its cycles on the embedding labels all have that order, since the action is translation in the abelian group. The two negative signs ensure that it preserves the K_q-sign on the K_d-type. Thus $\pi_p T_{\mathrm{hi}} \subset \Phi_0 \setminus T_{\mathrm{hi}}$. Choose in each cycle the half containing at least half of its high places. At least $h/2$ high places are then in the chosen halves.

For each such t, each column and each edge the high partition has $\nu_{t,1} \ge a$ by the word calculation in Lemma 108.11. At $\mu = \pi_p t$ the partition has $\nu_{\mu,1} \le b/(nY) + O(x^B)$ for a fixed B: this place is neither high nor distinguished, so its weight is $\lceil b/(nY)\rceil + O(mH_0+1)$ and its starting first coordinate has polynomial size. The rank s is polynomially bounded as well. Keep only the first high row and the first Frobenius power in the row contribution of Theorem 56.2. Since $d_p(\nu_{t,1}, s) \ge a/p - s$, the contribution of this place is at least $a/p - b/(nY) - O(x^B)$, increasing B if necessary. This is positive throughout $p \le Y$, since $(b/(nY) + x^B)Y/a = o(1)$. The constants are independent of the word, edge and translate.

Here $U_p = 0$. The factor two in $\mathcal{D}_u$ comes from the two integral test lattices. After division by hma, the lower bound from these primes is their sum of $(\log p)/p$ up to an additive loss

$$O\!\left(\frac{(b/(nY) + x^B)Y}{a}\right) = o(1),$$

by the Chebyshev bound again. The lemma gives the more precise lower bound $(1 - |S|/D)\log Y/4 - O_{\psi,\varepsilon}(\log^2(2x))$ for this normalized expression. Since D tends to infinity and $\log^2(2x) = o(\log Y)$, the theorem follows. □

Corollary 108.25 (Aligning the high cosets in the dyadic quotient)**.** *Let $e_2 = [K_\psi \cap C_{j_x} : \mathbb{Q}]$. One may choose the high cosets in the same fields so that the preceding inequality improves to*

$$\mathcal{D}_u \;\ge\; (\tfrac14(1 + e_2^{-1}) - o_{\psi,\varepsilon}(1))hma \log X'.$$

In particular the coefficient is $1/2 - o(1)$ for the conductor-one character.

Proof. Use a generator to identify G_j with $\mathbb{Z}/2^j$. Restriction identifies $A_0 = \mathrm{Gal}(K_\psi C_j/K_\psi)$ with e_2G_j, and $D \mid |A_0|$ on these intervals. In each coset of A_0, denote its common residue modulo e_2 by v, represented in $[0, e_2)$. Inside the quotient modulo e_2D, choose the high residues

$$v + t\, e_2 D/4, \qquad t = 0, 1, 2,$$

and also $t = 3$ in the identity coset. For large x the integer D is divisible by four. These are three or four distinct DA_0-cosets in the required coset, by the restriction identification. This is an allowed selection in Theorem 6.29. The character cancellations use only that T_{hi} is a union of DA_0-cosets, with the prescribed number in each A_0-coset, and that T_2 is balanced modulo D. The sizes and word weights per type are therefore those of Corollary 9.15 on this field.

Now take $M = e_2D/4$ and form S in G_j/MG_j. Its elements are differences $v - v'$, so $|S|$ is bounded in terms of ψ and $S \cap e_2(G_j/MG_j) = \{0\}$. For $\bar{p} \notin S$, the theorem's half-count argument still works. In addition, if $\bar{p}$ is a nonzero multiple of e_2, all high places of any one cycle lie in the same alternating half. Indeed if t' and t are high and $t' = \pi_p^n t$, their projection difference is $n\bar{p}$. It is a multiple of e_2 in S, so it is zero modulo M. The nonzero element $\bar{p}$ has even order in that quotient, forcing n even. Choose the half containing all these high places in each cycle.

The additional $h/2$ high places, beyond the original lower bound $h/2$, can thus be counted with the second weight $\mathbf{1}_{e_2(G_j/MG_j)} - \mathbf{1}_{\{0\}}$. Its mean is $1/e_2 - 1/M$, and its Fourier coefficient sum in absolute value is at most two: a subgroup indicator is the average of the characters annihilating it. Summing the lemma for the first and second weights gives the normalized lower bound

$$\frac{1}{4}\left(1 + \frac{1}{e_2} - \frac{|S|+1}{M}\right)\log Y - O_{\psi,\varepsilon}(\log^2(2x)).$$

Since M grows and $e_2 \le o$, this proves the result. □

These lower bounds enter the centered inequality by subtraction on its right hand side. The archimedean Gram determinants there are calculated on the same projected signed columns. Their required upper bound, the existence of enough independent columns, and the remaining strict cohomology lengths are open steps in the argument for Dirichlet GRH.

108.9 Rational divisors and the height of the signed period columns

Use the common terminal normalization of Corollary 108.22 on the polynomial-conductor family of Corollary 9.27. Let $\widetilde{Y}_1, \dots, \widetilde{Y}_{q_c}$ be the signed columns in this normalization. Write $\widetilde{G}_\sigma$ for the Gram matrix of their projections in the positive class metric at an embedding $\sigma : J \hookrightarrow \mathbb{C}$. Assume the projections are independent when a logarithm of a Gram determinant occurs.

At each unramified $p \ge 5$ inert in K_d let $g_{i,p} = \sum_{j=1}^{m} V_p(\nu_{ij})$. Extend U_p by zero as in Corollary 56.3. Define

$$b_i^{\mathrm{per}} = \prod_p p^{\,g_{i,p} - \lceil U_p \rceil}, \qquad \rho_x = 2\sum_p (\lceil U_p \rceil - U_p)\log p,$$

with both operations restricted to these primes. The data here are calculated before any values of period integrals are known.

Lemma 108.26 (An integer divisor for each individual column)**.** *Each b_i^{per} is a positive integer. Division by it leaves a gain of at least $\lceil U_p \rceil$ at each indicated prime on the individual column in adapted frames, before its specified scalar, stabilizer and projector denominators. For every fixed $A > 0$,*

$$\frac{2}{q_c}\sum_i \log b_i^{\mathrm{per}} = \mathcal{D}_u - \rho_x, \qquad 0 \le \rho_x = o_{\psi,\varepsilon,A}(b/x^A).$$

Proof. On each word the integral gains from the edges give $g_{i,p} \ge U_p$, by Theorem 56.2 and Lemma 108.11. At a prime inert in both quadratic fields this uses only $g_{i,p} \ge 0$. Since $g_{i,p}$ is an integer, all the exponents in the product are nonnegative. They have finite support by Corollary 56.3. The divisibility in adapted frames follows from the edge inclusions on each path in the proof of that corollary. Removing the rational integer subtracts the same valuation at every coefficient place over p. The terminal normalization is a unit at these primes.

Taking logarithms gives the identity from the definition of $\mathcal{D}_u$. A rounding term can occur only when $U_p > 0$, which forces $p \le 2nxX'$ by Theorem 9.22 and the actual word cutoff. Each term before multiplication by $2\log p$ is less than one. Chebyshev's bound gives $\rho_x = O(nxX')$, which is $o(b/x^A)$ since n is polynomial, $X' = \exp(x^{\varepsilon/3})$, and $b = \exp(x^{\varepsilon})$. □

Let G_σ° be the Gram matrix for the projections of $\widetilde{Y}_i/b_i^{\mathrm{per}}$. To specify all other terms, put

$$\begin{aligned} K_u = {} & \frac{1}{q_c}\sum_{i,j}\Big(d_{ij}\log \mathrm{rd}(E_x) - \tfrac{1}{2}\mathbf{1}_{d_{ij}\ \mathrm{odd}}\log d_0\Big) \\ & + \frac{2}{q_c}\sum_{p\in\mathcal{M}\setminus\{T\}}\sum_{i,j} J_{ij,p}^{\mathrm{mat}}\log p + \frac{1}{q_c}\sum_{i,j}\left(2fr_j + \sum_{\tau}\nu_{ij,\tau,1}\right)\log T \\ & + \frac{2}{q_c}\sum_i \log C_i + 2\log D + \log m_G. \end{aligned}$$

Here $f = [E_x^+ : \mathbb{Q}]$, r_j is the source rank on the edge and d_{ij} its total differential degree. The row-prime terms are the finite Fourier costs of Corollary 98.18; C_i, D, m_G clear the given scalar, class mass and projector denominators. For any such family of Gram matrices G use the real number

$$H(G) = \frac{1}{q_c[J:\mathbb{Q}]}\sum_\sigma \log\det G_\sigma + K_u.$$

Theorem 108.27 (A centered bound for the divided columns)**.** *With the specified tests on each E_x, for every fixed $A > 0$,*

$$\Delta_u \le H(G^\circ) - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x + o_{\psi,\varepsilon,A}(b/x^A).$$

The two heights satisfy the exact identity

$$H(\widetilde{G}) - H(G^\circ) = \frac{2}{q_c}\sum_i \log b_i^{\mathrm{per}}.$$

Proof. Projection commutes with multiplication of a column by a rational scalar. At every embedding the Gram determinant is therefore multiplied by $\prod_i (b_i^{\mathrm{per}})^{-2}$. This proves the second identity. At all row and terminal primes, and at primes ramified in E_x, the integers being removed are units. At the other finite places the precise subtraction is accounted for in Lemma 108.26. In particular there is no need to add the integer once more to the scalar denominators in K_u. Corollaries 9.27 and 56.3 give $\Delta_u \le H(\widetilde{G}) - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \mathcal{D}_u + o(b/x^A)$. Insert the two exact identities to obtain the result. □

Corollary 108.28 (A necessary height for independent projected columns)**.** *For the word weights and sparse sets on these intervals one has*

$$\begin{aligned} H(G^\circ) &\ge (\tfrac{1}{2} - o(1))hma\log X', \\ H(\widetilde{G}) &\ge (\tfrac{3}{4} - o(1))hma\log X'. \end{aligned}$$

For the allowed cosets in Corollary 108.25 the second coefficient is at least $\frac{3}{4} + 1/(4e_2) - o(1)$. The lower bounds concern the full quantity H, including K_u.

Proof. By Corollary 9.15, $2\Delta_u/(hmb\delta) = \mathcal{G}_\psi(x) + O(x^{1/2+\varepsilon/2})$, where the notation on the right denotes the fixed weighted sum of the $G_\chi(x)$ in (W). Each fixed $G_\chi(x) = o_\chi(x)$ by Proposition 3.2. Since $0 < \delta \le 1$, $\Delta_u = o(hma)$ uniformly over u. No estimate for zeros inside the critical strip is required beyond that proposition's zero-free boundary argument.

In $A_{\mathrm{sig}}(x)$ the three quadratic logarithmic derivatives are $O_{\psi,\varepsilon}(\log x)$ by Theorem 5.6. Also $\log n = O_\psi(\log x)$ and $\delta = |T_2|/n = o(1)$ by the representative bound in the proof of Theorem 9.26. Its displayed formula for the complementary main term therefore gives

$$A_{\mathrm{sig}}(x) = (\tfrac{1}{2} + o(1))hma \log X'.$$

The bound of Corollary 9.27 makes $W_{d_0}(u) = o(hma)$. Rearranging the centered inequality proves the first claim. The exact height difference, the rounding bound, and Theorem 108.24 now give the second. Apply Corollary 108.25 for the last coefficient. □

For comparison with the precision actually needed, the size conditions preceding (6.3) give $\mathcal{B}/(hmb) \ll_{\psi,\varepsilon} x^{1/2+2\varepsilon}$. Indeed the three contributions after division are bounded by $(2x/m)(1 + m^2/k)$, $2(n/h)mx^{\varepsilon}$, and $(h_2/h)(k/n^2)x^{2\varepsilon}$. Use $h \ge nx^{-\varepsilon/8}/2$, $h_2 \le n$, the stated bounds for m, k, and $m^2/k \le m/x$. Thus $x^{5\varepsilon}\mathcal{B}/(hma) \ll x^{-1/2+7\varepsilon}$. Proving a sufficiently precise upper bound for the divided Gram determinants at this scale still requires information on the actual signed periods.

108.10 First prime powers in the actual projected columns

Take the polynomial-conductor intervals of Theorem 9.26. Fix a node x and one of its translates u. Work with the ordinary word, its integral Weyl table and its projected small test lattice eS as in Theorem 108.15. Write $L = 2nx\exp(x^{\varepsilon/3})$ and let $\mathcal{P}$ consist of the unramified rational primes $5 \le p \le L$ which split in K_q and are inert in K_d. In the sum defining U_p in Section 108.4, denote the summands of a given odd exponent j, summed over all edges and high embeddings, by $U_{p,j}$. For $Z = x^{3/4}$ put

$$H_p = \sum_{\substack{j \ge 3 \text{ odd} \\ p^j \le Z}} U_{p,j}, \qquad g_p = \lfloor H_p \rfloor, \qquad B_p = \lceil hma/p \rceil.$$

All three numbers are computed from the input word. In particular $H_p = 0$ except possibly at the at most $x^{1/4}$ rational primes $p \le x^{1/4}$.

Lemma 108.29 (The omitted odd prime powers)**.** *At every node and translate,*

$$0 \le \sum_{p \in \mathcal{P}} (U_p - U_{p,1} - H_p)\log p \ll hma\, x^{-1/2}\log(2x).$$

Proof. The defining ramp is nonnegative and bounded by a/p^j per edge and high embedding. Put $P = Z^{1/3}$. For $p > P$ the entire sum of p^{-j} for $j \ge 3$ is at most $2p^{-3}$. Integral comparison of $\sum_{n>P}(\log n)n^{-3}$ gives $O(P^{-2}\log(2P))$. For $p \le P$ the first omitted odd power exceeds Z. Its tail has sum at most $2/Z$, so this range contributes $O(P\log(2P)/Z)$. Both bounds are $O(x^{-1/2}\log(2x))$. Multiplication by hma proves the claim without a hypothesis on primes in a short interval. □

For this fixed count choose any nonempty finite list of its ordinary signed columns having the word weights just used. One column suffices to define the quantities below. In a common normal coefficient field J of degree d let $\mathfrak{a}_c$ be the fractional ideal generated by all the pairings

of column c with eS. The zero ideal is allowed. Use C_c, D, m_G from Theorem 108.15, including the stabilizer factors and the denominator of the small projector. For $p \in \mathcal{P}$ define

$$\begin{aligned} t_{c,p} &= d^{-1} v_p(N_{J/\mathbb{Q}}\mathfrak{a}_c) + v_p(C_c D m_G), \\ v_p^* &= \min\Big\{B_p,\ \max\Big(0, \min_c t_{c,p} - g_p\Big)\Big\}, \\ N(x,u) &= \prod_{p\in\mathcal{P}} p^{dv_p^*}. \end{aligned}$$

Here $v_p(N0) = +\infty$. Thus $N(x,u)$ is defined even when an entire projected table vanishes. At a prime where all entries vanish the exponent used is exactly dB_p.

Theorem 108.30 (An integer upper comparison for one translate)**.** *The exponents dv_p^* are nonnegative integers. For the actual field and word at the node put*

$$M(x,u) = \frac{2}{Q\delta}\left(A_{\mathrm{sig}}(x) - W_{d_0}(u) - 2\sum_{p\in\mathcal{P}} H_p \log p\right).$$

Uniformly in the translate and the chosen columns,

$$\mathcal{G}_\psi(x) \ \le\ \frac{4\log N(x,u)}{dQ\delta} - M(x,u) + O_{\psi,\varepsilon}(x^{1/2+2\varepsilon}).$$

Enlarging J preserves $\log N(x,u)/d$. At any chosen subset $\mathcal{E} \subset \mathcal{P}$ one can use $v_p^* = B_p$ without reading a projected coefficient there. The resulting increase in the right side is at most

$$O_{\psi,\varepsilon}\left(x^{1+\varepsilon}\sum_{p\in\mathcal{E}}\frac{\log p}{p}\right) + O_{\psi,\varepsilon,A}(x^{-A}) \qquad (A>0).$$

Proof. The norm of a nonzero fractional ideal is positive rational. For its normalized valuation one has $d^{-1}v_p(N\mathfrak{a}_c) = \sum_{w|p}[J_w : \mathbb{Q}_p]v_w(\mathfrak{a}_c)/d$, where $v_w(p) = 1$. This follows from the usual product of residue degrees with integral ideal exponents. Apply Theorem 108.15 on the image lattice to the single column c. At every active prime it gives $t_{c,p} \ge U_p$. For a zero column this holds with infinite left side. Hence $t_{c,p} - g_p \ge U_{p,1}$ at every active prime. The other bound $B_p \ge U_{p,1}$ follows by dropping every threshold at the first power. At an inactive prime the first-power term is zero. We obtain $U_{p,1} \le v_p^*$ at all primes. All finite valuations in the definition have denominator dividing d, and the upper bound is an integer. This proves the integrality assertion. Ideal extension multiplies its norm valuation by the relative degree, since the local degrees add in a tower. Division by the new d therefore preserves all $t_{c,p}$, including for the zero ideal.

Insert the bound for $U_{p,1}$ and Lemma 108.29 in the exact centered identity of Theorem 9.26. The signs give $-A_{\mathrm{sig}}(x)+W_{d_0}(u)$ there. Multiplication by $2/(Q\delta)$ gives the displayed main term. Here $Q = hmb$, $a \le 2xb$ and $\delta \ge x^{-\varepsilon}$, so the normalized higher-power error is $O_{\psi,\varepsilon}(x^{1/2+2\varepsilon})$. The remainder $o(b/x^A)$ in that identity can use any fixed A. Finally Corollary 9.15 compares $2\Delta_u/(Q\delta)$ with $\mathcal{G}_\psi(x)$ at this individual translate, with an error $O_{\psi,\varepsilon}(x^{1/2+\varepsilon/2})$.

Replacing an exponent by its upper bound adds at most $4(Q\delta)^{-1}\sum_{\mathcal{E}}(hma/p+1)\log p$. In the term from the ceiling, the sum of logarithms is at most $L\log(2L)$. Since n grows polynomially and $b = \exp(x^\varepsilon)$, $L\log(2L)/(Q\delta) = O_{\psi,\varepsilon,A}(x^{-A})$ for every fixed A. This gives the claimed allowance for exceptional primes. □

Corollary 108.31 (Integral tests at one translate per scale)**.** *Use the nodes and weights of Theorem 6.40 on the polynomial-conductor intervals. At each node x_j choose one translate and*

let $N_j, d_j, M_j, Q_j, \delta_j$ denote the data of Theorem 108.30 there. Put $E_k = (\sigma-\theta)^{-k-1}$ and select an integer $H \geq 1$ with

$$HE_k \;\geq\; 1 + \sum_j d_j \sum_{p\in\mathcal{P}_j} B_{j,p}\log p.$$

For $l = 0, 1, 2$ form the positive rational integers

$$P_l = \prod_j N_j^{\lceil 4Hw_{lj}/(d_jQ_j\delta_j)\rceil}.$$

If for every rational frequency and arbitrarily distant common intervals at that frequency one has, simultaneously,

$$H^{-1}\log P_l - \sum_j w_{lj}M_j \leq B_\tau E_k \qquad (l = 0, 1, 2),$$

every open-strip zero of the family has real part at most θ, for $1/2 + 4\varepsilon \leq \theta < 1$. The choice of H uses only finite input data and the positive error allowed in the test.

Proof. Since $0 \leq \log N_j \leq d_j \sum_{p\in\mathcal{P}_j} B_{j,p}\log p$, rounding all exponents up changes each normalized sum by a number between zero and E_k. Multiply the node inequality by the positive weights. Equivalently, interpolate the resulting node majorants of $\mathcal{G}_\psi$ by Lemma 6.37. The interpolation error is $O_{\psi,\varepsilon}(x^{1/2+4\varepsilon})$ on a mesh with fixed constant. Its integral and that of the node error against the three kernels are $O_{\psi,\varepsilon}(E_k)$, by the gamma integral in Theorem 6.40. That theorem's pole argument now gives the conclusion. No identification between the coefficient fields of different nodes is needed to multiply their rational integers. □

The finite exponents retain the original signed projections and the ramified prime in the centering term. The estimate still needed is an upper bound on the logarithms of these integers after that centering, uniform along the specified intervals.

108.11 The projected period at primes between the square root and the scale

On the polynomial-conductor intervals fix the two auxiliary primes q, d_0 of Theorem 12.15. Let $\mathcal{P}_0$ be the rational primes prime to $2Fqd_0$ which split in K_q and are inert in K_d. Put $\mathcal{I}_x = \{p \in \mathcal{P}_0 : \sqrt{x} < p \leq x\}$. We give a comparison requiring projected integral coefficients only at these primes. The translate in those coefficients can be chosen separately at each prime.

Write $w_{\mathbf{1}} = 1,\ w_\eta = 1/(3o+1)$ for the other characters of A/A_0, and use the primitive representatives $\alpha_{ij\eta}$ of Theorem 6.23. Let $c_\psi(n)$ be as in Lemma 6.37. For these fixed auxiliary primes define the real constants

$$D^* = \sum_{\alpha_{ij\eta}\neq\mathbf{1}} (-1)^j w_\eta \operatorname{Re}\frac{L'(1,\alpha_{ij\eta})}{L(1,\alpha_{ij\eta})},$$

$$H^* = \sum_{p\in\mathcal{P}_0}\log p \sum_{\substack{v\geq 3\\ v\ \text{odd}}} \frac{c_\psi(p^v)}{p^v}.$$

The second series converges absolutely. Put

$$M_0(x) = x(\log x - 1 - \gamma - D^* - 4H^*) - 4x\sum_{\substack{p\in\mathcal{P}_0\\ p\leq\sqrt{x}}}\frac{c_\psi(p)\log p}{p}.$$

Proposition 108.32 (The centered first-power sum in a single range)**.** *Uniformly on these common intervals,*

$$\mathcal{G}_\psi(x) = 4\sum_{p\in\mathcal{I}_x} c_\psi(p)(x/p-1)\log p - M_0(x) + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}).$$

Proof. Form $\sum_{i,j,\eta}(-1)^j w_\eta \operatorname{Re} G_{\alpha_{ij\eta}}(x)$. The $i=j=0$ terms are exactly $\mathcal{G}_\psi(x)$. There are $3o$ other terms, each bounded by $O(x^{1/2+\varepsilon/2})$ by the auxiliary twist estimate on the common interval. In the defining ramps, $q, d_0 > x^{401}$ and so neither prime occurs. Coprimality of their conductors with F makes the sum over i,j at p^v equal to the multiplier $(1+\xi_q(p)^v)(1-\xi_d(p)^v)$. It is four exactly at odd v and primes with the indicated two signs and is zero elsewhere. The primes dividing $2F$ and the prime three split in K_d on sufficiently distant intervals, by its prescribed congruences. The remaining coefficient is $c_\psi(p^v)$. The principal correction is $-x\log x+(1+\gamma)x$ and the sum of the other linear corrections is xD^*. This gives the expression with all odd powers directly from the primitive character formula.

For $p\le\sqrt{x}$ at the first power, replacing $x/p-1$ by x/p costs $O(\sqrt{x}\log(2x))$ after summation since $c_\psi(n)<2$. The odd powers $v\ge 3$ contribute $4xH^*+O(x^{1/3}\log^2(2x))$. Indeed there are at most $x^{1/3}\log_2 x$ pairs with $p^v\le x,\ v\ge 3$, so the discarded unit in each ramp costs at most the stated error. For $p^v>x$, the tail of $\sum p^{-v}\log p$ is $O(x^{-2/3}\log(2x))$ by the two-range estimate in Lemma 108.29, now with cutoff $Z=x$. Substitution proves the formula. In this formula the arguments of the ramps are x. The ramified prime which appears at the much larger word arguments in Theorem 9.26 exceeds this cutoff. □

For $p\in\mathcal{I}_x$ let z_p denote the A-coordinate of its Frobenius. With T, T_2 of Theorem 6.29, set $k_p = h\delta c_\psi(p)$.

Lemma 108.33 (A translate-independent integral multiplicity)**.** *The integer k_p counts the high embeddings carried to the distinguished set by the first power at p, for every translate u. In particular $0\le k_p\le h$.*

Proof. In the product coordinates of Theorem 6.23, the required condition on $t\in T$ is $t+z_p\in u+T_2$. The sign pair is $(0,1)$, so the coordinate change in that theorem gives this condition also for odd ψ. Expanding the indicator on A gives the count

$$\frac{1}{n}\sum_{\theta\in\widehat{A}}\theta(z_p-u)\left(\sum_{t\in T}\theta(t)\right)\left(\sum_{z\in T_2}\overline{\theta(z)}\right).$$

If θ is nontrivial on A_0, the product of the two sums vanishes by the subgroup invariance and residue balance in Theorem 6.29. Otherwise the second sum is h_2, $\theta(u)=1$, and the first sum is hw_θ by the prescribed proportions. On these unramified primes the surviving Fourier sum is $c_\psi(p)$, since conjugate terms pair. Multiplication by hh_2/n proves the assertion. □

Put $X'=\exp(x^{\varepsilon/3})$, $a=\lceil xb\rceil$ and define two nonnegative integers from the common input at $p\in\mathcal{I}_x$ by

$$G_p=\left\lfloor m(h-k_p)\,(a/p-2b/(nX'))\right\rfloor,\qquad B_p=\lceil mk_pa/p\rceil.$$

Here x is sufficiently large. Choose any finite list of column and translate pairs from the actual ordinary words with these parameters. For a pair (u,c) write $t_{u,c,p}$ for the normalized valuation of its projected entry ideal, including C_cDm_G, in Theorem 108.30. Use one coefficient field of degree d for this finite list. At each p any subset of the pairs may be read. Put

$$a_p^*=\min\left\{B_p,\ \max(0,\min_{(u,c)\text{ read at }p} t_{u,c,p}-G_p)\right\},\qquad N_0(x)=\prod_{p\in\mathcal{I}_x}p^{da_p^*},$$

taking the minimum of an empty list to be $+\infty$. If the whole list is empty use $d = 1$.

Theorem 108.34 (A prime-by-prime choice of projected entries)**.** *The number $N_0(x)$ is a positive integer, and uniformly over these choices of actual projected columns,*

$$\mathcal{G}_\psi(x) \le \frac{4\log N_0(x)}{dQ\delta} - M_0(x) + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}).$$

Replacing any selections by the empty list on a set $\mathcal{E} \subset \mathcal{I}_x$ raises the right side by at most $O_\psi(\sum_{p\in\mathcal{E}} \log p)+O_{\psi,\varepsilon,A}(x^{-A})$ for every fixed $A > 0$. The three integral tests of Corollary 108.31 hold as a sufficient condition with N_0, M_0 in place of N, M, summing the rounding allowance only over $\mathcal{I}_x$ with the B_p of this subsection.

Proof. All target embeddings at the first power for the chosen signs belong to Φ_1. On its nondistinguished part the word threshold is at most $2b/(nX')$ eventually. In fact it differs from $b/(nX')$ by at most $2+2mH_0$, which is polynomial in x, as in Theorem 9.22. At a distinguished place the threshold lies between b and $b + \sqrt{b}$ by Corollary 9.15. Since $a/p \ge b$ on $\mathcal{I}_x$, there are $h - k_p$ nondistinguished ramps at every edge, each at least $a/p - 2b/(nX') > 0$. The other k_p ramps are at least $(a/p - b - \sqrt{b})_+$. Thus the full word sum U_p for *each* translate is at least

$$G_p + mk_p(a/p - b - \sqrt{b})_+.$$

This also holds if $U_p = 0$, when both terms on the right vanish. At active primes the single-column estimate of Theorem 108.15 applies to each $t_{u,c,p}$. At inactive primes the maximum with zero suffices for the distinguished lower bound. Indeed if a read translate is inactive, the displayed lower bound forces both $G_p = 0$ and $mk_p(a/p - b - \sqrt{b})_+ = 0$, independently of the translate. The upper cutoff B_p also dominates the latter quantity. This proves

$$mk_p(a/p - b - \sqrt{b})_+ \;\le\; a_p^* \;\le\; mk_p a/p + 1.$$

The exponents become integers after multiplication by d by the ideal norm formula of Theorem 108.30.

Using $k_p = h\delta c_\psi(p)$, $Q = hmb$, and $a \ge xb$, the lower inequality bounds $c_\psi(p)(x/p - 1)$ above by $a_p^*/(Q\delta) + c_\psi(p)/\sqrt{b}$. The sum of the errors times $\log p$ is $O_{\psi,\varepsilon,A}(x^{-A})$ for any fixed A, since $p \le x$. Proposition 108.32 gives the asserted upper bound. At a prime switched to its upper cutoff the increase is at most $mk_p(b + \sqrt{b}) + 1$ before normalization, by the same two inequalities. After division by $Q\delta$ and summation this gives the stated cost. Finally the ceiling and logarithm argument of Corollary 108.31 applies to these integers with exactly the same positive kernels and interpolation error. □

The geometric term now reads integral coefficients only in $(\sqrt{x}, x]$, with prime-dependent choices of translates on the same field. The outstanding upper bound compares its logarithm with $M_0(x)$ to square-root accuracy after averaging. The two node inequalities above by themselves allow a total excess of order $x\log(2x)$.

108.12 How many primes the integral comparison must detect

Fix ψ, ε and the polynomial-conductor intervals of Section 108.11. At a node put

$$s_p(x) = \frac{B_p - a_p^*}{Q\delta}, \qquad R(x) = 4\sum_{p\in\mathcal{I}_x} [c_\psi(p) - s_p(x)]_+ \log p, \qquad H_0(x) = \frac{4\log N_0(x)}{dQ\delta} - M_0(x).$$

Thus $s_p(x) = 0$ when no projected column improves the elementary upper bound B_p. We measure how often this can happen if the proposed upper comparison is to hold.

Lemma 108.35 (The excess in the upper comparison)**.** *Uniformly over all choices of entries, on every node,*

$$H_0(x) = \mathcal{G}_\psi(x) + R(x) + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}).$$

On the interior $2Y \le x \le Y^{3/2}/2$ *one also has*

$$4\sum_{p\in\mathcal{I}_x} c_\psi(p)\log p = (1+o_{\psi,\varepsilon}(1))x.$$

Both estimates are unconditional.

Proof. Write $B_p = mk_pa/p + \eta_p$ with $0 \le \eta_p < 1$. Since $a = xb + O(1)$, $k_p = h\delta c_\psi(p)$, the normalized upper cutoff differs from $c_\psi(p)x/p$ by at most $c_\psi(p)/(bp)+1/(Q\delta)$. The lower bound for a_p^* in Theorem 108.34 consequently gives

$$0 \le s_p(x) \le c_\psi(p)(1+b^{-1/2}) + 1/(Q\delta).$$

Replacing $c_\psi(p) - s_p(x)$ by its positive part costs $O_{\psi,\varepsilon,A}(x^{-A})$ after summing with $\log p$, for any fixed A. Subtract the prime sum of Proposition 108.32 from the definition of H_0 and use the cutoff estimate just proved. This gives the first formula.

For the second, during a given interval keep the auxiliary primes fixed. Consider the nonnegative ramp sum

$$F(y) = 4\sum_{p\in\mathcal{P}_0}\log p \sum_{v\ge1\ \text{odd}} c_\psi(p^v)(y/p^v-1)_+.$$

By the character expansion in Proposition 108.32,

$$F(y) = y\log y - (1+\gamma+D^*)y + o_{\psi,\varepsilon}(y) \qquad (Y \le y \le Y^{3/2}).$$

Indeed each $\mathcal{G}_\psi(y)/y$ tends to zero for fixed ψ. Every open-strip zero has real part strictly below one, and dominated convergence applies in the absolutely summable explicit formula. The auxiliary twists contribute $O_\psi(y^{1/2+\varepsilon/2})$ uniformly on the interval. The coefficient D^* may depend on the two primes but is constant across this interval.

The function F is convex. For $0 < \eta < 1/4$ its right derivative at an interior x is between the backward and forward difference quotients using steps ηx. For $y\log y$ both such slopes equal $\log x + 1 + O(\eta)$. The linear term is exact and the errors in the slopes are $o_{\psi,\varepsilon}(1)/\eta$, uniformly in the interior. Thus $xF'_+(x) - F(x) = x(1+O(\eta)+o_{\psi,\varepsilon}(1)/\eta)$. First let Y tend to infinity and then η to zero. A ramp at $n \le x$ contributes exactly one to $x(d/dx)_+(x/n-1)_+ - (x/n-1)_+$, also at the endpoint. It follows that $xF'_+(x) - F(x)$ is four times the sum of $c_\psi(p^v)\log p$ for $p^v \le x$, $p \in \mathcal{P}_0$, v odd. The primes below $\sqrt{x}$ at the first power cost $O(\sqrt{x}\log(2x))$, and all higher powers cost $O(x^{1/3}\log^2(2x))$ by the counting estimate in Proposition 108.32. This proves the second formula. □

Write $w_j = \int_U^{3U/2} K_k(u)\lambda_j(e^u)\,du$ for the unmodulated positive weights of Theorem 6.40, and put $L_k = (\sigma-1)^{-k-1}$.

Theorem 108.36 (A necessary bound on the undetected prime mass)**.** *Fix* $1/2 + 4\varepsilon \le \theta < 1$ *and a uniform mesh constant. There is a number* $\delta_* > 0$*, depending on the fixed family and* θ, ε*, with the following property. On any sequence of common intervals on which*

$$\sum_j w_j H_0(x_j) \le B(\sigma-\theta)^{-k-1},$$

one has

$$\sum_j w_j R(x_j) \ll_{\psi,\varepsilon,\theta,B} Y^{-\delta_*} L_k, \qquad 4\sum_j w_j \sum_{p\in\mathcal{I}_{x_j}} c_\psi(p)\log p = (1+o_{\psi,\varepsilon}(1))L_k.$$

Constants may also depend on the mesh constant. For any fixed $0<\lambda\le 1$, the fraction of this last weighted mass on which $s_p(x_j)\le(1-\lambda)c_\psi(p)$ is $O_{\psi,\varepsilon,\theta,B}(\lambda^{-1}Y^{-\delta_*})$. In particular the integral tests cannot hold along such intervals with only the elementary upper cutoffs at every node.

Proof. Let Φ_ψ be the transform in Lemma 2.3. The closed disk about σ of radius $\sigma-1$ lies in $\operatorname{Re} s\ge 1$, where there are no poles. The possible poles are locally finite in $\operatorname{Re} s>0$, and the transform is holomorphic at one. Compactness therefore allows a strictly larger pole-free closed disk of radius $r<\sigma$. Cauchy's inequality gives $|A_k(\sigma)|\ll_\psi r^{-k}$, at this single real center and for every k.

Apply Lemma 6.37 pointwise on each mesh cell with the interpolation weights, and integrate against K_k. It bounds the weighted sum of the nodal $\mathcal{G}_\psi(x_j)$ *from below* by the integral of $\mathcal{G}_\psi(e^u)$ on the window minus $O((\sigma-1/2-4\varepsilon)^{-k-1})$. The tail proof in Theorem 6.40 identifies that integral with $A_k(\sigma)+O_\psi(\sigma^{-k})$. Also $\sum_j w_j x_j^{1/2+\varepsilon}\ll(\sigma-\theta)^{-k-1}$ since adjacent node ratios are at most two on distant windows. Combining the hypothesis and Lemma 108.35 gives

$$0\le\sum_j w_j R(x_j)\ll_{\psi,\varepsilon,\theta,B} r^{-k}+(\sigma-\theta)^{-k-1}.$$

Both r and $\sigma-\theta$ exceed $\sigma-1$. As $k=\lceil 5\sigma\log Y/4\rceil$, division by L_k gives the asserted positive power saving.

For the denominator of the fraction, $\sum_j\lambda_j(x)x_j=x$ exactly. Thus $\sum_j w_j x_j=\int_U^{3U/2}K_k(u)e^u\,du=(1+o(1))L_k$. The omission of nodes outside the interior in the second assertion of the lemma is harmless here, both for x_j and for the prime mass, which is $O(x_j\log(2x_j))$ on the whole interval. Indeed an endpoint node outside the interior, or any such node near either end, has a hat function supported within $O(1)$ in log scale of one of the two endpoints, since adjacent ratios are at most two. On those strips $x_j\log(2x_j)\ll e^{2u}$ wherever its hat is nonzero. The tail estimates in Lemma 2.4 remain $O(\sigma^{-k})$ upon shifting their cutoff by a fixed distance in u, since this changes the exponential majorants in that proof only by constant factors. The sum of all the hats is one. Consequently the interior estimate of Lemma 108.35 gives the second formula after summation. Each prime in the stated exceptional set contributes at least $\lambda c_\psi(p)$ to the positive part defining R. Division of the first formula by the second gives the fraction. For the unmodulated test the three-weight hypothesis at $\tau=0$, $l=0$ is twice the hypothesis here, up to the upward rounding in Corollary 108.31. The rounding is nonnegative. This proves the last assertion for the integral tests as well. □

This gives a quantitative obstruction on the very projected lattices used for the comparison. A proof using these integral tests must recover almost all the indicated prime mass, to the displayed precision. The statement supplies a necessary condition on the valuations and leaves its verification on the period columns open.

108.13 The common divisor at every prime in the middle range

Keep the actual polynomial-conductor fields and the set $\mathcal{I}_x$ of Section 108.11. Here k is the initial rank of a theta word. Write $L_u\subset\Phi_1$ for the h_2 distinguished embeddings at translate u. At an edge i of a word w, write ν_i for its partition tuple and s_i for its target rank. Define

$g_{u,w,p} = \sum_{i=1}^{m} V_p(\nu_i)$, where V_p is the maximum over the two orientations of each even Frobenius cycle in Theorem 56.2.

Theorem 108.37 (Saturation before division of the period columns). *Fix ψ and $0 < \varepsilon < 1/52$. At every sufficiently large node and every $p \in \mathcal{I}_x$, for every actual word and every column on that word allowed in Theorem 108.34,*

$$t_{u,c,p} \;\geq\; g_{u,w,p} \;\geq\; \frac{mkh_2 b}{2p} \;>\; G_p + B_p.$$

Consequently the capped exponents in that theorem satisfy $a_p^* = B_p$, for every choice of columns or translates at each prime.

Proof. The word ranks and the number of parts are bounded by powers of x. On each of the m edges the first k parts at $\tau \in L_u$ equal $b + O(x^C)$. On Φ_0 all parts except possibly the first have size $O(bx^C/X' + x^C)$, with $X' = \exp(x^{\varepsilon/3})$. The first is bounded by $a + O(bx^C/X' + x^C)$. These estimates follow from the partition recurrence in the proof of Theorem 111.9. In detail, padding by zeros and subtracting the return shift cancels the base weight on each newly introduced part. On the original parts the two shifts at successive edges leave only half the rank difference. Their total and the offsets in the word letters are polynomial. Inserting the initial parts in (9.1) gives the stated bounds. In particular the *sum* of the parts at every $\mu \in \Phi_0$ is at most $2a$, since the polynomial number of errors is $o(b)$.

On the indicated signs the permutation $\pi = \sigma_p c_E$ interchanges Φ_0 and Φ_1. Choose Φ_1 as the source half on every cycle. For each $\tau \in L_u$, at the destination $\mu = \pi\tau$ retain just the k indices from τ, of exponent one and row numbers $1, \dots, k$. For each such index the list entry on edge i is at least

$$\max\{0, \lceil (\nu_{i,\tau,l} - s_i(p-1))/p \rceil\} \;\geq\; \frac{b - O(x^C)}{p} - s_i \;\geq\; \frac{3b}{4p}.$$

The last inequality holds uniformly because $p \leq x$ and $b = \exp(x^{\varepsilon})$. The sum of the largest k entries is at least the sum of these k entries. The penalty for choosing them is at most the sum of all parts at μ, hence at most $2a$. Since $pa/(kb) = O(x^2/k) = o(1)$, their net contribution is at least $kb/(2p)$. Distinct τ's yield distinct destinations. All other destinations contribute nonnegatively, by the empty choice. Summing first over destinations and then over the edges proves the second inequality.

For the first inequality, the operator at edge i maps the integral Weyl lattice into $p^{V_p(\nu_i)}$ times the next one, by Theorem 56.2. The integral frames match at consecutive vertices. Clearing the coefficient denominators, the finite stabilizers and the ordinary projector therefore gives a multiple of $p^{g_{u,w,p}}$ on every path, also after taking the signed sum and pairing with each integral test. These are exactly the denominators included in $t_{u,c,p}$. Corollary 56.3 verifies the same assertion at every coefficient place over p in the adapted frames. In an arbitrary marked product frame the scalar change has absolute norm one, as proved in Theorem 108.15. Thus the normalized rational valuation satisfies the asserted inequality in either frame. A zero column has valuation $+\infty$.

Finally the definitions of the two cutoffs give $G_p + B_p \leq mha/p + 1$. Since $h \leq n$, $h_2 \geq nx^{-\varepsilon}$, $a = xb + O(1)$, and $k \geq x^{9/2+10\varepsilon}$, the ratio of $mkh_2 b/(2p)$ to $mha/p + 1$ tends uniformly to infinity. This also proves the assertion for an empty list, by the convention in the definition of a_p^*. □

Corollary 108.38 (Size of the undivided comparison on the common intervals)**.** *On $2Y \le x \le Y^{3/2}/2$ one has $H_0(x) = (1+o(1))x$. For the unmodulated weights of Theorem 108.36,*

$$\sum_j w_j H_0(x_j) = (1+o(1))(\sigma-1)^{-K-1},$$

where $K = \lceil 5\sigma \log(Y)/4 \rceil$ is the order of the kernel. In particular the proposed upper test for the undivided columns fails on every sufficiently distant interval for any fixed $\theta < 1$ in the range of that theorem.

Proof. The preceding theorem gives $s_p(x) = 0$ for every $p \in \mathcal{I}_x$. Both identities of Lemma 108.35 now give the interior asymptotic since $\mathcal{G}_\psi(x) = o_\psi(x)$. For the weighted statement its proof and that of Theorem 108.36 give $4\sum_j w_j \sum_{p \in \mathcal{I}_{x_j}} c_\psi(p)\log p = (1+o(1))(\sigma-1)^{-K-1}$. The evaluation of this last sum there uses only the prime mass and the mesh. Also $\sum_j w_j x_j = (1+o(1))(\sigma-1)^{-K-1}$, and $\sup_{x \ge Y}(|\mathcal{G}_\psi(x)| + x^{1/2+\varepsilon})/x$ tends to zero. Apply the first identity of the lemma at every node. Finally $((\sigma-\theta)/(\sigma-1))^{K+1}$ tends to infinity. □

The failed comparison reads the valuations before removal of the rational divisors in Lemma 108.26. The centered count in Section 6.6 and the valuations after this division are separate open questions.

108.14 The residual valuation of a divided period column

We remove the divisor whose effect was exhibited in Theorem 108.37. On the same fields and at the same nodes, choose any finite list of signed projected columns. Each list member c is attached to one word $w = w(c)$ at one translate u. Scalar combinations within a fixed word and its physical weights have their prescribed denominator C_c. Write d for the common coefficient field degree, and $e_{u,c,p} = t_{u,c,p} - g_{u,w,p}$ for the residual valuation, with value $+\infty$ for a zero column. By Lemma 108.26 the integer

$$b^{\mathrm{per}}_{u,w} = \prod_p p^{g_{u,w,p} - \lceil U_p(u) \rceil}$$

divides each cleared column in the adapted frames. Here the product is over the unramified primes $p \ge 5$ inert in K_d, with the extension convention of that lemma. Division is by a rational scalar on the entire column. Let $t^\circ_{u,c,p}$ be the valuation with the original denominator corrections on this divided column. In the definition of a^*_p use t° in place of t, calling the resulting exponent a°_p. Write $e_p = \min_{(u,c)} e_{u,c,p}$ over the entries read at p, with the same empty-list convention. All sums in this subsection run over $p \in \mathcal{I}_x$.

Lemma 108.39 (The word threshold after centering)**.** *Put $A_p = Q\delta c_\psi(p)(x/p - 1)$. Then $A_p \ge 0$, and uniformly on the common intervals,*

$$\frac{1}{Q\delta} \sum_p \log p \, \max_u \left| \lceil U_p(u) \rceil - G_p - A_p \right| \ll_{\psi,\varepsilon} x^{\varepsilon} \log(2x).$$

Proof. By Lemma 108.33 exactly $k_p = h\delta c_\psi(p)$ high embeddings at each edge have distinguished first destination. There the threshold lies in $[b, b+\sqrt{b}]$. Since the positive part is 1-Lipschitz and $x/p \ge 1$, replacing each such ramp by $b(x/p-1)$ costs at most $\sqrt{b} + 1/p$. At each of the other $h - k_p$ destinations the threshold lies in $[0, 2b/(nX')]$, and $a/p \ge b$. Subtracting the unrounded expression defining G_p leaves between zero and $2mhb/(nX')$ from these destinations. The two roundings cost at most two altogether. Every remaining odd power $j \ge 3$ contributes at most

mha/p^j, and hence all of them together contribute at most $2mha/p^3$. These bounds apply to each translate with the same constants. Thus the absolute difference in the lemma is bounded by

$$mk_p(\sqrt{b}+1/p)+2mhb/(nX')+2+2mha/p^3.$$

After division by $Q\delta = hmb\delta$ and summation with $\log p$, the first three terms are $O_{\psi,\varepsilon,A}(x^{-A})$ for every fixed $A>0$. This uses $\delta \ge x^{-\varepsilon}$ and the superpolynomial growth of b and X'. For the last term use $a/b \le 2x$ and $\sum_{n>\sqrt{x}} n^{-3}\log n \ll x^{-1}\log(2x)$, by integration. This proves the bound. □

Theorem 108.40 (The excess remaining after the rational division)**.** *The number $N^\circ(x) = \prod_p p^{da_p^\circ}$ is a positive integer. Uniformly over the chosen words, translates and subsets of entries read at individual primes,*

$$\frac{\log N^\circ(x)}{dQ\delta} = \sum_p c_\psi(p)(x/p-1)\log p + \sum_p \min\{c_\psi(p), e_p/(Q\delta)\}\log p + O_{\psi,\varepsilon}(x^\varepsilon \log(2x)).$$

The same formula holds if every word at a given translate is divided by the common integer $B_u = \gcd_w b_{u,w}^{\mathrm{per}}$. In particular either normalization gives

$$\frac{4\log N^\circ(x)}{dQ\delta} - M_0(x) = \mathcal{G}_\psi(x) + 4\sum_p \min\{c_\psi(p), e_p/(Q\delta)\}\log p + O_{\psi,\varepsilon}(x^{1/2+\varepsilon}),$$

using the integer for the chosen normalization on the left.

Proof. Removing the word divisor subtracts exactly $g_{u,w,p} - \lceil U_p(u)\rceil$ from the normalized ideal valuation. Thus $t^\circ_{u,c,p} = \lceil U_p(u)\rceil + e_{u,c,p}$, where $e_{u,c,p} \ge 0$. This also holds for a zero column, in the extended sense. The finite norm valuations have denominator dividing d, so that capping at B_p and taking the maximum with zero gives a nonnegative exponent in $d^{-1}\mathbb{Z}$.

Put $C_p = Q\delta c_\psi(p)$. The unrounded upper cutoff is $mk_p a/p$; since $a = xb + O(1)$,

$$|B_p - (A_p + C_p)| \le 1 + mk_p/p.$$

After normalization and summation the right side is $O_{\psi,\varepsilon,A}(x^{-A})$ for any fixed A. For a nonempty selection with finite minimum, replace $\lceil U_p(u)\rceil - G_p$ by A_p before taking the minimum. The change is at most the maximum absolute error in the lemma. Taking the positive part and capping do not enlarge this error. Replacing the cap by $A_p + C_p$ contributes at most the cutoff error above. As $A_p, e_p \ge 0$, the resulting exponent is $A_p + \min(C_p, e_p)$. If the list is empty or only zero columns are read, both prescriptions take their upper cutoffs and only the cutoff error occurs. Sum over the primes and apply the lemma.

By Theorem 111.3 every quotient $b_{u,w}^{\mathrm{per}}/B_u$ divides the same D_x, for all u, w, with $\log D_x \ll x^C X'$. Thus the valuation using B_u lies between $t^\circ_{u,c,p}$ and $t^\circ_{u,c,p} + v_p(D_x)$. This bound persists under the prime-dependent minima and cutoffs, including when the list is empty. The increase of the normalized logarithm is bounded by $\log D_x/(Q\delta) = O_{\psi,\varepsilon,A}(x^{-A})$. This proves the first formula for both choices. Proposition 108.32 gives the second. □

The nonnegative excess in the last display yields the same sufficient three upper tests as Corollary 108.31 after replacing its integer and center by N°, M_0, and using its rounding procedure with the cutoffs B_p on $\mathcal{I}_x$. Indeed the displayed error integrates to at most the error allowed there. Even an exact evaluation $e_p = 0$ at every prime would recover the centered prime sum to this precision. An upper bound on that sum, or independently on the centered logarithm of the divided columns, is still required.

108.15 The two level projections on definite automorphic forms

Consider the family $E=\mathbb{Q}(\sqrt{-d},\mu_{q_1})$ of Corollary 11.30, and put $F=E^{+}$, $f=[F:\mathbb{Q}]$. Write r for the rank of the unitary group in this family. Its two distinct level primes p_0,p_1 lie in $[b^{10},b^{11}]$ and are one modulo Q^2, where $Q=(q_0d)^{s+6}$ is a multiple of q_1d. Thus each splits completely in E, by its action on cyclotomic roots. At either prime there are f places of F, with residue field $\mathbb{F}_{p_i}$; choose one of the two places of E over each. The two residue groups are

$$H_i=\mathrm{GL}_r(\mathbb{F}_{p_i})^{f},\qquad H=H_0\times H_1.$$

For a field $\mathbb{F}_q$ and $r\ge 2$, call a monic degree-r polynomial *allowed* if it is squarefree, has nonzero constant term, and has no root in $\{a\in\mathbb{F}_q^{\times}:a^{Q}=1\}$. Call it *primitive* if it is the minimal polynomial of a generator of $\mathbb{F}_{q^r}^{\times}$. Such a polynomial has degree r and is allowed. Green's parametrization attaches distinct irreducible characters to the squarefree degree-r polynomials of nonzero constant term. In this parametrization one fixes compatible generators for the character groups of the finite tori. For a polynomial with irreducible factor degrees $d_1,\dots,d_k$, take the one-box partition on each corresponding Frobenius orbit. The resulting character χ_c has degree

$$\chi_c(1)=\frac{\prod_{j=1}^{r}(q^j-1)}{\prod_{j=1}^{k}(q^{d_j}-1)}.$$

This is the squarefree case of Green's degree and classification theorems. [2, §8, Thms. 13–14]

In $\mathbb{C}[H_i]$ let e_i be the sum of central character idempotents for the f-tuples of allowed labels having at least one primitive member. Write $e=e_0e_1$, viewing each factor in $\mathbb{C}[H]$. We compute the fraction δ_i that e_i discards in the regular representation and then evaluate its rank on the definite automorphic space.

Proposition 108.41 (Plancherel mass at one level prime)**.** *Let $q=p_i$, $r\ge 2$, $q\ge 3$, and let A be the sum of $\chi_c(1)^2/|\mathrm{GL}_r(\mathbb{F}_q)|$ over allowed c. Put $n=q^r-1$. Then*

$$P=\frac{\varphi(n)}{rn}\prod_{j=1}^{r-1}(1-q^{-j}),\qquad \delta_i=1-A^{f}+(A-P)^{f}.$$

In particular

$$\delta_i\le f\left(\frac{2+\gcd(Q,q-1)}{q}+\frac{1}{q-1}\right)+\exp\!\left(-\frac{f}{4r(1+r\log q)}\right).$$

For every sufficiently large node of the stated family, $\delta_0+\delta_1\le b^{-8}$, uniformly for its prescribed level primes.

Proof. Counting ordered bases gives $|\mathrm{GL}_r(\mathbb{F}_q)|=q^{r(r-1)/2}\prod_{j=1}^{r}(q^j-1)$. Consequently each squarefree label has mass

$$\frac{\chi_c(1)^2}{|\mathrm{GL}_r(\mathbb{F}_q)|}=q^{-r}\frac{\prod_{j=1}^{r}(1-q^{-j})}{\prod_{j=1}^{k}(1-q^{-d_j})^2}\ge q^{-r}\Big(1-\frac{1}{q-1}\Big).$$

There are $\varphi(n)/r$ primitive labels. Indeed a generator of the multiplicative group cannot lie in a smaller subfield, and its r Frobenius conjugates have the same minimal polynomial. Substitution of $k=1,d_1=r$ in the exact mass formula gives their total mass P.

Among the q^r monic polynomials, exactly q^{r-1} have a repeated factor. To see this, unique factorization gives $\prod_{L}(1+z^{\deg L})=(1-qz^2)/(1-qz)$ for monic irreducibles L, by taking the

quotient of the two Euler products for all monic polynomials at z and z^2. Each prescribed root excludes q^{r-1} polynomials. There are $1+\gcd(Q,q-1)$ excluded roots including zero. Taking the union bound and the lower mass per allowed label gives

$$1-A\le (2+\gcd(Q,q-1))/q+1/(q-1).$$

This follows directly when the root-count lower bound is positive; otherwise the displayed upper bound is at least one. The ranks of distinct character idempotents add, and Plancherel masses multiply in a direct product. All labels being allowed has probability A^f; all being allowed with none primitive has probability $(A-P)^f$. These events give the asserted equality for δ_i.

For any positive integer n,

$$\frac{n}{\varphi(n)}=\prod_{p\mid n}\frac{1+p^{-1}}{1-p^{-2}}\le \zeta(2)\sum_{d\mid n}d^{-1}\le 2(1+\log n).$$

Also $\prod_{j=1}^{r-1}(1-q^{-j})\ge 1-1/(q-1)\ge 1/2$. Thus $P\ge 1/[4r(1+r\log q)]$. Using $1-A^f\le f(1-A)$ and $(A-P)^f\le e^{-fP}$ proves the inequality.

We verify both terms on the specified fields. Since E contains $\mathbb{Q}(\mu_{2^s})$ and lies in $\mathbb{Q}(\mu_{q_1 d})$, Corollary 11.30 gives

$$x^{11}/4\le 2^{s-2}\le f\ll_{q_0,d} x^{11}\log(2x).$$

Here $2^s=m^{22}$, $m\ge\sqrt{x}$. Also $s=O(\log x)$, $r\le 3x^5$, $(\log x)^6\le\log b\le x^{1/100}$, and $10\log b\le\log q\le 11\log b$. Thus Q grows at most polynomially in x for fixed q_0,d, whereas b exceeds every such polynomial eventually. The first term is at most b^{-9} for large x. For the second, $r(1+r\log q)\log b\ll x^{10+2/100}$; hence $f\ge 36r(1+r\log q)\log b$ for all sufficiently large x. Each $\delta_i\le 2b^{-9}$, proving the claim. □

Theorem 108.42 (Dimension and integral index on the definite endpoint)**.** *Let the unitary group of rank r over F be definite at all real places. Take the standard compact level at the two split primes and any fixed compact open level away from them. Let K impose principal reduction at both p_0,p_1. For algebraic automorphic forms V of level K and any fixed finite-dimensional archimedean weight, one has*

$$\dim(1-e_i)V=\delta_i\dim V,\qquad \dim(1-e)V\le(\delta_0+\delta_1)\dim V\le b^{-8}\dim V$$

at sufficiently large nodes. The idempotent e lies in $|H|^{-1}\mathbb{Z}[H]$. If Λ_ℓ is any H-stable full lattice in a characteristic-zero model of V over a finite extension of $\mathbb{Q}_\ell$, put

$$I_\ell=e\Lambda_\ell/(\Lambda_\ell\cap eV_\ell),\qquad d_\ell=\text{ramification index of the coefficient field.}$$

For arbitrary such coefficient fields and lattices at each prime,

$$\sum_\ell\frac{\operatorname{length}_{\mathcal{O}_\ell} I_\ell}{d_\ell}\log\ell\le b^{-8}\dim V\,\log|H|,\qquad \log|H|\ll_{q_0,d}x^{21}\log(2x)\log b.$$

Proof. Let $K^{(i)}$ restore the full compact group at p_i while retaining principal reduction at the other prime. Reduction of invertible integral matrices is onto, by lifting their entries and using the determinant. Hence $K^{(i)}/K=H_i$. Its right action on the finite level-K class set is free. Indeed, for a finite adelic representative g, the group of rational points lying in $gK^{(i)}g^{-1}$ is finite. The diagonal rational points are closed and discrete in the full adelic group, and both the indicated finite-adelic set and the entire archimedean group are compact. At a place over p_{1-i}, an element of this finite group is conjugate into $1+p_{1-i}M_r(\mathbb{Z}_{p_{1-i}})$. This group is torsion

free for odd $p = p_{1-i}$. In fact, if $C = 1 + p^a B$ with $a \geq 1$ and B nonzero modulo p, the first nonzero term of $C^u - 1$ has valuation a when $p \nmid u$, and that of $C^p - 1$ has valuation $a + 1$ by the binomial formula. Iteration rules out every finite order. Thus the rational element is the identity at one completion, and hence is the identity. If a lift in $K^{(i)}$ fixes the double coset of g, this trivial intersection puts the lift in K, proving freeness.

The level-K forms have the full archimedean weight space as fiber over each double coset, since its stabilizer is also trivial. The finite class set here is the usual class set of positive-definite Hermitian lattices at a compact open level. On a free H_i-orbit identify each fiber with the fiber over one base point by translation. The sections over this orbit then form copies of the regular representation of H_i. The coefficient at one of a group algebra projection is its normalized trace on that representation. This proves the single-prime rank formula. Since e_0, e_1 are commuting orthogonal projections (choose an invariant inner product), the kernel of their product is the sum of their kernels. Proposition 108.41 gives the joint bound.

For an irreducible character χ the coefficient of the central idempotent at h is $\chi(1)\overline{\chi(h)}/|H|$, with H here the whole product and χ a tensor product character. Its numerator is an algebraic integer. The indicated set of characters is stable under every automorphism of the character-value cyclotomic field. Indeed Green's formula takes a Galois power on the roots of unity to the same power on the torus character labels. This permutes Frobenius orbits and preserves their length, primitive order, and the property of being a degree-one root of order dividing Q. Thus summing the numerators gives rational algebraic integers, as asserted. In particular both $|H|e$ and $|H|(1 - e)$ preserve every H-stable lattice.

The map $\lambda \mapsto e\lambda$ identifies I_ℓ with the quotient of Λ_ℓ by the sum of its intersections with the two eigenspaces. Indeed it is onto, and $e\lambda$ lies in the lattice precisely when both components of λ do. Using $1 - e$ instead gives the same quotient. It follows that I_ℓ is killed by $|H|$ and can be generated by at most $\dim(1 - e)V$ elements, since the image of the lattice under $1 - e$ is free of that rank over the coefficient valuation ring. The elementary divisor theorem bounds its normalized length by $\dim(1 - e)V \, v_\ell(|H|)$. Sum and use integer factorization. Finally, $\log |H| \leq f r^2 \log(p_0 p_1)$ by counting all matrices, and the estimates for f, r, p_i above give the asserted order. □

The rank computation uses the other prime's level each time it recovers a regular representation. It therefore applies to the definite space with both markings on the same field as Corollary 11.30. Estimating the norm of a given projected Poincaré series further requires its mass in these isotypic components. Projection to the global subspace of theta lifts also introduces the congruences among automorphic representations.

108.16 The finite Bessel character at the two level primes

We compute the character average that occurs at the split places in the Bessel period of the shorter construction. Its Fourier transform counts vectors for which the first powers of a matrix are linearly independent. This count controls the effect of the finite level projector on the Bessel character itself.

Let V be an r-dimensional space over $k = \mathbb{F}_q$. Fix $a \geq 1$ and $c = r - 2a \geq 1$, and choose ordered blocks of sizes $1, \dots, 1, c, 1, \dots, 1$, with a lines on each side. Let $\mathfrak{p}$ be the block upper triangular algebra, $\mathfrak{u}$ its strictly block upper triangular ideal, $U = 1 + \mathfrak{u}$, and $G = \mathrm{GL}(V)$. Fix a nontrivial additive character ϑ of k. Choose D with blocks immediately below the diagonal as follows. On consecutive lines in each arm the map is the identity in the specified coordinates. The map from the last left line to the middle space is a vector w, and the map from the middle space to the first right line is a functional u. Suppose $\gamma = u(w) \neq 0$. Set its other blocks

to zero. Nonzero coefficients along either arm can be put into this form by rescaling the line bases, which preserves nonvanishing of the middle pairing. Define

$$\psi(1+X)=\vartheta(\mathrm{tr}(DX)) \quad (X\in\mathfrak{u}).$$

It is a group character since $\mathrm{tr}(DXY)=0$ for $X,Y\in\mathfrak{u}$. Extend it by zero to G, and let $\psi^{\#}(h)=|G|^{-1}\sum_{g\in G}\psi(g^{-1}hg)$.

Theorem 108.43 (The Fourier transform of the finite Bessel character)**.** *Set $F(X)=\psi^{\#}(1+X)$ on matrices where $1+X$ is invertible and set it to zero elsewhere. Use the transform*

$$\widehat{F}(A)=\sum_{X\in\mathrm{End}(V)}F(X)\overline{\vartheta(\mathrm{tr}(AX))}.$$

Write k_A for the fraction of vectors $v\in V$ for which $v,Av,\dots,A^{2a}v$ are independent. Then

$$\widehat{F}(A)=k_A/B_{q,r,a},\qquad B_{q,r,a}=\prod_{j=0}^{2a}(1-q^{j-r}),\qquad \sum_{\substack{h\in G\\ h\neq 1}}|\psi^{\#}(h)|^2\le B_{q,r,a}^{-1}-1.$$

Proof. Before averaging over conjugates, the function on matrices is $\vartheta(\mathrm{tr}(DX))1_{\mathfrak{u}}(X)$. Each $1+X$ in its support is invertible. The trace pairing identifies the annihilator of $\mathfrak{u}$ with $\mathfrak{p}$: testing one entry of a strict upper block tests the transposed lower block. Orthogonality of additive characters therefore gives transform $|\mathfrak{u}|1_{D+\mathfrak{p}}$. Fourier transformation commutes with conjugation. Hence $\widehat{F}(A)$ is $|\mathfrak{u}|$ times the probability that the matrix of A in a uniformly chosen basis belongs to $D+\mathfrak{p}$.

We count that probability. Record from a basis its first a columns $v_1,\dots,v_a$ and its last a coordinate functionals $\lambda_1,\dots,\lambda_a$, starting with the last functional and proceeding backwards. Write L_j for the span of the first j columns, R_j for the span of the first j of these functionals, and $W=\bigcap_i\ker\lambda_i$. Then $L_a\subset W$ and $C=W/L_a$ has dimension c. The basis identifies the original middle space with C, and thereby gives $w'\in C$ and $u'\in C^*$ with $u'(w')=\gamma$. Every tuple of independent frames with these annihilations and every such pair in C has the same number of basis lifts. Indeed one may choose a basis of the kernel of u', complement it by w', lift a resulting middle basis to W, and complete it by lifts of the basis dual to the λ_i. Equivalently G acts transitively on these data and the map from bases is equivariant.

The condition on the blocks of A in this basis says exactly

$$\begin{aligned} Av_j-v_{j+1}\in L_j,\qquad & \lambda_jA-\lambda_{j+1}\in R_j \quad (1\le j<a),\\ Av_a\in W,\qquad & [Av_a]=w' \text{ in } C,\\ (\lambda_aA)|_W=u'\circ(W\longrightarrow C).& \end{aligned}$$

For example the first line fixes one subdiagonal entry and kills all entries below it in the corresponding column or to its left in the corresponding row. The two last conditions fix the two blocks adjacent to C and kill the remaining blocks below them. Other entries belong to $\mathfrak{p}$.

Put $v=v_1,\lambda=\lambda_1$. The first line makes v_j a monic polynomial of degree $j-1$ in A applied to v. It gives the analogous monic polynomial on the right of λ for λ_j. The equations $\lambda_i(v_j)=0$ successively give $\lambda A^hv=0$ through $h=2a-2$, since every such h equals $i+j-2$ for $1\le i,j\le a$. The condition $Av_a\in W$ gives the next vanishing. Finally $\lambda_aA^2v_a=u'(w')=\gamma$ gives

$$\lambda A^hv=0 \quad (0\le h<2a),\qquad \lambda A^{2a}v=\gamma.$$

Conversely these moment equations force independence of $v,Av,\dots,A^{2a}v$, because a relation ending at an earlier power would express $A^{2a}v$ in terms of earlier powers. The same argument

works for $\lambda, \lambda A, \dots, \lambda A^{2a}$. Choose v_{j+1} freely in $Av_j + L_j$ and choose λ_{j+1} freely in $\lambda_j A + R_j$. The moment equations give all required annihilations, put Av_a in W, and make $\lambda_a A$ vanish on L_a. Its induced functional on C pairs with $[Av_a]$ to γ, so u', w' are determined. Each solution (v, λ) gives exactly $q^{a(a-1)}$ tuples, by multiplying the two sets of choices of q^j at each step.

For each of the $q^r k_A$ independent starting vectors there are exactly q^{r-2a-1} functionals satisfying the $2a+1$ moment equations. For the other vectors there are none. Thus $\widehat{F}(A)$ is a constant times k_A, with the constant independent of A. Its average over all matrices is $F(0) = 1$, by character orthogonality. For uniform independent A, v, the probability that $v \neq 0$ is $1 - q^{-r}$. Given j independent successive vectors and the previous values of A, the next value of A on the last of them is uniform on V: a linear map prescribed on the first $j-1$ independent vectors extends in equally many ways for each image of the j-th. Its chance of missing their span is $1 - q^{j-r}$. This computes the average of k_A as $B_{q,r,a}$, proving the transform formula. Since $0 \le k_A \le 1$, the average squared transform is at most $B_{q,r,a}^{-1}$. Finite Fourier Plancherel gives the sum of $|F(X)|^2$ as that average. Removing $X = 0$ subtracts exactly one and proves the estimate. □

Corollary 108.44 (Mass of the projected finite Bessel character)**.** *On the fields and levels of Proposition 108.41, put $a = m/2 - 1$, $c = r - 2a$, and $K_x = 8f(p_0^{-c} + p_1^{-c})$. Let Ψ be the product of the reduced Bessel characters extended by zero on H. For all sufficiently large x,*

$$\sum_{s \in H,\ s \neq 1} |\Psi^{\#}(s)|^2 \le K_x.$$

In the representation induced to H from the character of its Bessel unipotent subgroup, the fraction of the dimension discarded by e is

$$\delta + O(\sqrt{\delta(1-\delta)K_x}), \qquad \delta = 1 - (1-\delta_0)(1-\delta_1).$$

The implied constant is one. The same error bound with δ_i holds for the rank of $1 - e_i$. In particular all these fractions are $O(b^{-9})$.

Proof. We check the middle pairing, since its nonvanishing is used in the moment calculation. At a marked prime the split algebra is $\mathbb{Q}_p \times \mathbb{Q}_p$ with involution interchanging factors. Write the hyperbolic lines in the order $e_1, \dots, e_a$, the middle space, $e_a^*, \dots, e_1^*$. If g is the first matrix, the second equals $J^{-1}(g^{-1})^t J$, where J pairs the opposite lines and has a symmetric, unimodular middle block T. In the lattice basis the distinguished middle vector z satisfies $z^t T z = -N$, where $N > p_0, p_1$ is the separate prime level. Indeed the middle basis replaces one unit vector x_j of the form $\mathrm{diag}(1, \dots, 1, -N)$ by $x_j/\overline{\pi} + z/N$, with $\pi\overline{\pi} = N$. Its pair with z is -1, its norm is zero, and the two-by-two middle block has determinant -1. The reduction in the first matrix is the entire strictly block upper unipotent group, since entries can be lifted freely over $\mathbb{Z}_p$ and the second matrix is then determined.

The additive character reads the consecutive entries along $e_1, \dots, e_a, z$ in both components, with a unit coefficient in each. In fact the two completions are $\mathbb{Q}_p$, the different is a unit there, and division by $p_0 p_1$ has valuation -1 in each completion. In the first matrix this gives nonzero coefficients along the left arm and the last row evaluated at z. In the second matrix the inverse negates each first off-block entry, since every higher power of $g - 1$ has zero in that block. Transposition and J therefore give the reversed arm in the first matrix and, at its central end, the functional $-z^t T$. The two central blocks have pairing a nonzero scalar times $z^t T z = -N \bmod p$. This verifies precisely the hypothesis of Theorem 108.43 for the actual reduced character. Reductions in all factors are independent: take the specified integral

unipotent lift at each split place and the identity away from the marked primes. Thus the extended adelic character on its residue image is the stated product.

Conjugation averages and sums of squared absolute values factor over the product. The full sum is at most $\prod_{i=0}^{1} B_{p_i,r,a}^{-f}$, and the identity contributes one. Since $p_i \ge 3$, summing $-\log(1-y) \le 2y$ over $y = p_i^{j-r}$, $0 \le j \le 2a$, bounds the log of the product by $4f(p_0^{-c} + p_1^{-c})$. This quantity tends to zero by the scale bounds of Proposition 108.41, since $c = r - m + 2 \ge x^5 - 2\sqrt{x}$. Using $e^v - 1 \le 2v$ for small $v \ge 0$ gives K_x.

The induced character divided by its degree is exactly $\Psi^{\#}$, by the character formula for induction from a subgroup. Let $d = 1 - e = \sum_s d_s s$. Its identity coefficient is δ by the product formula for the regular trace. Since $d = d^* = d^2$, $\sum_{s \ne 1} |d_s|^2 = \delta - \delta^2$, by taking the coefficient at one in $d^* d$. The normalized trace of d on the induced space is $\sum_s d_s \Psi^{\#}(s)$. Its identity term is δ and Cauchy–Schwarz bounds the rest by $\sqrt{\delta(1-\delta)K_x}$. Taking $d = 1 - e_i$ gives the single-prime assertion. Finally $\delta \le 4b^{-9}$ by the proof of Proposition 108.41, and K_x is smaller than any fixed negative power of b on this family. □

These bounds and the odd-cocycle estimate can be used on the same entire interval in this construction. More precisely, fix q_0, d, take $V_0 \le x \le X = V_0^{3/2}$, and put

$$s_* = \lceil 22 \log_2(2\sqrt{X}) \rceil, \quad N_* = 3 q_0 d \, l_0(s_*) 2^{s_*}, \qquad t_* = \lceil 15 X^5 \rceil,$$

where $l_0(s)$ is the least power of three at least $27s$. For every possible tower exponent $1 \le s \le s_*$, put $E_s = \mathbb{Q}(\sqrt{-d}, \mu_{3 q_0 l_0(s) 2^s})$. Then one integer

$$J_I = \mathcal{J}_{N_*, t_*} \prod_{1 \le s \le s_*} h_{E_s}$$

controls the odd-cocycle lengths of Theorem 11.29, including weight zero, on every field used in the finite Bessel comparison above. Indeed $2^s = m^{22} \le (2\sqrt{X})^{22}$ at every node, and $l_0(s) \mid l_0(s_*)$ because both are powers of three. Hence all $E_s \subseteq \mathbb{Q}(\mu_{N_*})$ and all needed weights lie below t_*. Each $\log h_{E_s} \ll N_* \log(2N_*)$ by the class-number estimate in Corollary 11.30. Since $s_* = O(\log(2X))$ and $N_* \ll_{q_0,d} X^{11} \log(2X)$, the theorem gives $\log J_I \ll_{q_0,d} X^{54} \log^5(2X)$. The finite projector and its character estimate at a node use that node's own two places and field E_s. All the preceding bounds for them hold throughout the interval when V_0 is sufficiently large.

This evaluates the finite character in the Bessel term of the period. Extending it to the whole Poincaré Gram matrix requires bounds for rational elements outside the Bessel subgroup in the unfolded integral.

108.17 Rational orbital cancellation and the two frame filtrations

Retain the shorter construction and its levels from Corollary 108.44. Work at a single node x of the interval there. Write $F = E^+$, $a = m/2 - 1$, and $G = \mathrm{U}(h)$ over F. Use a Hermitian form conjugate linear in its first variable. The space over E is

$$V = L \oplus (X_0 \perp Ez) \oplus L^\vee, \qquad L = \langle e_1, \dots, e_a \rangle, \qquad h(z,z) = -N.$$

The first and last summands are opposite totally isotropic spaces, X_0 is totally positive definite, and $N > 0$. Put $n = a + 1$, $f_i = e_i$ for $i \le a$, $f_n = z$, and $T_i = \langle f_1, \dots, f_i \rangle$, with $T_0 = 0$ and $T = T_n$. Let U be the unipotent radical of the stabilizer of the flag $T_1, \dots, T_a$, and set $S = U \rtimes \mathrm{U}(X_0)$, with $\mathrm{U}(X_0)$ acting trivially on the other summands. Write $\sigma(u)$ for the sum of the entries in positions $(i, i+1)$ in the action on the frame, $1 \le i \le a$. In the final entry one

applies the last row on L to the vector z. This is additive on U, since a product of two strictly upper block matrices has zero in every first off-diagonal block. Put $\alpha = -1/(M\delta_E)$, with δ_E the chosen different generator and $M = p_0p_1$. The finite character is the standard finite adelic additive character of E applied to $\alpha\sigma(u)$.

Lemma 108.45 (A simultaneous endomorphism for two rational characters)**.** *For $\gamma \in G(F)$, set $H = U \cap \gamma U\gamma^{-1}$. Suppose the finite character takes equal values at u and $\gamma^{-1}u\gamma$ for every $u \in H(F)$. There is an E-linear endomorphism A of V such that, for $1 \le i < n$,*

$$Af_i - f_{i+1} \in T_i, \qquad A\gamma f_i - \gamma f_{i+1} \in \gamma T_i.$$

Proof. Let $\mathfrak{n} = \operatorname{Lie} U$ over F. Matrix logarithm and exponential are inverse finite polynomials on strict block unipotents and nilpotents in characteristic zero. They commute with conjugation. For the Hermitian adjoint $u^* = u^{-1}$ amounts to $(\log u)^* = -\log u$. Thus the logarithms of $H(F)$ are exactly the intersection of $\mathfrak{n}$ and $\gamma\mathfrak{n}\gamma^{-1}$ on F-points. Also $\sigma(\exp X) = \sigma(X)$, using the same entries on the Lie algebra. On rational arguments the finite character is the inverse of its archimedean factor, the exponential of the absolute trace times $2\pi i$, up to the common choice of sign. Consequently

$$\lambda(X) = \operatorname{Tr}_{E/F}(\alpha\sigma(X))$$

and its conjugate by γ, given by $X \mapsto \lambda(\gamma^{-1}X\gamma)$, agree on the intersection. Indeed if their difference were $d \neq 0$ at X, evaluating at $X/(2[F : \mathbb{Q}]d)$ would give absolute trace $1/2$.

The two functionals now define one F-linear functional on the sum of their domains. Extend it to the unitary Lie algebra by extending a basis, and extend scalars to E. This last Lie algebra becomes $\operatorname{End}_E(V)$. For instance each endomorphism is the sum of a skew-adjoint endomorphism and an imaginary scalar times a skew-adjoint endomorphism, by taking its skew-adjoint and self-adjoint parts. The scalar extension of $\mathfrak{n}$ is the entire strict upper block space: the adjoint preserves that space in the ordering with reversed dual lines, so the same decomposition applies. Represent the resulting functional by $\operatorname{tr}(B\,\cdot)$ using the nondegenerate ordinary matrix trace pairing.

On a skew-adjoint matrix, λ is the sum of $\alpha\sigma(X)$ and its conjugate. After scalar extension the conjugate term uses only the opposite arm: conjugation on the entries of a skew-adjoint matrix is obtained by negative transposition and the Hermitian Gram matrix, which interchanges the two outer flags and preserves the middle block. Thus it has zero coefficient on each entry from a later block into a left line. Testing an arbitrary entry of this kind in row i specifies precisely the part of Bf_i below that line. The only prescribed nonzero part is αf_{i+1}, including αz for the last left line. This gives $Bf_i - \alpha f_{i+1} \in T_i$. Applying the identical argument in the γ-translated basis gives the second relation for B. Take $A = \alpha^{-1}B$. □

Theorem 108.46 (Ranks and minors of agreeing rational frames)**.** *Write $C = (h(f_i, \gamma f_l))_{i,l\le n}$, let D_i be its leading minor of order i, set $D_0 = 1$, and put $\Delta = D_n\overline{D_n} - N^2 D_a\overline{D_a}$. If $T \cap \gamma T = 0$, then Δ is totally positive and $D_n \neq 0$. Otherwise $\Delta = 0$. In the latter case, under the character agreement of Lemma 108.45, there is a unique $0 \le j \le a$ with*

$$\dim_E(T \cap \gamma T) = n - j, \qquad \dim_E(L \cap \gamma L) = a - j, \qquad D_j \neq 0.$$

There is one $\zeta \in E^\times$, $\zeta\bar{\zeta} = 1$, such that for each $j < i \le n$ a common vector in $T_i \cap \gamma T_i$ has top coefficient ζ in the first frame and 1 in the second. None of the lower levels $i \le j$ has a nonzero common vector.

Proof. The joint Gram matrix of the $2n$ columns has off-diagonal blocks C, C^*. Both diagonal blocks are zero apart from their last diagonal entry $-N$. Its determinant is $(-1)^n\Delta$. Indeed the

terms using neither diagonal entry give $(-1)^n D_n \overline{D_n}$, those using both give $(-1)^{n-1} N^2 D_a \overline{D_a}$, and using exactly one leaves unequal numbers of rows and columns to match across the two blocks. A common vector gives linearly dependent columns. When the columns are independent, the span is nondegenerate at each real place. To see this, $T^\perp = L \oplus X_0$ is positive semidefinite with nullspace L. A vector in the radical of the joint span is orthogonal to both frames and has norm zero, so lies in $L \cap \gamma L = 0$; scalar extension preserves this intersection. The span contains z and a totally isotropic a-space orthogonal to z. Its negative index is at least $a + 1 = n$, and the ambient negative index is exactly n. The determinant therefore has sign $(-1)^n$ at every real place. This proves the assertions about Δ and D_n.

For the remaining assertions write $W = T \cap \gamma T$. For $0 \neq w \in W$ take the larger i of its two top indices. Write u, v for its two coefficients at i, temporarily allowing one to be zero. Applying A^{n-i} takes the two expressions into the top frame level, with coefficients u, v at $z, \gamma z$, by Lemma 108.45. The norm of this one vector is both $-Nu\bar{u}$ and $-Nv\bar{v}$. It follows that both coefficients are nonzero. In particular the two filtrations induced on W coincide. In their common level W_i, the map to the top coefficient in either frame has kernel W_{i-1}. Each nonzero successive quotient is therefore one-dimensional; its pair of coefficients spans a line generated by $(\zeta, 1)$ with $\zeta\bar{\zeta} = 1$. The map A takes such a pair unchanged to the next level for $i < n$. All nonzero quotients thus occur consecutively at the top and give the same ζ. Counting the quotients in W_n and W_a proves the two dimension formulas.

We have $T = T_j \oplus W$ and $\gamma T = \gamma T_j \oplus W$. If $y \in \gamma T_j$ is orthogonal to T_j, it is also orthogonal to $W \subset \gamma T$, since $\gamma T_j \subset \gamma L$. Hence $y \in T^\perp$. Its norm is zero, so $y \in L$ by positivity. Now $y \in W \cap \gamma T_j = 0$. The pairing of the two j-spaces is nonsingular, which proves $D_j \neq 0$. □

Proposition 108.47 (Cancellation over the stabilizer of a rational term)**.** *For class representatives $h_c, h_{c'} \in \mathrm{U}(X_0)(\mathbb{A}_f)$ write $\Gamma_c = U(F) \cap h_c K(M) h_c^{-1}$, where $K(M)$ is the principal level in the integral stabilizer. Put $\Gamma = \Gamma_c \cap \gamma \Gamma_{c'} \gamma^{-1}$. If the two rational characters disagree, the integral of the Bessel orbital kernel over every fiber $\Gamma \backslash H(F \otimes_{\mathbb{Q}} \mathbb{R})$ is zero. In particular every absolutely convergent full or cut orbital integral for this term vanishes.*

Proof. Write $H_\infty = H(F \otimes_{\mathbb{Q}} \mathbb{R})$ and use a similar notation for G_∞. In the unfolding, the two principal congruence groups act on γ by left and right multiplication; its stabilizer consists of the pairs $(u, \gamma^{-1} u \gamma)$ with $u \in \Gamma$. The finite character is trivial on Γ_c. Indeed the frame is fixed pointwise by h_c, and for $u \in U$ its first off-diagonal frame entries are unchanged by conjugation by h_c. These entries are integral, and at the levels are divisible by M; division by δ_E therefore puts their trace in the conductor of the finite character. The same argument applies to c'.

The quotient $\Gamma \backslash H_\infty$ is compact. Here are details for this particular type of arithmetic subgroup. Restrict scalars to $\mathbb{Q}$ and choose a rational flag of Lie ideals with successive central, one-dimensional quotients, successively taking rational central lines in nilpotent quotients. Exponential coordinates in a basis adapted to this flag give real coordinates on H_∞; in particular each successive central direction has group law addition on its quotient. For each rational basis element X, $\exp(dX) \in \Gamma$ for a sufficiently divisible positive integer d. This follows from the finite exponential polynomial: outside finitely many primes its coefficients and both class changes are integral, and at the exceptional primes it tends to the identity as the local divisibility of d increases. Reducing one coordinate at a time, starting at the outermost quotient, gives representatives in a product of compact intervals in exponential product coordinates. The subgroup is discrete, since its rational matrix entries lie in fixed fractional integer lattices at all finite places. This proves compactness and gives a finite invariant quotient measure.

For either Bessel function W, its defining equivariance on the left by the real unipotent is multiplication by the archimedean character χ_∞. For the full vector this is the character in

the Bessel integral (translate its unipotent integration variable); the cut vector is multiplied by a function constant on left U_∞-cosets. On a rational unipotent the character is the inverse of its finite factor, by triviality of the adelic additive character on E. It follows that $K_\gamma(y) = W(y) \otimes W(\gamma^{-1}y)^*$ transforms under $h \in H_\infty$ by the character $\eta(h) = \chi_\infty(h)/\chi_\infty(\gamma^{-1}h\gamma)$. It is trivial on Γ and nontrivial by rational disagreement. For fixed y, right invariance of measure on $\Gamma\backslash H_\infty$ shows that the integral of $K_\gamma(hy)$ there is unchanged by multiplication by any $\eta(h_0)$. Choosing $\eta(h_0) \neq 1$ proves the vanishing. Finally integrate the fiber integral on $H_\infty\backslash G_\infty$ using quotient Haar measure. Both groups are unimodular. This gives zero with compact cutoffs on the outer quotient, and for every absolutely convergent orbital integral. □

Corollary 108.48 (Elimination of the isotropic parabolic on a level slice)**.** *Suppose $h_c^{-1}\gamma h_{c'}$ is integral at all finite places and that, in the adapted basis at one of the split marks, its first diagonal entry reduces to* 1*. Every rational term preserving L either belongs to $S(F)$ or has the stabilizer cancellation of Proposition 108.47. In particular this holds on either slice obtained by requiring the residue to be in the Bessel image above p_i.*

Proof. For an agreeing term with $\gamma L = L$ the preceding theorem has $j = 0$, since $a \geq 1$. Starting at $i = 1$, its common vectors with coefficient pair $(\zeta, 1)$ prove inductively that $\gamma T_i = T_i$ and $\gamma f_i - \zeta f_i \in T_{i-1}$, also at $i = n$. The unitary element $\zeta^{-1}\gamma$ lies in the flag parabolic. In its Levi quotient the diagonal on the left lines is one, and the middle unitary transformation fixes z; hence it preserves $z^\perp = X_0$ inside the middle space. Removing that middle action leaves the unipotent radical. Thus $\gamma \in \zeta S(F)$.

Both class representatives fix e_1. Its line is a primitive lattice summand, so integrality at every finite place implies that the eigenvalue ζ is integral there. Since $\zeta\bar{\zeta} = 1$, all complex conjugates of ζ have absolute value one. Such an algebraic integer is a root of unity: the monic polynomials of its positive powers have bounded degree and bounded integer coefficients, so two of the powers coincide. At the selected component $E_w = \mathbb{Q}_{p_i}$, its residue equals the indicated first diagonal entry. The group $1 + p_i\mathbb{Z}_{p_i}$ is torsion-free for odd p_i, by the convergent injective p_i-adic logarithm there. Therefore $\zeta = 1$ and $\gamma \in S(F)$. All disagreeing terms have zero fiber integral by the proposition. Every element of the Bessel image has first diagonal entry one, as required. □

On these slices the agreeing terms outside $S(F)$ therefore have either $1 \leq j \leq a$ or $\Delta \neq 0$. Quantitative control requires their multiplicities and the sizes of their integrals.

108.18 Divisors of the frame discriminant at the marked primes

Continue with the two frames of Theorem 108.46. We work with pairs of classes in the actual level. Write $f = [F : \mathbb{Q}]$ in exponents, and write $\mathcal{F} = (f_1, \dots, f_n)$ for the frame matrix. A rational term γ occurs between c, c' only if $g = h_c^{-1}\gamma h_{c'}$ carries the standard finite lattice to itself. In these coordinates the two integral frames are $\mathcal{F}, g\mathcal{F}$. Indeed both class representatives fix the original frame, and the change to the lattice basis preserves its columns. Their Hermitian Gram matrices are therefore the rational Gram matrices of the preceding subsection. The lattice form is integral, including at N: the altered middle basis there has an integral two-by-two block with diagonal $0, -N$ and off-diagonal entries of norm one. In particular C has entries in $\mathcal{O}_E$ and $\Delta \in \mathcal{O}_F$.

Theorem 108.49 (Divisibility by the indices of the two frames)**.** *Suppose $\Delta \neq 0$. At a split place $v \mid p_0p_1$ of F, let $w, \bar{w}$ be the places of E over v. Form the matrix $B_w = (\mathcal{F}, g\mathcal{F})_w$ in the integral lattice at w. Let ℓ_w be the length of the saturation of its column lattice modulo that*

lattice, and let d_w be the dimension of the intersection of the two reduced frame spans. With normalized integral valuations one has

$$\operatorname{ord}_v \Delta \;\geq\; \ell_w + \ell_{\bar{w}} \;\geq\; d_w + d_{\bar{w}}.$$

In particular on the slice where the reduction belongs to the Bessel image at all places above p_i,

$$p_i^{2nf} \mid \mathrm{N}_{F/\mathbb{Q}}(\Delta), \qquad \left|\mathrm{N}_{F/\mathbb{Q}}(\Delta)\right|^{-s} \leq p_i^{-2nfs} \quad (s \geq 0).$$

If both slices apply, the two divisors multiply. On the same slice, for any rational cell, $p_i^{j}\mathcal{O}_E$ contains D_j for $0 \leq j \leq a$.

Proof. The two frames each have independent reduction: the first consists of primitive basis columns and the second is carried there by a lattice automorphism. Their joint reduction thus has rank $2n - d_w$. The columns are independent in characteristic zero by Theorem 108.46. Smith normal form over the local discrete valuation ring shows that the least valuation of any maximal minor of B_w is ℓ_w, the sum of its elementary divisor exponents. Exactly d_w of those exponents are positive. Write J_w for the integral matrix pairing the two components of the ambient lattice. The joint Gram over w is $B_{\bar{w}}^{t} J_w B_w$. Expanding its determinant twice by Cauchy–Binet shows that each summand contains a maximal minor of each B and an integral minor of J_w. Their determinant is $(-1)^n\Delta$, which proves the inequalities. This argument retains nontrivial elementary divisors at arbitrary powers of the mark.

If the reduction is in the Bessel image then it preserves the whole frame span in both components, so that $d_w = d_{\bar{w}} = n$. There are f places of F over the completely split prime p_i, each of residue degree one. Taking the field norm gives the divisibility assertion; total positivity makes the inverse power estimate immediate. For the last assertion each entry of the leading j-square Gram block vanishes modulo every w over p_i: a vector in the first isotropic arm pairs to zero with the whole frame. Expand the determinant at each such place. □

Proposition 108.50 (The positive vector measured by the discriminant)**.** *Put $u = \operatorname{adj}(C) e_n^{\mathrm{std}}$, where e_n^{std} is the last coordinate vector in E^n. Let $t = \gamma \mathcal{F} u$, and let q be the X_0-component of t in the rational orthogonal middle decomposition. Then*

$$N h(q, q) = \Delta.$$

At every marked component where the reduction preserves the frame span, the vector q belongs to the p_i^{a+1}-multiple of the local positive lattice of class c, provided the reduction preserves the span at its paired component as well.

Proof. The identity $Cu = D_n e_n^{\mathrm{std}}$ gives $t \in L^{\perp}$ and $h(z, t) = D_n$. Thus $t = l + q - (D_n/N) z$, where $l \in L$. On the other hand the last coordinate of u is D_a, so $h(t, t) = -N D_a \overline{D_a}$ by unitarity. Equating the two evaluations gives the formula at every embedding. In the marked lattice basis q is obtained by projecting $g\mathcal{F}u$ to its positive part, then carrying it back by the class representative. The projection is integral there: the only middle change of basis has denominators dividing N, a unit at the marks. The components of u are signed a-minors of the first a rows of C, whose entries vanish modulo p_i when the frame span is preserved in the specified components. Hence p_i^{a} divides u. Each positive projection of a column of $g\mathcal{F}$ vanishes modulo p_i also. Multiplication yields the claimed extra factor of p_i. □

We next allow the two reduced frame spans to share only one vector. In this situation there is a further divisor that can be retained after removing the elementary divisors of both isotropic arms. For a local lattice D, the saturation of a submodule is its fraction-field span intersected with D.

Theorem 108.51 (The two terminal columns at a meeting place)**.** *Assume $\Delta \neq 0$. At each marked component w form $B_{0,w}$ from the two isotropic arms, omitting the two last columns of the frames. Write $S_{0,w}$ for the saturation of its columns and let $\ell_{0,w}$ be their index length in $S_{0,w}$. In the free quotient by $S_{0,w}$, let b, b' be the images of the two terminal columns. The ideal generated by all determinants*

$$\det \begin{pmatrix} t_1(b) & t_1(b') \\ t_2(b) & t_2(b') \end{pmatrix}$$

as t_1, t_2 range over integral linear forms on the quotient has valuation $\ell_w - \ell_{0,w}$. Suppose the two finite characters agree on the intersection of their conjugate reduced unipotent groups. Then

$$\ell_w - \ell_{0,w} \geq 1 \qquad (d_w > 0).$$

Consequently the product of the norms of all meeting components divides the absolute finite index given by any such determinant chosen locally at every finite place, if the forms and columns are integral everywhere and the determinant is an idele.

Proof. Choose bases adapted to $S_{0,w}$ and its direct complement. The exterior product of the $2a$ isotropic columns is a uniformizer to the power $\ell_{0,w}$ times a primitive generator of the top exterior power of $S_{0,w}$, up to a unit. Wedging the two terminal columns into it uses only their images in the free quotient. Thus the minimum valuation of the maximal minors of B_w is $\ell_{0,w}$ plus the minimum valuation of a two-row minor of (b, b'). Arbitrary pairs of forms generate the same ideal as these minors, by determinant expansion. This proves the equality of valuations.

For the remaining assertion work over $\mathbb{F}_{p_i}$ at w, writing s for the local matrix. In this component the unipotent is $1+\mathfrak{n}$, with $\mathfrak{n}$ the full strict block upper algebra. Its intersection with its conjugate is $1+(\mathfrak{n}\cap s\mathfrak{n}s^{-1})$. As in the proof of Corollary 108.44, the two-component character in this single matrix realization is the additive exponential of a linear form on $X = u - 1$; its consecutive coefficients on the left frame are the same nonzero scalar. This description holds with $w, \bar{w}$ interchanged, using the other component of the additive character on the left. A character equality on the global reduced intersection gives one on each factor by taking identity components at all other places. Over the prime field the additive exponential is injective as a map to the p_i-th roots of unity. The two linear forms therefore agree on the intersection. Extend them to one functional on the full matrix algebra and represent it by ordinary matrix trace. Dividing its representing matrix by the left coefficient gives a matrix A with

$$Af_i \equiv f_{i+1} \pmod{T_i}, \qquad Asf_i \equiv sf_{i+1} \pmod{sT_i} \quad (i < n)$$

over the residue field. The prescribed lower blocks give these formulas just as in Lemma 108.45. This finite-field proof uses $u-1$ and holds independently of denominators in a matrix logarithm.

If the spans meet, write a nonzero common vector in the two frames. Let i be the largest index at which at least one coefficient is nonzero. Apply A^{n-i}. The two coefficients at the terminal positions are now the former coefficients at i, so they are not both zero. The resulting equality, modulo the span of both isotropic arms, is a nontrivial relation between the terminal columns. It remains a relation modulo the reduction of the larger saturated lattice $S_{0,w}$. Hence every two-row minor there vanishes. The index of an integral finite idele is the product of residue norms to its local valuations. Taking one factor from every meeting component proves the last assertion. □

The two estimates apply at each x on the interval in Corollary 108.44, with its own E, M and class lattices. They constrain an individual term and its terminal determinant. The second moment involves products of two amplitudes summed over all pairs in a residue class.

108.19 The exact real measure over a pair of frames

At a real place normalize z to have norm -1. Write $n = a{+}1$, $k = \dim X_0 \geq 1$ and $r = 2a{+}k{+}1$. In coordinates (u, x, t, v) of sizes $a, k, 1, a$ the Hermitian form has matrix J with quadratic form $u^*v + v^*u + x^*x - |t|^2$. Use the matching order of the dual lines here. Let $\mathcal{F} = (e_1, \dots, e_a, z)$ and $G = \mathrm{U}(J)$. For $g \in G$ put $Z(g) = \mathcal{F}^* J g \mathcal{F}$. For any n-square complex matrix Z set

$$\delta(Z) = |\det Z|^2 - |\det Z_{[1,a],[1,a]}|^2, \qquad \Omega = \{Z : \delta(Z) > 0\}.$$

Write $S_j = 2\pi^j/(j-1)!$ for the area of the unit sphere in $\mathbb{C}^j$ and $V_j = \prod_{l=1}^{j} S_l$. Lebesgue measure on complex entries is $d\Re\, d\Im$. On Hermitian matrices use the diagonal real and the strict upper complex entries, and on skew-Hermitian matrices use the imaginary parts of the diagonal and the same strict upper entries.

We specify Haar measures. On G use the invariant measure that at the identity is $1/V_r$ times Lebesgue measure in the skew-Hermitian tangent coordinates $J\,dg$. This is the compact-dual normalization: for $J = 1$ it gives probability measure on $\mathrm{U}(r)$, as verified below. Write P for the subgroup fixing every column of $\mathcal{F}$. Its elements are the transformations

$$(u, x, t, v) \longmapsto (u - B^*hx + (K - B^*B/2)v,\ hx + Bv,\ t,\ v),$$

where $h \in \mathrm{U}(k)$, $B \in \mathbb{C}^{k\times a}$ and $K^* = -K$. The formula follows by pairing the proposed images of the positive and dual columns with the fixed frame and with one another. Direct substitution also verifies the isometry. Give P the measure $ds = dB\,dK\,dh$ with dh probability Haar. This is left and right invariant: multiplication translates the central K variable by an expression in B, and the linear actions on B are unitary or the identity.

Theorem 108.52 (Disintegration on the open set of frame products)**.** *There are measurable representatives $g_Z \in G$ with $Z(g_Z) = Z$ for $Z \in \Omega$. For any nonnegative measurable function φ on G,*

$$\int_{Z(g)\in\Omega} \varphi(g)\, dg = \frac{V_k S_k}{V_r} \int_{\Omega} \delta(Z)^{k-1} \int_P \int_P \varphi(s' g_Z s)\, ds\, ds'\, dZ.$$

Both sides may be infinite. The inner integral is independent of the representative of the indicated frame products.

Proof. We first compute directly on the set of frames. For a complex r-by-n matrix Y of independent columns, the map $Y \mapsto Y^*JY$ is a submersion, even at a singular Gram value. In fact Y^*J is surjective and one can solve $Y^*J\,dY = H/2$ for any Hermitian variation H. Let $d\mu$ be the fiber density at $Y^*JY = -e_n e_n^*$, given in submersion coordinates by dividing Lebesgue measure dY by the specified measure on Hermitian matrices. Such coordinates define the same density on overlaps by the change of variables formula. It is invariant under G, since left multiplication preserves both measures and the Gram value.

Write $Z = \mathcal{F}^*JY$; it consists of the a dual rows of Y and the negative of its z-row. If Z is invertible pass to $Y' = YZ^{-1}$, with dual rows $(I_a, 0)$ and z-row $(0, -1)$. Let (η, ξ) be the last row of Z^{-1}, let (A, p) be the top rows of Y', and (B, x) its X_0-rows. Here η is a row and x a single column. The Gram equations for Y' are precisely

$$\begin{aligned} A + A^* + B^*B &= -\eta^*\eta, \\ p + B^*x &= -\eta^*\xi, \\ x^*x &= 1 - |\xi|^2. \end{aligned}$$

Indeed the required Gram is the negative outer product of the last row of Z^{-1}. The right side of the last equation is positive on Ω, since $\xi = \det Z_{[1,a],[1,a]} / \det Z$.

There are $a + k$ free complex rows before imposing the Gram equations. Their change from Y' to Y has real Jacobian $|\det Z|^{2(a+k)}$ with Z held fixed. On Hermitian matrices the change of prescribed Gram has Jacobian $|\det Z|^{2n}$. For a diagonal matrix with positive singular values s_i, the diagonal factors are s_i^2 and the complex off-diagonal factors contribute $(s_i s_j)^2$; unitary factors preserve the Hermitian Lebesgue measure. Dividing the two Jacobians yields $|\det Z|^{2(k-1)}$. Solving for $A + A^*$ leaves its skew part $K = (A - A^*)/2$ with factor 2^{-a} from the real diagonal variables. Each off-diagonal pair has Jacobian one, as one checks on the sum and half-difference of A_{ij} and $\overline{A_{ji}}$. Solving for p has Jacobian one. Polar coordinates for x with squared radius $1 - |\xi|^2$ give $S_k(1 - |\xi|^2)^{k-1}/2$ times uniform probability on the unit sphere. These calculations give the frame density on Ω in the coordinates $Z, B, K, \omega = x/\|x\|$:

$$d\mu = 2^{-n} S_k \, \delta(Z)^{k-1} \, dZ \, dB \, dK \, d\sigma(\omega).$$

For a fixed Z, take $B = K = 0$ and a fixed unit ω_0 to construct $Y_Z = Y'Z$ by the displayed equations. This is a frame of the required Gram, with independent columns since $\mathcal{F}^* J Y_Z = Z$. Such a frame is the image of $\mathcal{F}$ by an isometry: inside the orthogonal complement of its negative line choose duals to the a isotropic columns, then subtract one half of their mutual Gram matrix times the isotropic columns to make the duals isotropic. Complete the hyperbolic pairs with an orthonormal basis of their positive complement inside that orthogonal complement. These choices can be made measurably by using the first successful basis minors and Gram–Schmidt. They yield $g_Z \mathcal{F} = Y_Z$. Applying the formula for P to Y_Z gives exactly the free B, K and $\omega = h\omega_0$ above. The image of dh is sphere probability by rotation invariance. Consequently, in the frame density the integral over these three coordinates is the integral over the left copy of P.

It remains to compute the factor for lifting frames to G. In the space of complex square matrices let $d\nu_r$ be the fiber density of $g^* J g = J$ by the same Gram construction. It is invariant, and at the identity its density is 2^{-r} in the specified tangent coordinates: the derivative is the Hermitian part $J \, dg + (J \, dg)^*$, with unit Jacobian off the diagonal and a factor two on each real diagonal. Multiplication by J on the ambient entries preserves Lebesgue measure. Thus $dg = 2^r d\nu_r / V_r$. For the positive identity form, one can integrate its square Gram fiber by adding orthonormal columns successively. Orthogonality fixes complex coordinates with unit Jacobian; the remaining unit length equation in complex dimension j contributes $S_j/2$ by polar coordinates. The mass of that fiber is $V_r/2^r$, which verifies the stated normalization.

Disintegrate the square Gram fiber first in the columns of the frame and then in the remaining columns. Its conditional fiber over $\mathcal{F}$ can be computed there, since translation by any isometry has complex determinant of absolute value one. The equations pairing the remaining columns with $\mathcal{F}$ fix their dual rows and z-rows with unit Jacobian. For the a dual columns the remaining rows are $K - B^* B/2, B$, with factor 2^{-a} from their zero Gram as before. For the k positive columns, pairing with these duals fixes their top rows as $-B^* h$ with unit Jacobian; their remaining Gram equation is $h^* h = I_k$. Its density is $V_k \, dh / 2^k$ by the compact calculation. The ordering of these equations only permutes real or complex coordinates, so the full factor along the right copy of P in $d\nu_r$ is $2^{-a-k} V_k \, ds$. These successive changes of variables may first be made in local submersion charts for continuous compactly supported functions and then extended to nonnegative measurable functions by uniqueness of the resulting measures. Combining them multiplies the frame density by $2^{r-a-k} V_k / V_r$. Since $r - a - k = n$, this proves the formula. The same coordinates show that two representatives differ by left and right factors in P; Haar invariance proves the last assertion. □

Corollary 108.53 (The density at a rational term on the marked slices)**.** *For a rational term with $\Delta \neq 0$, write Z_v for the frame products at each normalized real place of the shorter construction. Set $\kappa_v = |M\Phi_v(\delta_E)|/(2\pi)$ and $b_{j,v} = \kappa_v^{2(n-j)}/N$ for $1 \le j \le n$. The coefficient of the two Haar integrals in the product of the open-cell densities at $(Z_v)_v$ is*

$$\left(\frac{V_k S_k}{V_r}\right)^f \left(\prod_{v,j} b_{j,v}\right)^{2(k-1)} \mathrm{N}_{F/\mathbb{Q}}(\Delta)^{k-1}.$$

For terms on the p_i-slice, its reciprocal is bounded above by

$$\left(\frac{V_r}{V_k S_k}\right)^f \left(\prod_{v,j} b_{j,v}\right)^{-2(k-1)} p_i^{-2nf(k-1)}.$$

Proof. The real frame is obtained from the rational frame by scaling the last vector by $1/\sqrt{N}$ and the lengths along consecutive left lines by successive additional factors κ_v. In the group coordinates this is the diagonal substitution making the middle norm -1 and the simple-root character have unit coefficient in the real wave. Indeed the archimedean coefficient in the wave $\exp(s - \bar{s})$ has absolute value $2\pi/|M\Phi_v(\delta_E)| = \kappa_v^{-1}$; the ratios of adjacent frame scales are κ_v. Phases of basis vectors cancel in absolute determinants. Thus each leading Gram determinant of order l has its absolute value multiplied by $\prod_{j \le l} b_{j,v}$. Since $b_{n,v} = 1/N$, subtraction of their squared absolute values at orders n and a gives $\delta(Z_v) = (\prod_j b_{j,v})^2\, \Phi_v(\Delta) > 0$. Multiply the densities of Theorem 108.52 and use Theorem 108.49 for the reciprocal bound. □

The density describes integration in the continuous Gram variable. Estimating a period at a specified rational value requires control of the function being integrated there.

108.20 Pointwise restriction at a prescribed real Gram matrix

We pass from integration in Z to evaluation at a single Z. Keep G, P, Ω, n, k, r and the Haar measures of Theorem 108.52. Write S for the real Bessel subgroup from Section 108.17. Its action on the ordered frame has image the group N_n of complex upper unitriangular matrices and kernel P. Indeed the upper unitriangular changes among the first a lines extend by their inverse adjoints on the dual lines. The possible changes $z \mapsto z + l$, $l \in L$, are all realized by the unipotent radical: for a middle-space column map D from the dual lines into $\mathbb{C}z$, the middle coordinates change by Dv, and the top line coordinates by $-D^* J_{\mathrm{mid}}(x,t) - D^* J_{\mathrm{mid}} Dv/2$. This preserves the form and the linear map $D \mapsto -D^* J_{\mathrm{mid}} z$ ranges over L. We use Lebesgue measure on the strict upper entries of N_n, which is Haar in these triangular coordinates. Together with ds on P this specifies Haar on S; equivalently it is Haar on its unipotent radical times probability Haar on $\mathrm{U}(k)$. All quotient measures below use these choices.

Put $c_G = V_k S_k / V_r$ and let $d = 2n^2$, the real dimension of the Gram variable. Use operator norm for complex matrices. For $Z_0 \in \Omega$ write $\xi_0 = (Z_0^{-1})_{nn}$. A real number P_0 will satisfy

$$P_0 \ge \max\{8r,\ \|Z_0\|,\ \|Z_0^{-1}\|,\ (1 - |\xi_0|^2)^{-1/2}\}.$$

Use the complex entry coordinates on matrices, regarded as d real coordinates. Let Q be the cube centered at Z_0 with coordinate half-width $\epsilon = P_0^{-16}/4$.

Lemma 108.54 (A section with quantitative control in a Gram cube)**.** *On Q one can choose the representatives g_Z of Theorem 108.52 real analytically, fixing a positive space of dimension $k-1$ pointwise. On Q, $\delta(Z)^{k-1} \ge \delta(Z_0)^{k-1}/2$. Both g_Z and g_Z^{-1} have analytic continuations of operator norm at most P_0^{32} to the polydisc of radius P_0^{-16} in the complexified real coordinates around Z_0.*

Proof. Use $Y = Y_Z$ constructed in Theorem 108.52, with the sphere direction a fixed positive coordinate vector. This takes values in the $2n$-space consisting of the two isotropic arms, z, and this one positive direction. Write $Y_a = (Y_1, \dots, Y_a)$ and put

$$U = \mathcal{F}(Z^*)^{-1}, \qquad w = Z^{-1}e_n, \qquad D = U_{[1,a]} + \tfrac{1}{2} Y_a w_{[1,a]} w_{[1,a]}^*,$$

where the subscript on U selects its first a columns. Since $Y^* J U = I$ and $U^* J U = -ww^*$, the columns of D are isotropic duals to Y_a and are orthogonal to Y_n. Set $t = U_n + Y_n$. This is orthogonal to every column of Y and has norm for J equal to $1 - |\xi|^2$, where $\xi = (Z^{-1})_{nn}$. Subtracting $Y_a D^* J t$ makes t orthogonal to D and preserves its norm. Send the source frame to Y, the duals to D, and the distinguished positive unit vector to $(t - Y_a D^* J t)/(1 - |\xi|^2)^{1/2}$. Extend by the identity on the other $k - 1$ positive directions. The stated Gram relations give an isometry and specify g_Z.

For the analytic estimates write $p = P_0$. Extend both Z and its entrywise conjugate using the same complexified real coordinates, with opposite signs on the imaginary part. Perturbations of either matrix have operator norm at most $2np^{-16}$. The inverse perturbation series bounds the new inverse norms by $2p$, and their changes by $4np^2 p^{-16}$. In particular the change in $1 - \xi\bar{\xi}$, relative to its central value, is at most $12np^4 p^{-16}$: at the center $|\xi_0| < 1$ and $1 - |\xi_0|^2 \geq p^{-2}$. Its analytic square root is thus single valued and its reciprocal has norm at most $2p$. On the real cube the variation of $\log|\det Z / \det Z_0|$ is at most $4n^2 p\, p^{-16}$, by the convergent matrix logarithm of $I + Z_0^{-1}(Z - Z_0)$. In $\log(\delta(Z)/\delta(Z_0))$ this term occurs twice, plus the logarithm of the relative change in $1 - |\xi|^2$. Multiplication by $k - 1 \leq p$ gives absolute value less than $\log 2$, since $p \geq 8r$. This proves the lower bound.

For clarity, on the complex polydisc the free blocks of YZ^{-1} are given by the previous Gram equations with $B = K = 0$. The row η has length at most $2p$, as does its continued conjugate. The formulas there give $\|Y\| \leq p^7$. The displayed formulas here then give $\|U\| \leq p^2$, $\|D\| \leq p^{12}$, $\|t\| \leq p^8$, and $\|Y_a D^* J t\| \leq p^{27}$. With $p \geq 8r$ these bounds give $\|g_Z\| \leq p^{32}$, also for the continued conjugate matrix. The analytic continuation of g_Z^{-1} is the continuation of $J g_Z^* J$ and satisfies the same bound. □

Lemma 108.55 (The quotient measure in the Gram cube)**.** *For every nonnegative measurable left S-invariant function v and every measurable $P' \subset P$,*

$$\int_Q \int_{P'} v(g_Z s)\, ds\, dZ \leq \frac{2}{c_G\, \delta(Z_0)^{k-1}} \int_{S\backslash G} v(g)\, d\dot{g}.$$

Proof. Theorem 108.52, disintegrated on the left by P, gives the measure on $P\backslash G$ over the open set as the image of $c_G \delta(Z)^{k-1} dZ\, ds$ under $(Z, s) \mapsto P g_Z s$. To verify this for a compactly supported function on the quotient one can multiply its lift by a nonnegative function of left-orbit integral one, using local sections and a partition of unity. The left integral in the theorem removes that factor. Monotone exhaustion gives the general measure identity. Use Lemma 108.54 and enlarge P' to P. It suffices to show that the part with $Z \in Q$ of each $P\backslash S$-fiber has measure at most one.

On a fiber meeting the cube choose a representative g_1 with $Z_1 = Z(g_1) \in Q$. For $h \in S$, $Z(hg_1) = ((h^{-1})|_T)^* Z_1$, since h preserves the form. Write $A = ((h^{-1})|_T)^*$, a lower unitriangular matrix. If $AZ_1 \in Q$, then $\|A - I\| \leq \|AZ_1 - Z_1\|\, \|Z_1^{-1}\| \leq 8n\epsilon P_0 < 1/4$. Haar on $P\backslash S$ is Lebesgue measure in the entries of A: inversion on a unipotent triangular group preserves Haar, and adjoint exchanges upper and lower coordinates with unit absolute Jacobian. Each available complex entry is in a disc of area less than one. The volume of their product is at most one. Integrating over $S\backslash G$ proves the lemma. □

Let $X_1, \dots, X_{r^2}$ be a real Hilbert–Schmidt orthonormal basis of $\operatorname{Lie} G$, with the real inner product $\Re \operatorname{tr}(X^{*}Y)$. Let $R_X W(g)$ be the derivative of $W(g\exp(tX))$ at zero, and $R_{\mathbf{i}}$ the successive derivatives in a word $\mathbf{i}$. Suppose W is a smooth, finite-dimensional Hilbert-space-valued function, with $W(hg) = \tau(h)W(g)$ for a unitary representation τ of S. For an integrable compactly supported scalar ψ set $T_\psi W(g) = \int_G \psi(y) W(gy)\, dy$. Right derivatives of W and the corresponding convolutions have the same left equivariance, so their squared norms descend to $S\backslash G$.

Theorem 108.56 (A restriction estimate for localized Bessel convolutions)**.** *Put $p = P_0$, $P' = \{s \in P : \|s\| \le p^{20}\}$, and take $\phi \in L^2(G)$ supported on $\|y\| \le p^{20}$. For every word of length at most d, suppose*

$$\|T_\psi(R_{\mathbf{i}} W)\|_{L^2(S\backslash G)} \le A p^{|\mathbf{i}|} \|\psi\|_{L^2(G)}$$

for all ψ supported on the same ball. Then

$$\int_{P'} \|T_\phi W(g_{Z_0} s)\|^2 \, ds \le \frac{p^{1000 n^2}}{c_G\, \delta(Z_0)^{k-1}} \, A^2 \|\phi\|_{L^2(G)}^2 .$$

There are no differentiability assumptions on the scalar test function ϕ.

Proof. The ball in G is compact: a unitary matrix for J satisfies $g^{-1} = J g^{*} J$, and G is closed in matrix space. The same holds for P'. In particular ϕ has finite L^1-norm. Differentiation in Z under its integral is valid by local uniform boundedness of the derivatives of W on compact sets.

We bound a mixed derivative in any $l \le d$ distinct real coordinates Z_j. Write $B_j(Z) = g_Z^{-1} \partial_j g_Z$. Differentiating $W(g_Z s y)$ uses the Lie element $\operatorname{Ad}(y^{-1}) \operatorname{Ad}(s^{-1}) B_j(Z)$ on its right. We describe the bounds after iteration, including the coefficients that depend on y. Express the adjoint maps and B_j in the fixed orthonormal basis. Each matrix entry of $\operatorname{Ad}(y^{-1})/p^{40}$ and of $\operatorname{Ad}(s^{-1})/p^{40}$ has modulus at most one. This follows from $\|g^{-1} X g\|_{\mathrm{HS}} \le \|g\|^2 \|X\|_{\mathrm{HS}}$ on the indicated balls. In each differentiated integral absorb the entries depending on y, after division by p^{40}, into ϕ. The resulting weights are ϕ times products of such entries; each has L^2-norm at most $\|\phi\|_2$ and is independent of s, Z.

More explicitly the mixed derivative is a sum of terms $c(Z,s)\, T_\psi(R_{\mathbf{i}} W)(g_Z s)$, with $|\mathbf{i}| \le l$. The sum of the suprema of $|c|$ on $Q \times P'$ is at most $p^{180 l}$. Here are estimates establishing this bound. On the polydisc of radius $3p^{-16}/4$, Cauchy's inequality and Lemma 108.54 give $\|B_j\| \le 4p^{80}$; each Lie coefficient, extended complex linearly from the real algebra, is bounded by p^{84}. For the latter assertion one may choose the orthonormal basis as J times the usual orthonormal skew-Hermitian basis. Its coordinates complexify to trace pairings of Hilbert–Schmidt norm one; any other real orthonormal basis is related by an orthogonal change of coordinates. A derivative of an existing coefficient costs at most $4(d+1)p^{16} \le p^{20}$, by shrinking the polydisc radius by $p^{-16}/(4(d+1))$ at each step. When the derivative hits W, the expansion uses at most r^6 summands, from the three Lie indices, of size at most p^{84+80} times the old coefficient. The factors depending on s stay in the coefficient and depend on no Gram variables. Thus the sum of suprema increases by at most $p^{20} + r^6 p^{164} \le p^{180}$ per derivative. The final polydisc still contains Q. This proves the indicated expansion and estimate by induction, starting from coefficient one.

Take the $L^2(Q \times P')$-norm. For each term, Lemma 108.55, applied to the squared norm of its convolution, bounds that norm after removal of c by

$$\left(\frac{2}{c_G \delta(Z_0)^{k-1}} \right)^{1/2} A p^{l} \|\phi\|_2 .$$

The triangle inequality gives the same expression times p^{180l} for the whole mixed derivative.

Finally, on an interval of length 2ϵ, evaluation at its center is bounded in norm by $(2\epsilon)^{-1}\int\|u(t)\|\,dt+\int\|u'(t)\|\,dt$, by the fundamental theorem of calculus. Apply this successively in all d coordinates with values in $L^2(P')$; equivalently one may iterate the averaging and signed integration operators of the one-variable identity. Cauchy–Schwarz bounds the result by $(2\epsilon)^{-d/2}$ times the sum, over coordinate subsets, of $(2\epsilon)^l$ times the corresponding mixed derivative norm. Since $2\epsilon=p^{-16}/2$, $l\le d=2n^2$ and $p\ge 8r$, the scalar factor after squaring is at most

$$2(2\epsilon)^{-d}\big(2^d p^{181d}\big)^2\le p^{1000n^2}.$$

This proves the claim. □

The estimate evaluates a real convolution at the prescribed Gram value, whereas Theorem 108.52 integrates over Gram values. Corollary 108.53 supplies the product of its inverse densities at the actual rational terms. The remaining factor A concerns the specific archimedean Bessel vector and its discrete-series norm.

108.21 The formal-degree bound for the real Bessel average

We evaluate the quotient-norm constant in Theorem 108.56 for the real function that enters the shorter construction. All measures are those of Theorem 108.52 and Section 108.20. In this subsection K is a maximal compact subgroup of $G=\mathrm{U}(a+k,a+1)$. Let π be its discrete series of Harish-Chandra parameter λ. Write d_π for its formal degree with respect to dg. Inner products are conjugate linear in the first variable. Let w be its unit lowest-K-type coefficient vector, $x\in G$, and $b=\pi(x)w$. Let τ be the unitary representation of S on the finite-dimensional value space V_τ, given by the simple-root character on the unipotent radical and the chosen type on $\mathrm{U}(k)$. For $v\in V_\tau$ and a K-finite vector u of π, set

$$c_u(g)=\langle b,\pi(g)u\rangle,\qquad W_u(g)=\int_S c_u(sg)\tau(s)^{-1}v\,ds.$$

In particular $c_w(g)=\langle w,\pi(x^{-1}g)w\rangle$; the freedom to choose the real point x is part of the Bessel average.

Proposition 108.57 (Orthogonality after averaging a discrete-series coefficient)**.** *Suppose $|\lambda_i-\lambda_j|>r-1$ for every noncompact root e_i-e_j. Then W_u exists, is smooth and bounded, and belongs in quotient norm to $L^1\cap L^2(S\backslash G)$. It transforms on the left by τ. For some real $m\ge 0$ one has*

$$m=\int_S\langle b,\pi(s)b\rangle\,\langle v,\tau(s)^{-1}v\rangle\,ds,$$
$$\langle W_u,W_{u'}\rangle_{S\backslash G}=\frac{m}{d_\pi}\langle u,u'\rangle.$$

In particular $m=d_\pi\|W_w\|_2^2$. For every compactly supported $\psi\in L^2(G)$,

$$\|T_\psi W_u\|_{L^2(S\backslash G)}\le\frac{\|W_w\|_2}{\sqrt{d_\pi}}\,\|u\|\,\|\psi\|_2.$$

Proof. We use the discrete-series integrability criterion in [3]. For root α the threshold is $\frac14\sum_\beta|\langle\alpha,\beta\rangle|$ in the same metric as the pairing with λ. In type A_{r-1} take roots of squared length two. The two roots $\pm\alpha$ contribute four; $4(r-2)$ further roots each contribute one. This gives $r-1$. The criterion for the semisimple derived group applies here since multiplication by the compact center is a finite covering and the absolute values of coefficients are invariant in that

central direction. Every K-finite coefficient is consequently in $L^1(G)$, including its derivatives: applying any Lie algebra word to a K-finite vector leaves it K-finite by covariance under the finite dimensional adjoint representation. Fixed left or right translations have the same property of integrability of all derivatives. For example a left derivative after left translation is the translation of the derivative conjugated by its group element. This permits $b = \pi(x)w$ at arbitrary x.

Here is the point that allows integration on the noncompact subgroup S. If c is such a coefficient or translate, then $\sup_g \int_S |c(sg)|ds < \infty$. Take a small coordinate box transversal to S in G/S at the identity and a smooth section $t \mapsto z(t)$ with $z(0) = 1$. Evaluation at zero of $c(z(t)sg)$ is bounded by a fixed sum of integrals of absolute mixed derivatives on this box, by the iterated one-dimensional formula used in Theorem 108.56. These derivatives are linear combinations with bounded coefficients of left-translation derivatives of c at $z(t)sg$. The coordinates $z(t)s$ disintegrate Haar as $j(t)\,dt\,ds$ with smooth positive j, bounded below on the closed box. Right translation by g preserves Haar. Thus integrating this point bound in s is bounded by a finite sum of global $L^1(G)$-norms, independent of g. It applies to the derivatives also. Differentiation of W_u along right translations now gives $W_{d\pi(X)u}$; for example the integrated Taylor remainder is bounded by a constant times the square of the increment using the same bound on the second derivative. The argument iterates. It proves smoothness and boundedness; changing the integration variable gives the equivariance. The quotient L^1-norm is at most $\|v\|\|c_u\|_1$ by disintegration. Boundedness gives L^2.

Expand the inner product of the two averages, write the second integration variable in S as hs, and unfold $sg = t$ over G. Absolute interchange follows from

$$\int_G |c_u(t)| \int_S |c_{u'}(ht)|\,dh\,dt < \infty$$

by the uniform restriction estimate just proved. Schur orthogonality, in the normalization defining d_π, gives

$$\int_G \overline{c_u(t)}\, c_{u'}(ht)\,dt = d_\pi^{-1} \langle b, \pi(h)b\rangle\, \langle u, u'\rangle.$$

The value-space inner product is $\langle v, \tau(h)^{-1}v\rangle$. This proves the formulas and the nonnegativity assertion by taking $u = u' = w$. The same computation holds for finite sums of translates of K-finite vectors, since the coefficient bounds used for Fubini still apply. Hence $u \mapsto W_u$ extends by density to a scaled isometry of norm $\|W_w\|_2$ (the zero map if $m = 0$), and intertwines π with right translation. Consequently $T_\psi W_u$ is the image of $\int \psi(y)\pi(y)u\,dy$. Testing this vector against a unit vector and using Cauchy–Schwarz and Schur orthogonality gives norm at most $\|u\|\|\psi\|_2/\sqrt{d_\pi}$. □

Lemma 108.58 (Derivatives of the lowest compact type)**.** *Suppose* $\max_i |\lambda_i| \le L$*, and let the real Lie basis be as in Theorem 108.56. For any word of length* $s \le d = 2n^2$*,*

$$\|d\pi(X_{i_s}) \cdots d\pi(X_{i_1})w\| \le \big[\, 4\sqrt{r}\,(L + 3r + d + 1)\,\big]^s.$$

Proof. Conjugate the form to a diagonal sign matrix by a unitary change of basis, preserving the Hilbert–Schmidt metric. In the Cartan decomposition choose orthonormal bases of the compact and noncompact parts. For $\Re\,\mathrm{tr}(XY)$ the corresponding Casimir is $\Omega_G = \sum_{\mathfrak{p}} X^2 - \sum_{\mathfrak{k}} X^2$; write $C_K = -\sum_{\mathfrak{k}} X^2$. Differentiated unitarity on smooth K-finite vectors gives

$$\sum_i \|d\pi(X_i)u\|^2 = \langle u, (2C_K - \Omega_G)u\rangle.$$

The sum is unchanged under real orthogonal change of basis. The Harish-Chandra parameter gives the usual Casimir scalar $\|\lambda\|^2 - \|\rho\|^2$, with roots of squared length two. On a compact

highest weight μ, C_K has scalar $\|\mu+\rho_c\|^2-\|\rho_c\|^2$. These formulas use the trace form, including the compact center. Equivalently, in the complex matrix basis the quadratic element is $\sum_{ij} E_{ij}E_{ji}$; commuting opposite root vectors into triangular order adds 2ρ to its linear term, which gives the indicated shifted-square formula. The compact calculation uses the same root pairs within the compact blocks.

The lowest compact type has highest weight $\lambda+\rho_n-\rho_c$, by the discrete-series lowest-type theorem [4]; each coordinate differs from the corresponding λ_i by at most r. All weights of this compact type have coordinates bounded in absolute value by $L+r$: within either unitary block the weights lie in the convex hull of the permutations of its highest weight, by the highest-weight theorem. Lie algebra application is a K-map from the tensor product with the adjoint representation. Each weight of that representation is zero or a root, and changes a coordinate by at most one. After t applications every possible compact highest weight has coordinates bounded by $L+r+t$. Since those of ρ,ρ_c are at most r, inserting these bounds in the displayed Casimir identity bounds each next derivative by $4\sqrt{r}(L+3r+t+1)\|u\|$, also for a sum of compact types by their orthogonality. Iteration proves the lemma. □

Theorem 108.59 (Pointwise real estimate with formal degree)**.** *Take the real Bessel average $W=W_w$ above, or a scalar multiple, with a parameter satisfying the gap in Proposition 108.57. With the Gram and support conditions of Theorem 108.56, choose also $P_0\ge 4\sqrt{r}(L+3r+d+1)$. Then*

$$\int_{P'}\|T_\phi W(g_{Z_0}s)\|^2\,ds\le\frac{P_0^{1000n^2}}{c_G\delta(Z_0)^{k-1}}\,\frac{\|W\|_2^2}{d_\pi}\,\|\phi\|_2^2.$$

For W scaled to squared quotient norm $\dim V_\tau$, the numerator of the formal-degree factor is $\dim V_\tau$.

Proof. Apply Proposition 108.57 with u a word in $d\pi(X_i)$ applied to w. Its Bessel average is the corresponding right derivative of W. The derivative lemma gives the hypothesis of Theorem 108.56 with $A=\|W\|_2/\sqrt{d_\pi}$, including the case $W=0$. The construction uses the doubled parameter 2λ; its separation for the undoubled entries is at least $500r^3$, which exceeds $r-1$. The test and the Gram matrix use the normalizations of the preceding theorem. Rescaling multiplies both squared norms by the same factor. □

This estimate holds for every translate x. The identity for m in Proposition 108.57 identifies the positive local period that must be bounded below to choose and control a nonzero normalized vector in a prescribed range of x.

108.22 One open-cell pairing at all infinite places

We estimate the real integral associated with a single rational term of nonzero discriminant. The estimate keeps the cut error as an integral with its original weight. We begin at one real place, retaining G,S,P,g_Z,c_G from the preceding subsections. A point y lies in the frame box of radius ρ if $\|y^{-1}f_j\|\le\rho$ for all normalized frame vectors. Set $\ell(y)=\inf_{h\in S}(\|hy\|+1)$. For $Z\in\Omega$ with $|Z_{ij}|\le\rho^2$ and $\rho\ge 8r$, put

$$H=\frac{\rho^{9n}}{\delta(Z)},\qquad T=H^{140n}.$$

Lemma 108.60 (Bounding a relative position by the two frames)**.** *If y is in the frame box and $\ell(g_Zy)\le\rho$, then $\|y\|\le T$ for the section of Lemma 108.54.*

Proof. We have $\|Z\| \le n\rho^2 \le \rho^3$, $\delta(Z) \le |\det Z|^2 \le \rho^{6n}$, and $H \ge \rho^{3n}$. The adjugate bound yields $\|Z^{-1}\| \le \rho^{3(n-1)}/\sqrt{\delta(Z)}$. Also $(1-|(Z^{-1})_{nn}|^2)^{-1/2} = |\det Z|/\sqrt{\delta(Z)}$. All the parameters needed for Lemma 108.54 are at most H^2, so $\|g_Z\| = \|g_Z^{-1}\| \le H^{64}$. Work in its fixed-coordinate $2n$-space V_*, spanned by the isotropic and dual arms, z and one positive middle coordinate. On this space form the square matrix $D = (\mathcal{F}, g_Z^{-1}\mathcal{F})$. Its joint Gram has off-diagonal block Z^*, so the determinant calculation of Theorem 108.46 at $N = 1$ gives $|\det D| = \sqrt{\delta(Z)}$. We have $\|D\| \le H^{65}$, whence $\|D^{-1}\| \le H^{131n}$ by the singular value product formula.

Choose $h \in S$ with $\|hg_Z y\| \le 2\rho$, and write $h^{-1}\mathcal{F} = \mathcal{F}B$ with B upper unitriangular. The two matrices $y^{-1}\mathcal{F}$ and $y^{-1}g_Z^{-1}\mathcal{F}B$ have norms at most $\sqrt{n}\rho$ and 2ρ, respectively. Their cross Gram is Z^*B, so $\|B\| \le 2\sqrt{n}\rho^2\|Z^{-1}\| \le H^4$. Since $\det B = 1$, $\|B^{-1}\| \le H^{4n}$. It follows that the restriction of $y^{-1}D$ has norm at most H^{5n} and $\|y^{-1}|_{V_*}\| \le H^{136n}$. The negative eigenspace of J is contained in V_*. Writing P_- for its orthogonal projection, any $A \in G$ satisfies $AA^* = J + 2AP_-A^*$ because $AJA^* = J$. This bounds $\|y^{-1}\|^2$ by $1 + 2H^{272n}$, and proves the claim. □

Lemma 108.61 (A set of shifts in the real frame stabilizer)**.** *Let $T \ge 2$, $p \ge 64r^3(2T)^8$, and $C_T = \{g : \|g\| \le T\}$. There is a compact inversion-invariant set $B_p \subset P$ of positive finite measure, with $\|s\| \le p^7$ on B_p, such that $\mathcal{A} = B_pC_T$ satisfies, for every W transforming on the left by a unitary representation of S,*

$$\frac{1}{\mathrm{vol}_P(B_p)} \int_{\mathcal{A}} \|W(y)\|^2 \, dy \le 2(2T^2)^{n(n-1)} \|W\|^2_{S\backslash G}.$$

Proof. In the coordinates of Theorem 108.52, take all $h \in \mathrm{U}(k)$ and balls $\|B\|_{\mathrm{HS}} \le p^3$, $\|K\|_{\mathrm{HS}} \le p^6$. The matrix formula gives the norm bound; inversion takes B to $-h^{-1}B$ and K to $-K$. To bound the integral by quotient disintegration, it suffices to bound the measure of $\{s \in S : sy \in \mathcal{A}\}$. On any left S-orbit meeting $\mathcal{A}$, choose a representative $y_0 \in C_T$. Any s with $sy_0 \in B_pC_T$ is in $B_p(S \cap C_{T^2})$, because $\|y_0^{-1}\| = \|y_0\| \le T$.

For $s_0 \in S \cap C_{T^2}$, both its frame matrix $A \in N_n$ and A^{-1} have norm at most T^2. A lift $l(A) \in S$ of norm at most $(2T^2 + 2)^3$ is obtained by extending the leading a-block of A diagonally as in Section 108.20 and following it by the shear that adds the top part of its last column to z. The diagonal factor has norm at most $1 + T^2$; the shear has norm at most $(1 + \|A\|)^2 + 1$ by its displayed action there. The element $s_0l(A)^{-1}$ is in P; its B-coordinate is a matrix block and its K-coordinate is the skew part of a block. Each has Hilbert–Schmidt norm at most $\sqrt{r}\,T^2(2T^2 + 2)^3 \le p$.

Multiplying B_p on the right by any such element of P increases the two radii by at most p and $p + p^4$. Indeed the central cross term in the product is the skew part of a product of the two B-coordinates, up to unitary factors. Its Hilbert–Schmidt norm is at most the product of their norms. The ratio of the enlarged P-volume to the original is thus at most $(1 + p^{-2})^{2ka}(1 + 2p^{-2})^{a^2} \le 2$ for the stated p. Integrate this bound over A using the Haar quotient $P\backslash S = N_n$. Each of its $n(n-1)/2$ complex coordinates has modulus at most T^2, so their volume is at most $(2T^2)^{n(n-1)}$. This bounds the fiber measure and proves the lemma. □

We now allow all f real places. Use one copy of these groups and measures for each place v, and product Haar on G_∞. Let W_v be any scalar multiple of the translated discrete-series Bessel average of Theorem 108.59 and $W = \bigotimes_v W_v$. Let $W_v^\flat$ multiply W_v by a bounded left S-invariant cutoff supported on $\ell(y) \le \rho_v$, and $W^\flat = \bigotimes_v W_v^\flat$. For $\gamma = (\gamma_v)_v$ define

$$K_\gamma(y) = W(\gamma y) \otimes W(y)^*, \qquad K_\gamma^\flat(y) = W^\flat(\gamma y) \otimes W^\flat(y)^*.$$

Write $Z_v = Z(\gamma_v)$ and $H_v = \rho_v^{9n}/\delta(Z_v)$.

Theorem 108.62 (A cut pairing at a rational Gram value)**.** *Suppose $n \ge 32$, $Z_v \in \Omega$, $\rho_v \ge 8r$, and $|(Z_v)_{ij}| \le \rho_v^2$. At each place assume the weight factor $4\sqrt{r}(L_v + 3r + 2n^2 + 1) \le H_v^2$. Let ϕ be a measurable scalar function on G_∞, bounded by one in absolute value and invariant under left translation by $\prod_v P$. Suppose that whenever $\phi(y)K_\gamma^\flat(y)$ is nonzero, each y_v is in the frame box of radius ρ_v. Then this cut integrand is supported in a compact set and*

$$\Big\| \int_{G_\infty} \phi(y) K_\gamma^\flat(y)\, dy \Big\|_{\mathrm{HS}} \le \left(\prod_v \frac{H_v^{10^5 n^4}\, \|W_v\|_{S\backslash G}^4}{d_{\pi_v}\, c_G\, \delta(Z_v)^{k-1}} \right)^{1/2} + \int_{G_\infty} |\phi(y)|\, \|K_\gamma(y) - K_\gamma^\flat(y)\|_{\mathrm{HS}}\, dy.$$

The inequality is valid also for a nonfactorizable weight ϕ.

Proof. First work locally. By Theorem 108.52 we may write $\gamma = s_L g_Z s_R$ with $s_L, s_R \in P$. Use $T = H^{140n}$, $p = H^{2000n}$; the bounds in Lemma 108.60 and $H \ge \rho^{3n}$ verify the hypotheses for the chart, formal degree and Lemma 108.61. Choose s in B_p with its normalized Haar restriction and set $t(s) = s_R^{-1} s^{-1}$. Any z in the frame box with $\ell(\gamma t(s) z) \le \rho$ has $s^{-1}z$ in the frame box, as s fixes the frame; also $\ell(g_Z s^{-1} z) = \ell(\gamma t(s) z)$. Thus $s^{-1}z \in C_T$ by Lemma 108.60, so $z \in \mathcal{A} = B_p C_T$. In particular, at all places together the nonzero cut integrand at $t(s)z$ has $z \in \prod_v \mathcal{A}_v$, since each $t(s)_v$ fixes the frame and the cutoff enforces the length condition. Taking any fixed shifts proves the compact support assertion.

For the full average consider the local map

$$\psi \longmapsto \left(s \longmapsto \int_{\mathcal{A}} \psi(z) W(g_Z s^{-1} z)\, dz \right)$$

from $L^2(\mathcal{A})$ to $L^2(B_p; V_\tau)$ with unnormalized measure there. Its squared operator norm is at most $p^{1000n^2} \|W\|_{S\backslash G}^2 / (d_\pi c_G \delta(Z)^{k-1})$ by Theorem 108.59. Indeed $\mathcal{A}$ has norm at most $p^7 T \le p^{20}$, B_p lies in the allowed ball of shifts, and inversion preserves its measure. Apply the tensor product of these operators to the scalar functions

$$\phi(z) \overline{\left(\bigotimes_v W_v(z_v) \right)_j} \quad \text{on} \quad \prod_v \mathcal{A}_v$$

for each value-space coordinate j. The operator bound multiplies: this follows first on Hilbert tensor products from the positive-operator bound for each factor, and then on product L^2 by density of finite sums of rectangle functions. Thus it holds for the indicated general ϕ. Sum the squared output norms in j. Left equivariance under s_L and $t(s)$ only inserts unitaries on the two value spaces. Since $\phi(t(s)z) = \phi(z)$, the resulting quantity is the integral over the shifts of

$$\left\| \int_{\prod_v \mathcal{A}_v} \phi(t(s)z) K_\gamma(t(s)z)\, dz \right\|_{\mathrm{HS}}^2.$$

On the input side the sum of squared norms is at most the product of $\int_{\mathcal{A}_v} \|W_v(z_v)\|^2\, dz_v$. Divide by the product of the shift volumes and apply Lemma 108.61. At each place the scalar loss, apart from $\|W_v\|_2^4 / (d_{\pi_v} c_G \delta(Z_v)^{k-1})$, is bounded by $2(2T^2)^{n(n-1)} p^{1000n^2} \le H^{10^5 n^4}$: with the chosen T, p its exponent to base H is at most $2{,}001{,}000\, n^3$, and $n \ge 32$.

For every shift the cut integral over $\prod_v \mathcal{A}_v$, after the indicated translation, is the original full-domain cut integral by its support and Haar invariance. Its difference from the corresponding integral of the full kernel is bounded by the displayed tail on G_∞. Integrate this triangle inequality against the probability measure of the shifts and use Cauchy–Schwarz and the preceding second-moment estimate. □

For the shorter construction take $\rho_v = b^{37n}$. Its cut vanishes past length b^{6n+2}, and the prescribed large-node scales have $n \geq 32$ and $r + L_v + 1 \leq b^4$; these bounds imply the numerical hypotheses above. For a near rational term the two frames at a common point are both in the box, so Cauchy–Schwarz there gives $|(Z_v)_{ij}| \leq \rho_v^2$, since J has operator norm one. The adjoint kernel $W(y) \otimes W(\gamma^{-1}y)^*$ is covered by replacing γ with γ^{-1} and conjugating ϕ. The two Gram matrices are adjoints and have the same δ. Corollary 108.53 evaluates the product of the density denominators on these very terms. To bound the global second moment one must also sum the rational orbits, their stabilizer weights, and the cut errors.

108.23 Summing over rational frame products and one normal vector

We return to the rational Hermitian space and to the integral class lattices of Section 108.18. In this subsection P is the rational group fixing every column of $\mathcal{F}$, and U_0 is the kernel of $P \to \mathrm{U}(X_0)$ on the positive middle space. Thus $P = U_0 \rtimes \mathrm{U}(X_0)$. Write $D = \mathrm{N}_{F/\mathbb{Q}}(\Delta)$ when $\Delta \neq 0$. It is a positive integer for any integral class position. We describe which rational orbits are counted by keeping the cross Gram matrix C and the first positive normal $q = q_\gamma$ of Proposition 108.50.

Proposition 108.63 (Rational frame-fixer orbits on the open cell)**.** *Each realized cross Gram matrix C with $\Delta \neq 0$ determines a single $P(F)$-double orbit. The orthogonal complement $Y_\gamma = (T + \gamma T)^\perp$ is totally positive of dimension $k-1$, and the simultaneous pointwise stabilizer of the two frames is $\mathrm{U}(Y_\gamma)$. Its projection to $\mathrm{U}(X_0)$ is the full stabilizer of q_γ. The orbits under $U_0(F)$ on the left and $P(F)$ on the right inside this double orbit are therefore distinguished exactly by q_γ.*

Proof. The joint Gram is nonsingular, with negative index n at every real embedding, by Theorem 108.46. These are all the negative directions in the ambient space. Thus its orthogonal complement is totally positive of dimension $r - 2n = k - 1$. Matching the two frames for terms of the same C gives an isometry of their joint spans. We recall an extension argument over E/F. Choose an orthogonal basis of the common abstract nondegenerate span with nonzero diagonal entries, by Hermitian diagonalization. Suppose two corresponding next vectors u, v have the same nonzero norm and are both orthogonal to the vectors already matched. For $y = u - v$ and $\beta = h(y, u) \neq 0$, the map $t \mapsto t - yh(y, t)/\beta$ preserves the form since $\beta + \bar{\beta} = h(y, y)$. It sends u to v and fixes the preceding vectors. If $\beta = 0$, then $h(u + v, u) = 2h(u, u) \neq 0$; apply the same formula with target $-v$, and then negate on the nondegenerate line Ev. Induction gives the desired rational extension. Since it matches both frames, it fixes $\mathcal{F}$ and matches the two images of $\mathcal{F}$. This proves the double-orbit assertion.

An element fixing both frames is the identity on their span, and is exactly an arbitrary unitary transformation on its orthogonal complement. The projection of Y_γ to X_0 is an isometry: $Y_\gamma \subset T^\perp = L \oplus X_0$ and its form is positive definite. Its image is orthogonal to q_γ, by pairing with $t = l + q_\gamma - (D_n/N)z \in \gamma T$ in Proposition 108.50. Since $h(q_\gamma, q_\gamma) = \Delta/N \neq 0$, dimension forces the image to be the whole hyperplane $q_\gamma^\perp$. Every unitary transformation of this hyperplane extends, through Y_γ, to the simultaneous stabilizer just described. Its action on X_0 fixes q_γ: it fixes t and z, and the middle projection on $L^\perp/L$ intertwines this action. Its image is therefore exactly $\mathrm{U}(q_\gamma^\perp)$, fixing q_γ pointwise.

Right multiplication by $P(F)$ fixes the frame image. On left multiplication by $p \in P(F)$, C stays fixed and q_γ changes to its image by the projection of p. The kernel is $U_0(F)$. Quotienting the left copy of $P(F)$ by this kernel and by the simultaneous stabilizer gives the stated classification. □

Let Λ_c be the integral class lattice in V, and write Q_c for its projection to X_0 in the rational decomposition. It is a full $\mathcal{O}_E$-lattice there. The two hyperbolic arms are primitive direct lattice summands, also after the class change. Removing their coordinates does not affect Q_c.

Lemma 108.64 (Counting integral positive normals without a loss at N)**.** *Fix C and c. Let $\mathfrak{a}$ be the ideal generated by the coordinates of* $\operatorname{adj}(C)e_n^{\mathrm{std}}$*. For an integral term on the p_i-slice, $q_\gamma \in p_i\mathfrak{a}Q_c$. More generally, if the normals being counted belong to $\mathfrak{b}Q_c$ for a nonzero integral ideal $\mathfrak{b}$, their number, when nonzero, is at most*

$$9^{kf}\bigl(D/\mathrm{N}_{E/\mathbb{Q}}\mathfrak{b}\bigr)^k.$$

On the p_i-slice one may take $\mathrm{N}_{E/\mathbb{Q}}\mathfrak{b} \ge p_i^{2nf}$.

Proof. For each middle lattice vector $x+\alpha z$ one has $N\alpha \in \mathcal{O}_E$, by pairing with z. For two such vectors the difference between $Nh(x,x')$ and $Nh(x+\alpha z, x'+\alpha' z)$ is $\overline{N\alpha}\,N\alpha'$. Hence $Nh(Q_c,Q_c) \subseteq \mathcal{O}_E$. In particular if $0 \neq u \in \mathfrak{b}Q_c$, then the totally positive $Nh(u,u)$ is in the F-ideal $\mathrm{N}_{E/F}\mathfrak{b}$. Indeed $\overline{\mathfrak{b}}\mathfrak{b}$ is the extension of this ideal, as one checks at split, inert and ramified primes. Thus $\mathrm{N}_{F/\mathbb{Q}}(Nh(u,u)) \ge \mathrm{N}_{E/\mathbb{Q}}\mathfrak{b}$.

Each projected column of $\gamma\mathcal{F}$ is in Q_c, and is in p_iQ_c at the places above p_i on the indicated slice, by the frame-span preservation there. Hence $q_\gamma \in p_i\mathfrak{a}Q_c$. The first a rows of C vanish modulo every component at p_i; their a-minors generate $\mathfrak{a}$. Thus $p_i\mathfrak{a} \subseteq p_i^n\mathcal{O}_E$.

Set $R = D^{1/(2f)}$ and $m = (\mathrm{N}_{E/\mathbb{Q}}\mathfrak{b})^{1/(2f)}$. At embedding v, use the Euclidean norm from Nh and rescale it by $R/\sqrt{\Phi_v(\Delta)}$. The scale factors have product one. By the first paragraph, distinct points of $\mathfrak{b}Q_c$ are at distance at least m in the maximum of the rescaled local norms: their squared norms have product at least m^{2f}. Each of the proposed normals has local radius exactly R at every place. If a normal exists then $R \ge m$, by comparison with zero. Disjoint open balls of radius $m/2$ around these normals lie in the ball of radius $R+m/2$. Volume in this real $2kf$-dimensional normed space bounds the number by $(1+2R/m)^{2kf} \le (3R/m)^{2kf}$, proving the claim. □

For fixed archimedean vectors write $t = 10^5n^4$, $\rho = b^{37n}$, $A_0 = \prod_{v,j} b_{j,v}$ with the scales of Corollary 108.53, and

$$B_x = \rho^{9nft}A_0^{-2(t+k-1)}\prod_v \frac{\|W_v\|_2^4}{d_{\pi_v}c_G}.$$

By that corollary the square of the first term on the right of Theorem 108.62 is exactly $B_xD^{-(t+k-1)}$.

Theorem 108.65 (A sum over the first normal on either marked slice)**.** *Fix c,c' and a marked slice p_i. Sum $B_xD^{-(t+k-1)}$ once for each distinct pair (C,q_γ) from a near integral open-cell term on this slice. With $\rho = b^{37n}$, the sum is at most*

$$B_x(1+2N\rho^2)^{2fn^2}\,9^{kf}\,p_i^{-2nfk}.$$

If the entire residue matrix is fixed at both levels, the first parenthesis can be replaced by $1+2N\rho^2/M$. For sufficiently large nodes of the shorter construction the first bound is at most B_xb^{-18nkf}.

Proof. A near term has normalized Gram entries of modulus at most ρ^2. Before the rescaling at v, the (j,l)-entry therefore has modulus at most

$$R_{jl,v} = N\rho^2\kappa_v^{-(2n-j-l)}.$$

The geometric mean over v is at most $N\rho^2$: the different generator has absolute norm $|\mathrm{disc}(E)|$, so $(\prod_v \kappa_v)^{1/f} = M|\mathrm{disc}(E)|^{1/(2f)}/(2\pi) \geq 1$. To count possible integral entries of C rescale each complex plane by $(\prod_v R_{jl,v})^{1/f}/R_{jl,v}$. The factors have product one. Distinct algebraic integers are separated by at least one in the rescaled maximum norm, since the absolute field norm of their difference is a nonzero integer. The same disjoint-ball argument as above bounds the count by $(1+2N\rho^2)^{2f}$. When the residue is fixed modulo both levels, all possible entries in the same position are congruent modulo M: in class coordinates they are the entries of $\mathcal{F}^* J g \mathcal{F}$ in the same integral Gram model. The separation is then at least M. Multiply these bounds over the n^2 positions.

For any chosen C, Lemma 108.64 counts its first normals by $9^{kf} D^k p_i^{-2nfk}$. Multiplying by the summand leaves $D^{-(t-1)} \leq 1$. This proves both bounds. For the last estimate the separate prime is chosen with $N \leq b^{60n}$, the level satisfies $p_i \geq b^{10}$, and $b \geq 9$. Thus the Gram count costs at most b^{270fn^3}. The family has $n = m/2 \leq \sqrt{x}$ and $k = r - 2n + 1 \geq x^5 - 2\sqrt{x}$, so $k \geq 270n^2$ eventually. The Gram count and 9^{kf} then cost at most b^{2nkf}, giving the assertion. □

For clarity, the finer rational orbit data can be described without counting them. On a fixed C, retain the second normal $q_{\gamma^{-1}}$ and the isometry between the two normal hyperplanes induced by Y_γ: send the positive projection of $t \in Y_\gamma$ to that of $\gamma^{-1}t$. These two projections are onto their respective hyperplanes by Proposition 108.63. If both actions are restricted to $U_0(F)$, the remaining data are exactly the two normals and this isometry. Indeed the stabilizer in the two copies of $\mathrm{U}(X_0)$ of the two normals and the isometry is the graph of $\mathrm{U}(Y_\gamma)$; each of its elements lifts to the unitary on the full space acting as the identity on the joint frame span. This is exactly the image of the stabilizer of γ. Passing further to principal unipotent congruence groups can split these orbits again. Theorem 108.65 counts its specified pairs once; the second moment weights those finer lifts.

108.24 The discriminant of two integral terms with a common link

We prove the arithmetic discriminant gain for two open-cell terms in the same level position whose positive links agree. Keep the original class lattices $\Lambda = \Lambda_c$ and $\Lambda' = \Lambda_{c'}$ in the indefinite space. Write $R = \mathcal{O}_E$, $\Lambda_X = \Lambda \cap X_0$ and $\Lambda_L = \Lambda \cap L = \bigoplus_{i=1}^a Re_i$. Use primes on subscripts for the corresponding sublattices of Λ'. A term γ carries Λ' isometrically to Λ. Let $H_\gamma = q_\gamma^\perp \subset X_0$ and $j_\gamma : H_\gamma \to X_0$ be the isometry induced by the two positive projections of $Y_\gamma = (T + \gamma T)^\perp$ in Proposition 108.63.

For $x \in H_\gamma$ its unique lift to Y_γ has the form $t_\gamma(x) = x + l_\gamma(x)$ with $l_\gamma(x) \in L$. Its inverse image under γ has positive component $j_\gamma x$ and isotropic component in L. The integral lattice of the term is

$$L_\gamma = \{x \in \Lambda_X \cap H_\gamma : l_\gamma(x) \in M\Lambda_L\}.$$

For a positive Hermitian lattice A of rank e define $d(A)$ as its real covolume at the complex embeddings, divided by $(2^{-f}\sqrt{|\mathrm{disc}(E)|})^e$. Use Euclidean volumes for its positive form at each embedding. This gives $d(R) = 1$ for the usual form, and passage to an index i sublattice of full rank multiplies d by i. Equivalently, in any rational basis with Gram G_0 and determinant ideal $\mathfrak{d}$ of the lattice coordinates, one has $d(A) = \mathrm{N}_{E/\mathbb{Q}}(\mathfrak{d})\mathrm{N}_{F/\mathbb{Q}}(\det G_0)$. Indeed this holds in the basis lattice by the real Jacobian at the complex embeddings, and the lattice index gives the general formula after clearing denominators. It agrees with the square root of the index of the local Gram determinant.

Theorem 108.66 (An exact discriminant formula for the level lift lattice)**.** *For an integral term with $\Delta_\gamma \neq 0$, define the finite index*

$$I_\gamma = [R^n : \{(h(\gamma f_j, y))_{j=1}^n : y \in M\Lambda_L + \Lambda_X\}].$$

Then the lattice L_γ has rank $k-1$ and

$$d(L_\gamma) = \frac{M^{2af}\,\mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma)}{I_\gamma}.$$

Proof. Let P here denote the n-by-a matrix $P_{ji} = h(\gamma f_j, e_i)$, and put $Bx = (h(\gamma f_j, x))_j$. All these values are integral on the lattices in question. The matrix P is the first a columns of C^* and has rank a, since C is invertible. Set $u = \mathrm{adj}(C)e_n^{\mathrm{std}}$ and let $\mathfrak{a} = (u_1, \dots, u_n)$. The functional $u^* : R^n \to \overline{\mathfrak{a}}$ is onto. Its kernel is $Y = R^n \cap \mathrm{span}_E(P)$, since $Cu = D_n e_n^{\mathrm{std}}$. The a-minors of P generate $\overline{\mathfrak{a}}$. Smith normal form at each prime gives $[Y : PR^a] = \mathrm{N}_{E/\mathbb{Q}}\mathfrak{a}$. Moreover $u^*Bx = h(q_\gamma, x)$ by the construction of the normal. Write $J = h(q_\gamma, \Lambda_X)$, a nonzero integral ideal contained in $\overline{\mathfrak{a}}$.

Set $A = MPR^a + B\Lambda_X$. It has rank n because $u^*A = J \neq 0$. For $x \in \Lambda_X \cap H_\gamma$ one has $Bx \in Y$, and $l_\gamma(x)$ has coordinates solving $Pl = -Bx$. Therefore $x \mapsto Bx \bmod MPR^a$ identifies $(\Lambda_X \cap H_\gamma)/L_\gamma$ with $(A \cap Y)/MPR^a$. Surjectivity follows because the B-part of any element in $A \cap Y$ is annihilated by u^*. The exact sequences for u^* give

$$[R^n : A] = [Y : Y \cap A]\,[\overline{\mathfrak{a}} : J],$$
$$[\Lambda_X \cap H_\gamma : L_\gamma] = \frac{M^{2af}\,\mathrm{N}_{E/\mathbb{Q}}J}{I_\gamma}.$$

This also proves the asserted rank.

It remains to compute the discriminant of the saturated hyperplane lattice. Project Λ_X orthogonally to Eq_γ. At embedding v, in a unit vector coordinate on this line, its projected coordinates are the embeddings of J divided by $\sqrt{\Phi_v h(q_\gamma, q_\gamma)}$. Covolume in an exact sequence of real Euclidean lattices is the product of the kernel covolume and the projected covolume; this follows by lifting a basis of the image and wedging with a basis of the kernel. Hence

$$d(\Lambda_X \cap H_\gamma) = d(\Lambda_X)\,\frac{\mathrm{N}_{F/\mathbb{Q}}h(q_\gamma, q_\gamma)}{\mathrm{N}_{E/\mathbb{Q}}J}.$$

Finally $d(\Lambda_X) = N^f$ in this construction. Before the class change all middle basis vectors except the distinguished one are ordinary. For that coordinate the integral middle lattice uses $x_*/\bar{\pi} + z/N,\ z$, where the finite idele $\pi\bar{\pi} = N$. A combination has zero z-part precisely when the coefficient of the first vector is divisible by N, so the pure positive coordinate lattice there is $\pi\widehat{R}$. Its ideal norm is N^f by $\pi\bar{\pi} = N$. A finite unitary class change in X_0 has determinant of absolute ideal norm one, since its conjugate times itself is one. Thus it preserves this discriminant. Inserting $h(q_\gamma, q_\gamma) = \Delta_\gamma/N$ cancels the factors of N. Multiplying the last two index formulas finishes the proof. □

Let

$$t_\gamma = [\,\Lambda \cap (L + \gamma L) : \Lambda_L + \gamma\Lambda'_L\,].$$

For a fixed residue class write $z(s)$ for the product of the norms of all marked components where the two reduced full frame spans meet. If character agreement has only been checked above one rational mark, one may form the product just over the components there. In particular on the p_i-Bessel slice the factor $z_i = p_i^{2f}$ is always available.

Lemma 108.67 (A surviving factor after both isotropic arms are removed)**.** *For an agreeing finite residue class, $t_\gamma z(s)$ divides I_γ. On a p_i-Bessel slice $t_\gamma z_i$ divides I_γ even without the agreement at the other mark.*

Proof. The lattice Λ is Hermitian unimodular. Its hyperbolic blocks are unimodular, the changed two-dimensional middle Gram has determinant -1, and the remaining block is the identity. The frame lattice $\Lambda \cap T$ is generated by the columns of $\mathcal{F}$. Its orthogonal lattice is $K = \Lambda \cap T^{\perp} = \Lambda_L + \Lambda_X$. By the perfect pairing and saturation of K, the dual of K identifies with $\Lambda/(\Lambda \cap T)$ via the Hermitian pairing. This is a local statement about free modules at a prime and its conjugate: an integral linear functional on a saturated direct summand extends over the whole ambient lattice. Its kernel in the pairing is the annihilator of K, whose rational span is T.

Enlarging $M\Lambda_L + \Lambda_X$ to K gives an index dividing I_γ. In the indicated perfect dual bases the evaluation map on K is the conjugate transpose of the columns of $\gamma\mathcal{F}$ in $\Lambda/(\Lambda \cap T)$. The index of its image in R^n is therefore the product of residue norms to the index lengths ℓ_w of the full joint frame columns at all finite primes. Indeed append the primitive first frame in adapted lattice coordinates; saturation of the combined columns is saturation of the projected columns in the quotient. Hermitian transpose exchanges conjugate places, which preserves the product of norms. Likewise t_γ is the product using the lengths $\ell_{0,w}$ of the two isotropic arms. At every place $\ell_w - \ell_{0,w} \geq 0$, by wedging in the two terminal columns after saturation as in Theorem 108.51; this part of that argument works over any local discrete valuation ring. At each agreeing meeting component the difference is at least one by that theorem. On a Bessel slice the terminal congruence $gz \equiv z \bmod L$ in both components gives the required dependence after saturation directly. Multiplication yields the divisibility assertions. □

Theorem 108.68 (The common-link discriminant gain for integral pairs)**.** *Suppose γ, η are open-cell terms in the same integral class position and with the same residue at both marked levels. Suppose their positive link graphs agree: $H_\gamma = H_\eta$ and $j_\gamma = j_\eta$ there. Put $L_{12} = L_\gamma \cap L_\eta$. Then*

$$\bigl(z\, d(L_{12})\bigr)^2 \leq M^{4af}\, \mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma)\, \mathrm{N}_{F/\mathbb{Q}}(\Delta_\eta)$$

for $z = z(s)$ under finite character agreement, or $z = z_i$ on a p_i-Bessel slice. In particular the bound applies to open terms with the same full link: imposing zero z-coordinate in both middle components of the full link gives precisely the positive graph above.

Proof. We first show $[L_\gamma : L_{12}] \mid t_\eta$. For $x \in L_\gamma$, put $x' = j_\gamma x = j_\eta x$, and write

$$t_\gamma = x + l_\gamma, \quad \gamma^{-1} t_\gamma = x' + l'_\gamma, \qquad t_\eta = x + l_\eta, \quad \eta^{-1} t_\eta = x' + l'_\eta.$$

The subscripts refer to the unique rational lifts. The given finite residue equality implies $(1 - \eta\gamma^{-1})\Lambda \subset M\Lambda$: at all primes over M it is the congruence in the common class basis, and at the other primes M is a unit. Hence

$$u(x) = \frac{t_\gamma - \eta\gamma^{-1} t_\gamma}{M} = \frac{l_\gamma - l_\eta}{M} - \eta\, \frac{l'_\gamma - l'_\eta}{M} \;\in\; \Lambda \cap (L + \eta L).$$

The two summands in $L + \eta L$ are uniquely determined, since $T \cap \eta T = 0$. The class of $u(x)$ modulo $\Lambda_L + \eta\Lambda'_L$ vanishes precisely when $l_\eta \in M\Lambda_L$. In one direction use uniqueness and $l_\gamma \in M\Lambda_L$; in the other, after removing the integral first summand from $u(x)$, the remaining integral vector belongs to $\Lambda \cap \eta L = \eta\Lambda'_L$. This yields the required injection of the quotient and the divisibility. Interchanging the terms gives $[L_\eta : L_{12}] \mid t_\gamma$.

By Theorem 108.66 the two resulting bounds are

$$d(L_{12}) \leq t_\eta\, M^{2af} \mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma)/I_\gamma,$$
$$d(L_{12}) \leq t_\gamma\, M^{2af} \mathrm{N}_{F/\mathbb{Q}}(\Delta_\eta)/I_\eta.$$

Multiply and use Lemma 108.67 on each term. The two isotropic torsion indices cancel, leaving z^{-2}. □

This comparison uses the same lattices as the open-cell integral and the same two levels. When the two positive graphs differ, their common lattice has smaller rank. Bounding and counting those pairs, as well as the terms of zero frame discriminant, is still required for the full second moment.

108.25 The discriminant gain when two open-cell links differ

We now allow distinct positive link graphs. We work first with all middle coordinates and impose the two terminal equations only after comparing the links. Write $R = \mathcal{O}_E$, use the discriminant d of Section 108.24, and put $A = \mathcal{F}_{[1,a]}$, the first a frame columns. For an open-cell term γ set

$$P_\gamma = (A, \gamma A), \quad Z_\gamma = (z, \gamma z), \quad V_\gamma = (L + \gamma L)^\perp.$$

The columns of P_γ are independent and $\dim_E V_\gamma = k + 1$. Every $t \in V_\gamma$ has coordinates $t = l + x + uz$ and $\gamma^{-1}t = l' + x' + u'z$, with $l, l' \in L$, $x, x' \in X_0$. On $D = X_0 \oplus E$ use the positive form $h_D((x, c), (y, d)) = h(x, y) + N\bar{c}d$. Then (x, u') and (x', u) have equal norm, by unitarity of γ. The first coordinate pair determines t: if it is zero then the other pair vanishes by positivity and $t \in L \cap \gamma L = 0$. Dimension now shows that the pairs are related by a unitary map $J_\gamma : D \longrightarrow D$, sending (x, u') to (x', u). Its graph is the full link; its restriction with both last coordinates zero is the positive graph of Proposition 108.63.

Take $\Lambda = \Lambda_c$, $\Lambda' = \Lambda_{c'}$ as before. Let $K^0_\gamma = \Lambda \cap V_\gamma$ and let K^+_γ be the sublattice imposing $l \in M\Lambda_L$. We identify both lattices with their first pair of coordinates in D. Let $\ell_\gamma(t) = (h(z, t), h(\gamma z, t)) = (-Nu, -Nu')$, and set

$$Q_\gamma = |\mathrm{N}_{E/\mathbb{Q}} D_n(\gamma)|, \quad D_\gamma = \mathrm{N}_{F/\mathbb{Q}}(\Delta_\gamma), \qquad G_\gamma = M^{2af} \max\{Q_\gamma / N^f, D_\gamma\}.$$

Lemma 108.69 (The full link and its two terminal linear forms)**.** *For each open integral term,*

$$d(K^0_\gamma) = \frac{Q_\gamma}{N^{2f} t_\gamma}, \qquad d(K^+_\gamma) \le \frac{M^{2af} Q_\gamma}{N^{2f} t_\gamma}.$$

At an embedding of F in $\mathbb{R}$, the product of the squares of the two nonzero singular values of ℓ_γ, using h_D and the usual metric on $\mathbb{C}^2$, is $N^2\Delta_\gamma / |D_n(\gamma)|^2$ at that embedding. Each one of the two component functionals has norm $\sqrt{N}$.

Proof. Write H for the ambient Hermitian matrix in rational coordinates, $P = P_\gamma$, $Z = Z_\gamma$, $E_0 = P^*H$ and $E_1 = Z^*H$. On $\ker E_0 = V_\gamma$ the positive graph metric is exactly the restriction of

$$H' = H + N^{-1} E_1^* E_1.$$

Indeed the indefinite norm is $h(x, x) - N|u|^2$ and the added term is $N(|u|^2 + |u'|^2)$. At an archimedean embedding form $B_0 = \left(\begin{smallmatrix} H' & E_0^* \\ E_0 & 0 \end{smallmatrix}\right)$. Its absolute determinant is $|\det H|\, |D_n|^2 / N^2$. To see this, start with the Hermitian block matrix with first row of blocks H, E_0^*, E_1^*, the two lower diagonal blocks 0 and $-NI_2$, and zero between the two lower blocks. Eliminating $-NI_2$ gives $N^2 \det B_0$. Eliminating H gives, up to sign, $\det H$ times the determinant of the joint frame Gram with N added at the two terminal diagonal entries. Its two diagonal frame blocks are now zero and its off-diagonal blocks are C, C^*. Its absolute determinant is $|D_n|^2$.

Here is the lattice consequence of this bordered calculation. The map E_0 sends Λ to a full lattice in R^{2a} of index t_γ, since Λ is Hermitian unimodular and evaluation on the columns of P has the elementary divisors of those columns at the conjugate primes. Choose a rational basis of $\ker E_0$ and then lifts of the standard basis of the target. Call the resulting basis matrix T_0. In this basis $E_0 T_0 = (0, I_{2a})$; expansion along the border gives $\det(H'|_{\ker E_0}) = |\det T_0|^2 |\det B_0|$,

where the left side denotes the determinant in the chosen kernel basis. In these coordinates the determinant ideal of Λ is the product of those of K_γ^0 and $E_0\Lambda$, by the short exact sequence, or by choosing local bases. The basis change contributes the inverse ideal of $\det T_0$. The formula for d in the preceding subsection therefore gives $d(K_\gamma^0) = Q_\gamma/(N^{2f}t_\gamma)$: the factor consisting of the norm of the original determinant ideal of Λ times $|\mathrm{N}_F \det H|$ is one by unimodularity. Imposing the congruence on l has index at most $|\Lambda_L/M\Lambda_L| = M^{2af}$.

For the singular values border B_0 a second time by E_1 on the original ambient coordinates and zero elsewhere; give this new border zero diagonal block. Its absolute determinant is $|\det H|\Delta_\gamma$, because the term $N^{-1}E_1^*E_1$ can be removed by row operations, and the remaining determinant after eliminating H is the joint Gram determinant of the two frames. Divide by $|\det B_0|$. In coordinates $E_0T_0 = (0, I)$ the upper ambient part of B_0^{-1} is the inverse kernel metric on the kernel and zero on the remaining coordinates, as is seen by solving its bordered linear equations. Thus this ratio is the determinant of the Gram of the two functionals on V_γ in the dual positive metric, proving the formula. Finally each functional is N times the last coordinate of a unitary identification with D, whose last coordinate line has squared length N. □

Proposition 108.70 (Congruence on the quotient by the common graph)**.** *For open terms γ, η in one class position with the same residue at both levels, put $I = \ker(J_\gamma - J_\eta)$ in D, $c = \operatorname{codim}_D I$ and $T_i = K_i^+ \cap I$ for $i = \gamma, \eta$. Then*

$$d(T_\gamma)d(T_\eta) \le (2/M)^{2cf} d(K_\gamma^+)d(K_\eta^+).$$

Proof. On the two source spaces consider the cross pairing

$$B(u, v) = h_D(u, v) - h_D(J_\gamma u, J_\eta v).$$

Its left and right kernels are both I, since the two maps are unitary and defined on all of D. Its operator norm is at most two. On $K_\gamma^+ \times K_\eta^+$ it is integral and divisible by M. Indeed if $t_1, t_2 \in \Lambda$ are the two lifts, expansion of the swapped terminal coordinates gives $B(u, v) = h(t_1, t_2) - h(\gamma^{-1}t_1, \eta^{-1}t_2)$. Now $\gamma^{-1}t_1, \gamma^{-1}t_2 \in \Lambda'$, and $(\gamma^{-1} - \eta^{-1})t_2 \in M\Lambda'$ by the shared residue. This proves the congruence.

Let B_i denote the orthogonal projection of K_i^+ to $I^\perp$ for h_D. Rationality of the projection makes these full lattices in the c-dimensional quotient. There B is nondegenerate and $B(B_\gamma, B_\eta) \subset MR$. Taking the determinant of the pairing gives

$$M^{2cf} \le 2^{2cf} d(B_\gamma)d(B_\eta).$$

Explicitly, in rational bases the determinant, times the conjugate determinant ideal of the first lattice and the determinant ideal of the second, generates an ideal contained in M^cR, by taking exterior powers at every finite prime. At an infinite place the absolute determinant of the pairing is at most 2^c times the square root of the product of the two positive Gram determinants. Take the absolute field norm and use the formula for d. Finally $d(K_i^+) = d(T_i)d(B_i)$ by orthogonal projection. The empty determinant is one if $c = 0$. □

Theorem 108.71 (The discriminant of any pair of open terms in one block)**.** *In the setting of Proposition 108.70, let $I_X = \{u \in I : u \in X_0,\ J_\gamma u \in X_0\}$. The common integral lattice of the two positive links is $L_{12} = I_X \cap L_\gamma \cap L_\eta$. Then*

$$d(L_{12})^2 \le (2/M)^{2cf} G_\gamma G_\eta.$$

For $c = 0$ the finer bound of Theorem 108.68 also applies with its meeting-place factor. All common dimensions in this assertion are allowed, including zero.

Proof. The two terminal maps ℓ_i coincide on I, since the input and output coordinates there agree. Their common kernel in I is I_X. Write $T_{i,X} = T_i \cap I_X$ and s for the rank of this terminal map on I. It has integral values on T_i by its definition on the integral lifts. We give bounds $d(T_{i,X}) \le A_i d(T_i)$.

If $s = 0$, take $A_i = 1$. If $s = 1$, choose a nonzero one of the two component functionals on I, so its kernel there is I_X. Its image lattice is a nonzero ideal inside R. Its real Jacobian on the orthogonal line to its kernel at each place is at most N by Lemma 108.69. Thus one can take $A_i = N^f$. If $s = 2$, use the two components together. Their image lattice in R^2 has discriminant at least one for the standard positive metric. The squared exterior norm on a subspace is bounded by that on the entire space, and Lemma 108.69 gives $A_i = N^{2f} D_i/Q_i$. In both positive ranks the covolume step is the same: the discriminant of the projected lattice orthogonal to the kernel is the image discriminant divided by the product of the real Jacobians on that complement. Divide $d(T_i)$ by the projected discriminant to get the kernel discriminant. Together with Lemma 108.69 all three cases give $A_i d(K_i^+) \le G_i/t_i$.

The lattice $T_{i,X}$ is $L_i \cap I_X$ in the positive coordinates: both terminal components vanish and its lift is integral with the indicated level condition on the isotropic part. We claim $[T_{\gamma,X} : L_{12}] \mid t_\eta$, and the reverse bound with the terms exchanged. For $u \in T_{\gamma,X}$ the positive images $j_\gamma u, j_\eta u$ coincide. Therefore the class of $(1-\eta\gamma^{-1})t_\gamma(u)/M$ in $(\Lambda \cap (L+\eta L))/(\Lambda_L + \eta\Lambda'_L)$ is defined, with kernel exactly L_{12} by the decomposition in the proof of Theorem 108.68. This argument uses agreement of images at u, which holds for every $u \in I_X$. Multiplying the resulting index inequalities and applying Proposition 108.70 gives

$$d(L_{12})^2 \le t_\gamma t_\eta A_\gamma A_\eta (2/M)^{2cf} d(K_\gamma^+) d(K_\eta^+) \le (2/M)^{2cf} G_\gamma G_\eta.$$

At $c = 0$ the two full unitary maps are equal and their positive graphs agree. □

The factors in this estimate come from congruence on the actual two class lattices. It covers each pair with nonzero frame discriminants. To apply it in a second moment one still needs the multiplicities of the integral pairs and estimates for the other cell ranks.

108.26 The pair discriminant at every rational cell rank

Keep the Hermitian space, the two class lattices and the level $M = p_0 p_1$ of the preceding subsection. We treat the terms for which the two frames have a common vector. Their rational lifts can have a common isotropic summand. We first remove that summand over $R = \mathcal{O}_E$.

Suppose that the characters agree on the rational stabilizer. For γ with $\Delta_\gamma = 0$, let j be its rank in Theorem 108.46, put $U = L \cap \gamma L$, and write $U_R = U \cap \Lambda$. Let $A_j = (e_1, \dots, e_j)$. A bar on a vector in $U^\perp$ denotes its image in $U^\perp/U$. Define

$$t_\gamma = [\, \Lambda \cap (L + \gamma L) : \Lambda_L + \gamma\Lambda'_L \,],$$
$$i_\gamma = [\, \Lambda_L/U_R : \bar{A}_j R^j \,], \qquad i'_\gamma = [\, \gamma\Lambda'_L/U_R : \overline{\gamma A_j} R^j \,].$$

The columns in each denominator span its numerator over E. In fact the common flag from Theorem 108.46 gives $L = \langle A_j \rangle \oplus U$ and $\gamma L = \langle \gamma A_j \rangle \oplus U$. The index i_γ can equally be computed from the first j isotropic rows of the frame product matrix inside the lattice generated by all its isotropic rows. Indeed evaluation of $l \in \Lambda_L$ against the second frame annihilates U_R, because $U \subset \gamma L$. Its restriction to A_j has rank j by $D_j \neq 0$, so its kernel is exactly U_R. Evaluation against the first frame proves the corresponding description of i'_γ. All indices here are absolute orders of finite quotients.

Lemma 108.72 (The two terminal vectors in the isotropic quotient)**.** *The lattice $\bar{\Lambda} = (\Lambda \cap U^\perp)/U_R$ is Hermitian unimodular. In $\bar{V} = U^\perp/U$ the subspaces $\bar{L}$ and $\overline{\gamma L}$ pair perfectly over E.*

Their sum has an orthogonal complement W of dimension $k+1$. If $d_* = \det C_{\{1,\dots,j,n\},\{1,\dots,j,n\}}$, then

$$d_* = -N\zeta D_j,$$

where ζ is the scalar for this term in Theorem 108.46. We include $j = 0$ by $D_0 = 1$. The two terminal linear forms $t \mapsto h(z,t), h(\gamma z, t)$ on W have joint rank one.

Proof. The module U_R is saturated in Λ. Locally it is a direct summand. Since Λ is self-dual, evaluation on U_R is onto its conjugate dual. Choose a local basis of U_R and dual lifts. Their joint Gram matrix has blocks $\left(\begin{smallmatrix} 0 & I \\ I & B \end{smallmatrix}\right)$, whose determinant is a unit. The summand they generate splits off orthogonally by integral projection. Its contribution to $U_R^{\perp}/U_R$ is zero, and the remaining summand is self-dual. At a split place take bases and dual lifts in the paired components of the local ring $R \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$. The same block matrix is invertible there. This argument applies also above two. Local self-duality gives the assertion about $\bar{\Lambda}$.

In the displayed bases of the two arms the cross pairing has determinant $D_j \neq 0$. The arms have zero intersection and are each isotropic. Together they span a nondegenerate $2j$-space in the $(2j+k+1)$-space $\bar{V}$. Both terminal vectors are orthogonal to U. Subtract their orthogonal projections onto the sum of the two arms. For $\bar{z}$ the subtracted vector belongs to $\bar{L}$, since $\bar{z} \perp \bar{L}$ and the two arms pair perfectly. For $\overline{\gamma z}$ it belongs to $\overline{\gamma L}$. The resulting vectors $z_0, z_1 \in W$ both have norm $-N$. The common vector at the terminal level of the two flags gives $\gamma z - \zeta z \in L + \gamma L$, whence $z_1 = \zeta z_0$. Changing each terminal vector by its indicated arm is a row or column operation of determinant one on the subframe matrix. Its remaining terminal entry is $h(z_0, z_1) = -N\zeta$. This proves the formula and shows that the two linear forms on W are proportional and nonzero. □

For any rank use the positive space $D = X_0 \oplus E$, with squared norm $h(x,x) + N\bar{u}u$. Representatives of a class of W may be taken in $(L+\gamma L)^{\perp}$. As in the open case write

$$t = l + x + uz, \qquad \gamma^{-1}t = l' + x' + u'z.$$

Changing t by a vector of U leaves both middle components fixed. The pairs (x, u') and (x', u) have equal positive norm by unitarity. If one vanishes, both vanish and $t \in U$. Since $\dim_E W = \dim_E D$, the first pair identifies W with D, and the second differs from it by a unitary map J_γ. Write K_γ^0 for the image in D of $\Lambda \cap (L+\gamma L)^{\perp}$. Write K_γ^+ for the image of those representatives that satisfy $l \in M\Lambda_L$. The lattice L_γ of the positive link is obtained by imposing $u = u' = 0$ on K_γ^+.

Proposition 108.73 (The determinant of the integral link in lower rank)**.** *For $0 \le j \le a$ in the preceding notation,*

$$d(K_\gamma^0) = \frac{|\mathrm{N}_{E/\mathbb{Q}} d_*|}{N^{2f} t_\gamma i_\gamma i'_\gamma}, \qquad [K_\gamma^0 : K_\gamma^+] \le M^{2jf}.$$

Each terminal functional has norm $\sqrt{N}$ for the positive metric at every real place.

Proof. In $\bar{V}$ take $P = (\bar{A}_j, \overline{\gamma A_j})$, $Z = (\bar{z}, \overline{\gamma z})$, and write H for the Hermitian matrix. The lattice K_γ^0 identifies with $\bar{\Lambda} \cap \ker(P^* H)$. Indeed any integral lift of such a class is already orthogonal to both arms. The positive metric on this kernel is the restriction of $H + N^{-1}(Z^*H)^*(Z^*H)$, as is seen by substituting the two expressions for t. The bordered determinant with border P^*H therefore has absolute value $|\det H|\,|d_*|^2/N^2$ at each embedding. Explicitly, adjoin the two rows Z^*H with diagonal block $-NI_2$. Eliminating that block contributes N^2. Eliminating

H instead gives $\det H$ times, up to sign, the Gram matrix of the two subframes with both terminal diagonal entries changed to zero. The latter determinant has absolute value $|d_*|^2$.

The image of P^*H on $\bar{\Lambda}$ has index $t_\gamma i_\gamma i'_\gamma$ in R^{2j}. By self-duality, local elementary divisors of the evaluation map are those of the columns of P at the conjugate prime. Their product of absolute norms is the index of PR^{2j} in its saturation. Saturating the sum of the two full arm lattices costs precisely t_γ, since U_R lies in both arms and a lift of a class in their rational sum belongs to $L+\gamma L$. Inside the sum the projected arms are disjoint. Their separate indices therefore multiply to $i_\gamma i'_\gamma$. Apply the basis-and-image calculation of Lemma 108.69 to this border. The ambient determinant ideal times the absolute norm of the ambient Gram determinant is one by unimodularity. This proves the first identity, including the empty border when $j = 0$.

In the class lattice the top isotropic coordinates are an integral direct summand, so for an integral representative the component l belongs to Λ_L. Assign its class modulo $M\Lambda_L + U_R$ to the element of K^0_γ. This is independent of the representative and has kernel K^+_γ. The target is the reduction modulo M of the rank j projective module Λ_L/U_R. Its order is M^{2jf}, by local freeness. Finally the functionals have values $-Nu, -Nu'$. Each is N times the last coordinate of a unitary identification with D, so has squared norm N. □

For an open term write $j = n = a + 1$, $d_* = D_n$, $i_\gamma = i'_\gamma = 1$. For a term of any cell rank put

$$G_\gamma = M^{2\min(j,a)f} \max\left\{ \frac{|\mathrm{N}_{E/\mathbb{Q}} d_*|}{N^f i_\gamma i'_\gamma}, \mathrm{N}_{F/\mathbb{Q}} \Delta_\gamma \right\}.$$

The second quantity in the maximum vanishes at every lower rank.

Theorem 108.74 (The common lattice at the two split levels)**.** *Take two integral terms γ, η between the same class lattices, with the same residue at both levels, each satisfying the rational frame conclusion of Theorem 108.46. Set $I = \ker(J_\gamma - J_\eta)$, $c^+ = \operatorname{codim}_D I$ and*

$$I_X = \{ y \in I : y \in X_0,\ J_\gamma y \in X_0 \}, \qquad L_{12} = I_X \cap L_\gamma \cap L_\eta.$$

Then L_{12} has full rank in I_X, and

$$d(L_{12})^2 \le (2/M)^{2c^+ f} G_\gamma G_\eta.$$

Under finite character agreement, if $c^+ = 0$ and one term is open, the left side can be multiplied by $z(s)^2$, the square of the product of the norms of the meeting places. In all cases

$$k - \dim_E I_X \ \le\ c^+ + \min\left\{ \mathbf{1}_{\Delta_\gamma \neq 0}, \mathbf{1}_{\Delta_\eta \neq 0} \right\}.$$

Proof. The comparison of full links in Proposition 108.70 applies with the quotient representatives just constructed. Indeed its cross pairing is $h(t_1, t_2) - h(\gamma^{-1} t_1, \eta^{-1} t_2)$. Its expression in middle coordinates is independent of the representatives, and choosing integral ones puts its values in MR by the two residue congruences. On the quotient by I this pairing is nondegenerate and has operator norm at most two. Thus for $T_i = K^+_i \cap I$ one has the same determinant inequality

$$d(T_\gamma) d(T_\eta) \le (2/M)^{2c^+ f} d(K^+_\gamma) d(K^+_\eta).$$

On I the two terms have the same terminal forms. If these forms have joint rank s, then $\dim I_X = k + 1 - c^+ - s$. On all of D their rank is one for a lower-rank term by Lemma 108.72 and two for an open term by Lemma 108.69. This proves the dimension bound. Put $T_{i,X} = T_i \cap I_X$. If either term has lower rank, then $s \le 1$. At rank zero $T_{i,X} = T_i$. At rank one choose a nonzero

component on I. Its image on T_i is a nonzero integral ideal by evaluation on integral lifts. Its real Jacobian on the line transverse to the kernel is at most N at every embedding. The orthogonal projection formula for covolume therefore yields $d(T_{i,X}) \le N^f d(T_i)$. Multiplying the preceding inequality by the costs of these two cuts gives

$$d(T_{\gamma,X})d(T_{\eta,X}) \le (2/M)^{2c^+ f}\frac{G_\gamma G_\eta}{t_\gamma t_\eta}.$$

Here Proposition 108.73 supplies the factor for each lower-rank term. An open term in a mixed pair uses the first estimate of Lemma 108.69, so its minor term in the maximum suffices. If both terms are open, the same inequality, allowing $s = 2$, was proved in Theorem 108.71 before the final passage to the common integral lattice.

It remains to justify that passage when the lifts are nonunique. For $x \in T_{\gamma,X}$, choose a level lift $t_\gamma = x + l_\gamma$, $l_\gamma \in M\Lambda_L$. Choose any rational η-lift $t_\eta = x + l_\eta$ to the same positive graph point over I_X. The two inverse images have the same positive coordinate x', with respective top components $l'_\gamma, l'_\eta \in L$. Then

$$v = \frac{(1-\eta\gamma^{-1})t_\gamma}{M} = \frac{l_\gamma - l_\eta}{M} - \eta\frac{l'_\gamma - l'_\eta}{M} \in \Lambda \cap (L + \eta L).$$

The numerator is divisible by M on Λ because the two reductions agree. Any two level lifts for γ differ by a vector in $M(\Lambda \cap L \cap \gamma L)$. Consequently the class of v modulo $\Lambda_L + \eta\Lambda'_L$ is well defined and additive.

Suppose it is zero and choose $a_0 \in \Lambda_L, b_0 \in \Lambda'_L$ with $v = a_0 + \eta b_0$. The difference $d_0 = (l_\gamma - l_\eta)/M - a_0$ lies in $L \cap \eta L$ by the displayed expression. Replacing t_η by $t_\eta + Md_0$ gives a lift with top component $l_\gamma - Ma_0 \in M\Lambda_L$. It is integral, since $x \in \Lambda \cap X_0$ for the chosen γ-lift. Conversely, if an η-level lift exists, its use in the formula makes the first summand integral. The remaining summand belongs to $\Lambda \cap \eta L = \eta\Lambda'_L$. The kernel of the class map is therefore precisely L_{12}. This proves $[T_{\gamma,X} : L_{12}] \mid t_\eta$, and interchanging the terms gives $[T_{\eta,X} : L_{12}] \mid t_\gamma$. These finite indices prove the rank assertion. Multiplication cancels the two torsion indices and gives the asserted inequality.

Finally $c^+ = 0$ makes the two maps J equal everywhere. Their joint terminal ranks coincide, so if either term is open both are open. Their positive graphs agree, and Theorem 108.68 gives the stated meeting factor. □

The theorem proves the arithmetic pair estimate used in `New/Spanning/RemainingTerms/Gain.lean` on the class lattices of the shorter construction. Summing the orbital integrals also requires the number and the analytic weights of pairs with a given common lattice.

108.27 The weighted count of lattices in the marked classes

The common lattice in Theorem 108.74 lies in $\Lambda_X = \Lambda_c \cap X_0$. Its discriminant enters the period estimate with weight $d(L)^{-k}$. We sum this weight over every lattice of a prescribed corank in the actual source classes. The prime N in this calculation is the norm of the negative terminal vector with sign reversed. It is distinct from the two primes dividing M.

Put $R = \mathcal{O}_E$, $f = [E : \mathbb{Q}]/2$, and $\rho_E = |\operatorname{disc} E|^{1/(2f)}$. We first work in an arbitrary positive Hermitian unimodular lattice $\mathcal{U}$ of rank k over R. For an E-subspace V put $d_V = d(\mathcal{U} \cap V)$, with the normalized covolume of Section 108.24.

Lemma 108.75 (The sum over rational subspaces). *For $0 \le \ell \le k$ and $k \ge 1$,*

$$\sum_{\operatorname{codim}_E V = \ell} d_V^{-k-1} \le (25fk\rho_E)^{f(k+1)\ell}.$$

Proof. For a lattice P in its rational span, write $P^{\#}$ for its Hermitian dual there. The determinant-ideal formula for normalized covolume gives $d(P^{\#}) = 1/d(P)$. Orthogonal projection of $\mathcal{U}$ onto $V^{\perp}$ has dual $\mathcal{U}^{\#} \cap V^{\perp} = \mathcal{U} \cap V^{\perp}$. Indeed pairing a vector of $V^{\perp}$ with a projected vector gives its original pairing. The projection is rational and its image is a lattice, so taking its double dual recovers it. Multiplicativity of covolumes on the kernel and the orthogonal projection now gives $d_V = d_{V^{\perp}}$. Also $d_V \geq 1$, because the integral lattice in V is contained in its dual. For $\ell = 0$ the asserted sum is one.

Use the Euclidean norm $\|y\|^2 = \sum_{v|\infty} h_v(y,y)$ on all the complex embeddings in the CM type. Put $P = \mathcal{U} \cap V^{\perp}$ and $m = 2f\ell > 0$. Its real covolume is $(2^{-f}\rho_E^f)^{\ell} d_V$. Let $\lambda_1, \dots, \lambda_m$ be its Euclidean successive minima. We record the ball case of Minkowski's inequality: $\omega_m \prod_i \lambda_i \leq 2^m \operatorname{covol}(P)$, where ω_m is the unit ball volume. Choose successively a shortest lattice vector outside the real span of the ones already chosen, and orthonormalize their flag. The lengths at these choices are the successive minima, since i independent vectors of length at most λ_i cannot lie in a space of dimension $i-1$. In these orthonormal coordinates dilate axis i of the unit ball by λ_i. A nonzero lattice vector whose last nonzero coordinate is i has length at least λ_i, while a vector in the open ellipsoid with that last coordinate has length strictly less than λ_i. The half-sized open ellipsoid thus has disjoint translates by P. Comparing volume with a fundamental domain proves the inequality.[1]

Next choose $x_1, \dots, x_\ell$ in P by successively minimizing length outside their previous E-span. The real span at all embeddings of an E-space of dimension $i-1$ has dimension $2f(i-1)$. Hence $\|x_i\| \leq \lambda_{2f(i-1)+1}$. The unit ball contains a cube of side $2/\sqrt{m}$. Multiplying the minima in groups of $2f$ therefore gives

$$\prod_{i=1}^{\ell} \|x_i\|^{2f} \leq m^{m/2} (2^{-f}\rho_E^f)^{\ell} d_V = (f\ell\rho_E)^{f\ell} d_V.$$

The tuple determines V by taking its orthogonal complement.

For nonzero $y \in \mathcal{U}$, the absolute norm to $\mathbb{Q}$ from F of the totally positive algebraic integer $h(y,y)$ is at least one. The arithmetic-geometric mean inequality gives $\|y\|^2 \geq f$. Distinct lattice points are thus at least one apart. In dimension $D = 2fk$ the number in a ball of radius $t \geq 1$ is at most $(2t+1)^D$, by packing balls of radius $1/2$. For $p = 2f(k+1)$, integrate pt^{-p-1} against the count of nonzero vectors of norm at most t. This gives

$$\sum_{0 \neq y \in \mathcal{U}} \|y\|^{-p} \leq 3^{2fk}(k+1) \leq 25^{f(k+1)}.$$

Choose one of the preceding tuples for each V. Raising the product inequality to $k+1$ bounds d_V^{-k-1} by $(f\ell\rho_E)^{f(k+1)\ell} \prod_i \|x_i\|^{-p}$. Sum over all tuples of nonzero lattice vectors and use $\ell \leq k$. □

At a node in the shorter construction $N \equiv 1 \bmod q_1 d$, so N splits completely in $E \subseteq \mathbb{Q}(\mu_{q_1 d})$. At its f pairs of split places the distinguished middle coordinate imposes one row congruence at one member of each pair. More explicitly, in the middle basis $x_*/\bar{\pi} + z/N, z$ of Section 108.24, a pure X_0-vector with distinguished coordinate v_* has those two coefficients $\bar{\pi} v_*$ and $-v_*/\pi$, where $\pi\bar{\pi} = N$. Their integrality is equivalent to $v_* \in \pi R$ locally. The other pure coordinates are integral. For a class of a definite isometry g, this condition is read in g^{-1} coordinates.

Proposition 108.76 (The average over the terminal rows)**.** *Let $\mathcal{C}$ be the finite set of the definite classes at the source level. Forget the level of a class and retain its underlying unimodular*

[1] For the successive-minima inequality for general symmetric convex bodies, see R. P. Bambah, A. Woods and H. Zassenhaus, Journal of the Australian Mathematical Society **5** (1965), 453–462.

lattice $\mathcal{U}$, up to isometry. To average an isometry-invariant sum on the classes over this $\mathcal{U}$, one can average uniformly over $(N^k-1)^f$ tuples of nonzero residue covectors when the summand depends only on their common kernel in $\mathcal{U}$. This common kernel is the marked lattice Λ_X. For a fixed rank $r=k-\ell$ sublattice L of $\mathcal{U}$, put $P=\mathcal{U}\cap EL$, $i=[P:L]$. The fraction of row tuples for which L belongs to the kernel is at most

$$\frac{iN^{f\ell}}{(N^k-1)^f}.$$

Proof. Fix a finite adelic representative of the coarse class and let K be its standard integral level in lattice coordinates. The finer classes over it are the orbits on K/K_{src} under the finite automorphism group of $\mathcal{U}$ in these coordinates. This follows by writing two representatives as gk, gk' in the double quotient and cancelling g. Finiteness of the automorphism group follows from discreteness on a real Euclidean lattice and compactness of its isometry group. It acts freely on these cosets. In fact stabilizing a finer coset gives a finite-order integral matrix congruent to the identity at the odd split prime p_0, where the source level is principal. The torsion calculation in Theorem 108.42 applies: if $1+p^aB$ has a nonzero first residue B mod p, prime-to-p powers preserve its depth and the p-th power increases the depth by exactly one. Such a matrix has infinite order. Thus every orbit has the same size and a uniform average over finer classes equals the coset average for an invariant sum.

The image of K in the split component at each chosen place over N is $\mathrm{GL}_k(\mathbb{F}_N)$: lift an invertible residue matrix integrally and use its inverse transpose in the conjugate component. These choices are independent at different places. Under $k\in K$ the distinguished row becomes the distinguished row of k^{-1} in the fixed coarse coordinates. The subgroup K_{src} fixes this row pointwise modulo the chosen prime. Indeed its embedded elements preserve the middle lattice and fix z. For an integral X_0-matrix A, the coefficient along z in the image of a middle basis vector x_j with $j\neq *$ is $-A_{*j}/\pi$. In the image of $x_*/\bar{\pi}+z/N$ the coefficient is $(1-A_{**})/N$. Hence on the chosen component the required row is the original unit row modulo π. Preservation at the conjugate place is compatible with the inverse transpose. The coordinate description before the proposition identifies the pure lattice with the row kernel. Transitivity on nonzero rows and equal fiber sizes now give the uniform row average.

At a chosen place choose a local basis of $\mathcal{U}$ beginning with a basis of the saturation P. By local elementary divisors take L to be generated there by $\varpi^{a_j}e_j$, $1\le j\le r$. A residue covector annihilating L must have zero j-th entry if $a_j=0$. Allowing also the zero covector bounds its number by $N^{\ell+\#\{j:a_j>0\}}$. The product of $N^{\#\{j:a_j>0\}}$ over the chosen places is at most the global index i, by $\#\{j:a_j>0\}\le\sum_j a_j$. Division by the total number of rows proves the bound. □

Theorem 108.77 (The corank sum on the period classes)**.** *Write $n=a+1$, $H=b^{n^3f}$, $\tau=b^{kf/4}$. At every sufficiently large node of the shorter construction on $V_0\le x\le V_0^{3/2}$, and for every $0\le\ell\le k$,*

$$\sum_{c\in\mathcal{C}}\sum_{\substack{L\subseteq\Lambda_c\cap X_0\\ \mathrm{rank}_R\,L=k-\ell,\ d(L)\le H}} d(L)^{-k}\le\frac{|\mathcal{C}|}{N^{kf}}\tau^{\ell+1}.$$

Here L ranges over all R-lattices of the indicated rank in their rational spans, with the zero lattice of discriminant one at rank zero.

Proof. First fix a coarse lattice and average as in Proposition 108.76. Group the lattices L by $V=EL$, and write $d=d_V$ and $i=[\mathcal{U}\cap V:L]$. Then $d(L)=di$, $d,i\ge1$. Put $\mu=2a+4$,

$r = k - \ell$. When $di \le H$ one has

$$i(di)^{-k} \le H^{\mu+1} d^{-k-1} i^{-r-\mu}.$$

Indeed after division the right side is at least $d^{\mu} i^{\ell}$ times the left side.

For a rank r projective R-lattice P the sum of $[P : L]^{-s}$ over its full-rank sublattices equals $\prod_{j=0}^{r-1} \zeta_E(s-j)$ for $s > r$. Locally at a prime of norm q, project onto the last coordinate and fix the intersection with the first $r-1$ coordinates. If that intersection has index t and the last image is $\varpi^m R_w$, there are t choices for the first coordinates of a lift of ϖ^m, and the total index is tq^m. The local series is thus $(1-q^{-s})^{-1}$ times the rank $r-1$ series at $s-1$. Every finite index sublattice is determined by these choices at its finitely many exceptional primes. Conversely their intersection inside P realizes the choices, by the Chinese remainder theorem. This proves the formula including nonfree P. At $s = r+\mu$ it is at most 2^r for our fields. Indeed $\zeta_E(t) \le \zeta(t)^{2f}$ by the Euler factors, and $\log \zeta(t) \le 2^{-t}(1 + 2/(t-1))$ for $t > 1$, by summing the tail by integration. For $t = \mu + 1 = 2a + 5$ and $(a+1)2f \le 2^{2a}$ these inequalities give $\log \zeta_E(t) < \log 2$.

Combining these bounds with Lemma 108.75 and $(N^k - 1)^{-f} \le 2^f N^{-kf}$ bounds the conditional mean over each coarse class by

$$N^{-kf} 2^{f+k} H^{2a+5} \{ N^f (25 f k \rho_E)^{f(k+1)} \}^{\ell}.$$

All terms summed are nonnegative, so finite partial sums justify the changes of order. The bound holds with the same constant for every coarse class, hence for the unweighted mean on $\mathcal{C}$.

We verify the sizes on the indicated interval. Write $m = 2n$ for the number of lifts and r_0 for the target rank. Here $\sqrt{x} \le m \le 2\sqrt{x}$, $x^5 \le r_0 \le 3x^5$, $k = r_0 + 1 - m$, and $2^s = m^{22}$. The cyclotomic field $\mathbb{Q}(\mu_{2^s}) \subseteq E$ gives $f \ge 2^{s-2}$. The inclusion $E \subseteq \mathbb{Q}(\mu_{q_1 d})$ and $q_1 d \ll_{q_0, d} x^{11} \log(2x)$ also give $2f, \rho_E \le q_1 d$. For the latter inequality the cyclotomic discriminant formula bounds its root discriminant by its conductor, and the relative discriminant formula can only lower the root discriminant in a subfield. Consequently $k \le f$, $(a+1)2f \le 2^{2a}$, and $k \ge 1000(2n)^9$ for large x. The construction imposes $x^{10^6} \le x_1$, $x_1^{10^6} \le b$ and $\log N \le 30 m \log b$. Thus $(200 f k \rho_E)^{1000} \le b$, $N \le b^{60n}$ and $b \ge 1000$ throughout the interval for large V_0. We obtain $2^{f+k} \le b^f$ and

$$1 + n^3(2a+5) \le k/4, \qquad 60n + (k+1)/1000 \le k/4.$$

The first inequality absorbs the factor $2^{f+k} H^{2a+5}$ in τ. The second absorbs the factor inside braces in τ. Multiplication by $|\mathcal{C}|$ proves the claim. □

The sum includes every common lattice of the preceding theorem. To pass from lattices to orbital pairs one must also count the distinct rational terms over a fixed lattice and residue label. Multiple such terms can have the same lattice.

108.28 Rational frames over a lattice and the weighted pair count

We count the multiplicity with which a lattice in Theorem 108.77 can occur. Fix the first class c and write $\Lambda = \Lambda_c$. For the image of the frame under a term from c' write its blocks in the original rational coordinates as

$$\gamma \mathcal{F} = \begin{pmatrix} A_\gamma \\ X_\gamma \\ v_\gamma \\ B_\gamma \end{pmatrix}.$$

The four blocks have rows along $L, X_0, Ez, L^\vee$, respectively. The entries of v_γ are the scalar coefficients along z. We order the last block by the duals of the basis of the first block. The cross products $C = \mathcal{F}^* h \gamma \mathcal{F}$ specify both $B_\gamma = C_{[1,a],*}$ and $v_\gamma = -C_{n,*}/N$. Each column of the frame is integral in Λ, and its top and bottom coordinates are integral. Two frames with the same C have their positive blocks differing by columns of Λ_X, since the terminal and bottom parts cancel and the top part can be subtracted integrally.

Let B_{res} be the image of the integral finite Bessel group in the product of the two level groups. Choose one representative for each coset sB_{res} and write a residue $s = s_0 t$ with $t \in B_{\mathrm{res}}$; call t its label. A term is counted once per double coset under the two rational pointwise unipotent frame fixers of principal level. We count cosets admitting a near representative. At both y and $\gamma^{-1} y$, near means that the points lie in the reduced frame domain and the Bessel length at every real place is at most b^{6n+2}. We take the rank j of a surviving term as in Theorem 108.46, with $j = n$ in the open case. Write $\delta = D_j \neq 0$ and form the n-by-n matrix Y from the first j rows of C followed by the remaining rows of the identity. Then $\det Y = \delta$.

Lemma 108.78 (Bounded invariants of near frames)**.** *At sufficiently large nodes of the interval of Theorem 108.77 there are at most $b^{270 n^3 f}$ possible matrices C for near integral terms. For fixed j, C, at most $b^{502 n^4 f}$ arrays occur as the top coordinates of the columns $\gamma \mathcal{F} \operatorname{adj}(Y) e_i$, $j < i \leq n$. These columns belong to T, and are unchanged by every pointwise frame fixer on the left. Moreover the gauge of Theorem 108.74 satisfies $G_\gamma \leq H = b^{n^3 f}$.*

Proof. Use the normalized frame at a real place, with scale factors $|s_h| = \kappa_v^{n-h}/\sqrt{N}$ from Corollary 108.53. We have $1 \leq \kappa_v \leq b^{23}$ for all these places. Indeed the chosen different generator, up to a factor of absolute value one, is $\sqrt{-d}\, \Phi_{q_1}'(\zeta_{q_1})$, where here $\Phi_j(T)$ denotes a cyclotomic polynomial. If $u = \operatorname{rad}(q_1)$, then u is fixed and $\Phi_{q_1}(T) = \Phi_u(T^{q_1/u})$. This identity follows by comparing the orders of the roots and the degrees. The derivative at a primitive root therefore has magnitude $(q_1/u)|\Phi_u'(\xi)|$ for some primitive u-th root ξ. These finitely many derivative values are nonzero. The bounds on q_1, b, M in the preceding subsection now give $2 \leq |\Phi_v(\delta_E)| \leq b$ and the stated bounds on $\kappa_v = M|\Phi_v(\delta_E)|/(2\pi)$.

For clarity, the reduced domain is defined by taking the strict upper coefficients of the Gram–Schmidt triangular matrix in rational frame coordinates in a set of residue representatives modulo MR in $\mathbb{C}^f$. One can take all representatives to have maximum modulus at most b^{26}. Choose a rational basis from the spanning integral elements $\zeta_{q_1}^i, \sqrt{-d}\, \zeta_{q_1}^i$. A half-open parallelepiped for M times their integer span has maximum modulus at most $2 f M \max(1, \sqrt{d}) \leq b^{26}$. For each coset of MR choose the lexicographically first of its finitely many points there. This gives a measurable set by comparison among finitely many translates.

At a reduced point the normalized triangular coefficient at (i, l), $i < l$, is the rational one times s_l/s_i, of modulus ratio at most one. The diagonal coefficients are one. Let F_y denote the matrix of normalized columns $y^{-1} f_h$. For a point in the cut choose a Bessel element u_0 with $\|u_0 y\| \leq b^{7n}$, using the infimum defining the Bessel length and $n \geq 2$. Its action on the frame is upper unitriangular. Column operations of that kind preserve the successive orthogonalized columns of F_y. Their lengths are bounded by $\|(u_0 y)^{-1}\| = \|u_0 y\|$, by orthogonalization after the change of frame. Hence each column of F_y has length at most $n b^{26} b^{7n} \leq b^{37n}$. This applies to both boxed points at the same place. Their cross products are bounded by b^{74n} since the normalized Hermitian matrix has operator norm one. Removing the two scale factors costs at most $N \leq b^{60n}$, so $|\Phi_v C_{il}| \leq b^{134n}$. Integral tuples of h entries of maximum modulus R_0 at all embeddings number at most $(1 + 2R_0)^{2fh}$. In fact the product formula gives maximum distance at least one between distinct tuples, and balls of radius $1/2$ pack inside the ball of radius $R_0 + 1/2$ in the maximum norm. With $h = n^2$ this proves the first count.

For $j < n$, the kernel of evaluation against A_j on γT equals $T \cap \gamma T$. It contains that space because $A_j \subset L$, and it has dimension $n - j$ by $D_j \neq 0$. This puts each indicated column in T. Its top coordinates are integral since Y is integral. To bound them, the unitriangular matrix U of u_0 on the normalized frame satisfies $\|U\| \le b^{7n}\|F_y\| \le b^{44n+1}$. Here the normalized fixed frame consists of Euclidean orthonormal basis vectors and $u_0 y F_y$ has coefficient matrix U. Since $\det U = 1$, for all w

$$\|w\| \le b^{46n^2}\|F_y w\|.$$

Indeed $\|Uw\| \le b^{7n}\|F_y w\|$ and $\|U^{-1}\| \le \|U\|^{n-1}$. All entries of $\operatorname{adj} Y$ and $\det Y$ have modulus at most b^{135n^2}, by determinant expansion and $n \le b$. Express the shared column both in the fixed and in the second frame at y. Its second-frame coefficients before scaling are entries of $\operatorname{adj} Y$. The ratio of any two frame scales is at most b^{23n}. The preceding coefficient inequality and the column lengths at $\gamma^{-1}y$ bound every fixed rational coefficient by

$$b^{46n^2}\, n b^{37n+23n+135n^2} \le b^{250n^2}.$$

Packing applies to the at most an top entries, proving the second count. For $j = n$ the array is empty.

For the gauge, $0 \le \Phi_v(\Delta) \le |\Phi_v(D_n)|^2$. In the lower ranks Lemma 108.72 gives $|\Phi_v(d_*)| = N|\Phi_v(D_j)|$. In the open case $d_* = D_n$. All finite row indices are at least one. Since $M \le b^{22}$, the bound for either term in the gauge is $G_\gamma \le b^{(44a+60n+270n^2)f} \le b^{n^3 f}$ for sufficiently large n. □

Proposition 108.79 (Multiplicity over a fixed positive lattice)**.** *Fix c, t and a corank ℓ lattice $L_0 \subseteq \Lambda_X$, $d(L_0) \le H$. The number of near terms (c', γ) of these ranks and this label with $L_0 \subseteq L_\gamma$ is at most*

$$b^{2000n^4(\ell+1)f}.$$

Proof. First, equality of frames up to a left level move identifies terms of one label. Indeed, after the move, the relative rational isometry fixes the first frame pointwise. Write it as uh in $P = U_0 \rtimes \mathrm{U}(X_0)$ as in Proposition 108.63. The relative finite matrix after the two class changes fixes the same frame in lattice coordinates. Its residue thus lies in B_{res}; the two residues belong to the same coset $s_0 B_{\mathrm{res}}$. Equal labels imply equality of the residues. The relative finite matrix is integral and congruent to one at M. On the middle quotient of the integral lattice $(\Lambda \cap L^\perp)/\Lambda_L$ it induces the class translate of $h \oplus 1_{Ez}$, still integral and congruent to one. Extending this middle action by the identity on the two integral arms shows that the class translate of h belongs to the definite source level, the preimage of the principal level under this extension. Thus the two source classes c' coincide. With their representatives equal, h is an automorphism at that level and is the identity by Proposition 108.76. The relative rational element is consequently a pointwise unipotent level move on the right. This proves the identification in the original double cosets.

Fix j, C and the shared array of Lemma 108.78. Write $B = C_{[1,a],*}$. We give all the level moves used below. In coordinates (p, y_0, q) on the two isotropic arms and the middle block, with middle Gram H_0, one such move is

$$(p, y_0, q) \longmapsto (p - D^* H_0 y_0 - \epsilon D^* H_0 D q,\ y_0 + Dq,\ q).$$

Here $D = M(\eta_1, \dots, \eta_a)$ with columns in $M\Lambda_X$ and $\epsilon = -\zeta_3 \in R$, so that $\epsilon + \bar{\epsilon} = 1$. Substitution in the Hermitian form verifies unitarity. The move fixes L and z pointwise. It preserves Λ and is one modulo M, since any η_i pairs integrally with the middle lattice. Its inverse is integral by triangularity. We may also use the central move $p \mapsto p + Sq$ with $S = -S^*$ in $M\ \mathrm{Mat}_a(R)$.

For each term choose an R-linear lift $x \mapsto x + Mw(x)$, $x \in L_0$, orthogonal to the second frame, with $w(x) \in \Lambda_L = R^a$. Such lifts exist by the definition of L_γ; they can be chosen linearly because the surjection of the module of level lifts onto L_0 splits over the projective module L_0. Each row of w is a functional on L_0. The submodule of functionals $x \mapsto h(\eta, x)$, $\eta \in L_0$, has index $d(L_0)^2$, by Hermitian duality in its rational span. Grouping the lifts by these classes costs at most $d(L_0)^{2a}$. Within one group the first move makes w equal to a fixed representative, because it changes its rows by $-h(\eta_i, x)$. For the resulting frames,

$$X_\gamma^* h_X x = -MB^* w(x) \qquad (x \in L_0),$$

where h_X in matrix expressions denotes the positive Gram matrix. Their positive columns therefore differ from the fixed frame's positive columns by vectors of $P_0 = \Lambda_X \cap (EL_0)^\perp$, a lattice of rank ℓ. Record these differences modulo $M\delta P_0$ in each column. There are at most $|\mathrm{N}_{E/\mathbb{Q}}(M\delta)|^{\ell n}$ records.

Within one record put $V = X_\gamma - X_{\gamma'}$. All columns of VY^{-1}/M belong to Λ_X, by the adjugate formula and the congruence. If $j < n$, its columns $i > j$ vanish because the corresponding combinations of the two frames lie in T. If $j = n$, only the last column needs attention. The vector $\gamma \mathcal{F} C^{-1} e_n$ has bottom coordinates zero and z-coordinate $-1/N$. It belongs to γT, where the form is negative semidefinite at every embedding. Thus its X_0-component has norm at most $1/\sqrt{N}$ at each embedding in the positive Hermitian space. The same holds for the second term. The last column of VC^{-1} has norm at most $2/\sqrt{N} < 1$ everywhere. It is in Λ_X. A nonzero such vector has a totally positive integral norm with product at least one, so this column also vanishes. These arguments apply after the preceding moves, since the moves fix the first frame. Multiplying VY^{-1} by Y now gives $V = DB$, with the columns of D in $M\Lambda_X$: use the first $\min(a, j)$ columns and fill the others with zeros. The first type of move, with the appropriate sign, thus makes the entire positive block equal throughout this record.

Only the top coordinates can still differ. Record their residues modulo $M\delta\bar{\delta}$ entry by entry. This costs at most $|\mathrm{N}_{E/\mathbb{Q}}(M\delta\bar{\delta})|^{an}$. Let Ξ be the top difference of two frames with one record. The equal Gram matrices of the frames give $B^*\Xi = -\Xi^* B$. Moreover Ξ vanishes on $\ker B$. For $j < n$ this kernel is spanned by $Y^{-1}e_i$ for $i > j$, since B has rank j by Theorem 108.46 and the first j rows are independent. The shared top entries are the same and have been fixed through all moves. For $j = n$, B has rank a, so B^* is injective and the Gram identity gives $\Xi \ker B = 0$.

Let R_0 be Y^{-1} times the n-by-a rectangular diagonal matrix with ones only in the first $\min(a, j)$ positions. Then $BR_0B = B$: the first $\min(a, j)$ rows are preserved and span the row space. Put $P = BR_0$, $T_0 = \Xi R_0$, so $T_0 B = \Xi$, since $1 - R_0 B$ has image in $\ker B$. The matrix

$$S = T_0 - T_0^* - P^* T_0$$

is skew-Hermitian since $P^*T_0 = R_0^* B^* \Xi R_0$ is. Multiplying the Gram identity by R_0^* on the left gives $T_0^* B = -P^* \Xi$, hence $SB = \Xi$. Each entry of R_0 is integral after multiplication by δ. The congruence on Ξ therefore puts S in $M \operatorname{Mat}_a(R)$. The central move aligns the two frames, proving they represent one term. This argument also covers $j = 0$, when the shared columns are all the columns.

There are at most $n + 1$ choices of j. The two norms of moduli used above are each at most $b^{600n^2 f}$ by $|\Phi_v \delta| \le b^{135n^2}$ and $M \le b^{22}$. Multiplying the two counts in Lemma 108.78, the lift count, and the two residue counts bounds the base-b logarithm, divided by f, by

$$2 + 270n^3 + 502n^4 + 2an^3 + 600n^2(\ell n + an) \le 2000n^4(\ell + 1).$$

The first term absorbs $n + 1$. All the choices were made inside a fixed c, t, L_0, as required. $\square$

Theorem 108.80 (The sum over near pairs in agreeing residue blocks)**.** *On the interval and fields of Theorem 108.77, sum over ordered pairs of near surviving terms in the same class position (c,c') and with the same agreeing residue at both levels. At corank $\ell = k - \dim I_X$ the sum satisfies*

$$\sum_{c,c'} \sum_{\text{such pairs of corank } \ell} d(L_{12})^{-k} \le |B_{\mathrm{res}}| \, \frac{|\mathcal{C}|}{N^{kf}} \, \tau^{\ell+2}.$$

Every such pair satisfies $d(L_{12}) \le H$ and $\ell \le n^3$.

Proof. The integral positive lattice L_{12} has $d(L_{12}) \ge 1$. By Theorem 108.74 and Lemma 108.78,

$$1 \le d(L_{12}) \le H(2/M)^{c^+ f}.$$

Since $M/2 \ge b^{19}$, this gives $c^+ \le n^3/19$. The dimension assertion of Theorem 108.74 then gives $\ell \le c^+ + 1 \le n^3$. Assign to each ordered pair its first class, the common residue label and the actual lattice $L_0 = L_{12}$. Its two terms both contain this lattice. For each (c, t, L_0) their number is bounded by the square of the bound in Proposition 108.79. This remains an upper bound when the second class is required to be the same for both. Since $k \ge 1000(2n)^9$ and $\ell \le n^3$, the square is at most $b^{kf/4} = \tau$. Sum the weight $d(L_0)^{-k}$ using Theorem 108.77 and take the $|B_{\mathrm{res}}|$ labels. This proves the claim. □

The discriminant estimate and the weighted count now apply to the same pairs in the unfolded period. The absolute sizes of their orbital integrals and the contribution of the discarded cut tails still enter the second moment.

108.29 Projected frame volumes and the integral over positive shifts

Let V denote the complex Hermitian space at one infinite place of the shorter construction, in the normalized coordinates of Corollary 108.53. Its form is $h(v,w) = v^* H w$, where $H = H^* = H^{-1}$ exchanges the two isotropic arms, is 1 on the k-dimensional positive summand X_0, and is -1 on the unit terminal vector z. Put $F = (e_1, \dots, e_a, z)$, $T = \operatorname{im} F$, $L = \langle e_1, \dots, e_a \rangle$, $n = a + 1$. For an isometry y, let $\Omega = y \ker(H + 1)$. It is a negative space of dimension n. For another isometry g, put $Z = F^* H g F$. The ratio of projected frame volumes is

$$\nu_g(y) = \frac{|\det(\Omega_b^* H g F)|}{|\det(\Omega_b^* H F)|},$$

where Ω_b denotes any basis matrix of Ω. A basis change multiplies both volumes by the same positive factor. Both determinants are nonzero. Indeed $T^\perp = L \oplus X_0$ is positive semidefinite, as is $gT^\perp$, whereas Ω is negative definite.

We use the flags of Theorem 108.46. Let j be the rank of a surviving term, $i = \min(a, j)$, $p = i + 1$, $F_i = (e_1, \dots, e_i, z)$, and $B = F_i^* H g F_i$. For $j = n$ this is the full invertible Gram matrix. For $j \le a$ invertibility follows from Lemma 108.72 and the nonzero leading minor. Write X for projection on the positive middle summand.

Lemma 108.81 (A determinant bound in a negative plane)**.** *There are a matrix ω with p columns in Ω and an invertible lower triangular matrix $D = F_i^* H \omega$, depending only on y, Z, i, such that $\omega^* H \omega = -I_p$. Put $\omega' = g F_i B^{-1} D$, $V_g = X(\omega' - \omega)$ and let w be the last column of V_g. Then*

$$|\det B| \det(I_k + V_g V_g^*) \le 2^p |\det D|^2 \nu_g(y).$$

Suppose $\|y^{-1} f\| \le \rho$ for each column f of F and the Bessel length of y is at most θ. For $d = D_{pp}$ we then have

$$|\det D| \le \rho^p, \qquad \frac{|d|^2}{1 + \|w\|^2} \le \theta^{2(j-i)}.$$

Proof. The map $F^{*}H : \Omega \to \mathbb{C}^{n}$ is an isomorphism by the nondegeneracy just proved. Take the preimages ω_{0} of the columns of $F^{*}HgF_{i}B^{-1}$. Their pairings with F_{i} form the identity. Apply Gram–Schmidt in the positive metric $-h|_{\Omega}$, beginning with the last column. Each new column is a combination of that column and later ones, with a positive real diagonal coefficient. Thus $\omega = \omega_{0}D$ has the stated triangular properties and depends continuously on y for fixed Z. Continuity follows also by using the basis $y\Omega_{-}$ for a fixed basis Ω_{-} of $\ker(H+1)$ in the initial linear system.

Put $U = L \cap gL$. In the open case $U = 0$ since Z is invertible. In the lower ranks the common triangular flag gives $a - i$ independent vectors in U, one at each index from $i+1$ to a. The leading i-minor implies $\dim U \le a - i$ by pairing L with gL. The columns of ω are orthogonal to U. Indeed $F^{*}H\omega = F^{*}H\omega'$, and the vectors of $U \subset T$ pair to zero with gF_{i}. In both frames replace the omitted columns $e_{i+1}, \dots, e_{a}$ and their images by the same triangular common vectors. Each replacement has determinant of modulus one, since the two leading coefficients at every replaced index have modulus one. Extend ω to a basis of Ω. In the two pairing matrices the common columns have a zero block in the first p rows, and they have identical entries in the other rows. Expansion along these columns cancels their common determinant, which is nonzero by invertibility of the full matrices. Hence

$$\nu_{g}(y) = \frac{|\det(\omega^{*}HgF_{i})|}{|\det D|}.$$

This also covers the full frame, where the omitted block is empty.

The columns of $\omega' - \omega$ belong to $T^{\perp}$. Their Gram matrix is $V_{g}^{*}V_{g}$, because L is the radical of $T^{\perp}$. The form restricted to gT is negative semidefinite. On writing $K = -\omega^{*}H\omega'$, expansion therefore gives

$$K + K^{*} = I + V_{g}^{*}V_{g} - \omega'^{*}H\omega' \succeq I + V_{g}^{*}V_{g}.$$

If $P = (K + K^{*})/2$, then $P^{-1/2}KP^{-1/2} = I + \mathrm{i}S$ with $S = S^{*}$. Each eigenvalue of $I + \mathrm{i}S$ has modulus at least one, so $|\det K| \ge \det P$. Monotonicity of determinant on positive definite matrices follows by conjugating the larger matrix by the inverse square root of the smaller one. Also $\det(I + V_{g}^{*}V_{g}) = \det(I + V_{g}V_{g}^{*})$ by the singular values. Substituting $|\det K| = |\det D|^{2}\nu_{g}(y)/|\det B|$ proves the determinant bound.

The columns of $y^{-1}\omega$ are Euclidean orthonormal in $\ker(H+1)$. Each row of D has length at most the Euclidean length of the corresponding $y^{-1}f$, by the projection inequality. Hadamard's inequality gives the first bound on D. If $j \le a$, take a common terminal vector with coefficients $\zeta, 1$ along z, gz, respectively, where $|\zeta| = 1$. Subtract the common vectors of indices $a, \dots, i+1$, in that order, to kill the coefficients along $ge_{a}, \dots, ge_{i+1}$. This gives a vector $w_{0} \in T \cap g\,\mathrm{im}\,F_{i}$ with the same terminal coefficients. Now $F_{i}^{*}Hw_{0} = -\zeta e_{p}^{\mathrm{std}}$, so the last column of ω' is $-(d/\zeta)w_{0}$ by invertibility of B. It has squared Hermitian norm $-|d|^{2}$ and is orthogonal to the corresponding column of $\omega' - \omega$. Subtracting the latter column gives the last column of ω and hence $-1 = -|d|^{2} + \|w\|^{2}$.

If $j = n$, triangularity gives $\omega_{p} \perp L$, where ω_{p} denotes the last column. The Bessel length is one plus the infimum of $\|uy\|$ over the Bessel unipotent group, so some such u has $\|uy\| \le \theta$. As $u^{-1}z - z \in L$, Cauchy–Schwarz after applying y^{-1} gives $|d| = |h(u^{-1}z, \omega_{p})| \le \|(uy)^{-1}z\| \le \theta$. Here $\|(uy)^{-1}\| = \|uy\|$ since H is a unitary involution. This proves the second bound in every rank. □

Proposition 108.82 (The integral of the real majorant)**.** *Fix a representative g_{0}, its Z, j, and a point y in the preceding bounds. For $A \in \mathbb{C}^{k\times i}$ consider all surviving g with this Z, j and*

$$X(gF_{i}) = X(g_{0}F_{i}) + AB_{[1,i],*}.$$

There is a single nonnegative smooth function $\varphi_y(A)$ majorizing $\nu_g(y)^{-(k+i)}$ for every such g, with

$$\int_{\mathbb{C}^{k\times i}} \varphi_y(A)\,dA \le \frac{(2\rho^2)^{pi}(2^p\theta^{2(j-i)})^k}{|\det B|^{k+i}} \qquad (k\ge 7).$$

Here dA is the ordinary real Lebesgue measure in complex orthonormal coordinates. The majorant is measurable in y.

Proof. Write $D=\left(\begin{smallmatrix} D_1 & 0 \\ * & d\end{smallmatrix}\right)$. The shift in $X(\omega')$ is $AB_{[1,i],*}B^{-1}D=(AD_1,0)$. Thus $V_g=(C_1+AD_1,w)$ with C_1,w independent of A,g. Put $G=I_k+ww^*$, $s=k+i$. The lemma gives the stated domination with

$$\varphi_y(A)=\left(\frac{2^p|\det D|^2}{|\det B|}\right)^s \det\!\big(G+(C_1+AD_1)(C_1+AD_1)^*\big)^{-s}.$$

All its ingredients depend continuously on y for fixed frame products and fixed g_0. Changing variables to $W=G^{-1/2}(C_1+AD_1)$ gives an integral of the determinant factor equal to

$$|\det D_1|^{-2k}(\det G)^{-k}\pi^{ki}\prod_{t=0}^{i-1}\frac{t!}{(k+t)!}.$$

Indeed the real Jacobian for left multiplication by $G^{1/2}$ is $(\det G)^i$, and right multiplication by D_1^{-1} contributes $|\det D_1|^{-2k}$. To integrate the remaining factor, for any positive P use $\det(P+vv^*)=\det P(1+v^*P^{-1}v)$. At exponent $s'>k$ one column integrates to $\pi^k\Gamma(s'-k)/\Gamma(s')$ times $(\det P)^{1-s'}$. This follows by writing the negative power as its gamma integral and integrating the Gaussian on $\mathbb{C}^k$; nonnegativity justifies the exchange. Starting at $s'=k+i$ and repeating for all columns yields the displayed product. For $k=7$, $\pi^k/k!<1$, and the ratio decreases thereafter because $\pi/(k+1)<1$. Each $\pi^k t!/(k+t)!\le 1$. Now $\det G=1+\|w\|^2$ and $|\det D|=|\det D_1|\,|d|$. The two bounds of the lemma give the claimed integral after collecting the i-th and k-th powers. For $i=0$ the integral over the zero space is evaluation and the empty product is one. □

Theorem 108.83 (Mean over a translate of the marked column lattice)**.** *Take the fields, marked classes and interval of Theorem 108.77. At every sufficiently large node put $\rho=b^{37n}$, $\theta=b^{6n+2}$, $\tau=b^{kf/4}$. Fix a near surviving term of rank $j\ge 1$, put $e=j-i$, and set*

$$P_j=\prod_{v:E^+\hookrightarrow\mathbb{R}}|\Phi_v(D_j)|=|\mathrm{N}_{E/\mathbb{Q}}D_j|^{1/2}.$$

Choose one embedding Φ_v of E over each real place v. Use the positive columns $Q\in X^i_{0,\infty}$ in the original rational basis. A shift by MQ times the first i isotropic rows of the rational cross-product matrix induces a normalized shift $A_v=Q_vS_v$ at place v. Let $\Psi_y(Q)=\prod_v\varphi_{y_v}(Q_vS_v)$. For $\mathcal{A}=(\Lambda_c\cap X_0)^i$, any shift t, and any fundamental parallelepiped $\mathcal{P}$ for $\mathcal{A}$,

$$\frac{1}{\operatorname{vol}\mathcal{P}}\int_{\mathcal{P}}\sum_{\lambda\in\mathcal{A}}\Psi_y(Q+t+\lambda)\,dQ \;\le\; \frac{\tau}{c_E^{ki}N^{fi}}\left[\left(\frac{b^{13}}{M^2}\right)^{if}\frac{N^{ef}}{P_j}\right]^k, \qquad c_E=2^{-f}\sqrt{|\mathrm{disc}(E)|}.$$

The bound holds uniformly for the reduced points y in the Bessel length cut. The measure on each $X_{0,v}$ is Euclidean volume for its positive form.

Proof. The bounds on y_v required for the proposition hold at every place by the proof of Lemma 108.78. Write $s_{h,v}$ for the frame scale of column h, of modulus $\kappa_v^{n-h}/\sqrt{N}$. Put $r_v=$

$\prod_{h=1}^{i}\kappa_v^{n-h}$. Row h of the normalized cross products acquires a factor $\bar{s}_{h,v}$, and each column acquires its own scale. The latter scales also multiply the shifted image columns. Thus

$$S_v=\operatorname{diag}_{h\le i}(M/\bar{s}_{h,v}),\qquad |\det S_v|=\frac{(M\sqrt{N})^i}{r_v},\qquad |\det B_v|=\frac{r_v^2}{N^j}|\Phi_v(D_j)|.$$

For the last equality, the unscaled subframe has determinant of modulus $N|\Phi_v(D_j)|$ in closed rank and $|\Phi_v(D_j)|$ in open rank. The product of the squared subframe scales, including the terminal scale, is r_v^2/N^{i+1}.

The integral in the Q_v coordinates is multiplied by $|\det S_v|^{-2k}$. The factor r_v^{2k} cancels exactly and $r_v^{-2i}\le 1$, since $\kappa_v\ge 1$. By the proposition the resulting bound at v is at most

$$\left[\frac{(2\rho^2)^pN^j}{|\Phi_v(D_j)|}\right]^i\left[\frac{b^{13i}N^e}{M^{2i}|\Phi_v(D_j)|}\right]^k.$$

Indeed $2^p\theta^{2e}\le b^{13i}$. In closed rank use $i\ge 1$ and $b\ge 4$. In open rank $n=i+1$, and $2(6n+2)+(i+1)/2\le 13i$ when $i\ge 33$, as holds at these nodes. Since D_j is a nonzero algebraic integer, $P_j\ge 1$. The product of the first brackets to the power i is at most τ, since $2\rho^2\le b^{75n}$, $N\le b^{60n}$, and $(75np+60nj)i\le 135n^3\le k/4$.

Tonelli’s theorem and translation of each summand identify the integral of the periodization over $\mathcal{P}$ with the integral of Ψ_y on the whole real vector space. The translates tile it up to null sets for every t. By the covolume formula of Section 108.24 and $d(\Lambda_c\cap X_0)=N^f$, the volume of $\mathcal{P}$ is $c_E^{ki}N^{fi}$. Dividing the product of the local bounds by this volume proves the assertion. □

To pass from this translation mean to an orbital contribution one must also count the integral frames with the same positive shift and integrate their cutoffs against the Bessel functions.

108.30 The orbital Schur bound and the contribution of the closed cells

Return to the rational frame $\mathcal{F}$ and the fields and interval of Theorem 108.83. Write U for the Bessel unipotent group and U^0 for the subgroup that fixes $\mathcal{F}$ pointwise. For a class c let Γ_c be the rational elements of U acting integrally on Λ_c and as the identity modulo M. Write $\Gamma_c^0=\Gamma_c\cap U^0$. The full unipotent covolume

$$v_U=\operatorname{vol}(\Gamma_c\backslash U_\infty)$$

is independent of c. Indeed the finite lattice in U of class c is conjugate to the one at the identity by its representative in $\mathrm{U}(X_0)(\mathbb{A}_{F,f})$. The determinant of this conjugation on the independent positive coordinates of the Lie algebra, over $F=E^+$, is a power of $\mathrm{N}_{E/F}\det_E h_c=1$; the remaining graded coordinates are fixed. Exponential coordinates identify U with its Lie algebra as a variety. Thus the finite rational approximation and product measure argument in Lemma 70.23 applies with this U and this principal level. It expresses the displayed volume as the volume of the adelic unipotent quotient divided by the volume of the finite open lattice. Both volumes are positive and finite.

We describe the cutoffs used in the orbital integral. The image of Γ_c on $\mathcal{F}$ is exactly the upper unitriangular group with entries above the diagonal in MR, where $R=\mathcal{O}_E$. The congruence gives the inclusion, since the two isotropic arms are split summands. Every elementary entry between two isotropic vectors lifts by the elementary matrix on L and its inverse adjoint on the dual arm. To lift an entry from the terminal vector z to e_h, choose a vector t of the middle lattice with $h(t,z)$ any specified element of R. Indeed z is part of a basis locally in the marked

lattice, so evaluation on it maps the integral dual onto R; the middle lattice is Hermitian unimodular. Set $\xi = -\zeta_3 \in R$, so $\xi + \bar{\xi} = 1$. The map

$$v \longmapsto v + M\{e_h h(t,v) - t h(e_h,v)\} - M^2 \xi h(t,t) e_h h(e_h,v)$$

is the required element of Γ_c. In the pairing of two transformed vectors the terms linear in M cancel. The square of the first-order term contributes $M^2 h(t,t)\overline{h(e_h,v)}h(e_h,w)$, canceled by the two last terms since $\xi+\bar{\xi}=1$. Higher terms vanish by $e_h \perp t, e_h$. The map and its isometric inverse preserve the lattice and the level. Every required triangular matrix is a product of the elementary ones by elimination.

Let $R_{\mathrm{red}}(y)$ be the reduced-set indicator from Lemma 108.78. In rational frame coordinates the upper unitriangular Gram–Schmidt coefficient matrix of $y^{-1}\mathcal{F}$ changes by right multiplication by the inverse of the frame matrix of u when y is replaced by uy, $u \in \Gamma_c$. A unique matrix in the image just described puts its strict upper entries into the chosen representatives modulo MR. Reduce in increasing order of the superdiagonal, using elementary column additions. Each addition affects only its chosen entry and larger superdiagonals, and at the first superdiagonal of a difference the two choices would differ by MR. This proves uniqueness. Choose a measurable nonnegative p_c whose translates under Γ_c^0 sum to one. For example, a discrete subgroup acting on a Lie group by left multiplication has a Borel fundamental set: cover its quotient by countably many open sets admitting local sections and restrict the sections to the successive disjoint differences. Set

$$\beta_c = p_c R_{\mathrm{red}}, \qquad \beta_c^{\square} = \beta_c \, \mathbf{1}_{\{\ell(y_v) \le \theta \ \forall v\}}.$$

Summing first over Γ_c^0 in each coset gives $\sum_{u \in \Gamma_c} \beta_c(uy) = 1$. In particular the argument below applies to the class cutoffs for any choice of this partition on the pointwise fixer.

Let $W^\flat = \bigotimes_v W_v^\flat$ be the cut Bessel functions, with cut of absolute value at most one supported at length at most $\theta = b^{6n+2}$. Use the squared local quotient norm bound $\|W_v^\flat\|^2_{U_v \backslash G_v} \le d_v$, where d_v is the dimension of the value space. This is the normalization of Theorem 108.59: the uncut average has squared norm d_v when scaled there, and cutting decreases it. The quotient by U_v gives the same norm as the quotient by the Bessel group since the remaining compact group has Haar mass one and acts unitarily. The zero function also satisfies the bound. Write $d_B = \prod_v d_v$.

For a rational term γ in a fixed block (c, s, c'), let $\mathcal{O}_\gamma = \Gamma_c^0 \gamma \Gamma_{c'}^0$ as a set of rational isometries. The sum of the absolute cut amplitudes on this orbit is bounded by the nonnegative integral

$$A_\gamma = \sum_{\delta \in \mathcal{O}_\gamma} \int_{G_\infty} \beta_c(y) \beta_{c'}(\delta^{-1} y) \|W^\flat(y)\| \, \|W^\flat(\delta^{-1} y)\| \, dy.$$

The product of the two norms is exactly the Hilbert–Schmidt norm of the corresponding simple tensor. The finite residue and finite character have already been fixed in this expression.

Lemma 108.84 (Positive shifts and the multiplicity at each shift)**.** *Let $j \ge 1$ be the rank of γ, $i = \min(a, j)$, and let $H_i \subset H_{\mathrm{all}}$ be the R-span of the first i isotropic rows inside the span of all isotropic rows of $C = \mathcal{F}^* h \gamma \mathcal{F}$. Put $i_\gamma = [H_{\mathrm{all}} : H_i]$, as in Lemma 108.72, with $i_\gamma = 1$ in open rank. The orbit breaks up into at most i_γ^k sets. In each one choose a base term δ_0. The changes in the positive frame block there are $M\lambda C_{[1,i],*}$ with unique parameters $\lambda \in (\Lambda_c \cap X_0)^i$.*

Fix a reduced point y in the length cut and $\widehat{y} = uy$, $u \in U_\infty^0$. For a fixed parameter in one of these sets, the sum of $\beta_{c'}^{\square}(\delta^{-1}\widehat{y})$ over its terms is at most $\tau = b^{kf/4}$.

Proof. Right multiplication by $\Gamma_{c'}^0$ fixes the image frame. A left pointwise unipotent adds to its positive block a multiple of its bottom block $C_{[1,a],*}$. In fact the coefficient of its h-th row is

the positive component of $(u-1)e_h^\vee$, where $e_h^\vee$ is the dual isotropic basis vector. For $u \in \Gamma_c^0$ this component belongs to $M\Lambda_X$, $\Lambda_X = \Lambda_c \cap X_0$: the bottom and terminal changes are zero since the frame is fixed, and the integral top component can be subtracted. Thus all the changes belong to $M(\Lambda_X \otimes_R H_{\mathrm{all}})$ in the positive block. The submodule formed with H_i has index i_γ^k. This can be checked prime by prime, where Λ_X is free of rank k; both tensor products embed in X_0^n by extending to E, using local freeness. The first i rows are independent by the nonzero leading minor (in open rank they are rows of an invertible matrix). They are a basis of H_i and give the unique parameter relative to any base representative of the coset. All other blocks except the top are now fixed, since C fixes the terminal and bottom blocks.

For a nonzero cutoff the normalized columns $\widehat{y}_v^{-1}\delta F_v$ have lengths at most $\rho = b^{37n}$. Compare any two contributing terms of this fiber. Their rational frame difference is only in the top block and has entries $t_{hj} \in R$, since the top arm splits off integrally. Moreover $\widehat{y}_v^{-1}F_v = y_v^{-1}F_v$, and the bound in the proof of Lemma 108.78 gives $\|y_v^{-1}F_v w\| \geq b^{-46n^2}\|w\|$. A difference of normalized columns of index j' has coefficient $\Phi_v(t_{hj'})s_{j',v}/s_{h,v}$ on this fixed frame. Hence $|\Phi_v(t_{hj'})| \leq 2\rho\, b^{46n^2+23n} \leq b^{47n^2}$ at every embedding for sufficiently large n. Relative to one contributing frame, the packing bound for integral entries of Lemma 108.78 bounds the number of full frames in the fiber by $(1+2b^{47n^2})^{2fan} \leq b^{96n^4 f} \leq \tau$. An empty fiber contributes zero.

Terms with identical full frame differ on the right by an element of $\Gamma_{c'}^0$. Indeed their quotient fixes $\mathcal{F}$ and is principal on $\Lambda_{c'}$, since their residues coincide. Its action on the positive middle quotient preserves $\Lambda_{c'} \cap X_0$ and is the identity modulo M there: apply the quotient isometry to a positive vector, then subtract its top component as above. The definite automorphism group is finite, and the principal subgroup at either odd level prime is torsion free by the argument in Proposition 108.76. The positive middle action is therefore trivial, leaving exactly the pointwise unipotent fixer. The sum of the cutoffs for this full frame is bounded by the sum of the translates of $p_{c'}$ under $\Gamma_{c'}^0$, which is one. Multiplication by the frame count proves the claim. □

Theorem 108.85 (Schur's inequality on a rational orbit)**.** *For a near surviving orbit of rank $j \geq 1$ put $e = j - i$, $\mathcal{V}_i = c_E^{ki} N^{fi}$, and let P_j be as in Theorem 108.83. Then*

$$\frac{A_\gamma}{v_U d_B} \leq \frac{\tau^2}{\mathcal{V}_i}\left[\left(\frac{b^{13}}{M^2}\right)^{if} \frac{N^{ef}\sqrt{i_\gamma i'_\gamma}}{P_j}\right]^k.$$

Here i'_γ is the row index for γ^{-1}.

Proof. Write $s_0 = k + i$ and define

$$F_\gamma(y) = \sum_{\delta \in \mathcal{O}_\gamma} \beta_{c'}^{\square}(\delta^{-1}y) \prod_v \nu_{\delta_v}(y_v)^{-s_0}.$$

It is measurable, nonnegative and left Γ_c^0-invariant: left multiplication reindexes the orbit and fixes the original frame. At a reduced y in the cut, fix $u \in U_\infty^0$. For $Q \in X_{0,\infty}^i$, choose a real point $T(Q) \in U_\infty^0$ whose inverse adds $Q_v S_v$ times the first i bottom rows to the positive block at v. Such a continuous choice follows from the explicit matrix before Theorem 108.52: take $h = 1, K = 0$ and the negative of $Q_v S_v$, padded by zero columns, for B. At $uT(Q)y$ the volume ratio for δ equals the one at y for $T(Q)^{-1}u^{-1}\delta$, since $u, T(Q)$ fix the first frame. The Gram matrix is unchanged and all the common vectors of the flags retain their coefficients. In any one of the sets of Lemma 108.84, take $u^{-1}\delta_0$ as the real base frame. The difference of its positive block from that of $T(Q)^{-1}u^{-1}\delta$, with the latter minus the former, is $(Q_v + \Phi_v\lambda)S_v$ times the first i Gram rows. Indeed the shift coming from u^{-1} is common to the two frames as their bottom rows are identical.

The majorants of Proposition 108.82 can thus be taken the same throughout the fiber over λ. Lemma 108.84 bounds the remaining cutoff sum there by τ, also at $uT(Q)y$. Summing over all λ, applying Theorem 108.83, and then summing over the at most i_γ^k sets yields

$$\frac{1}{\mathcal{V}_i}\int_{\mathcal{P}} F_\gamma(uT(Q)y)\,dQ \le C_\gamma, \qquad C_\gamma = \frac{\tau^2}{\mathcal{V}_i}\left[\left(\frac{b^{13}}{M^2}\right)^{if}\frac{N^{ef}i_\gamma}{P_j}\right]^k.$$

The proof of the translation mean uses only the Gram value, the flags, and the real base frame. It therefore applies to the base $u^{-1}\delta_0$ even when u is irrational.

Integrate first on U_∞^0-orbits in G_∞ using Haar disintegration. On such an orbit the reduced indicator, the length cut and $\|W^\flat\|^2$ are constant. The remaining cutoff p_c integrates an invariant function by the quotient measure on $\Gamma_c^0\backslash U_\infty^0$. Right multiplication by each $T(Q)$ preserves this measure, since a unipotent group is unimodular. Averaging this identity over $Q\in\mathcal{P}$ and using the displayed bound therefore gives

$$\int_{G_\infty}\beta_c^{\square}(y)\|W^\flat(y)\|^2F_\gamma(y)\,dy \le C_\gamma\int_{G_\infty}\beta_c(y)\|W^\flat(y)\|^2\,dy \le C_\gamma v_U d_B.$$

For the last inequality integrate first over U_∞. The Γ_c-translates of β_c sum to one; its integral on each such orbit is v_U. The remaining quotient norm factors over the places. All the changes of order apply to nonnegative functions by Tonelli's theorem.

Apply Cauchy–Schwarz on the product of counting measure on the orbit and Haar measure on G_∞. In one squared factor insert $\prod_v \nu_{\delta_v}(y_v)^{-s_0}$, and in the other insert its reciprocal. Use the measure with density $\beta_c^{\square}(y)\beta_{c'}^{\square}(\delta^{-1}y)$; the cut on both vectors permits these boxed cutoffs in A_γ. The first squared factor is the integral just estimated. For the other change variables by $y\mapsto\delta^{-1}y$ and invert the orbit. The projected volume ratio is then inverted by its determinant definition. The inverse has Gram matrix C^*, the same P_j, j, and row index i'_γ. Thus $A_\gamma^2 \le (v_U d_B)^2 C_\gamma C_{\gamma^{-1}}$. Taking the square root proves the formula. □

Corollary 108.86 (The squared contribution of all positive closed ranks)**.** *In each agreeing residue block (c,s,c'), sum A_γ once per surviving orbit with $1\le j\le a$, and denote that sum by $A_{\mathrm{cl}}(c,s,c')$. At all sufficiently large nodes,*

$$\sum_{c,c',\,s\ \text{agreeing}}\left(\frac{A_{\mathrm{cl}}(c,s,c')}{v_U}\right)^2 \le \tau^{-40}|B_{\mathrm{res}}|\,|\mathcal{C}|\,d_B.$$

This controls also the squared norms of the signed sums on these cells, by the triangle inequality.

Proof. Only near orbits can contribute, since both frames belong to the reduced length cut wherever the integrand is positive. The common lattice and gauge of Theorem 108.74 do not change under independent level moves on either frame: left moves preserve the positive component of every lift with zero bottom component, changing its top component by an integral multiple of M; right moves have the identical property on the second lift. Thus the pair estimate applies to representatives of these orbits. Alternatively choose the near representatives when first expanding the square. For $j\le a$ we have $|\mathrm{N}_{E/\mathbb{Q}}d_*| = N^{2f}P_j^2$ and $\Delta_\gamma = 0$. The bracket of the theorem becomes

$$\left(\frac{b^{13}}{M}\right)^{if}\frac{N^{f/2}}{\sqrt{G_\gamma}}.$$

Moreover $c_E \ge 1$: the relative discriminant formula for $E/\mathbb{Q}(\mathrm{i})$ gives $|\mathrm{disc}(E)| \ge 4^f$. In particular $\mathcal{V}_i \ge 1$. For two closed terms of ranks $i, i' \ge 1$, the positive corank ℓ satisfies $\ell \le c^+$

in Theorem 108.74. Its estimate therefore gives

$$\frac{A_\gamma A_\eta}{(v_U d_B)^2} \le \tau^4\, b^{-14kf} N^{kf} (2/M)^{\ell kf} d(L_{12})^{-k},$$

using $M \ge b^{20}$. By Theorem 108.80 the sum over all ordered pairs is at most

$$|B_{\mathrm{res}}|\, |\mathcal{C}|\, d_B^2\, \tau^6 b^{-14kf} \sum_{\ell \ge 0} [\tau (2/M)^{kf}]^\ell$$

after division by v_U^2. The series is at most two for large nodes. Finally $d_B \le b^{2kf}$. To see this on the actual fields, the chosen diagram for the source at v has only columns of heights one and k; the number of single boxes is $t_v = xb$ at the designated places and zero elsewhere. Removing columns of full height removes only a determinant twist. The remaining symmetric power has a basis of degree-t_v monomials on k variables, so its dimension is at most $(t_v+1)^k \le b^{2k}$, since b exceeds every fixed power of x. Multiply over v. As $b^{-14kf} = \tau^{-56}$ and $d_B \le \tau^8$, the asserted inequality follows from $2\tau^{-42} \le \tau^{-40}$. □

The rank-zero terms, the open terms for which the discriminant exceeds the minor bound, and the removal of the Bessel length cut still enter the full second moment.

108.31 Compact rotation of the terminal vector in the open cell

In open rank the last column of the real Schur estimate lies in a complex unit ball. We integrate its contribution in the direction of the positive middle space. This integration takes place after the arithmetic unipotent average, so it requires no compact rotation of the integral lattice.

Lemma 108.87 (The angular integral of the terminal column)**.** *At one real place in open rank use the notation of Lemma 108.81. Write $d = D_{nn}$, $v_0 = gFB^{-1}e_n^{\mathrm{std}}$, $q = Xv_0$ and $x = X\omega_n/d$. Then*

$$t^2 := \|q\|^2 = \frac{\Phi_v(\Delta_\gamma)}{|\Phi_v(D_n)|^2} \le 1, \qquad s^2 := \|x\|^2 = 1 - |d|^{-2} < 1.$$

As the point y varies to hy, $h \in \mathrm{U}(X_0)$, the quantity D is fixed, x varies to hx, and the factor $|d|^2/(1+\|w\|^2)$ in the integral of the majorant becomes $(1 + t^2 - 2\Re\langle q, hx\rangle)^{-1}$. For probability Haar dh and

$$c = \tfrac{1}{2}(1 + t^2 + \sqrt{(1+t^2)^2 - 4t^2 s^2}), \qquad r_0 = ts/c,$$

its k-th power has the exact mean

$$\int_{\mathrm{U}(X_0)} (1 + t^2 - 2\Re\langle q, hx\rangle)^{-k}\, dh = \frac{c^{-k}}{1 - r_0^2} \le |d|^2.$$

Proof. In open rank $B = F^* H g F$. Thus $F^* H v_0 = e_n^{\mathrm{std}}$. The bottom components of v_0 vanish and its normalized terminal component is -1. Its squared Hermitian norm can also be computed in the gF frame. It is $-|(B^{-1})_{nn}|^2$, since only the last column of that frame has nonzero norm. The cofactor formula and the scale of the terminal vector give

$$|(B^{-1})_{nn}| = \frac{N|\Phi_v(D_a)|}{|\Phi_v(D_n)|}.$$

Equating the two norms gives the formula for t^2. The subframe is the whole frame, and $F^* H \omega_n = d e_n^{\mathrm{std}}$. The same computation using $h(\omega_n, \omega_n) = -1$ gives the formula for s^2. The last column of ω' is dv_0, hence $w = d(q - x)$. Using $1 - s^2 = |d|^{-2}$ now gives the indicated

denominator. The negative basis construction commutes with h, as h preserves both the metric and the ordered frame. This proves the transformation law for D, x.

For $st > 0$, the unit vector in the direction of hx is uniform on the sphere in $\mathbb{R}^{2k}$. Indeed the push-forward of probability Haar is the unique rotation-invariant probability under the transitive unitary action, and normalized Euclidean surface measure has this invariance. Choosing the axis in the direction of q makes the denominator equal to $c(1 + r_0^2 - 2r_0u_1)$, where u_1 is the real first coordinate on this sphere. The identities $c(1 + r_0^2) = 1 + t^2$, $cr_0 = ts$, and $c \geq 1$ follow from the quadratic formula and $t, s \leq 1$. In particular $r_0 \leq s < 1$.

For completeness, at dimension $m = 2k > 2$ the mean of $\|z-u\|^{2-m}$ over surface probability on the unit sphere is one for $\|z\| < 1$. It is a radial harmonic function there, by differentiation on compact subballs, and its value at zero is one. The radial harmonic differential equation and regularity at the origin force it to be constant. Differentiate at $z = r_0e_1$ to obtain $\int (r_0 - u_1)\|r_0e_1 - u\|^{-m}\, du = 0$. On multiplying $\|r_0e_1 - u\|^{-m}$ by $1 + r_0^2 - 2r_0u_1$ and integrating, the result is one. Consequently its mean without this multiplier is $(1 - r_0^2)^{-1}$. This proves the formula. For $st = 0$ the denominator is constant and gives the same value directly. Finally $c \geq 1$, $r_0 \leq s$ give the bound. □

Theorem 108.88 (A compact average in the orbital integral)**.** *For every near surviving orbit of rank $j \geq 1$ on the interval of Theorem 108.85 one has*

$$\frac{A_\gamma}{v_U d_B} \leq \frac{\tau^2}{\mathcal{V}_i} \left[\frac{2^{(i+1)f}}{M^{2if}} \frac{N^{(j-i)f} \sqrt{i_\gamma i'_\gamma}}{P_j} \right]^k .$$

Proof. In the exact integral in Proposition 108.82 retain the factor $R_v(y)^k = (|d|^2/(1+\|w\|^2))^k$. The factors $\pi^{ki} \prod_{t=0}^{i-1} t!/(k+t)!$ are at most one, and $|\det D|^{2i} \leq \rho^{2pi}$. After the change of coordinates in Theorem 108.83 the local bound before estimating R_v is

$$\left[\frac{(2\rho^2)^p N^j}{|\Phi_v(D_j)|} \right]^i \left[\frac{2^p N^{j-i}}{M^{2i} |\Phi_v(D_j)|} \right]^k R_v(y)^k.$$

For $j \leq a$ one has $R_v = 1$ by the common terminal vector. In open rank $q = X(gFB^{-1}e_n^{\mathrm{std}})$ is independent of the orbit representative and of the shifts in the positive columns used in Lemma 108.84. In fact left multiplication by a pointwise unipotent preserves the positive component of any vector with zero bottom component, and right multiplication fixes the input frame. The matrix B is unchanged. Also D and x are constant on U_∞^0-orbits of y, by the same equivariance of the triangular basis and the zero bottom component of ω_n. Thus at a fixed y the factor $\prod_v R_v(y_v)^k$ is common to every fiber in the forward Schur integral.

To justify averaging this factor, first perform the integration over $\Gamma_c^0 \backslash U_\infty^0$ exactly as in Theorem 108.85. The right translations by $T(Q)$ give the same bound with the displayed local factors. After this step the remaining measure on $U_\infty^0 \backslash G_\infty$ is quotient Haar times

$$\mathrm{vol}(\Gamma_c^0 \backslash U_\infty^0)\, R_{\mathrm{red}}(y)\, \mathbf{1}_{\mathrm{cut}}(y)\, \|W^\flat(y)\|^2.$$

Indeed integration of $p_c(uy)$ over u gives that covolume for each y, by its partition identity. This covolume is finite by the same rational unipotent argument used for v_U. The displayed density is invariant under left $\prod_v \mathrm{U}(X_{0,v})$: the compact group fixes the frame, preserves the length, and acts unitarily on the values of the Bessel function. It preserves the quotient Haar measure since it normalizes U_∞^0 and preserves its Haar measure. For the latter assertion its modulus is a continuous homomorphism from a compact group to the positive reals, hence is

one. We may therefore replace the factor $\prod_v R_v(y_v)^k$ in the integral by its compact mean. Tonelli gives the product of the local means.

In open rank the lemma bounds each mean by $|d|^2 \le \theta^2$ on the cut. The product of the first displayed brackets raised to i, times θ^{2f}, is at most $b^{(135n^3+12n+4)f} \le \tau$ at these nodes, since $P_j \ge 1$. The same estimate applies in closed rank with the terminal factor one. The cutoff multiplicity costs another τ, the coset count gives i_γ^k, and the translation mean divides by $\mathcal{V}_i$. The integral of the remaining squared Bessel norm against β_c is at most $v_U d_B$. This proves that the forward squared factor divided by $v_U d_B$ is at most

$$\frac{\tau^2}{\mathcal{V}_i}\left[\frac{2^{pf}N^{(j-i)f}i_\gamma}{M^{2if}P_j}\right]^k.$$

Apply the same result to the inverse orbit and take the geometric mean as in Theorem 108.85. □

Corollary 108.89 (Open terms with bounded product of normal lengths)**.** *In each agreeing block sum A_γ over open surviving orbits for which the normalized real normal vectors satisfy*

$$\prod_v \|q_v\| = \frac{\sqrt{\mathrm{N}_{F/\mathbb{Q}}\Delta_\gamma}}{P_n} \le \frac{(M/b)^{af}}{N^{f/2}}.$$

Call this sum $A_*(c,s,c')$. Then

$$\sum_{c,c',\,s\ \text{agreeing}} \left(\frac{A_*(c,s,c')}{v_U}\right)^2 \le \tau^{-3a}|B_{\mathrm{res}}|\,|\mathcal{C}|\,d_B.$$

Proof. For an open orbit $i = a,\ i_\gamma = i'_\gamma = 1$. Write $G_0 = M^{2af}P_n^2/N^f$. Then the gauge in Theorem 108.74 satisfies

$$G_\gamma/G_0 = \max\{1, N^f\,\mathrm{N}_{F/\mathbb{Q}}\Delta_\gamma/P_n^2\}.$$

Our condition gives $\sqrt{G_\gamma/G_0} \le (M/b)^{af}$. By the theorem, $\mathcal{V}_a \ge 1$, and $2^{a+1} \le b^{a/4}$ for $b \ge 2^8,\ a \ge 1$, it follows that

$$\frac{A_\gamma}{v_U d_B} \le \tau^2\left[b^{-3af/4}\frac{N^{f/2}}{\sqrt{G_\gamma}}\right]^k.$$

Two open terms in the same block can have positive corank $\ell = c^+ + 1$; the estimate of Theorem 108.74 always gives $c^+ \ge \max(0, \ell - 1)$. Applying that theorem to the product of these bounds leaves the factor $(2/M)^{\max(0,\ell-1)kf}d(L_{12})^{-k}$. The factor N^{kf} cancels upon summing by Theorem 108.80. Thus the normalized second moment is at most

$$|B_{\mathrm{res}}|\,|\mathcal{C}|\,d_B^2\,\tau^{6-6a}\sum_{\ell\ge 0}\tau^\ell(2/M)^{\max(0,\ell-1)kf}.$$

The sum is $1 + \tau/(1-\tau(2/M)^{kf}) \le 3\tau$. Since $d_B \le \tau^8$ by Corollary 108.86, the result follows from $3\tau^{15-6a} \le \tau^{-3a}$. This inequality holds at all sufficiently large nodes. □

For larger products of normal lengths the compact mean still gives Theorem 108.88. The common-lattice estimate then involves the larger discriminant term in G_γ; bounding those blocks on the scale of the full period requires further information.

108.32 A stationary position at every real place

Fix a node $V_0 \le x \le V_0^{3/2}$ and a real place of the field $F = E^+$ used in Theorem 108.88. We construct a translate for the local period m of Proposition 108.57. Use normalized real coordinates as in Corollary 108.53, so the last vector z has Hermitian norm -1 and the character on the frame is $\exp(\sigma - \bar\sigma)$, where σ is the sum of its entries immediately above the diagonal. Write $a = n-1$, $r = k+2n-1$. The basis of the normalized space consists of a isotropic vectors f_j, the positive middle space X_0 of dimension k, the line of z, and the dual vectors $f_j^\vee$. It is Euclidean orthonormal. The Hermitian form has matrix H with $H^2 = 1$ and $\langle f_i, H f_j^\vee \rangle = \delta_{ij}$.

Proposition 108.90 (The compact weights at the given node)**.** *Order the coordinates by decreasing Harish-Chandra parameter. The weight operator L of the highest vector in the lowest compact type is zero on the first $k-1$ positive coordinates. On the remaining $2n$ coordinates the signs are $+,-,\dots,+,-$, and the weights are distinct integers $\mu_0 > \cdots > \mu_{2n-1}$ in $[-3B, -B]$. Here $B = \kappa + r^8$, where κ is the weight at this place. The compact value representation has weight operator $L' = \operatorname{diag}(0,\dots,0,t)$ after a permutation, where $t = xb$ at the designated a-places and $t = 0$ elsewhere. For all sufficiently large nodes, $D_0 := 3B + t \le b^2$.*

Proof. The undoubled row parameter has entries $k/2 - 1 - i$ for $0 \le i < k-1$. At $i = k-1+j$ its entry is $-\kappa - 2(j+1)r^8$, $0 \le j < 2n$, by the shifted weight formula $\kappa_j = \kappa + (2j+1)r^8$. The first block has positive sign and the last block alternates starting with a positive sign. The prescribed scales have $r = O(x^5)$, $2n \le 2\sqrt{x}$, $\log b \ge (\log x)^6$, and $\log x_1 \le 10^{-6} \log b$. The weight κ is either b or xb/x_1, with $x_1 \mid b$. Thus $2(2n)r^8 + r = o(\kappa)$, and the row is in decreasing order. Also $\kappa \le xb$, $t \le xb$, whence $D_0 \le b^2$.

For signs ϵ_i and row entries λ_i, the formula $L = \lambda + \rho_{\mathrm{n}} - \rho_{\mathrm{c}}$ recalled in Lemma 108.58 gives twice the difference between λ_i and the compact weight at position i as

$$\sum_{j>i} \epsilon_i \epsilon_j - \sum_{j<i} \epsilon_i \epsilon_j .$$

Indeed a compact root contributes with the opposite sign to a noncompact root. The alternating block has sign sum zero, so the difference on the first block is $(k - 2 - 2i)/2 = \lambda_i$. On the last block the correction has absolute value at most $(r-1)/2$. Its entries are integers since r is odd. The gap $2r^8$ gives strict decrease, the first entry is at most $-\kappa - r^8$, and the last is at least $-\kappa - 4nr^8 - (r-1)/2 \ge -3B$. Finally the diagram defining the compact Bessel value space counts columns strictly below height k with last weight zero. Its only columns at those heights have height one, with the indicated multiplicity t. Thus it is the symmetric power with the stated highest weight. □

Theorem 108.91 (A stationary flag with a uniform length bound)**.** *Write the defining space in sign coordinates and let p be a Euclidean unitary identification of its sign form with H in the normalized space. There are independent vectors $v_0, \dots, v_a$, a positive isometric embedding $I : \mathbb{C}^k \to \mathbb{C}^r$, and $g \in \mathrm{U}(H)$ such that*

$$gpv_j = f_{j+1}\ (0 \le j < a), \qquad gpv_a = z, \qquad gpI = \text{the prescribed inclusion of } X_0,$$
$$Lv_j = v_{j+1}\ (j < a), \qquad LI - IL' \text{ has image in } \mathbb{C}v_0, \qquad \inf_{s \in S} (1 + \|sg\|_{\mathrm{op}}) \le b^{5n}.$$

Here S is the subgroup acting by upper unitriangular matrices on the ordered frame $(f_1, \dots, f_a, z)$.

Proof. Set $P(T)=\prod_{j=0}^{2n-1}(T-\mu_j)$, $Q(T)=T-t$. For a polynomial A define $[A]$ to be zero on the first $k-1$ coordinates and to have tail coordinates

$$[A]_j=\frac{A(\mu_j)}{\sqrt{|P'(\mu_j)(\mu_j-t)|}}.$$

The sign of $P'(\mu_j)$ is $(-1)^j$, and $\mu_j<t$. Thus the Hermitian product of two such vectors is

$$-\sum_j \frac{\overline{A(\mu_j)}\,C(\mu_j)}{P'(\mu_j)(\mu_j-t)}.$$

Put $v_h=[T^hQ]$ for $0\le h\le 2a$ and $u=\sqrt{P(t)}[1]$. Lagrange interpolation shows that $\sum_j R(\mu_j)/P'(\mu_j)$ is the coefficient of T^{2n-1} in R whenever $\deg R\le 2n-1$. Apply this to $R=T^{h+l}Q$. For $h+l<2a$ the Hermitian product of v_h,v_l vanishes, and for $h+l=2a$ it is -1. Applying the formula to T^h gives $\langle u,v_h\rangle_\epsilon=0$ for $h\le 2a$. Partial fractions give $\sum_j[P'(\mu_j)(t-\mu_j)]^{-1}=1/P(t)$, so u has Hermitian norm one. The first $a+1$ vectors v_h are independent since a polynomial of degree at most a vanishing at all $2n$ nodes is zero.

Let I take the first $k-1$ standard vectors to themselves and the last one to u. Multiplication by L acts as multiplication by T on $[A]$. In particular $Lu=tu+\sqrt{P(t)}v_0$, which proves the weight assertions. The Gram matrices now agree on the indicated frames and middle spaces. To extend their identification to an isometry, choose vectors dual to the independent isotropic arms in the orthogonal complement of $I(\mathbb{C}^k)\oplus\mathbb{C}v_a$. This complement is nondegenerate. Subtract from the dual vectors the arms times half the Hermitian Gram matrix of the duals. Their pairings with the arms stay equal to the identity and their mutual pairings become zero. The resulting bases give g with the required exact images.

We bound the length using a representative of Sg. Since the nodes and t are separated integers, $|P'(\mu_j)|\ge 1$ and $1\le t-\mu_j\le D_0$. Thus $|(v_h)_j|\le D_0^{h+1/2}$ and $|u_j|\le D_0^n$. Put $K_0=\sqrt{r}\,D_0^{2n}$. Apply Euclidean Gram–Schmidt without rescaling to $pv_0,\dots,pv_a$ and call the resulting columns b_h. Their lengths are at most K_0. The Hermitian product of b_h with pv_{2a-h} is still -1, since all earlier columns pair to zero with it. Consequently $\|b_h\|\ge K_0^{-1}$. For $h<a$ their isotropic duals $Hb_h/\|b_h\|^2$ have lengths at most K_0. The arms and these duals are pairwise Euclidean orthogonal, by Gram–Schmidt, $H^2=1$, and isotropy of the span of the arms. Their joint span is H-stable. The terminal vector b_a lies in its Euclidean orthogonal complement, by Gram–Schmidt and Hermitian orthogonality to the arms. Its Hermitian norm is -1.

The complement has signature $(k,1)$. Write $b_a=b_+ + b_-$ in the positive and negative eigenspaces of H. A unitary identification of those eigenspaces followed by the real two-dimensional boost with diagonal entries $\|b_-\|$ and off-diagonal entries $\|b_+\|$ sends the standard negative unit vector to b_a. Choose the directions of the two components as axes; if $b_+=0$ only a unitary identification is needed. Since $\|b_-\|^2-\|b_+\|^2=1$, this is an isometry of norm at most $2K_0$, extended by the identity in the remaining positive directions. Combining it with the map of the standard arms and duals to the orthogonal columns just constructed gives an isometry g_1^{-1} of norm at most $2K_0$. Its inverse has the same norm since H is a Hermitian involution. The frames $g^{-1}(f_1,\dots,f_a,z)$ and $g_1^{-1}(f_1,\dots,f_a,z)$ differ by an upper unitriangular change of columns. Hence $g_1g^{-1}\in S$. Finally $1+2K_0\le b^{5n}$ by $D_0\le b^2$ and the stated growth of the ranks. This proves the uniform assertion. □

Corollary 108.92 (An exact trace identity for the period phase)**.** *For $s\in S$, set $Y=p^*g^{-1}sgp-1$, and $Y'=h-1$, where h is the induced action on X_0. Then*

$$\operatorname{tr}Y=\operatorname{tr}Y',\qquad \operatorname{tr}(LY)-\operatorname{tr}(L'Y')=\sigma(s)-\overline{\sigma(s)}.$$

In particular the integrand for m in Proposition 108.57, at the translate g and the indicated unit highest weight vectors, has vanishing first derivative on S at the identity.

Proof. Use the three-block Witt basis $A, W, A^\vee$, with A the arms $v_0, \dots, v_{a-1}$, $W = I(\mathbb{C}^k) \oplus \mathbb{C}v_a$, and the isotropic duals constructed in the theorem. The middle form is $H_W = \mathrm{diag}(1_k, -1)$. Both Y and its Hermitian adjoint $Y^\dagger$, which equals $p^* g^{-1} s^{-1} g p - 1$, are upper block triangular. Indeed the isometries preserve A and $A^\perp$. The action on the middle quotient fixes the terminal vector, and therefore restricts to h on its orthogonal complement. The AA blocks of $Y, Y^\dagger$ are strictly upper triangular and the middle block of Y is $\mathrm{diag}(Y', 0)$. The last diagonal block of Y is the conjugate transpose of the first block of $Y^\dagger$. This gives the ordinary trace.

For the weighted trace, L_{AA} is the shift $v_j \mapsto v_{j+1}$ up to the last arm, whose image is $v_a \in W$. The only entry of L_{WA} is this last map. Also $L_{A^\vee A} = 0$ and the compression of L to the I-part of W is L'. Thus the contractions $\mathrm{tr}(L_{AA} Y_{AA})$ and $\mathrm{tr}(L_{WA} Y_{AW})$ add to $\sigma(s)$, including the terminal simple root. Since $L = L^\dagger$, the corresponding contractions on the dual blocks are the complex conjugates of the same contractions with $Y^\dagger$. Explicitly, $L_{A^\vee A^\vee} = L_{AA}^*$, $L_{A^\vee W} = L_{WA}^* H_W$, and $Y_{WA^\vee} = H_W^{-1} ((Y^\dagger)_{AW})^*$. They give $\overline{\sigma(s^{-1})}$. All remaining contractions vanish except the middle trace $\mathrm{tr}(L'Y')$. The entries immediately above the diagonal add under multiplication of upper unitriangular matrices, so $\sigma(s^{-1}) = -\sigma(s)$.

For the derivative, the normalized matrix coefficient of a torus weight vector of weight L has derivative $Z \mapsto \mathrm{tr}(LZ)$ on the unitary Lie algebra. On the diagonal torus this follows by differentiating its character. The complex-linear extension vanishes on off-diagonal matrix units by torus conjugation, since the coefficient is invariant under that conjugation. The same argument applies to L'. Differentiating the trace identity now cancels both coefficient derivatives against the derivative of the inverse simple-root character. □

The point and the trace identity hold at every place with the same upper bound on the length. To normalize the global test vector one still needs a lower bound for the integral m; this requires controlling the coefficient away from its stationary region.

108.33 Localization and the positive contribution near the stationary point

At the translate of Theorem 108.91 write $G_s = p^* g^{-1} s g p$, $h_s = \mathrm{pr}_{X_0} s$ and $J_s = G_s \oplus h_s$. Put $Y = G_s - 1$, $Y' = h_s - 1$, and let C_s be the matrix retaining the entries of G_s between coordinates of opposite signs and zero elsewhere. Let $\varphi(s) = \sigma(s) - \overline{\sigma(s)}$, and let

$$\Theta(s) = \langle w, \pi(g^{-1} s g) w \rangle \overline{\langle e, \tau_0(h_s) e \rangle} \exp(-\varphi(s)).$$

Here τ_0 is the compact value representation and w, e are the unit highest vectors used above. The unsigned norm $\| \cdot \|_2$ on matrices in this subsection is the Hilbert–Schmidt norm. The integral of Θ on S is m in Proposition 108.57.

Lemma 108.93 (The phase in exponential coordinates). *If $\|J_s - 1\|_{\mathrm{op}} < 1$, then*

$$\mathrm{tr}(L \log G_s) - \mathrm{tr}(L' \log h_s) = \varphi(s)$$

for the power-series logarithms at the identity.

Proof. Let $D(A, A') = \mathrm{tr}(LA) - \mathrm{tr}(L'A')$. By Corollary 108.92 and additivity of φ, $D(G_s^j - 1, h_s^j - 1) = j\varphi(s)$ for each integer $j \geq 0$. For every $l \geq 2$, expand $(G_s - 1)^l, (h_s - 1)^l$ by the binomial theorem. The coefficients sum to zero, so one may replace G_s^j, h_s^j by $G_s^j - 1, h_s^j - 1$. The l-th finite difference of the linear function $j\varphi(s)$ vanishes. This gives $D(Y^l, (Y')^l) = 0$. Termwise application of D to the two convergent series proves the formula. □

Theorem 108.94 (Localization modulo the common compact subgroup)**.** *The common first $k-1$ positive coordinates define a subgroup $K_c = \mathrm{U}(k-1)$ of S acting by the same unitary matrix on those coordinates in both factors of J. The function Θ is invariant under left translation by K_c. If $\|C_s\|_2 = c \le 1/6$, there is $u \in K_c$ with*

$$\|G_{us} - 1\|_2^2 \le 13c^2, \qquad \|h_{us} - 1\|_2^2 \le 11c^2.$$

In particular K_c consists precisely of the elements of S whose first component is in the maximal compact subgroup of the sign model.

Proof. An element of $\mathrm{U}(k-1)$ on the common coordinates extended by the identity on their complement fixes every v_h and has the stated action on I, so it defines a member of S. It fixes w: the highest weight is zero on the common block, and each adjacent compact root there has highest weight difference zero. Its root $\mathrm{SU}(2)$ therefore fixes the highest vector by the weight string formula. These root groups and the diagonal torus generate the block. It fixes e in the symmetric power, and σ vanishes there since the frame is fixed pointwise. This gives the asserted invariance.

Let P_c project to the common coordinates and put $P_o = 1 - P_c$, $\nu_i = -L_{ii}$. For each row of the sign-unitary matrix G_s, preservation of the sign form gives the identity

$$2\Re Y_{ii} = 2\sum_j |(C_s)_{ij}|^2 - \sum_j |Y_{ij}|^2.$$

In fact $\sum_j \epsilon_i\epsilon_j |(G_s)_{ij}|^2 = 1$; expanding the diagonal entry $1 + Y_{ii}$ gives the identity. The same formula holds for columns. By the exact phase identity, $\Re\,\mathrm{tr}(\nu Y) = -\Re\,\mathrm{tr}(L'Y') = t(1 - \Re(h_s)_{kk}) \ge 0$. Multiply the row identities by ν_i and sum. Since $B \le \nu_i \le 3B$ off the common block and $\nu_i = 0$ on it, the row and column estimates give

$$\|P_o Y\|_2^2 \le 6c^2, \qquad \|Y P_o\|_2^2 \le 6c^2.$$

Also the unweighted row identities summed for both factors and the ordinary trace identity give $\|Y'\|_2^2 = \|Y\|_2^2 - 2c^2$.

Write $A = (G_s)_{cc}$. From the cc-block of $G_s^* \epsilon G_s = \epsilon$ we have $\|A^*A - 1\|_2 \le \|P_o Y P_c\|_2^2 \le 6c^2$, using $\|D^*\epsilon D\|_2 \le \|D\|_2^2$. Let $A = U|A|$ be its polar decomposition. The bound on c implies invertibility, and each singular value satisfies $|s_j - 1| \le |s_j^2 - 1|$. Thus $\||A| - 1\|_2 \le 6c^2$. Left translate by the member of K_c acting by U^*. The new cc-block is $|A|$ and the noncompact part has unchanged norm. The preceding row and column estimates apply at the new point. Bounding the entries outside cc by the sum of those two estimates gives a squared norm at most $12c^2 + 36c^4 \le 13c^2$. The unweighted trace identity gives the second bound. Taking $c = 0$ shows that the indicated element is the inverse of s, since $s \mapsto G_s$ is faithful. □

Theorem 108.95 (Positive mass in a neighborhood of the identity)**.** *Put $d_* = r + k$, $\beta = b^3$, $\eta = (200\beta^5)^{-1}$. With Haar measure on the actual Bessel subgroup let*

$$\Omega_\eta = \{s : \|J_s - 1\|_{\mathrm{op}} \le \eta\}, \qquad \mathcal{B}_\beta = \{s : \|J_s\|_{\mathrm{op}} \le 1 + \beta\}.$$

At every real place, for all sufficiently large nodes,

$$\Re\Theta(s) \ge \tfrac{1}{2} \quad (s \in \Omega_\eta), \qquad \int_{\Omega_\eta} \Theta(s)\, ds \ge \tfrac{1}{2}\beta^{-16 d_*^2}\,\mathrm{vol}(\mathcal{B}_\beta) > 0.$$

On the product of the f real groups the integral on $\prod_v \Omega_{\eta,v}$ is at least $2^{-f}\beta^{-16 f d_*^2}\prod_v \mathrm{vol}(\mathcal{B}_{\beta,v})$.

Proof. For $s \in \Omega_\eta$, $Z = \log G_s$ has $\|Z\|_{\mathrm{op}} \le 2\eta$ and $\|Z\|_2 \le 2\sqrt{r}\,\eta$. It is in the real Lie algebra: conjugation of the logarithm by the sign involution after taking adjoints gives the logarithm of G_s^{-1}, which equals $-Z$ on this neighborhood. In an orthonormal real Lie basis the sum of the absolute values of its coordinates is at most $r\|Z\|_2$. Lemma 108.58 at length one and the unitary path $\exp(uZ)$, $0 \le u \le 1$, give

$$|\langle w, \pi(g^{-1} s g) w\rangle - 1| \le 8r^2(\max_i |\lambda_i| + 3r + 2n^2 + 1)\eta.$$

Here the sign basis is carried to the normalized group by p, which preserves the matrix norms. On the symmetric power the compact coefficient is $(h_s)^t_{kk}$, so its difference from one has modulus at most $t\eta$, also for $t = 0$. Finally

$$|\exp(-\varphi(s)) - 1| \le |\varphi(s)| \le (3rB + t)\eta$$

by the trace identity and the fact that φ is imaginary. Each coefficient has modulus at most one. Their product differs from one by at most the sum of these three errors. The sum is less than $1/2$ for all sufficiently large nodes since $r = O(x^5)$, $n \le \sqrt{x}$, $3B + t \le b^2$, $\max_i |\lambda_i| \le 3xb$, and $\log b \ge (\log x)^6$.

We compare the two volumes by packing in complex matrices of size d_* with the operator norm. The graph $s \mapsto J_s$ is proper: its first projection is the conjugate of the inclusion of the closed subgroup S. For $R = 1 + \beta$, take a maximal η/R-separated set J_{s_i} in the image of $\mathcal{B}_\beta$. Disjoint ambient balls of radius $\eta/(2R)$ about them lie in a ball of radius $R + \eta/(2R)$. Comparing Lebesgue volumes in real dimension $2d_*^2$ bounds their number by $(1 + 2R^2/\eta)^{2d_*^2} \le \beta^{16 d_*^2}$ for $\beta \ge 10^4$. Every J_s in that image is within η/R of some J_{s_i}. Since a sign-unitary matrix and its inverse have equal operator norm, $\|J_s J_{s_i}^{-1} - 1\|_{\mathrm{op}} \le \eta$. The right translates $\Omega_\eta s_i$ cover $\mathcal{B}_\beta$. Both sets have finite volume, and each contains a neighborhood of the identity. This proves the asserted comparison of volumes.

The group S is unimodular: its unipotent radical is unimodular and its compact factor preserves Haar on the radical. The set Ω_η is inversion invariant by the sign adjoint identity for $J_s - 1$. The two unitary coefficients and the character take conjugate values at inverses, so the integral on Ω_η is real. The lower bound follows by integrating the first assertion, and the product bound follows by Fubini on the product of these compact sets. □

In the full period the complementary region can contribute with either sign. The localization theorem describes where the integrand can have small noncompact displacement; estimating its phase and decay there and beyond the compact neighborhood is needed to bound the complete integral.

108.34 Decay of the period away from the maximal compact subgroup

We estimate the part of the actual local integral on which the displacement C_s of Theorem 108.94 is large. Put $D = (r + k)^2$, $T = D \log \beta$ and $\delta_* = (T r^{1/4})^{-1}$, with $\beta = b^3$ as before. Use coordinates in decreasing order of λ. Write $\alpha = \lambda + \rho$, $\mu = \lambda + \rho - 2\rho_c$, and $K = \mathrm{U}(p) \times \mathrm{U}(q)$ for the maximal compact subgroup of the sign unitary group. For an integer weight v and an invertible matrix A put

$$m_v(A) = (\det A)^{v_{\min}} \prod_{v_{\min} < j \le v_{\max}} \Delta_{\{i : v_i \ge j\}}(A),$$

where Δ_B denotes the principal minor on B and the empty minor is one. We first fix precisely the coefficient formula used in the estimate.

Proposition 108.96 (The compact integral in matrix coordinates)**.** *Denote the highest-vector coefficient of π in sign coordinates by c_λ. Write $y = k_0 z$ by polar decomposition, $k_0 \in K$, $z > 0$, and normalize Haar on K to have mass one. Then*

$$c_\lambda(y) = \frac{1}{M_\mu} \int_K \frac{m_\mu(k_0 u)\overline{m_\mu(u)}}{m_{-\alpha}(u^* z u)}\, du, \qquad M_\mu = \int_K |m_\mu(u)|^2 du = (\dim V_\mu)^{-1}.$$

The germ at the identity of c_λ on the complex matrix group is

$$\mathcal{F}(Z) = \frac{1}{M_\mu} \int_K \frac{|m_\mu(u)|^2}{m_{-\alpha}(u^* Z u)}\, du.$$

In this germ $\mathcal{F}(b_- Z b_+) = \mathcal{F}(Z)$ for block diagonal unitriangular b_-, b_+ that are lower and upper triangular, respectively, in the chosen order.

Proof. Use the compact-Cartan discrete-series realization by cohomological induction from the Borel for the positive roots of λ. The torus character has weight $\lambda - \rho$, an integral weight here, and the good-range inequalities are $\lambda_i - \lambda_j > 0$ for $i < j$. The bottom compact type has weight μ, as in Lemma 108.58. On positive Hermitian matrices the Flensted-Jensen integral for this compact type is the average of the conjugates of its highest-line projection times the inducing character multiplied by the determinant character of the nilradical. In Cartan polar coordinates its value at $k_0 z$ acquires the left factor $\sigma_\mu(k_0)$. These are the compact-Cartan case of the bottom-layer coefficient theorem and its duality formula. We use B. Sun, *Matrix coefficients of cohomologically induced representations*, Compos. Math. 143 (2007), Theorems 2.1, 6.3 and formula (26).

Here are the characters and projections in that formula. The determinant character adds 2ρ to the inducing weight, giving α. For a positive Hermitian matrix $A = u^* Z u$, take its unique factorization $A = n_0 d n_0^*$ with n_0 upper unitriangular and d positive diagonal. It follows from successive Schur complements starting at the last entry. Products of the last entries of d are exactly the trailing principal minors of A. Since α is strictly decreasing, $m_{-\alpha}(A) = \prod_i d_i^{-\alpha_i}$. Thus the torus factor is $1/m_{-\alpha}(A)$. The compact highest coefficient is $m_\mu(u)$: in either sign block take the tensor product of the exterior highest vectors for its columns and the determinant power of its last weight. The individual coefficients are their principal minors, and the product of unit highest vectors generates the component of highest weight μ. Taking the matrix entry of the conjugated highest-line projection, and replacing the integration variable by its inverse, gives the indicated numerator. Schur orthogonality gives $M_\mu = 1/\dim V_\mu$. In the polar decomposition the positive factor and the unitary factor belong to the sign group: the involution $y \mapsto \epsilon y^{-*} \epsilon$ fixes y and preserves both parts, so uniqueness gives the assertion.

For the local formula consider the operator-valued integral before taking the highest entry. It extends holomorphically to $\|Z - 1\|_{\mathrm{op}} < 1$, since every compressed $u^* Z u$ is then invertible and the compact integral is locally uniformly holomorphic. On positive matrices the duality formula gives covariance under $Z \mapsto a Z a^*$ for $a \in K_{\mathbb{C}}$, with factors $\sigma_\mu(a), \sigma_\mu(a)^*$ on the two sides. Consequently its germ is covariant under independent left and right $K_{\mathbb{C}}$-multipliers. Indeed both sides are holomorphic in the left multiplier, the right multiplier and the argument; the conditions that the right multiplier be the adjoint of the left and that the argument be Hermitian give a totally real set of full real dimension. Uniqueness of the Taylor expansion gives the identity of germs. At a nearby real point $y = k_0 z$ this covariance gives precisely the left factor in the polar formula, so the asserted scalar germ is the one for c_λ. Upper compact unitriangular matrices fix the highest vector. Lower ones fix its dual covector on the right of that covector. The two multipliers therefore leave its matrix entry unchanged. The identity extends along connected parameter paths on which the three arguments of the covariance identity stay in the ball of holomorphy. □

Lemma 108.97 (Decay from the opposite-sign parameter gaps)**.** *Take $H_0 = r^8$ and set $A_y = (y + \epsilon y \epsilon)/2$, the block diagonal part of y. For the parameters of Proposition 108.90,*

$$|c_\lambda(y)|^2 |\det A_y|^{H_0} \le 1, \qquad |c_\lambda(y)|^4 \le (1 + \|y - A_y\|_2^2)^{-H_0}, \qquad |c_\lambda(y)| \le (2/\|y\|_{\mathrm{op}})^{H_0}.$$

Proof. First suppose $y = z > 0$. Then $\epsilon z \epsilon = z^{-1}$, $\det z = 1$, and $A_z = (z + z^{-1})/2$ is positive. Jacobi's complementary-minor identity gives $\Delta_B(z) = \Delta_{B^c}(z)$, since z^{-1} is the conjugate of z by a diagonal sign matrix. Set $S_j = \{i : \alpha_i \ge j\}$. All coordinates in $D_j = S_{j-H_0} \setminus S_j$ have the same sign whenever that set is nonempty. Indeed opposite signs have λ-gap at least H_0: consecutive tail entries are $2r^8$ apart and the first block is separated from the negative tail by $\kappa - O(r) > r^8$. Adding the decreasing coordinates of ρ increases the gaps.

For disjoint B, C the positive principal minors satisfy $\Delta_{B\cup C}(Z) \le \Delta_B(Z)\Delta_C(Z)$ by the Schur complement formula and monotonicity of positive determinants. On D_j the compressions of A_z and z coincide. Applying the inequality first to A_z and then to the complementary minor of z gives

$$\frac{\Delta_{S_{j-H_0}}(A_z)}{\Delta_{S_j}(A_z)} \le \Delta_{D_j}(z) = \Delta_{D_j^c}(z) \le \Delta_{S_j}(z)\, \Delta_{S_{j-H_0}^c}(z).$$

Multiply over all integers j. All but finitely many ratios are one. The numerator on the left shifts the denominator by H_0 positions, giving $(\det A_z)^{H_0}$. On the right, complements have the same minor, and $\det z = 1$. The product is $m_\alpha(z)^2 = m_{-\alpha}(z)^2$, again by complementary minors.

Conjugation by $u \in K$ preserves $\det A_z$. For $y = k_0 z$ it also gives $|\det A_y| = \det A_z$, since $A_y = k_0 A_z$. The preceding inequality thus bounds the denominator in Proposition 108.96 below by $|\det A_y|^{H_0/2}$ at every integration point. Cauchy–Schwarz and invariance of Haar bound the integral of the absolute numerator by M_μ. This proves the first inequality.

Put $C = y - A_y$. The diagonal blocks of $y^* \epsilon y = \epsilon$ give $A_y^* A_y = 1 + C^* C$. Hence $|\det A_y|^2 \ge 1 + \operatorname{tr}(C^* C)$ by the nonnegative eigenvalues of $C^* C$. Finally the eigenvalues of z occur in reciprocal pairs, and the largest is $\|y\|_{\mathrm{op}}$. Each contributes $(s + s^{-1})/2 \ge 1$ to $\det A_z$, with the largest and its reciprocal together contributing at least $(\|y\|_{\mathrm{op}}/2)^2$ when $\|y\|_{\mathrm{op}} > 1$. For norm one the bound is immediate. These facts give the last two estimates. □

Theorem 108.98 (Absolute integral outside the compact tube)**.** *At every real place of the same fields and at every sufficiently large node in the prescribed interval,*

$$\int_{\|C_s\|_2 \ge \delta_*} |\Theta(s)|\, ds \le \beta^{-100D} \operatorname{vol}(\mathcal{B}_\beta).$$

Proof. On this set the compact coefficient and the character have modulus at most one. The lemma gives $|\Theta| \le \exp(-H_0 \delta_*^2/8)$, since $\log(1 + \delta_*^2) \ge \delta_*^2/2$. Here $H_0 \delta_*^2 / T = r^{15/2}/T^3 \to \infty$ uniformly: $r \in [x^5, 3x^5]$, $r + k \le 2r$, and $\log \beta = 3 \log b \le 3x^{1/100}$. The uniform upper bound is thus at most e^{-300T}. By taking the geometric mean with the global bound from the lemma and unitarity, we get on that set

$$|\Theta(s)| \le e^{-150T} \min\{1, (2/\|J_s\|_{\mathrm{op}})^{H_0/2}\}.$$

The operator norm of J_s equals that of G_s, since $\|h_s\|_{\mathrm{op}} = 1 \le \|G_s\|_{\mathrm{op}}$.

The integral of this minimum is at most $2 \operatorname{vol}(\mathcal{B}_\beta)$. Indeed the portion inside $R = 1 + \beta$ costs at most that ball's volume. For $U \ge 1$, the packing argument of Theorem 108.95 with separation $1/U$ covers $\{\|J_s\|_{\mathrm{op}} \le U\}$ by at most $(1 + 2U^2)^{2D}$ translates of $\{\|J_s - 1\|_{\mathrm{op}} \le 1\}$, which is contained in $\mathcal{B}_\beta$. On the shell $2^j R < \|J_s\|_{\mathrm{op}} \le 2^{j+1} R$, $j \ge 0$, the cost divided by $\operatorname{vol}(\mathcal{B}_\beta)$ is at most

$$12^{2D} R^{4D} (2/R)^{H_0/2}\, 2^{j(4D - H_0/2)}.$$

The first factor, including the power of $2/R$, is at most $1/2$ eventually since $H_0/D \to \infty$ and $R \to \infty$. The common ratio is also at most $1/2$. Summing proves the asserted bound, using $2e^{-150T} \le \beta^{-100D}$. □

This tail is smaller than the positive integral on Ω_η in Theorem 108.95. To compare the entire period with that positive integral, it remains to bound cancellation in the tube $\|C_s\|_2 < \delta_*$. Theorem 108.94 places its integrand within $4\delta_*$ of identity coordinates after compact translation.

108.35 A lower bound for the full real Bessel period

We prove that the signed integral in the tube of Theorem 108.98 retains a fixed fraction of the mass estimated in Theorem 108.95. The estimates below have absolute constants independent of both the dimension and the number of minors in a weight. For a projection P, $\det_P A$ denotes the determinant of PAP acting on its range, and $[P, A] = PA - AP$.

Lemma 108.99 (Expansion of a compressed product of exponentials)**.** *Let P be an orthogonal projection and let $\|A_j\|_2 \le d \le 1/40$ for $1 \le j \le 3$. Set $u_0 = \max_j \|[P, A_j]\|_2$, $F_P(A, B) = \operatorname{tr}(PA(1-P)BP)$. Then the logarithm at the identity satisfies*

$$\log \det_P(e^{A_1}e^{A_2}e^{A_3}) = \operatorname{tr}(P\textstyle\sum_j A_j) + \frac{1}{2}\sum_j F_P(A_j, A_j) + \sum_{j<l} F_P(A_j, A_l) + E,$$
$$|E| \le 16000\, d\, u_0^2.$$

Proof. Introduce a scalar t in all three exponents and expand the trace of the logarithm on $P\mathbb{C}^r$. The compression differs from its identity by at most $e^{3|t|d} - 1$ in operator norm. Its logarithmic series converges for $|t| \le 1$. Expand the exponentials and the logarithm in powers of t. The sum of absolute scalar coefficients of words of total degree l before inserting projections is bounded by the coefficient of z^l in $-\log(2 - e^{3z})$. This follows by replacing the coefficients of the logarithm by their absolute values: the majorizing series for its argument minus the identity is $e^{3z} - 1$. The coefficients are nonnegative and the series has value less than one at $z = 1/10$; hence this bound is at most 10^l.

Insert $1 = P + (1 - P)$ at each unseparated junction between letters. There are at most 2^l resulting paths per word. Those staying in the range of P give exactly the trace logarithm of the product of the three compressed exponentials, namely of $e^{PA_1P}e^{PA_2P}e^{PA_3P}$ on that range. Its determinant is the exponential of the sum of their traces. Their contribution is thus precisely the linear term. Every other path crosses the boundary of P at least twice, since it begins and ends in its range. The two crossing factors have Hilbert–Schmidt norm at most u_0; all other factors have operator norm at most d. The absolute trace is bounded by $u_0^2 d^{l-2}$ by the Hilbert–Schmidt Cauchy–Schwarz inequality. In degree two the crossing terms are exactly the two sums displayed in the statement, from a square in one exponential or from two exponentials in their indicated order. The sum in degrees $l \ge 3$ is bounded by $\sum_{l \ge 3} 20^l d^{l-2} u_0^2 \le 16000 d u_0^2$. This also gives absolute convergence of the remainders. □

Proposition 108.100 (A Gaussian estimate in the shrinking neighborhood)**.** *Use the notation of Proposition 108.96. For an integer weight v put $Q_v(A) = \sum_{i,j} |v_i - v_j||A_{ij}|^2$. Suppose Z is in the sign unitary Lie algebra and $\|Z\|_2 \le 8\delta_*$. For $q = Q_\alpha(Z)/4$ and all sufficiently large nodes,*

$$\left| c_\lambda(e^Z) e^{-\operatorname{tr}(\mu Z)} - e^{-q} \right| \le \tfrac{1}{20} e^{-q} + e^{-200T}.$$

Here μ in a trace denotes the diagonal matrix of that weight. If Z' is skew-Hermitian on the compact value space of dimension k in its defining representation, with $\|Z'\|_2 \le 8\delta_*$, the

corresponding estimate is

$$\left|((e^{Z'})_{kk})^{t}e^{-tZ'_{kk}}-e^{-q'}\right|\le \tfrac{1}{20}e^{-q'}+e^{-200T},\qquad q'=\tfrac{t}{2}\sum_{j\neq k}|Z'_{kj}|^{2}.$$

Proof. Put $d=8\delta_{*}$, which tends to zero with $dT\to 0$. Let U,V be the strict lower and strict upper parts of the compact (equal-sign) blocks of Z. For $(A_1,A_2,A_3)=(-U,Z,-V)$ all norms are at most d. Write $Y=e^{-U}e^{Z}e^{-V}$. Proposition 108.96 permits using $\mathcal{F}(Y)$ for $c_\lambda(e^{Z})$. In fact all partial products with the exponents scaled between zero and one differ from the identity by at most $e^{3d}-1<1$ in operator norm, so the germ identity extends to the matrices used here.

Fix a trailing coordinate projection P in decreasing order of λ and $u\in K$. Put $R=uPu^{*}$, $s_0=\|[P,Z]\|_2$, $h_0=\|[P,u]\|_2=\|R-P\|_2$. Since taking either triangular mask can only delete entries from $[P,Z]$, one has $\|[P,A_j]\|_2\le s_0$ and $\|[R,A_j]\|_2\le s_0+2dh_0$. Apply Lemma 108.99 with projection R. The first term is $\operatorname{tr}(R\operatorname{diag}Z)$: the sum of the three arguments retains just the diagonal and the opposite-sign entries, and R is block diagonal by sign. Its difference from $\operatorname{tr}(PZ)$ is at most dh_0^2, since $\sum_i|R_{ii}-P_{ii}|=2\operatorname{tr}(P-PR)=h_0^2$.

The quadratic term may be evaluated at P with error at most $Cdh_0(s_0+dh_0)$ for an absolute C. Indeed the crossing blocks $RA_j(1-R)$ and $PA_j(1-P)$ differ in Hilbert–Schmidt norm by at most $2dh_0$, by changing one projection at a time. The reverse crossing blocks have the same bound. Their individual norms are at most s_0+2dh_0 at R and s_0 at P. Expanding each of the six trace products gives the estimate.

For $i\in P$, $j\notin P$, necessarily $i>j$. If the signs differ, only $F_P(Z,Z)/2$ contributes this pair, giving $Z_{ij}Z_{ji}/2=|Z_{ij}|^2/2$. If the signs agree, the two square terms from U,V vanish on this pair. The four remaining coefficients of $Z_{ij}Z_{ji}$ are $1/2,-1,1,-1$, in order from the square in Z and the three mixed terms. Here $Z_{ji}=-\overline{Z_{ij}}$, so the result is again $|Z_{ij}|^2/2$. Summing gives $s_0^2/4$, since Z_{ij},Z_{ji} have the same modulus. The lemma and the preceding estimates yield, with an absolute constant,

$$\log\det_{P}(u^{*}Yu)=\operatorname{tr}(PZ)+s_0^2/4+O\big(d(s_0^2+h_0^2)\big).$$

For the weight $-\alpha$ all the projections in its definition are trailing. For any weight v, summing $\|[P,A]\|_2^2$ over its level projections gives $Q_v(A)$: the number of levels separating i from j is $|v_i-v_j|$. Also the sum of the projections is $\operatorname{diag}(v)-v_{\min}1$. The determinant factor restores the constant since $\det Y=e^{\operatorname{tr}Z}$. It follows that

$$\frac{1}{m_{-\alpha}(u^{*}Yu)}=\exp\{\operatorname{tr}(\mu Z)-q+E(u)\},\qquad |E(u)|\le C_1d\{q+Q_\mu(u)+r^2\}$$

with absolute C_1. To pass from α to μ, the difference $2\rho_c$ has entries of modulus at most r. Thus the difference of the linear traces is at most r^2d and $Q_\alpha(u)\le Q_\mu(u)+2r^2$ by unitarity of u.

The weight $m(u)=|m_\mu(u)|^2$ satisfies $0\le m(u)\le e^{-Q_\mu(u)/2}$. For one of its projections, the squared compressed determinant is the determinant of $1-Pu^{*}(1-P)uP$ on the range of P. Its eigenvalues lie in $[0,1]$, so it is at most $\exp(-\|(1-P)uP\|_2^2)$. Unitarity gives $2\|(1-P)uP\|_2^2=\|[P,u]\|_2^2$. Multiply over levels, using $|\det u|=1$. By the compact Weyl dimension formula $M_\mu^{-1}=\dim V_\mu\le(3B+r)^{r(r-1)/2}\le e^{T}$: every factor is the ratio of a highest-weight difference plus a root height to that height, at least one in the denominator, and all differences are at most $3B$.

After removing $e^{\operatorname{tr}(\mu Z)}$ the error from e^{-q} is consequently bounded by the integral of $m(u)e^{-q}|e^{E(u)}-1|/M_\mu$. Where $q+Q_\mu(u)\le 1000T$, the bound on E is $o(1)$ uniformly since

$r^2 \le T$ and $dT \to 0$. This portion costs at most $e^{-q}/20$ eventually. On the complement $|E| \le (q+Q_\mu(u))/4$ eventually. The unnormalized integrand is at most $2\exp(-(q+Q_\mu(u))/4)$. Integration against probability Haar and division by M_μ costs at most $2e^{-249T} \le e^{-200T}$. This proves the first assertion.

For the compact value coefficient take the last coordinate projection and the single exponential $e^{Z'}$ in Lemma 108.99. Its linear term, after multiplication by t, is tZ'_{kk}, and its quadratic term is $-q'$ by skew-adjointness. The error has modulus at most $C_2 dq'$, because $\|[P,Z']\|_2^2 = 2\sum_{j\neq k}|Z'_{kj}|^2$. If $q' \le 1000T$ the error in the exponential is $o(1)$ relative to $e^{-q'}$. Otherwise its absolute contribution is at most $2e^{-q'/2} \le e^{-200T}$. This includes $t=0$, when the coefficient is constantly one. □

Theorem 108.101 (Uniform lower bound for the complete real period)**.** *At the translate g of Theorem 108.91 and at each real place of the same fields, for all sufficiently large nodes $V_0 \le x \le V_0^{3/2}$, the complete integral of the Bessel coefficient satisfies*

$$m = \int_S \Theta(s)\,ds = d_\pi \|W_w\|^2_{L^2(S\backslash G)} \;\ge\; \tfrac{1}{4}\beta^{-16D}\,\mathrm{vol}(\mathcal{B}_\beta) > 0.$$

In particular the product of these full local periods is at least $4^{-f}\beta^{-16fD}\prod_v \mathrm{vol}(\mathcal{B}_{\beta,v})$.

Proof. If $\|C_s\|_2 < \delta_*$, Theorem 108.94 gives $u \in K_c$ with the same integrand and both components of $J_{us}-1$ of Hilbert–Schmidt norm at most $4\delta_*$. Their logarithms Z, Z' have Hilbert–Schmidt norms at most $8\delta_*$ by the logarithmic series and submultiplicativity. Their compact weight traces are imaginary. By Lemma 108.93 the two linear phases in Proposition 108.100 cancel the character in Θ. Its value is therefore the product of the first normalized coefficient in that proposition with the conjugate of the second.

Put $A = e^{-q}, A' = e^{-q'}$, $e_0 = e^{-200T}$. If the two factors are within $A/20+e_0$ and $A'/20+e_0$ of A, A', expanding their product bounds its distance to AA' by $(2/20+1/400)AA'+4e_0$, since $A, A', e_0 \le 1$. In particular $\Re\Theta(s) \ge -4e_0$ throughout the tube. The tube lies in $\mathcal{B}_\beta$: the compact translation is unitary in both factors, so it preserves $\|J_s\|_{\mathrm{op}}$, and $\|J_{us}\|_{\mathrm{op}} \le 1+4\delta_*$.

The integral is real and equals the squared-norm expression by Proposition 108.57. Integrate its real part first on Ω_η using Theorem 108.95. On the rest of the tube the negative contribution is at most $4e^{-200T}\,\mathrm{vol}(\mathcal{B}_\beta)$. The remaining set contributes at worst the negative of the absolute integral in Theorem 108.98. Consequently

$$\frac{m}{\mathrm{vol}(\mathcal{B}_\beta)} \ge \tfrac{1}{2}e^{-16T} - 4e^{-200T} - e^{-100T} \ge \tfrac{1}{4}e^{-16T}$$

eventually. Every constant above is independent of the place, and the uniform growth bounds used are those of the given nodes. Multiplication gives the final assertion. □

The real test vector can now be multiplied by a positive scalar to give it squared quotient norm equal to the dimension of its value space. Passage to an integral projected theta period still requires the global comparison.

108.36 Normalized test vectors and their truncation

Continue with the real translates constructed in Theorem 108.91. Haar on the compact factor of $S = U \rtimes \mathrm{U}(k)$ is probability Haar, so an S-invariant scalar function has the same integral on $U\backslash G$ and $S\backslash G$. All vector norms below descend in this way.

Lemma 108.102 (Integrals of an operator norm bound)**.** *At one real place let g be its stationary translate and let $\phi(t) = (2/\|t\|_{\mathrm{op}})^{9r^2}$ on G. Write $V = \mathrm{vol}(\mathcal{B}_\beta)$ for the S-volume in Theorem 108.101. Uniformly for all $y \in G$, at all sufficiently large nodes,*

$$\int_S \phi(g^{-1}sy)\,ds \le b^{r^2}V, \qquad d_\pi \int_G \phi(t)\,dt \le b^{38r^2}.$$

Proof. Put $P = 9r^2$ and $R \ge 1$. In the part of the matrix coset $g^{-1}Sy$ of operator norm at most R, take a maximal $1/R$-separated set. As in Theorem 108.95 its cardinality is at most $(1+2R^2)^{2r^2}$. If $A = g^{-1}sy$ is within $1/R$ of $A_i = g^{-1}s_iy$ in this set, then

$$\|g^{-1}ss_i^{-1}g - 1\| = \|(A - A_i)A_i^{-1}\| \le 1.$$

Here and below in this proof undecorated matrix norms are operator norms. The corresponding ss_i^{-1} belongs to $\mathcal{B}_\beta$, since its first component has operator norm at most two and its second is unitary. Thus the S-volume with $\|g^{-1}sy\| \le R$ is at most $(1+2R^2)^{2r^2}V$ by right invariance. For $2^j \le \|g^{-1}sy\| < 2^{j+1}$, $j \ge 0$, multiply the volume bound by 2^{P-jP}. The sum is at most

$$C_rV, \qquad C_r = 2^{P+1}12^{2r^2} \le b^{r^2}$$

eventually, since the common ratio is 2^{4r^2-P}. This is uniform in y.

Taking the whole group in the packing argument gives $\int_G \phi \le C_r\,\mathrm{vol}_G\{\|t-1\| \le 1\}$: the relative points AA_i^{-1} now lie in that set. Cover this last set, of norm at most two, by right translates of the ball $\{\|t-1\| \le \eta\}$ using separation $\eta/2$. The number required is at most $(1+8/\eta)^{2r^2} \le \beta^{12r^2}$, since $\eta = (200\beta^5)^{-1}$ and $\beta \to \infty$. The coefficient derivative estimate in the proof of Theorem 108.95 applies to every t on this small ball. In particular $|c_\lambda(t)| \ge 1/2$, with the coefficient transported by the sign frame. By Schur orthogonality $d_\pi \int_G |c_\lambda(t)|^2\,dt = 1$. It follows that d_π times the small ball's volume is at most four. The claim follows by $C_r \le b^{r^2}$ and $\beta = b^3$. □

Theorem 108.103 (A normalized local vector and its cut)**.** *At each real place put $d_v = \dim V_{\tau_v}$, $\theta = b^{6n+2}$, $j = 3r^3f$, and $l_b(y) = \max\{1, \ell(y)/b^{6n}\}$. There is a positive scalar multiple F_v of the stationary Bessel average and a continuous cut $F_v^\flat$ such that*

$$\|F_v\|_{U\backslash G}^2 = d_v, \qquad \|F_v^\flat\|_{U\backslash G}^2 \le d_v, \qquad \|F_v - F_v^\flat\|_{U\backslash G}^2 \le d_v b^{-2r^3f}.$$

They agree for $\ell \le \theta/2$, and $F_v^\flat$ vanishes for $\ell \ge \theta$. Both have the prescribed S-equivariance. There is a bounded, measurable, S-invariant nonnegative function A_v such that

$$l_b(y)^j \max\{\|F_v(y)\|, \|F_v^\flat(y)\|\} \le A_v(y), \qquad (\sup A_v)\int_{U\backslash G} A_v \le b^{400r^2}d_v.$$

The full vector satisfies the reproducing identity

$$F_v(y) = d_\pi \int_G \overline{c_\lambda(h)}F_v(yh)\,dh.$$

Proof. Write m for the full integral in Theorem 108.101. By that theorem $c = (d_v d_\pi/m)^{1/2}$ is finite and positive, and $F_v = cW_w$ has the asserted norm. Let $F_v^\flat = \zeta(\ell(y))F_v(y)$, where ζ equals one up to $\theta/2$, is linear from one to zero on $[\theta/2, \theta]$, and is zero thereafter. The length is continuous: for $t \in G$, $\ell(yt) \le \ell(y)\|t\|$ and $\ell(y) \le \ell(yt)\|t^{-1}\|$. These follow by taking the infima and using $\|t\|, \|t^{-1}\| \ge 1$. They give continuity as $t \to 1$, and the length is left S-invariant by definition. Equivariance, continuity, cut support and the cut norm bound follow.

For $h = g^{-1}sy$, the length bound on g gives $\ell(y) = \ell(gh) \le b^{5n}\|h\|$. Hence $\ell(y)/b^{6n} \le \|h\|/2$ for large nodes. The gap $H_0 = r^8$ of Lemma 108.97 exceeds $9r^2 + j$, uniformly, since $f \ll_{q_0,d} x^{11}\log(2x)$ and $r \ge x^5$. It follows that

$$l_b(y)^j |c_\lambda(h)| \le (2/\|h\|)^{9r^2}.$$

When $l_b(y) > 1$ one has $\|h\| \ge 2$ and uses coefficient decay with exponent $9r^2 + j$. When $l_b(y) = 1$ use decay if $\|h\| \ge 2$, and unitarity if $\|h\| \le 2$.

Take $A_v(y) = c\int_S \phi(g^{-1}sy)\,ds$. Integration of the last estimate, using the unitarity of the action on the values, gives the pointwise bound for F_v and its cut. Tonelli gives measurability and disintegration over the quotient gives $\int_{U\backslash G} A_v = c\int_G \phi$; right invariance on S gives its left invariance as a function of y. By Lemma 108.102 and the lower bound for m,

$$(\sup A_v)\int A_v \le c^2 b^{r^2} V \int_G \phi \le 4 d_v\, \beta^{16(r+k)^2} b^{39r^2} \le b^{400r^2} d_v.$$

Outside $\ell \le \theta/2$ we have $l_b(y) \ge b$. There $\|F_v - F_v^\flat\| \le b^{-j}A_v(y)$; inside, the difference vanishes. Integrating its square gives $d_v b^{-6r^3 f + 400r^2} \le d_v b^{-2r^3 f}$ for all sufficiently large nodes.

Finally $d_\pi \int_G \overline{c_\lambda(h)}\pi(h)\,dh$ is the orthogonal projection onto the line of w: Schur orthogonality gives the matrix entries of this operator against arbitrary vectors, and the coefficient is integrable by Proposition 108.57. The map $u \mapsto W_u$ in that proposition intertwines π with right translation and is bounded into the quotient L^2-space. Applying it to the projection identity on w proves reproduction in L^2. The convolution is continuous by boundedness of F_v, integrability of its kernel, and dominated convergence. The quotient Haar measure has full support, so the continuous representatives agree everywhere. □

Corollary 108.104 (The cut at all real places of the given fields)**.** *With $d_B = \prod_v d_v$, $F = \bigotimes_v F_v$ and $F^\flat = \bigotimes_v F_v^\flat$, one has*

$$\|F\|^2_{U_\infty\backslash G_\infty} = d_B, \qquad \|F^\flat\|^2_{U_\infty\backslash G_\infty} \le d_B, \qquad \|F - F^\flat\|_{U_\infty\backslash G_\infty} \le f b^{-r^3 f}\sqrt{d_B}.$$

These specific nonzero full local tests and their cuts can be used in Theorems 108.62 and 108.88.

Proof. Product Haar and Tonelli multiply the squared quotient norms. In the difference of the tensors replace one factor at a time. In each of the f terms the changed factor contributes at most $b^{-r^3 f}\sqrt{d_v}$ and all the others at most $\sqrt{d_{v'}}$. The triangle inequality proves the estimate. The integrability, equivariance, support, choice of coefficient and local norm required for the cited estimates are those just verified. The bounds, including the exponent containing f, hold at the nodes of Theorem 108.101. □

108.37 The truncation error after summation over rational translates

Use the full tensor F, its cut $F^\flat$, and the functions A_v from Theorem 108.103. Fix a source class c and a residue s at the two marked primes. Denote by $\mathcal{T}(c, s, c')$ the rational isometries γ for which $h_c^{-1}\gamma h_{c'}$ preserves the finite lattice and reduces to s. Throughout this subsection write

$$L(y) = \prod_{v\mid\infty} l_b(y_v), \qquad J = r^3 f, \qquad d_0 = 2nrf, \qquad \rho = b^{37n}.$$

The function β_c is the reduced partition from the proof preceding Lemma 108.84. Thus it is nonnegative, is supported on reduced frames, and its Γ_c-translates sum to one.

Lemma 108.105 (Counting all translates at a prescribed length)**.** *For $u \geq 1$, uniformly in c, s, y, at every sufficiently large node one has*

$$\sum_{c'} \sum_{\substack{\gamma \in \mathcal{T}(c,s,c') \\ L(\gamma^{-1}y) \leq u}} \beta_{c'}(\gamma^{-1}y) \;\leq\; b^{6n^2 d_0} u^{2d_0}.$$

Consequently the sum of $\beta_{c'}(\gamma^{-1}y)L(\gamma^{-1}y)^{-J}$ over all these classes and all their translates is at most $2b^{12n^3 rf}$.

Proof. Write e_j for the rational frame columns, including the terminal column $e_n = z_0$ of norm $-N$. Write $\widehat{e}_{j,v} = s_{j,v}\Phi_v(e_j)$ for their unit Euclidean normalizations in the real model of Corollary 108.53. The transformed frame we count has columns $y_v^{-1}\gamma_v \widehat{e}_{j,v}$. At $x = \gamma^{-1}y$ in the indicated support, $\ell(x_v) \leq b^{6n}u$. Choose $w_v \in S_v$ with $\|w_v x_v\| \leq 2b^{6n}u$. In the normalized model these unitary isometries of the indefinite form satisfy $\|g^{-1}\| = \|g\|$. The successive orthogonalized columns of $x_v^{-1}\widehat{e}_{j,v}$ are unchanged by the unitriangular column operations induced by w_v^{-1}. Their lengths are therefore at most $2b^{6n}u$. All strict upper coefficients recovering the original columns are bounded by b^{26} on the reduced set, as shown in Lemma 108.78. Every transformed column has length at most $2nb^{26+6n}u \leq \rho u$.

Two distinct transformed frames have distance at least $(N\rho u)^{-1}$ in the maximum of the column norms over the places. To prove this, take two translates γ_1, γ_2, possibly from different right classes, set $x = \gamma_1^{-1}y$, $\delta = \gamma_1^{-1}\gamma_2$, and assume the distance is strictly smaller. For each pair i, j the entry $h(e_i, (\delta - 1)e_j)$ belongs to $\mathcal{O}_E$. Indeed the two finite class lattices have the same frame, and δ takes the second lattice isometrically onto the first, where the form is integral. At v the normalized pairing has modulus at most

$$\|x_v^{-1}\widehat{e}_{i,v}\| \, \|x_v^{-1}(\delta_v - 1)\widehat{e}_{j,v}\| < 1/N,$$

since the defining Hermitian matrix has operator norm one. Restoring the two scale factors costs at most N. Thus the algebraic integer has absolute value less than one at every embedding, and its norm shows that it is zero.

It follows that $(\delta - 1)e_j \in T^{\perp} = L_0 \oplus X_0$, where here L_0 denotes the isotropic arm spanned by $e_1, \ldots, e_a$. Its Hermitian norm is zero: expand $h(\delta e_j, \delta e_j) = h(e_j, e_j)$ and use its orthogonality to e_j. The definiteness of X_0 then puts the difference in L_0. Write it as $\sum_{h \leq a} t_{hj} e_h$; each t_{hj} is integral because the arm is a direct lattice summand. In any nonzero column choose the largest h with $t_{hj} \neq 0$. Multiplication by w_v, upper unitriangular on this arm, leaves that top coefficient unchanged. In normalized coordinates its modulus is $|s_{j,v}/s_{h,v}| \, |\Phi_v(t_{hj})|$. On the other hand

$$\|w_v(\delta_v - 1)\widehat{e}_{j,v}\| \leq \|w_v x_v\| \, \|x_v^{-1}(\delta_v - 1)\widehat{e}_{j,v}\| < \frac{2b^{6n}}{N\rho}.$$

Both scale ratios are at most b^{23n}. Thus all conjugates of t_{hj} have modulus less than one, a contradiction. This proves the separation.

The ambient real dimension of the tuple of columns is d_0. Balls of radius $(2N\rho u)^{-1}$ about distinct frames have disjoint interiors in the maximum norm. Comparing their volumes with the concentric enlargement of the ball of radius ρu bounds their number by $(1 + 2N\rho^2 u^2)^{d_0}$. This is at most $(b^{6n^2}u^2)^{d_0}$, since $N \leq b^{60n}$ and n tends to infinity. The first paragraph of the proof of Proposition 108.79 identifies the multiplicity at a single frame: equality of the residue implies that the relative map induces a principal-level isometry on the definite middle quotient. Hence the right classes agree, and their definite principal-level automorphism is trivial. The relative maps then lie in $\Gamma_{c'}^0$. The weights in the fiber add to at most one, being distinct terms in the partition by $\Gamma_{c'}$. This proves the first assertion without a restriction on the left

point y. For the second use $L^{-J} = \int_L^\infty J u^{-J-1}\,du$ and Tonelli's theorem. The extra factor is $J/(J-2d_0) \le 2$ at large nodes. □

Theorem 108.106 (Absolute truncation error in a full rational row)**.** *Write $K_\gamma(y) = F(y)F(\gamma^{-1}y)^*$ for the matrix outer product, and $K_\gamma^\flat$ for its cut counterpart. Both kernels are absolutely integrable after summation over every row (c,s) with weights $\beta_c(y)\beta_{c'}(\gamma^{-1}y)$. With the same weights put*

$$E_{csc'} = \sum_{\gamma\in\mathcal{T}(c,s,c')} \int_{G_\infty} \beta_c(y)\beta_{c'}(\gamma^{-1}y)\|K_\gamma(y) - K_\gamma^\flat(y)\|_{\mathrm{HS}}\,dy.$$

On the fields and interval of Theorem 108.101, for every sufficiently large node,

$$\sum_{c'} E_{csc'} \le \tfrac{1}{2} v_U b^{-k^3 f}.$$

Proof. Put $A(y) = \prod_v A_v(y_v)$, $a_* = \prod_v \sup A_v$. At $z = \gamma^{-1}y$ each kernel norm is at most $A(y)a_* L(y)^{-3J}L(z)^{-3J}$ by Theorem 108.103. The cut matrix is the full matrix multiplied by the product of all the scalar cutoffs at y, z, a real number in $[0,1]$. If the difference is nonzero, some local length exceeds $b^{6n+2}/2$. Thus $L(y)L(z) \ge b$ for large b, and

$$\|K_\gamma(y) - K_\gamma^\flat(y)\|_{\mathrm{HS}} \le b^{-2J} A(y) a_* L(z)^{-J}.$$

For the full and cut kernels separately the same upper bound holds with the factor b^{-2J} omitted. The last factor is summable against the right weights by Lemma 108.105. For the left weight one has

$$\int_{G_\infty} \beta_c(y) A(y)\,dy = v_U \prod_v \int_{U_v\backslash G_v} A_v.$$

Indeed at fixed y integration of $\beta_c(uy)$ over U_∞ unfolds to the integral of one over $\Gamma_c\backslash U_\infty$, of volume v_U. The A_v are left U_v-invariant. Tonelli's theorem and Theorem 108.103 consequently bound the sum of the error norms by

$$2 v_U d_B\, b^{(-2r^3+12n^3r+400r^2)f}.$$

This also proves the asserted absolute convergence, by deleting the gain b^{-2J}. Finally $d_B \le b^{2kf}$, $k \le r$, and $12n^3r + 400r^2 + 2k = o(r^3)$ on this interval because $n \le \sqrt{x}$ and $r \ge x^5$. The stated bound follows. The estimates also hold for arbitrary subsets of rational terms by nonnegativity. □

Corollary 108.107 (The error in the residue second moment)**.** *Uniformly on the same interval the array of differences between full and cut block sums, normalized by v_U, has squared Hilbert–Schmidt norm at most*

$$v_U^{-2} \sum_{c,s,c'} E_{csc'}^2 \le |\mathcal{C}| b^{-k^3 f} \le \tau^{-40} |B_{\mathrm{res}}|\,|\mathcal{C}| d_B.$$

The bound persists when each block is restricted to a subset of its rational translates or multiplied termwise by scalars of modulus at most one.

Proof. For each c, s, the sum of the squares of the nonnegative errors is at most the square of their sum. At a split place of F above p_i the number of possible residue matrices is at most $p_i^{r^2}$; there are f such places above each mark. Therefore there are at most $M^{fr^2} \le b^{22fr^2}$ residues. Apply Theorem 108.106 and use $k^3 \ge 22r^2$ for all sufficiently large nodes, since $k = r - 2n + 1$. The last inequality follows from $\tau = b^{kf/4}$ and $|B_{\mathrm{res}}|, d_B \ge 1$. The triangle inequality gives the assertion for signed matrix sums. It bounds their difference as an array and hence also the absolute difference of their Hilbert–Schmidt norms. □

109 Marked determinants and signed coefficient matrices

109.1 One marked period determinant at all complex embeddings

Fix one of the fields E_x, one translate and $s \geq 1$ signed columns of common small type in Theorem 108.6. Use its projected tests and its mass metric, including the same normalization of the columns on both sides. In particular one may use the divided columns $\widetilde{Y}_j/b_j^{\mathrm{per}}$ of Theorem 108.27. Suppose the projected columns are independent. Then their Gram determinants are positive at every complex embedding: the rational quadrature formula expresses them as positive sums of squared absolute values of algebraic minors, and a nonzero minor stays nonzero at every embedding.

Write $Z(t_1, \dots, t_s)$ for the determinant obtained by pairing the columns with the projections of the indicated small tests. Let $\mathcal{V}$ be the finite list of row tests $t_{a,g}$ in the quadrature formula, with positive rational weights q_t. Put

$$q_* = \sum_{t \in \mathcal{V}} q_t = r_\mu \sum_a |\Gamma_a|^{-1}, \qquad L_0 = g_1 g_2 T, \qquad M = \max(1, \max_a |\Gamma_a|).$$

Fix one coefficient place w_p above each of $p = g_1, g_2, T$. Denote by b_p the least valuation at that place on the image of the integral test lattice under the projector, at exterior degree s. The b_p here are valuations of period ideals; they may include the chosen column denominators. The row functional of each $t \in \mathcal{V}$ has norm at most $\sqrt{M}$.

Lemma 109.1 (Perturbations attaining all three finite minima)**.** *There are tests $u_1, \dots, u_s$ of functional norm at most $3L_0\sqrt{M}$ at every complex embedding such that*

$$v_{w_p} Z(u_1 + L_0 v_1, \dots, u_s + L_0 v_s) = b_p \qquad (p = g_1, g_2, T)$$

for any v_i integral in the small test lattices at all three chosen places. In particular the assertion holds for integral linear combinations of tests of $\mathcal{V}$.

Proof. For each p choose s frame rows attaining b_p, possible by Theorem 108.3. For the first row, the second row, and so on, combine the three lifts by the Chinese remainder coefficients between zero and $L_0 - 1$ equal to one at their own rational prime and zero at the others. These are exactly the tests of Corollary 108.7, giving the displayed norm bound and $v_{w_p} Z(u_1, \dots, u_s) = b_p$. On perturbing the rows, each additional determinant term contains at least one factor L_0. Its remaining rows are all permissible at w_p, so before that factor its valuation is at least b_p. It therefore cannot change the least valuation of the original term. This argument uses only $v_{w_p}(L_0) > 0$, including for ramified coefficient places. □

Theorem 109.2 (One algebraic period determinant at all embeddings)**.** *Let J be a number field of definition for the periods in these tests and put $d = [J : \mathbb{Q}]$. There is one determinant $z \in J$ of the given projected columns such that $v_{w_p}(z) = b_p$ at the three chosen places and, for every $\sigma : J \hookrightarrow \mathbb{C}$,*

$$\frac{s!}{(4q_*)^s} \det G_\sigma \;\leq\; \frac{|\sigma z|^2}{L_0^{2s}} \;\leq\; \bigl((3 + 2d^2)^2 M\bigr)^s \det G_\sigma.$$

All row coefficients used in the construction are rational integers applied to the preceding semilocally integral tests.

Proof. At each embedding select a tuple $(t_1^\sigma, \dots, t_s^\sigma)$ of quadrature tests maximizing the absolute determinant. Its value $B_\sigma > 0$ satisfies

$$B_\sigma^2 \geq \frac{s!}{q_*^s} \det G_\sigma.$$

Indeed the Gram formula sums over unordered subsets of size s. The sum of the products of their weights is at most $q_*^s/s!$, by expanding the s-th power of the sum and retaining only tuples with distinct indices. This bound needs no lower estimate for an individual quadrature weight.

Choose common rows r_i inductively. Before step i, for each σ use the first $i-1$ selected rows and complete them with $t_i^\sigma, \dots, t_s^\sigma$. Their determinant at σ is nonzero, initially by definition. With the other rows fixed call the determinant as a functional of row i by l_σ, in this embedding. We choose

$$r_i = u_i + L_0 \sum_{\tau : J \hookrightarrow \mathbb{C}} c_\tau t_i^\tau, \qquad c_\tau \in \{0, 1, \dots, 2d\}.$$

Choose the c_τ for this step independently and uniformly for the counting argument. Fix σ and the choices other than c_σ. Consecutive values of $l_\sigma(r_i)$ differ by the nonzero number $L_0 l_\sigma(t_i^\sigma)$. At most one choice lies in the open complex disk of radius $L_0|l_\sigma(t_i^\sigma)|/2$ about zero, by the triangle inequality. The fraction failing for any embedding is at most $d/(2d+1) < 1$. Hence a single choice keeps the new determinant at least $L_0/2$ times the previous one in absolute value at each embedding. Iteration gives $|\sigma Z(r_1, \dots, r_s)| \geq (L_0/2)^s B_\sigma$, proving the lower estimate.

The choices are integral at the marks, so the lemma gives all three valuations. Each selected row has functional norm at most $(3 + 2d^2)L_0\sqrt{M}$ by the triangle inequality. Apply the wedge of the row functionals to the wedge of the projected columns at σ. Its norm is at most the product of the functional norms, and the squared norm of the column wedge is $\det G_\sigma$. This proves the upper estimate for the very same $z = Z(r_1, \dots, r_s)$. □

Corollary 109.3 (A uniform comparison for divided signed columns)**.** *There is a finite family $\mathcal{Z} \subset J$ of actual period determinants all attaining the three b_p, such that, with $m_\sigma = L_0^{-s} \max_{z \in \mathcal{Z}} |\sigma z|$,*

$$\frac{s!}{q_*^s} \det G_\sigma \;\leq\; m_\sigma^2 \;\leq\; (16M)^s \det G_\sigma.$$

For the base small level on the polynomial-conductor fields in Theorem 108.27, take $s = q_c$ and the divided columns. With its K_u, define

$$H_{\mathcal{Z}} = \frac{2}{s[J : \mathbb{Q}]} \sum_\sigma \log m_\sigma + K_u.$$

Then $H_{\mathcal{Z}} = H(G^\circ) + O_{\psi,\varepsilon}(x^C)$ for a fixed C, uniformly on that interval. The error is independent of the coefficient-field degree and is $o_{\psi,\varepsilon,A}(b/x^A)$ for every fixed $A > 0$.

Proof. Form the list of determinants with rows $u_i + L_0\epsilon_i t_i$, where $t_i \in \mathcal{V}$ and $\epsilon_i = \pm 1$ vary independently. The lemma supplies the finite assertions and every row has norm at most $4L_0\sqrt{M}$. At a fixed embedding choose the maximizing tuple in the preceding proof. Average these determinants multiplied by $\prod_i \epsilon_i$ over all choices of the signs. Multilinearity kills every term containing an unperturbed row and leaves precisely $L_0^s Z(t_1, \dots, t_s)$. The maximum of the absolute values is at least the absolute value of that average. The bound for B_σ and the exterior norm upper bound give both inequalities.

After taking logarithms the upper error in $H_{\mathcal{Z}} - H(G^\circ)$ is at most $\log(16M)$; the lower error is at worst $-\log^+ q_*$, where $\log^+ q = \max(0, \log q)$. We verify their sizes. Corollary 108.7 bounds $\log M$ by a fixed power of x. The Weyl dimension formula for r_μ has at most $[E_x : \mathbb{Q}]n^2$ factors, each at most the largest width plus n, where n is the small rank. The widths are $O(xb + x^{C_1})$ and the degrees and ranks are polynomially bounded, as in Theorem 108.6. Thus $\log r_\mu \ll x^{C_2}$.

The class mass is bounded above by the number of small classes. Put $N = [E_x : \mathbb{Q}]n$ and take a cyclotomic conductor N_E for E_x from Corollary 9.27. The full-level bound in the proof

of Lemma 99.17 is $\log h_{\max} \ll N^4 \log(2NN_E)$, obtained by recording the trace-lattice Gram matrix and the matrix of a primitive field generator in a bounded integral basis. Passing to the base residue markings adds at most $L_0^{N^2}$ classes per full-level class. Indeed at each of the marked rational primes the kernel of full residue reduction on the underlying integral lattice lies in its small level, and a residue matrix has at most p^{N^2} possibilities. The small objects and their row frames are those in Section 97.4. The logarithms of N_E, L_0 have polynomial bounds on the interval, by Corollaries 9.27 and 108.7. Hence $\log^+ q_* \ll x^{C_3}$. If additional row levels are imposed, the same exact inequalities hold, with the logarithm of their further index added to this class-count bound. Finally $b = \exp(x^{\varepsilon})$ dominates every fixed power of x. This proves the claimed error. □

Consequently $H_{\mathcal{Z}}$ can replace $H(G^{\circ})$ in Theorem 108.27 at its stated precision on the base level. This compares the archimedean and marked finite tests on the given signed periods. The absolute upper estimate for their values in Section 6.6 and the required independence for the chosen input columns remain to be established.

109.2 Translation averages of the signed theta columns

Fix E_x, a translate, and ordinary words in the directed tower of Theorem 9.13 with a common small test space. Work at one complex embedding. Write V for the dual of this projected test space, with the Hermitian norm of Theorem 108.6. An initial table for a word gives a vector in V by the signed path formula of Theorem 60.32. We compute a family of functions whose values at the origin are precisely these vectors. The other values will locate the cancellation which is possible at the origin.

On edge j choose one reference theta vacuum Θ_j on its principally polarized trace lattice Λ_j. Use orthonormal matrix coordinates for its Hermitian metric at the CM embeddings, so the metric weight on the edge is $\exp(-c_j|z|^2)$, $c_j > 0$. On a conjugated edge use the conjugate complex space and coordinates throughout. The finite expansion of Proposition 100.13 expresses each class section, in the common matrix coordinates, as a finite sum of $U_{a_j}\Theta_j$ times scalars. Here a_j is a torsion class modulo Λ_j, and U is the Hermitian translation of Lemma 54.5. Indeed the lattice intertwiner is a sum over the quotient by the intersection of two lattices, and a finite position condition is a finite character sum. Products of the resulting translations are translations times roots of unity. Choose one lift for each residue a_j, absorbing the change of its scalar in the coefficient. Finitely many such sections have the same multiplier on a sublattice $\Gamma_j = L_j\Lambda_j$. One may take L_j divisible by all the translation orders, since the ratio of two period multipliers on Λ_j is their commutator character.

Lemma 109.4 (Inner products of translated derivatives)**.** *On one edge let*

$$f_{\alpha,a}(w) = \partial_z^{\alpha}(U_w U_a \Theta)(0).$$

With probability Haar on $\mathbb{C}^d/\Gamma$ the functions, interpreted with their common unitary multiplier in the variable w, satisfy

$$\int f_{\alpha,a}(w)\overline{f_{\beta,b}(w)}\,dw = \mathbf{1}_{\alpha=\beta}\mathbf{1}_{a=b}c^{|\alpha|}\alpha!\,\|\Theta\|^2.$$

The norm on the right uses probability Haar and the theta metric on $\mathbb{C}^d/\Lambda$.

Proof. For a holomorphic section H, the derivative at zero is $e^{-c|w|^2/2}\prod_l(\partial_{w_l} - c\bar{w}_l)^{\alpha_l}H(w)$. In the metric on the torus the adjoint of $\partial_{w_l} - c\bar{w}_l$ is $-\partial_{\bar{w}_l}$. The opposite-face boundary terms cancel by the multiplier law. Commuting this lowering operator across the raising operators in

the second factor gives c for a matching coordinate. If a lowering operator remains it kills the holomorphic section. If only raising operators remain, move one in the other direction and use holomorphy of the first section. The surviving factor is therefore $\mathbf{1}_{\alpha=\beta} c^{|\alpha|}\alpha!$ times the inner product of the two undifferentiated sections.

For $a \neq b$ modulo Λ, unimodularity of the polarization gives a period $\lambda \in \Lambda$ whose commutators with U_a and U_b differ. Otherwise $a - b$ would belong to the lattice dual to Λ for the alternating form, which is Λ. The period translation acts unitarily on the common cover and has the two sections as eigenvectors with distinct unitary eigenvalues. Their inner product is zero. A translate has the same metric norm as Θ by translation invariance. Passing to a finite cover preserves the probability integral of its squared magnitude. □

For column i introduce independent matrix variables D_j, one matrix at each CM embedding on each edge. Expand its prescribed path formula with these rules. Keep the actual value of the initial table at the composite covector of each path, including the condition that its components be nonzero. At each edge make the polynomial substitution $X = D_j^{\mathsf{t}} Y_{[r]}$ after the specified determinant twist, as in (54.22). Keep the monomials in D_j unevaluated. Apply the terminal coefficient extraction, the small test and its projection, and include the class masses and all finite scalars. On expanding the theta sections, group together paths with the same tuple $a = (a_j)_j$ of torsion residues. The coefficient on D^{α} in that group is a vector $v_{i,\alpha,a} \in V$, since it is linear in the small test. The coupling index on the last two maps is summed once. These vectors require finite scalar and polynomial operations. In particular, the signed path sum takes place before any absolute values. Any column divisor or normalization can be included in its coefficients.

Put $Y_i(\boldsymbol{w}) = \sum_{\alpha,a} v_{i,\alpha,a} \prod_j f_{\alpha_j,a_j}(w_j)$, summing the finite expansion. Then $Y_i(0)$ is the original projected column. Indeed replacing each monomial in D_j by its undivided derivative is exactly the class operator. This replacement commutes with polynomial operations on the other edge variables and with linear terminal restriction. It is the construction in `Formalism/LocalToGlobal/KernelLattices/ThDistC.lean`, line 27. The variable $\boldsymbol{w}$ translates the sections at the chosen representatives. This definition also makes sense when such a translation requires a change of adelic level to be realized as an automorphic test.

Theorem 109.5 (Covariance for the signed paths). *Write $h = \prod_j \|\Theta_j\|^2$ and $q_\alpha = \prod_j c_j^{|\alpha_j|}\alpha_j!$. For a pair of columns in these common coordinates one has the operator identity*

$$\int Y_i(\boldsymbol{w}) Y_l(\boldsymbol{w})^{*}\, d\boldsymbol{w} = h \sum_{\alpha,a} q_\alpha\, v_{i,\alpha,a} v_{l,\alpha,a}^{*}.$$

The integral is over the product of the common covers. If i, l belong to distinct binary words of Theorem 9.13 at this translate, the integral vanishes. For $i = l$ its range is exactly $\operatorname{span}\{v_{i,\alpha,a}\}$.

Proof. Expand the finite sums and use the lemma on each edge. This proves the formula. On the diagonal its quadratic form is a positive sum of squared absolute values. Its kernel is the orthogonal complement of the indicated span.

To see what happens between words, fix a test at either end of the operator identity, a torsion tuple a, and every differential monomial on the edges after the first. The remaining polynomial in the first matrix is a sum of matrix coefficients of one compact column type. Indeed in rank k the first substitution evaluates an input flag polynomial of the first partition ν at $D_1^{\mathsf{t}} Y_{[k]}$. Right multiplication $D_1 \mapsto D_1 g$ replaces its argument by $g^{\mathsf{t}} D_1^{\mathsf{t}} Y_{[k]}$. The translates of flag polynomials of column weight ν under this action have the Schur type indexed by ν, by the highest-weight realization in Section 54.10 (equivalently by the Cauchy decomposition of matrix polynomials). The transpose convention gives the same indexing at every word.

Changing the two matrix frames on this edge acts by left and right invertible matrices and preserves its isotypic component. Contractions at later vertices and the finite signed sums are linear in these first-edge polynomials.

At a fixed CM embedding the first partition is obtained by subtracting the half-sum of positive roots from the initial parameters and adding the common input determinant shift. This is the weight calculation in Lemma 108.11. The shifts on this edge and the translate are independent of the bits. Formula (9.1) shows that at a differing bit the corresponding entries of the partitions differ by Δ. They therefore index inequivalent representations of $U(k)$. The Fischer inner product on the first matrix is invariant under unitary changes of its columns, by its Gaussian integral formula. Its pairing on inequivalent types vanishes by Schur's lemma. The factors $c_1^{|\alpha_1|}$ merely rescale each homogeneous degree; different degrees already pair to zero. Summing the pairings over the fixed data gives the claimed zero operator. □

Corollary 109.6 (Independence under separate torsion translations)**.** *For s prescribed columns let $V_i = \operatorname{span}\{v_{i,\alpha,a}\}$. There are torsion points $\boldsymbol{w}_i$ of the product of common covers for which $Y_1(\boldsymbol{w}_1), \dots, Y_s(\boldsymbol{w}_s)$ are independent if and only if*

$$\dim \sum_{i \in I} V_i \ \geq |I| \qquad (I \subseteq \{1, \dots, s\}).$$

The orders in these covers may be powers of any one fixed prime.

Proof. The span of all values of Y_i is V_i. Containment in V_i follows from the expansion, and a functional annihilating all the values lies in the kernel of the diagonal covariance. The same reasoning and continuity apply to values on the dense set of prime-power grids. A change of lift in the cover changes a value by a common unit scalar and preserves its line.

Independent choices from the V_i require the displayed inequalities. For sufficiency use induction on s, the case $s = 1$ being immediate. If equality holds for a nonempty proper subset I, choose its vectors by induction and divide the ambient space by their span, which is $\sum_{i \in I} V_i$. The dimension inequalities for $I \cup J$ give the required inequalities for the remaining indices in the quotient. Lift their independent choices. If every proper nonempty subset has strict inequality, choose one nonzero vector in the last space and quotient the ambient space by the line it spans. Each of the other nonempty sums loses at most one dimension and induction applies. Finally express each chosen vector as a finite linear combination of torsion values. The nonzero exterior product expands into exterior products of those values, one of which must be nonzero. □

At the prescribed first and later tests the point in this family is $\boldsymbol{w} = 0$ for every word. Orthogonality in the integral above allows vanishing at that point. It determines the signed coefficient spaces against which the fixed evaluation must be tested.

109.3 An upper bound at the prescribed theta tests

Retain the actual signed coefficient vectors of Theorem 109.5. The arithmetic test evaluates every translated column at the origin. To bound that evaluation, group the terms by their torsion tuple before sampling. Each group has a multiplier on the reference trace lattice itself. Thus the length of the shortest dual vector needed in the estimate comes from the original Hermitian lattice, even when the common cover used to compare different multipliers is large.

Let d_j be the complex dimension of the matrix coordinates at edge j, including all CM embeddings, and put $D = 2 \sum_j d_j$. Write K for the largest total degree $\sum_j |\alpha_j|$ among monomials

of the prescribed columns, taking zero for an empty list of monomials. Write $f_0 = [E_x : K_d]$. Define

$$N = \max\left\{1, \left\lceil 8\sqrt{\frac{\pi\ \mathrm{rd}(E_x)}{2f_0}(K+2D+2)}\right\rceil\right\}, \qquad \beta = \tfrac{3}{2}N^D.$$

For each column i, let $\mathcal{A}_i$ be the finite set of tuples a in its expansion. Write $Y_{i,a}$ for the part of Y_i indexed by a.

Lemma 109.7 (Evaluation for a fixed tuple of torsion shifts)**.** *Let $B : V \to W$ be a linear map to a finite-dimensional Hermitian space. For each i, a,*

$$\|BY_{i,a}(0)\|^2 \le \beta \int \|BY_{i,a}(\boldsymbol{w})\|^2\, d\boldsymbol{w} = \beta h \sum_{\alpha} q_\alpha \|Bv_{i,\alpha,a}\|^2.$$

Here the integral uses probability Haar on the product of the reference tori $\mathbb{C}^{d_j}/\Lambda_j$.

Proof. In one coordinate of W, remove the unitary metric factor $\exp(-\sum_j c_j|w_j|^2/2)$ from the expression for $BY_{i,a}(\boldsymbol{w})$. The result is a sum of raising derivatives of the fixed holomorphic product section $\bigotimes_j U_{a_j}\Theta_j$. All terms have this section's multiplier. The commutator in Lemma 109.4 implies that $K+1$ antiholomorphic derivatives in any common direction annihilate the sum. Indeed each such derivative either removes one raising operator or acts on the holomorphic section and gives zero.

Scale the edge coordinates by $\sqrt{c_j}$ to make the common metric parameter one. For the scaled trace lattice on each edge the alternating form is $\pi^{-1}\operatorname{Im}\langle z, w\rangle$. Unimodularity identifies its Euclidean dual with its multiple by i/π, up to a sign. The lower bound of Theorem 53.3 for the squared length of a nonzero trace vector in the scaled lattice is $2\pi f_0/\operatorname{rd}(E_x)$. Hence the real dual minimum of the product has length at least $\sqrt{2f_0/(\pi\operatorname{rd}(E_x))}$. These are the normalizations in the proof of Theorem 54.34. Changing a unitary multiplier by a character leaves this period lattice and the adjoint and commutator identities intact.

Apply Lemma 54.33 with this minimum, dimension D, and degree K. The grid average of the squared magnitude is at most $3/2$ times its integral. The origin is one of the N^D grid points, and every summand is nonnegative. This proves the inequality in each coordinate of an orthonormal basis. Summation gives the assertion for W. Orthogonality of the derivative indices at fixed a gives the stated integral. □

At an embedding σ use its own small-space metric, finite coefficients $v^{(\sigma)}_{i,\alpha,a}$, vacuum norm product h_σ, and differential factors $q_{\alpha,\sigma}$. To form these coefficients one transports the original scalar coefficients together with the phases of the translated germs. Theorem 53.6 supplies a phase common to every derivative order of each such germ. This also allows the fixed marked product normalization of Proposition 100.13. For $\boldsymbol{a} \in \prod_{i=1}^{s}\mathcal{A}_i$, put

$$E_{\boldsymbol{a},\sigma} = h_\sigma^s \sum_{\alpha_1,\dots,\alpha_s}\left(\prod_{i=1}^{s} q_{\alpha_i,\sigma}\right)\left\|v^{(\sigma)}_{1,\alpha_1,a_1}\wedge\cdots\wedge v^{(\sigma)}_{s,\alpha_s,a_s}\right\|^2, \qquad B_\sigma = \sum_{\boldsymbol{a}}\sqrt{E_{\boldsymbol{a},\sigma}}.$$

The sum over each derivative index can be taken over its finite support. The factors inside the exterior product include the finite masses, coupling signs, terminal restriction and finite projection of the original data.

Theorem 109.8 (A bound for the original exterior period)**.** *For $s \ge 1$, the Gram matrix $G_\sigma(0)$ of the actual columns at the prescribed tests satisfies*

$$\det G_\sigma(0) \ \le\ \beta^s B_\sigma^2.$$

The formula applies in every rank. The right side uses undifferentiated reference vacuum norms and finite coefficient operations.

Proof. Fix $\boldsymbol{a}$ and consider the exterior product of the s corresponding column parts, with a separate variable $\boldsymbol{w}_i$ in each part. With all the other factors fixed, exterior multiplication is a linear map of the remaining vector into $\bigwedge^s V$. Applying the lemma in that variable bounds the squared norm at zero by β times its integral. Repeating this argument and using Fubini bounds the squared norm with every variable zero by β^s times the product-torus integral. Expanding the exterior product by multilinearity, two distinct lists of multi-indices are orthogonal in at least one variable by Lemma 109.4. The diagonal terms give precisely $E_{\boldsymbol{a},\sigma}$.

At the origin, multilinearity also expresses the actual exterior product as the sum of these products over all $\boldsymbol{a}$. The triangle inequality bounds its norm by $\beta^{s/2}B_\sigma$. The squared exterior norm is the Gram determinant, including when all maximal minors vanish. □

Corollary 109.9 (The finite coefficients in the centered estimate)**.** *Use the divided period columns of Theorem 108.27 and include each division in $v^{(\sigma)}_{i,\alpha,a}$. On the polynomial-conductor interval one has $\log\beta = o_{\psi,\varepsilon,A}(b/x^A)$ for every fixed $A>0$, uniformly in the translate and word list. If these columns are independent at the prescribed tests, their centered count satisfies*

$$\Delta_u \le K_u + \frac{2}{s[J:\mathbb{Q}]}\sum_{\sigma}\log B_\sigma - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x + o_{\psi,\varepsilon,A}(b/x^A).$$

Proof. The number of embeddings of E_x, the ranks and the number of edges have polynomial bounds in x. On edge j the total degree is the sum of the sizes of its partitions. Their largest entries are $O(xb+x^C)$, by the construction of the word weights and the rank shifts in Lemma 108.11. Thus D and $\log(2+K)$ are polynomially bounded. The conductor bound of Corollary 9.27 gives the same for $\log \mathrm{rd}(E_x)$, using the conductor–discriminant formula. These bounds are uniform at the conjugate trace tori and independent of s. They give $\log\beta \le C'x^{C''}$, which is $o(b/x^A)$ since $b=\exp(x^\varepsilon)$. Independence makes all B_σ positive by the theorem. Substitute the logarithmic inequality in the definition of $H(G^\circ)$ and apply Theorem 108.27. □

This upper bound evaluates the prescribed tests themselves. It leaves the sizes of the indicated signed finite coefficient sums and vacuum norms to be compared with $A_{\mathrm{sig}}(x)$ at the precision required in Section 6.6.

109.4 The fixed projected period and its coefficient matrices

We evaluate the archimedean contribution at the original small tests on E_x in the product coordinates of Lemma 53.8. The coefficient of the total derivative degree will be explicit. The remaining expression involves positive matrices built from signed polynomials in a finite-dimensional space.

Fix an embedding σ of a common coefficient field J, and s projected columns in the same Hermitian space V. At edge j of column i let n_{ij} be the complex dimension of its product coordinates, k_{ij} its differential degree, and $c_{ij,\sigma}=\pi/A_{ij,\sigma}$ the metric parameter of its marked elliptic curve C^σ_{ij}. Use conjugate coordinates on conjugate edges. Put $\alpha! = \prod_{j,h}\alpha_{jh}!$. The finite expansion of the fixed column is

$$Y_i = \sum_{a\in\mathcal{A}_i}\sum_{\substack{\alpha\\ |\alpha_j|=k_{ij}}} v_{i,a,\alpha}\prod_{j,h} J_{\alpha_{jh}}(U_{a_{jh}}\Theta_{t_{ij}}), \qquad J_m(F)=F^{(m)}(0).$$

Indeed, on each edge expand each class section by the product formula preceding Theorem 100.13. The linear change of matrix coordinates in each term replaces the edge polynomial by a homogeneous polynomial of the same degree k_{ij}, the size of its partitions. Apply the signed path formula of Theorem 109.5 with the same terminal coefficient and the same projection. Collect now by the full tuple a of elliptic translations and by the derivative indices α

in product coordinates. This gives the vectors $v_{i,a,\alpha}$ by polynomial operations, including the chain-rule coefficients and all normalizing scalars. In the last two maps there is one sum over their common intermediate class. The phases of the transported sections are included before collecting coefficients. Thus the displayed identity uses the germ of each prescribed section at its given point. We suppress σ on the vectors and take any required rational column division in them. Empty expansions give zero columns.

For every nonempty $\mathcal{A}_i$ choose positive reals $\mu_i(a)$ with sum one. Form the positive semidefinite operator

$$B_i(\mu_i) = \sum_{a,\alpha} \frac{\alpha!}{\mu_i(a)} v_{i,a,\alpha} v_{i,a,\alpha}^{*} \quad \text{on } V.$$

Here vv^{*} takes h to $v\langle v, h\rangle$, with the inner product linear in its second argument. For any positive semidefinite operators $B_1, \dots, B_s$ define

$$D_s(B_1, \dots, B_s) = [t_1 \cdots t_s] \det_V \left(I + \sum_{i=1}^{s} t_i B_i \right).$$

The square brackets take the coefficient of exactly the specified monomial.

Theorem 109.10 (An upper bound at the fixed projected tests)**.** *Write $G_\sigma = (\langle Y_i, Y_j\rangle)_{i,j}$. For each column set*

$$T_{i,\sigma} = \prod_{j} \left[\left(\frac{2 \exp(L_{C_{ij}^{\sigma}})}{g_{ij,\sigma}(t_{ij}^{\sigma})^2} \right)^{n_{ij}} c_{ij,\sigma}^{k_{ij}} H_{n_{ij}}(k_{ij}) \right],$$

where L_C is from Theorem 56.29 and $H_n(k)$ is from Lemma 54.8. Then

$$Y_i Y_i^{*} \;\le\; T_{i,\sigma} B_i(\mu_i), \qquad \det G_\sigma \;\le\; \left(\prod_i T_{i,\sigma} \right) D_s(B_1, \dots, B_s).$$

The inequalities are valid for every choice of the probabilities, including at a rank-deficient projected table.

Proof. For a single elliptic coordinate z, Hermitian translation and Lemma 56.21 give

$$|(U_w \Theta_t)(z)|^2 \le \exp(c|z|^2) \sup_{C} g^2/g(t)^2 \le \exp(c|z|^2)\, 2\exp(L_C)/g(t)^2$$

with every lift w of the prescribed shift. Take the product over the n coordinates of an edge and denote the holomorphic product by F. Its homogeneous Taylor part of order k has squared Fischer norm $\sum_{|\alpha|=k} |\partial^\alpha F(0)|^2/\alpha!$. The sphere bound in Lemma 54.8 bounds this by $(2e^{L_C}/g(t)^2)^n c^k H_n(k)$. Multiplying over the independent edge variables gives a uniform upper bound $T_{i,\sigma}$ for the sum of the squared jet products divided by $\alpha!$, for each fixed tuple a. For $h \in V$, apply Cauchy–Schwarz to $\langle h, Y_i\rangle$ with factors $\sqrt{\alpha!/\mu_i(a)}$ and their reciprocals. The first squared sum is $\langle h, B_i h\rangle$. The second is at most $\sum_a \mu_i(a) T_{i,\sigma} = T_{i,\sigma}$. This proves the operator inequality.

To see the effect on the determinant, write each B_i as a sum of rank-one operators uu^{*}. Cauchy–Binet, applied to the matrix with columns $\sqrt{t_i}\, u$, shows that D_s is the sum of $\|u_1 \wedge \cdots \wedge u_s\|^2$ over one column from each list. In particular it is nonnegative, additive in each operator, and increasing in each for the positive semidefinite order. For the list $Y_i Y_i^{*}$ its value is precisely $\det G_\sigma$. The second assertion follows from the first. If an original list is empty take its $B_i = 0$. □

For a positive operator B on V let $e_s(B)$ be the sum of all products of s distinct eigenvalues, counted with multiplicity. Put $e_s(B) = 0$ if $s > \dim V$. Define

$$\mathcal{C}(B_1, \dots, B_s) = \inf_{\substack{\lambda_i > 0 \\ \prod_i \lambda_i = 1}} e_s\left(\sum_i \lambda_i B_i\right).$$

This expression can be bounded from above by choosing any positive scalings of product one. The optimization loses at most a fixed constant per column in the logarithmic estimate.

Lemma 109.11 (Capacity of the coefficient matrices)**.** *For all $s \geq 1$,*

$$\frac{s!}{s^s}\mathcal{C}(B_1, \dots, B_s) \;\leq\; D_s(B_1, \dots, B_s) \;\leq\; \mathcal{C}(B_1, \dots, B_s).$$

If $\mathcal{C} > 0$, its logarithm is the infimum of a convex function of the real variables $\log \lambda_i$ with zero sum.

Proof. Write $n = \dim V$ and assume $s \leq n$, the other case being immediate. The polynomial $P(t) = e_s(\sum_i t_i B_i)$ is homogeneous of degree s. Its coefficients are nonnegative by the same rank-one expansion, and $[t_1 \cdots t_s]P = D_s$. Every positive evaluation therefore bounds $D_s \prod t_i$ from above. On writing $t_i = e^{y_i}$, the logarithm is the log of a sum of exponentials of affine functions. Its Hessian is the covariance matrix of their exponents with their normalized positive weights, hence is positive semidefinite whenever $P \neq 0$.

For the reverse bound we give Gurvits's capacity argument [5]. The polynomial $\det(zI + \sum_i t_i B_i)$ is nonzero when all its variables have positive imaginary part, since the imaginary part of its quadratic form is positive definite. A polynomial with this nonvanishing property is called stable. Its derivatives and real specializations are stable or zero. For a derivative in one variable of degree $m > 0$, its leading coefficient, as a polynomial in that variable, is the locally uniform limit upon setting that variable to iR and dividing by $(iR)^m$ as $R \to \infty$. It is nonzero as a polynomial and hence has no zeros in the product of upper half-planes by Hurwitz's theorem. The univariate degree thus stays m there, and the logarithmic derivative has strictly negative imaginary part by factoring the polynomial into linear terms. This proves the derivative assertion. Real specialization follows by taking a limit from the upper half-plane. The version of Hurwitz used on a product follows by restricting to complex lines through a putative zero inside a small polydisk and using the univariate result. Since P is the coefficient of z^{n-s}, differentiation in z and setting $z = 0$ give its stability up to the zero case.

Let P now denote any homogeneous stable polynomial of degree q in q variables with nonnegative real coefficients. Its capacity is $\inf_{t>0} P(t)/\prod t_i$, equivalent by homogeneity to the infimum at $\prod t_i = 1$. Assume its capacity is positive. Freeze the first $q-1$ variables at positive real values t'. The resulting polynomial $R(x)$ has only nonpositive real roots by stability, real coefficients, and positivity on $x > 0$. Its degree m is positive since a positive capacity excludes a constant in x. A root at zero of multiplicity at least two would likewise force capacity zero. If zero is a simple root, $\inf_{x>0} R(x)/x = R'(0)$. If $R(0) > 0$, write $R(x) = R(0)\prod_{\nu=1}^{m}(1 + x/b_\nu)$ with $b_\nu > 0$. The arithmetic–geometric mean bounds it above by $R(0)(1 + xR'(0)/(mR(0)))^m$. Minimizing this upper bound divided by x gives

$$R'(0) \;\geq\; g_m \inf_{x>0} R(x)/x, \qquad g_1 = 1, \qquad g_m = ((m-1)/m)^{m-1} \quad (m \geq 2).$$

The same inequality holds for the simple-root case. The numbers g_m decrease with m, so $g_m \geq g_q$. Divide by the product of the fixed t' and take their infimum. The derivative at zero retains capacity at least g_q times the old capacity. It is again stable and homogeneous

with nonnegative coefficients. Repeat in all variables. The final scalar is the coefficient of their product and $\prod_{q=1}^{s} g_q = s!/s^s$. This proves the lower bound, also valid when the capacity is zero. □

Corollary 109.12 (The finite matrix term on the period interval)**.** *On the polynomial-conductor interval of Theorem 108.27 take its $s = q_c$ divided columns. Form $B_{i,\sigma}$ from the actual projected coefficient vectors above and put $\mathcal{C}_\sigma = \inf_{\mu_i} \mathcal{C}(B_{1,\sigma}(\mu_1), \dots, B_{s,\sigma}(\mu_s))$. Set*

$$L_{d_0} = \log(\sqrt{d_0}/(2\pi)) - \gamma_{\mathrm{E}} + D_{\xi_{d_0}}, \qquad \bar{k} = \frac{1}{s} \sum_{i,j} k_{ij}.$$

If the columns are independent at the prescribed tests, then, uniformly on these fields and translates, for every fixed $A > 0$,

$$\begin{aligned} \Delta_u \ &\le K_u + \bar{k} L_{d_0} + \frac{1}{s[J:\mathbb{Q}]} \sum_{\sigma} \log \mathcal{C}_\sigma - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x \\ &\quad + o_{\psi,\varepsilon,A}(b/x^A). \end{aligned}$$

Proof. The two preceding results bound $\det G_\sigma^\circ$ by $\mathcal{C}_\sigma \prod_i T_{i,\sigma}$, also after the infimum since its prefactor is independent of all probabilities. A positive determinant gives $\mathcal{C}_\sigma > 0$. For each edge, Theorem 53.22 gives mean $\log c_{ij,\sigma} = L_{d_0}$ over J. The elliptic curves and nonzero marks in the product presentation are the transported copies of the marked CM model; complex conjugation permutes its embeddings. By Lemma 56.26 and Theorem 56.29, the mean log of the first factor inside the square brackets defining $T_{i,\sigma}$, including its exponent, is

$$n_{ij}\big(\log 2 + 1 + \log(\sqrt{d_0}/(2\pi)) + D_{\xi_{d_0}} - 2b_K(t_{ij})\big).$$

Here $b_K(t_{ij}) \ge 0$. The upper bound after dropping this nonpositive term is $O_{\psi,\varepsilon}(n_{ij} x^{\varepsilon/4})$ by Corollary 53.23 and its bounds on $d_0, D_{\xi_{d_0}}$. Also $\log H_{n_{ij}}(k_{ij}) \le n_{ij}(1 + \log(1 + k_{ij}/n_{ij}))$ by the sphere lemma. The sum of the coordinate dimensions along a word and the log of its largest degree plus two are bounded by powers of x, uniformly in the word, by the rank and weight bounds in Corollary 109.9. Thus, after division by s, all these upper errors together are $O_{\psi,\varepsilon}(x^C)$ for some C depending only on the fixed parameters. Since $b = \exp(x^\varepsilon)$ this is $o(b/x^A)$. Insert the height bound into Theorem 108.27. □

Consequently the archimedean term at these tests has an upper bound in terms of the signed finite coefficient matrices and the explicit degree contribution $\bar{k} L_{d_0}$. Section 6.6 still requires a sufficiently small bound relative to $A_{\mathrm{sig}}(x)$, a nonzero determinant at the fixed tests, and the exceptional strict-length estimate on this same interval.

109.5 A metric bound for the fixed signed coefficients

In Corollary 109.12 the signed coefficient vectors come from the original projected tests. We express their contribution using a single positive Hermitian operator at each embedding. This also gives an upper bound from any chosen orthogonal decomposition of the actual test space.

Lemma 109.13 (Selecting a degree in a positive determinant)**.** *Let B be a positive semidefinite Hermitian operator in finite dimension. For every integer $s \ge 1$,*

$$e_s(B) \ \le \ \inf_{t>0} \frac{\det(I + tB)}{t^s} \ \le \ \frac{e^s s!}{s^s}\, e_s(B).$$

The constant is independent of the dimension.

Proof. Write $P(t) = \det(I + tB) = \sum_{j=0}^{r} d_j t^j$, where r is the rank and $d_j = e_j(B) > 0$ for $0 \le j \le r$. All roots of P are negative real numbers. Every positive evaluation bounds its degree-s term from above. For $r < s$ let $t \to \infty$ to obtain zero. Suppose $s \le r$. The ratios $b_j = j d_j / d_{j-1}$ decrease weakly with j. Indeed, by Rolle's theorem each derivative of P also has all its roots negative, with multiplicities allowed. The log of a positive polynomial with such a factorization has nonpositive second derivative on the positive real axis and at zero. Applied to $P^{(j-1)}$ at zero this gives $(j! d_j)^2 \ge (j-1)! d_{j-1} (j+1)! d_{j+1}$, as required. Take $t = s / b_s$. For $j > s$, each ratio of successive $t^j d_j$ is at most s/j, and for $j \le s$ the ratio is at least s/j. Multiplying the inequalities starting from $t^s d_s$ in either direction gives

$$\frac{t^j d_j}{t^s d_s} \le \frac{s!}{s^s} \frac{s^j}{j!}.$$

Sum over $j = 0, \dots, r$ and use the exponential series. □

Fix the embedding and abbreviate its test space to V, of dimension n. Retain the complete vectors $v_{i,a,\alpha}$ of Section 109.4, including the division of each column when using G_σ°. For a positive definite Hermitian operator H define

$$d_{ia}(H) = \sum_{\alpha} \alpha! \, \langle v_{i,a,\alpha}, H v_{i,a,\alpha} \rangle,$$

$$\Psi_\sigma(H) = \operatorname{tr} H - \log\det H - n + s + 2 \sum_{i} \log\left(\sum_{a \in \mathcal{A}_i} \sqrt{d_{ia}(H)} \right).$$

Theorem 109.14 (One Hermitian metric for the signed coefficient matrices)**.** *Let $D_{\mathrm{mix}} = \inf_\mu D_s(B_1(\mu_1), \dots, B_s(\mu_s))$, with one probability distribution on the original tuples for each word as in Theorem 109.10. Then*

$$D_{\mathrm{mix}} \;\le\; \mathcal{C}_\sigma \;\le\; \exp\Big(\inf_{0 < H \le I} \Psi_\sigma(H) \Big) \;\le\; \frac{e^s s!}{s^s} \mathcal{C}_\sigma \;\le\; e^s D_{\mathrm{mix}}.$$

The infimum of Ψ_σ over all positive definite operators has the same value. A zero coefficient column gives zero in all three expressions in exponential scale.

Proof. Assume each word has some nonzero coefficient. Define

$$Q = \inf_{\mu,\, \lambda_i > 0} \frac{\det(I + \sum_i \lambda_i B_i(\mu_i))}{\prod_i \lambda_i}.$$

Write $\lambda_i = t \lambda_i'$, with $\prod_i \lambda_i' = 1$. For fixed λ' and μ apply Lemma 109.13 to $\sum_i \lambda_i' B_i(\mu_i)$. Taking the remaining infima gives $\mathcal{C}_\sigma \le Q \le (e^s s!/s^s) \mathcal{C}_\sigma$. Lemma 109.11 gives the two inequalities involving D_{mix}.

For any positive semidefinite W,

$$\log\det(I + W) = \min_{H > 0} \{ \operatorname{tr}(H(I+W)) - \log\det H - n \}.$$

Indeed the difference between the right-hand expression and the left-hand side is the sum of $u - 1 - \log u$ over the positive eigenvalues u of $(I+W)^{1/2} H (I+W)^{1/2}$. It is nonnegative and equality holds for $H = (I+W)^{-1} \le I$. In $\log Q$ insert this identity and take the infima in the opposite order, all being infima over independent sets. Once H and μ are fixed, minimizing in the i-th scalar gives

$$\inf_{\lambda_i > 0} \{ \lambda_i \operatorname{tr}(H B_i(\mu_i)) - \log \lambda_i \} = 1 + \log \operatorname{tr}(H B_i(\mu_i)).$$

Next use $\operatorname{tr}(HB_i(\mu_i)) = \sum_a d_{ia}(H)/\mu_i(a)$. Cauchy–Schwarz gives the minimum, in the sense of an infimum, as $(\sum_a \sqrt{d_{ia}(H)})^2$. For positive d_{ia} use weights proportional to their square roots; zero terms can be assigned arbitrarily small positive total mass. Substitution gives $\log Q = \inf_H \Psi_\sigma(H)$, including the value $-\infty$. The same proof starting with $0 < H \le I$ gives the restricted formula since that set contains every minimizer $(I+W)^{-1}$. For a zero column the mixed coefficient vanishes and in Q its scalar can be taken arbitrarily large. This gives the asserted limiting interpretation. □

Corollary 109.15 (An estimate using any chosen test metric)**.** *In the setting of Corollary 109.12 with independence at the fixed tests, take any $0 < H_\sigma \le I$ on each actual projected test space. Then*

$$\begin{aligned}\Delta_u \ &\le K_u + \bar{k}L_{d_0} + \frac{1}{s[J:\mathbb{Q}]}\sum_\sigma \Psi_\sigma(H_\sigma) - A_{\mathrm{sig}}(x) + W_{d_0}(u) - \rho_x \\ &\quad + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

In particular let Π_b be the orthogonal projections of any decomposition of V, of ranks m_b, and let $0 < h_b \le 1$. For $H = \sum_b h_b \Pi_b$ the entire metric term is

$$E_{ia,b} = \sum_\alpha \alpha! \, \|\Pi_b v_{i,a,\alpha}\|^2,$$

$$\Psi_\sigma(H) = s + \sum_b m_b(h_b - 1 - \log h_b) + 2\sum_i \log \sum_a \left(\sum_b h_b E_{ia,b}\right)^{1/2}.$$

Proof. Insert $\log \mathcal{C}_\sigma \le \Psi_\sigma(H_\sigma)$ in Corollary 109.12. The spectral decomposition of the chosen H gives the displayed expression, keeping the projection and all path sums in each vector before taking its squared norm. For comparison with the exact mixed coefficient, the previous theorem shows that optimization over all metrics costs at most one in the normalized log per column. This is $o(b/x^A)$ on the indicated intervals for every fixed A. □

The finite metric problem is convex along the positive definite matrix paths $H(t) = H_0^{1/2}(H_0^{-1/2}H_1H_0^{-1/2})^t H_0^{1/2}$. Indeed diagonalizing the middle matrix expresses each $d_{ia}(H(t))$ as a sum of exponentials of linear functions of t with nonnegative coefficients. Positive such sums are log-convex by Hölder's inequality. Their square roots and the sums of these square roots remain log-convex by the same inequality. The trace of $H(t)$ is convex by the same diagonalization, and $\log\det H(t)$ is affine. This gives the asserted convexity for nonzero columns. Estimating these signed coefficient energies at the required size and proving independence at the original tests remain arithmetic questions.

110 Terminal Fourier components in the period height

110.1 The connected half of the terminal Fourier index

Use the common terminal prime T and the unrefined integral Weyl lattices of Theorem 98.14. Thus $T \ge 5$ is good and splits completely in the fields of the terminal test. On an edge with source matrix rank r, the local condition at each chosen terminal half is $\xi' \bar{X} = \xi$, where the two covectors at its vertices are nonzero. The factor T^{-r} in Lemma 108.20 is given by the actual normalized average over $u \in \mathbb{F}_T^r$. We calculate its contribution before summing over the intermediate covectors.

Lemma 110.1 (Summing a connected frequency block)**.** *Fix a coefficient place $w \mid T$, one such block, and its output covector ξ'. Suppose the terminal frequencies in that block belong*

to the connected CM half at w. In a rational orthonormal product basis adapted to their subspace, the complete multiplier with the denominator T^r on the product vacuum has integral undivided derivatives in all the constrained coordinates. This remains true term by term after the translations at the other finite primes. The choices of bases can be imposed independently on all blocks over T.

Proof. At the residue position lattice, $u \mapsto (\bar{X} \mapsto \xi' \bar{X} u)$ injects $\mathbb{F}_T^r$ into its dual matrix space: some entry of ξ' is a unit, and that row of $\bar{X}$ can vary freely. Lift ξ' to a primitive integral covector and take a basis of the domain of u. The resulting r rank-one functionals extend to an integral basis of the dual lattice. Indeed projection to the row with that unit entry gives a left inverse after division by the unit. Take the dual basis on the position side. Pairing corresponding vectors in these two bases gives an orthonormal basis in the split Hermitian model, as in the proof of Theorem 98.14. Do this at every required block and approximate the combined local basis to precision T^2 by a rational orthonormal basis. The Cayley approximation of Theorem 100.14 permits this open congruence at both halves. Its vacuum intertwiner has index prime to T.

In these product coordinates the r modulation directions are in distinct elliptic coordinates modulo the integral periods, each generating one pure CM T-line. We verify the phases needed when their terms are summed. At each coordinate choose the primitive period ω_1 of Lemma 56.41 to agree with the chosen local generator after multiplication by T, to precision T^2; explicitly prescribe $2\omega_1$ modulo T^2 times the period lattice. The primitive vector lifting argument in that lemma permits precisely these two split congruences. The difference from the original local modulation, after division by T, lies in T times the local period lattice. On the orbit of the T-local vacuum under all these modulations this difference acts trivially. It fixes the vacuum and its pairings with all points of denominator T are integral. The local Heisenberg cocycle is trivial for these pairs as well: the split form pairs integral coordinates with the dual coordinates, and the two factors of T cancel. This uses the product coordinates generating the trace lattice at T.

Here is the same check for the rational lift at the other primes. In a scalar product coordinate write the period $\omega_1 = \delta l$, with $l \in \mathcal{O}_K$ in the standard elliptic lattice $\delta\mathcal{O}_K$. The rational lift of the generator is $2l/T$. Its Hermitian norm $4l\bar{l}/T^2$ is an even integer, since the opposite component of l is divisible by T^2. Thus the corrected translation $W(2jl/T)$ in the proof of Theorem 53.13 agrees with $U_{2j\omega_1/T}$. At primes other than T the lift is integral, including its factor two at the dyadic prime. Its period action on the reference vacuum is trivial there in the corrected convention; the correction is trivial on pairs of integral periods. The product-coordinate directions have zero mutual pairing. Equivalently, on this orbit the physical modulation for a generator is a scalar times the indicated cyclic translation, and any change of scalar compatible with the group law is a character. Such a character only shifts the Fourier label. This verification accounts for the possible phases before any arithmetic triangle inequality is used.

The terminal multiplier on the reference product is therefore the product of r complete cyclic projections S_{k_j} of Lemma 56.41, with the additive labels supplied by ξ. The remaining factors are the original scalar vacua. Operators supported at other finite primes commute with the entire terminal multiplier, by the factorization of the finite Heisenberg action. A prime-to-T translation applied after a cyclic projection can be represented on its common cover by a point of prime-to-T order. Indeed the map from the common cover to the original curve has kernel of order T, so gives an isomorphism on prime-to-T torsion. At T choose a lift that is integral in the common period lattice. On reducing a rational lift modulo that lattice, the period acts by the character of the specified section, a root of unity. This gives exactly the $U_{b_0} S_{k_j}$ of

Theorem 56.42, up to a unit. Thus all undivided derivatives are integral, including those at a zero label. The other-prime vacuum intertwiners are finite sums of such translations with coefficients integral at T. The unit comparison in Lemma 98.13 permits returning to a common locally integral frame. □

Theorem 110.2 (The terminal cost in the prescribed Weyl lattices)**.** *For an edge j on column i and a coefficient place $w \mid T$ put $B_w = \{\tau : P_{\tau,w} \in S_T\}$, using the conjugate coefficient place on a conjugated edge. Remove the separately specified nonterminal scalar factors, and include the Fourier denominators at T. Each matrix entry of the terminally restricted class operator between the integral Weyl lattices has valuation at least*

$$-\sum_{\tau \in B_w} (r_j + \nu_{ij,\tau,1}).$$

The bound holds for the complete indicator on the original matrix lattice with its specified covectors.

Proof. By the split position and frequency identification in Lemma 108.20, the primary shift at v lies over the conjugate of its selected position prime. Consequently it is in the étale CM half at the coefficient place when $P_{\tau,w} \in S_T$, and in the connected half otherwise. This is the conjugate annihilator computation of Theorem 108.21. The map $\tau \mapsto v = P_{\tau,w} \cap F$ is bijective.

Keep the Fourier expansion in the blocks indexed by B_w. It costs T^{-r_j} at each such block. Fix one tuple of its frequencies. Use the bases of Lemma 110.1 in all connected blocks. In each étale block with nonzero frequency choose the first dual coordinate equal to its primitive rank-one functional as in Theorem 98.14. These choices are simultaneous since they are in distinct local blocks. The connected blocks with their denominators now have integral derivatives of all orders. In an étale block only its first coordinate can cost one in valuation per derivative, and none do if its frequency is zero.

In a physical block τ each derivative matrix is an integral combination of regular operators and, for a matching nonzero étale frequency, the extra term $U_\tau \partial_a$ in its first coordinate, where U_τ has rank one. The regular operators are ordinary derivatives on the integral coordinates and $T\partial_a$ on the first coordinates with a possible loss. This decomposition follows by reduction of the adapted basis at T. Derivatives entering such a first coordinate from a different block have coefficients in $T\mathcal{O}_w$. Each flag minor is affine along the rank-one direction U_τ, even after the source matrix substitution. A Weyl test has $\nu_{ij,\tau,1}$ flag minors in each product. Expanding in the separate rank-one variables therefore leaves at most that many extra derivatives in block τ. This is precisely the differential argument of Theorem 98.14, now on the product with the connected blocks already summed. It applies to each integral source generator and dual test. Restoring the denominator on the étale blocks and summing their finitely many terms proves the claim. □

Corollary 110.3 (A smaller terminal term on the common intervals)**.** *In Theorem 108.27 the terminal summand of K_u can be replaced by*

$$\frac{\log T}{q_c} \sum_{i,j} \left(f r_j + \sum_{\tau} \nu_{ij,\tau,1} \right).$$

Here $f = [E_x^+ : \mathbb{Q}]$ and q_c is the number of columns in the specified independent list. The saving in the upper bound for Δ_u is $(f \log T/q_c) \sum_{i,j} r_j$.

Proof. For a fixed τ, the two conjugate halves at T are exchanged by conjugation of the coefficient places; equivalently the pullbacks $P_{\tau,w}$ cover the primes of E_x over T with their

degree weights. Exactly half of this weight has $P_{\tau,w} \in S_T$. The theorem thus contributes at most $\frac{1}{2}(fr_j + \sum_\tau \nu_{ij,\tau,1}) \log T$ per edge in the averaged denominator estimate. The two factors in the Gram calculation give the asserted coefficient. Each bound was uniform in the covectors and in the coupled intermediate class. The nonarchimedean triangle inequality therefore carries it through the prescribed class sums and compositions. The same projector denominator, scalar division by b_i^{per}, and archimedean matrix in Theorem 108.27 then give the claim on each of its original fields and intervals. □

The corollary reduces the terminal contribution for any independent list admitted in that theorem. Establishing the required projected rank and uniform upper bound for the resulting Gram height remains necessary in Section 6.6.

110.2 Cancellation of the explicit terminal index in the height

Fix the nonzero covectors at the vertices of a path as in Lemma 108.20. This time perform the full cyclic Fourier sums of Lemma 110.1 at every terminal position. For a given class, choose its rational orthonormal basis adapted to the r_j frequency directions at each of the f positions on edge j. The construction of that lemma applies to both CM halves. It identifies the terminal section on each of the fr_j constrained coordinates with one S_k; the good-reduction orientation was used only in evaluating its finite valuation. Expand the other local operators on the product vacuum as finite sums of translations at primes distinct from T. Return the scalar for each choice of basis to the original normalization according to Theorem 100.3. It has absolute value one at all complex embeddings.

Lemma 110.4 (Norm and pointwise bound for the complete projection)**.** *Let $h = \|\Theta_t\|_{\mathrm{av}}^2$ on the original normalized elliptic curve, with the metric density $\exp(-c|z|^2)$, $c = \pi/A$, and probability Haar measure. For every k, the average squared norm of S_k on its elliptic period lattice is h/T. The same holds for $U_b S_k$ for any torsion lift b. At every complex embedding these normalized factors, formed with the original finite Fourier multipliers, satisfy*

$$|(U_b S_k)(z)|^2 \le e^{c|z|^2}\, \frac{B_K h}{T}, \qquad B_K = e \max\{1, 2\sqrt{|D_K|}/\pi\},$$

using the area and reference norm at that embedding. The corresponding bound for an unprojected translated elliptic factor is $e^{c|z|^2} B_K h$.

Proof. On the common cover by Γ of Lemma 56.41, all T summands have squared norm h by translation invariance and invariance of Haar averages under finite covers. They are mutually orthogonal: the unitary periods from the original lattice act on them by distinct characters. Consequently the normalized sum with coefficient magnitudes $1/T$ has squared norm h/T. Its metric density descends to $\mathbb{C}/\Lambda'$. Both lattices have area A; translating the function again preserves its average norm.

For transport under embeddings we give the scalar calculation. Take $N = 2M$ as in Theorem 56.35, with a common M divisible by T killing every shift under consideration, enlarging it to kill a lift from the common cover if necessary. Under an embedding the rational quotient in that theorem transports exactly when $[w/(2N)]$ is transported. For $w = 2\omega_1/T$ this point is $[\omega_1/(TN)]$. Its conjugate is represented by a primitive period ω_1' divided by TN, since its exact order is TN. To lift a primitive residue pair modulo TN, complete it to a unimodular matrix modulo each prime power by elimination at a unit entry, and combine the elementary matrices by the Chinese remainder theorem. Lifting them gives an integral unimodular matrix. Semilinearity for CM preserves the condition of belonging to the prescribed pure line at level T^2, with the prime transported. Use the multiples jw' of this same representative $w' = 2\omega_1'/T$ in all the

terms, and a representative b' transporting $[b/(2N)]$. The composition phase is transported as well:

$$\exp(\pi i E(b,w)) = e_{N^2}([b/(2N)],[w/(2N)])^2.$$

The defining multiplier for U_bU_{jw} uses this phase or its inverse to power j. Functoriality of the Weil pairing and additivity of division points thus give a common translate of a full cyclic sum on the conjugate curve. The character ζ_T^{-k} conjugates to another T-th root of unity. This proves the norm assertion there including its relative phases.

For the pointwise estimate write either elliptic period lattice as γI, where I is a fractional $\mathcal{O}_K$-ideal. Every nonzero $\alpha \in I$ satisfies $|\alpha|^2 = |N_{K/\mathbb{Q}}\alpha| \geq N(I)$, since $(\alpha)I^{-1}$ is integral. The area of I is $\sqrt{|D_K|}N(I)/2$. Thus the shortest period has square length at least $2A/\sqrt{|D_K|}$, uniformly for the old ideal and for the new ideal $\mathfrak{p}^{-1}\bar{\mathfrak{p}}I$. Choose a disk of squared radius $r^2 = \min(A/\pi, A/(2\sqrt{|D_K|}))$, which embeds up to its boundary in the elliptic torus. At any chosen center apply the mean-value inequality to the squared modulus of the holomorphic Hermitian translate. Its integral over the disk is at most $e^{cr^2}A$ times its squared average norm. Division by πr^2 bounds the metric value there by B_K times that norm. Undoing the metric gives the formulas. □

For the exact expansion at an embedding σ, let a index the remaining data of a summand of column i: the covectors, the class path with its coupled intermediate vertex, and the translations outside T. Allow separate indices for different choices of product coordinates. After expanding the original homogeneous differential and the prescribed projection, write $v'_{i,a,\alpha}$ for the resulting coefficient vector, including the frame scalar, class masses and signs. Use the divided columns if the Gram matrix is G°. The scalar factors differentiated in that summand are the complete U_bS_k and the unprojected $U_b\Theta_t$ just constructed. The Fourier coefficient $1/T$ is already inside each S_k. For positive $\mu_i(a)$ of sum one put

$$B'_i = \sum_{a,\alpha} \frac{\alpha!}{\mu_i(a)} v'_{i,a,\alpha}(v'_{i,a,\alpha})^*.$$

An empty sum gives zero. Let n_{ij}, k_{ij} be the number of elliptic coordinates and the prescribed total derivative degree of edge j. Set $h_{ij,\sigma}$ equal to the squared average norm of its original marked elliptic factor, $c_{ij,\sigma} = \pi/A_{ij,\sigma}$, and

$$Q_{i,\sigma} = \prod_j (B_K h_{ij,\sigma})^{n_{ij}} c_{ij,\sigma}^{k_{ij}} H_{n_{ij}}(k_{ij}), \qquad d_i^T = f\sum_j r_j.$$

Here $H_n(k)$ is the explicit sphere factor of Lemma 54.8. Use the corresponding factor norms in the product if the elliptic marks differ by coordinate.

Theorem 110.5 (Archimedean estimate after the terminal sums)**.** *For the prescribed Gram matrix at any embedding and a list of s columns, the coefficient matrices just defined satisfy*

$$\det G_\sigma \;\leq\; T^{-\sum_i d_i^T} \left(\prod_i Q_{i,\sigma}\right) D_s(B'_1,\dots,B'_s).$$

Proof. Exactly fr_j scalar factors on edge j are projected, since the fixed nonzero ξ' gives r_j independent functionals at each position. Their product, including the unconstrained coordinates, is bounded in squared magnitude by

$$T^{-fr_j}(B_K h_{ij,\sigma})^{n_{ij}} \exp(c_{ij,\sigma}\|z\|^2)$$

by Lemma 110.4. Lemma 54.8 bounds the sum of the squared derivatives of degree k_{ij} divided by their multi-index factorials by this constant times $c_{ij,\sigma}^{k_{ij}} H_{n_{ij}}(k_{ij})$. Taking the product over the

separate edge variables bounds the corresponding sum for each fixed a by $T^{-d_i^T}Q_{i,\sigma}$. Vector Cauchy–Schwarz with weights $\alpha!/\mu_i(a)$, as in Theorem 109.10, gives $Y_iY_i^* \le T^{-d_i^T}Q_{i,\sigma}B_i'$. Monotonicity and multilinearity of the mixed determinant D_s give the claimed bound. □

Corollary 110.6 (The explicit terminal Fourier index cancels)**.** *Write $K_u^{(0)}$ for K_u in Theorem 108.27 with its displayed terminal summand omitted. For an independent list of q_c divided projected columns, the upper bound in that theorem on the same fields and intervals holds with its $H(G^\circ)$ replaced by*

$$K_u^{(0)} + \frac{\log T}{q_c}\sum_{i,j,\tau}\nu_{ij,\tau,1} + \frac{1}{q_c[J:\mathbb{Q}]}\sum_{\sigma:J\hookrightarrow\mathbb{C}}\log\left(\left(\prod_i Q_{i,\sigma}\right)D_{q_c}(B_1',\dots,B_{q_c}')\right).$$

Proof. Use Corollary 110.3 at the finite places. The terms involving fr_j there sum to $q_c^{-1}\sum_i d_i^T\log T$. Theorem 110.5 subtracts exactly this number in the averaged logarithm of the Gram determinant. The stated first-row term and the displayed coefficient bound give the result. □

The matrices B_i' still include all allowed intermediate covectors and depend on T. Bounding their contribution and exhibiting the required rank for the original signed projection are separate arithmetic questions. The cancellation here concerns the specified terminal normalization in the height on E_x.

110.3 Rank of the terminal derivative table on nonzero covectors

Let $p = T$ and use a connected terminal frequency block in Lemma 110.1. Keep the nonzero output covector ξ' fixed. After choosing the basis of its r-dimensional frequency subspace, the input covector is a linear functional on the Fourier parameter u. Write $k = (k_1,\dots,k_r) \in \mathbb{F}_p^r$ for its coordinates, so that its original coefficient is $\exp(-2\pi i\sum_h k_h u_h/p)$. Use the compatible cyclic translations in the proof of that lemma. The input condition at the vertex is $k \neq 0$. First consider a single product term from the translations at the primes other than T.

Lemma 110.7 (The moment of a connected Fourier projection)**.** *For each scalar coordinate h there is a nonzero residue β_h, depending on the differential and the frequency generator, such that for every prime-to-p translate U_b from the common cover and all $n \ge 0$, $k_h \in \mathbb{F}_p$,*

$$\overline{J_n(U_bS_{k_h})} = \overline{J_0(U_bS_{k_h})}\,(\beta_hk_h)^n.$$

We use $0^0 = 1$. The factor of order zero is a unit for $k_h \neq 0$. It is a unit also at $k_h = 0$ if the image of b on $\mathbb{C}/\Lambda'$ differs from the mark t'.

Proof. Write $z' = \alpha z$ and $S_0 = c\Theta_{t'}'$ as in the proof of Theorem 56.42. In the connected orientation $v(\alpha) = 1$ and c is a unit. The points $[\lambda_k]$ in Lemma 56.41 are in the étale p-line on the new curve and $[\lambda_k] = k[\lambda_1]$. Indeed the period pairing with the chosen frequency identifies Λ/Γ with the character group, and $\Gamma \subset \Lambda'$. Use this identification after scaling the curve by α. The prime-to-p point b contributes zero to the additive quantity L_1 in Theorem 56.33, where the annihilator can be chosen with p-part exactly p. Multiplication of its residue formula by $\overline{\alpha/p}^{\,n}$ gives the assertion with

$$\beta_h = \overline{\alpha/p}\,L_1(\alpha\lambda_1).$$

This is nonzero by full étale order. The scalar relating the two translated sections multiplies both the derivative and the value by the same unit, so the equality retains the original phase. At

label zero the point on the new curve has prime-to-p order. Injectivity of prime-to-p reduction and the order-zero divisor calculation in Theorem 56.33 give the last assertion. □

Fix prime-to-p displacements b_h. For a finite list $\mathcal{A}$ of multi-indices use the actual rectangular table

$$M_{\alpha,k} = \prod_{h=1}^{r} J_{\alpha_h}(U_{b_h} S_{k_h}), \qquad \alpha \in \mathcal{A}, \quad k \neq 0.$$

Define $R(0) = 0$ and $R(m) = 1 + ((m-1) \bmod (p-1))$ for $m > 0$, and apply R coordinatewise. Put $\mathcal{B} = R(\mathcal{A}) \subseteq \{0, \dots, p-1\}^r$.

Theorem 110.8 (Rank on the full set of nonzero terminal covectors)**.** *If each b_h misses its mark at label zero, then the rank of the reduction of M is*

$$|\mathcal{B}| - \mathbf{1}_{\{0,p-1\}^r \subseteq \mathcal{B}}.$$

The unshifted product vacuum satisfies the hypothesis. For arbitrary b_h, restrict to the labels $(\mathbb{F}_p^{\times})^r$. On these columns the residue rank is the number of distinct componentwise classes of α modulo $p-1$. In either case some minor of the stated size is a unit at this coefficient place when the rank is positive.

Proof. By Lemma 110.7, scaling a row by $\prod_h \beta_h^{-\alpha_h}$ and a column by the inverse of its product of order-zero values leaves the matrix of the functions $k \mapsto k^{\alpha}$. These scalings are invertible on the indicated labels. On all of $\mathbb{F}_p$ one has $X^m = X^{R(m)}$ as functions. The monomials with each exponent in $\{0, \dots, p-1\}$ form a basis of all functions on $\mathbb{F}_p^r$ over the residue field. For $r = 1$ this is polynomial interpolation at p distinct elements. Tensor products give the result for every r.

On deleting the origin the only functions in the kernel are multiples of its indicator, whose unique expansion in this basis is $\prod_h (1 - X_h^{p-1})$. It lies in the span of the selected monomials precisely when all its 2^r monomials are selected, proving the first formula. At the unshifted point each new mark is nonzero two-torsion. On the multiplicative torus the same interpolation uses the exponents $0, \dots, p-2$ in each coordinate and positive or zero powers reduce modulo $p-1$. The ordinary rank criterion over the residue field gives a nonzero minor; the original matrix is integral, so this minor lifts to a unit. □

Corollary 110.9 (All homogeneous orders at one terminal block)**.** *Assume the order-zero unit hypotheses of Theorem 110.8. If the rows run over all monomials of one degree $d > 0$, the residue rank on $k \neq 0$ is the number of $b \in \{0, \dots, p-1\}^r$ for which*

$$0 < |b| \leq d, \qquad |b| \equiv d \pmod{p-1}.$$

For $d < p$ it is $\binom{d+r-1}{r-1}$. For $d \geq r(p-1)$ it is $(p^r - 1)/(p-1)$. Degree zero has rank one. More generally, for any prescribed linear combinations of these monomial rows with integral coefficients, their residue rank is the dimension of the span of the corresponding reduced polynomials evaluated on $k \neq 0$, after replacing each variable by $\beta_h X_h$.

Proof. For $d > 0$ every reduced monomial has at least one positive exponent. Conversely one can increase any positive exponent of b by $d - |b|$ when this is a nonnegative multiple of $p-1$. This proves the counting formula and shows that its span excludes the delta function at zero. For $d < p$ no exponents change. For $d \geq r(p-1)$, every nonconstant reduced monomial of the given scalar weight occurs. On the punctured vector space all functions satisfying $F(tk) = t^d F(k)$ have dimension $(p^r - 1)/(p-1)$, one arbitrary value on each projective

line. Extend such functions by zero at the origin. Their unique reduced polynomials have zero constant term, and only monomials of weight $d \bmod (p-1)$ occur by uniqueness under scalar substitution. This proves the large-degree formula. In the last assertion the same unit factor from the order-zero values multiplies every row in a given column, so the polynomial computation also applies to the stated linear combinations. □

These are the covectors in the terminal indicator $\xi' \bar{X} = \xi$. When two adjacent class operators share a covector, the global period contracts their two tables in that index. The rank calculation supplies integral minors before that contraction; the rank of the contracted, signed projection in Section 6.6 requires its actual coefficients.

110.4 The signed covector sum on a connected terminal block

Fix a field, word, edge and terminal prime $p = T$ of Theorem 9.13. Work at a coefficient place where the chosen terminal frequency block is connected. On a conjugated edge make this choice on its conjugate. Keep the output covector $\xi' \neq 0$ and the unrefined frames fixed. The input at this block ranges over $k \in \mathbb{F}_p^r \setminus \{0\}$, where r is the incoming rank. Let κ be the residue field of a coefficient valuation ring containing the finitely many scalars in the calculation. Bars denote reduction to this field.

Use the complete cyclic projections and their Fourier labels in Lemma 110.7. In a product term γ of the outgoing section let $b_{\gamma,h}$ be the prime-to-p displacement in its h-th constrained coordinate. Set

$$W_\gamma(k) = \prod_{h=1}^{r} \overline{J_0(U_{b_{\gamma,h}} S_{k_h})}.$$

Any phase multiplying the whole translated product and depending on k is included in $W_\gamma(k)$ by its residue. This convention keeps the same phase in every differential order. The formulas below also allow $W_\gamma(k) = 0$.

Lemma 110.10 (The Weyl symbol in the input covector)**.** *In the adapted basis one can take the following reduction simultaneously for all input covectors. Evaluate an integral Weyl symbol on the translated product at this edge. Leave the unconstrained product derivatives as formal commuting variables D^0. Its residue after evaluation in the r constrained coordinates is*

$$W_\gamma(k)\, Q(D^0, k), \qquad \deg_k Q \le \nu_{\tau,1},$$

where τ is the matching physical block and $\nu_{\tau,1}$ is the first row length of its partition. The polynomial Q is obtained by integral polynomial substitution and reduction. One may then apply any integral linear functional to the coefficients in the complementary derivatives.

Proof. Extend a primitive lift of ξ' to a basis of the dual target space over $\mathbb{Z}_p$. Tensor this basis with a basis of the source. It gives a basis of the dual position matrix lattice, whose first r vectors are the functionals $X \mapsto \xi' X u_h$. The dual position basis consists of the dual simple tensors. In either basis the first r matrices have one common tensor factor. Consequently every linear combination of these matrices has rank at most one. This also holds after transposition. Use these choices in the product basis construction of Lemma 110.1, with the same approximation modulo p^2. The rational basis is fixed as k varies. In the physical chain rule the matrices of the first r directions reduce to $U_1, \dots, U_r$ with this common factor in the matching block. They reduce to zero in every other block. Indeed reduction selects the position or the dual position component of the split basis at the given embedding. Distinct local blocks are independent summands of the split lattice. Taking the tensor basis on both sides ensures the rank property for either choice of component.

For a monomial in the product derivatives, Lemma 110.7 replaces each constrained derivative variable in its residue by $\beta_h k_h$, with the common factor $W_\gamma(k)$ taken only once. Linearity gives this rule for the original symbol, including at a zero label. Thus in the matching physical matrix the substitution is

$$D_\tau^0 \longmapsto D_\tau^0 + \sum_{h=1}^{r} U_h \beta_h k_h.$$

The correction is an outer product with one factor constant over the polynomial ring, so it has rank at most one. By Lemma 108.17 a source generator and a dual test use $P_\tau([D_\tau A_\tau^{\mathsf{t}}, 0])$, followed by coefficient extraction in the elementary parameters of A_τ. Right multiplication and zero padding preserve the common outer factor. Each flag minor is affine in k because two columns from the outer product give zero in its determinant expansion by columns. Expansion by rows gives the same conclusion when the common factor is on the other side. Every term in the flag polynomial has $\nu_{\tau,1}$ minors. The symbols at the other embeddings are constant in k. The product, parameter extraction and integral linear combinations give the asserted degree bound. All reductions take place in the polynomial coefficients before evaluating complementary derivatives, which proves the last assertion. □

To apply the formula to a contraction, choose integral coordinates for the incoming table at the shared vertex. In expanding the sum, the index γ can record a coordinate of that table, a product term of the outgoing section, and any fixed coupling at the other marked primes. Write $Y_{\gamma,i}(k)$ for the incoming values for column i, including any assigned signed scalars. The adjacent map is allowed to depend arbitrarily on k. The expansion uses the unrefined stabilizer masses of Theorem 60.32. For reduction, take a common scalar for the finite list of incoming values and another for all evaluations of the complementary derivative monomials and remaining coefficients. Choose them so the two lists are integral. Such scalars exist by finiteness. They give the same overall normalizing factor in every summand. The assertion concerns the resulting integral matrix and its specified normalization. In particular it supplies no bound for the cost of those two scalars.

Theorem 110.11 (Contraction on the lines of the covector space)**.** *For each line L in $\mathbb{F}_p^r$ choose $v_L \in L \setminus \{0\}$. For $0 \le e \le D = \nu_{\tau,1}$ form the signed moments*

$$B_{\gamma,i,e}(L) = \sum_{a \in \mathbb{F}_p^\times} a^e W_\gamma(a v_L)\, \overline{Y_{\gamma,i}(a v_L)}.$$

For a fixed outgoing Weyl row j, expand the polynomial of Lemma 110.10 after the complementary evaluations as $\sum_{e=0}^{D} Q_{j,\gamma,e}(X)$ in its homogeneous parts. The residue of the matrix after summing over the entire shared covector in this block has entry

$$\sum_{\gamma, L} \sum_{e=0}^{D} Q_{j,\gamma,e}(v_L)\, B_{\gamma,i,e}(L).$$

For several edge shapes the same formula uses their individual degree bounds. It applies with any remaining indices fixed, and subsequently to their original finite linear combinations with integral coefficients. A nonzero minor of this matrix gives a minor with unit determinant in the normalized integral matrix.

Proof. The indicator of the edge is $\xi' \bar{X} = k$ in the fixed frames. By its complete Fourier expansion every outgoing product term has precisely the cyclic scalar labels used in the lemma. All dependence on the same covector from the previous edge is in the values $Y_{\gamma,i}(k)$. Holding the common coupling fixed until the two incident factors have been multiplied retains the signs

of that summand. Apply the lemma after the indicated denominator clearing. For $k = av_L$ the homogeneous part of degree e takes the value $a^e Q_{j,\gamma,e}(v_L)$. Summing over a, then over the lines, uses each admissible k exactly once. Replacing v_L by tv_L multiplies the two factors by t^e and t^{-e}. This proves the identity independently of the representatives. The minor assertion follows by reduction of the determinant. □

Corollary 110.12 (Visible moments at the prescribed word weights)**.** *On the original common intervals $D + 1 < p - 1$ for every sufficiently large x, uniformly in the chosen word, edge and place. On a fixed line let s_L be the dimension over κ of the span of the functions $a \mapsto W_\gamma(av_L)$ for a list of outgoing product terms. When a common scalar incoming table on this line varies, all normalized Weyl rows in this list together have residue rank at most $\min(p-1, s_L(D+1))$, with D the maximum first row length on the list. Here any additional coefficients varying with a must be included in the functions defining s_L.*

For one unshifted product and $D + 1 < p - 1$, the common kernel of the $D + 1$ moments on the line has dimension $p - D - 2$. Its intersection with the ordinary augmentation condition $\sum_a Y(av_L) = 0$ has dimension at least $p - D - 3$. Every member of this intersection has an integral lift satisfying the same augmentation condition before reduction.

Proof. By Lemma 108.11 and the weight formulas following (9.1), $D \ll xb + x^C$ for a fixed C, where $b = \exp(x^\varepsilon)$. Theorem 71.17 gives $\log p = (17 + o(1))m \log b$ and $m \to \infty$. Hence $(D + 1)/(p - 1) \to 0$. On a single line the row functions in the theorem lie in the sum, over $0 \le e \le D$, of a^e times the indicated s_L-dimensional space. The dimension of that sum proves the upper bound.

For the unshifted product every $W_\gamma(av_L)$ is nonzero by Theorem 110.8. Multiplication by this weight is an invertible diagonal operator. The functions $1, a, \dots, a^D$ are independent on $\mathbb{F}_p^\times$, since a polynomial of degree at most $D < p - 1$ vanishing there is zero. Subtracting the rank $D+1$ of the moment map from $p-1$ gives the kernel dimension. One more linear equation removes at most one dimension. Lift all but one value arbitrarily and take the last to be the negative of their sum. This lifts the given residue vector with exact augmentation. □

The moments concern the actual shared index on one outgoing block. On that block even the unshifted integral test can lose many incoming residue directions after contraction. To obtain the projected rank in Section 6.6 one must establish survival for the prescribed incoming columns with their actual weights and the remaining class sums.

110.5 Scalar orbits through the signed theta chain

Fix $p = T$, the word and the terminal generator of Theorem 9.13. Use the original initial tables whose values sum to zero on each nonzero scalar orbit at each $v \mid T$, with the other coordinates held fixed. Extend these tables by zero at any zero covector. The local character decomposition in Corollary 106.26 is integral at T. We compute a gain in the finite estimate each time such an input passes through a terminal incidence condition in the chain.

At a fixed coefficient place $w \mid p$ let ω denote the Teichmüller lift of $\mathbb{F}_p^\times$, and write $d = p-1$. Use the convention $H(ak) = \omega(a)^t H(k)$ for a scalar character of an incoming table. We take $1 \le t < d$. For a fixed nontrivial additive character ψ put

$$g(t) = \frac{d - t}{d}, \qquad G_t(c) = \sum_{a \in \mathbb{F}_p^\times} \omega(a)^t \psi(-ac).$$

One has $G_t(0) = 0$ and $v_w(G_t(c)) = g(t)$ for $c \neq 0$, in the normalization $v_w(p) = 1$. Here is a direct check of the sign convention in Theorem 104.12. Changing $c \neq 0$ changes the sum

by a root of unity, so use the additive terms ζ^a for a primitive p-th root ζ. Let $\lambda = \zeta - 1$, with $v_w(\lambda) = 1/d$ by $\prod_{a=1}^{p-1}(1-\zeta^a) = p$. The coefficient of λ^n after expanding $(1+\lambda)^a$ is $\sum_{a=1}^{p-1}\omega(a)^t\binom{a}{n} \in \mathbb{Z}_p$. For $0 \le n < d-t$ its reduction vanishes by summing powers of exponents from t to $t+n < d$ on $\mathbb{F}_p^\times$. For $n = d-t$ the reduction is $-1/n!$. Thus the term of index $d-t$ has uniquely smallest valuation. The assertion at zero follows by summing the nontrivial scalar character.

Proposition 110.13 (The orbit sum at each edge of a fixed character)**.** *At each terminal place prescribe a nontrivial scalar character χ_v of the initial table, in the convention just given. At every vertex choose representatives for the lines of nonzero covectors. At the terminal vertex choose the actual marked covector for its line. On a local edge with consecutive representatives η, η' and source dimension r, the full signed period in this character may be computed by summing over the intermediate lines and using the local multiplier*

$$\begin{aligned}\mathcal{K}_{\eta,\eta',\chi_v}(X) &= \sum_{a\in\mathbb{F}_p^\times} \chi_v(a)\,\mathbf{1}_{\eta'\bar{X}=a\eta} \\ &= p^{-r}\sum_{u\in\mathbb{F}_p^r} \psi(\eta'\bar{X}u)\sum_{a\neq 0}\chi_v(a)\psi(-a\eta u)\end{aligned}$$

on the integral matrix lattice. The terms with $\eta u = 0$ vanish. For one product vacuum in the adapted coordinates of Lemma 110.4, the complete local section for this multiplier has average squared norm

$$(p-1)p^{-r}\prod_{h=1}^{r} h_h$$

on the constrained coordinates, where h_h is the corresponding original elliptic average squared norm. This holds at every complex embedding and after any common translation on these coordinates.

Proof. Write the covector at the j-th vertex as $a_j\eta_j$. Its incidence with the next vertex is $\eta_{j+1}\bar{X}_j = (a_j/a_{j+1})\eta_j$. Take $a_m = 1$ at the marked final covector. Substitution of $s_j = a_j/a_{j+1}$ is a bijection on the m scalar variables, and $\chi_v(a_0) = \prod_{j=0}^{m-1}\chi_v(s_j)$. Thus all scalar sums separate with exactly the first multiplier displayed above. Perform the substitution independently at each terminal place. The path formula of Theorem 60.32 has already used the unrefined stabilizer masses. The matrix cosets at other places, their signed coefficients, and the coupling coset common to the last two edges are fixed during this substitution. The theta section is linear in each local test. Summing over the other data now gives the stated formula for the signed period. The finite Fourier expansion of the indicator gives the second equality. Scalar character orthogonality removes the terms specified.

For the complex norm the section is $\sum_{a\neq 0}\chi_v(a)\prod_h S_{a\eta_h}$. Work on the product of the common covers for the cyclic projections. In each coordinate the p frequency translates of the vacuum are orthogonal with equal squared norm h_h, by their distinct period characters. Their normalized finite Fourier transform therefore gives p orthogonal sections of squared norm h_h/p. Distinct a give distinct label vectors since $\eta \neq 0$. Taking tensor products and then summing squared norms proves the formula. Translations are unitary on the common covers. At a complex embedding the finite Fourier characters and their phases transport as in Lemma 110.4. The scalar characters continue to have absolute value one. □

Theorem 110.14 (Divisibility from the incoming scalar orbits)**.** *Use the unrefined integral Weyl lattices of Theorem 110.2. For column i and edge j let B_w be the embeddings with étale frequency at the effective coefficient place on that edge, and $v(\tau) = P_{\tau,w} \cap E_x^+$. Suppose the*

incoming table has valuations at least h in the common lattice and scalar characters $\chi_v(a) = \omega(a)^{t_v}$. After summing all input covectors, the terminal part of this edge has entries of valuation at least

$$h - \sum_{\tau \in B_w} (r_j + \nu_{ij,\tau,1} - g(t_{v(\tau)})).$$

The separately specified nonterminal scalars and class masses are excluded here. For every separately zero-sum input the same bound holds with $1/(p-1)$ at each occurrence of $g(t_v)$. These gains add through all the consecutive edges of the original signed period.

Proof. Keep the output covectors and the data away from T fixed. In each étale frequency block use the second formula of Proposition 110.13. For each input line the incoming value at its representative is integral after the common input bound is removed. At a nonzero frequency pairing the scalar sum contributes precisely $g(t_v)$ in valuation. The reciprocal p^{r_j} contributes $-r_j$. Its remaining modulation $X \mapsto \psi(\eta' \bar{X} u)$ has rank one, as a frequency matrix, and is independent of the input scalar. Keep the complete cyclic projections in the connected blocks. There each term with a fixed input scalar is integral in every constrained differential order by Lemma 110.1. The sum with the character coefficients remains integral.

The adapted bases of Theorem 110.2 apply to the product of these choices. At an étale block only a single rank-one direction can cost a valuation of one per derivative, up to $\nu_{ij,\tau,1}$ derivatives. The scalar Gauss factors multiply the entire differential symbol. Their valuation gain survives that estimate and the return to the common frame. Summing over the remaining frequencies and input lines by the valuation inequality proves the bound.

For a fixed matrix coset the coefficient map reads the zero-extended table at $\xi' \bar{X}$. Scaling ξ' by a scales this pullback by a, so scalar types and separate zero sums propagate, including at a fixed coupling coset. Conjugation preserves nontriviality of the character. Induction adds the gains along the chain. The masses and denominators are those of Theorem 108.21.

The scalar projectors at w have unit denominators, powers of $p-1$. They preserve the input bound and the refined lattice on each line, since a scalar preserves the covector kernel. The zero-sum condition kills the trivial character at each place. Every remaining summand gains $g(t_v) \geq 1/(p-1)$, and addition preserves this bound. □

Corollary 110.15 (A gain for the divided projected columns)**.** *For a list of q_c independent projected columns with separately zero-sum initial tables, one may subtract*

$$\frac{f \log T}{q_c (T-1)} \sum_{i,j} 1$$

from the upper bound for their contribution to Δ_u in Corollary 110.6. If all initial tables have the quadratic scalar character at every terminal place, one may subtract $(f \log T/(2q_c)) \sum_{i,j} 1$ instead.

Proof. Scaling by z_i^{-1} as in Theorem 108.21 puts the initial values in the unrefined lattice with the same scalar type. Apply the final lowering and column division of Theorem 108.27. For each τ the places with étale frequency have total normalized degree $1/2$ by Corollary 110.3. Summing their gains for both period ideals and dividing by q_c gives the saving. The quadratic exponent is $(T-1)/2$ at every completion. Theorem 110.5 supplies the archimedean bound. □

The determinant estimate assumes independence. Section 6.6 still requires that independence and the full height bound on the prescribed projections.

110.6 Mean Gauss valuations in the projected period height

Keep the terminal prime $p = T$, the word, the marked generators, and the integral frames of Theorem 110.14. Put $F = E_x^+$ and $f = [F : \mathbb{Q}]$. For a given column fix a place v of F above p. Its residue field is $\mathbb{F}_p$. The scalar action on nonzero input covectors is by $\mathbb{F}_p^\times$. Use a finite Galois coefficient field J containing E_x, the test data and all scalar character values. The probability of a coefficient place $w \mid p$ is $[J_w : \mathbb{Q}_p]/[J : \mathbb{Q}]$. This is the weighting in the global height, and enlarging J gives the same average by the tower formula for local degrees.

At the effective coefficient place of an edge let $\mathcal{E}_v$ be the event that the block belonging to v has étale frequency. For a nontrivial scalar character χ let $g_w(\chi) = (p - 1 - t)/(p - 1)$, where $\chi = \omega_w^t$, $1 \le t < p - 1$, in the convention of Proposition 110.13. The valuation of p is one. On a conjugated edge use the effective place and character of that edge in both the event and g_w.

Lemma 110.16 (The smallest mean on a set of coefficient places of weight one half)**.** *One has* $\Pr(\mathcal{E}_v) = 1/2$*. If* χ *has order* $m > 1$*, then*

$$\mathbb{E}_w[\mathbf{1}_{\mathcal{E}_v} g_w(\chi)] \ge A(m).$$

Here $A(2) = 1/4$*, and for* $m > 2$*,*

$$A(m) = \frac{1}{m\varphi(m)} \sum_{\substack{1 \le a < m/2 \\ (a,m)=1}} a \;\ge\; \frac{1}{12}.$$

These are the exact lower bounds when only the marginal distribution of $g_w(\chi)$ *and the weight of the event are known.*

Proof. For fixed w, v there is a unique embedding τ in the CM type with $\tau^{-1}w \cap F = v$, by complete splitting and the bijection in Theorem 110.2. Conjugation on J fixes the totally real field $\tau(F)$ pointwise. Thus the same τ occurs for the conjugate place. On $\tau(E_x)$ it is CM conjugation, so the two pullbacks to E_x are the partners above v. Exactly one belongs to S_T. Conjugation preserves the local degree. The same pairing applies when the effective place at an edge is conjugated. This gives the half weight for each individual v.

Fix a generator of $\mathbb{F}_p^\times$. Its value under χ is a primitive m-th root. Since $m \mid p - 1$, p splits completely in $\mathbb{Q}(\zeta_m)$. Reduction at all its places takes that value through the primitive m-th roots in $\mathbb{F}_p$, each once. The total local degree over every such place in J is the same. Thus $g_w(\chi)$ is uniformly distributed on $\{a/m : 1 \le a < m,\ (a, m) = 1\}$. For $m > 2$ precisely half of these values lie below $1/2$, by pairing a with $m - a$. Among events of weight $1/2$ the integral is smallest when it puts all its mass on those values. For $m = 2$ the function is identically $1/2$.

For the numerical bound let μ be the Möbius function, $\mathrm{rad}(l) = \prod_{q \mid l} q$, and let $\omega(l)$ count the distinct prime divisors of l. For $m > 2$ the endpoint $m/2$ makes no contribution and inversion gives

$$m\varphi(m)A(m) = \sum_{d \mid m} \mu(d) d \sum_{1 \le b \le m/(2d)} b.$$

Writing $u = m/d$, the inner sum equals $(u^2 + 2u)/8$ for even u and $(u^2 - 1)/8$ for odd u. The quadratic terms together give $m\varphi(m)/8$. For odd m, the remaining terms sum to $-\prod_{q \mid m}(1 - q)/8$. For $4 \mid m$, each squarefree $d \mid m$ leaves even u, and the remaining terms vanish by $\sum_{d \mid m} \mu(d) = 0$. For $m = 2l$ with $l > 1$ odd, the even-u linear terms vanish by the same cancellation over the divisors of l. The odd-u corrections then give $2\prod_{q \mid l}(1 - q)/8$. Consequently

$$8A(m) = \begin{cases} 1 - (-1)^{\omega(m)} \mathrm{rad}(m)/m^2, & m \text{ odd}, \\ 1, & 4 \mid m, \\ 1 + (-1)^{\omega(l)} \mathrm{rad}(l)/l^2, & m = 2l,\ l > 1 \text{ odd}. \end{cases}$$

In both odd arguments l or m is at least three and $\operatorname{rad}(l) \le l$. This proves the lower bound. It is attained by the displayed marginal formula when $m = 6$. □

Theorem 110.17 (A terminal saving independent of the order of a pure scalar character)**.** *Let q_c divided projected columns be independent as in Corollary 110.6. Suppose each initial table belongs to a nontrivial scalar character at each $v \mid T$, of order m_{iv} on column i. From the upper bound in that corollary one may subtract*

$$\frac{2\log T}{q_c}\sum_{i,j}\sum_{v\mid T} A(m_{iv}) \;\ge\; \frac{f\log T}{6q_c}\sum_{i,j} 1.$$

Here j runs over the edges of the word for column i. If the columns can be paired with matching edges, terminal halves and effective places, and inverse scalar characters at every v, the saving is at least $(f\log T/(2q_c))\sum_{i,j} 1$.

Proof. At a coefficient place w, the improvement at v from Theorem 110.14 is $\mathbf{1}_{\mathcal{E}_v} g_w(\chi_{iv})$ in valuation. That theorem applies to the complete edge after the incoming scalar sum, with the common class at two consecutive edges held fixed until their contributions are multiplied. The finite bounds therefore add over edges. Averaging over w, multiplying by $\log T$, and including both factors in the Gram bound gives the prefactor $2\log T/q_c$, as in Corollary 110.15. Lemma 110.16 bounds each remaining average. There are exactly f places v because T splits completely in F.

For a matching pair one has $g_w(\chi) + g_w(\chi^{-1}) = 1$ at the same effective place, including for quadratic χ. The event is the same at the paired edges, hence its mean contribution summed over the two columns is $1/2$. The saving per column averaged over that pair follows after the same factor of two in the Gram bound. □

Corollary 110.18 (Bounds in terms of the scalar support of the prescribed input)**.** *For an independent list with separately zero-sum initial tables, let s_{iv} be the number of distinct scalar characters occurring at v in column i, over a splitting field. In the first bound of Theorem 110.17 each $A(m_{iv})$ may be replaced, for a general input, by*

$$\max\left\{\frac{1}{2(T-1)},\; \frac{1}{3136\,s_{iv}^2}\right\}.$$

For a single supported character the larger bound of the theorem applies directly.

Proof. The projectors onto scalar characters have denominators dividing a power of $T-1$, which is a unit at T. They preserve the integral lower bound of the input. Each component propagates through the incidence condition with its scalar type as in Theorem 110.14. For their sum use the minimum of the gains over the $s = s_{iv}$ supported characters at this place. A nonzero column has $s \ge 1$ and augmentation excludes the trivial character. Every g_w is at least $1/(T-1)$, which proves the first option after averaging over the half event.

For $0 < y < 1$ and any character order $m \ge 2$, its marginal distribution gives

$$\Pr\{g_w(\chi) \le y\} = \frac{\#\{1 \le a \le my : (a,m) = 1\}}{\varphi(m)} \le y + \frac{2^{\omega(m)}}{\varphi(m)}.$$

Indeed Möbius inversion replaces each integer part by its argument with an error of at most one per squarefree divisor. The probability is zero if $my < 1$. For all m, $\sqrt{m}\,2^{\omega(m)}/\varphi(m) \le 6$. At a prime power q^a the factor is at most $2\sqrt{q}/(q-1)$, which is less than one for $q \ge 7$. The product of the three possible factors at $2, 3, 5$ is $\sqrt{30} < 6$. Thus in both ranges $\Pr\{g_w(\chi) \le y\} \le 7\sqrt{y}$.

Set $y = (28s)^{-2}$. The union of the exceptional events for all supported characters has weight at most $1/4$. The étale event has weight $1/2$, leaving weight at least $1/4$ where all the gains exceed y. Its contribution to the mean is at least $y/4 = 1/(3136s^2)$. Linearity of the average and the valuation inequality for the sum prove the claim. □

These bounds retain the original scalar supports and the same integral projections. The archimedean coefficient term and the first-row terms in Corollary 110.6, as well as independence for the columns needed in Section 6.6, remain in the comparison.

110.7 The CM half in the terminal character average

Use the actual CM field $K = \mathbb{Q}(\sqrt{-D})$ of the common period construction, with D the auxiliary prime congruent to seven modulo sixteen. Its discriminant is $-D$. Write λ for its primitive quadratic Dirichlet character of conductor D. The CM embeddings of E_x extend one chosen embedding of K by Theorem 52.10. The terminal prime is $T = \pi\bar{\pi}$ with $\pi \in \mathcal{O}_K$, by Theorem 9.13. In particular the condition $P_{\tau,w} \mid \pi$ in Theorem 110.2 depends only on the restriction of the effective coefficient place w to K. It specifies one of the two primes of K above T, the same one for all τ on this edge. Denote this event by $\mathcal{E}$.

Lemma 110.19 (The joint distribution of the scalar exponents and the CM half)**.** *At one place $v \mid T$ let the nonempty support of a separately zero-sum input consist of characters ρ^e for e in a set $I \subseteq \{1, \dots, m-1\}$, where ρ generates the group spanned by the support and has order $m \mid T-1$. Write $[k]_m$ for the least nonnegative residue modulo m and put*

$$H_I(a) = \min_{e \in I} \frac{[ea]_m}{m} \qquad (a \in (\mathbb{Z}/m\mathbb{Z})^\times).$$

Then the mean $\mathbb{E}_w[\mathbf{1}_{\mathcal{E}} \min_{e\in I} g_w(\rho^e)]$ in the terminal estimate is

$$\begin{cases} \dfrac{1}{2\varphi(m)} \displaystyle\sum_{(a,m)=1} H_I(a), & D \nmid m, \\[2ex] \dfrac{1}{\varphi(m)} \displaystyle\sum_{\substack{(a,m)=1 \\ \lambda(a)=\epsilon}} H_I(a), & D \mid m. \end{cases}$$

The sums use one complete residue system. In the second line the sign is determined at any single coefficient place w_0 on the chosen half by $\epsilon = \lambda(m g_{w_0}(\rho))$. If $I = -I \bmod m$, either line equals the first expression.

Proof. Let $a(w) = m g_w(\rho)$, a unit residue as in Lemma 110.16. The definition of the Gauss valuation uses the negative of the scalar character exponent modulo $T-1$. Multiplying that exponent by e and taking its positive fractional representative gives $g_w(\rho^e) = [ea(w)]_m/m$. These numerators are nonzero since the characters in the support are nontrivial.

The prime T splits completely in both K and $\mathbb{Q}(\zeta_m)$. Its primes in the compositum of the two fields are thus indexed simply transitively by the Galois group. Each receives the same total local-degree weight from the coefficient field. The restriction to K fixes $\mathcal{E}$, and the restriction to $\mathbb{Q}(\zeta_m)$ fixes $a(w)$. If $D \nmid m$, then D is unramified in $\mathbb{Q}(\zeta_m)$, whereas it ramifies in K. Hence the intersection is $\mathbb{Q}$, and these two uniform choices are independent. If $D \mid m$, then K is the quadratic subfield of $\mathbb{Q}(\zeta_D)$ cut out by λ. For example, the quadratic Gauss sum at D has square $-D$ and transforms by λ. A Galois element sending the prime at w_0 to the prime at w fixes K exactly when its exponent on ζ_m is in $\ker \lambda$. The inverse of that exponent multiplies $a(w_0)$ to give $a(w)$. This proves the coset formula and the sign rule. For inverse-stable support

$H_I(-a) = H_I(a)$. The two cosets are interchanged by -1, since K is imaginary, proving the last assertion. The argument uses the effective place throughout, so it applies equally to a conjugated edge. □

Theorem 110.20 (The exact mean for a pure scalar character at the terminal prime)**.** *For a character of order $m > 1$ the mean $G = \mathbb{E}_w[\mathbf{1}_{\mathcal{E}} g_w(\chi)]$ is $1/4$ when $D \nmid m$. For $D \mid m$ it is*

$$G = \frac{1}{4} - \frac{\epsilon h(K)}{w_K \varphi(m)} \prod_{\substack{q \mid m \\ q \text{ prime}, q \neq D}} (1 - \lambda(q)),$$

where $h(K)$ is the class number, $w_K = |\mu(K)|$, and ϵ is as in the lemma for $\rho = \chi$. In particular every character of even order has $G = 1/4$. For every order one has, uniformly in v, the word and the edge,

$$|G - 1/4| \le \frac{h(K)}{w_K (D-1)} \ll \frac{\log(2D)}{\sqrt{D}}.$$

Proof. Use $I = \{1\}$ in the lemma. The mean of a/m over all unit residues is $1/2$ by symmetry, including $m = 2$. For $D \mid m$, replace the indicator of the coset by $(1 + \epsilon\lambda(a))/2$. Its correction to $1/4$ is $\epsilon/(2m\varphi(m))$ times $\sum_{1 \le a \le m,\ (a,m)=1} a\lambda(a)$.

Put $B_\lambda = D^{-1} \sum_{a=1}^{D} a\lambda(a)$. For R a multiple of D, periodicity and $\sum_{a=1}^{D} \lambda(a) = 0$ give $\sum_{b=1}^{R} b\lambda(b) = R B_\lambda$. Inclusion-exclusion for the distinct prime divisors of m other than D now gives

$$\frac{1}{m} \sum_{\substack{1 \le a \le m \\ (a,m)=1}} a\lambda(a) = B_\lambda \prod_{\substack{q \mid m \\ q \text{ prime}, q \neq D}} (1 - \lambda(q)).$$

Indeed a squarefree divisor u of those primes contributes $\mu(u) u \lambda(u) \sum_{b \le m/u} b\lambda(b)/m = \mu(u)\lambda(u) B_\lambda$. The identity $B_\lambda = -2h(K)/w_K$ is the imaginary quadratic class number formula at zero. In detail, $L(0, \lambda) = -B_\lambda$ follows by splitting the Dirichlet series into Hurwitz zeta functions and using $\zeta(0, a/D) = 1/2 - a/D$. The factorization $\zeta_K = \zeta L(\cdot, \lambda)$, $\zeta(0) = -1/2$, and the analytic class number formula $\zeta_K(0) = -h(K)/w_K$ give the asserted identity.

The prime two splits in K, as $-D \equiv 1 \bmod 8$. Therefore $2 \mid m$ kills the correction whenever $D \mid m$. For an odd m divisible by D, each factor at $q \neq D$ has magnitude at most two, while $\varphi(q^a) \ge 2$ for every odd prime power. The remaining denominator $\varphi(D^a)$ is at least $D - 1$. This gives the first uniform bound. For the second, Minkowski gives an integral ideal of norm at most $2\sqrt{D}/\pi$ in each class of K. The number of quadratic ideals of norm b is at most the divisor function of b, by comparison of the split, inert and ramified Euler factors with those of $\zeta(s)^2$. Summing up to R gives at most $R(1 + \log R)$ ideals, by counting pairs of positive integers with product $\le R$. This bounds $h(K)$ as required. □

Corollary 110.21 (The terminal contribution on the original common intervals)**.** *For an independent list of q_c divided projected columns whose input is in one nontrivial scalar character at every terminal place, one may subtract at least*

$$\frac{f \log T}{q_c} \left(\frac{1}{2} - \frac{2h(K)}{w_K (D-1)} \right) \sum_{i,j} 1$$

from the upper bound of Corollary 110.6. If every order is even or is prime to D, the coefficient in parentheses may be taken to be $1/2$. For the family of Theorem 12.15 it is at least $1/2 - o(1)$ uniformly as the intervals tend to infinity. For a general separately zero-sum input the allowable subtraction on any block is obtained by multiplying the finite sum in Lemma 110.19 by $2 \log T/q_c$.

Proof. Apply Theorem 110.14 at each effective coefficient place and average the gains as in Theorem 110.17. For the pure inputs use the exact means just calculated. There are $f = [E_x^+ : \mathbb{Q}]$ blocks per edge, and the two period factors in the determinant give the stated multiplier. For a sum of characters use the integral scalar idempotents at T and the smallest of the supported gains on that block. The joint-distribution lemma evaluates its average. In the selection in Theorem 12.15, $D \equiv -1$ modulo all odd primes up to a parameter tending to infinity, so the prime D tends to infinity as well. The preceding uniform class number estimate proves the limiting statement on these same fields. □

This determines the averaged scalar-character factor of the terminal Gauss sums. The first-row derivative costs, the archimedean coefficient matrices and the required projected rank still enter the bound for the full period determinant.

110.8 Compact frame pairings in the projected period

At a fixed archimedean embedding, the upper bound of Theorem 110.5 uses the Fischer inner products of the polynomials in the differential variables. We compute these inner products with the initial forms and the terminal tests kept in the formula. This extends the scalar calculation at the identity frame in Theorem 54.30. It also gives cross terms between paths with different intermediate coefficient maps.

For one embedding of the CM type let $r \le s$ be consecutive ranks. Give the space H^- of input flag polynomials the norm

$$\|Q\|^2 = \int_{U(r)} |Q(u)|^2 \, du,$$

with Haar probability, and use the corresponding norm in rank s. Restriction to the unitary group determines a polynomial, by Zariski density. A determinant twist on this group preserves the norm. If the input has column weight λ, let n be the twist giving the partition $\nu = \lambda + (n, \dots, n)$ used in the differential symbol. The function $Q \det^n$ extends to a polynomial of this column weight on the full matrix space. In (54.22), view the differential matrix as an independent s-by-r matrix Z. Its polynomial operator is

$$(C(Z)Q)(Y) = (Q \overset{n}{\det})(Z^{\mathsf{t}} Y_{[r]}).$$

Write H^+ for the compact coefficient space of column weight $(\nu, 0, \dots, 0)$ in rank s, and $d_+ = \dim H^+$. For several CM embeddings use the tensor products of these spaces and maps, with a separate Z_τ at each embedding. Set

$$c = \prod_{\tau} \prod_{h=1}^{r} (r - h + 1)_{\nu_{\tau,h}}, \qquad (a)_k = \prod_{l=0}^{k-1} (a + l), \qquad (a)_0 = 1.$$

Throughout this subsection A^* is the Hermitian adjoint in the stated metrics.

Lemma 110.22 (The covariance of one polynomial substitution)**.** *Give the entries of the matrices Z_τ independent standard complex Gaussian distributions of density $\pi^{-1} e^{-|z|^2}$. For every endomorphism A of H^-, one has*

$$\mathbb{E}\, C(Z) A C(Z)^* = \frac{c}{d_+} \operatorname{tr}(A)\, I_{H^+}.$$

Proof. First use one embedding and $A = QQ^*$. For $Y \in U(s)$ the columns of $Y_{[r]}$ are orthonormal. Each row of $Z^{\mathsf{t}} Y_{[r]}$ is a standard Gaussian row of length r by unitary invariance, and

different rows are independent. Write this square Gaussian matrix as uB by Gram–Schmidt, with upper triangular B of positive diagonal. Conditional on B, left unitary invariance makes u Haar. Write $t_h = B_{hh}$. Conditional on the first $h-1$ Gaussian columns, t_h^2 is a sum of $r-h+1$ independent exponential variables of mean one, by projecting the next column to the orthogonal complement. Its moment of order ν_h is $(r-h+1)_{\nu_h}$, as follows from the generating function $(1-z)^{-(r-h+1)}$. The flag identities under upper triangular substitution give

$$|(Q \overset{n}{\det})(uB)|^2 = |Q(u)|^2 \prod_{h=1}^{r} t_h^{2\nu_h}.$$

Successive conditioning from the last column gives $\mathbb{E}|(C(Z)Q)(Y)|^2 = c\|Q\|^2$, independent of Y. Integrating over Y gives the trace of $\mathbb{E}(C(Z)QQ^*C(Z)^*)$.

This operator commutes with left translation by $U(s)$, since changing Y to gY changes Z to $g^{\mathsf{t}}Z$, which has the same law. The column-weight space H^+ is irreducible under this left action. Indeed the Cauchy decomposition of matrix polynomials, used in Section 54.10, pairs one row module with the matching column module. Selecting the right upper-unipotent invariants of the given torus weight selects its unique highest line. Thus Schur's lemma makes the operator scalar, and its trace fixes the scalar. Polarization gives the assertion for $Q_1Q_2^*$; these maps span all endomorphisms. For several embeddings, independence gives the tensor product identity on simple tensors of endomorphisms, which span the full endomorphism space. □

Fix one of the words and the embedding σ. Expand the actual column as in Theorem 110.5, keeping the two last edges' shared class or coupling coset as a single choice. For a summand γ, write x_γ for its initial coefficient vector. At edge j denote the two compact spaces by H_j^-, H_j^+, the latter dimension by d_j, and the constant in the lemma by c_j. Write $M_{\gamma j} : H_j^+ \to H_{j+1}^-$ for the intervening coefficient transport, and $L_\gamma : H_m^+ \to V$ for the final linear map to the actual projected test space. Include the scalar of this summand in L_γ, including its phase, class measure and any column division. One may use an identity map at an intermediate vertex. Determinant characters on frames are included in the maps or in the weights of the adjacent spaces. Use conjugate spaces and conjugate variables on the conjugated factors. More explicitly, conjugation of a product conjugates every preceding coefficient and variable in that product. Moving the bars to these factors gives linear maps between the corresponding conjugate spaces and preserves the Gaussian law and the adjoint formulas.

Group summands only when their ordered scalar elliptic derivative functions and the unitary changes from product variables to physical matrix variables agree on every edge, at this embedding. Write a for such a group. Finer groups, including individual summands, are allowed. The definition permits covectors and adapted bases to vary throughout the original sum. In one group the coefficient polynomial of path γ has the form

$$P_\gamma(Z) = L_\gamma C_m(Z_m) M_{\gamma,m-1} C_{m-1}(Z_{m-1}) \cdots M_{\gamma 1} C_1(Z_1) x_\gamma.$$

In fact keeping the derivatives formal in the class formula gives exactly the determinant multiplication and substitution in (54.22). Expanding translations in the finite local tests changes the scalar elliptic functions to which one applies these derivatives. The remaining coefficient operations are the initial evaluation, the class transports, and the terminal lowering and projection of Theorem 60.32. All are independent of the formal differential variables. An orthonormal product basis from Lemma 53.8 changes the variables by a unitary matrix on the full edge space. One common such change in a group preserves all its Fischer inner products, even when it mixes matrix coordinates.

Theorem 110.23 (The signed coefficient matrix for a group of paths)**.** *Return to the product variables and write the sum of P_γ in one group a as $\sum_\alpha v_{a,\alpha} Z^\alpha$. Its positive semidefinite coefficient matrix is*

$$\begin{aligned} S_a &= \sum_\alpha \alpha!\, v_{a,\alpha} v_{a,\alpha}^* \\ &= \frac{\prod_{j=1}^m c_j}{d_m} \sum_{\gamma,\delta \in a} \operatorname{tr}(x_\gamma x_\delta^*) \left(\prod_{j=1}^{m-1} \frac{\operatorname{tr}(M_{\gamma j} M_{\delta j}^*)}{d_j} \right) L_\gamma L_\delta^*. \end{aligned}$$

In particular the sum retains the cross terms after the signed projection. At a vertex where the two transports are the actions of elements g_γ, g_δ in the same unitary representation of dimension d_j, the trace factor there is its character at $g_\gamma g_\delta^{-1}$ divided by d_j.

Proof. For a standard complex Gaussian entry, $\mathbb{E}[Z^b \bar{Z}^e] = \mathbf{1}_{b=e} b!$ by polar integration. Independence therefore identifies the first line with the expectation of the outer product of the sum of the path polynomials with itself. For a pair of summands start with $x_\gamma x_\delta^*$. Integrate Z_1 by Lemma 110.22. Its value is a scalar multiple of the identity of H_1^+. Transport on the two sides changes that identity to $M_{\gamma 1} M_{\delta 1}^*$. Its trace enters upon integrating Z_2. Each subsequent integration performs the same operation, since the edge variables are independent. The last transports take the last identity to $L_\gamma L_\delta^*$. Multiplying the constants and summing over the ordered pairs proves the formula. For a unitary action its adjoint is the action of the inverse, giving the character assertion. □

Corollary 110.24 (A bound for the prescribed determinant without derivative indices)**.** *For each column i attach its index to the preceding data. Use $Q_{i,\sigma}$ and d_i^T from Theorem 110.5. Put*

$$A_{i,\sigma} = \sum_\gamma \|x_\gamma\| \, \frac{\|L_\gamma\|_{\mathrm{HS}}}{\sqrt{d_{im}}} \prod_{j<m} \frac{\|M_{\gamma j}\|_{\mathrm{HS}}}{\sqrt{d_{ij}}}, \qquad F_i = \prod_{j=1}^m c_{ij}.$$

The endpoint norm uses the given metric on V. For any positive probabilities $\mu_i(a)$ on the groups of each column, the bound of Theorem 110.5 holds with $B_i' = \sum_a S_{ia}/\mu_i(a)$, computed by Theorem 110.23. In particular the actual Gram matrix at the prescribed tests satisfies

$$\det G_\sigma^\circ \;\le\; \prod_i (T^{-d_i^T} Q_{i,\sigma} F_i A_{i,\sigma}^2).$$

Both assertions apply on the original period fields for every translate and every word. Furthermore, for every fixed $B > 0$, uniformly on the intervals of Theorem 108.27,

$$\log F_i = \sum_{j,\tau,h} \nu_{ij,\tau,h} (\log \nu_{ij,\tau,h} - 1) + o_{\psi,\varepsilon,B}(b/x^B),$$

where a zero part contributes zero.

Proof. Within one group the scalar elliptic germs are common. The Cauchy–Schwarz estimate for their derivatives in the proof of Theorem 110.5 therefore applies with the summed coefficient polynomial. Giving each group its own probability yields the first assertion by the same operator inequality and monotonicity of the mixed determinant.

For a single path take the trace of the theorem's identity. Its square root is $\sqrt{F_i}$ times precisely this path's summand in $A_{i,\sigma}$. The jet bound on that path multiplies this square root by at most $(T^{-d_i^T} Q_{i,\sigma})^{1/2}$. Sum over the paths by the triangle inequality for the norm in V. Hadamard's inequality for their resulting column norms proves the displayed bound, including dependent lists.

For $a \ge 1$ and $k \ge 0$, $(a)_k = k!\binom{a+k-1}{a-1}$. The binomial factor lies between one and $(a+k)^{a-1}$. Comparison of the increasing function $\log t$ with its integrals over successive unit intervals gives $\log(k!) = k\log k - k + O(\log(2+k))$ for $k \ge 1$. Thus each log factor differs from the asserted main term by $O(a\log(2+a+k))$, including $k = 0$. The number of embeddings and edges and their ranks are bounded by powers of x. All parts satisfy $k \ll xb + x^C$ on the given words, by Lemma 108.11 and the partition ordering. Since $b = \exp(x^{\varepsilon})$, the total error is bounded by a fixed power of x, which gives the stated uniform error. □

The factorials in this estimate are explicit for the signed columns at the original tests. The endpoint maps, initial forms and class sums still affect their height. Section 6.6 also requires a lower bound for the rank of the evaluated periods and the uniform cohomology estimate on the same intervals.

111 Common divisors of the projected columns

111.1 A common divisor for the signed theta columns

Use the polynomial-conductor intervals of Theorem 108.27. Recall $b = \exp(x^{\varepsilon})$, $X' = \exp(x^{\varepsilon/3})$ and $w'' = b/(nX')$ from the construction of (9.1). At a fixed x and translate u, let w run through the chosen binary words. Write $\nu_{w,j,\tau,l}$ for part l of the partition on edge j at the embedding τ. The two ranks on that edge are $r_j \le s_j$, and $r_* = \max_j r_j$. Let $\mathcal{P}_x$ consist of the unramified primes $p \ge 5$ in E_x which are inert in the quadratic field K_d. Put, as in Lemma 108.26,

$$g_{w,p} = \sum_j V_p(\nu_{w,j}), \qquad b^{\mathrm{per}}_{u,w} = \prod_{p\in\mathcal{P}_x} p^{\,g_{w,p}-\lceil U_p\rceil}.$$

Here U_p depends on u and is zero in the nonsignal case specified in Corollary 56.3. The exponents are nonnegative. We determine the logarithm of their common divisor up to an error smaller than b/x^A for every fixed $A > 0$.

Lemma 111.1 (A uniform cutoff for all row gains)**.** *For all sufficiently distant intervals, throughout each interval and for all translates and words,*

$$g_{w,p} = 0 \qquad (p \in \mathcal{P}_x, \quad p > P_x = 12 r_* n x X').$$

For $p \le P_x$ and any two words at the same translate one has $|g_{w,p} - g_{w',p}| \ll_{\psi,\varepsilon} x^C$ with an exponent independent of p, u, w, w'.

Proof. The first-row formula in Lemma 108.11 is $\nu_{w,j,\tau,1} = u_1(\tau) + e_{j,\tau} - (r_j + s_j - 1)/2$. The least base value in (9.1) and its following weight formulas is at least w'', the rank shifts are bounded by a power of x, and $u_1(\tau) \ge 0$. The largest u_1 is at most $\lceil xb\rceil + O(k)$, and $e_{j,\tau} \le 2b$ eventually. Since w'' grows faster than any fixed power of x, these facts give

$$\min_{w,j,\tau} \nu_{w,j,\tau,1} \ge b/(2nX'), \qquad \max_{w,j,\tau,l} \nu_{w,j,\tau,l} \le 3xb.$$

For two words at this translate every part at the same position differs by at most Δ. Indeed the only varying input entries are the interior u_i, each changing by Δ when its bit changes. Subsequent steps add scalars independent of the word to the old entries and pad by entries independent of the word. This is the recurrence for the ordered weights in the proof of Lemma 108.11.

Fix a single edge of ranks r, s. For one of the allowed choices H of alternating halves of the even cycles of $\pi = \sigma_p c_E$, let μ lie in the opposite halves. For each positive odd a write

$h = \pi^{-a}\mu$. The list for μ in the definition of V_p has entries $D_{l,a} = d_{p^a}(\nu_{h,l}, s)$, $1 \le l \le r$. For integers $M \ge 2,\ s \ge 1,\ t \ge 0$,

$$0 \le d_M(t,s) = \max(0, \lceil (t - s(M-1))/M \rceil) \le \lfloor t/M \rfloor.$$

The upper bound follows by writing $t = Mq + l'$ with $0 \le l' \le M - 1 \le s(M-1)$. Thus the sum of the whole list is at most $3rxb\,p/(p^2-1) \le 6rxb/p$. Every nonempty choice of list entries in the maximum must pay at least $\nu_{\mu,1}$ from the opposite partition. For $p > P_x$ this already exceeds the sum of the list. All maxima are zero, proving the cutoff.

For the variation bound write each sorted-list maximum as a maximum over subsets I of $\sum_{(l,a)\in I} D_{l,a} - \sum_{l \le |I|} \nu_{\mu,l}$, where parts beyond r are zero. Use the same finite list of indices $p^a \le 3xb$ for both words, padding by zeros. The change of $D_{l,a}$ has absolute value at most $\lceil \Delta/p^a \rceil$, by the ceiling formula. The change of the subtracted sum is at most $r\Delta$. Consequently every subset value changes by at most

$$r\Delta + r \sum_{\substack{a \text{ odd} \\ p^a \le 3xb}} \lceil \Delta/p^a \rceil \ \le\ r(\Delta + 2\Delta/p + 1 + \log(3xb)/\log p).$$

Taking maxima preserves a uniform absolute difference bound. Sum over μ, take the maximum over H, and sum over the edges. The numbers of embeddings and edges, the ranks, Δ and $\log(3xb)$ are bounded by powers of x. □

There is also a formula without ceilings for the main term. For one cycle of even length e and a given opposite embedding μ, use the infinite list

$$t_{l,a} = \nu_{\pi^{-a}\mu,l}/p^a \quad (1 \le l \le r, \quad a \ge 1 \text{ odd}).$$

Write T_k for the sum of its k largest entries, $T_0 = 0$, and T_∞ for the sum of all its entries. Set $A_k = \sum_{l \le k} \nu_{\mu,l}$ for $0 \le k \le r$. Define $\overline{V}_p(\nu)$ by taking the same sum over opposite embeddings and maximum over H as for V_p, using at each μ the rational number

$$\max\left\{ \max_{0 \le k < r} (T_k - A_k),\ T_\infty - A_r \right\}.$$

For an empty collection of even cycles take both gains to be zero.

Proposition 111.2 (A finite evaluation of the main term)**.** *In this formula one has*

$$T_\infty = \frac{\displaystyle\sum_{\substack{1 \le a < e \\ a \text{ odd}}} p^{-a} |\nu_{\pi^{-a}\mu}|}{1 - p^{-e}}.$$

For $1 \le k < r$, T_k is obtained from the entries with $a < ke$. For the actual words, $\overline{V}_p(\nu) = 0$ for $p > P_x$, and, for all p,

$$0 \le \overline{V}_p(\nu) - V_p(\nu) \le fr\big(s(1 + \log(3xb)/\log p) + 2\big),$$

where $f = [E_x^+ : \mathbb{Q}]$.

Proof. Group the terms by a mod e. Each such residue and row gives a geometric progression of ratio p^{-e}, proving the sum formula. A positive entry beyond its first k terms has k larger predecessors in the same progression. Zero progressions can be ignored or padded by zeros. This proves the finite rule for T_k. Among subsets of size at least r, the penalty is always A_r,

and their sums increase to the sum of the entire list. This proves that the displayed maximum is the supremum over all finite subsets. The cutoff proof just given also bounds the sum of this larger list by $6rxb/p$.

Increasing a nonnegative summable list by a total amount a_0 increases the supremum by at most a_0. For $p^a \le 3xb$, each $t_{l,a}$ exceeds $D_{l,a}$ by at most s, by the ceiling formula. For $p^a > 3xb$ the old entries are zero. If $J = \lfloor \log(3xb)/\log p \rfloor$, the sum over $a > J$ for any one row is at most $3xb\,p^{-J-1}/(1-p^{-1}) < 2$. Adding these bounds for at most f lists of r rows proves the inequality for each orientation and hence for their maxima. □

Theorem 111.3 (One integer for every theta word at a translate)**.** *At each x, u put $B_u = \gcd_w b^{\mathrm{per}}_{u,w}$. There is a positive integer D_x, valid for every translate, such that*

$$B_u \mid b^{\mathrm{per}}_{u,w} \mid B_u D_x, \qquad \log D_x \ll_{\psi,\varepsilon} x^C X'.$$

For any reference word w_0 in the family the common divisor satisfies the uniform estimate

$$0 \;\le\; \sum_{\substack{p\in\mathcal{P}_x\\ p\le P_x}} \left(\sum_j \overline{V}_p(\nu_{w_0,j}) - U_p \right) \log p - \log B_u \;\ll_{\psi,\varepsilon} x^C X'.$$

On enlarging D_x within this bound, the explicit compact factorials of Corollary 110.24 also have a common factor F_u^- with $F_u^- \mid F_w \mid F_u^- D_x$.

Proof. For $p \in \mathcal{P}_x$, set the exponent of p in D_x equal to the maximum over translates of $\max_w g_{w,p} - \min_w g_{w,p}$. Give other primes exponent zero. By Lemma 111.1 each exponent is polynomially bounded in x and the support is contained in $p \le P_x$. The elementary bound $\sum_{p\le P_x} \log p \le P_x \log(2P_x)$ gives the required logarithmic estimate, with an enlarged C. Subtracting the same $\lceil U_p \rceil$ from every exponent at a translate proves the two divisibilities.

The exponent of p in B_u is $\min_w g_{w,p} - \lceil U_p \rceil$. The difference from the coefficient of $\log p$ in the displayed main term is the sum of three nonnegative quantities,

$$\sum_j (\overline{V}_p(\nu_{w_0,j}) - V_p(\nu_{w_0,j})), \qquad g_{w_0,p} - \min_w g_{w,p}, \qquad \lceil U_p \rceil - U_p.$$

The first is bounded by a power of x by the proposition, the second by the lemma, and the third by one. At primes above the cutoff both gains and U_p vanish, since $0 \le \lceil U_p \rceil \le g_{w,p}$ there. Summation proves the absolute estimate.

For the compact factor take componentwise minima and maxima of the parts over words and form F_u^-, F_u^+ from the same rising factorial products. Each F_w/F_u^- divides F_u^+/F_u^-. At any position this last ratio introduces at most Δ positive integer factors, all at most $r_* + 3xb$. There are polynomially many positions and translates, by Proposition 6.4 and the rank bounds in Corollary 109.9. Multiplying the previous D_x by these ratios for all translates increases its logarithm by at most a polynomial in x. □

Corollary 111.4 (The common division on each projected minor)**.** *Let Y° have columns $\widetilde{Y}_i/b^{\mathrm{per}}_{u,w_i}$ and let Y^B have columns $\widetilde{Y}_i/B_u$, with the same integral period evaluations and the same projection. In any one system of test coordinates at any finite place, write I_h for the fractional ideal generated by the size-h minors. For every $h \ge 1$,*

$$D_x^h I_h(Y^\circ) \;\subseteq\; I_h(Y^B) \;\subseteq\; I_h(Y^\circ).$$

For every independent list of q_c columns in the original positive test metric and at every archimedean embedding,

$$0 \le \frac{1}{q_c} \log \frac{\det G^B_\sigma}{\det G^\circ_\sigma} \le 2 \log D_x = o_{\psi,\varepsilon,A}(b/x^A) \qquad (A > 0 \text{ fixed}).$$

Proof. The change multiplies column i by the positive integer $b^{\mathrm{per}}_{u,w_i}/B_u$ dividing D_x. Each minor is therefore multiplied by a divisor of D_x^h, proving both ideal inclusions even when the ideal is zero. The Gram determinant is multiplied by the square of the product of the column factors. Take logarithms and use $x^{A+C}X'/b \to 0$. The row-gain construction and the common integer use the fields and weights of these very columns. □

This supplies a division compatible with taking linear combinations across words. The finite expression in the theorem computes its size to the precision stated. The comparison of the remaining signed period height with the prime count, its required rank, and the exceptional strict lengths are still needed in Section 6.6.

111.2 Large primes in the common period divisor

Continue with the common integer B_u of Theorem 111.3. We evaluate the contribution of the primes above a fixed power of x using the Frobenius group of the same period field. Put $G = \mathrm{Gal}(E_x/\mathbb{Q})$, $d_G = |G|$, and let S_x contain the ramified rational primes and $2, 3$, with no other primes. Write N_x for the conductor of the field. On these intervals N_x, d_G are bounded by powers of x, by Theorem 9.26 and the field construction. For $g \in G$ put

$$A_g(t) = \sum_{\substack{p \le t,\ p \notin S_x \\ \mathrm{Frob}_p = g}} \frac{\log p}{p}, \qquad y_0 = \exp((\log(2x))^4).$$

Lemma 111.5 (Reciprocal prime sums in the period field)**.** *Uniformly for $t \ge y_0$ and $g \in G$,*

$$A_g(t) = \frac{\log t}{d_G} + C_g + O_{\psi,\varepsilon}(e^{-c\sqrt{\log t}})$$

with some $c > 0$ independent of x, u, g. If χ^* induces the character χ of G, define $D_{\mathbf{1}} = -\gamma$ and $D_\chi = -L'(1,\chi^*)/L(1,\chi^*)$ for $\chi \ne \mathbf{1}$ in this lemma. The constant is

$$C_g = \frac{1}{d_G} \sum_{\chi \in \widehat{G}} \overline{\chi(g)} \left(D_\chi - \sum_{p \in S_x} \frac{\chi^*(p)\log p}{p} - \sum_{p} \frac{\chi^*(p)^2 \log p}{p(p - \chi^*(p))} \right).$$

Proof. The real primitive characters with an auxiliary factor have no exceptional zero near one by Theorem 5.6. The other real primitive characters have conductor dividing the fixed $8F$. Thus the zero-free bound, the zero count and the explicit formula in the proof of Lemma 7.1 apply uniformly on these fields. To check the lower endpoint here, truncate the zero sum in (3.1) at $\exp(\sqrt{\log t})$. The low-zero contribution after division by t is at most

$$O_{\psi,\varepsilon}\left(\log^3(2N_x) \exp\left(-c_1 \min\left\{ \sqrt{\log t},\ \frac{\log t}{\log(2N_x)} \right\} \right) \right).$$

The tail is $O((\log(2N_x) + \sqrt{\log t})e^{-\sqrt{\log t}})$ by the inverse-square ordinate bound of that lemma. Its elementary terms after division by t are $O_{\psi,\varepsilon}((1 + \log t + \log^2(2N_x))/t)$, including the error from the shifted line. Since $\log(2N_x) \ll \log(2x)$ and $\log t \ge (\log(2x))^4$, these bounds give $|G_{\chi^*}(t)| \ll te^{-c_2\sqrt{\log t}}$ for some $c_2 > 0$.

For $h = te^{-c_2\sqrt{\log t}/3}$, take the forward difference quotient of R_{χ^*} from (1.1). It differs from $\sum_{n \le t} \Lambda(n)\chi^*(n)/n$ by at most $(h+1)\log(2t)/t$, by bounding the additional terms with $t < n \le t + h$. The two G_{χ^*}-values in the quotient contribute $O(e^{-c_3\sqrt{\log t}})$ for a smaller $c_3 > 0$. The difference quotient of the main term at the trivial character is $\log t - \gamma + O(h/t)$, and for the nontrivial characters it is exactly D_χ. To pass to unramified primes, remove all powers of

exponent at least two and the first powers in S_x. Every prime in S_x is already below y_0 for large x. The sum of all higher powers of one prime gives the last summand of the formula for C_g. The part with $p^a > t$, $a \ge 2$, in absolute value, is $O(\log(2t)/\sqrt{t})$. For $p \le \sqrt{t}$ the geometric tail is at most $2\log p/t$ per prime. For larger p sum $2\log p/p^2$ by comparison with the integral over all integers. Finally character orthogonality selects $\mathrm{Frob}_p = g$. Averaging over d_G characters preserves the absolute uniform error. □

Choose any one reference word at the fixed translate. Set $S_{j,\tau,k} = \sum_{l=1}^{k} \nu_{w_0,j,\tau,l}$, $S_{j,\tau,0} = 0$, and $S_{j,\tau} = S_{j,\tau,r_j}$. For g inert in K_d write $\pi_g = gc_E$ acting on the embedding labels as in Theorem 56.2. For each even cycle C of this permutation, let C_0, C_1 be its two alternating halves. Put

$$f_{j,g,\mu}(z) = \max_{0\le k\le r_j} \{S_{j,\pi_g^{-1}\mu,k} z - S_{j,\mu,k}\},$$

$$H_g(z) = \sum_{j,C} \max_{b'=0,1} \sum_{\mu\in C_{b'}} f_{j,g,\mu}(z) - \mathbf{1}_{g|K_q=1} \sum_{j,t\in T_{\mathrm{hi}}} [az - e_{j,\pi_g t}]_+.$$

Here $a = \lceil xb \rceil$, and j runs through all edges in the last sum as well. Set $H_g = 0$ when g splits in K_d. Let L_u denote the distinguished small-place set of (9.1), and let O_u consist of the embedding labels outside both T_{hi} and L_u. For g inert in K_d put

$$J_g = \sum_{j,C} \max_{b'=0,1} \sum_{\mu\in C_{b'}\cap O_u} S_{j,\pi_g^{-1}\mu} - \mathbf{1}_{g|K_q=1}\, a \sum_j \#\{t \in T_{\mathrm{hi}} : \pi_g t \notin L_u\}.$$

Give J_g value zero for the other g. All these weights use the original selected field, high set, distinguished set and translate.

Proposition 111.6 (The three prime ranges of the divisor)**.** *There is an exponent C, depending only on the fixed parameters, for which $|J_g| \ll bx^C$. For $z \ge 0$, H_g is continuous and piecewise affine with $H_g(0) = 0$ and Lipschitz constant at most $O(bx^C)$. For any sufficiently large fixed M, uniformly for all g and $x^M < t \le y_0$,*

$$H_g(1/t) = J_g/t + O_{\psi,\varepsilon}(bx^C/X').$$

At primes $p > x^M$ in $\mathcal{P}_x$, the difference between $\sum_j \overline{V}_p(\nu_{w_0,j}) - U_p$ and $H_{\mathrm{Frob}_p}(1/p)$ is $O(bx^C/p^3)$.

Proof. Keep only $a' = 1$ in the lists $t_{l,a'}$ of Proposition 111.2. The largest k entries of the resulting list come from the first k parts at $\pi_g^{-1}\mu$. This gives $f_{j,g,\mu}(1/p)$, including the choice $k = r_j$. The omitted entries have total at most $3r_j xb\, p^{-3}/(1-p^{-2})$ per opposite embedding. The same bound with $ap^{-3}/(1-p^{-2})$ applies to the omitted higher odd powers of each pair in U_p, because all $e_{j,\tau}$ are positive. The halves can be chosen independently on different cycles. The supremum over subsets and maximum over halves change by no more than the uniform error. All multiplicities are polynomial in x, giving the assertion at primes.

The constants in the affine functions have nonpositive sign before taking their maxima, and the choice $k = 0$ is available. Each maximum is Lipschitz with constant bounded by its largest slope. All slopes, summed over their indices, are $O(bx^C)$ by Lemma 111.1. The same bound applies to the subtracted ramps, proving the assertions about H_g and J_g.

At a high place the first part is at least $b/2$ for large x, since u_1 contains $\lceil xb \rceil$. At a place in L_u this lower bound follows from $B_\tau \ge \lceil b \rceil$ in the formula for $e_{j,\tau}$. At the remaining places the full partition size is $O(bx^C/X')$, since $B_\tau \le \lceil b/(nX') \rceil + 1$ there and the other contributions

to the first part are polynomial. These statements use the rank shifts in Lemma 108.11. For $t > x^M$ with M sufficiently large, each incoming partition sum divided by t is less than $b/2$. Thus at the first two kinds of destination every f is zero. At $\mu \in O_u$, choosing $k = r_j$ and dropping the penalty give lower and upper bounds within $S_{j,\mu}$ of $S_{j,\pi_g^{-1}\mu}/t$.

In the subtracted ramps the weights $e_{j,\tau}$ on L_u are at least b. At the remaining places, including the high places, $e_{j,\tau} = O(bx^C/X')$ by the explicit base values. Therefore the ramp vanishes when its destination is in L_u; at the other destinations it differs from a/t by $O(bx^C/X')$. Taking the orientations and the sums proves the middle-range formula, with the polynomial factors absorbed in the exponent C. □

Theorem 111.7 (The absolute size from polynomially many primes and field constants)**.** *For every fixed $A > 0$ one can take a fixed $M = M(\psi, \varepsilon, A)$ and $z_0 = x^M$ such that, uniformly in the translate and the reference word,*

$$\begin{aligned}\log B_u = {} & \sum_{\substack{p \in \mathcal{P}_x \\ p \le z_0}} (g_{w_0,p} - \lceil U_p \rceil) \log p \\ & + \sum_{g \in G} J_g \left(\frac{\log y_0}{d_G} + C_g - A_g(z_0) \right) \\ & + \frac{1}{d_G} \sum_{g \in G} \int_{y_0}^{P_x} H_g(1/t)\, dt + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

The integrals use only finitely many affine pieces and ordinary logarithms. At the same precision the prime series in all the C_g may be truncated at $p \le z_0$, retaining the logarithmic derivatives at one in Lemma 111.5.

Proof. Eventually $z_0 < y_0 < P_x$. By Theorem 111.3 the change from B_u to the divisor of the reference word costs $O(x^C X')$ in the logarithm. Keep its exact exponents at $p \le z_0$. At the other primes the replacement by $\sum_j \overline{V}_p - U_p$ costs $O(x^C X')$ by the nonnegative error estimates in the same theorem. On $z_0 < p \le P_x$ the first-power approximation in the proposition costs in total

$$O\left(bx^C \sum_{n > z_0} \frac{\log n}{n^3} \right) = O\left(bx^C (1 + \log z_0)/z_0^2 \right).$$

Take M large enough that this is $o(b/x^A)$.

Up to y_0, replacement by J_g/p costs at most $O(bx^C y_0 \log(2y_0)/X') = o(b/x^A)$. This follows from $\log X' = x^{\varepsilon/3}$, which dominates every fixed power of $\log x$. The remaining middle sum is exactly $\sum_g J_g(A_g(y_0) - A_g(z_0))$. Substitute the lemma at y_0. Its error, even after summation over G, is at most $bx^{C'} e^{-c(\log(2x))^2} = o(b/x^A)$.

On $[y_0, P_x]$ put $W_g(t) = tH_g(1/t)$. The Lipschitz estimate and $H_g(0) = 0$ give $|W_g| \ll bx^C$ and $|W_g'(t)| \ll bx^C/t$ on each smooth piece. The sum over primes in this last range is the Stieltjes integral of W_g against A_g on $(y_0, P_x]$, summed over g. Partial summation with Lemma 111.5 replaces this by the integral in the statement. The constants C_g cancel between endpoints and the integral of the derivative. The error is bounded by

$$O\left(d_G bx^C (1 + \log P_x) e^{-c(\log(2x))^2} \right).$$

The pieces are continuous at every junction, so partial summation adds no interior boundary errors. On each piece of $H_g(1/t)$ the integrand is $a_1 + a_2/t$. For each μ the candidates are $r_j + 1$ lines in $1/t$. Their pairwise crossings give a finite subdivision. On each resulting interval the two half-cycle sums cross at most once unless they coincide. Taking the union of these

subdivisions uses a polynomial number of breakpoints in the numbers of edges, embeddings and ranks.

Finally the absolute tail of the prime series in each C_g after z_0 is $O((1+\log z_0)/z_0)$, by comparison with the sum of $\log n/n^2$. All primes of S_x are below z_0 upon enlarging M. Another enlargement absorbs this last error after multiplication by $\sum_g |J_g|$. This gives the claimed accuracy for the original common divisor on every field of the interval. □

The arithmetic information for this divisor now occurs in the primes of size at most x^M, the field constants at one and the explicit integrals. Its value alone gives neither the rank of the signed periods nor the upper bound for their remaining height in Section 6.6.

111.3 The terminal restriction and the two factorial norms

At one translate use the incoming analytic row and the definite word w of (9.1). Both use the same field, embeddings and m edges. Two metric normalizations enter at the last definite edge. We first identify their conversion in the coefficient matrix of Theorem 110.23.

Lemma 111.8 (The endpoint in highest-vector normalization)**.** *Let V_v be a compact flag module of dimension d_v. Use the Hermitian Fischer form in which its highest polynomial p_v has norm one. The squared Haar frame norm of Section 110.8 is $1/d_v$ times this squared Fischer norm. This holds also after a determinant twist and for products over the CM embeddings.*

In the integral frames let $L : V_l \to V_m$ be the actual terminal restriction with its normal-coordinate scalar removed. Here the indices l, m denote the two weights. Write $d_l = \dim V_l$. For endpoint tests $R_\gamma, R_\delta : V_m \to V$ in the same frames and a fixed metric on V, one has

$$\frac{(R_\gamma L)(R_\delta L)^{*,\mathrm{Haar}}}{d_l} = s_w\, R_\gamma R_\delta^{*,\mathrm{F}}, \qquad s_w = \prod_\tau c_\tau^2 \kappa_\tau.$$

The adjoints on the right use the normalized Fischer metric on V_m. The identity holds if both composites on the left are precomposed with the same unitary frame change.

Proof. Evaluation at the identity is the inner product with p_v by Lemma 98.4. The coefficients of p_v are real, so the identity gives the Hermitian version with the form linear in its first entry. Unitary row translation preserves the inner product on the linear variables and hence the Fischer metric on their symmetric tensors. Thus evaluation on unitary frames pairs the vector with the orbit of p_v. The Haar average of the rank-one orthogonal projections onto these unit translates commutes with the unitary action. The flag space is irreducible, as in Lemma 110.22. The average has trace one, so it equals I/d_v. Pairing with an arbitrary vector gives the first assertion. A determinant has absolute value one on the unitary group. Fubini and polarization give the product assertion, with d_v the product dimension.

For a map from V_l to V, changing the domain metric from Fischer to Haar multiplies its adjoint by d_l. Hence the quotient in the display equals $R_\gamma L L^{*,\mathrm{F}} R_\delta^{*,\mathrm{F}}$, with Fischer metrics on both coefficient spaces. Theorem 74.28 identifies $LL^{*,\mathrm{F}}$ with $s_w I$. A common unitary frame change cancels against its adjoint. □

For the endpoint factor in Corollary 110.24, taking the trace gives

$$\frac{\|R_\gamma L\|_{\mathrm{HS,Haar}}}{\sqrt{d_l}} = \sqrt{s_w}\, \|R_\gamma\|_{\mathrm{HS,F}}.$$

The tests R_γ can contain the contraction at the small class, its finite projection and the path scalar from Theorem 60.32. A change of terminal frame can be kept in the polynomial path

or included in this calculation when it is common to the two factors. In the unnormalized column $L_z = Z_w^{\mathrm{norm}} L$, so the squared endpoint has the additional factor $|\sigma Z_w^{\mathrm{norm}}|^2$. In $\widetilde{Y}_w = (z_w Z_w^{\mathrm{norm}})^{-1} Y_w$ it cancels. The factors z_w^{-1} and $1/b_{u,w}^{\mathrm{per}}$ still enter the divided column once each. By Corollary 82.33 the absolute logarithm of s_w, and the sum of its absolute finite log valuations, are $o(b/x^A)$ for every fixed $A > 0$, even with suprema over the words and translates.

Let $D(t) = t \log t - t$ for $t > 0$, with $D(0) = 0$. Let F_w be the product of rising factorials in Corollary 110.24. Define $\mathcal{H}_{\mathrm{in}}$ to be the product, over all real places and edges of the incoming row, of $H(\alpha)H(\beta)H(\gamma)$ in Proposition 82.20. Thus each incoming hook product is taken in Gaussian coordinates of dilation $1/\pi$. Write $h = |T_{\mathrm{hi}}|$, $h_2 = |L_u|$, $a = \lceil xb \rceil$ and $Y = X' = \exp(x^{\varepsilon/3})$.

Theorem 111.9 (The factorials on the given row and word)**.** *Set $N_w = \sum_{j,\tau,l} \nu_{w,j,\tau,l}$, the total definite degree. On the intervals of Theorem 108.27, uniformly in u, w, there is a fixed C such that*

$$N_w = m(ha + kh_2 b) + O_{\psi,\varepsilon}(bx^C/Y + x^C).$$

For every fixed $A > 0$ one has

$$\begin{aligned} \log F_w &= m\{hD(a) + kh_2 D(b)\} + o_{\psi,\varepsilon,A}(b/x^A), \\ \log \mathcal{H}_{\mathrm{in}} &= m(k-1)h_2 D(b) + o_{\psi,\varepsilon,A}(b/x^A), \\ \log \frac{F_w}{(b_{u,w}^{\mathrm{per}})^2 \mathcal{H}_{\mathrm{in}}} &= m\{hD(a) + h_2 D(b)\} - 2\log B_u + o_{\psi,\varepsilon,A}(b/x^A). \end{aligned}$$

Here B_u is the common integer in Theorem 111.3.

Proof. At the first edge the old highest weights are u_l minus the corresponding half-sum entries, of absolute difference $O(k)$ from u_l. The shift at edge j adds $e_{j,\tau} - s_j/2$ to form its partition. For the next input one pads by zeros and subtracts $e_{j,\tau} - r_j/2$. Each old entry consequently decreases by $(s_j - r_j)/2$. A newly introduced entry begins at $-e_{j,\tau} + r_j/2$. This is the recurrence of Lemma 108.11. All rank shifts and jH_0 are polynomial in x, and $e_{j,\tau} = B_\tau + (2j-1)H_0$. It follows that at every edge the original k parts are $u_l(\tau) + B_\tau + O(x^C)$. Every other part is $O(x^C)$, because the two occurrences of B_τ cancel.

At a high place the top part equals $a + O(bx^C/Y + x^C)$. The remaining parts have size $O(bx^C/Y + x^C)$, including the part with initial entry $-\lceil xb/Y \rceil - (k-1)/2$. At a distinguished place the original k parts all equal $b + O(x^C)$. Outside these two sets all parts are $O(bx^C/Y + x^C)$. These bounds follow from the three explicit base values after (9.1) and its interior bit entries. Summing the bounds proves the formula for N_w after absorbing the polynomial number of indices into C.

Apply the last assertion of Corollary 110.24. Replacing a part listed above by a, b, or zero as appropriate changes its D-value by $O(bx^{C'}/Y + x^{C'})$. Indeed the integral of $|\log t|$ on $(0,1)$ is one, and on $[1, 3xb+2]$ the derivative is bounded by $\log(3xb+2)$. The number of terms is polynomial and Y grows faster than every fixed power of x. This proves the first logarithmic estimate.

For the incoming hook products the proof of Proposition 82.20 gives exactly $k-1$ parts $B_\tau + O(x^C)$ per edge and place. The other parts are polynomially bounded. Its hook formula contributes polynomial total error to $m(k-1)\sum_\tau D(B_\tau)$. At the distinguished places $B_\tau = b + O(1)$. At all other places $B_\tau = O(bx^C/Y + x^C)$, whose contribution falls within the stated error. Subtract the resulting two estimates and use the uniform bound $|\log b_{u,w}^{\mathrm{per}} - \log B_u| = o(b/x^A)$ in Theorem 111.3. This proves the last formula. □

Corollary 111.10 (An upper bound for the integral divisor in degree N_w)**.** *For all words and translates on these intervals,*

$$0 \le \log b_{u,w}^{\mathrm{per}} \le N_w(\log P_x + O(1)), \qquad \frac{2\log b_{u,w}^{\mathrm{per}}}{\log F_w} = O_{\psi,\varepsilon}(x^{-2\varepsilon/3}),$$

where $P_x = 12r_*nxY$*.*

Proof. In the subset formula for V_p drop all penalties and keep all entries. At odd exponent j' each source place occurs at most once, and each entry is at most $\nu_{w,j,\tau,l}/p^{j'}$ by Lemma 111.1. Summing gives $g_{w,p} \le N_w p/(p^2-1)$. The signal subtraction is nonnegative, and the cutoff is P_x. For $y \ge 2$, the bound $\sum_{p\le y} \log p/p \le \log y + O(1)$ follows from $\sum_{p\le n} \lfloor n/p \rfloor \log p \le \log(n!)$ and $\sum_{p\le n} \log p = O(n)$, with $n = \lfloor y \rfloor$. The difference $p/(p^2-1) - 1/p$ contributes a convergent prime sum after multiplication by $\log p$. This gives the first estimate.

Let N be the polynomially bounded number of part positions. Since one first part is at least $b/(2nY)$ by Lemma 111.1, we have $\log(N_w/N) \ge \log b - \log Y - O(\log x)$. Each $(r-l+1)_v$ is at least $v!$, whose logarithm is at least $D(v)$ by integral comparison (also for $v = 0$). Convexity now gives $\log F_w \ge N_w(\log(N_w/N) - 1)$. Finally $\log b = x^{\varepsilon}$ and $\log P_x = x^{\varepsilon/3} + O(\log x)$. The second estimate follows. □

The comparison above evaluates the polynomial factors at both sets of prescribed highest weights. A relation between the full period norms must also account for the class sums and for the normalizations accompanying these polynomial factors.

111.4 The leading term of the integral period divisor

On the same fields and intervals, put $Y = X'$, let $o = \mathrm{ord}(\psi)$, and put

$$M_x = mkh_2 b, \qquad c_\psi = \frac{1}{2} - \frac{1}{8 \cdot 2^{v_2(o)}}.$$

We compute the leading term of $\log B_u$ by counting the destinations of the distinguished places under Frobenius. This also gives the sign and size of the comparison with the incoming hook products in Theorem 111.9.

Use $G = \mathrm{Gal}(E_x/\mathbb{Q})$. For g inert in K_d write $\pi_g = gc_E$ acting on the CM type $\Phi = \Phi_0 \sqcup \Phi_1$, where $T_{\mathrm{hi}} \subset \Phi_0$ and $L_u \subset \Phi_1$. Write $O_u = \Phi \setminus (T_{\mathrm{hi}} \cup L_u)$. For each even cycle C of π_g let C_0, C_1 be its alternating halves. Define the integer

$$e_u(g) = \sum_{C \text{ even}} \max_{i=0,1} \#\{t \in L_u \cap C_i : \ \pi_g t \in O_u\}.$$

Set $e_u(g) = 0$ when g splits in K_d. In particular $0 \le e_u(g) \le h_2$.

Lemma 111.11 (The prime sum at the scale of the distinguished places)**.** *Uniformly in u and the reference word on these intervals,*

$$\frac{\log B_u}{M_x \log Y} = \frac{1}{|G|h_2} \sum_{g\in G} e_u(g) + o_{\psi,\varepsilon}(1).$$

Proof. Write $t_0 = \exp((\log(2x))^4)$ and $t_1 = Y\exp(-(\log(2x))^2)$. Eventually $t_0 < t_1 < P_x$. By Theorem 6.29, $h \le n$, $h_2 \ge nx^{-\varepsilon}$, and $k \ge x^{9/2}$. Thus $mha/M_x = O(x^{1+\varepsilon}/k) = o(1)$. The degree formula of Theorem 111.9 gives $N_w = (1+o(1))M_x$. By Theorem 111.3, we may replace B_u by $b_{u,w}^{\mathrm{per}}$, with logarithmic error $o(b/x)$.

At each contributing prime the exponent is between zero and $N_w p/(p^2-1)$ by the proof of Corollary 111.10. At $p \le t_0$ the contribution is therefore $O(M_x(1+\log t_0))$. At $t_1 < p \le P_x$ it is $O(M_x(1+\log(P_x/t_1)))$. Indeed partial summation of the Chebyshev bound $\sum_{p\le t}\log p = O(t)$ gives $\sum_{y<p\le z}\log p/p \ll 1+\log(z/y)$. Both bounds are $o(M_x \log Y)$.

It remains to evaluate the middle range. Denote the first-power weight of Proposition 111.6 by H_g, and the full size of a partition on edge j at μ by $S_{j,\mu}$. For $t_0 < p \le t_1$, all incoming sizes divided by p are smaller than $b/2$, since all sizes are $O(bx^C)$ for a fixed C. At any destination in $T_{\mathrm{hi}} \cup L_u$ the first part is at least $b/2$ eventually by the weight calculation of Theorem 111.9. Its first-power maximum is therefore zero. At a destination in O_u, taking the full list or dropping the penalty bounds its maximum between $S_{j,\pi_g^{-1}\mu}/p - S_{j,\mu}$ and $S_{j,\pi_g^{-1}\mu}/p$.

Each source in L_u has total size $kb + O(x^C)$ at every edge, by the same weight calculation. At a high source the total is $a + O(bx^C/Y + x^C)$. At every ordinary source or destination it is $O(bx^C/Y + x^C)$. For a fixed choice of halves we may therefore keep just the distinguished sources with ordinary destinations, giving each weight kb/p. The total error from all the high sources is at most $mha/p + O(bx^{C'}/Y + x^{C'})$, including the polynomial number of positions. The other substitution errors and the dropped destination penalties are $O(bx^{C'}/Y + x^{C'})$. The subtracted first-power signal is at most mha/p. These bounds hold for any choice of halves. Maximization on each cycle at each edge now selects exactly $e_u(g)$ such sources per edge, since π_g switches the two halves. We obtain

$$H_g(1/p) = \frac{mkb}{p}\, e_u(g) + O\left(\frac{mha}{p} + \frac{bx^{C'}}{Y} + x^{C'}\right) = \frac{mkb}{p}\, e_u(g) + o(M_x/p).$$

The last error is uniform, since $p/Y \le \exp(-(\log(2x))^2)$.

By Proposition 111.6 the higher odd powers add $O(bx^C/p^3) = o(M_x/p)$ uniformly in this range. Replacing the integer row gains and the signal ceiling by their continuous versions costs $O(x^C Y)$ in the total prime sum by Theorem 111.3: the two rounding errors there are separately nonnegative before taking their difference, so the bound also holds on this subinterval. Consequently the middle sum differs from $mkb\sum_{t_0<p\le t_1} e_u(\mathrm{Frob}_p)\log p/p$ by $o(M_x \log Y)$. Here only the unramified primes outside S_x are used, as in Lemma 111.5.

Apply that lemma to $A_g(t_1) - A_g(t_0)$ for each g. The constant cancels. After multiplication by $e_u(g)/h_2$ and summation, the error is $O(|G|e^{-c(\log(2x))^2}) = o(1)$, since $|G|$ is polynomial in x. Finally $\log(t_1/t_0)/\log Y \to 1$, proving the assertion. □

Theorem 111.12 (The leading constant for the common integral divisor)**.** *On the intervals of Theorem 108.27 one has*

$$\log B_u = (c_\psi + o_{\psi,\varepsilon}(1))M_x \log X', \qquad \log b^{\mathrm{per}}_{u,w} = (c_\psi + o_{\psi,\varepsilon}(1))M_x \log X',$$

uniformly for every translate and every word. In particular $c_\psi = 3/8$ for every character of odd order.

Proof. Write $A = \mathrm{Gal}(K_\psi C_{j_x}/\mathbb{Q})$ and $A_0 = \mathrm{Gal}(K_\psi C_{j_x}/K_\psi)$. The two quadratic coordinates are independent of A and of each other by the ramification at q, d_0 in Theorem 9.26. Each sign pair occurs with weight $1/4$, and at fixed signs the translation in A is uniform. Indeed multiplying by c_E adds a fixed element to these coordinates, including a possible component in A. Identify each of Φ_0, Φ_1 with A. Then $|A| = n$, A_0 is cyclic of two-power order, and A/A_0 is cyclic of order o.

First take g inert in K_d and split in K_q. Then π_g switches Φ_0, Φ_1. Every cycle is even and its two halves are its intersections with those sets. All distinguished sources can be selected,

except those sent into T_{hi}. Each such source is sent uniformly through Φ_0 by the group average. The conditional average of $e_u(g)/h_2$ is therefore $1 - h/n = 1 - o(1)$, by Theorem 6.29.

Now take g inert in both quadratic fields. Then π_g preserves each of Φ_0, Φ_1 and translates Φ_1 by a uniform $v \in A$. Only its action there matters. Temporarily count all distinguished sources in the better half of each even cycle, including those sent to L_u. Call the resulting count $e^0(v)$. Removing those destinations changes each maximum by at most the number removed, so

$$0 \le e^0(v) - e_u(g) \le \#\{t \in L_u : t + v \in L_u\}.$$

The average of the right side over v is h_2^2/n, by counting all ordered pairs of distinguished places. From the construction in Theorem 6.29, the original distinguished set has integer representatives in $[0, |A_0|/(8L))$, with $L \to \infty$. Hence $h_2/n = O(1/L + 1/n) = o(1)$, also after translation.

Translate within Φ_1 to view L_u in A_0. That same construction makes it exactly balanced in the D residue classes modulo DA_0, with D a power of two tending to infinity. This means that every such class contains the same number of its elements. Fix the coset of v in A/A_0, and let r be its order there. Visits of a v-cycle to A_0 are spaced by r steps. If r is even, every such cycle is even and all its visits to L_u have the same parity. Thus $e^0(v) = h_2$.

For odd r, multiplication by r permutes A_0, so rv is uniform there. Outside a fraction $1/D$, its subgroup is $2^j A_0$ with $2^{j+1} \mid D$. A v-cycle meeting A_0 then meets it in one coset of $2^j A_0$ and is even. Successive visits differ by its generator rv and are r steps apart. Their parities therefore distinguish the two classes modulo $2^{j+1}A_0$. Balance modulo DA_0 gives equal counts in these classes, so $e^0(v) = h_2/2$. On the exceptional fraction use $0 \le e^0(v) \le h_2$.

The fraction of elements of odd order in the cyclic quotient of order o is $2^{-v_2(o)}$. Averaging the last two cases and restoring the excluded destinations shows that the conditional mean of $e_u(g)/h_2$ when both signs are inert is

$$1 - \frac{1}{2 \cdot 2^{v_2(o)}} + O(1/D + h_2/n).$$

The other two sign pairs contribute zero. The full mean is $c_\psi + o(1)$ uniformly in u. Apply Lemma 111.11 and Theorem 111.3. □

Corollary 111.13 (The integral divisor in the two norm factors)**.** *For the original row and every prescribed word,*

$$\log \frac{F_w}{(b^{\mathrm{per}}_{u,w})^2 \mathcal{H}_{\mathrm{in}}} = -(2c_\psi + o_{\psi,\varepsilon}(1))\, M_x \log X'.$$

For an embedding σ write $A^0_{i,\sigma}$ for the sum of path norm products in Corollary 110.24 on $\widetilde{Y}_i$, before division by b^{per}_{u,w_i}. For q_c words the actual Gram determinant of the divided columns satisfies

$$\begin{aligned} \det G^\circ_\sigma \;&\le\; \exp\bigl(-(2c_\psi + o_{\psi,\varepsilon}(1))\, q_c M_x \log X'\bigr) \\ &\quad \times \prod_{i=1}^{q_c} T^{-d_i^T} Q_{i,\sigma}\, \mathcal{H}_{\mathrm{in}}(A^0_{i,\sigma})^2. \end{aligned}$$

The bound includes lists of deficient rank.

Proof. Since $\log a \ll_\varepsilon x^\varepsilon$, the ratio of $m(hD(a) + h_2 D(b))$ to $M_x \log X'$ is $O(x^{2\varepsilon/3}(xh/h_2 + 1)/k) = o(1)$ by the size inequalities at the start of Lemma 111.11. Theorems 111.9 and 111.12 now give the first formula. Each path norm scales by the column divisor, so $A_{i,\sigma} = A^0_{i,\sigma}/b^{\mathrm{per}}_{u,w_i}$. Substitute into Corollary 110.24 and sum the uniform logarithmic error over the q_c factors. □

Section 6.6 still requires control of the hook, elliptic and class factors together, sufficient rank at the prescribed tests, and the strict local length bound.

111.5 The size of the divisor from five Frobenius powers

At a node and a prime $p \in \mathcal{I}_x$, decompose $\pi = \sigma_p c_E$ into cycles C. Each cycle has even length. Put $T_C = C \cap T_{\mathrm{hi}}$, $L_C = C \cap L_u$. For either all positive odd j, or the truncated set $J_9 = \{1,3,5,7,9\}$, write J for the set of exponents and define

$$F_C(J) = \sum_{t \in T_C} \left(\sum_{\substack{j \in J \\ \pi^j t \notin L_C}} \frac{a}{p^j} + \mathbf{1}_{\pi t \in L_C} (a/p - b)_+ \right),$$

$$R_C(J) = \sum_{\mu \in C \cap \Phi_0} \left(kb \sum_{\substack{j \in J \\ \pi^{-j}\mu \in L_C}} p^{-j} - a \mathbf{1}_{\mu \in T_C} \right)_+ .$$

We suppress J when all positive odd powers are included. The first number uses the sources on Φ_0, the second the sources on Φ_1. Both are explicit in the Frobenius permutation and the two prescribed sets of embeddings.

Theorem 111.14 (The maximizing orientation in the middle prime range)**.** *For all sufficiently large nodes and every $p \in \mathcal{I}_x$, every actual maximizing orientation for $V_p(\nu_i)$ uses Φ_1 as source on each cycle meeting L_u. For every fixed $A > 0$, uniformly in p, u, w, the divisor of the word in Lemma 108.26 satisfies*

$$v_p(b^{\mathrm{per}}_{u,w}) = m \sum_{C \colon L_C \neq \varnothing} (R_C - F_C) + o_{\psi,\varepsilon,A}(b/x^A).$$

Proof. On one edge replace the partition at $t \in T_{\mathrm{hi}}$ by $(a, 0, \ldots)$, the partition at $\tau \in L_u$ by the rectangle with k parts b, and all other partitions by zero. These model rows are allowed to have real entries. The estimates in the proof of Theorem 111.9 give a total absolute row error $O_{\psi,\varepsilon}(bx^{C_0}/X' + x^{C_0})$, also when summed over all the edges and embeddings. Here C_0 is a fixed constant.

For either orientation first pass to the list with entries $\nu_{\pi^{-j}\mu,l}/p^j$. Proposition 111.2 bounds the error by a polynomial on each edge, also on each individual cycle with its orientation fixed, as its proof compares the unsorted lists entry by entry. The contribution at a destination is the supremum over finite subsets of the sum of their entries less the sum of that many leading destination parts, padded with zeros. Perturbing the incoming lists changes this supremum by at most their total absolute change. The cost changes by at most the sum of absolute changes to the destination parts. Each source row appears with total incoming weight at most $\sum_{j \text{ odd}} p^{-j} \le 1$. These facts bound the total change due to the model rows by twice their total absolute error. The resulting errors, including the polynomial loss, are $o_{\psi,\varepsilon,A}(b/x^A)$ for any fixed A.

In the orientation with source on Φ_0, a destination outside L_C can retain its entire list at zero cost. At a destination in L_C, a selection of fewer than k entries pays b for each. At most one first-power entry is nonzero and all entries with $j \ge 3$ are smaller than b, since $p > \sqrt{x}$. A selection of at least k entries costs kb, more than the entire incoming sum $ap/(p^2-1)$. Thus the value is exactly F_C. For source on Φ_1, any nonempty selection pays the same cost, a at destinations in T_C and zero elsewhere. Comparing the whole incoming list with the empty

choice gives R_C. Taking a supremum permits infinite lists by monotone convergence of their entry sums.

If $L_C \neq \varnothing$, the distinct destinations of its first powers show $R_C \geq |L_C|(kb/p - a)$. Also $F_C \leq |T_C|ap/(p^2-1) \leq 2na/p$. The ratios $ap/(kb) = O(x^2/k)$ and $na/(kb) = O(nx/k)$ tend uniformly to zero. Hence $R_C - F_C \geq kb/(2p)$ on each such cycle for large x, a gap exceeding the errors above on every edge. The maxima on different cycles are independent. If L_C is empty the model has $R_C = 0$ and $F_C \geq 0$, so its maximum equals F_C, whether or not the actual model perturbation breaks a tie.

It remains to subtract $\lceil U_p(u) \rceil$. In the word definition replace the distinguished thresholds by b and the other thresholds by zero. At $p^j \leq 3xb$ this costs at most $\sqrt{b} + 2b/(nX')$ per term by the same threshold bounds as in Lemma 108.39. There are polynomially many such terms. Beyond this cutoff the sum of all the model ramps and the original ramps is $O(mh)$, by a geometric tail. The model first and higher powers sum to exactly $m \sum_C F_C$, since $a/p^3 < b$. The total error, including the ceiling, is $o_{\psi,\varepsilon,A}(b/x^A)$. Subtract this from the actual maximized sum over edges to obtain the assertion. □

Corollary 111.15 (Five odd powers determine the normalized logarithm)**.** *Choose a translate $u(p)$ at each $p \in \mathcal{I}_x$. For the common integer B_u of Theorem 111.3,*

$$\frac{1}{Q\delta} \sum_{p \in \mathcal{I}_x} v_p(B_{u(p)}) \log p = \frac{m}{Q\delta} \sum_{p \in \mathcal{I}_x} \log p \sum_{C:\, L_C \neq \varnothing} (R_C(J_9) - F_C(J_9)) + O_{\psi,\varepsilon}\Big(x^{-1/2+(10+1/8)\varepsilon} \log(2x) \Big).$$

The error tends to zero. The same formula holds for any selection of individual word divisors, with the word allowed to depend on the prime as well.

Proof. At a fixed translate $v_p(B_u)$ is the minimum over the words of $v_p(b^{\mathrm{per}}_{u,w})$. The main term of the theorem is independent of the word and its error is uniform. Its contribution to the normalized logarithm is $O_{\psi,\varepsilon,A}(x^{-A})$ for any chosen A, by using a larger exponent in the theorem before summing over $p \leq x$.

Deleting the terms $j \geq 11$ in the reverse sums costs at most $2kbh_2/p^{11}$ in total per edge. For each j a source has a unique destination, and taking the positive part can only reduce the change. The forward tail is at most $2ha/p^{11}$ by the same counting. Both estimates remain valid upon restricting the cycles in the corollary. By integration, $\sum_{n>\sqrt{x}} n^{-11} \log n \ll x^{-5} \log(2x)$. After including m and normalizing by $Q\delta$, the two bounds carry the respective factors $2kn/h$ and $2a/(b\delta)$. From Theorem 6.29 these are $O(x^{9/2+(10+1/8)\varepsilon})$ and $O(x^{1+\varepsilon})$. This gives the displayed error. Its exponent is negative for $0 < \varepsilon < 1/52$. □

This evaluation determines the divisor removed in Section 108.14 over the indicated primes. The logarithmic sum of residual valuations in Theorem 108.40 also involves the images of the actual projected maps.

112 Signed terminal frequencies and their complex norms

112.1 The connected terminal Fourier projection at arbitrary precision

Fix a translate and a place above the terminal prime $p = T$ on the fields of Section 6.6. We evaluate the connected frequency coordinates of Lemma 110.1 in the original signed period. Write $\mathcal{O}$ for the valuation ring of a finite local coefficient field containing the data, enlarged

when necessary, and normalize $v(p) = 1$. Congruences below are modulo $p^M\mathcal{O}$, including when the coefficient field is ramified. For $k \in \mathbb{F}_p$ let $[k]$ be its representative in $\{0, \dots, p-1\}$.

In one scalar coordinate put $F_{b,k} = U_b S_k$, with the complete cyclic projections and prime-to-p displacements of Theorem 56.42. Write $D = d/dz$ in the original frame and $J_i(F) = (D^i F)(0)$. On the quotient elliptic curve C' let $z' = \alpha z$ be the coordinate for its good differential. The connected orientation gives $v(\alpha) = 1$. The displacement for label one gives a point Q on C' of exact p-order whose reduction still has order p. Label k gives $[k]Q$, by the period pairing in Lemma 56.41. Use Z for the elliptic logarithmic derivative in the good differential, as defined in Lemma 56.32.

Lemma 112.1 (A common center for all prime-to-p translates)**.** *The scalar $\theta = \alpha Z(Q)$ is a unit. For every b, k and every $d \geq 0$,*

$$B_d(b,k) = \big((D - \theta[k])^d F_{b,k}\big)(0) \;\in\; p^d\mathcal{O}.$$

In particular the same center $\theta[k]$ works for all displacements at the other primes. The assertion holds when the section vanishes at the origin.

Proof. In the coordinate z' the section is a scalar times the translate of the marked theta on C'. The scalar includes the phase of the translation and has valuation zero in the connected orientation by Theorem 56.42. Let P_0 be the sum of its two-torsion mark and the point induced by b. Its order is prime to p. The point that enters the logarithmic derivative at label k is $P_k = P_0 + [k]Q$. For $k \neq 0$ its reduction is different from the origin because its p-primary part has order p. Lemma 56.32 gives $v(Z(Q)) = -1$, which proves the unit assertion.

We need the first logarithmic derivative to precision p in $\mathcal{O}$. Use the integral extension of C' by the additive group constructed in Lemma 56.32. Over points reducing away from the origin it has an integral coordinate s_0. Addition of a vertical scalar translates that coordinate, and the unique torsion lift $\widehat{P}$ of a torsion point has coordinate $Z(P)$. There is an integral lift g_Q of Q with $s_0 = 0$. Thus $\widehat{Q} = g_Q + \iota(Z(Q))$, where ι denotes the vertical inclusion. The torsion lift of P_0 is integral. Indeed, start with an integral lift in any chart at P_0, multiply it by an order m of P_0 prime to p, and subtract from that lift the resulting integral vertical scalar divided by m. The corrected lift is killed by m. Uniqueness follows from the absence of torsion in the characteristic-zero additive kernel. It also shows that adding torsion lifts lifts the sum. Consequently

$$\widehat{P}_k = \widehat{P}_0 + [k]g_Q + \iota([k]Z(Q)).$$

For $k \neq 0$ the sum of the first two terms is integral and belongs to the s_0 chart over P_k. Hence $Z(P_k) - [k]Z(Q) \in \mathcal{O}$. Returning to z gives

$$(D \log F_{b,k})(0) - \theta[k] = \alpha\big(Z(P_k) - [k]Z(Q)\big) \in p\mathcal{O}.$$

Here $J_0(F_{b,k})$ is a unit by Theorem 56.42.

In a good Weierstrass model for C' the second logarithmic derivative in z' is $-u_{\Lambda'} - \wp(P_k)$, and the subsequent ones are $-\partial^{j-2}\wp(P_k)$ for $j \geq 3$, with ∂ the invariant derivation in that differential. These are integral by the logarithmic derivative calculation of Theorem 56.33. Indeed P_k reduces away from the pole, $u_{\Lambda'}$ is integral on the good model, and its invariant derivation preserves regular functions locally. In the original coordinate the derivative of order j acquires α^j. Thus the logarithmic derivatives at zero of the formal series $e^{-\theta[k]z}F_{b,k}(z)$ belong to $p^j\mathcal{O}$ for all $j \geq 1$. Its undivided derivative of order d is its constant value times the sum over the partitions of a set of d elements, multiplying for each block the logarithmic derivative of its size. This formula follows by differentiating, either adding a singleton or enlarging one

of the existing blocks. Each product lies in $p^d\mathcal{O}$, proving the assertion for nonzero labels without dividing by a factorial. At $k = 0$ the center is zero and Theorem 56.42 gives directly $v(J_d(F_{b,0})) \geq d$, including at a zero of the section. □

For s such connected coordinates use variables $Z_1, \dots, Z_s$, units θ_i, and a tuple of labels $\mathbf{k}$. Write $c_i = \theta_i[k_i]$. For a polynomial f with coefficients in $\mathcal{O}$, its Hasse derivative $f^{[\mathbf{d}]}(c)$ is the coefficient of $X^{\mathbf{d}}$ in $f(c + X)$. It is integral even when ordinary derivatives divided by factorials would be inconvenient to evaluate. For displacements b_i write $B_{i,d} = B_d(b_i, k_i)$ in the corresponding frames.

Theorem 112.2 (The signed terminal calculation modulo every power of p)**.** *At every integer precision $M \geq 1$ one has*

$$f(D_1, \dots, D_s) \prod_i F_{b_i,k_i}(z_i)\Big|_{z=0} \equiv \sum_{|\mathbf{d}|<M} f^{[\mathbf{d}]}(c) \prod_{i=1}^{s} B_{i,d_i} \pmod{p^M\mathcal{O}}.$$

The operator on the left annihilates, modulo p^M, all polynomials in

$$I^M, \qquad I = (p,\ Z_1^p - \theta_1^{p-1}Z_1, \dots, Z_s^p - \theta_s^{p-1}Z_s).$$

For the full set of labels the finite coefficient ring has the description

$$\mathcal{O}[Z_1, \dots, Z_s]/I^M \ \simeq \prod_{\mathbf{k}\in\mathbb{F}_p^s} \mathcal{O}[Z_1, \dots, Z_s]/(p, Z_1 - c_1, \dots, Z_s - c_s)^M,$$

where the centers in each factor use its own label. At fixed labels only the $\binom{s+M-1}{s}$ Taylor coefficients with $|\mathbf{d}| < M$ enter, and the coefficient with index $\mathbf{d}$ is needed only modulo $p^{M-|\mathbf{d}|}$.

Proof. Expand f as $\sum_{\mathbf{d}} f^{[\mathbf{d}]}(c)(Z - c)^{\mathbf{d}}$. The variables act on independent scalar factors, so evaluation of a centered monomial gives exactly $\prod_i B_{i,d_i}$. Its valuation is at least $|\mathbf{d}|$ by the lemma. This proves the congruence, the precision of each coefficient, and vanishing on the M-th power of each fixed-label ideal at that label. Counting monomials of total degree less than M gives the stated binomial coefficient.

For different label tuples the ideals $(p, Z_1 - c_1, \dots, Z_s - c_s)$ are comaximal, since some pair of centers differs by a unit. Their intersection is I. Indeed modulo p the polynomial $Z_i^p - \theta_i^{p-1}Z_i$ is the product of the distinct linear factors $Z_i - \theta_i[k]$ with unit differences. The Chinese remainder theorem in each successive variable identifies the quotient by these polynomials with the product of the evaluation rings $\mathcal{O}/p$ at the tuples. This proof also applies when $\mathcal{O}/p$ has nilpotents. For pairwise comaximal ideals their product is their intersection, and their powers are still pairwise comaximal. Taking the M-th powers thus gives the asserted product decomposition and the common annihilation. □

For example, in one coordinate and for every $n \geq 1$ the calculation at precision two is

$$J_n(F_{b,k}) \equiv c^n J_0(F_{b,k}) + nc^{n-1}\big(J_1(F_{b,k}) - cJ_0(F_{b,k})\big) \pmod{p^2\mathcal{O}}.$$

Here $c^0 = 1$ also when $k = 0$. This retains the first correction to the residue formula at the same terminal character.

Corollary 112.3 (Elementary divisors at the prescribed projection)**.** *Fix the frames and a finite matrix of the signed projected columns and tests of Theorem 60.32. Expand each actual entry over its class paths and their shared covectors. At the chosen place evaluate the other factors, and choose a common scalar clearing the coefficients of the remaining polynomials in*

the connected derivatives, including the incoming values, class measures, and any prescribed column divisions. Let H be this integral matrix. In each term apply the theorem to its connected coordinates, and denote the resulting signed matrix by H_M. At every precision M the elementary-divisor valuations of H and H_M, each truncated at M, agree. Zero elementary divisors are assigned valuation $+\infty$. In particular the number with valuation strictly below M is a lower bound on the rank of the original projected columns.

Proof. The integral substitution in (54.22) is a polynomial in the commuting scalar derivatives in the exact chosen frames. After the indicated clearing it remains integral when all other factors have been evaluated. For conjugated edges use the conjugated coefficient data at their corresponding places. Lemma 110.1 gives precisely the complete cyclic projections used above, up to the recorded phases. Apply the congruence to each product term, keeping a shared covector fixed in both adjacent factors until they have been multiplied. Summing with the original coefficients gives $H = H_M$ modulo p^M entry by entry. No error is multiplied by the number of paths in this assertion since $p^M\mathcal{O}$ is an ideal.

Their cokernels after reduction modulo p^M are identical. If ϖ is a uniformizer and $e = v_\varpi(p)$, Smith reduction of a matrix over $\mathcal{O}$ presents this finite cokernel as a sum of modules of lengths $\min(v_\varpi(d_i), eM)$, together with the free target rows reduced modulo p^M. The matrix dimensions are the same, so uniqueness of elementary divisors gives precisely the claimed truncated list on dividing by e. Every divisor of valuation less than M is nonzero in characteristic zero. Clearing by a nonzero common scalar preserves the characteristic-zero rank. □

The calculation applies on the same interval as Corollary 15.23. An estimate for the number of surviving elementary divisors and the height of the projected period over that whole interval is still required in Section 6.6.

112.2 Complex norms on the nonzero terminal frequencies

Fix the terminal prime $p = T = \varpi\bar{\varpi}$ of Theorem 9.13, an output covector, and one of the translated product terms in Lemma 110.1. We work at a complex embedding and keep the point of evaluation at zero. In one constrained elliptic coordinate the lattice is $\Lambda = \delta I$, with I a fractional ideal of the full ring of K_d. Write A for its area, $c = \pi/A$, and Θ for the marked theta section. Its norm uses the Hermitian metric and Haar probability on $\mathbb{C}/\Lambda$. Interchange $\varpi, \bar{\varpi}$ if necessary so that the frequency subgroup is $\varpi^{-1}\Lambda/\Lambda$. Choose its generator w to precision p^2 as in Lemma 56.41. With the translation operators and phases of that lemma, put

$$S_k = p^{-1}\sum_{j=0}^{p-1} e^{-2\pi i k j/p} U_{jw}\Theta, \qquad y_k = J_P(U_b S_k), \qquad J_P(H) = \sum_{n=0}^{K} a_n \partial_z^n H(0).$$

Here $P(Z) = \sum_{n=0}^{K} a_n Z^n$ is any complex polynomial and b is the fixed prime-to-p shift of the product term, lifted to the common cover. Give these polynomials the squared norm $\mu(P) = \|\Theta\|^2 \sum_{n=0}^{K} c^n n! |a_n|^2$.

Lemma 112.4 (Energy on a split cyclic subgroup)**.** *Let λ be the shortest nonzero length of the real dual lattice $\Lambda^\vee$, and put $\rho = \lambda/\sqrt{c}$. Define*

$$E = 50\left(\frac{2\sqrt{K+6}}{\pi\sqrt{p}\,\rho}\right)^6, \qquad N = \max\{1, \lceil 8\sqrt{K+6}/\rho \rceil\}, \qquad B = 3N^2/2.$$

Then

$$\left|\sum_{k\in\mathbb{F}_p} |y_k|^2 - \mu(P)\right| \le E\mu(P), \qquad \max_{k\in\mathbb{F}_p} |y_k|^2 \le (B/p)\mu(P).$$

Also $\rho^2 \ge 2/(\pi\sqrt{|\operatorname{disc} K_d|})$.

Proof. Finite Fourier orthogonality gives $\sum_k |y_k|^2 = p^{-1}\sum_j |J_P(U_bU_{jw}\Theta)|^2$. Commuting the two translations changes each summand inside the absolute value by a scalar of modulus one, common to all derivative orders. The last mean therefore samples the periodic function $g(z) = |J_P(U_zU_b\Theta)|^2$ on $\varpi^{-1}\Lambda/\Lambda$. Its Haar mean is $\mu(P)$. Indeed the metric representative of order n is obtained by applying $(\partial_z - c\bar z)^n$ to the holomorphic section and inserting the metric factor. The adjoint is $-\partial_{\bar z}$. Commuting a lowering operator past a raising operator contributes c. Any leftover lowering operator kills a holomorphic section, so different orders are orthogonal and the squared norm in order n is $c^n n!\|\Theta\|^2$, as in Lemma 109.4.

The Fourier estimate proved in Lemma 54.33 gives $|\widehat{g}(u)| \le \mu(P)(2\sqrt{c(K+6)}/(\pi|u|))^6$ for $0 \ne u \in \Lambda^\vee$. Its hypothesis holds because each lowering operator reduces the raising order by one, so $K+1$ lowering operators kill the whole section. A unitary character in the period multiplier cancels in the metric pairing used in the proof. Averaging retains only the dual of $\varpi^{-1}\Lambda$, which is $\bar\varpi\Lambda^\vee$. Thus the shortest surviving nonconstant frequency has length $s = \sqrt{p}\,\lambda$. In $2^h s \le |u| < 2^{h+1}s$ there are at most $25\cdot 2^{2h}$ such frequencies, by packing disjoint disks of radius $s/2$. Summing the sixth-power bound and using $25\sum_{h\ge 0} 2^{-4h} < 50$ proves the first inequality.

For the pointwise estimate pass to $\Gamma = \bar\varpi\Lambda$. The p sections $U_{jw}\Theta$ on that cover are eigenvectors for a common unitary period translation with distinct eigenvalues. Indeed the pairing of a suitable period of Λ with the generator w is a primitive p-th root. These vectors are orthogonal and have the same norm as Θ for probability measure. Hence $\|S_k\|^2 = \|\Theta\|^2/p$ for every k. By Lemma 56.41, S_k also has a theta multiplier on $\Lambda' = (\bar\varpi/\varpi)\Lambda$, differing from that of S_0 by a unitary character. Its squared metric norm on this torus agrees with the one on the cover. The rotation from Λ to Λ' preserves c and λ. The same adjoint calculation gives mean $\mu(P)/p$ for $|J_P(U_zU_bS_k)|^2$ on the new torus. Lemma 54.33 with the indicated N bounds its mean on the full N-torsion grid by $3\mu(P)/(2p)$. Its N^2 summands include the nonnegative value $|y_k|^2$ at zero, giving the assertion. The identical grid argument on the original torus gives $|J_P(U_b\Theta)|^2 \le B\mu(P)$.

Finally the planar dual is $i\Lambda/A$. If $0 \ne u \in I$, then $(u)I^{-1}$ is an integral ideal, so $|u|^2 \ge N(I)$. The area is $|\delta|^2 N(I)\sqrt{|\operatorname{disc} K_d|}/2$. Consequently $\rho^2 = \min_{0\ne z\in\Lambda}|z|^2/(\pi A)$ has the stated lower bound. □

In one scalar coordinate there is also an estimate after subtracting the mean on $\mathbb{F}_p^\times$. Write $\bar y = (p-1)^{-1}\sum_{k\ne 0} y_k$. Since $\sum_k S_k = \Theta$, the squared norm lost in this subtraction is exactly

$$(p-1)|\bar y|^2 = \frac{|J_P(U_b\Theta) - y_0|^2}{p-1} \le \frac{B(1+p^{-1/2})^2}{p-1}\,\mu(P).$$

Here the triangle inequality uses the two pointwise bounds just proved. Together with the cost $|y_0|^2 \le B\mu(P)/p$ of deleting zero, this gives a lower bound for the actual one-coordinate augmentation, with the original phases of the Fourier sum.

Theorem 112.5 (Rank and norms on nonzero terminal covectors)**.** *For one fixed translated product term let r_0 be the number of constrained coordinates at the chosen split block. In coordinate h use $S_{h,k_h}, b_h, c_h, K_h, E_h, B_h$ as above. On the polynomials $P = \sum_\nu a_\nu Z^\nu$ of degree at most K_h in Z_h, put*

$$\|P\|_{\mathrm F}^2 = \Bigg(\prod_h \|\Theta_h\|^2\Bigg)\sum_\nu |a_\nu|^2 \prod_h c_h^{\nu_h}\nu_h!.$$

Let $T_b P(k)$ be $P(\partial)$ applied at zero to $\prod_h (U_{b_h} S_{h,k_h})(z_h)$. Then

$$\prod_h (1 - E_h - B_h/p)_+ \, \|P\|_{\mathrm{F}}^2 \;\le\; \sum_{0 \neq k \in \mathbb{F}_p^{r_0}} |T_b P(k)|^2 \;\le \prod_h (1 + E_h) \, \|P\|_{\mathrm{F}}^2,$$

where $u_+ = \max(u, 0)$. At the word weights of Theorem 9.13 these factors lie between $1 - \eta$ and $1 + \eta$, with $\eta \le p^{-1+o(1)}$ uniformly as the interval recedes. In particular the evaluation on the nonzero covectors is injective on the prescribed polynomial subspace.

Proof. On one coordinate Lemma 112.4 bounds the squared singular values of the full evaluation map above by $1 + E_h$ in the indicated polynomial metric. Deleting the label zero costs at most $(B_h/p)\mu(P)$ for every input, so the map on $\mathbb{F}_p^\times$ has squared singular values at least $(1 - E_h - B_h/p)_+$, including zero eigenvalues if necessary. On a Cartesian product of label sets the evaluation map is the tensor product of the one-coordinate maps. This follows on monomials by differentiation in separate variables, and extends by linearity. The displayed polynomial metric is the tensor product metric. Diagonalizing the positive Gram operators gives the products of the one-coordinate eigenvalues. The nonzero covectors contain $(\mathbb{F}_p^\times)^{r_0}$ and are contained in the full grid. Adding squared absolute values can only increase the quadratic form. This proves both bounds, including for polynomials with mixed terms.

For the uniform statement take K_h up to the total degree of the prescribed polynomial substitution on the edge. At each CM embedding its degree is the size of its partition and is at most the row rank times the first row length, by the determinant shift in Lemma 108.11. The number of embeddings and all ranks are bounded by fixed powers of x. The first row is $O(x \exp(x^{\varepsilon}) + x^{C})$ by the weight calculation in Corollary 110.12. Also d_0 is polynomially bounded on the conductor intervals of Theorem 12.15, so the discriminant estimate in Lemma 112.4 gives $\rho_h^{-2} \ll x^{C'}$ at every embedding. It follows that $B_h \ll x^{C''} \exp(x^{\varepsilon})$ and $E_h \ll (B_h/p)^3$. The terminal size estimate in Theorem 71.17 gives $\log p \gg m x^{\varepsilon}$ with $m \to \infty$. Hence $r_0 \max_h (E_h + B_h/p) \le p^{-1+o(1)}$. The lower product is at least one minus that sum once all factors are positive. The upper product is at most $\exp(\sum_h E_h)$, which gives the stated error. Restricting a positive definite quadratic form to the prescribed subspace proves injectivity there. □

Corollary 112.6 (Gram determinants and pairing with an incoming table)**.** *Let $P_1, \dots, P_q$ be linearly independent prescribed polynomials for the same product term, with coefficients independent of k. Write G_{F} for their polynomial Gram matrix and G_{term} for the Gram matrix of their values on all nonzero covectors. On the above intervals*

$$\left| \frac{1}{q} \log \frac{\det G_{\mathrm{term}}}{\det G_{\mathrm{F}}} \right| \le 2\eta \qquad (\eta \le 1/2).$$

On the span of the given inputs let T be this evaluation map, and let Π be orthogonal projection onto its image in the space of tables on nonzero covectors, with counting measure. For every incoming table Y one has

$$(1 - \eta)\|\Pi Y\|^2 \le \|T^* Y\|_{\mathrm{F}}^2 \le (1 + \eta)\|\Pi Y\|^2.$$

Proof. Apply the theorem to every linear combination of the P_i. Conjugating the resulting Gram inequalities by $G_{\mathrm{F}}^{-1/2}$ puts each eigenvalue of the relative Gram matrix in $[1 - \eta, 1 + \eta]$. Taking logarithms and using $-\log(1 - \eta) \le 2\eta$ proves the determinant bound. The positive eigenvalues of TT^* are the eigenvalues of T^*T, and TT^* is zero on the orthogonal complement of the image. Its quadratic form on Y is $\|T^* Y\|_{\mathrm{F}}^2$, proving the second assertion. For a bilinear incoming contraction take the complex conjugate of the incoming coefficients in the Hermitian pairing. □

Corollary 9.34 permits these estimates at the very primes for which the full strict groups vanish throughout the weight interval. For the signed period of Section 6.6, one must still bound the projections of the prescribed incoming tables and the cancellation among terms with different shifts and class paths.

112.3 The scalar-orbit projection of the terminal derivatives

In a terminal block the scalar $a \in \mathbb{F}_p^{\times}$ acts on an entire covector by $k \mapsto ak$. The incoming tables in Proposition 110.13 have mean zero on every such orbit. We estimate the part of an outgoing theta table that survives pairing with this space. All translations, phases and derivatives below use the fixed product of Theorem 112.5 at the original point.

Lemma 112.7 (Translation by a short lift of a frequency)**.** *In one scalar coordinate write $y = (J_P(U_b S_k))_{k\in\mathbb{F}_p}$ and $E_+ = E(K+1)$, with E as in Lemma 112.4. There is a unitary operator W on the space of these tables, independent of P, such that*

$$(Wf)(k) = \gamma e^{2\pi i a k/p} f(k+d), \qquad |\gamma| = 1, \quad a \neq 0,$$
$$\|(I-W)y\| \le 4\sqrt{(K+1)/p}\sqrt{1+E_+}\,\|P\|_{\mathrm{F}}.$$

Here $a, d \in \mathbb{F}_p$ and the norm on tables uses counting measure.

Proof. Choose a shortest nonzero $v \in \Lambda$ and set $h = v/\varpi$. Its length is smaller than that of any nonzero period, so $h = aw + \ell$ with $a \not\equiv 0 \pmod p$ and $\ell \in \Lambda$. The disk of radius $|v|/2$ embeds up to its boundary in the torus of area A. Thus $c|h|^2 = \pi|v|^2/(pA) \le 4/p$. In the finite Fourier sum defining S_k, cyclically shifting the summation index gives $U_w S_k = e^{2\pi i k/p} S_k$. A period translation U_ℓ on the summand indexed by j acts by its original period sign times a character of j, by the commutator calculation in Lemma 56.41. Thus it sends S_k to that sign times S_{k+d} for a fixed d. Composing these two translations and commuting past U_b introduces a constant phase independent of k. This proves that $J_P(U_h U_b S_k) = (Wy)(k)$ with the indicated unitary W.

To estimate the difference, let $H_t = U_{th} U_b S_k$ for $0 \le t \le 1$. The translation law in the common coordinate is $U_u H(z) = \exp(-c\bar{u}z - c|u|^2/2) H(z+u)$. Differentiating along this real segment yields $\dot{H}_t = h\partial_z H_t - c\bar{h} z H_t$. Hence $dJ_P(H_t)/dt = J_Q(H_t)$, where $Q(Z) = hZP(Z) - c\bar{h}P'(Z)$. In the factorial metric, $\|ZP\|_{\mathrm{F}} \le \sqrt{c(K+1)}\|P\|_{\mathrm{F}}$ and $\|cP'\|_{\mathrm{F}} \le \sqrt{cK}\|P\|_{\mathrm{F}}$, by shifting each monomial in the norm formula. The complex sampling lemma applies along the segment. Its proof for translated sections uses only unitarity of the multiplier and therefore permits an arbitrary complex displacement. Also $U_{th}U_b$ differs from U_{th+b} by a common phase. The vector of the derivatives in t thus has norm at most $\sqrt{1+E_+}\|Q\|_{\mathrm{F}}$. Integrate from zero to one and insert the bound on $|h|\sqrt{c}$. □

Proposition 112.8 (Loss at a single multiplicative character)**.** *For every character χ of $\mathbb{F}_p^{\times}$, including the trivial character, put $u_\chi(0) = 0$ and $u_\chi(k) = \chi(k)/\sqrt{p-1}$ for $k \neq 0$. If $t = \sqrt{(K+1)/p} \le 1/2$, then*

$$|\langle u_\chi, y\rangle|^2 \le 150(1+E_+)t\,\|P\|_{\mathrm{F}}^2.$$

Proof. For every integer j prime to p, the operator W^j is, up to a unit scalar, a shift by jd and a modulation by $ja \neq 0$. For a nontrivial χ the finite-field translation and modulation bound of Gurevich, Hadani and Sochen therefore gives $|\langle u_\chi, W^j u_\chi\rangle| \le 2/\sqrt{p}$. In their standard split torus the chosen vector is precisely the normalized character on $\mathbb{F}_p^{\times}$ extended by zero, with the counting inner product [6, §I-B and (I.1)]. For the trivial character the same inner product, up

to a phase, is the sum of the nonconstant additive character of frequency ja over all k except 0 and $-jd$, divided by $p-1$. The complete sum vanishes, so its modulus is at most $2/(p-1)$. For $p \mid j$ the unitary bound one suffices.

Write $s = 1 - t$. By unitarity the inverse of $I - sW^*$ is its convergent geometric series. Squaring the norm and grouping pairs of exponents by their difference gives

$$\|(I - sW^*)^{-1}u_\chi\|^2 \le \frac{1}{1-s^2}\left(1 + \frac{4}{\sqrt{p}\,t} + \frac{2s^p}{1-s^p}\right) \le \frac{6}{t}.$$

Indeed $\sqrt{p}\,t \ge 1$, $s^p \le e^{-pt} \le e^{-\sqrt{p}}$, $p \ge 5$, and $1 - s^2 \ge t$. On the other hand the sampling lemma gives $\|y\| \le \sqrt{1+E_+}\|P\|_{\mathrm{F}}$. Applying Lemma 112.7 in $I - sW = tI + s(I - W)$ gives $\|(I - sW)y\| \le 5t\sqrt{1+E_+}\|P\|_{\mathrm{F}}$. Pairing this last vector with $(I - sW^*)^{-1}u_\chi$ recovers the original inner product in absolute value. Cauchy–Schwarz proves the assertion. □

For $D = \mathbb{F}_p^{r_0} \setminus \{0\}$ let A be the orthogonal projection onto the tables with mean zero on each line of nonzero multiples. In particular

$$(Af)(k) = f(k) - (p-1)^{-1} \sum_{a \in \mathbb{F}_p^\times} f(ak).$$

The same a acts on every coordinate. Use T_b for the evaluation map of Theorem 112.5 on D.

Theorem 112.9 (Rank after the scalar-orbit projection)**.** *On every sufficiently distant common interval, at every complex embedding and for one fixed translated product of the marked terminal block,*

$$(1 - \eta')\|P\|_{\mathrm{F}}^2 \le \|AT_bP\|^2 \le (1 + \eta')\|P\|_{\mathrm{F}}^2, \qquad \eta' \le p^{-1/2+o(1)}.$$

This holds for every linear combination of the prescribed polynomial rows. In particular their rank is preserved by the evaluation followed by scalar augmentation. More generally, on removing the same J specified scalar character types from every line, the lower bound has loss at most $p^{-1+o(1)} + Jp^{-1/2+o(1)}$.

Proof. Let $D_0 = (\mathbb{F}_p^\times)^{r_0}$. Both D_0 and its complement in D are unions of whole scalar orbits. Hence restricting to D_0 commutes with the projections onto individual types, and hence also with deleting them. On D_0 the evaluation is $T_1 \otimes T_R$, where T_1 is scalar evaluation on the first punctured coordinate and T_R is the tensor product for the others. For $r_0 = 1$ use the one-dimensional trivial space as the empty tensor product. Let P_χ be orthogonal projection to the character χ in the first coordinate. It has rank one. Proposition 112.8 gives, in the order of Hermitian forms,

$$T_1^* P_\chi T_1 \le \delta I, \qquad \delta = 150(1 + E_{1,+})\sqrt{(K_1+1)/p}.$$

Write Q_β for the orthogonal type projections of the simultaneous scalar action on the remaining coordinates. These sum to the identity. On the product grid the projection to total type α is $\sum_\chi P_\chi \otimes Q_{\alpha\chi^{-1}}$, since characters multiply on tensor products. Pulling this projection back by $T_1 \otimes T_R$ gives an operator bounded by

$$\delta I \otimes \sum_\chi T_R^* Q_{\alpha\chi^{-1}} T_R = \delta I \otimes T_R^* T_R.$$

This uses positivity of each second factor and therefore applies also to polynomials coupling different coordinates. By the scalar sampling lemma $\|T_R\|^2 \le \prod_{h>1}(1 + E_h)$. The lower

bound on the unprojected product grid in the proof of Theorem 112.5 is $\prod_h (1 - E_h - B_h/p)_+$. Subtract at most $J\delta \prod_{h>1}(1 + E_h)$ for J removed types, then add the nonnegative squared norm on $D \setminus D_0$. The upper bound follows by contractivity of an orthogonal projection and the upper bound on D in the same theorem. In the word range $K_h + 1 = p^{o(1)}$, $E_{h,+} \le p^{-3+o(1)}$, and the lower product is at least $1 - p^{-1+o(1)}$. These estimates are uniform in the word, the shift and the embedding. Taking $J = 1$ and removing the trivial character gives the result for A. □

Corollary 112.10 (Pairing with the full space of separately augmented tables)**.** *Take a fixed product term at all split terminal blocks and impose the zero-mean condition independently at each block, holding the other covectors fixed. Evaluation followed by orthogonal projection to this space has all squared singular values in $[1 - p^{-1/2+o(1)}, 1 + p^{-1/2+o(1)}]$ on the prescribed polynomial subspace with its product factorial metric. The same bounds hold for the Rayleigh quotients of its Gram matrix relative to the polynomial Gram matrix on every independent list of rows.*

Proof. Each block uses its own scalar group. Averaging in independent variables commutes, so the projection onto the common zero-mean space is the tensor product of the single-block projections. The product evaluation also factors. Diagonalizing the positive Gram operators multiplies their eigenvalue bounds. The number of blocks is bounded by a fixed power of x, hence is $p^{o(1)}$ on these intervals. Multiplying the bounds in the theorem preserves the stated error. Restriction to any polynomial subspace preserves the form inequalities. For an incoming vector with the separate zero-mean conditions, moving the orthogonal projection across the pairing leaves that vector fixed. Thus this is the Gram matrix for pairing with an orthonormal basis of the entire stated space. □

The prime choices in Corollary 9.34 apply here. An individual prescribed incoming column can be perpendicular to the span of these projected rows. The estimate in Section 6.6 still requires control of those columns after the intermediate classes, shifts and their coefficients are summed.

112.4 Signed sums of torsion translates at the marked prime

Two translates which have the same displacement on the common elliptic cover must be added with their original phases. We estimate the cross terms for distinct displacements on that cover, still at the origin used to define the period. For one constrained coordinate use $\Gamma = \bar{\varpi}\Lambda$ and $\Xi = \varpi^{-1}\Lambda$. Write $L_b P = (J_P(U_b S_k))_{k \in \mathbb{F}_p}$. We use the same marked section and factorial metric for all b. A translation by a period of Γ acts on every S_k by the same scalar: such a period commutes with all U_{jw} and acts on Θ by its period sign. Thus one can group equal displacements modulo Γ, incorporating the translation phase in their coefficients.

Lemma 112.11 (Inner products for distinct torsion displacements)**.** *Let b, b' be distinct modulo Γ with $M(b - b') \in \Gamma$, and let P, Q have degree at most K. Take a basis h_1, h_2 of Ξ and set $H = \max_i |h_i|$. For every integer $j \ge 1$,*

$$|\langle L_b P, L_{b'} Q\rangle| \le (1 + E(K + j))(MH\sqrt{c(K + j)})^j \|P\|_{\mathrm{F}} \|Q\|_{\mathrm{F}}.$$

One can take $cH^2 \le (4 + \pi\sqrt{|\operatorname{disc} K_d|})/p$.

Proof. Write $e(h, b)$ for the unit scalar in $U_h U_b = e(h, b) U_b U_h$. The associated alternating form, up to sign, is $\operatorname{Im}(\bar{h}b)/A$ in the exponential $e^{2\pi i t}$. The annihilator of Ξ under this pairing

is Γ. Indeed the area normalization makes Λ self-dual for the alternating form, and moving the factor ϖ^{-1} across it conjugates that factor. As $b - b' \notin \Gamma$, for some basis vector h the number $\xi = e(h, b - b')$ is a nontrivial M-th root of unity. In particular $M \geq 2$ and $|\xi - 1| \geq 2\sin(\pi/M) \geq 4/M$.

Form the scalar function of a real variable

$$F(t) = \left\langle \left(J_P(U_{th}U_bS_k)\right)_k, \left(J_Q(U_{th}U_{b'}S_k)\right)_k \right\rangle,$$

using the inner product linear in its first argument. Since $h = aw + \ell$ with $\ell \in \Lambda$, U_h acts on the list $(S_k)_k$ by one unitary matrix, a modulation and a label translation up to a common phase, as in Lemma 112.7. Also $U_{nh} = U_h^n$. Moving this translation past the two shifts and canceling the common unitary in the pairing gives $F(n) = \xi^n F(0)$ for each nonnegative integer n.

In the first argument, differentiation in t applies the polynomial operator $P \mapsto hZP - c\bar{h}P'$. On degree at most $K+j-1$ its norm is at most $2|h|\sqrt{c(K+j)}$, by the factorial norm calculation in that lemma. The same holds in the second argument. Applying the product rule j times and the full-grid sampling bound in Lemma 112.4 at the translated points gives

$$\sup_t |F^{(j)}(t)| \leq (1 + E(K+j))(4|h|\sqrt{c(K+j)})^j \|P\|_{\mathrm{F}}\|Q\|_{\mathrm{F}}.$$

The j-th forward difference of F at zero is $(\xi - 1)^j F(0)$. By iterating the fundamental theorem of calculus it is also the integral of $F^{(j)}(t_1 + \cdots + t_j)$ on the unit j-cube. Division by $|\xi - 1|^j$ proves the claim.

For the basis take a shortest period $v_1 \in \Lambda$, with length s. It is primitive, so it extends to a lattice basis. Subtract multiples of v_1 from the second vector to obtain v_2 with parallel component of magnitude at most $s/2$. Its perpendicular component has magnitude A/s. The disk argument gives $cs^2 \leq 4$, and the ideal norm bound in Lemma 112.4 gives $s^2/A \geq 2/\sqrt{|\operatorname{disc} K_d|}$. Consequently $c|v_2|^2 \leq 1 + (\pi/2)\sqrt{|\operatorname{disc} K_d|}$. Take $h_i = v_i/\varpi$. □

In each coordinate h, take a nonempty finite set B'_h of displacements in one coset of the M_h-torsion on $\mathbb{C}/\Gamma_h$, with one lift per displacement. For the tuple $\boldsymbol{b} \in \prod_h B'_h$ let $P_{\boldsymbol{b}}$ be the polynomial obtained after summing all coefficients belonging to that tuple. Coefficients may be signed or complex and must be independent of the shared frequency k. Give the collection the squared norm $\|\boldsymbol{P}\|^2 = \sum_{\boldsymbol{b}} \|P_{\boldsymbol{b}}\|_{\mathrm{F}}^2$. Let $S\boldsymbol{P}$ be the sum of the corresponding product evaluations $T_{\boldsymbol{b}}P_{\boldsymbol{b}}$ on the nonzero covectors. We use A for subtraction of the scalar-orbit means as in Theorem 112.9.

Theorem 112.12 (Norm and rank of a signed sum with torsion displacements)**.** *Take the marked primes, embeddings and derivative degrees on the common intervals of Corollary 9.34. The following bounds hold uniformly for the cosets, the sets within them and the grouped coefficient polynomials.*

1. *If $0 < \sigma < 1/2$ is fixed and all $M_h \leq p^{1/2-\sigma}$, then $\big|\|S\boldsymbol{P}\|^2 - \|\boldsymbol{P}\|^2\big| \leq p^{-2\sigma+o(1)}\|\boldsymbol{P}\|^2$.*
2. *If $0 < \sigma < 1/4$ is fixed and all $M_h \leq p^{1/4-\sigma}$, then $\big|\|AS\boldsymbol{P}\|^2 - \|\boldsymbol{P}\|^2\big| \leq p^{-2\sigma+o(1)}\|\boldsymbol{P}\|^2$.*

For uniformly $M_h = p^{o(1)}$ the respective errors are $p^{-1+o(1)}$ and $p^{-1/2+o(1)}$. These estimates also hold for a product of marked blocks, with each covector nonzero and the scalar-orbit projections imposed separately. In each case the evaluated rank of a list of rows equals the rank of its list of grouped coefficient arrays.

Proof. First work in a single coordinate and write $n = |B'_h| \le M^2$. On the direct sum over the shifts, the full-grid map is $L : (P_b)_b \mapsto \sum_b L_b P_b$. Lemma 112.4 bounds each diagonal block of $L^*L - I$ in operator norm by $E(K)$. The off-diagonal blocks have norm at most $u_j = (1 + E(K + j))(MH\sqrt{c(K + j)})^j$ by Lemma 112.11. Using $\sum_{b \ne b'} v_b v_{b'} \le (n-1)\sum_b v_b^2$ on the norms of the inputs gives, for every $j \ge 1$,

$$\|L^*L - I\| \le E(K) + (n-1)u_j.$$

Deleting the label zero costs at most nB/p in the squared operator norm of that evaluation functional. This follows by Cauchy–Schwarz across the n shifts from the pointwise bound, with B of Lemma 112.4. In the punctured coordinate the squared norm of the projection onto any one multiplicative character is bounded, as a form on the direct sum, by $150n(1 + E(K + 1))\sqrt{(K + 1)/p}$ times its input form. This is the same Cauchy–Schwarz argument applied to Proposition 112.8.

On the full or componentwise punctured grid the map on the arrays is the tensor product of these one-coordinate direct-sum maps. Indeed on a tensor of lists the product of the sums is the sum over all shift tuples. Such tensors span the array space, with exactly the stated direct-sum polynomial metric. The tensor bounds, restriction to nonzero covectors and scalar-character calculation in Theorems 112.5 and 112.9 therefore apply with the three bounds just obtained. In particular the cost of removing the trivial total character is at most the one-character loss in one coordinate times the upper squared norm in the other coordinates. This follows by summing the orthogonal character projections on those other coordinates, as in the proof of the latter theorem.

For the first range take a fixed integer $j \ge 1/\sigma$. Uniformly $\sqrt{c(K + j)}H \le p^{-1/2+o(1)}$ by the basis bound and the growth estimates of Theorem 112.5. Then $nu_j \le p^{1-2\sigma-j\sigma+o(1)}$, and the zero-label loss is at most $p^{-2\sigma+o(1)}$. For the second range take $j = 2$. The off-diagonal error is at most $M^4 p^{-1+o(1)} \le p^{-4\sigma+o(1)}$, and the character loss is at most $M^2 p^{-1/2+o(1)} \le p^{-2\sigma+o(1)}$. The other errors are smaller. When $M = p^{o(1)}$, taking $j = 2$ gives the two sharper rates as stated. The number of coordinates and blocks is bounded by a fixed power of x, hence is $p^{o(1)}$. Multiplying the upper and lower factors preserves these rates. The relative error tends to zero in each range. Applying the lower inequality to every linear combination of the rows proves the rank assertion, including when the original list is dependent. □

Corollary 112.13 (Displacements at the two auxiliary primes)**.** *Fix an integer $C \ge 1$. Within one fixed product model of a marked block, group the signed terms by the cosets, in the common covers, of the $(g_1 g_2)^C$-torsion in each constrained coordinate. Within each group the ranks of the signed polynomial arrays and of their separately augmented terminal tables agree. The logarithms of the two Gram determinants on any independent list of q arrays differ by at most $qp^{-1/2+o(1)}$. The other displacements, the chosen cosets and the lifts of their base points may vary with the interval.*

Proof. Proposition 6.30 puts each g_i between $(kb)^8$ and twice that quantity, with $b = \exp(x^\varepsilon)$. Thus $\log(g_1 g_2)^C \ll_C x^\varepsilon + \log x = o(\log p)$ by Theorem 71.17. The order is coprime to p. Projection of the common cover to the original elliptic curve has degree p and so is an isomorphism on torsion of this order. Explicitly, multiplication by that order is invertible on its kernel, allowing adjustment of any lift to a unique torsion lift. Differences of the indicated displacements therefore satisfy the coset hypothesis in the same cover as the Fourier labels. The theorem now applies to the signed sums within each group. On an independent list, conjugating the two Gram inequalities by the inverse square root of the polynomial Gram matrix bounds each relative eigenvalue within $p^{-1/2+o(1)}$ of one. Summing their logarithms gives the assertion. □

Corollary 112.14 (Contraction against the actual refined initial class)**.** *Fix a word c, an initial coarse class and one unit-basis matrix coefficient of its infinity type ρ, as in Proposition 64.29. At the first edge fix the output covectors and product coordinates. Collect any set of signed path contributions whose first-edge displacements meet the hypotheses of part 2 of Theorem 112.12 in these coordinates. The nonterminal cosets, the remaining edges and the test scalars may be summed within this collection, keeping each shared index once. For each output test i let $\boldsymbol{P}_i$ be its grouped array, including its arithmetic coefficients and class masses. Let G_{F} be the Gram matrix of these arrays, and let G_{in} be the Gram matrix of the resulting period functionals, restricted to the separately zero-mean tables on this actual initial class and matrix coefficient. Use the dual of the automorphic probability norm on the functionals. Then*

$$(1-\eta)R_cG_{\mathrm{F}} \ \le G_{\mathrm{in}} \le \ (1+\eta)R_cG_{\mathrm{F}}, \qquad R_c = M_H(\dim\rho)(p^k-1)^f,$$

where k is the initial unitary rank, $f = [E_x^+ : \mathbb{Q}]$, and η is the augmented error in that theorem. On summing over distinct initial coarse classes or orthogonal matrix coefficients, the corresponding Gram matrices add with the same bounds.

Proof. By Proposition 64.29 the M_H initial classes at the two flag primes have trivial stabilizers. Each has exactly $N = (p^k-1)^f$ refined classes indexed by the nonzero covectors, by Corollary 9.14. They have equal Haar mass $1/(M_HN)$ in probability measure, since refinement divides each compact open class into equal-measure cosets. Schur orthogonality gives archimedean squared norm $1/\dim\rho$ for the chosen matrix coefficient. Hence the squared norm of an arbitrary table ϕ supported on this class and coefficient is $R_c^{-1}\sum_k|\phi(k)|^2$, where here the summation index runs through the tuples of nonzero covectors. All such values can be prescribed independently. Imposing the zero means restricts this space precisely to the image of the product projection A.

In the path identity of Theorem 60.32 the initial argument is read once at its input refinement. With the next covectors fixed, Lemma 108.20 expresses the first incidence at that input by the cyclic projections in our coordinates. All choices on later edges are independent of the first input label once the common indices are fixed. Summing their coefficients with the actual phases therefore gives on this collection the functional $\phi \mapsto \sum_k(S\boldsymbol{P}_i)(k)\phi(k)$. This formula absorbs the weights of the arithmetic sum into its coefficient array, so the use of counting measure in writing it introduces no second class mass. Independent variations in the unconstrained coordinates may be evaluated first by linearity.

For any table y, the squared dual norm of $\phi \mapsto \sum_k y(k)\phi(k)$ on this subspace is $R_c\|Ay\|^2$. Indeed, in counting measure its Riesz vector is $\bar{y}$, and the zero-mean projection commutes with conjugation. Scaling the source metric by $1/R_c$ scales the dual metric by R_c. Apply the identity to every linear combination of the tests and use Theorem 112.12. Disjoint initial classes have disjoint supports and distinct basis matrix coefficients are orthogonal by Schur orthogonality. Dual norms on their orthogonal direct sum therefore add in squares, as claimed. □

Distinct intermediate classes can still contribute to the same initial class, with different first output covectors, displacements or product models. Their functionals then add before the dual norm is taken. The lower bound for this full sum and its comparison with the incoming analytic row are still required in Section 6.6.

112.5 Inner products of residue pullbacks through different class paths

In Theorem 60.32 the matrices on two paths through different intermediate classes can be compared in their common initial and final frames. We calculate their inner product when the final covector runs over its full table. We then count the markings at which their initial

evaluations can lie on the same line. Throughout, $p = T$ is the split terminal prime already chosen on the interval.

At one half let U, V be vector spaces over $\mathbb{F}_p$ of dimensions k, n. For a nontrivial complex character χ of $\mathbb{F}_p^\times$, let $H_\chi(U^\vee)$ be the space of tables on $U^\vee$ satisfying $h(a\xi) = \chi(a)h(\xi)$ for $a \neq 0$. All tables vanish at zero and use counting measure. For a linear map $C : U \to V$ write $P_C h(y) = h(yC)$. This is an operator into $H_\chi(V^\vee)$. We use a Hilbert–Schmidt inner product linear in the second argument.

Theorem 112.15 (The Gram matrix of two prescribed residue maps)**.** *For any two maps $C, D : U \to V$, one has*

$$\langle P_C, P_D \rangle_{\mathrm{HS}} = \frac{1}{p-1} \sum_{t \in \mathbb{F}_p^\times} \chi(t)\, p^{n - \operatorname{rank}(D - tC)}.$$

For $C \neq 0$ its diagonal entry is $a_C = p^n(1 - p^{-\operatorname{rank} C})/(p-1)$. Suppose C, D are nonzero and projectively distinct as matrices. Let h be the rank of $D - ZC$ over $\mathbb{F}_p(Z)$, and $\delta = \min_{t \neq 0} \operatorname{rank}(D - tC)$. Then

$$\frac{|\langle P_C, P_D \rangle_{\mathrm{HS}}|}{\sqrt{a_C a_D}} \;\leq\; \min\left(1, \frac{h p^{-\delta}}{1 - p^{-1}}\right), \qquad \delta \geq 1.$$

The inner product is zero if the ranks at all nonzero scalars are equal. For $D = tC$, $P_D = \chi(t) P_C$.

Proof. Choose one representative u of each line in $U^\vee$. The functions supported on their respective lines, taking the values $(p-1)^{-1/2}\chi(a)$ at au, form an orthonormal basis. Compute the inner product by applying both operators to this basis and summing over $y \in V^\vee$. A pair of nonzero evaluations contributes only when $yD = tyC$. There is then exactly one such t, and the contribution is $\chi(t)/(p-1)$. For each fixed t, the equation has $p^{n - \operatorname{rank}(D - tC)}$ solutions. Those with $yD = yC = 0$ must be removed for each t. Their number is independent of t and $\sum_{t \neq 0} \chi(t) = 0$, giving the formula. Setting $D = C$ gives the diagonal. Homogeneity proves the assertion for scalar multiples.

A nonzero h-rowed minor of $D - ZC$ is a polynomial of degree at most h. Every specialization of smaller rank is a root, and every specialization has rank at most h because the larger minors vanish identically. Subtract the constant p^{n-h} in the character sum. At most h summands then survive, each of modulus at most $p^{n-\delta}$. The two diagonal entries are at least $p^n(1 - p^{-1})/(p-1)$. Division and the Hilbert–Schmidt Cauchy–Schwarz inequality give the estimate. A zero specialization would make the matrices proportional. The assertion about constant rank follows from the same character sum. □

Corollary 112.16 (A bound for signed sums on the full terminal table)**.** *Fix common endpoints and frames. For a finite list of paths let $C_{i,v} \neq 0$ be the composite residue matrices at the selected halves $v \mid \pi_T$, with f such halves. Prescribe a nontrivial character χ_v at each half. Put $Q_i = \bigotimes_v P_{C_{i,v}}$, $a_i = \prod_v a_{C_{i,v}}$. Let A_i be their coefficient maps between fixed finite-dimensional Hilbert spaces, including all finite signs and assigned class masses. In a fixed coupling on the last two edges the same shared index is used in both maps. Matrices differing only by local nonzero scalars may first be collected, multiplying each coefficient by the corresponding character values. For the resulting list put*

$$d_{ij} = \#\{v : [C_{i,v}] \neq [C_{j,v}]\}, \qquad b = \min\{1, k/(p-1)\}, \qquad e = \max_i \sum_{j \neq i} b^{d_{ij}}.$$

Then

$$(1-e)\sum_i a_i\|A_i\|_{\mathrm{HS}}^2 \;\le\; \Big\|\sum_i Q_i\otimes A_i\Big\|_{\mathrm{HS}}^2 \;\le\; (1+e)\sum_i a_i\|A_i\|_{\mathrm{HS}}^2.$$

In particular if every two tuples differ at least at d halves, a list of M terms has $e\le(M-1)b^d$.

Proof. For the full final table replace the marked covector in the path recursion of Theorem 60.32 by its argument y. The incidence conditions now give the initial covector $(y_vC_{i,v})_v$. A zero entry gives a zero value. The coefficient of a fixed path, including its matrix cosets, is held fixed in this extension. Evaluation at the prescribed marking recovers the original summand. A path with a zero composite at a half can be discarded before this calculation. Across several final classes the calculation applies in each pair of endpoint frames separately.

Inner products of tensor products multiply. Each unequal matrix line therefore contributes a normalized inner product of modulus at most b by Theorem 112.15, since $h\le k$. At the other halves the modulus is one. Expand the squared norm, writing $z_i=\sqrt{a_i}\|A_i\|_{\mathrm{HS}}$. The absolute value of the sum of off-diagonal terms is at most $\sum_{i\ne j} b^{d_{ij}}z_iz_j$. The inequality $2z_iz_j\le z_i^2+z_j^2$ bounds this by $e\sum_i z_i^2$. This also verifies the estimate for arbitrary signed coefficients without an independence assumption on the paths. □

Proposition 112.17 (Markings at which two paths can cancel)**.** *For prescribed C,D at one half, choose y uniformly among nonzero final covectors. The probability that both yC,yD are nonzero and on the same line is*

$$\frac{1}{p^n-1}\sum_{t\ne0}\big(p^{n-\operatorname{rank}(D-tC)}-p^{n-\operatorname{rank}[C\ D]}\big),$$

where $[C\ D]:U\oplus U\to V$ has both sets of columns. For nonproportional C,D the probability is at most $\min\{1,(p-1)p^{-h}+hp^{-\delta}\}$, with h,δ as above. For independent markings at several halves the exact probabilities multiply. For a list of paths the fraction of markings at which any two surviving paths read the same tuple of input lines is at most the sum of these products over unordered pairs.

Proof. A surviving collision determines t uniquely. The common kernel of the two pullbacks has exactly $p^{n-\operatorname{rank}[C\ D]}$ elements, including zero. Removing it gives the numerator. Among nonzero y the probability of annihilating a fixed matrix of rank j is $(p^{n-j}-1)/(p^n-1)\le p^{-j}$. Apply this with $j=h$ at all regular scalars and with $j\ge\delta$ at the at most h exceptional scalars. Independence here refers solely to uniform counting on a product of finite sets. Multiplication gives the product formula, followed by the union bound. At any particular marking outside this union, different surviving tuples define orthogonal evaluation functionals on the homogeneous initial table, since their line supports are disjoint. □

At the particular Bessel marking the formula tests the actual covectors $y_v=(\bar\pi \bmod v)\overline{e_N^\vee}$. A counting bound for the excluded markings leaves their membership to be checked at this prescribed tuple. Even within one surviving tuple of input lines the signed arithmetic coefficients still add before the absolute value is taken.

112.6 Initial lines at the prescribed marking and their heights

Fix $E=E_x$, its $f=[E^+:\mathbb{Q}]$ selected terminal places S, and the ordinary initial frames of the signed period. Put $n=[E:\mathbb{Q}]=2f$ and $p=T=\varpi\bar\varpi$, where S consists of the places above the chosen generator ϖ in the imaginary quadratic field. Thus p splits completely in E,

$\varpi\mathcal{O}_E = \prod_{v\in S} v$, and every complex conjugate of ϖ has absolute value $\sqrt{p}$. We give bounds at the specified marking for the same field and prime used in Corollary 9.34.

Write $\mathrm{red}_S(P)$ for the tuple of projective reductions of an E-point $P = [z]$. At each finite place divide the coordinates by any local scalar that gives them maximum absolute value one before reducing. The resulting line is independent of the choice. Use the absolute projective height

$$H_2(P) = \left(\prod_w \|z\|_w^{n_w} \right)^{1/n},$$

where the finite norms are maxima with $|p|_w = p^{-1}$ at $w \mid p$, the infinite norms are Euclidean, and n_w is the local degree. In particular the weight of a complex place is two. The product formula gives independence from the global representative.

Lemma 112.18 (Separation of reductions at the actual terminal halves)**.** *For distinct projective points P, P', the number $c(P, P')$ of places in S where their reductions agree satisfies*

$$c(P, P') \log p \le n \log \big(H_2(P) H_2(P') \big).$$

In particular reduction on S is injective among points of height strictly less than $p^{1/4}$. The same assertion holds for matrices regarded as points in projective space. In affine coordinates $P = [1 : z_2 : \cdots : z_k]$ with entries in $\mathcal{O}_E$ of absolute value at most B at every embedding, two distinct such points satisfy the stronger bound $c(P, P') \log p \le n \log(2B)$. Here $B \ge 1$.

Proof. For representatives z, z' choose a nonzero minor $m = z_i z'_j - z_j z'_i$. At a finite place, $|m|_w \le \|z\|_w \|z'\|_w$. At an agreeing place in S, reduction after the indicated local scalings gives an additional factor p^{-1}. At each infinite place the determinant inequality on the two coordinate pairs gives the same bound without that factor. Multiply over all places using $\prod_w |m|_w^{n_w} = 1$ and $n_v = 1$ on S. The factor $p^{-c(P,P')}$ proves the assertion. If all $f = n/2$ places agree, the product of the heights is at least $p^{1/2}$. For integral affine points choose an index with $z_j - z'_j \neq 0$. Every agreeing place divides this algebraic integer and its absolute norm is at most $(2B)^n$. This proves the last claim. □

Theorem 112.19 (Short representatives and many separated initial lines)**.** *Let N_* be the cyclotomic modulus of the interval. In the fixed residue coordinates of the initial class every tuple of lines in $\prod_{v\in S} \mathbb{P}^{k-1}(\mathbb{F}_p)$ is represented by an integral row in E^k with every entry, in every complex embedding, of absolute value at most $nN_*\sqrt{p}/2$. Its projective height is at most $\sqrt{k}\, nN_*\sqrt{p}/2$.*

Fix $0 < \theta < 1$ and $k \ge 2$. There is a set of integral affine rows whose distinct reductions differ at more than θf of the actual halves. With

$$L = \left\lfloor \frac{p^{(1-\theta)/2}}{4nN_*} \right\rfloor$$

one can take exactly $(L+1)^{n(k-1)}$ rows. On the common intervals the logarithm of this number, divided by $f(k-1)\log p$, is $1 - \theta + o(1)$, uniformly in x.

Proof. Put $K = \mathbb{Q}(\zeta_{N_*})$, which contains E by Theorem 12.15. The traces to E of the powers of ζ_{N_*} span E over $\mathbb{Q}$. Indeed the powers span K, and trace is surjective because its restriction to E is multiplication by $[K : E]$. Select a rational basis $b_1, \dots, b_n$ among these traces. They are algebraic integers with $|\sigma b_j| \le [K : E] \le N_*$ in every embedding.

The Chinese remainder theorem supplies an integral lift z of any tuple in $\prod_{v\in S} \mathbb{F}_p$. Expand z/ϖ in the basis b_j, with rational coordinates c_j. Subtract $\varpi \sum_j m_j b_j$ from z, rounding each

c_j to a nearest integer m_j. This preserves every prescribed residue and leaves absolute value at most $\sqrt{p}\sum_j |\sigma b_j|/2$ at every embedding. Choose a nonzero representative of each desired line at each place and apply this construction to each coordinate. The resulting integral row is locally primitive at every selected place. Its finite height factors are at most one, giving the height bound. This argument only uses a rational basis of integral elements.

For the separated family take first entry one and, independently for each other entry, any sum $\sum_j m_j b_j$ with $m_j \in \{0, \dots, L\}$. Linear independence gives the stated count. Every nonconstant entry is bounded by $B = nN_*L$, which is at least one on sufficiently distant intervals. A nonzero difference of entries has norm at most $(2B)^n$, and $2B < p^{(1-\theta)/2}$. Lemma 112.18 gives $c < (1-\theta)f$. Finally n, N_* have polynomial bounds in V, while $\log p = (17 + o(1))m \log b$ with $b = \exp(x^{\varepsilon})$ by Theorem 71.9. Thus $\log(4nN_*) = o(\log p)$ uniformly. Substituting in L proves the logarithmic count. □

Proposition 112.20 (Contraction and the integral image on a prescribed set of lines)**.** *Fix a coarse initial class a, a tuple of nontrivial scalar characters with product χ on $D = (\mathbb{F}_p^{\times})^f$, and a finite set $\mathcal{B}$ of distinct tuples of input lines. Take the coefficient field to contain the character values. For each $B \in \mathcal{B}$ fix any representative q_B in the actual residue frames. For the prescribed final marking and projected test F, set*

$$W_{B,F} = \sum_{\gamma\colon [\eta_\gamma]=B} \chi(t_\gamma) W_{\gamma,F}, \qquad \eta_\gamma = t_\gamma q_B, \quad t_\gamma \in D.$$

The sum is over every surviving path starting at a with the indicated tuple, including all intermediate classes and the coupled last edges of Theorem 60.32. On the homogeneous integral initial tables supported on $\mathcal{B}$ the image of the period as an $\mathcal{O}_J$-module is exactly

$$\sum_{B \in \mathcal{B}} W_{B,F}(D_{a,B}),$$

with $D_{a,B}$ the actual coefficient lattice on the line orbit. On a fixed orthonormal matrix coefficient let $w_{B,F}$ denote the scalar value of the functional there. The squared dual norm on this complex source in the automorphic probability metric is exactly

$$\frac{R_c}{(p-1)^f} \sum_{B \in \mathcal{B}} |w_{B,F}|^2, \qquad R_c = M_H(\dim \rho)(p^k - 1)^f.$$

For multiple tests the identity applies to every linear combination. It therefore gives a positive-semidefinite lower bound on the Gram matrix of the full projected period on the ordinary initial source.

Proof. A homogeneous table on the orbit of q_B has values $\phi(tq_B) = \chi(t) z_B$. The initial class has trivial stabilizer, and its lattice on each orbit is constant as shown before Theorem 77.20. The unit character values make integrality equivalent to $z_B \in D_{a,B}$, including at places dividing $p-1$. Every such choice is allowed on each orbit independently. Nontriviality at each half makes every local scalar sum zero. Substitution in the path identity gives exactly $\sum_B W_{B,F}(z_B)$, proving the image formula. For one orthonormal coefficient, Corollary 112.14 gives the squared source norm $(p-1)^f R_c^{-1} \sum_B |z_B|^2$. Taking the dual norm proves the second formula. Restriction to a subspace can only decrease a dual norm, also for each linear combination of the tests. This is the asserted matrix inequality. □

Corollary 112.21 (Paths with small projective height at the fixed marking)**.** *Work in common residue coordinates at the initial class. Suppose the marked rows of a collection of paths are the reductions of nonzero rows P_i over E of height less than $p^{1/4}$. Distinct projective P_i then*

contribute to disjoint terms of the sum in Proposition 112.20. In particular this holds for compatible integral matrix lifts $A_{i,j}$ giving the prescribed nonzero terminal pullbacks, if the reductions of the projective products are the marked input lines and

$$H_2(y_i)\prod_{j=0}^{m-1} H_2(A_{i,j}) < p^{1/4}.$$

Here y_i lifts the marked terminal row and the matrix height uses the entries as projective coordinates. The same conclusion holds for the entire separated family in Theorem 112.19.

Proof. Matrix multiplication satisfies the maximum-entry norm inequality at finite places by the ultrametric inequality. At infinite places the Euclidean norm for rows and the Frobenius norm for matrices are submultiplicative, by Cauchy–Schwarz. Taking the weighted products bounds the height of the nonzero product row by the displayed left side. The separation lemma gives distinct tuples, and the integral and norm formulas use disjoint input orbits on these tuples. □

Thus all actual tuples admit bounded integral coordinate lifts, and many tuples admit mutually separated smaller ones. The signed coefficient $W_{B,F}$ at each tuple still includes paths of every height reducing to it. A lower estimate for those coefficient sums and their comparison with the lattice indices across the tests are required for Section 6.6.

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[1] C. Bayer and J. Teichmann, *The proof of Tchakaloff's theorem*, arXiv:math/0502473 (2005). Authors' preprint.

[2] J. A. Green, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447. doi:10.1090/S0002-9947-1955-0072878-2.

[3] D. Miličić, Asymptotic behaviour of matrix coefficients of the discrete series, *Duke Math. J.* **44** (1977), 59–88, pp. 59–60. Article.

[4] H. Hecht and W. Schmid, A proof of Blattner's conjecture, *Invent. Math.* **31**, 129–154.

[5] L. Gurvits, *Van der Waerden/Schrijver–Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials*, Electron. J. Combin. **15** (2008), R66. Author's preprint.

[6] S. Gurevich, R. Hadani and N. Sochen, *The finite harmonic oscillator and its applications to sequences, communication, and radar*, IEEE Trans. Inform. Theory **54** (2008), 4239–4253, equation (I.1) and §I-B. Authors' paper.