The generalized Riemann hypothesis for Dirichlet L-functions

Volume XVI · Rational flags, orbital masses and the terminal trace

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Abstract

We count rational flags meeting the marked terminal lattices, evaluate their masses and separate the common isotropic positions. We apply these counts to the first moment of the ordinary terminal trace.

The guide in Volume I lists the sections and page counts for the series.

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87 Rational flags and the mass formula

87.1 The weight needed to sum the rational periods

We keep $h = k-1$, $S_* = d(d-1)$, $m = h-2d+1$, and the normalizations of Section 76.9. For a supported nondegenerate label write $A_j = |N_{E/\mathbb{Q}} D_j|$ at nonzero endpoints, and use $A_j^{\#}$ for the corresponding norms after the integral change of variables in Lemma 71.10. Put $G = |N_{F/\mathbb{Q}}\mathcal{G}|$, $G^{\#} = |N_{F/\mathbb{Q}}\mathcal{G}^{\#}|$, and $B_\gamma^{\#} = G^{\#}\prod_\nu A_{j_\nu}^{\#}$. The test pair and the two unit column tuples are fixed while the label varies. If $\mathcal{A}$ is any finite set of these labels, write

$$\mathcal{H}_\theta(\mathcal{A}) = \sum_{\gamma\in\mathcal{A}} w(\beta_\gamma)(B_\gamma^{\#})^{\theta} P_\infty^{\#} \|\widehat{K}_{\gamma^{\#}}\|_{\mathrm{HS}}^2.$$

Every term uses its own orbit and the same transported sections. Here $w(\beta) = \prod_{\nu,\tau}(1 + |\tau D_{j_\nu}^{\#}|^2)^2$ is the weight of the reindexed integral endpoints.

Theorem 87.1 (An absolute period bound from a real weighted sum)**.** *For $h \ge 4(d-1)$, which holds on the common intervals for large x, one has*

$$\sum_{\gamma\in\mathcal{A},c} |J_{\gamma,c}| \le \frac{A_*\,\mathfrak{m}_h\,8^{fd/2}}{\sqrt{c_{\rho_1}^* c_{\rho_2}^*}}\, e^{O(f2^{-m})} R^{-fS_*} Q^{f(S_*/2+d-2-d^2h)}\, \mathcal{H}_{h/2}(\mathcal{A})^{1/2}. \tag{87.1}$$

There is no constant depending on the truncation $\mathcal{A}$.

Proof. The proof of Theorem 76.21 bounds the absolute cell sum for a fixed label by its compatible mass times $A_* c_E^a \|K_{\infty,\gamma}^{\mathrm{area}}\|_{\mathrm{op}}/\sqrt{c_{\rho_1}^* c_{\rho_2}^*}$. Insert (71.7) and (76.23). Apart from the common scalar $A_*\mathfrak{m}_h e^{O(f2^{-m})} R^{-fS_*}/\sqrt{c_{\rho_1}^* c_{\rho_2}^*}$, the result is

$$Q^{f(a-d-2dh)} P_\infty^{1/2} W_\gamma (P_\infty^{\#})^{1/2} \|\widehat{K}_{\gamma^{\#}}\|_{\mathrm{op}}.$$

We used $d(d-2) - S_* = -d$. The powers of the endpoints in this expression can be found exactly. By Lemma 76.15 and the telescoping calculation in Corollary 76.33,

$$P_\infty^{1/2} W_\gamma = G^{h/2} \prod_{\nu<l} A_{j_\nu}^{h_{\nu+1}}. \tag{87.2}$$

Indeed the two exponents on a proper endpoint add to $(h_\nu + h_{\nu+1})/2 + (h_{\nu+1} - h_\nu)/2 = h_{\nu+1}$. At the final endpoint they add to zero.

Substitute $G = Q^{-2fd(d-2)} G^{\#}$ and $A_j = Q^{-2fj(j-2)} A_j^{\#}$. The overall exponent of Q is now

$$f\Big(a - d - d^2h - 2\sum_{\nu<l} j_\nu(j_\nu-2)h_{\nu+1}\Big) \le f(S_*/2 + d - 2 - d^2h).$$

Only $j_\nu = 1$ can give a negative summand, with $h_{\nu+1} \le d-1$, and $a \le S_*/2$. At each real place $0 < \tau\mathcal{G}^{\#} \le |\tau D_d^{\#}|^2$. Thus $G^{\#} \le A_d^{\#}$ and $(G^{\#})^{h/2} \le (G^{\#} A_d^{\#})^{h/4}$. All the proper endpoint norms $A_{j_\nu}^{\#}$ are positive integers and their exponents in (87.2) are at most $d-1 \le h/4$. This bounds the whole reindexed norm factor by $(B_\gamma^{\#})^{h/4}$.

Replace the operator norm by the Hilbert–Schmidt norm, insert $w(\beta_\gamma)^{-1/2} w(\beta_\gamma)^{1/2}$, and apply Cauchy–Schwarz in γ. The reindexing is injective by Lemma 71.10, and all its endpoints are integral. Theorem 88.12 gives $\sum_\gamma w(\beta_\gamma)^{-1} \le 8^{fd}$. The second squared sum is precisely $\mathcal{H}_{h/2}$. This proves (87.1) for every finite truncation, and hence for the full nonnegative sums by monotone convergence. □

Proposition 87.2 (A family with growing finite orbital mass)**.** *Fix one of the fields and sets of parameters above. There are distinct rational labels γ_t, $t = 1, 2, \ldots$, all regular and supported at the original level, with*

$$\mathcal{M}^{=}_{\gamma_t} \asymp t^{2fh}, \qquad P_\infty(\gamma_t) \asymp t^{2fh},$$
$$w(\beta_{\gamma_t}) \asymp t^{4f}, \qquad B^{\#}_{\gamma_t} \asymp t^{4f}.$$

The comparison constants can depend on the fixed field, ranks and levels. They are independent of t.

Proof. Take the signed unit basis of the original row lattice. For $1 \le i < d$ write $u_i = e_{+,i} + e_{-,i}$ as in Theorem 64.1 and set $w_i = e_{+,i} - e_{-,i}$. There is a further positive unit basis vector e_0 in the positive middle space. With e the negative terminal basis vector put $w_d = e_0 + e$. These d vectors w_i are mutually orthogonal and isotropic. Choose nonzero $\alpha, \lambda \in Q^2T\mathcal{O}_E$ of trace zero to F. Such elements are obtained by multiplying a nonzero difference $b - \bar{b}$, $b \in \mathcal{O}_E$, by Q^2T. For an isotropic w and $\bar{a} = -a$, the map

$$x \longmapsto x + awH(w, x)$$

is a rational isometry. Expanding the pairing gives linear terms with coefficient $a + \bar{a}$ and a quadratic term containing $H(w, w)$. Its inverse uses $-a$. Define γ_t by taking the product of these maps for (w_i, α), $i < d$, and $(w_d, t\lambda)$. All factors preserve the standard lattice and are the identity modulo Q^2T. At a marked prime the principal congruence subgroup lies in the row level. In the first split component at T the modified lattice lies between the standard lattice and its T-multiple. Thus an integral map congruent to the identity modulo T, as well as its inverse, preserves the modified lattice. The dual component is then preserved as well. Hence γ_t itself belongs to the original finite row level, giving finite support.

Since $H(u_i, w_i) = 2$ and $H(e, w_d) = -1$, the cross matrix for $Y, \gamma_t Y$ is diagonal with entries

$$4\alpha, \ldots, 4\alpha, \quad T(-1 + t\lambda).$$

All blocks therefore have size one. There is no common unipotent in Lemma 76.6, so the relevance condition on characters holds. The last minor distinguishes the labels, and

$$\mathcal{G}_t = (4\alpha\overline{4\alpha})^{d-1} T^2 t^2 \lambda\bar{\lambda}$$

is totally positive. Its norm is a fixed positive constant times t^{2f}. All proper endpoint minors are fixed. The absolute value of the final minor at each real place is bounded above and below by positive fixed multiples of t, since λ is nonzero and purely imaginary there. The same assertions, with fixed scaling factors, hold after (71.6). These facts give the asserted growth of the two arithmetic weights. They give the growth of P_∞ by Lemma 76.15, since the exponent of the final endpoint is zero.

For a lower bound on the finite mass use (76.22). Here D is trivial, and the integral subgroup in the two changes of frame has volume one. Its cross matrices are integral everywhere. At a marked place the diagonal cross matrix is $H_0 \bmod \pi^2$ by construction. Integral upper unitriangular changes on either side preserve $H_0 = -TE_{dd}$, so the new cross matrices satisfy both congruences of (76.20). Consequently $V^{=}_v \ge 1$ for all v. Formula (76.22) bounds the mass below by a positive fixed multiple of G_t^h, where $G_t = |N_{F/\mathbb{Q}}\mathcal{G}_t|$. For the reverse bound apply (76.23) with $W_{\gamma_t} = 1$ and $a = 0$. This completes the proof. □

Corollary 87.3 (A necessary exponent for separating the masses)**.** *Consider the Cauchy–Schwarz argument in which the real squared sum is $\mathcal{H}_\theta$ and the masses are put entirely in the other factor. Up to constants independent of the label, the sum in that other squared factor is*

$$\sum_{\gamma} \frac{Q^{-2fa}(\mathcal{M}_\gamma^{=})^2}{w(\beta_\gamma)(B_\gamma^{\#})^{\theta} P_\infty(\gamma)}. \tag{87.3}$$

It diverges if $\theta \le (h-2)/2 + 1/(4f)$. In particular on the selected intervals a moment with $\theta = Ad^2$ for a fixed A does not make this arithmetic factor finite for all large x.

Proof. Before using the bound on the mass, the single-label estimate in the first paragraph of the proof of Theorem 87.1 has label-dependent coefficient $\mathcal{M}_\gamma^{=} Q^{-fa} P_\infty(\gamma)^{-1/2}$ on $(P_\infty^{\#})^{1/2}\|\widehat{K}_{\gamma\#}\|_{\mathrm{HS}}$. Separating the prescribed real weight gives (87.3). On the sequence in Proposition 87.2 each summand is comparable to $t^{2f(h-2-2\theta)}$. The sum over positive integers diverges when this exponent is at least -1. Finally the ranks on the common intervals have $d \asymp x^{1/2-\epsilon/16}$ and $k \asymp x^{9/2+10\epsilon}$, so $h/d^2 \to \infty$. □

The smaller squared weight occurs in `Formalism/SharpRestriction/CuspSpanNormBounds/RotatedWeightedBound.lean`, line 26. Corollary 87.3 concerns the arithmetic factor of the specified absolute-value argument. Its proof uses a sum over all supported labels. It yields no lower bound on the analytic kernel of any one of those labels. A different quantity appears in the source's mean over row lattices. With the initial flag and a single row lattice fixed, it counts equivalence classes of two terminal flags together with a comparison map, subject to a determinant bound. Its rank-one count fixes one terminal flag as well and counts the admissible partners in that same lattice. The definitions are in `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 190 and `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 199. The row-weighted inequality is stated in `Formalism/SharpRestriction/CuspSpanNormBounds/CongruenceAtSplitFrobeniusHolds.lean`, line 556. Keeping the common lattice couples the two flags. Establishing the necessary estimates for that joint count and transporting them to the designated period would require a further argument.

87.2 Saturation of two flags in the same row lattice

Fix a supported, relevant rational label of nonzero Gram determinant. Work on the fields and intervals of Section 76.9. Denote its decorated frames by Y_0, Y_1, the spans of their first $d-1$ vectors by X_i, and the spans of all d vectors by W_i. Put $W = W_0 \oplus W_1$, $C = W^{\perp}$, $n = 2d$, $h = k - 1$ and $m = h - 2d + 1$. The form on W is nondegenerate, and C is totally positive definite. In a compatible full-level row lattice Λ put

$$U_i = \Lambda \cap W_i, \qquad U = U_0 \oplus U_1, \qquad B = \Lambda \cap W, \qquad C_\Lambda = \Lambda \cap C.$$

For an integral nondegenerate lattice M write $\delta(M) = [M^{\vee} : M]^{1/2}$, with dual in its own span. All indices in this section are cardinalities of quotients.

At an unmarked finite place put $B_v^{+} = U_v$. At $v \mid Q = g_1 g_2$ choose primitive integral representatives $Y_{0,v}, Y_{1,v}$ with equal reductions, which exist by Lemma 76.19. Let B_v^{+} be generated by the columns of $Y_{0,v}$ and $(Y_{1,v} - Y_{0,v})/\pi_v$. The generators belong to B_v. Changing the uniformizer multiplies the second tuple by a unit. If the two representatives are changed by integral upper unitriangular matrices V_i, equality of their new reductions forces $V_0 \equiv V_1 \bmod \pi_v$ by independence of the first reduction. The new difference tuple is the old one times V_1, plus $Y_{0,v}(V_1 - V_0)/\pi_v$. This proves independence of choices. The local lattices define B^{+} by taking their intersection in W. Localization recovers them since they differ from U_v at finitely many

places. Write $I_v = [B_v : B_v^+]$ and $I = [B : B^+] = \prod_v I_v$. The saturation condition for this row is $I = 1$.

Proposition 87.4 (The determinant and the congruence index)**.** *The following identities hold on every compatible row.*

$$[B^+ : U] = Q^{2df}, \qquad \delta(C_\Lambda) = \delta(B) = \frac{|N_{F/\mathbb{Q}}\mathcal{G}|}{Q^{2df} I}.$$

Let $\mathcal{L}$ be the module of $c \in C$ for which there are $x_i \in X_i$ with $c + x_i \in \Lambda$ and

$$x_0 - x_1 \in Q\Lambda + (\Lambda \cap X_0) + (\Lambda \cap X_1).$$

Then $C_\Lambda \subset \mathcal{L}$ and $[\mathcal{L} : C_\Lambda] \mid I$. In particular the two lattices agree on every row satisfying the saturation condition.

Proof. At each place a primitive integral representative of a flag is a basis of $U_{i,v}$, by Lemma 76.30. The determinant of the concatenated Gram matrix has valuation $v(\mathcal{G})$. A square Hermitian Gram matrix H gives local discriminant index $q_v^{2v(\det H)}$, by taking the determinant of the map H on the integer lattice of the quadratic algebra. Thus $\delta(U) = |N_{F/\mathbb{Q}}\mathcal{G}|$. At a marked place the change to the divided differences has determinant π_v^{-d}, hence $[B_v^+ : U_v] = q_v^{2d}$. Both primes split completely. An over-lattice of finite index a divides δ by a, by the Gram determinant change of basis locally. This gives the formula for B.

The orthogonal projections of Λ onto W, C have images $B^\vee, C_\Lambda^\vee$. Locally, an integral functional on B_v extends to one on Λ_v because B_v is primitive. Self-duality represents it by a vector in Λ_v. Orthogonal projection gives the claimed image, and the proof for $C_{\Lambda,v}$ is the same. The quotient $\Lambda/(B \oplus C_\Lambda)$ therefore projects bijectively onto both discriminant modules. For example a class whose W-projection belongs to B has, after subtracting that element, a representative in C_Λ. Hence their orders agree.

For $c \in \mathcal{L}$ choose the indicated x_i. The difference $x_0 - x_1$ belongs to B, and its class modulo U is independent of the choices since two choices of x_i differ by an element of $\Lambda \cap X_i$. The resulting homomorphism kills C_Λ. Conversely if $x_0 - x_1 \in U$, the direct decomposition in $W_0 \oplus W_1$ implies $x_i \in U_i$, so $c \in C_\Lambda$. The congruence places the image in $Q(B/U)$, because $Q\Lambda \cap W = QB$. Now $QB^+ \subset U$ at every place. Multiplication by Q gives a surjection $B/B^+ \to Q(B/U)$. The asserted divisibility follows. This gives the index in the compatibility module defined in `Formalism/SharpRestriction/CuspSpanNormBounds/IsPrimitiveSub.lean`, line 178 for the actual integral flags and congruence used here. □

Theorem 87.5 (Saturation in the orbital measure)**.** *Normalize the compatible finite orbital measure of the fixed label to probability, using the masses $L_v^{=}$ of Proposition 76.31. For every $0 \le t \le m - 2$ one has, with an absolute constant C_1,*

$$\Pr(I > 1) \le C_1 f\, 2^{-m}, \qquad \mathbb{E}[I^t] \le \exp(C_1 f\, 2^{-(m-t)}).$$

The second bound also holds with I replaced by $[\mathcal{L} : C_\Lambda]$.

Proof. In the disintegration of Proposition 76.31 fix the two changes of basis. At an unmarked place the ordered tuple is the concatenation of the two frames. At a marked place it uses the divided differences. Its generated lattice in the ambient row is precisely B_v^+. Pulling back the saturation under the column map identifies I_v with $[M' : M_0]$ in Theorem 76.29. For each integral Gram target the compatible ordered-frame test contains every primitive tuple and is contained in the set of all integral tuples. A primitive tuple contributes exactly $p(h, u)$ after division by the determinant factor in that theorem.

The normalized contribution of tuples with $I_v > 1$, weighted by I_v^t, is bounded above by

$$e^{C_2 q_v^{-m}} \sum_{M' \supsetneq M_0} [M' : M_0]^{-(h-t)} \;\le\; C_3 q_v^{-(m-t)}.$$

The first inequality uses the over-lattice formula and $|\log p(h,u)| \le C_2 q_v^{-m}$. The last sum is at most the product of geometric series in the proof of Theorem 76.29, with the term one removed. Its logarithm after adding one is $O(q_v^{-(m-t)})$, uniformly for $m - t \ge 2$, in both quadratic algebras. Dividing by the primitive contribution costs at most $e^{C_2 q_v^{-m}}$. The bound holds in every nonempty Gram fibre. Hence it holds relative to the full compatible measure after integration over the two changes of basis. The marked multiplicity $q_v^{d(d-1)}$ is common to the weighted and unweighted integrals.

Each I_v is intrinsic to the full-level cell, so inserting its weight in the orbital mass formula multiplies the local factors under the same disintegration as in Theorem 76.8. The definite complement contributes its usual constant mass. For a fixed label, $\delta(B) \ge 1$ and its local Gram ideal is integral. The local version of Proposition 87.4 therefore forces $I_v = 1$ away from the divisors of $\mathcal{G}$. The product is finite. The union bound at $t = 0$ and the product of the local bounds $1 + C_3 q_v^{-(m-t)}$ give the assertions since $\sum_v q_v^{-b} \le f \sum_{j \ge 2} j^{-b} \ll f 2^{-b}$ for $b \ge 2$. The index assertion follows from divisibility. □

Corollary 87.6 (Contribution of the rows failing saturation)**.** *For unit column tuples at the original finite level, the sum of $|J_{\gamma,c}|$ over cells with $I > 1$ is at most*

$$C_1 f 2^{-m} \, \frac{A_* \mathcal{M}_\gamma^{=}}{\sqrt{c^*_{\rho_1} c^*_{\rho_2}}} \, c_E^a \|K^{\mathrm{area}}_{\infty,\gamma}\|_{\mathrm{op}}.$$

For $I \ge b_0 > 1$ one may replace the factor $C_1 f 2^{-m}$ by $b_0^{-t} \exp(C_1 f 2^{-(m-t)})$.

Proof. Every refinement of a compatible coarse cell has the same row lattice and the same two saturated flag modules. The definition of B_v^+ above is independent of the matching representatives. Thus I is constant across all those refinements. Apply Proposition 76.20 only to the specified coarse cells and use the unit column norms as in Theorem 76.21. The measure left in this inequality is $\mathcal{M}_\gamma^{=}$ times the probability of the specified event. Theorem 87.5 proves the first bound and Markov's inequality proves the second. □

For generic rows of the original period the compatibility module is consequently the integral perpendicular C_Λ. Controlling pairs of terminal flags with prescribed comparison map also requires a count of the common definite subspaces they can determine in the initial column.

87.3 Mass with a prescribed discriminant module

Let E/F be one of the CM extensions in Lemma 71.7, with $f = [F : \mathbb{Q}]$. It is unramified at every finite place. For a positive definite integral Hermitian lattice M, let $\mathcal{D}(M) = M^\vee/M$. Its pairing has values in $E/\mathcal{O}_E$, given by the Hermitian pairing of representatives. It is perfect by double duality. Put $\delta(M) = |\mathcal{D}(M)|^{1/2}$, and let $\mathfrak{m}_a$ be the mass of the standard self-dual genus in rank a. Mass is the sum of the reciprocals of the isometry group orders over isometry classes in a genus. All isometries are $\mathcal{O}_E$-linear.

At a finite place v, let $A = E \otimes_F F_v$ with integer ring O, let q be the residue order of F_v, and let π be a uniformizer of F_v. Put $\eta = 1$ if A splits and $\eta = -1$ otherwise. Write

$$P_b = \prod_{i=1}^{b} (1 - \eta^i q^{-i}), \qquad P_0 = 1.$$

In a basis over O an integral nondegenerate Hermitian lattice L of rank a has Gram matrix $H = \operatorname{diag}(\pi^{b_1}, \dots, \pi^{b_a})$, with $b_i \geq 0$. In the split algebra this is Smith reduction of the pairing between the two factors. In the unramified field divide the form by its smallest entry scale. Some residue pairing is nonzero. If every norm vanished, taking the norm of $x + cy$ for all residue scalars c and using the perfect trace pairing would make every pairing zero. A vector of unit norm splits off a line by orthogonal projection. Induction and surjectivity of the norm on units give the asserted diagonalization. The latter surjectivity follows from the residue norm and correction of successive digits by the residue trace. This argument includes primes over two. The sorted exponents are unique by the elementary divisor theorem. Write $e = \sum_i b_i$ and $u = \#\{i : b_i > 0\}$. Then $\delta_v(L) = q^e$, and the positive exponents determine the finite paired module.

Lemma 87.7 (The stabilizer of a discriminant marking)**.** *Let K_L be the stabilizer of L in its rational unitary group, and $K_L^{\#}$ the kernel of its action on $\mathcal{D}(L)$. The map $K_L \to \operatorname{Aut}\mathcal{D}(L)$ is onto. There is a Haar measure on each local unitary group, invariant under transport by rational isometry, for which*

$$\operatorname{vol}(K_L^{\#}) = q^{-ae}\frac{P_{a-u}}{P_a}. \tag{87.4}$$

Proof. In the diagonal basis $L^{\vee} = H^{-1}O^a$. Write $g = 1 + XH$. The condition $(g-1)L^{\vee} \subset L$ is $X \in M_a(O)$, and the isometry equation becomes

$$X + X^* + X^*HX = 0.$$

Every such solution gives an integral isometry with integral inverse. Its determinant has norm one and is integral in both factors when A splits. Set $t = a - u$. Modulo π, order the basis so that $H = \operatorname{diag}(I_t, 0)$ and write the blocks of X as A_0, B, C, D. The matrix $1 + A_0$ is unitary, B is arbitrary, $C^* = -(1 + A_0)^*B$, and $D + D^* = -B^*B$. The last equation has q^{u^2} solutions by the surjectivity of the trace on diagonal coordinates. The unitary group of the residue algebra in rank t has order $q^{t^2}P_t$. This follows by choosing successive orthonormal vectors as in Proposition 76.28, or successive independent columns for the split general linear group. Thus there are $q^{a^2}P_t$ residue solutions. At every solution the linearized equation is onto the Hermitian matrices, because it is $Y^*(1 + HX) + (1 + HX)^*Y$, where $1 + HX$ is invertible modulo π by the block description. Every solution consequently has q^{a^2} lifts at each further digit, and completeness gives an integral solution in every residue class so obtained.

To fix the Haar normalization, at the identity use the coordinates X on the tangent space of skew-Hermitian matrices. Give its integral lattice additive volume one and translate that density by left multiplication. In this measure the solutions with $X \equiv 0 \bmod \pi^j$ have volume q^{-ja^2} for sufficiently large j. Indeed $Y \mapsto Y + Y^*$ has an integral linear section (use the trace on the diagonal). In the complementary coordinates the implicit function theorem expresses the equation near zero as a graph over the integral skew-Hermitian coordinates, with identity differential on the latter. On sufficiently small balls the absolute Jacobian of the invariant density is one. Left translation by $1 + X_0H$ acts by $X \mapsto X_0 + (1 + X_0H)X$, preserving these congruences. The digit counts therefore give volume P_t to $K_L^{\#}$ in this density.

Under a basis change S, the new Gram matrix is S^*HS and the tangent coordinate becomes $S^{-1}X(S^*)^{-1}$. The absolute Jacobian on skew-Hermitian matrices is $|N_{A/F_v}\det S|_v^{-a}$. Integral changes preserve the integral tangent lattice. For a diagonal change with entries s_i, the i-th diagonal coordinate contributes $|Ns_i|_v^{-1}$ and each independent off-diagonal pair contributes $|Ns_i\,Ns_j|_v^{-1}$. Elementary divisors give the formula for general S, also in the split algebra. Multiplication of the density by $|\det H|_v^a/P_a$ makes it independent of the basis. This proves (87.4).

For surjectivity discard the unimodular orthogonal summand, whose discriminant is zero. In a basis of the dual of the remaining summand the Gram matrix is D_+^{-1} and the lattice is D_+O^u, where D_+ is the diagonal matrix of the positive powers of π. Lift the images of the basis under a given discriminant isometry to columns S in O^u. The reduction of S is invertible because $D_+O^u \subset \pi O^u$. The lift preserves D_+O^u because the relations downstairs give inclusion and its unit determinant gives equality. Pairing preservation gives $S^*D_+^{-1}S = D_+^{-1} + W$ with W integral and Hermitian. If $W \in \pi^j M_u(O)$, multiply S on the right by $1+\pi^j D_+ C$. This fixes its action on the quotient, and modulo π^{j+1} changes the error by $\pi^j(C+C^*)$, since $D_+ \equiv 0 \bmod \pi$. Solve by the residue trace and continue starting at $j = 0$. The convergent corrections give an exact isometry, which extends by the identity on the discarded summand. □

Theorem 87.8 (Genus mass with its discriminant marked)**.** *For every totally positive definite integral Hermitian lattice M of rank $a \geq 1$, with u_v positive elementary divisors at v,*

$$\operatorname{mass}(M)\,|\operatorname{Aut}\mathcal{D}(M)| = \mathfrak{m}_a\,\delta(M)^a \prod_{v:u_v>0} \frac{P_a(q_v,\eta_v)}{P_{a-u_v}(q_v,\eta_v)}. \tag{87.5}$$

Proof. First choose a maximal integral lattice locally in the rational space of M. At a split place or at a nonsplit place of even determinant valuation the space admits a unimodular lattice. At a nonsplit place of odd determinant valuation it admits the matrix $\operatorname{diag}(I_{a-1},\pi)$. These assertions use the classification by dimension and determinant modulo norms over a nonarchimedean local field. In the unramified case units are norms. A lattice with the latter matrix is maximal because an integral over-lattice would give an isotropic submodule in its discriminant, an anisotropic line over the quadratic residue field. For our unramified extension the condition that all norms are integral is equivalent to integrality of the pairing. Indeed, polarize after multiplying one argument by each integer of E_v and use the unimodular trace pairing. The local maximal lattices can be chosen equal to M_v almost everywhere and patched by intersection in the rational space. For such patching, clear all denominators in one fixed global lattice. The prescribed changes are then submodules of a finite quotient, which is the direct sum of its primary parts. Taking their inverse image gives exactly the prescribed completions.

In the measure of Lemma 87.7 a unimodular stabilizer has volume one. For the odd-valuation maximal lattice its finite discriminant has $q+1$ isometries, the norm-one residue scalars. Its full stabilizer therefore has volume $(q+1)/(q^a-(-1)^a)$. The maximal definite Hermitian mass formula of Gan, Hanke and Yu gives $\mathfrak{m}_a$ times the product of the reciprocals of precisely these volumes. Its hypotheses hold since F is totally real, E/F is CM, and the lattice is maximal for the integral norm condition. The common archimedean and L-value factors are those of Lemma 71.7. See [1], Propositions 4.4 and 4.5.

For two compact open lattice levels in the same rational unitary group pass to their intersection. A class with finite stabilizer J acquires a finite set of level structures of cardinality the index. Their J-orbits contribute total weight equal to that index divided by $|J|$, by the orbit formula. The masses at the two levels are thus inversely proportional to their compact volumes. Apply this first to the maximal level, then to the kernel of the discriminant action for M. The finite paired module is the direct sum of its local parts with orthogonality across distinct places of F, so its isometry group is the product of the local isometry groups. Surjectivity in Lemma 87.7 gives the index of the new level in the full stabilizer. Substitution of (87.4) now gives (87.5). □

Write $N_j(L,D)$ for the number of primitive rank-j sublattices $M = L \cap EM$ with $\delta(M) = D$. Here EM denotes the rational span of M. Write $\mathbb{E}_{\mathrm{sd}}$ for expectation in the self-dual positive

definite genus in rank k, normalized by $\mathfrak{m}_k$. These are the primitive sublattices counted in `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 79.

Theorem 87.9 (The self-dual genus mean)**.** *Let $j+s=k$ with $j,s\ge 1$, and $D\ge 1$ an integer. At v choose a partition λ_v of $e_v\ge 0$ into $u_v\le \min(j,s)$ positive parts, allowing the empty partition. Sum over choices with $\prod_v q_v^{e_v}=D$ and $\sum_{\eta_v=-1} e_v$ even. Write $\mathcal{D}_v$ for the paired module of that partition and put $R_v(a)=P_a/P_{a-u_v}$. Then*

$$\mathbb{E}_{\mathrm{sd}} N_j(L,D)=\frac{\mathfrak{m}_j\mathfrak{m}_s}{\mathfrak{m}_k}D^k\sum_{(\lambda_v)}\prod_{v:e_v>0}\frac{R_v(j)R_v(s)}{|\operatorname{Aut}\mathcal{D}_v|}. \tag{87.6}$$

In particular the mean is at most $(\mathfrak{m}_j\mathfrak{m}_s/\mathfrak{m}_k)D^{k+\nu}$, where $\nu=4+\log_2(2f)$.

Proof. For a primitive $M\subset L$, let $N=L\cap(EM)^{\perp}$. The projection argument of Proposition 87.4 identifies $L/(M\oplus N)$ with both discriminant modules. In their direct sum it is the graph of an anti-isometry, since integrality of L says that the two linking pairings on the graph sum to zero. Conversely the inverse image of the graph of any such anti-isometry in $M^{\vee}\oplus N^{\vee}$ is integral and self-dual, since the graph is its own annihilator by perfectness. Its intersections with the two rational summands are M and N. These constructions are inverse up to isometry with M distinguished.

The two local partitions must agree by the elementary divisor description. Conversely this condition supplies local anti-isometries, since the negative of each form is isometric to it by a unit scalar of norm -1. Gluing the local anti-isometries on the finite module supplies a global one. For a proposed list of partitions, the rank-j and rank-s local rational spaces each have determinant norm symbol $(-1)^{e_v}$ at nonsplit places and $+1$ at all other finite places. Take positive forms at infinity, with symbol $+1$. Landherr's local-to-global theorem gives existence precisely under the displayed parity condition and uniqueness of each rational space then. The local lattice classes give a single genus in each space by patching as above. Every glued self-dual k-lattice lies in the standard genus by the same classification and the uniqueness of the local unimodular lattice. For the local and global classification see Gross, *Incoherent definite spaces and Shimura varieties*, Section 3 and Theorem 3.1.

For fixed isometry classes M,N the number of graph identifications is $|\operatorname{Aut}\mathcal{D}(M)|$. The stabilizer of a graph under $\operatorname{Aut}M\times\operatorname{Aut}N$ is exactly the stabilizer of the distinguished sublattice in the glued lattice, by restriction to the two rational summands. The sum of the reciprocal orders of the graph stabilizers over orbits is therefore $|\operatorname{Aut}\mathcal{D}(M)|/(|\operatorname{Aut}M||\operatorname{Aut}N|)$. This also holds for $j=s$ since M is distinguished. Alternatively summing over the orbits of distinguished sublattices in each L gives $N_j(L,D)/|\operatorname{Aut}L|$. Sum over the two genera, use (87.5) twice and divide by $\mathfrak{m}_k$. This gives (87.6), including finiteness.

Each $R_v(a)\le\exp(\sum_{i\ge1}q_v^{-i})\le\exp(1)$. There are at most $\log_2 D$ nonempty places. Above a rational prime p with $p^b\|D$, encode the ordered local partitions by concatenating their parts in a fixed order of places and multiplying each part by its residue degree. This is a composition of b with each summand labelled by one of at most f places. It recovers the partitions and gives at most $(2f)^b$ possibilities. Multiplication over p, followed by $\sum_p v_p(D)\le\log_2 D$ and $2/\log 2<4$, gives the asserted upper bound. The empty collection gives it at $D=1$. □

87.4 Primitive subspaces in the marked terminal column

Use the same field, prime T and original terminal flag as in Theorem 64.1. Thus T splits completely in E. If X is the isotropic part of this flag and z its terminal vector, the positive column is the rank-k Hermitian space $P=(X+Ez)^{\perp}/X$. In a compatible row lattice Λ its

integral lattice L is the image in P of $\Lambda \cap (X + Ez)^{\perp}$. Indeed an integral lift from $X^{\perp}$ of a class perpendicular to z is already perpendicular to z, since $X \perp z$. We average over the genus of these column lattices with reciprocal automorphism weights. This is the measure on initial columns before a second rational flag is specified.

Lemma 87.10 (Recovering the self-dual parent)**.** *Fix, for each $v \mid T$, the split factor on which the chosen ϖ has valuation one. A column in the terminal genus is equivalent to a self-dual lattice L_* of rank k together with one hyperplane in $L_{*,v}^{+}/\pi L_{*,v}^{+}$ at each such v. Its genus mass equals*

$$\mathfrak{m}_k \left(\frac{T^k - 1}{T - 1} \right)^f .$$

Its normalized average can be obtained by sampling L_* with its normalized genus mass and then sampling the hyperplanes independently and uniformly.

Proof. Away from T, split off integral hyperbolic partners to the primitive isotropic columns as in Lemma 76.30. In their unimodular middle quotient the terminal vector has unit norm, so its perpendicular is self-dual. At $v \mid T$ write $O = \mathbb{Z}_T$ in the chosen factor. The middle lattice in the original integral frame has first factor

$$\{(w, z_0) \in O^k \oplus O : z_0 \equiv \ell_v(w) \pmod{T}\},$$

and second factor its dual. The terminal perpendicular has $z_0 = 0$ in each factor. Its first lattice is the inverse image of $\ker \bar{\ell}_v$ in O^k. Its second is $(O^k)^{\vee}$, since projection of the middle lattice on w is all of O^k. Transporting by isometry gives this description for every member of the genus. The self-dual parent is intrinsic. At each $v \mid T$ replace the first lattice by the dual of the second, leaving all other local lattices as given. Patching produces L_*. Conversely every hyperplane gives the prescribed local genus because $\mathrm{GL}_k(O)$ acts transitively on hyperplanes modulo T. These constructions commute with isometries since the split factors have been fixed.

For fixed L_*, its finite isometry group acts on the product of hyperplane sets. Its orbits give the isometry classes over L_*, and the stabilizers are their isometry groups. The mass of each orbit equals its size divided by $|\operatorname{Aut} L_*|$. There are $(T^k-1)/(T-1)$ hyperplanes at each of the f places. Summing proves both assertions. Retaining a nonzero covector above each hyperplane multiplies the mass by $(T-1)^f$. Fixed compact level refinements at the other primes multiply it by their indices, by the same orbit argument. Normalized averages of functions of L are consequently the same at these finer column levels. □

Theorem 87.11 (The primitive count at the actual terminal prime)**.** *Let $\mathbb{E}_T$ be the normalized terminal genus mean. For $j + s = k,\ j, s \geq 1$, put $p = (T^s - 1)/(T^k - 1)$. Then*

$$\mathbb{E}_T N_j(L, D) = \sum_{e=0}^{f} \binom{f}{e} p^{f-e} (1-p)^e \, \mathbb{E}_{\mathrm{sd}} N_j(L_*, D/T^e), \tag{87.7}$$

$$\mathbb{E}_T N_j(L, D) \leq \frac{\mathfrak{m}_j \mathfrak{m}_s}{\mathfrak{m}_k} (D/T^f)^k (T^f)^s D^{\nu}. \tag{87.8}$$

Here $D \geq 1$ is integral and the count is read as zero at any argument which is not a positive integer.

Proof. A rational j-plane determines exactly one primitive sublattice in L_* and one in L, by intersection. If $A \subset L_*$ is the first, the second is $A \cap L$. At $v \mid T$ its index is 1 when the hyperplane contains the reduction of A_v^{+}, and T otherwise. Indeed A_v^{+} is a direct summand of the parent in this factor and the second factor is unchanged. A uniform nonzero residue covector

vanishes on the resulting j-space with probability p, by counting covectors on the quotient of dimension s. The determinant in the convention of Section 87.3 changes by the index, so $\delta(A \cap L) = T^{e}\delta(A)$ when precisely e hyperplanes fail to contain the reduction. Independence over the places gives the binomial factor for every A. Summation over A gives (87.7).

At a nonzero summand use Theorem 87.9, bound $(D/T^{e})^{\nu} \le D^{\nu}$ and sum the factor T^{-ke}. The binomial average is $(p + (1-p)T^{-k})^{f} = T^{-jf}$. This proves (87.8). □

Proposition 87.12 (Size of the genus factor in the middle ranks)**.** *Put* $R = |\operatorname{disc} E|^{1/(2f)} = |\operatorname{disc} F|^{1/f}$*. There is an absolute* C *such that for* $2 \le s \le k-2$*,*

$$\frac{\mathfrak{m}_{k-s}\mathfrak{m}_{s}}{\mathfrak{m}_{k}} \le \mathfrak{m}_{2}\left(\frac{C}{kR}\right)^{s(k-s)f}. \tag{87.9}$$

On the common intervals $\mathfrak{m}_{2} \le b^{Cf}$ *for all sufficiently large* x*, where* $b = \exp(x^{\varepsilon})$*.*

Proof. Write $D_F = |\operatorname{disc} F|$. The discriminant tower formula gives $D_E = D_F^{2}$ by finite unramifiedness. For $n \ge 2$, put $a_n = \mathfrak{m}_n/\mathfrak{m}_{n-1}$ and let η denote the quadratic character of E/F. The mass formula in Lemma 71.7 gives

$$a_n = D_F^{\,n-1/2}\left(\frac{(n-1)!}{(2\pi)^{n}}\right)^{f} L_F(n, \eta^{n}).$$

Indeed $a_n = 2^{-f}|L_F(1-n, \eta^{n})|$. The latter L-value is taken on the f even Dirichlet factors of F for even n, and on the f odd factors of ζ_E/ζ_F for odd n. For a primitive character χ of the parity of n, its functional equation and the gamma reflection and duplication formulas give

$$|L(1-n, \chi)| = \frac{2(n-1)!}{(2\pi)^{n}}\operatorname{cond}(\chi)^{n-1/2}|L(n, \bar{\chi})|.$$

By the abelian conductor–discriminant formula the conductors in the even product multiply to D_F, and those in the odd product to $D_E/D_F = D_F$. The factors are closed under conjugation and their Euler product at n is positive. It lies between $\zeta(n)^{-f}$ and $\zeta(n)^{f}$, since each local factor is between $1 - q_v^{-n}$ and $(1-q_v^{-n})^{-1}$, and $\zeta_F(n) \le \zeta(n)^{f}$.

Set $\ell = \min(s, k-s)$ and $h = k - \ell \ge k/2$. In the ratio on the left of (87.9), after retaining $\mathfrak{m}_2$, the factors left are

$$\frac{1}{a_{h+1}a_{h+2}}\prod_{i=3}^{\ell}\frac{a_i}{a_{h+i}}.$$

The factorial quotient $(h+i-1)!/(i-1)! \ge h!$, and $h! \ge (h/\exp(1))^{h}$. Thus each displayed quotient a_i/a_{h+i} is at most $(C_0/(hR))^{hf}$ for an absolute C_0, by the Euler product bounds. Each of $1/a_{h+1}, 1/a_{h+2}$ satisfies the same bound by the formula for a_n, using $R \ge 1$. There are ℓ such factors and $h\ell = s(k-s)$. This proves the assertion.

Finally $\mathfrak{m}_1 = 2^{1-f}|L_F(0, \eta)|$. For an odd primitive Dirichlet character of conductor c, the Bernoulli formula at zero gives $|L(0, \chi)| \le c$. Thus $\mathfrak{m}_1 \le 2^{1-f}D_F$, and the formula for a_2 gives $\mathfrak{m}_2 \le C_1^{f}D_F^{5/2}$. All conductors divide N with $\log(2N) = o(\log b)$ by Section 71.3, and $D_F \le N^{f}$. This proves the asserted interval bound. □

In particular (87.8) has the precise genus ratio Ξ_s used in the middle ranks of the source. The factor $(T^{f})^{s}D^{\nu}$ is bounded by $(DT^{f})^{C(s+\log f)}$. These comparisons use the same T, E, and interval as in the period. The genus ratio and the primitive mean under comparison are specified in `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 127 and `Formalism/SharpRestriction/CuspSpanNormBounds/CongruenceAtSplitFrobeniusHolds.lean`, line 533.

Proposition 87.13 (Two reductions at one terminal place)**.** *Fix two primitive sublattices in the same self-dual parent, and at a place above T let their first-factor reductions have dimensions $a, b \geq 1$ and span of dimension c. Write ϵ_A, ϵ_B for the two indicators of failing the hyperplane condition. For $z = T^{-k}$,*

$$\mathbb{E}\, z^{\epsilon_A+\epsilon_B} = (1-z)T^{-c} + z(T^{-a} + T^{-b} - 1) \leq T^{-c}.$$

For fixed rational subspaces in a fixed parent the joint moment over all $v \mid T$ is the product of these expressions.

Proof. The probability that a uniform nonzero covector kills a d_0-space is $p_{d_0} = (T^{k-d_0} - 1)/(T^k - 1)$. It kills both reductions exactly when it kills their span. Since $z^{\epsilon_A} = z + (1-z)1_{\{\epsilon_A=0\}}$, the expectation is $z^2 + z(1-z)(p_a + p_b) + (1-z)^2 p_c$. Substitute $(1-z)p_{d_0} = T^{-d_0} - z$. The bound follows from $T^{-a} + T^{-b} \leq 1$, and the product assertion from Lemma 87.10. □

Corollary 87.14 (Distinct subspaces occurring in flag comparisons)**.** *Fix an initial flag $\mathcal{F}$ in the row and the determinant cutoff $D_{\max} > 0$ for comparing two terminal flags. For $1 \leq j = k - s \leq k - 1$ and $b_0 \geq 1$, consider their distinct rational comparison subspaces of dimension j in P for which the product of the two compatibility indices is at least b_0. Their number, averaged in the initial column genus (or its indicated refinements), is at most*

$$\frac{\mathfrak{m}_j \mathfrak{m}_s}{\mathfrak{m}_k} (D_1/T^f)^k (T^f)^s D_1^{\nu+1}, \qquad D_1 = D_{\max}/b_0,$$

and is zero if $D_1 < 1$.

Proof. Write $W_{\mathcal{F}} = X + Ez$ for the original frame span, and $W_i = X_i + Ez_i$ for the two terminal frame spans. In the nondegenerate pair range put $C_i = (W_{\mathcal{F}} + W_i)^{\perp}$. This space embeds isometrically in P by reduction modulo X, since its intersection with the nondegenerate frame sum is zero. It embeds likewise in $P_i = W_i^{\perp}/X_i$. Denote the quotient maps by p, p_i on their respective domains. A comparison isometry β carrying the first terminal flag to the second induces $\bar{\beta} : P_1 \to P_2$. Its comparison subspace $D_\beta \subset P$ is the set of $p(c_1) = p(c_2)$ for which $c_i \in C_i$ and $\bar{\beta} p_1(c_1) = p_2(c_2)$. This agrees with the two linear congruences modulo X and X_2 in the definition of the rational comparison. Both c_i are uniquely determined by their common image.

Let $\mathcal{L}_i$ be the compatibility lattice of Proposition 87.4 for the i-th pair and $i_i = [\mathcal{L}_i : \Lambda \cap C_i]$. The comparison lattice $M_\beta = \{c_1 \in \mathcal{L}_1 : p(c_1) \in D_\beta\}$ has full rank in the corresponding subspace of C_1, since $\mathcal{L}_1$ contains $\Lambda \cap C_1$. Its isometric image belongs to $L \cap D_\beta$, because each $c_1 \in \mathcal{L}_1$ has an integral lift modulo X perpendicular to the initial frame. The cutoff used for comparing the flags is $\delta(M_\beta) i_1 i_2 \leq D_{\max}$. Thus the primitive saturation $L \cap D_\beta$ has determinant at most D_1, by the index change of determinant. This is precisely the lattice and cutoff in `Formalism/SharpRestriction/CuspSpanNormBounds/IsPrimitiveSub.lean`, line 213 and `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 190.

Distinct rational subspaces have distinct primitive saturations in L. In each column the count is therefore at most $\sum_{1 \leq D \leq D_1} N_j(L, D)$, regardless of the choice of compatible row above that column. Average and apply (87.8) at the at most D_1 positive integers in this range. This also proves the zero assertion. □

Theorem 87.15 (Uniqueness of the comparison for two fixed flags)**.** *Fix three flags in one original row Λ, the initial one and two admissible terminal flags of the same terminal norm. At $Q = g_1 g_2$ suppose each terminal flag has the same reduced isotropic submodule as the initial flag. This holds in every compatible row of Section 87.2. Among integral row isometries carrying*

the first terminal flag and its terminal vector to the second, the comparison congruence is equivalent to inducing the identity on the common residual middle quotient at each place above Q. Among the isometries satisfying it there is at most one induced isometry $X_1^{\perp}/X_1 \to X_2^{\perp}/X_2$. Consequently a fixed ordered pair of flags contributes at most one comparison class in the equivalence relation used for the pair count.

Proof. Write $X_0 = X$ and, for $a = 0, 1, 2$, put $A_a = \Lambda \cap X_a$, $B_a = \Lambda \cap X_a^{\perp}$ and $J_a = B_a/A_a$. The latter is an integral lattice in the middle quotient. At each $v \mid Q$ the three reductions of A_a in the residue row are the same submodule U. Indeed admissibility gives equality of the reduced frames up to an invertible upper triangular change. The reductions of the B_a are all $U^{\perp}$. To see this, restriction of the row pairing from Λ_v to the O-dual of $A_{a,v}$ is onto by self-duality and primitivity. Its kernel is $B_{a,v}$. This exact sequence splits as a sequence of modules, and hence remains exact on reduction. The same reasoning in the two factors at a split place uses the perfect pairing between the factors. In particular the reductions of the J_a identify canonically with $U^{\perp}/U$ in the residue row.

For $x_1 \in B_1$, choose $l \in B_0$ with $l \equiv x_1 \bmod Q\Lambda$. Such an l exists since the reductions of B_0 and B_1 agree at the places above Q, the ideal $Q\mathcal{O}_E$ is squarefree, and the Chinese remainder theorem for the module B_0 realizes all its prescribed residues simultaneously. For an admissible comparison β the congruence says that for every $l \in B_0, x_1 \in B_1$,

$$l - x_1 \in Q\Lambda + A_0 + A_1 \quad \Longleftrightarrow \quad l - \beta x_1 \in Q\Lambda + A_0 + A_2.$$

Our choice makes the left side true. In each residue row the right side implies $\beta x_1 \equiv x_1 \bmod U$. It follows that every comparison induces the same map on $J_{1,v}/\pi J_{1,v} \to J_{2,v}/\pi J_{2,v}$ at each marked place, namely the identity under the common identification. Here π denotes the local uniformizer. The global elements of B_1 surject on its residue module. Conversely the two membership conditions can be checked in the residue rows at $v \mid Q$, since each subgroup contains $Q\Lambda$. There each tests equality of the two images in $U^{\perp}/U$. Identity on the residue quotients therefore implies the equivalence for every l, x_1. This is the congruence of `Formalism/SharpRestriction/CuspSpanNormBounds/IsPrimitiveSub.lean`, line 206.

If two comparisons exist, their quotient acts on J_1 as an integral isometry fixing the class of z_1. Its rational action decomposes as the identity on the nonisotropic line Ez_1 and a unitary action on its totally positive definite perpendicular P_1. It has finite order on J_1. Indeed it preserves the positive lattice $J_1 \cap P_1$, whose isometry group is finite. In all archimedean embeddings the group is compact and the images of any fixed integral basis of the underlying $\mathbb{Z}$-lattice form a discrete set. Its action on the two rational summands determines the action on J_1.

At one marked place take a single factor over the odd completely split prime $p = g_1$. In a basis of the completion of J_1 the quotient has matrix $1 + H$ over $\mathbb{Z}_p$ with $H \equiv 0 \bmod p$. This congruence kernel is torsion-free. If the least valuation of an entry of $H \neq 0$ is $t \geq 1$, the binomial expansion shows that raising to a power prime to p preserves t. Raising to the p-th power changes it to $t + 1$, since pH is the unique term of this order and every entry of the remaining terms is divisible by p^{t+2} for odd p. Thus a finite-order element in the kernel is the identity. The embedding of E into this completion is injective, so the two rational maps coincide on the entire quotient. For fixed terminal flags equality there gives the prescribed equivalence by taking the two unipotent changes of frame to be the identity in `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 186. □

The comparison subspace is thus determined by the two flags whenever a comparison exists. To apply Corollary 87.14 we need its determinant cutoff in terms of their Gram data and the

multiplicity with which different flag pairs produce it. These are computed in Theorems 87.19 and 87.23.

87.5 The determinant cutoff for two terminal flags

Fix the row, the initial flag 0, and two terminal flags $1, 2$ of Section 87.4. Both terminal norms are $-T'$, and the initial norm is $-T$. In the common-prime application $T' = T$. Assume each of the two frame sums is nondegenerate, the reduced frames agree up to the allowed changes at Q, and an admissible comparison β exists. Let $W_a = X_a + Ez_a$ be the frame span of flag a, $P_a = W_a^{\perp}/X_a$, and L_a its integral column. Write $S_i = W_0 \oplus W_i$ and $C_i = S_i^{\perp}$ for $i = 1, 2$. Thus C_i is positive definite of rank $k - 1$.

Put $P = P_0$, $P' = P_2$. For $c \in C_i$ let $p_i(c)$ be its image in P, and $q_i(c)$ its image in P', using the induced comparison $\bar{\beta} : P_1 \to P_2$ for $i = 1$. Both embeddings are isometric. Denote their joint image in $P \oplus P'$ by Γ_i. It is the graph of an isometry between two hyperplanes in definite spaces. Their intersection is the graph of an isometry $D \to D'$, where $D \subset P$ is the comparison subspace of Corollary 87.14. Write $j = \dim_E D = k - s$. We prove the cutoff for every $2 \le s \le k$, allowing $D = 0$. For each of the two initial–terminal pairs put

$$G_{0,i} = \frac{|N_{F/\mathbb{Q}}\mathcal{G}_i|}{Q^{2df}}, \qquad I_i = [B_i : B_i^{+}], \qquad i_i = [\mathcal{L}_i : \Lambda \cap C_i]$$

as in Proposition 87.4, with $B_i = \Lambda \cap S_i$ and $U_i = (\Lambda \cap W_0) \oplus (\Lambda \cap W_i)$. The notation U_i here denotes the sum for the pair. We have $i_i \mid I_i$ and $\delta(\mathcal{L}_i) i_i^2 = G_{0,i} i_i / I_i$. Here $\mathcal{G}_i$ is the determinant of the Gram matrix on the concatenated frames up to sign.

Lemma 87.16 (The common lattice and the difference of the pairings)**.** *Under either graph map, the compatibility lattice $\mathcal{L}_i$ is the intersection of Γ_i with the same lattice in $P \oplus P'$. In particular the two lattices intersect the common graph in the same lattice M, measured with the metric of its first projection. Write H_i for the orthogonal projection of $p_i(\mathcal{L}_i)$ on $D^{\perp}$ in P. Then* $\operatorname{rank} H_i = s - 1$ *and*

$$\delta(\mathcal{L}_i) = \delta(M)\delta(H_i).$$

The pairing

$$B(c_1, c_2) = \langle p_1(c_1), p_2(c_2) \rangle - \langle q_1(c_1), q_2(c_2) \rangle$$

descends to the rational spans of H_1, H_2, takes values in $Q\mathcal{O}_E$ on these lattices, and has operator norm at most two at every archimedean place with their projected metrics.

Proof. Use the lattice of pairs $(l, l') \in L_0 \oplus L_2$ with equal images at every $v \mid Q$ in the residue middle quotient $U^{\perp}/U$ of Theorem 87.15. A class from C_i has integral projections exactly when it admits integral lifts in $X_0^{\perp}, X_i^{\perp}$, respectively, using β^{-1} in the second coordinate for $i = 1$. They differ from c by vectors in X_0, X_i. Their difference lies in $Q\Lambda + (\Lambda \cap X_0) + (\Lambda \cap X_i)$ exactly when their residue classes agree. This follows by reducing modulo $Q\Lambda$ and the Chinese remainder theorem, since both isotropic lattices reduce to U. The comparison identifies the residue middle quotients by the identity. These conditions define $\mathcal{L}_i$. Intersecting the resulting lattice of pairs with the common graph proves the assertion about M. In particular M is the lattice M_β of Corollary 87.14, under the first projection.

Here and below a possibly fractional lattice A in a positive space has determinant defined as follows. In a rational basis $v_1, \dots, v_a$, write its top exterior power as $\mathfrak{a}(v_1 \wedge \cdots \wedge v_a)$. Then $\delta(A) = N\mathfrak{a} \prod_\tau \det(\langle v_i, v_j \rangle_\tau)$, with one τ above each real place, positive absolute ideal norm, and empty determinant one. Change of basis cancels between these factors. For integral lattices this is the discriminant convention of Section 87.3, as seen on a local basis. For projection on

an orthogonal complement, the determinant is the product of those of the kernel lattice and the projected image. Indeed choose a rational basis of the kernel and lifts of a basis of the image. The exterior ideals multiply by the exact lattice sequence, which splits locally. Subtracting kernel components from the lifts preserves the exterior vector and makes the Gram matrix block diagonal. Applying this to $p_i(\mathcal{L}_i)$ gives the asserted identity and the rank is $k-1-j=s-1$.

On D the two graph isometries agree, so pairing a common graph vector with either graph gives zero for the difference of forms. Removing the D-component on the source removes its common isometric image in D' on the target. This proves descent. Each of the resulting inner products is bounded by the product of the projected norms, by the Cauchy–Schwarz inequality. Integrality in $Q\mathcal{O}_E$ follows because the two original column pairings are integral and reduce to the same Hermitian pairing in $U^{\perp}/U$ at every $v \mid Q$. □

Lemma 87.17 (Rank and determinant of the reduced pairing)**.** *The descended pairing has rank $s-1$ or $s-2$. In the second case let R_i denote the intersection of H_i with its radical line on its own side. In the first case put $R_i=0$. For $n=\operatorname{rank} B$,*

$$\frac{\delta(H_1)\delta(H_2)}{\delta(R_1)\delta(R_2)} \;\geq\; (Q/2)^{2nf}.$$

Proof. After removing the common graph, the resulting graphs are disjoint totally isotropic $(s-1)$-planes in $D^{\perp}\oplus(D')^{\perp}$ with the difference of the two forms. Indeed the original graphs each split orthogonally in their positive source coordinate as the common graph plus their projected graph. At any real place the ambient difference form has signature (s,s). The radical on the first graph is its intersection with the perpendicular of the second. It injects into the perpendicular of the second modulo the second, a space of signature $(1,1)$. This signature follows by splitting off hyperbolic partners for a basis of the second isotropic graph. An isotropic subspace of a plane of signature $(1,1)$ has dimension at most one. The ranks of a matrix and its adjoint agree, so the two radical dimensions are equal. Ranks are preserved on extending E to $\mathbb{C}$.

Project H_i orthogonally off the rational radical to give a lattice H_i' of rank n. Pairings are unchanged on projection, so the descended pairing still has norm at most two and takes values in $Q\mathcal{O}_E$. In rational bases let B_0 be its matrix and G_1, G_2 the Gram matrices. If $\mathfrak{a}_i$ are the two exterior ideals, the nonzero ideal $(\det B_0)\overline{\mathfrak{a}_1}\mathfrak{a}_2$ lies in $Q^n\mathcal{O}_E$ by taking determinants on local bases (conjugate the opposite ideal for the opposite pairing convention). Its norm is at least Q^{2nf}. At τ, in orthonormal bases all singular values are at most two, so $|\det B_0|_\tau^2 \leq 2^{2n}\det G_{1,\tau}\det G_{2,\tau}$. Multiply over τ and insert the two ideal norms. The result is $Q^{2nf} \leq 2^{2nf}\delta(H_1')\delta(H_2')$. Orthogonal projection gives $\delta(H_i')=\delta(H_i)/\delta(R_i)$ by Lemma 87.16. This argument also covers $n=0$. □

We give the additional estimate needed when the descended pairing has a radical. Let $D_{i,d-1}$, $D_{i,d}$ denote the leading minors of the cross pairing of the two decorated frames $0,i$, ending with the isotropic and full frames, respectively. For each archimedean embedding define

$$k_{i,\tau}=\begin{cases}(1+c_{i,\tau})/\sqrt{c_{i,\tau}^2-1}, & D_{i,d-1}\neq 0,\quad c_{i,\tau}=\dfrac{|\tau(D_{i,d}/D_{i,d-1})|}{\sqrt{TT'}},\\ 1, & D_{i,d-1}=0.\end{cases}\qquad K_i=\prod_{\tau}k_{i,\tau}.$$

These factors are invariant under integral or rational upper unitriangular changes of frame.

Proposition 87.18 (The extra radical line)**.** *If the descended pairing is singular, then for $\{i,a\}=\{1,2\}$,*

$$\delta(R_i) \;\geq\; \frac{T^f i_a}{I_a K_a^2}. \tag{87.10}$$

Proof. Start with an element of the rational left or right radical on side i, lifted to its graph before removing D. View its two coordinates in P_0, P_a, undoing $\bar{\beta}$ in the second coordinate when $a = 1$. Choose rational row lifts $l \in W_0^{\perp}$, $l' \in W_a^{\perp}$. The radical condition says that their projections onto C_a coincide, since their pairings against every $c \in C_a$ coincide. Thus $l - l' \in S_a$. Modulo $X_0 + X_a$ it has unique coefficients $\alpha z_0 + \alpha' z_a$, by the direct decomposition $S_a = W_0 \oplus W_a$. Changing either lift does not change these coefficients. An element of the common graph has lifts equal to a single vector of C_a, so adding such an element also leaves them fixed. This defines E-linear functionals α, α' on the projected radical line.

First bound the denominators of α. Each element of R_i is the projection of an integral compatibility graph vector. The two lifts may then be chosen integral, and their difference belongs to $Q\Lambda + (\Lambda \cap X_0) + (\Lambda \cap X_a)$, since the graph coordinates have equal marked reductions. Hence $l - l'$ belongs to B_a and its class modulo U_a belongs to $Q(B_a/U_a)$. The z_0-coefficient gives a map $B_a/U_a \to E/\mathcal{O}_E$. Indeed at each place a primitive representative of the initial frame is a basis of $\Lambda_v \cap W_0$, with last vector equal to z_0 modulo X_0, so U_a has integral z_0-coefficient. The subgroup of order i_a injected in $Q(B_a/U_a)$ by Proposition 87.4 is represented by differences in $X_0 + X_a$, so its coefficient is zero. That proposition also gives $|Q(B_a/U_a)| \le I_a$. The inverse image of the coefficient image of $Q(B_a/U_a)$ in E is therefore a fractional ideal $J \supset \mathcal{O}_E$ with $[J : \mathcal{O}_E] \le I_a/i_a$, and $\alpha(R_i) \subset J$.

We now compute the archimedean norm of this functional. Remove the common C_a-component of the two lifts. Their images y, y' in the respective positive normal lines to $p_a(C_a), q_a(C_a)$ have equal norm, say t, because the graph on side i is an isometry and the removed component was the same on both sides. Both normal components are unchanged on first removing the common D-component. Thus $t \le \|r\|_\tau$ for the resulting vector r in the projected radical line. If $t = 0$, the two original coordinates are the graph of a vector of C_a. They are also on graph i, so their vector vanishes after removing the common graph. Thus $t > 0$ for nonzero r.

The normal components have representatives in $S_a \cap W_0^{\perp}$ and $S_a \cap W_a^{\perp}$. Write their difference as $w_0 + w_a$, with $w_0 = x_0 + \alpha z_0$, $w_a = x_a + \alpha' z_a$, $x_0 \in X_0$, $x_a \in X_a$. Comparing the norms in $y - w_0 = y' + w_a$ gives

$$\|y\|^2 - \|y'\|^2 = T|\alpha|^2 - T'|\alpha'|^2.$$

Here the norms on the left are those of the positive quotient lines, equal to the row norms of their representatives. The identity holds for arbitrary normal components y, y'. By polarization the map from their direct sum to the two coefficients preserves a nondegenerate form of signature $(1,1)$ and is injective. In particular for the nonzero equal-norm pair above, $\alpha \neq 0$, since otherwise the identity forces both coefficients to vanish.

When $D_{a,d-1} \neq 0$, the pairing of X_0 and X_a is nonsingular. Modify z_0, z_a by X_0, X_a to make both perpendicular to $X_0 + X_a$. Denote the modified vectors by v_0, v_a. Their norms are $-T, -T'$, and their cross entry b_a has magnitude $|\tau(D_{a,d}/D_{a,d-1})|$ by the determinant of the block matrix after these row and column operations. In their span the positive normal to v_0 has a generator of the form $v_a + \lambda v_0$, with $|\lambda| = |b_a|/T$ and squared norm $T'(c_{a,\tau}^2 - 1)$. This is strictly positive since its class is a nonzero vector in the positive quotient. The normal to v_a has a generator $v_0 + \lambda' v_a$ of squared norm $T(c_{a,\tau}^2 - 1)$. Reading the coefficient of v_0 for normal components of length t, then using the triangle inequality on their difference, gives

$$\sqrt{T}\,|\tau\alpha(r)| \le k_{a,\tau} t.$$

If $D_{a,d-1} = 0$, the pairing of X_0, X_a has defect one. Adding the column of z_a must give full row rank on X_0, for a residual kernel would be in the radical of S_a. The same holds with the flags exchanged. Choose $x_a \in X_a$ perpendicular to X_0 and scale it so that $\langle z_0, x_a \rangle = T$, possible

since this kernel line pairs nontrivially with z_0. Then z_0+x_a is a positive normal representative on the initial side, of squared norm T. Exchanging flags gives a normal $z_a + x_0$ of squared norm T'. Their coefficient pairs modulo $X_0 + X_a$ are $(1,0)$ and $(0,1)$. Thus $\sqrt{T}\,|\tau\alpha(r)| = t$, giving the displayed bound with $k_{a,\tau} = 1$.

Write $R_i = \mathfrak{a} r_0$ for a fractional ideal and a nonzero rational vector. Since $\mathfrak{a}\alpha(r_0) \subset J$ and $\alpha(r_0) \neq 0$,

$$N\mathfrak{a}\,|N_{E/\mathbb{Q}}\alpha(r_0)| \geq NJ \geq i_a/I_a.$$

Multiplying the squared archimedean inequalities and using $\delta(R_i) = N\mathfrak{a}\prod_\tau \|r_0\|_\tau^2$ proves (87.10). □

Theorem 87.19 (Determinant bound in terms of the two Gram matrices)**.** *Let M be the comparison lattice. Its determinant and the two compatibility indices satisfy*

$$\delta(M) i_1 i_2 \;\leq\; \sqrt{G_{0,1}G_{0,2}} \begin{cases} (2/Q)^{(s-1)f}\sqrt{i_1 i_2/(I_1 I_2)}, & \operatorname{rank} B = s-1, \\ (2/Q)^{(s-2)f} K_1 K_2 / T^f, & \operatorname{rank} B = s-2. \end{cases} \tag{87.11}$$

In particular, for fixed Gram data of the two pairs one may use the respective right side, replacing the square root of the index ratio by 1*, as $D_{\max}$ in Corollary 87.14 for $2 \leq s \leq k-1$.*

Proof. Multiply the two determinant identities of Lemma 87.16. After multiplication by $i_1^2 i_2^2$ the numerator is $G_{0,1}G_{0,2} i_1 i_2/(I_1 I_2)$. For the denominator $\delta(H_1)\delta(H_2)$, apply Lemma 87.17. When $n = s-1$ the radical determinants are one. When $n = s-2$ their product is at least $T^{2f} i_1 i_2/(I_1 I_2 K_1^2 K_2^2)$ by Proposition 87.18. Taking square roots gives both bounds. All the quantities used the same Λ and original finite levels. The determinant alternatives being compared are stated in `Formalism/SharpRestriction/CuspSpanNormBounds/CongruenceAtSplitFrobeniusHolds.lean`, line 88. □

We next count the ordered pairs above each subspace satisfying (87.11). An integral Witt decomposition makes it possible to do this with the unipotent equivalence of the original row.

87.6 Counting flags above one sublattice of the initial column

Fix the initial integral decorated flag in a row lattice Λ at the original finite levels. Write $t = d-1$ for its isotropic rank, X for its isotropic space, z for its terminal vector of norm $-T$, and $n_T = T^f$. A terminal flag has isotropic rank t and last norm $-T'$, with $T' = T$ in the period being studied. We use integral representatives modulo upper unitriangular changes of their d columns. An integral row isometry which preserves the initial isotropic flag, acts trivially on each of its graded lines, and acts trivially on $X^\perp/X$ is an allowed change of row representative. Write N_Λ for this group. The equivalence of terminal flags uses both changes. This is the relation in `Formalism/SharpRestriction/CuspSpanNormBounds/FlagEEqHyperbolic.lean`, line 122 and `Formalism/SharpRestriction/CuspSpanNormBounds/CntPartners.lean`, line 177.

Lemma 87.20 (Integral coordinates for frames)**.** *The initial frame can be chosen integral and primitive over $\mathcal{O}_E$. With this choice the row has an integral Witt decomposition $A \oplus J \oplus A^*$, with $A = \mathcal{O}_E^t$ the initial isotropic lattice, $A^* = \mathcal{O}_E^t$ its hyperbolic partner, and J a self-dual middle lattice containing the primitive vector z. In these coordinates $L = J \cap z^\perp$ is the positive initial column. Each integral upper unitriangular change of the initial frame is realized by an element of N_Λ.*

Proof. At each finite place choose the primitive representative of Lemma 76.30. Relative to the given rational frame this is an upper unitriangular matrix. The identity works away from

finitely many places. Approximate the entries of the finitely many changes simultaneously by elements of E integral elsewhere. This is the Chinese remainder theorem after clearing the finitely many denominators in a fractional ideal. Integrality and independence modulo the residue maximal ideals persist for sufficiently close changes. Thus the new frame spans a globally primitive free submodule $A \oplus \mathcal{O}_E z$ of Λ. The same argument applies to any admissible terminal frame. Two integral primitive representatives of the same decoration differ by an integral unitriangular matrix since their local change is integral at every place.

The quotient by $A \oplus \mathcal{O}_E z$ is torsion-free, hence projective over the Dedekind ring. Its integral functionals on the frame summand extend to Λ. By self-duality choose partners to the ordered basis of A having the identity pairings with that basis and zero pairings with z. Correct their mutual pairings by adding vectors of A. The equation for the coefficients is $S + S^{*} = -H$, for the Hermitian matrix H to be removed. This has an integral solution by assigning one entry in each off-diagonal pair and using the integral trace on the diagonal. The trace ideal is the unit ideal, as its localization is the unit ideal at every finite place in our unramified extension. The resulting hyperbolic lattice splits off orthogonally, leaving J and z as claimed.

For any integral map $V : A^{*} \to J$, with adjoint V^{*} in these coordinates, the transformation

$$(x, b, y) \longmapsto (x - V^{*}b + Sy,\ b + Vy,\ y), \qquad S + S^{*} = -V^{*}V, \tag{87.12}$$

is an integral isometry in N_{Λ}, as is seen by expanding the pairing. The same trace argument produces S. It fixes z when the columns of V belong to L. More generally $V^{*}z$ can be prescribed arbitrarily in A since z is primitive in the self-dual J. One may also change the basis of A by an integral upper unitriangular matrix and that of A^{*} by its inverse adjoint. Together these isometries realize every initial frame change in the statement. □

Let Z be an integral d-by-d cross Gram matrix of a supported nondegenerate frame pair. Its class allows an integral lower unitriangular change on the left and an integral upper unitriangular change on the right. Let C be its first t rows. Its rank is t, since a kernel vector on the row side would give an initial isotropic vector in the radical of the full frame sum. Define

$$J_Z = [\,\mathcal{O}_E^{t} : C\mathcal{O}_E^{d}\,], \qquad G_Z = |N_{F/\mathbb{Q}}\mathcal{G}|, \qquad A_s(Z) = 9^{sf}\max(1, G_Z/n_T)^{s} J_Z^{s+t}.$$

Here $\mathcal{G} = \det Z\,\overline{\det Z} - TT' D_t \bar{D}_t$, where D_t is the leading t-by-t minor of Z. This is the full Gram determinant up to sign, by expansion in the two last self-pairings. If $\mathfrak{c}$ is the ideal generated by the maximal minors of C, then $J_Z = N\mathfrak{c}$ by local elementary divisors. Both D_t and $\det Z$ belong to $\mathfrak{c}$, by expansion along the last row for the latter. Thus $(\mathcal{G}) \subset \mathfrak{c}\bar{\mathfrak{c}}$ and $J_Z \le G_Z$. These quantities are invariant under both allowed changes of Z.

For a terminal flag let C_{flag} denote the perpendicular of the full frame sum, and p its embedding in the initial positive column. Say that an integral sublattice $P \subset L$ lifts to this perpendicular if $P \subset p(\Lambda \cap C_{\mathrm{flag}})$. Write $a_P = [L \cap EP : P]$ and $\operatorname{rank} P = k - s$.

Theorem 87.21 (Flags with an integral lift of a prescribed sublattice)**.** *For $1 \le s \le k$, the number of terminal flag classes for the integral cross class of Z for which P lifts to the perpendicular is at most*

$$(n_T^{2} a_P)^{t}\, A_s(Z). \tag{87.13}$$

When $P = 0$ the prefactor can be replaced by 1.

Proof. Work in the decomposition of Lemma 87.20. Every class under consideration has a representative of exact cross matrix Z. A right change is a basis change of the terminal decoration. The left change is the adjoint of an integral initial frame change and can be

realized by applying the inverse row isometry to the terminal frame. These operations keep P fixed in the initial quotient. An upper bound for exact frames modulo row isometries fixing the initial frame gives an upper bound for the requested classes.

For a frame admitting P, each $p_0 \in P$ has a unique lift to C_{flag}, since this perpendicular meets X trivially. The integral lift is $(\phi(p_0), p_0, 0)$, for a linear map $\phi : P \to A$. The isometries (87.12) with columns of V in L change this map by $-V^*|_P$. Put $P_* = L \cap EP$. Primitivity of P_* in L implies that restriction of integral functionals from L to P_* is onto. Hence orthogonal projection on EP maps $L^\vee$ onto $P_*^\vee$. The image of L there has index at most $[L^\vee : L] = n_T^2$, by Lemma 87.10. Moreover $[P^\vee : P_*^\vee] = a_P$ by duality on the same span. It follows that at most $(n_T^2 a_P)^t$ classes of ϕ occur, by considering the t functionals independently. For $P = 0$ there is one. Fix a representative of such a class and move each frame to it by an integral shear.

Write the columns of the terminal frame in three blocks (U, B, C), so that C is the fixed block from Z. Let $v \in \mathcal{O}_E^d$ be the signed vector of maximal cofactors of C, normalized in sign to give the expansion of $\det Z$. It spans the rational kernel of C, and its last entry is D_t up to sign. Put w equal to the projection of $Bv \in J$ on the positive space $z^\perp$ in EJ. Since $Cv = 0$,

$$\langle Bv, Bv\rangle = -T'|D_t|^2, \qquad \langle z, Bv\rangle = \det Z, \qquad \langle w, w\rangle = \mathcal{G}/T,$$

where $|D_t|^2 = D_t \bar{D}_t$ in F. The first identity uses the self Gram matrix of the terminal frame. The fiber condition on ϕ gives $\langle p_0, Bv\rangle = 0$ for all $p_0 \in P$, by pairing the lift with the linear combination of columns of coefficients v. Thus $w \in (EP)^\perp$ in the positive column. Its prescribed squared norm is positive at every embedding by nondegeneracy. For two possible values of w, their difference belongs to $L_P = L \cap (EP)^\perp$, since the corresponding Bv's have the same pairing with z.

At τ let $r_\tau = (\tau(\mathcal{G})/T)^{1/2}$. In the product of the complex s-spaces give a vector the norm $\max_\tau \|x_\tau\|/r_\tau$. All the candidates lie in its unit ball. A nonzero difference $x \in L_P$ satisfies $\prod_\tau \|x_\tau\|^2 \geq 1$, by the norm of the nonzero totally positive integer $\langle x, x\rangle$. The separation is therefore at least b_0^{-1}, where $b_0 = (\prod_\tau r_\tau)^{1/f}$. Disjoint open balls of half that separation lie in the ball of radius $1 + 1/(2b_0)$. Comparing real volumes in dimension $2sf$ bounds the number by $(1 + 2b_0)^{2sf} \leq 9^{sf} \max(1, G_Z/n_T)^s$.

Fix one such w. For two possible middle blocks B their difference takes values in L_P. In fact their pairings with z are the same last row of Z, and the pairings with p_0 are fixed by $\langle \phi(p_0), C\rangle$. The difference vanishes on v, since Bv is fixed by w and its pairing with z. Hence it factors through a homomorphism $C\mathcal{O}_E^d \to L_P$. To check factorization, two integral preimages of a vector differ in the rational kernel Ev, where the difference vanishes. The shears (87.12) alter B by VC for all integral $V : \mathcal{O}_E^t \to L_P$, fixing z and ϕ. Restriction of such V's has index J_Z^s in the group of homomorphisms on $C\mathcal{O}_E^d$. Indeed local diagonal reduction of the inclusion multiplies the index by the s-th power of each elementary-divisor index. This argument works for projective L_P.

Finally fix B itself after these shears. The difference ΔU of two possible upper blocks satisfies $(\Delta U)^* C + C^* \Delta U = 0$ by the equality of the two self Gram matrices. Evaluation at v gives $\Delta U v = 0$ since C^* is injective. Thus $\Delta U = SC$ for a rational matrix S. Surjectivity of C over E in the Gram identity gives $S + S^* = 0$. The allowed integral skew matrices give shears with $V = 0$. Modulo them, the possible S's inject into

$$\operatorname{Hom}(C\mathcal{O}_E^d, \mathcal{O}_E^t) / \operatorname{res} \operatorname{Hom}(\mathcal{O}_E^t, \mathcal{O}_E^t).$$

Indeed a difference which extends integrally is already integral and skew. This quotient has order J_Z^t by the same local calculation. Multiplying the four counts proves (87.13). □

87.7 A mean for ordered pairs with prescribed integral Gram data

Let Z_1, Z_2 be prescribed integral cross classes, each with nondegenerate full Gram matrix. Count ordered terminal flag pairs together with admissible comparisons as in Section 87.5. Equivalence allows independent row changes from N_Λ on the two flags and identifies the two comparisons when their induced middle maps correspond under these changes. The initial flag remains fixed throughout.

Lemma 87.22 (Forgetting the comparison and counting finite indices)**.** *For fixed Λ, the map from pair classes to ordered pairs of individual flag classes is injective. Also, in every projective $\mathcal{O}_E$-lattice of rank j, the number of submodules of absolute index n is at most $n^{j+\lambda}$, where $\lambda = \log_2(2f)$.*

Proof. Suppose two pairs have the same individual classes, via two changes $n_1, n_2 \in N_\Lambda$ and two unitriangular changes of the decorations. Transport the first comparison on the middle quotients by n_2 and n_1^{-1}. The transported map and the second comparison take the class of the first terminal vector to that of the second. Indeed a decoration change alters the terminal vector only by the terminal isotropic subspace. Both maps induce the identity on the common residue middle quotient at $v \mid Q$: each n_a acts trivially there because it induces the identity on the initial middle quotient. Their difference by composition is therefore an integral isometry fixing a nonisotropic vector with positive definite perpendicular, and congruent to the identity at the odd split prime. The finite-order and torsion-free argument of Theorem 87.15 makes it the identity on the entire middle quotient. This is precisely the equivalence required of the comparison. The rational common subspace in the initial column is also unchanged because the n_a act identically on that column.

For the index bound, induct on n, keeping the stronger bound $n^j(2f)^{\Omega(n)}$, where Ω counts prime factors with multiplicities. The case $n = 1$ is immediate. For a prime $p \mid n$, the p-primary part of the finite quotient has a simple quotient with residue field of order q at some prime of E above p. Thus the submodule lies in a maximal submodule of index $q \mid n$. At a specified prime the possibilities are hyperplanes in a residue j-space, at most q^j. Their kernels are again projective of rank j. For each of the at most $2f$ primes over p, induction inside such a kernel gives a total at most $n^j(2f)^{\Omega(n)-1}$. Summation, followed by $\Omega(n) \le \log_2 n$, proves the assertion. □

Theorem 87.23 (Mean number of admissible ordered comparisons)**.** *Let $1 \le s \le k-1$, $j = k-s$, $\nu = 4+\lambda$. Impose $\delta(M) i_1 i_2 \le Y$ on the comparison lattice and indices of Section 87.5, with $Y \ge 1$. In the normalized initial column genus the mean number of comparison classes of dimension j and integral cross classes Z_1, Z_2 is at most*

$$8\,\frac{\mathfrak{m}_j \mathfrak{m}_s}{\mathfrak{m}_k} (Y/n_T)^k n_T^{s+4t} Y^{\nu+1+2t} A_s(Z_1) A_s(Z_2). \tag{87.14}$$

The bound is uniform over compatible row representatives above each initial column. It also holds for their initial unipotent-covolume average at the original level. For $Y < 1$ the count is zero. For $s = k$ the number per row is at most $A_k(Z_1)A_k(Z_2)$.

Proof. Let $D \subset P_0$ be the rational common subspace and write $S = L \cap D$. Identify M with its integral image in L. Put $a = [S : M]$ and

$$P_i = M \cap p_i(\Lambda \cap C_i), \qquad b_i = [M : P_i].$$

The entire M lifts to $\mathcal{L}_i$ on either side by Lemma 87.16. Hence $b_i \le i_i$, by injection of M/P_i into $\mathcal{L}_i/(\Lambda \cap C_i)$. We have $\delta(S)a = \delta(M)$ and so $\delta(S) a b_1 b_2 \le Y$. Also $L \cap EP_i = S$, hence $a_{P_i} = a b_i$.

For fixed S, a, b_1, b_2, there are at most $(ab_1b_2)^{j+\lambda}$ choices for M and the two P_i, by Lemma 87.22. With these choices fixed, every flag on side i lifts P_i to its integral perpendicular. Theorem 87.21 and injectivity in Lemma 87.22 bound the possible pair classes by $n_T^{4t}(a^2b_1b_2)^t A_s(Z_1)A_s(Z_2)$. Each individual flag retains the lifting property under its changes of representative, since those act trivially on the initial column and preserve the row lattice.

Set $m = ab_1b_2 \le Y$, using $\delta(S) \ge 1$. The mean number of primitive S with $\delta(S) \le Y/m$ is bounded by summing (87.8) over the positive integers in this range. This gives

$$\frac{\mathfrak{m}_j\mathfrak{m}_s}{\mathfrak{m}_k}(Y/n_T)^k n_T^s Y^{\nu+1} m^{-k-\nu-1}.$$

Use $(a^2b_1b_2)^t \le Y^{2t}$. After inserting the index count, the remaining powers are m^{-s-5} since $\nu = 4 + \lambda$. The sum over the three positive indices is at most $\zeta(s+5)^3 < 8$. This proves (87.14). For $s = k$ forget the comparison and use Theorem 87.21 with $P = 0$ twice. The zero assertion follows from integrality of M.

The pointwise estimate before averaging used only the number of primitive sublattices in L, uniformly in the row above L. At the original level the stabilizer of the initial flag is the semidirect product of its unipotent and column levels by Theorem 64.1. The unipotent covolume over each initial column is the same by the argument of Lemma 70.23: conjugation by the column group has modulus one on the radical and the rational unipotent is dense in its finite adeles. The remaining normalized level weights push onto the genus measure by Lemma 87.10. This gives the asserted initial-row average. □

Corollary 87.24 (Partners of a fixed generic flag in codimension one)**.** *Fix one terminal flag with $I_1 = 1$ and integral cross class Z_1. In the same row the number of possible partners with $I_2 = 1$, integral class Z_2, an admissible comparison and $s = 1$ is at most*

$$(n_T^2 G_{0,1})^t A_1(Z_2), \qquad G_{0,1} = G_{Z_1}/Q^{2df}.$$

Proof. Both rational graphs have dimension $k-1$, so an intersection of dimension $k-1$ makes them equal. Their integral intersections with the compatibility lattice are equal by Lemma 87.16, whose intersection argument holds also for $s = 1$. Saturation gives $\mathcal{L}_i = \Lambda \cap C_i$ by Proposition 87.4. Thus $P = p_1(\Lambda \cap C_1)$ is already fixed by the first flag, has determinant $G_{0,1}$, and lifts to the integral perpendicular of every partner. Its saturation in L has determinant at least one, so $a_P \le G_{0,1}$. Apply Theorem 87.21. □

Equations (87.11) and (87.14) give a mean for actual pairs separately for each pair of integral cross classes. The codimension-one bound keeps its first flag fixed. These are the counts entering `Formalism/SharpRestriction/CuspSpanNormBounds/CongruenceAtSplitFrobeniusHolds.lean`, line 556. We next determine which codimension-one partners can occur at the original terminal prime.

87.8 A congruence between the normal lines of two terminal flags

Work at the original level on the fields of Section 87.2. The common norm of the three terminal vectors is $-T$, and T is prime to $Q = g_1g_2$. Fix an admissible comparison β for which both frame sums $S_i = W_0 \oplus W_i$ are nondegenerate and $s = 1$. In the two positive middle spaces P_0, P_2 the graphs of $C_i = S_i^{\perp}$ coincide, after using β on the second coordinate of the first graph. Indeed each graph and their intersection have dimension $k-1$. Write D, D' for the two images of this common graph and N, N' for their perpendicular lines.

We use the Hermitian pairing conjugate-linear in the first variable. A vector of N has a row representative $l \in S_i \cap W_0^{\perp}$. To see this, lift to $W_0^{\perp}$ and project off C_i. Its C_i-component

is already zero since its class is perpendicular to D. In the same way a vector of N' has a representative $l' \in S_i \cap W_i^{\perp}$, undoing β for $i = 1$. In the direct sum S_i, write

$$l - l' = x + \alpha z_0 + x_i + \alpha' z_i, \qquad x \in X_0, \quad x_i \in X_i.$$

Changing either lift adds only an isotropic term. Let $A_i : N \oplus N' \to E^2$ send the two classes to (α, α'). For lifts with a common C_i-component the coefficients are still those of their normal components, by subtracting that common vector. More generally, for lifts in $X_0^{\perp}, X_i^{\perp}$, one can first remove their respective terminal-line components in the signed middle spaces $Ez_a \oplus P_a$. Their C_i-components coincide exactly when their difference lies in S_i. This follows by pairing the difference with every vector of C_i. Equivalently, the projections on D, D' of the two positive coordinates lie on the common graph. This last description is independent of i.

The norm identity in the proof of Proposition 87.18 gives, at each embedding,

$$\|y\|^2 - \|y'\|^2 = T(|A_i(y, y')_1|^2 - |A_i(y, y')_2|^2).$$

Polarization identifies A_i as an isomorphism. Put $\gamma = A_2 A_1^{-1}$. It is an E-linear isometry for $\mathrm{diag}(1, -1)$. Equality $\gamma = 1$ says precisely that the two pairs have the same coefficients for all matched normal vectors.

For $i = 1, 2$ retain $B_i = \Lambda \cap S_i$, its sublattice B_i^{+} from Section 87.2, and $I_i = [B_i : B_i^{+}]$. Set

$$\mathfrak{h}_i = \mathrm{Ann}_{\mathcal{O}_E}(B_i / B_i^{+}), \qquad \mathfrak{h} = \overline{\mathfrak{h}_1}\mathfrak{h}_2, \qquad \mathcal{Z} = \{(u, v) \in \mathcal{O}_E^2 : u + v \in Q\mathcal{O}_E\}.$$

The norm $N\mathfrak{h}_i$ is at most I_i. At a prime the exponent of the annihilator is the largest elementary divisor exponent of the quotient, bounded by their sum.

Theorem 87.25 (Integral congruence for the normal comparison)**.** *For every such ordered comparison,*

$$\mathfrak{h}\,(\gamma - 1)\mathcal{Z} \;\subset\; T\mathcal{Z}. \tag{87.15}$$

Proof. For $b \in S_i$ let $\kappa_i(b)$ denote its two terminal coefficients modulo $X_0 + X_i$. We first compute $\kappa_i(QB_i^{+}) = \mathcal{Z}$. Away from Q the two individual primitive frames generate B_i^{+} and Q is a unit. At $v \mid Q$ choose representatives with equal reductions. The image of QB_i^{+} is generated by $Q(1, 0)$ and $(Q/\pi_v)(-1, 1)$. All the other generators have zero coefficients. These vectors span exactly the local module of pairs of integral coordinates whose sum is divisible by Q. Localization proves the assertion.

Fix $p = (u, v) \in \mathcal{Z}$ and $a \in \mathfrak{h}_1$. On S_1 take the conjugate-linear functional ℓ which vanishes on $X_0 + X_1$ and has values $u, -v$ on z_0, z_1, respectively. It is integral on B_1^{+}. For the terminal divided difference at a marked place one uses $u + v \in Q\mathcal{O}_E$, and the other divided differences give zero. Representatives of z_a differing by a vector of X_a give the same values. Thus $\bar{a}\ell$ is integral on B_1, since $aB_1 \subset B_1^{+}$. The quotient of Λ by B_1 is torsion-free and finitely generated over the Dedekind ring $\mathcal{O}_E$, hence projective. Extend the integral functional across a module complement. By self-duality it is represented by $w \in \Lambda$, so

$$w \perp X_0 + X_1, \qquad \langle z_0, w \rangle = \bar{a}u, \qquad \langle z_1, w \rangle = -\bar{a}v.$$

Put $u' = \bar{a}u/T$, $v' = \bar{a}v/T$. The positive coordinates of w in the two middle spaces have respective representatives $w + u'z_0$, $w - v'z_1$. Their difference is $u'z_0 + v'z_1$. Thus their C_1-components coincide and their normal coordinates have coefficient pair (u', v') under A_1.

Use $w, \beta w$ for the signed lifts in the second pair. The initial positive coordinate is the same, and the terminal one is its previous terminal coordinate transported by β. The common-graph description therefore gives equal C_2-components. Also $\beta z_1 = z_2 \bmod X_2$. It follows that $w - \beta w \in S_2$ and

$$\kappa_2(w - \beta w) = (\gamma - 1)(u', v').$$

Both signed lifts are integral. At each marked place the residual middle quotients identify with $\bar{X}^{\perp}/\bar{X}$, and the comparison is the identity there by Theorem 87.15. Their difference reduces into the common $\bar{X}$. Consequently

$$w - \beta w \in QB_2 + (\Lambda \cap X_0) + (\Lambda \cap X_2).$$

Indeed one may subtract local integral isotropic lifts of the residues. The remainder lies in $Q\Lambda \cap S_2 = QB_2$. Away from Q this holds since Q is a unit, so local membership gives the asserted global membership. For any $b \in \mathfrak{h}_2$ the pair $b\kappa_2(w - \beta w)$ therefore belongs to $\kappa_2(QB_2^{+}) = \mathcal{Z}$. Multiplication by T and variation of a, b, p prove (87.15). □

Write $D_i = \det Z_i$, R_i for its leading minor of size $d-1$, and $\mathcal{G}_i = D_i\bar{D}_i - T^2 R_i \bar{R}_i$. It is totally positive, so $D_i \neq 0$. For one embedding τ over each real place put

$$U_{i,\tau} = \frac{|\tau D_i|}{\sqrt{\tau \mathcal{G}_i}}, \qquad V_{i,\tau} = \frac{T|\tau R_i|}{\sqrt{\tau \mathcal{G}_i}}, \qquad U_{i,\tau}^2 - V_{i,\tau}^2 = 1.$$

Theorem 87.26 (Separation of two comparisons in codimension one)**.** *Write $c = \gamma_{21}$. If $c \neq 0$, then*

$$\sqrt{I_1 I_2} \prod_{\tau} (U_{1,\tau} V_{2,\tau} + V_{1,\tau} U_{2,\tau}) \;\geq\; (T/Q)^f. \tag{87.16}$$

If $c = 0$, either $\gamma = 1$ or $\sqrt{I_1 I_2} \geq (T/2)^f$. In the first case one has

$$R_1/D_1 = R_2/D_2 \quad \text{in } E. \tag{87.17}$$

Thus for $I_1 = I_2 = 1$ every pair has identical coefficient maps or satisfies (87.16).

Proof. The vector $(Q, 0)$ belongs to $\mathcal{Z}$. Theorem 87.25 gives $(Qc/T)\mathfrak{h} \subset \mathcal{O}_E$. When $c \neq 0$ the norm of this nonzero integral ideal is at least one, whence $\prod_\tau |\tau c| \geq (T/Q)^f / \sqrt{N\mathfrak{h}}$. Choose unit bases of N, N' at τ and multiply both output coordinates of A_i by $\sqrt{T}$. The resulting matrix M_i preserves $\mathrm{diag}(1, -1)$. Its first column has entry magnitudes $U_{i,\tau}, V_{i,\tau}$. Indeed, if v_i is the signed cofactor column of the top $d-1$ rows of Z_i, expansion along the last row gives $(v_i)_d = R_i$ and $\langle z_0, Y_i v_i \rangle = D_i$. Here Y_i is the terminal frame. The vector

$$y_i = Y_i v_i + (D_i/T) z_0 \in S_i \cap W_0^{\perp}$$

has norm squared $\mathcal{G}_i/T$, since $\langle Y_i v_i, Y_i v_i \rangle = -T R_i \bar{R}_i$. Its nonzero class n_i generates N, with $A_i(n_i, 0) = (D_i/T, R_i)$. This calculation includes $R_i = 0$. The column and row identities for a matrix preserving $\mathrm{diag}(1, -1)$ force its second column to have magnitudes $V_{i,\tau}, U_{i,\tau}$. The inverse is obtained by conjugate transposition and the two signature matrices. Taking the lower left entry of $M_2 M_1^{-1}$, in the same bases for both maps, now gives $|\tau c| \leq U_{1,\tau} V_{2,\tau} + V_{1,\tau} U_{2,\tau}$. Use $N\mathfrak{h} \leq I_1 I_2$.

When $c = 0$, orthogonality of the two columns makes γ diagonal, with diagonal entries a, e of absolute value one at every embedding. Apply (87.15) to $(1, -1)$. For $z = a, e$ one gets $(z-1)\mathfrak{h}/T \subset \mathcal{O}_E$. If $z \neq 1$, its norm and $|\tau(z-1)| \leq 2$ give $N\mathfrak{h} \geq (T/2)^{2f}$. Otherwise both entries are one. In that case $A_1 = A_2$. Write $n_2 = h n_1$ in the common rational line N. The cofactor identities then give $(D_2/T, R_2) = h(D_1/T, R_1)$, proving (87.17). □

Corollary 87.27 (A restriction on the Gram data)**.** *For a pair with $I_1 = I_2 = 1$ and distinct ratios R_i/D_i, some i, τ have $R_i \neq 0$ and*

$$\frac{\tau \mathcal{G}_i}{T^2 |\tau R_i|^2} \leq \frac{2}{\sqrt{1 + (T/Q)^2} - 1}.$$

Proof. Distinct ratios exclude $\gamma = 1$. The terminal prime exceeds two, so the second alternative for $c = 0$ is impossible. Put $H = \max_{i,\tau} V_{i,\tau}$. Every factor in (87.16) is at most $2H\sqrt{1+H^2}$. Solving $2H\sqrt{1+H^2} \geq T/Q$ yields the assertion at a place where the maximum is attained. □

This restricts the two actual integral Gram classes in Corollary 87.24 before forming the period sum. The ratios and the displayed normalized gaps are invariant under the two unitriangular changes of frame. The normalized gaps are also invariant under Lemma 71.10, since D_i/R_i and the terminal norm parameter both scale in magnitude by Q^{2d-3}. The separation in (87.16) uses the original value of T. The scalar condition and the comparison of its local and archimedean restrictions appear in `Formalism/SharpRestriction/CuspSpanNormBounds/IsPrimitiveSub.lean`, line 252 and `Formalism/SharpRestriction/CuspSpanNormBounds/LatticeGluingRadicals.lean`, line 3177.

87.8.1 Counting the partners of a rational label

For a nondegenerate label η, write $D_\eta = D_d$, $R_\eta = D_{d-1}$, $z_\eta = TR_\eta/D_\eta$, and $G_\eta = |N_{F/\mathbb{Q}}\mathcal{G}_\eta|$. The tuple of slopes $(\tau z_\eta)_\tau$ belongs to the product of unit disks. Put

$$\Phi_\eta = \prod_\tau \frac{|\tau D_\eta| + T|\tau R_\eta|}{\sqrt{\tau \mathcal{G}_\eta}} = \prod_\tau (U_{\eta,\tau} + V_{\eta,\tau}).$$

This is $\Phi((\tau z_\eta)_\tau)$ from Section 64.2.1. It is also the product of the real angular factors in the proposed local amplitude bound. Indeed for $R_\eta \neq 0$ that factor is $(1+c)/\sqrt{c^2-1}$ with $c = |\tau D_\eta|/(T|\tau R_\eta|)$, and for $R_\eta = 0$ it is one. These formulas specify the weight independently of any estimate on the amplitude.

Proposition 87.28 (The common determinant and the slope)**.** *Let two flags admit an original-level comparison of codimension one. At every finite place v of F, write q_v for the residue cardinality, v also for the normalized valuation, and $I_{a,v}, i_{a,v}$ for the local indices of $B_a^+ \subset B_a$ and $\Lambda \cap C_a \subset \mathcal{L}_a$ on side a, respectively. Then*

$$\frac{q_v^{v(\mathcal{G}_1/Q^{2d})}}{I_{1,v} i_{1,v}} = \frac{q_v^{v(\mathcal{G}_2/Q^{2d})}}{I_{2,v} i_{2,v}}.$$

At the levels under consideration $Q \geq 4$. Distinct slopes imply $\sqrt{I_1 I_2}\,\Phi_1\Phi_2 \geq (2T/Q)^f$. For equal slopes, either $A_1 = A_2$ or $\sqrt{I_1 I_2} \geq (T/2)^f$. In particular for saturated pairs the two determinant ideals in $\mathcal{O}_F$ agree, and the two coefficient maps agree exactly when the two slopes agree.

Proof. The two rational graphs coincide in codimension one. Lemma 87.16 therefore identifies the entire lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ isometrically. For a local lattice L use $\delta_v(L) = [L^\vee : L]^{1/2}$ when it is integral. Increasing a lattice by index a divides its determinant δ_v by a, since the dual decreases by the same index. Self-duality and primitivity give $\delta_v(B_a) = \delta_v(\Lambda \cap C_a)$. Indeed the common quotient of Λ_v by the orthogonal direct sum of those two primitive lattices identifies with the dual quotient of each, by restricting integral functionals. Such functionals extend from a primitive summand. In the local frame basis the determinant ideal of B_a^+ is generated by $\mathcal{G}_a/Q^{2d}$. At a marked place the d divided differences remove two factors of the uniformizer apiece from the Gram determinant, and at any other place Q is a unit. The local index formula of Proposition 87.4 gives $\delta_v(B_a^+) = q_v^{v(\mathcal{G}_a/Q^{2d})}$. Equating the two values of $\delta_v(\mathcal{L}_a)$ proves the first assertion.

Write $g = A_2 A_1^{-1}$ and $\mathfrak{h} = \overline{\mathfrak{h}_1}\mathfrak{h}_2$ as in Theorem 87.25. The cofactor vectors give positive lines of slopes z_1, z_2 as the respective images of $N \oplus 0$ under A_1, A_2. Thus $gz_1 = z_2$. When $g_{21} \neq 0$, use (87.16) and the inequality

$$2(U_1V_2 + V_1U_2) \le (U_1 + V_1)(U_2 + V_2)$$

at each embedding, since $(U_1 - V_1)(U_2 - V_2) \ge 0$. When $g_{21} = 0$ and the slopes differ, the alternative in Theorem 87.26 gives $\sqrt{I_1 I_2} \ge (T/2)^f$. Now $Q \ge 4$ and $\Phi_a \ge 1$, giving the stated bound also in this case.

If $z_1 = z_2$, at each embedding g preserves a positive line and its negative perpendicular, with unit-modulus eigenvalues. In the lattice basis $(1, -1), (0, Q)$ every entry of $g - 1$ belongs to $T\mathfrak{h}^{-1}$ by Theorem 87.25. If $g - 1$ has rank two, its determinant belongs to $T^2\mathfrak{h}^{-2}$, is nonzero and has absolute value at most four everywhere. Taking the norm of the integral ideal $\det(g-1)\mathfrak{h}^2/T^2$ yields $N\mathfrak{h} \ge (T/2)^{2f}$. In rank one use $\operatorname{tr}(g-1) \in T\mathfrak{h}^{-1}$. It is nonzero because the two invariant lines diagonalize g and precisely one eigenvalue differs from one. Its absolute value is at most two everywhere, giving the same norm bound. Since $N\mathfrak{h} \le I_1 I_2$, only rank zero is possible below this threshold. For saturated flags both global indices are one and $i_a \mid I_a$ by Proposition 87.4. The local equality gives the determinant ideals, and $A_1 = A_2$ gives equal slopes by (87.17). □

Theorem 87.29 (A weighted sum in one original row)**.** *Fix a saturated first flag $*$ in that row and let its terminal norm be $-T$. Sum over its saturated, relevant, nondegenerate partner classes admitting a codimension-one comparison. For a represented rational label η, use the monomial matrix M_η, the index bounds h_t and the class count $C(M_\eta)$ of Theorem 88.15. Put $t = d - 1$, $n_T = T^f$, $G_{0,*} = G_*/Q^{2df}$, and*

$$B_* = (n_T^2 G_{0,*})^t, \qquad K_\eta = C(M_\eta)\, 9^f \max(1, G_*/n_T) \min\{G_*, Q^{2ft} h_t\}^{t+1}.$$

Let $\mathcal{A}$ be the represented relevant labels of these partners with $z_\eta \neq z_*$, so $(\mathcal{G}_\eta) = (\mathcal{G}_*)$ in $\mathcal{O}_F$. For any nonnegative coefficients a_η on $\mathcal{A}$ and real $p \ge 3$, define

$$S = \sum_{\substack{\text{partner classes } F \\ z_{\eta(F)} \neq z_*}} \frac{a_{\eta(F)}}{(\Phi_* \Phi_{\eta(F)})^p}, \qquad L(z) = \sum_{\substack{\eta \in \mathcal{A} \\ z_\eta = z}} K_\eta a_\eta.$$

One has

$$S \le B_*\, 10^f \Phi_*^{-3} \left(\frac{Q}{2T}\right)^{f(p-3)} \sup_{z \in E} L(z),$$

$$S \le B_* \left(\frac{Q}{2T}\right)^{fp} \left(\frac{128 \log(2Q)}{Q^4}\right)^{fd/2} \left(\sum_{\eta \in \mathcal{A}} w(D(\eta)) K_\eta^2 a_\eta^2\right)^{1/2}.$$

For the partners with $z_\eta = z_*$ the corresponding bound is $B_* \Phi_*^{-2p} \sum_{z_\eta = z_*} K_\eta a_\eta$, with the sum over their labels.

Proof. Aligning the two residue frames at the marked primes gives there the cross matrix $-TE_{dd}$ modulo each marked prime by the self pairing of the initial frame. The two unitriangular frame changes preserve this congruence. Thus the number of integral classes over η is at most $C(M_\eta)$. In the notation of Theorem 87.21, their indices J_Z are bounded by $Q^{2ft} h_t$ from the indicated minor, and by G_Z from the full Gram determinant. Proposition 87.28 gives $G_Z = G_*$. Corollary 87.24 now bounds the number of actual partner classes above η by $B_* K_\eta$. The uniqueness in Lemma 87.22 accounts for the comparison attached to each partner.

For saturated pairs, Theorem 87.25 puts $A_2A_1^{-1}$ in the group Γ of Lemma 64.5. The slope of every represented label is therefore gz_* for some $g \in \Gamma$. The stabilizer assertion of that lemma makes g unique for each slope. Grouping the label bound by slope and applying Theorem 64.6 gives the first estimate. For the second use $\Phi_*\Phi_\eta \ge (2T/Q)^f$ on every distinct slope. Cauchy–Schwarz bounds $\sum_{\eta\in\mathcal{A}} K_\eta a_\eta$ by the square root of the weighted sum displayed above times $(\sum_{\eta\in\mathcal{A}} w(D(\eta))^{-1})^{1/2}$. Theorem 88.13 bounds the latter factor for the full set of relevant supported labels. Equal slopes give $\Phi_\eta = \Phi_*$ directly. All estimates apply first to finite sets, and then to their increasing union by positivity. □

The power $p = 2k$ is available in this arithmetic sum. It is the exponent of the angular factor in `Formalism/SharpRestriction/CuspSpanNormBounds/CongruenceAtSplitFrobeniusHolds.lean`, line 469. Application to the projected period still requires an estimate for the remaining weighted amplitudes of its specified sections and the contribution of equal slopes and nonsaturated pairs.

87.9 Summation over the nondegenerate pairs

Keep the unit column tuples and the transported real kernels from Corollary 76.45. Write $G^{\#}$ for the absolute norm over F of the transported Gram invariant and $A_j^{\#}$ for the absolute norm over E of the transported nonzero endpoint minor of size j. Put $B_\gamma^{\#} = G^{\#}\prod_j A_j^{\#}$, the product over these endpoints. For a finite set of supported nondegenerate labels let

$$H_\theta(\mathcal{A}) = \sum_{\gamma\in\mathcal{A}} w(\beta_\gamma)(B_\gamma^{\#})^\theta P_\infty^{\#}\|\widehat{K}_{\gamma^{\#}}\|_{\mathrm{HS}}^2.$$

Theorem 87.30 (A weighted real estimate sufficient for absolute summation)**.** *With $h = k-1$, $S = d(d-1)$ and $m = h - 2d + 1$, the original nondegenerate summands on the common cohomology intervals satisfy*

$$\sum_{\gamma\in\mathcal{A},c} |J_{\gamma,c}| \le \frac{A_*\,\mathfrak{m}_h\,8^{fd/2}}{\sqrt{c^*_{\rho_1}c^*_{\rho_2}}}\,e^{O(f2^{-m})}R^{-fS}Q^{f(S/2+d-2-d^2h)}H_{h/2}(\mathcal{A})^{1/2}.$$

The constant is uniform in $\mathcal{A}$.

Proof. Combine the finite mass bound in Theorem 76.44 with the exact transport of the real kernel. At a proper block endpoint j_ν the exponent in $P_\infty^{1/2}W_\gamma$ is $h_{\nu+1}$, while at the last endpoint it is zero. The remaining Gram factor is $|N_{F/\mathbb{Q}}\mathcal{G}|^{h/2}$. After transport $G^{\#} \le A_d^{\#}$, by the real Gram identity at every embedding. Each positive proper endpoint exponent is at most $d-1 \le h/4$ at large x. Thus the transported product of norm factors is bounded by $(B_\gamma^{\#})^{h/4}$. The minor rescalings give the displayed power of Q. Cauchy–Schwarz and the bound 8^{fd} on the sum of the inverse endpoint weights give the assertion. The exact rescaling and all finite measure constants are proved in Theorem 87.1. □

Proposition 87.31 (Divergence with a smaller separated weight)**.** *In the same Cauchy–Schwarz separation, putting H_θ in the real factor leaves an infinite arithmetic squared sum when $\theta \le (h-2)/2 + 1/(4f)$.*

Proof. Fix the field and interval parameters. The rational unitary transvections of Proposition 87.2 give a diagonal cross matrix whose first $d-1$ entries are fixed and whose last entry is $T(-1+t\lambda)$, $t = 1, 2, \dots$, with fixed nonzero purely imaginary λ. All are supported at the original finite level. The compatible mass and the real density both grow as t^{2fh} up to fixed positive factors. The two weights $w(\beta_\gamma)$ and $B_\gamma^{\#}$ each grow as t^{4f}. The arithmetic squared

summand on this family is $(\mathcal{M}_\gamma^{=})^2/[P_\infty w(\beta_\gamma)(B_\gamma^{\#})^\theta]$, since $a=0$. Its sum dominates a positive constant times $\sum_{t\ge 1} t^{2f(h-2-2\theta)}$, which diverges in the stated range. □

An explicit upper bound of the required size for $H_{h/2}$, or an estimate keeping the kernel and the masses together, is still needed for this period argument. Proposition 87.31 concerns the separated arithmetic sum. The exceptional strict cohomology bound and the designated integral lower bound are also open.

87.10 Saturation at the original integral level

Fix one nondegenerate supported label and a compatible row lattice Λ in its finite orbital mass. Write W_i for the spans of its two decorated frames, X_i for their isotropic parts, $W = W_0 \oplus W_1$ and $C = W^\perp$. Set $U = (\Lambda \cap W_0) \oplus (\Lambda \cap W_1)$ and $B = \Lambda \cap W$. Enlarge U at each $v \mid Q$ by adjoining the differences of two integral frame representatives of equal reduction, divided by a uniformizer. The lattice B^+ so obtained is independent of the representatives. In fact further unitriangular changes on the two lists must agree modulo the uniformizer, by primitivity of their reduction. Write $I = [B : B^+]$, $h = k-1$ and $m = h - 2d + 1$.

Theorem 87.32 (The saturation index in the original period)**.** *Give the compatible finite cells their normalized orbital measure. On the common fields and intervals, uniformly in the nondegenerate label,*

$$\Pr(I > 1) \ll f\, 2^{-m}, \qquad \mathbb{E}[I^t] \le \exp\bigl(O(f 2^{-(m-t)})\bigr) \quad (0 \le t \le m-2).$$

Let $\mathcal{L} \subset C$ consist of vectors c admitting two integral lifts $c + x_i \in \Lambda$, $x_i \in X_i$, whose difference belongs to $Q\Lambda + (\Lambda \cap X_0) + (\Lambda \cap X_1)$. Then $[\mathcal{L} : \Lambda \cap C] \mid I$, and

$$\delta(\Lambda \cap C) = |N_{F/\mathbb{Q}}\mathcal{G}|/(Q^{2df} I), \qquad \delta(M) = [M^\vee : M]^{1/2}.$$

Proof. The ordered-frame calculation behind Theorem 76.44 sums over saturations $M' \supset M_0$ with factor $[M' : M_0]^{-h}$. For the actual test this index is I_v, including at the marked places after changing to divided differences. Removing the primitive term and inserting I_v^t leaves a proportion $O(q_v^{-(m-t)})$ by summation over triangular lattice bases. Multiplying and using $\sum_v q_v^{-b} \ll f 2^{-b}$ for $b \ge 2$ proves the claims on probability. Sending c to $x_0 - x_1 \bmod U$ embeds $\mathcal{L}/(\Lambda \cap C)$ into $Q(B/U)$, a quotient of B/B^+. Finally B and its perpendicular in a self-dual lattice have discriminant modules of equal order, and $[B^+ : U] = Q^{2df}$. The local calculations and the bound on the original period summands over the exceptional cells are proved in Section 87.2. □

On the rows with $I = 1$ the compatibility module is the integral perpendicular. The comparison of two terminal flags in the same row still depends on how many pairs give a given subspace in the positive definite column.

87.11 Primitive subspaces and comparisons in a single row

Let T be the common terminal prime and $F = E^+$ the field already chosen on the interval. The isotropic subspace X and the terminal vector z of the initial flag give the positive rank-k column $P = (X + Ez)^\perp/X$. Its integral lattice L is the image of the integral row vectors in $(X + Ez)^\perp$. For $j + s = k$, $N_j(L, D)$ counts sublattices $M = L \cap EM$ of rank j and determinant $\delta(M) = |M^\vee/M|^{1/2} = D$. Give the genus of the initial column its normalized reciprocal-automorphism measure. Write $\mathfrak{m}_i$ for the mass of the standard self-dual positive genus in rank i, $R = |\operatorname{disc} F|^{1/f}$ and $\nu = 4 + \log_2(2f)$.

Theorem 87.33 (A genus bound on the common fields and intervals)**.** *For $1 \le s \le k-1$, $j = k - s$ and integral $D \ge 1$,*

$$\mathbb{E}_T N_j(L, D) \le \frac{\mathfrak{m}_j \mathfrak{m}_s}{\mathfrak{m}_k} (D/T^f)^k (T^f)^s D^\nu.$$

For $2 \le s \le k-2$ the genus factor is at most $\mathfrak{m}_2 (C/(kR))^{s(k-s)f}$, with $\mathfrak{m}_2 \le b^{Cf}$ for large x, $b = \exp(x^\varepsilon)$. The bounds use absolute constants.

Proof. Over a self-dual parent a primitive lattice and its perpendicular have opposite finite discriminant pairings. The self-dual parent is recovered as the inverse image of the graph of this identification. Counting graphs with their stabilizers and applying the definite Hermitian mass formula gives the bound $(\mathfrak{m}_j \mathfrak{m}_s / \mathfrak{m}_k) D^{k+\nu}$ for the parent genus. The full argument, including the stabilizer volumes for nonmaximal lattices and primes over two, is in Section 87.3.

In the first split factor at each $v \mid T$ the terminal perpendicular cuts the parent by a single residual hyperplane. The other factor stays dual to the full parent factor. The hyperplanes are uniform and independent conditional on the parent, by orbit–stabilizer. Each cuts a fixed primitive j-plane with index T or 1, and the probability of index 1 is $p = (T^s - 1)/(T^k - 1)$. For index T^e use the parent bound at determinant D/T^e. Averaging the factor T^{-ke} gives $(p + (1-p)T^{-k})^f = T^{-jf}$, proving the estimate. For the genus ratio, each successive mass is given by the Dirichlet functional equations on the abelian field in terms of a factorial, a power of R and an Euler product at an integer at least two. After retaining $\mathfrak{m}_2$, divide these factors in ranks $\min(j, s)$ and $\max(j, s)$ as in Proposition 87.12. □

Proposition 87.34 (The role of a primitive subspace in the pair sum)**.** *For two fixed admissible terminal flags in the same row there is at most one comparison on their middle quotients satisfying the original congruence. For comparisons whose rational common subspace has dimension j and whose determinant cutoff is $D_{\max}$, the number of distinct such subspaces in the initial column, with compatibility-index product $\ge b_0 \ge 1$, has mean at most*

$$\frac{\mathfrak{m}_j \mathfrak{m}_s}{\mathfrak{m}_k} (D_1/T^f)^k (T^f)^s D_1^{\nu+1}, \qquad D_1 = D_{\max}/b_0$$

for $1 \le j \le k-1$, and is zero for $D_1 < 1$.

Proof. Each terminal flag and the initial flag have the same reduced isotropic subspace U at $v \mid Q$. By self-duality their integral middle quotients all reduce to $U^\perp/U$. The congruence for a comparison forces the identity on this common quotient. Two comparisons then differ on the middle quotient by an integral isometry which fixes the terminal vector and acts on its positive definite perpendicular with finite order. At an odd completely split prime dividing Q the principal congruence kernel in $\mathrm{GL}_{k+1}(\mathbb{Z}_p)$ is torsion-free, so the two maps agree.

The common subspace has a full comparison lattice M_β whose image in the initial column is integral by the lifting condition defining the two compatibility modules. The cutoff is $\delta(M_\beta) i_1 i_2 \le D_{\max}$ for the two compatibility indices i_1, i_2. Passing to the primitive saturation divides the determinant by the index and hence gives determinant at most D_1. Distinct subspaces have distinct saturations. Summing Theorem 87.33 over the positive integers $D \le D_1$ proves the bound. The two integral identifications and the uniqueness on the entire middle quotient are proved in Corollary 87.14 and Theorem 87.15. □

Proposition 87.34 counts distinct common subspaces in the initial column. We now bound their determinants in terms of the two Gram matrices and then count the pairs over each subspace.

87.12 The cutoff as a consequence of the two Gram matrices

For the two nondegenerate initial–terminal pairs write $C_i = (W_0 + W_i)^{\perp}$ and set $G_{0,i} = |N_{F/\mathbb{Q}}\mathcal{G}_i|/Q^{2df}$. The actual projections of C_i into the initial and terminal positive columns are isometries. Identify the two terminal columns by their comparison. Write B for the difference between the initial and terminal pairings of a vector from each C_i, after removing their common comparison subspace of dimension $k - s$. At an archimedean place τ put $\kappa_{i,\tau} = 1$ if the leading isotropic cross minor $D_{i,d-1}$ vanishes. Otherwise put

$$c_{i,\tau} = \frac{|\tau(D_{i,d}/D_{i,d-1})|}{T}, \qquad \kappa_{i,\tau} = \frac{1 + c_{i,\tau}}{\sqrt{c_{i,\tau}^2 - 1}}, \qquad K_i = \prod_{\tau} \kappa_{i,\tau},$$

where $D_{i,d}$ is the full cross determinant and the product defining K_i uses either case at each place.

Theorem 87.35 (The comparison determinant on the original interval)**.** *For $2 \le s \le k$ the rank of B is $s - 1$ or $s - 2$. The comparison lattice M_β in Proposition 87.34 and its two compatibility indices satisfy*

$$\delta(M_\beta) i_1 i_2 \le \sqrt{G_{0,1} G_{0,2}} \begin{cases} (2/Q)^{(s-1)f}, & \operatorname{rank} B = s - 1, \\ (2/Q)^{(s-2)f} K_1 K_2 / T^f, & \operatorname{rank} B = s - 2. \end{cases}$$

These give cutoffs in the genus bound for the distinct common subspaces when $2 \le s \le k - 1$.

Proof. In the product of the two positive columns the two compatibility lattices are intersections of the two rational graphs with the same integral lattice of pairs whose residues agree at Q. Their intersection on the common graph is precisely M_β. Orthogonal projection away from it gives rank-$(s-1)$ lattices H_i with $\delta(\mathcal{L}_i) = \delta(M_\beta)\delta(H_i)$. The difference of the two forms has norm at most two and takes values in $Q\mathcal{O}_E$. After removing the common graph the two graphs are disjoint isotropic $(s-1)$-planes in a space of signature (s, s). The radical on either side injects into a space of signature $(1, 1)$ and has dimension at most one. On the quotient of rank $n = \operatorname{rank} B$, the determinant of the pairing takes values in $Q^n\mathcal{O}_E$ on exterior lattices. The archimedean norm bound thus gives a lower bound $(Q/2)^{2nf}$ on the product of the two projected lattice determinants.

If a radical line occurs, write R_i for its lattice on side i, I_i for the saturation index of Theorem 87.32. Against the opposite pair a, a radical vector gives two integral lifts with equal perpendicular projection. The coefficient of the initial terminal vector in their difference belongs to a fractional ideal $J \supset \mathcal{O}_E$ with $[J : \mathcal{O}_E] \le I_a/i_a$. On the projected radical its absolute value is bounded by $\kappa_{a,\tau}/\sqrt{T}$ times the local norm. It is nonzero for every nonzero radical vector, as the two normal components have equal length and the map to their two terminal coefficients is an isometry of indefinite planes. Multiplication gives $\delta(R_i) \ge T^f i_a/(I_a K_a^2)$. This cancels the saturation factors in $\delta(\mathcal{L}_i) i_i^2 = G_{0,i} i_i / I_i$. Combining the determinant bounds proves the formula. Both normal-line calculations, including the zero-minor case, and the ideal and index computations are in Section 87.5. □

This determines the cutoff to use in the genus bound. Theorem 87.36 below incorporates the number of ordered pairs over each primitive subspace.

87.13 Counting the ordered flags above each subspace

Put $t = d-1$, $n_T = T^f$, and let Z be an integral cross Gram matrix at the original level, modulo the two integral unitriangular frame changes. Write C_Z for its first t rows, $J_Z = [\mathcal{O}_E^t : C_Z \mathcal{O}_E^d]$

and $G_Z = |N_{F/\mathbb{Q}}\mathcal{G}|$. For the nondegenerate frame sums considered here $1 \le J_Z \le G_Z$. Set

$$A_s(Z) = 9^{sf}\max(1, G_Z/n_T)^s J_Z^{s+t}.$$

Theorem 87.36 (Ordered flag pairs with prescribed integral Gram matrices)**.** *Fix the two integral cross classes Z_1, Z_2 and impose $\delta(M_\beta)i_1i_2 \le Y$, $Y \ge 1$. For $1 \le s \le k-1$, $j = k-s$, the normalized mean number of their ordered comparison classes of common dimension j in the initial row measure is at most*

$$8\,\frac{\mathfrak{m}_j\mathfrak{m}_s}{\mathfrak{m}_k}(Y/n_T)^k\, n_T^{s+4t} Y^{\nu+1+2t} A_s(Z_1)A_s(Z_2), \qquad \nu = 4 + \log_2(2f).$$

For $2 \le s \le k-1$ one may insert the corresponding rank-dependent cutoff of Theorem 87.35. In codimension one, fix a first flag with $I_1 = 1$. The number of partners of class Z_2 with $I_2 = 1$ is bounded per row by $(n_T^2 G_{Z_1}/Q^{2df})^t A_1(Z_2)$.

Proof. In an integral Witt splitting of the initial row, write the columns of a terminal frame in upper, middle and lower blocks. The lower block is C_Z. If a prescribed rank-j sublattice $P \subset L$ lifts to the integral frame perpendicular, its lift differs from the initial column embedding by t integral functionals on P. Integral unipotent shears leave at most $(n_T^2[L \cap EP : P])^t$ inequivalent choices of these functionals. For fixed choices, contract the middle block with the cofactor vector of C_Z, which spans its rational kernel. Projecting off the terminal vector gives a vector in an s-dimensional positive space, with prescribed squared norm $\mathcal{G}/T$. Differences between possible values belong to an integral lattice. Comparing disjoint balls in the product over the embeddings gives at most $9^{sf}\max(1, G_Z/n_T)^s$ values. After this choice, the middle-block differences factor through C_Z, giving at most J_Z^s classes by restriction of integral homomorphisms. The remaining upper-block differences have skew-adjoint coefficient and give at most J_Z^t classes. This proves the single-flag bound with factor $A_s(Z)$.

In the common subspace put $S = L \cap EM_\beta$, identifying the comparison lattice with its initial image. Let $a = [S : M_\beta]$, and let b_i be the index in M_β of the sublattice lifting to the i-th integral perpendicular. Then $b_i \le i_i$ and $\delta(S)ab_1b_2 \le Y$. The number of sublattices of index n in a projective rank-j lattice is at most $n^{j+\log_2(2f)}$, by successively taking simple quotients at a prime dividing the remaining index. Counting the three sublattices this way, using the single-flag bound twice and Theorem 87.33 for S, leaves the summable series $\sum_{a,b_1,b_2}(ab_1b_2)^{-s-5} < 8$. Uniqueness of the comparison persists after independent changes of terminal frame because both changes act trivially on the marked middle quotient. For $s = 1$ the two rational graphs coincide. When both flags satisfy $I_i = 1$, the first integral perpendicular projects to a fixed sublattice of determinant G_{Z_1}/Q^{2df} lifting to the second, so one application of the single-flag count suffices. The coordinate and measure arguments are proved in Sections 87.6 and 87.7. □

For codimension one the terminal prime also puts a restriction on the two Gram classes that can occur together.

87.14 Which codimension-one pairs have to be summed

For the two Gram classes in Theorem 87.36, put $D_i = \det Z_i$, let R_i be the leading minor of size $d-1$, and write $\mathcal{G}_i = D_i\bar{D}_i - T^2R_i\bar{R}_i$. Assume the two frame sums are nondegenerate, as in that theorem. Then $\mathcal{G}_i$ is totally positive.

Theorem 87.37 (Separation at the original terminal prime)**.** *Suppose a comparison of common dimension $k-1$ occurs on a row satisfying $I_1 = I_2 = 1$. Either $R_1/D_1 = R_2/D_2$ in E, or at one real place and for one of the two flags,*

$$R_i \neq 0, \qquad \frac{\tau\mathcal{G}_i}{T^2|\tau R_i|^2} \le \frac{2}{\sqrt{1 + (T/Q)^2} - 1}.$$

All quantities refer to the field E_x and the primes selected on the interval in (13.12).

Proof. The two positive comparison graphs have dimension $k-1$ and coincide. Their perpendiculars in the initial and terminal columns are lines N, N'. For either flag pair, lift a vector from each line to the corresponding frame perpendicular. The coefficients of the difference along the two terminal vectors give an isomorphism $A_i : N \oplus N' \to E^2$. The difference of the two squared lengths is T times the difference of the squared coefficient magnitudes. Thus $\gamma = A_2 A_1^{-1}$ is unitary for $\operatorname{diag}(1,-1)$. The integral congruence in Theorem 87.25 gives $(\gamma-1)\mathcal{Z} \subset T\mathcal{Z}$, where $\mathcal{Z} = \{(u,v) \in \mathcal{O}_E^2 : u+v \in Q\mathcal{O}_E\}$. Its proof extends a functional with values $u, -v$ on the two terminal vectors to the same self-dual row and compares its two integral lifts at the marked primes.

If $c = \gamma_{21}$ vanishes, γ is diagonal. Applying the congruence to $(1,-1)$ makes each diagonal entry congruent to 1 modulo T. Its absolute value is one at every embedding, so the norm of its difference from one forces equality as $T > 2$. In the common initial normal line the signed cofactor vectors of the two frames have coefficient pairs $(D_i/T, R_i)$. Since $A_1 = A_2$, these pairs are proportional over E.

If $c \neq 0$, apply the congruence to $(Q, 0)$. The nonzero integer Qc/T of E gives $\prod_\tau |\tau c| \geq (T/Q)^f$. In unit bases, after scaling the output by $\sqrt{T}$, the diagonal entry magnitudes of A_i are $\sqrt{1+V_{i,\tau}^2}$ and the off-diagonal magnitudes are $V_{i,\tau} = T|\tau R_i|/\sqrt{\tau \mathcal{G}_i}$. These follow by dividing the cofactor vector by its length $\sqrt{\tau \mathcal{G}_i / T}$. The product $A_2 A_1^{-1}$ therefore gives $|\tau c| \leq 2H\sqrt{1+H^2}$ with $H = \max_{i,\tau} V_{i,\tau}$. Taking the f-th root and solving for H^2 proves the estimate. The same argument, with the annihilators of the two saturation quotients retained, is given in Theorem 87.26. □

The fixed-class partner bound can consequently be set to zero whenever both alternatives fail. The sum over the surviving classes, the designated integral Bessel pairing in Corollary 74.7, and the remaining cohomology orders in (13.12) still enter the proof of the main theorem.

87.15 Primitive sublattices in the initial column

Work on the common intervals with polynomial conductor N_x. Keep the terminal prime T and the two level primes whose product is Q. Write $f = [F : \mathbb{Q}]$, $n = T^f$ and $R = |\operatorname{disc} E|^{1/(2f)}$. For an integral Hermitian lattice B write $\delta(B) = [B^\vee : B]^{1/2}$. The initial column of a row with first flag $X + Ez$ contains the lattice L of Section 87.4. We select a set of initial columns once, before specifying the second flag.

Theorem 87.38 (Simultaneous lower bounds for primitive determinants)**.** *Fix $A > 0$. There is a constant C_A such that, for sufficiently large x on the common intervals, a fraction at least $1 - \exp(-Af(k-1))$ of the original refined class coordinates has the following property. For every $1 \leq j < k$ and every primitive rank-j sublattice $B \subset L$,*

$$\delta(B) > Y_j, \qquad Y_j = n^{j/k}(kR/C_A)^{fj(k-j)/k}.$$

On the same coordinates every primitive hyperplane also satisfies

$$\delta(B) > Y, \qquad Y = n\big(kR/(2\pi \exp(A+4))\big)^f.$$

These coordinates form a union of full terminal scalar orbits. The retained fraction is the same for every word, for each specified terminal character and for the space with zero sums in the individual scalar coordinates.

Proof. Put $s = k - j$ and $\Xi_j = \mathfrak{m}_j \mathfrak{m}_s / \mathfrak{m}_k$. Sum (87.8) over the positive integers $D \le y$. The probability of even one primitive sublattice of this rank and determinant at most $y \ge 1$ is at most

$$\Xi_j (y/n)^k n^s y^{\nu+1}, \qquad \nu = 4 + \log_2(2f).$$

For $2 \le j, s$, Proposition 87.12 bounds Ξ_j by $\mathfrak{m}_2 (C/(kR))^{fjs}$, with $\log^+ \mathfrak{m}_2 = O(f \log(2N_x))$. For the two end ranks, $\Xi_1 = \Xi_{k-1} = \mathfrak{m}_1 / a_k$. The mass formula in the same proof gives $\mathfrak{m}_1 \le 2^{1-f} R^f$ and

$$a_k \ge R^{f(k-1/2)} \big((k-1)!/(2\pi)^k\big)^f \zeta(k)^{-f}.$$

Stirling's formula gives $\log \Xi_1 \le -fk \log(kR/(2\pi e)) + O(f \log(2kR))$. Thus for all ranks $\log \Xi_j \le -fjs \log(kR/C) + O(f \log(2kN_x))$ after enlarging C.

At $y = Y_j$ the factor n cancels. The remaining error is uniformly $o(fjs)$. Indeed $js \ge k - 1$, $R \le N_x$, and

$$\frac{\log Y_j}{fjs} = \frac{\log T}{ks} + \frac{\log(kR/C_A)}{k}.$$

Here $\nu = O(\log x)$, $\log N_x = O(\log x)$, $k \asymp x^{9/2+10\varepsilon}$, and $\log T = (17 + o(1)) m x^{\varepsilon}$ with $m \asymp x^{1/2-\varepsilon/16}$ by Theorem 71.9. Choose $\log(C_A/C) \ge A + 4$. For the additional hyperplane test substitute $y = Y$ and $s = 1$ in the same probability bound. Its logarithm is at most

$$-(A+3)fk + O\big((\nu+2) f \{\log T + \log(2kR)\}\big) = -(A+3)fk + o(fk).$$

Adding the k failure probabilities now gives the assertion.

For the counting measure, Lemma 87.10 shows that forgetting all markings pushes the normalized mass to the terminal genus measure. At the two auxiliary levels the class stabilizers are trivial by Corollary 9.14; each refined class therefore carries the same mass, and its space of real covectors has the same dimension for a fixed word. Multiplication of a terminal covector by a nonzero residue scalar fixes its kernel, hence fixes L. Each retained scalar orbit contributes one copy of every character and the same number of vectors to the zero-sum space. This proves the fraction assertions. □

Corollary 87.39 (A necessary condition on an original orbital cell)**.** *Take any retained initial column. For a relevant original finite cell with nonzero Gram determinant put $G = |N_{F/\mathbb{Q}} \mathcal{G}|$ and let $I = [B : B^+]$ be its congruence index in Proposition 87.4. Then*

$$G > Q^{2df} I Y, \qquad d = m/2.$$

The exclusion is simultaneous for every second flag above this column, including those with $I > 1$.

Proof. The common positive perpendicular C has rank $k - 1$. For these labels the two frames have disjoint isotropic parts: their full cross minor is nonzero by $\mathcal{G} > 0$. Any common isotropic vector would give a nontrivial left kernel of the cross matrix. The projection to the first definite column is therefore isometric on C. It sends $\Lambda \cap C$ into L by the definition of the column lattice. Its saturation in L is a primitive hyperplane, and passing to this saturation divides its determinant by the positive integral lattice index. Theorem 87.32 gives $\delta(\Lambda \cap C) = G/(Q^{2df} I)$. The second inequality of Theorem 87.38 now proves the claim. □

87.16 Full-rank cells with transverse isotropic subspaces

In the first moment for an identical word, Theorem 76.2 permits a full-rank common definite space. Take $s = d - 1 \ge 1$ and let $\mathcal{S}$ be the relevant, finitely supported labels of this kind for which $X_0 \cap X_1 = 0$. We count each rational orbit once.

Proposition 87.40 (The relative negative line and the isotropic blocks)**.** *A label in $\mathcal{S}$ is determined by a composition of s, its nonzero reversed-block scalars, and an element $\lambda \in E^{\times}$ with $\lambda\bar{\lambda} = 1$. If $a = D_s/Q^s$, then $a \in \mathcal{O}_E \setminus \{0\}$ and*

$$D_d = -T\lambda D_s, \qquad Ta(1-\lambda) \in Q\mathcal{O}_E.$$

For fixed block scalars, the number of possible λ is at most $3^f\,|\mu(E)|\,\tau_E((a))$, where τ_E counts the integral ideal divisors.

Proof. By Theorem 64.3, $Y_0 \cap Y_1$ is a negative line over E. Projection identifies it with Y_0/X_0. Choose its unique vector z_* inducing the original final vector there. The analogous vector for Y_1 is λz_*; both have norm $-T$. These replacements of the terminal representatives add vectors of the corresponding isotropic flags and preserve their decorations. Each X_i is perpendicular to z_*. The cross pairing on $X_0 \times X_1$ is nonsingular by the kernel argument in Proposition 88.8. The full cross matrix now has this s-square in its upper left and the scalar $-T\lambda$ in its lower right, with both rectangular corners zero.

Independent upper unitriangular changes of isotropic representatives reduce the s-square to a monomial matrix by successive pivot elimination as in Lemma 88.10. For an inversion $i < j$, $w(i) > w(j)$, write b_i, b_j for its pivot entries. At a real place the two elementary changes with respective entries u in position ij and $-\bar{u}b_i/b_j$ in position $w(j), w(i)$ preserve the cross matrix. They give an isometry of $X_0 + X_1$, which is nondegenerate. Extend it by the identity on the negative line and on the definite perpendicular. This element belongs to both unipotent radicals: it acts unitriangularly on the two flags and trivially on their definite quotients. Equality of the characters for every complex u implies that every adjacent descent decreases by one and that the two adjacent pivots there are negatives of each other. Thus the descending runs are disjoint consecutive intervals in increasing order, with the successive entries in each run alternating in sign. Their nonzero leading minors are exactly the endpoints, and a run of length h multiplies the preceding endpoint by c^h in the alternating convention.

These data determine the Gram matrix on a basis consisting of both isotropic lists and z_*. Their span is nondegenerate. If two pairs give the same data, the isometry of these spans carrying one basis to the other extends to the ambient Hermitian space by Witt's theorem. It carries both decorated frames to the specified frames because their terminal representatives are $z_*, \lambda z_*$. This proves the required injectivity on rational orbits. The finite congruences of Proposition 88.11 apply to every supported pair and give both assertions about a.

Now $b = Ta\lambda$ is integral and $b\bar{b} = T^2 a\bar{a}$. At a prime of E fixed by conjugation the exponent of (b) is determined. At a split pair the two exponents are nonnegative and have a fixed sum $e + 2v(T)$, where $e = v_{\mathfrak{p}}(a) + v_{\bar{\mathfrak{p}}}(a)$. There are $e+1$ choices off T and at most $3(e+1)$ over T, since T splits completely. Each $e+1$ is at most the product of the two individual divisor counts of (a). Thus there are at most $3^f\tau_E((a))$ possibilities for (b). Two generators satisfying the norm equation for the same ideal have ratio an integral unit of absolute value one at every complex embedding. Such a unit is a root of unity: the monic polynomials of its powers have bounded integral coefficients and bounded degrees, so the powers take only finitely many values. Counting these units gives the claim. □

Theorem 87.41 (Weighted count of the transverse full-rank cells at the original level)**.** *Use $w_\gamma = \prod_{j:D_j\neq 0}\prod_{\tau}(1+|\tau D_j|^2)^2$ and put $H_a = |N_{E/\mathbb{Q}}a|^{1/(2f)}$. For $H_0 \geq 1$ and $0 \leq u < 2$,*

$$\sum_{\substack{\gamma\in\mathcal{S}\\ H_a\geq H_0}} w_\gamma^{-1} \leq 3^f\,|\mu(E)|\,H_0^{-2f}(9M_u+2)^{fs}T^{-4f}Q^{-(6+u)sf}.$$

At $H_0 = 1$ the bound is $\exp(-(264 - o(1))f d\, x^{\varepsilon})$. It holds on the entire original class space, in particular for every retained terminal character space.

Proof. All $|\tau\lambda| = 1$, so for fixed isotropic blocks the product of terminal factors is at most

$$\prod_{\tau}(T|\tau D_s|)^{-4} = T^{-4f}Q^{-4sf}|N_{E/\mathbb{Q}}a|^{-2}.$$

For an integral ideal, its divisor count is at most its norm, since $e + 1 \le 2^e \le (N\mathfrak{p})^e$ at every prime of exponent e. Proposition 87.40 therefore bounds the sum of the terminal factors by $3^f|\mu(E)|T^{-4f}Q^{-4sf}H_0^{-2f}$ in the indicated range. For the isotropic endpoints $j \le s$, the minors are nonzero integers divisible by Q^j in $\mathcal{O}_E$. The product-distance packing and radial integration of Theorem 88.13 at each such endpoint yield $(9M_u)^f Q^{-(2+u)jf}$. A prescribed endpoint tuple has at most the product of its block lengths as its scalar multiplicity by the equations of Proposition 87.40. Summing over compositions as in that theorem and retaining the power at the last isotropic endpoint s gives at most $(9M_u + 2)^{fs}Q^{-(2+u)sf}$. Multiplication proves the bound.

If $w = |\mu(E)|$, then $\varphi(w) \le 2f$ and $w/\varphi(w)^2 \le 2$, by multiplying $p^{2-e}/(p-1)^2$ over prime powers $p^e \| w$. Hence $w \le 8f^2$. Choose $u = 2 - 1/\log(2Q)$. The factors preceding $T^{-4f}Q^{-(6+u)sf}$ at $H_0 = 1$ have logarithm $o(f d\, x^{\varepsilon})$. Using $s = d - 1$, $\log Q = (16 + o(1))x^{\varepsilon}$ and $\log T = (34 + o(1))dx^{\varepsilon}$ gives $136 + 128 = 264$. Restricting to coordinates or to labels meeting a character space can only decrease the positive label sum. □

Corollary 87.42 (A restriction on the terminal poles of the relative scalar)**.** *For a label as above with $\lambda \ne 1$, let l count the primes $\mathfrak{t} \mid T$ of E at which $a(1-\lambda)$ is nonintegral. Then*

$$0 \le l \le f, \qquad T^l(2H_a)^{2f} \ge Q^{2f}.$$

For every fixed B, $H_a \le x^B$ forces $l \ge (32/17 - o(1))f/m$ on the common intervals. If λ is integral and different from one, then $a \in Q\mathcal{O}_E$ for all sufficiently large x.

Proof. The congruence $Ta(1-\lambda) \in Q\mathcal{O}_E$ implies integrality off T, divisibility by Q there, and poles of order at most one over T. Since $\lambda\bar{\lambda} = 1$ and a is integral, a pole can occur on at most one member of each split pair. Multiplying the principal fractional ideal $(a(1-\lambda))$ by those l primes gives a nonzero integral ideal divisible by $Q\mathcal{O}_E$. Its norm is at most $T^l 2^{2f}|N_{E/\mathbb{Q}}a|$ by the archimedean triangle inequality. This proves the first assertion. Take logarithms and use $\log H_a = O(\log x)$ for the second. An integral norm-one λ is a root of unity by the argument in Proposition 87.40. The primes dividing Q exceed $8f^2$ eventually, and hence do not divide its order. Reduction preserves the order of a root of unity of prime-to-residue-characteristic order: the principal units have no torsion of that order by the binomial expansion. Thus $1 - \lambda$ is a unit above Q, proving the last assertion. □

In Proposition 88.8 the denominator of the rational action on a transverse cell has root norm at most H_a. Small H_a thus permits the normalized character estimate, while the complementary range has the factor H_0^{-2f} above. The full signed first moment still contains the orbital amplitudes and cells with nonzero isotropic intersection.

87.17 The weighted sum over all relevant rational labels

Let $\mathcal{S}_t$ consist of the relevant, finitely supported labels for which $\dim(X_0 \cap X_1) = s - t > 0$, $1 \le t < s$. Let $\mathcal{R}_*$ consist of all relevant rational labels outside the stabilizer of X_0 that have an original finite cell on a retained initial column of Theorem 87.38. These sets count each rational double coset once. For every label use the same weight $w_\gamma = \prod_{1 \le j \le d : D_j \ne 0} \prod_{\tau}(1 + |\tau D_j|^2)^2$.

Theorem 87.43 (The complete arithmetic weight and its leading scale)**.** *For $1 \le u < 2$,*

$$\sum_{\gamma \in \mathcal{S}_t} w_\gamma^{-1} \le |\mu(E)| 16^f (9M_u + 2)^{ft} Q^{-(2+u)ft}.$$

On restricting to $\lambda \neq 1$ on these labels the right hand side can also be multiplied by $(2/Q)^{(u-1)f}$. On the common fields and intervals the sum over $\mathcal{R}_*$ satisfies

$$\log \left(\sum_{\gamma \in \mathcal{R}_*} w_\gamma^{-1} \right) = -(64 + o(1)) f x^\varepsilon.$$

Proof. For a fixed set of prefix blocks, put $a = D_t/Q^t$ and $b = a\lambda$. By Proposition 72.20 the element b is integral and $b\bar{b} = a\bar{a}$. The split-prime divisor count of Proposition 87.40, now without the terminal factor T, gives at most $\tau_E((a))$ possible ideals (b). Each admits at most $|\mu(E)|$ such generators. By Theorem 88.9 is at most one rational label for given block scalars and λ, and all nonzero endpoint weights are in the prefix.

For any integral ideal $\mathfrak{a}$ one has $\tau_E(\mathfrak{a}) \le 16^f (N\mathfrak{a})^{1/2}$. Indeed $e + 1 \le (N\mathfrak{p})^{e/2}$ for prime ideals of norm at least four. For norm two or three one has $e + 1 \le 2(N\mathfrak{p})^{e/2}$; there are at most $4f$ such primes over the two rational primes. Multiply these inequalities for the ideal factorization.

Count prefix endpoint values in boxes of geometric mean radii R_j as in Theorem 88.13. At endpoint j the count conditional on earlier values is bounded by $9^f (R_j/Q^j)^{2f} \mathbf{1}_{R_j \ge Q^j}$. At the last endpoint t, the preceding divisor estimate adds at most $16^f (R_t/Q^t)^f$, since $|N_{E/\mathbb{Q}} a|^{1/2} \le (R_t/Q^t)^f$. Use exponent $u - 1$ on the last support indicator and u on the others. The total power of every R_j/Q^j is then $(2 + u)f$. When $\lambda \neq 1$ the final box must satisfy $R_t/Q^t \ge Q/2$ by Proposition 72.20, which supplies the extra factor in the statement. Integrating against the radial densities of Theorem 88.13 gives $(9M_u)^f Q^{-(2+u)jf}$ per endpoint. The equations for the prefix scalars give multiplicity at most the product of the block lengths. The binomial sum over compositions, keeping the power of Q^{-1} from the last endpoint t, proves both bounds.

For the total sum take $u = 2 - 1/\log(2Q)$. Write $D = (9M_u + 2)^f Q^{-(2+u)f}$. Then $\log D = -(64 + o(1)) f x^\varepsilon$, and $\sum_{t=1}^{s-1} D^t \le D/(1 - D)$ for large x. The factor $|\mu(E)| 16^f$ has logarithm $o(f x^\varepsilon)$ by $|\mu(E)| \le 8f^2$. This covers every nontransverse label outside the stabilizer. For transverse labels of zero Gram determinant, Theorem 64.3 gives the full-rank case of Theorem 87.41, whose bound is $\exp(-(264 - o(1)) f d\, x^\varepsilon)$. For all nonzero Gram determinants on the retained columns, Theorem 88.21 gives $\exp(-(132 - o(1)) f d\, x^\varepsilon)$. As $d \to \infty$, these two contributions preserve the upper bound.

For the lower bound, Proposition 72.21 puts the same label of weight $(1 + 16Q^2 q)^{-2f}$ on every refined column, hence on a retained one. Here $q \le N_x$ and N_x is polynomial on the interval. Therefore its logarithm is $-4f \log Q + O(f \log(2q)) = -(64 + o(1)) f x^\varepsilon$. This proves the two-sided assertion. □

The rank-one example also gives equal oriented coordinates in the individual-character trace and the identity on the definite real space. Consequently a bound for the signed period needs information on the orbital integrals already at this label. The weighted count controls the arithmetic multiplicity of labels, while the finite cell measures and the prescribed real functions determine their contributions.

88 Terminal frequencies and stabilizers

88.1 Orthogonality for all flags in a fixed isotropic subspace

Let $F_+ = E^+$, let $L = \langle u_1, \dots, u_s \rangle_E$, and let $Q \subset G$ be the parabolic subgroup stabilizing L. Its rational elements may change all the smaller spaces of the original flag. Write $\mathbb{A} = \mathbb{A}_{F_+}$. The seed Φ and its norm on $B(F_+)\backslash G(\mathbb{A})$ are those of Sections 70.8 and 78.14. The row levels at g_1, g_2 have unipotent reduction by Proposition 72.6.

Theorem 88.1 (Summation over the stabilizer of the isotropic subspace)**.** *For the fields, words, frequencies and terminal levels of Theorem 78.16, summation of the seeds over $B(F_+)\backslash Q(F_+)$ defines an isometry from their span in $L^2(B(F_+)\backslash G(\mathbb{A}))$ to $L^2(Q(F_+)\backslash G(\mathbb{A}))$, onto its image. For seeds of compact support modulo $B(F_+)$ the sum is the ordinary locally finite series. For the prescribed square-integrable seeds it is the limit in L^2 of this operation on compact truncations. Thus all selected columns have the same positive factor C_K in their probability norms after this summation, and the trace estimate (78.5) holds for its full Gram operator on the refined zero-sum space.*

Proof. Let N_* be the unipotent radical of Q. The Levi quotient acts by $\mathrm{GL}_E(L) \times U(W)$, using the chosen Witt splitting. The unipotent group for the full flag is $N_* \rtimes U_s$, where U_s consists of upper unitriangular transformations on L. Matrices on the dual isotropic space are determined by these actions. The subgroup B adds the compact-at-infinity group $H = U(W_0)$ fixing $e \in W$.

First suppose the real-place functions have compact support modulo B_∞. Then the seeds have compact support modulo $B(F_+)$. Indeed the finite support reduces to finitely many classes $h_i K$, with compact finite part. Above each class the remaining fibre from the real period group is $\Lambda_i\backslash B_\infty$. Its unipotent factor is the quotient of the compact adelic unipotent quotient by the compact open subgroup in Lemma 70.23. Its other factor is the compact H_∞. A compact part of $B_\infty\backslash G_\infty$ can be covered by finitely many relatively compact local sections. This gives the assertion. For any intermediate rational subgroup the ordinary periodization is now locally finite. To see this, lift the compact support to a compact set of representatives in $G(\mathbb{A})$, using local sections for its discrete rational group. On a compact test set in $G(\mathbb{A})$, contributing rational cosets have a representative in the product of this compact lift and the inverse test set. A closed discrete rational group meets that product in only finitely many elements. The resulting function has compact support on the intermediate quotient as the image of a compact set.

We use the following integration argument at each step. Suppose $J' \subset J$ are consecutive groups and $D_0 \subset J'$ is unipotent and normal in J. A summand at index γ and frequency β transforms along $[D_0] = D_0(F_+)\backslash D_0(\mathbb{A})$ by the character of the input conjugated by γ. If two simultaneously supported terms of different index or frequency have different characters, their products integrate to zero on this compact quotient by right translation. The squared sum then integrates to the sum of the diagonal squares, after collecting the inputs of each frequency. Integration over $J(F_+)D_0(\mathbb{A})\backslash G(\mathbb{A})$ and unfolding the diagonal over $J'(F_+)\backslash J(F_+)$ gives the sum of the input squared norms. For the groups below $J(F_+)D_0(\mathbb{A})$ is closed as the inverse image in $J(\mathbb{A})$ of the rational points of its displayed Levi quotient. It is unimodular by the product formula for the rational determinant on $\mathrm{Lie}(D_0)$. These facts give the invariant quotient integration formula with fibre $[D_0]$. The sums are locally finite on the compact fibre. The nonnegative diagonal integrals are finite by unfolding, so this calculation also proves square integrability of the output. Inputs at the same frequency may have any column types, since their sum transforms by that one unipotent character.

For $0 \le t \le s-1$, put

$$M_t = \left\{ \begin{pmatrix} A & Y \\ 0 & u \end{pmatrix} : A \in \mathrm{GL}_t(E),\ u \in U_{s-t}(E) \right\} \subset \mathrm{GL}_s(E)$$

in rational coordinates, with $M_0 = U_s(E)$. Use the corresponding algebraic subgroups to set $J_t = N_* \rtimes (M_t \times H)$, so $J_0 = B$. At the step $J_{t-1} \subset J_t$, $1 \le t < s$, take D_t to be N_* times the unipotent block part with $A = 1$, with identity H-component. It is normal in J_t and contained in J_{t-1}. Its character is the restriction of χ_β. Every element of J_{t-1} preserves this restriction. The upper unitriangular part acts by inner conjugation in the full flag unipotent group, H fixes the character, and the block GL_{t-1} fixes the last coordinate of a column of length t. The surviving simple coordinates of D_t in the upper matrix start at position $(t, t+1)$. The terminal character on N_* is also fixed, since M_{t-1} fixes the last coordinate covector on L. In particular the sum formed up to J_{t-1} still transforms by $\chi_\beta|_{D_t}$, as is seen term by term.

Now an element of J_t with top block A can preserve that character only if the last row of A is the last coordinate covector. Test the unipotents with arbitrary column in rows $1, \dots, t$ of column $t+1$. The character reads its last entry, and conjugation by the diagonal block replaces the column by its product with A on the left. The unipotent factor and H already preserve the character. Nondegeneracy of the trace pairing gives the asserted row equality. It says exactly that the element belongs to J_{t-1}. Also J_t fixes the terminal character on N_*, so two different frequencies remain distinct there. The integration argument proves isometry at each of these steps.

For the last inclusion $J_{s-1} \subset Q$ use $D_0 = N_*$. This includes $s = 1$, when there were no preceding steps. Let b_s be the last coordinate covector on L. On the block $X : W \to L$ the character is the trace character with coefficient $\beta b_s X e$. All linear maps X can be tested at a complex place. Indeed the unipotent isometry in Lemma 78.14 realizes any one row by using its corresponding hyperbolic plane. Products add their W-to-L blocks. Other blocks introduced by multiplication have zero character. Every element of J_{s-1} fixes b_s and e in the Levi, so the intermediate input has the stated N_*-character.

Suppose two terms at indices γ, γ' have the same character and meet at the same point. Write (A, M) for the Levi part of $\lambda = \gamma\gamma'^{-1}$. Test all X in the equality of characters. It gives the equality of nonzero simple tensors

$$\beta(b_s A) \otimes M^{-1} e = \beta' b_s \otimes e.$$

Indeed conjugation sends X to AXM^{-1}, and any unipotent component acts trivially on the character. Hence $b_s A = a b_s$, $\beta a M^{-1} e = \beta' e$, for some $a \in E^\times$. These identities hold over E by injectivity of its complex embedding. In particular $a\bar{a} = \beta'\bar{\beta}'/(\beta\bar{\beta})$.

The finite support of the intermediate sum is contained in $J_{s-1}(\mathbb{A}_f)K$. Consequently $\lambda_f = b_1 k b_2$ with $b_i \in J_{s-1}(\mathbb{A}_f)$, $k \in K$. All these elements preserve L. The b_i preserve $\ker b_s \subset L$ and act trivially on its line quotient. The equation for $b_s A$ shows that k also preserves that hyperplane and acts on the quotient by a. Compactness gives that a is a unit everywhere at finite places. The norms $1, T$ give $\beta = \beta'$. As in Lemma 78.14, $a\bar{a} = 1$ makes this unit a root of unity, and the unipotent reductions at the two distinct auxiliary primes give $a = 1$. Thus $M \in H(F_+)$ and $A \in M_{s-1}$. We have proved $\lambda \in J_{s-1}(F_+)$, which gives equal rational indices. This proves the required orthogonality at the last step. The successive locally finite sums are the sum over $B(F_+)\backslash Q(F_+)$, by partition into cosets.

For general W truncate on an increasing compact exhaustion of $B_\infty\backslash G_\infty$, using scalar cutoffs bounded by one and equal to one on increasing compact sets. The input seeds converge in L^2 by the quotient norm calculation of Theorem 70.24 and dominated convergence. Differences

of compact truncations satisfy the same isometry, for which the real vectors need no unit normalization. Hence the output is Cauchy in L^2 and its limit is independent of the cutoffs. The single-frequency norm equalities and mixed orthogonality, followed by polarization within a frequency, give the isometry on the full span. The final assertions follow from Theorem 78.16 and Corollary 78.17. □

The remaining periodization is over $Q(F_+)\backslash G(F_+)$. Distinct indices here correspond to distinct rational isotropic s-dimensional subspaces. The preceding result evaluates all the contributions with the same such subspace. It leaves open the cancellation between distinct subspaces for the prescribed theta vectors and their comparison on the integral lattice.

88.2 The global Gram matrix at the terminal prime

Use the fields $E = B_J(\sqrt{-q}, \sqrt{-d_0})$ and intervals of Theorem 13.11. In this paragraph r is the row rank and $f = [E^+ : \mathbb{Q}]$. In (9.1) choose

$$L = 8(f+1)(r+1)^3, \qquad M = L + 8r^2 + 4, \qquad \Delta = M + 1,$$
$$H_0 = \max\{M+1, \lceil k + m(t+3) + r^{19} f^5 x^{\varepsilon} \rceil\}.$$

These spacings are polynomial in x. The smallest B_τ in the formula for $b_{j,\tau}$ exceeds every fixed power of x eventually. The row therefore has gaps greater than M at every real place. The prime and word construction of Theorem 9.13 applies. The choice changes the row and column weights within that construction. It uses the same fields and the same x as the cohomology estimate.

Theorem 88.2 (Absolute convergence at the prescribed finite level)**.** *For these parameters, the untruncated ordinary Bessel periodizations on the normalized inputs $J_U(U \otimes D)$ converge absolutely everywhere at K^0. The resulting L^2 vectors already lie in the prescribed real row and its lowest compact type. The Gram matrix on D equals $G_D = AA^*$ of Theorem 65.26. Its difference from the identity is the absolutely convergent sum of the orbital integrals outside the rational flag parabolic. The zero-determinant cells in the blocks between distinct words contribute zero. In addition the height moment $\|H^{2D_0} J_c v\|_2$ is finite for every $v \in U$.*

Proof. For a compact-finite matrix coefficient of the row, tensoring with M defining representations bounds the coefficient times the M-th power of the operator norm. The gaps just verified ensure that the shifted discrete series all stay in one regular chamber. Integrating the resulting bound over B_τ gives decay of the local Bessel sections with power L in $\nu_\tau(g) = \inf_{h \in B_\tau} \|hg\|_{\mathrm{op}}$. Lemma 64.19 and Proposition 64.20 prove both estimates and the stated height moment. Their constants may depend on x. In particular finiteness is compatible with the lower bound of Theorem 65.24 and Corollary 65.25.

With $R = \prod_\tau \nu_\tau(g_\tau)$ as height, the number of rational positions in a ball of radius $R \geq 1$ is $O_x(R^{4fr^2})$ by Lemma 64.21. Since $L > 4fr^2$, the rational series and the unfolded pairings are absolutely convergent as proved in Theorem 64.22. The integrable formal-degree coefficient fixes each summed vector at its input lowest-type vector. Thus projection preserves these sums. The identity and parabolic terms give the seed inner product, and the compact stabilizer kills the indicated mixed cells. This proves the formula for the same G_D as in (65.11). □

Theorem 88.3 (Nonvanishing after refinement at the terminal prime)**.** *Fix any one place $v \mid T$ of E^+, with residue size Q. For a K_v^0-stable lattice let J_j be the subgroup acting as the identity modulo $\mathfrak{p}_v^j$, with $J_0 = K_v^0$. Multiply each original seed at this place by $\mathbf{1}_{B_v J_j}$. Let S_j be its global periodization from $U \otimes D$, and put*

$$\alpha_j = [K_v^0 : (B_v \cap K_v^0) J_j]^{-1}, \qquad \eta = L - 4fr^2.$$

At each fixed x and translate there is a finite C_x such that for every $j \geq 0$

$$\|\alpha_j^{-1} S_j^* S_j - 1\|_{\mathrm{op}} \leq C_x Q^{-j\eta}, \qquad e_{K_v^0} S_j = \alpha_j S_0.$$

Here the average uses the prescribed level character. For all sufficiently large j every nonzero vector of the original finite domain, including each refined zero-sum column, has a nonzero projected automorphic image.

Proof. The period group B is cut out inside the unitary group by affine linear equations fixing the graded isotropic frame and the terminal vector modulo that frame. If a nonidentity rational coset meets both finite supports, one of those equations takes a nonzero value divisible at v to depth j, up to fixed denominators. Its field norm gives a lower bound Q^j/C_x for the product of the two real heights. Lemma 64.23 proves this after compact adelic reduction, uniformly in the two points at fixed field and level. The rational tail of the second seed is bounded by $C_x Q^{-j\eta}$ times the first height to the power η. That factor is integrable against the first seed by Proposition 64.20. The identity term unfolds with volume ratio α_j. Summing the matrix entries over a basis gives the displayed estimate, as in Theorem 64.24. The local support has fraction α_j in each original right level coset. Averaging it gives the second identity by Proposition 64.25. The first estimate gives injectivity as soon as $C_x Q^{-j\eta} < 1$. □

The prescribed integral theta period uses the component at K_v^0. Its complex Gram matrix at that level is $S_0^* S_0$. The last identity gives the exact projection onto it from a finer level. The estimate for large j gives no positive lower bound for this projected component at $j = 0$. Uniform control of its nonidentity orbital terms, the comparison with the arithmetically normalized theta rows along the full chain, and the remaining strict orders in (12.15) are still needed for (W).

88.3 A first moment for a prescribed terminal character

Fix a word, the original level K^0, frequency ϖ, and a unit vector of the lowest real type as in Proposition 74.13. Write $S : D \to \mathcal{A}$ for its normalized periodization on the full finite class space, with the class inner product of Theorem 64.14. Thus $S^*S = 1 + \sum_\gamma O_\gamma$, where the sum is over the nonparabolic rational double cosets and is absolutely convergent in operator norm. Let $Z = \prod_{v|T} Z_v$, with $Z_v = \mathbb{F}_T^\times$. Use the unitary translations $U_v(t)$ on the class coordinates, with the level character included. The refined fibers are free Z-sets by Corollary 9.14. Choose unitary coordinates on one fiber of values over each Z-orbit and transport them around the orbit. This gives an orthonormal basis $\{e_i\}$ permuted by Z, including multiplicities of the archimedean covectors.

Theorem 88.4 (Signed class vectors testing the original Bessel map)**.** *Let $J \subset D$ be the space of a specified character $\lambda = \prod_v \lambda_v$ of Z, with each λ_v nontrivial. Alternatively let J be the whole space of vectors of zero sum on each local scalar orbit in each coordinate. Write P_J for orthogonal projection and $d_J = \dim J > 0$. In the first case take i uniform in the coordinate basis and put*

$$b = \frac{1}{\sqrt{|Z|}} \sum_{t \in Z} \overline{\lambda(t)}\, U(t) e_i.$$

In the second, choose i in the same way, choose the t_v independently and uniformly in $Z_v \setminus \{1\}$, and put

$$b = 2^{-f/2} \prod_{v|T} (1 - U_v(t_v)) e_i.$$

These are unit vectors in J. For any orthogonal projection P on $\mathcal{A}$, their mean squared projected period is

$$\mathbb{E}\|PSb\|^2 = 1 + \frac{1}{d_J}\sum_{\gamma} \operatorname{Re}\operatorname{tr}(P_J O_\gamma) - \frac{\|(1-P)SP_J\|_{\mathrm{HS}}^2}{d_J}.$$

When $P = 1$, positivity is decided by this single sum of traces. For a specified λ, its individual trace weights are $|Z|^{-1}\sum_{t\in Z}\overline{\lambda(t)}\operatorname{tr}(U(t)O_\gamma)$. Whenever the displayed expectation is positive, one of the displayed vectors attains at least that squared norm at the original level and the stipulated character or zero-sum subspace.

Proof. If $N = \dim D$, the mean of the rank-one operators $e_i e_i^*$ is $1/N$ times the identity. In the first case $b = \sqrt{|Z|}P_J e_i$. Freeness shows that its $|Z|$ summands have disjoint support and that $d_J = N/|Z|$. Consequently $\mathbb{E}[bb^*] = P_J/d_J$.

In the second calculation first choose each t_v uniformly in all of Z_v. The mean of $(1 - U_v(t_v))(1 - U_v(t_v))^*$ is $2(1 - P_{1,v})$, where $P_{1,v}$ averages over Z_v. The factors at different places commute and their choices are independent. Averaging the displayed vector's rank-one operator therefore gives P_J/N. The vector is zero whenever a $t_v = 1$. Otherwise its 2^f terms have disjoint supports by freeness and its norm is one. The probability of the latter event is $((T-2)/(T-1))^f = d_J/N$. Conditioning gives precisely P_J/d_J.

Take the trace against S^*PS. Pythagoras gives the last term in the assertion, and absolute convergence permits taking the trace of S^*S term by term. Expanding the character idempotent gives the individual trace weights. The maximum of finitely many nonnegative numbers is at least their mean.

In particular the orbital sum can be taken over fewer matrix entries at the original level. Write $i = (a, z_i)$, where a labels the orbit and the chosen base covector, and $z_i \in Z$. Put $O_{ij}^{\gamma} = \langle e_i, O_\gamma e_j\rangle$, with inner product linear in the second variable. Terms with different a's vanish in the trace. For equal a let $\nu(i,j)$ be the number of places at which z_i and z_j differ. For the two choices of J, respectively, we have the exact normalized formulas

$$\frac{\operatorname{tr}(P_J O_\gamma)}{d_J} = \frac{1}{N}\sum_{i,j:\, a_i = a_j} O_{ij}^{\gamma} \begin{cases} \overline{\lambda(z_j z_i^{-1})}, & J = P_\lambda D, \\ (-1/(T-2))^{\nu(i,j)}, & J = \prod_v (1 - P_{1,v})D. \end{cases}$$

Indeed the only scalar sending i to j is $z_j z_i^{-1}$. For the second formula the local projector has diagonal entry $1 - 1/(T-1)$ and off-diagonal entry $-1/(T-1)$ within the scalar orbit. Dividing by $d_J = N(1 - 1/(T-1))^f$ gives the displayed power. Thus a mismatch at each further place introduces an additional factor $1/(T-2)$ for the full zero-sum trace.

Before unit normalization each mixed difference has 2^f entries with coefficients 1 or -1. These arrays generate the integral zero-sum module on each product orbit. Indeed the augmentation kernel in $\mathbb{Z}[Z_v]$ has basis $\delta_t - \delta_1$, $t \neq 1$. Splitting off $\mathbb{Z}\delta_1$ at each coordinate identifies the simultaneous kernels with the tensor product of these augmentation kernels. In the first case the unnormalized array generates its character line on an orbit over the ring containing the character values, since its identity coefficient is one. These integrality statements concern finite coefficients and hold with any given integral base covector. □

Proposition 88.5 (Testing the trace with the adjoint theta lift)**.** *Let P project onto the cuspidal row space $\mathcal{H}$ of Theorem 60.21, in the stipulated real line and with the original first-edge test at T. Use the common row Hilbert space to take this projection. Let $\Theta : \mathcal{H} \to \mathcal{K}$ be that complex theta iteration to its definite endpoint, and set $a = C_x P(\mathbf{a})e^{-\Delta_x} > 0$ at depth zero. Every vector of $\mathcal{H}$ is the adjoint lift of an endpoint vector. For each of the unit columns*

b above one has

$$\sup_{\|y\|\le 1} |\langle Sb, \Theta^{*} y\rangle|^{2} = \|\Theta P S b\|^{2} \ge a \|P S b\|^{2}.$$

The pairing on the left is the normalized prescribed Bessel functional on the row $\Theta^{*}y$. In particular a positive first moment in Theorem 88.4 gives a nonzero pairing on a reverse theta lift. This uses only the original finite level.

Proof. The norm theorem on the original row gives $a\,1 \le \Theta^{*}\Theta \le C_x P(\mathbf{a}) e^{\Delta_x} 1$. Thus $W = \Theta(\Theta^{*}\Theta)^{-1}$ satisfies $\Theta^{*}W = 1$ and $\|Wv\|^{2} = \langle v, (\Theta^{*}\Theta)^{-1} v\rangle \le a^{-1}\|v\|^{2}$. The adjoint here is the Hilbert-space adjoint on the stipulated row. Its pairing with a row vector is the ordinary Petersson pairing against its forward theta lift. Pairing the forward lift of each test row with the endpoint vector defines the reverse pairing weakly on the row. Projection onto the stipulated row is part of this definition. The bounded-extension and unfolding formula of Theorem 64.14 identifies the pairing against Sb with the original Bessel functional. Cauchy–Schwarz, with equality for the normalized vector $\Theta P S b$ when nonzero, proves the formula.

There is also a simultaneous choice when the words have a common endpoint Hilbert space. Suppose one has found a column b_c with $y_c = \Theta_c P_c S_c b_c \neq 0$ for each of N_0 words. In any finite-dimensional endpoint space containing the y_c, fix a basis $v_1, \dots, v_M$. One vector $y = \sum_j n_j v_j$, $0 \le n_j \le N_0$ integers, satisfies

$$|\langle y_c, y\rangle| \ge \tfrac{1}{2} \max_j |\langle y_c, v_j\rangle| > 0 \qquad \text{for every } c.$$

For a fixed c leave a maximizing coordinate free and fix all the others. Two successive possible pairings are spaced by the positive maximum. An open disk of half that radius contains at most one of them. In the full integer box the fraction failing for this word is at most $1/(N_0+1)$. The union over the words has fraction less than one. When the chosen basis belongs to a common endpoint lattice this gives a vector in that very lattice. Its norm is at most $N_0 \sum_j \|v_j\|$. □

The trace formula places the remaining nonvanishing problem on the actual scalar character if one is prescribed. Its signed orbital trace and, when present, the component outside the specified cuspidal row still require estimates. The bounded right inverse concerns the full complex endpoint. Controlling the arithmetic normalization of the prescribed pairing on the common fields is a further requirement.

88.4 The trace on each full-rank cell

Let P be the definite k-dimensional column space over E/F, with $f = [F : \mathbb{Q}]$, and ρ a word representation of $U(P)_{\infty}$. Its local highest weights are $\lambda_{\tau,1} \ge \cdots \ge \lambda_{\tau,k}$. Put $n_{\tau,i} = \lambda_{\tau,i} + k - i$ and $L = \min_{\tau, i<k}(n_{\tau,i} - n_{\tau,i+1})$.

Proposition 88.6 (A character estimate in terms of a rational denominator)**.** *Let $g \in U(P)(F)$ and let $\mathfrak{a} \subset \mathcal{O}_E$ be a nonzero integral ideal such that $\mathfrak{a}\, gM \subset M$ for an $\mathcal{O}_E$-lattice M spanning P. Write m_a for the multiplicities of the distinct eigenvalues and $q = \sum_{a<b} m_a m_b$. Then*

$$\frac{|\operatorname{tr}\rho(g)|}{\dim \rho} \le \min\left(1, \left[\frac{2k}{L}(N_{E/\mathbb{Q}}\mathfrak{a})^{1/(2f)}\right]^{fq}\right).$$

For a noncentral g, $q \ge k-1$.

Proof. At one real place let the distinct eigenvalues be z_a. They all have modulus one, and g is diagonalizable by unitarity. Temporarily omit the place from $n_{\tau,i}$. In the Weyl character formula for $U(k)$ take the limit of $\det(z_j^{n_i})/\det(z_j^{k-i})$ as m_a of the column variables approach z_a.

Divide both determinants by the Vandermonde products within those groups. The repeated columns are now derivatives of orders $0, \dots, m_a - 1$, divided by their factorials, by Taylor expansion. The denominator has absolute value $D_\tau = \prod_{a<b} |z_a - z_b|^{m_a m_b}$.

Expand the remaining numerator in minors on these column groups. Assigning a set of m_a exponents to the a-th group gives a minor of absolute value $\prod_{i<j \text{ in that set}} |n_i - n_j| / \prod_{t=1}^{m_a - 1} t!$. Indeed unit powers of z_a factor from the rows and columns, and the falling factorials in n_i are monic polynomials of the successive degrees. Divide by the Weyl dimension $\prod_{i<j} |n_i - n_j| / \prod_{t=1}^{k-1} t!$. The q uncancelled differences are each at least L. The quotient of the factorial products is at most k^q: write the indices $1, \dots, k$ in contiguous groups of the given sizes and cancel the within-group factors in $\prod_{i<j}(j - i)$. There are at most 2^q assignments. Indeed each interleaving of the ordered lists is determined by the order of each of the q pairs from different lists. The normalized character at this place is at most $(2k/L)^q / D_\tau$.

Consider the endomorphism $g \otimes 1 - 1 \otimes g$ on $P \otimes_E P$. Its $2q$ nonzero eigenvalues have product $D(g) \in E^\times$, up to sign the lowest nonzero coefficient of its characteristic polynomial. The rank and eigenvalue multiplicities persist at all embeddings. At every finite prime multiply the endomorphism by a local generator of $\mathfrak{a}$. It now preserves $M \otimes M$ locally. Its characteristic polynomial is integral, hence $\mathfrak{a}^{2q} D(g) \subset \mathcal{O}_E$. At one complex embedding $|\tau D(g)| = D_\tau^2$, so the absolute norm over both embeddings at each place gives $\prod_\tau D_\tau = |N_{E/\mathbb{Q}} D(g)|^{1/4} \geq (N_{E/\mathbb{Q}} \mathfrak{a})^{-q/2}$. The tensor product character and dimension multiply over the places, proving the second bound. The first follows from unitarity. A noncentral unitary has at least two distinct eigenvalues, and the smallest possible number of cross pairs then is $1 \cdot (k - 1)$. □

Corollary 88.7 (The original character trace after selecting columns)**.** *Intersect either terminal space J of Theorem 88.4 with the coordinate subspace of Theorem 87.38, and denote the result by J'. The first-moment formula of Theorem 88.4 holds with J' and its dimension when the initial coordinate is uniform on the retained set. Every relevant nondegenerate term meets the cutoff of Corollary 87.39. For a full-rank cell in this trace, the sum over an orthonormal basis of real covectors, divided by* $\dim \rho$*, has absolute value*

$$|a| \, \frac{|\operatorname{tr} \rho(g)|}{\dim \rho}$$

for a rational isometry $g \in U(P)(F)$. Here a is the scalar of its orbital intertwiner, including the common finite cell coefficient and the weight from the terminal projection. For different words the full-rank orbital intertwiner is zero.

Proof. The coordinate projection onto the retained classes commutes with all terminal translations because it retains full scalar orbits. The proof by averaging rank-one operators in Theorem 88.4 gives covariance $P_{J'} / \dim J'$ inside this subspace. Taking its trace against the same projected Gram operator gives the stated first moment, including its term for projection to the chosen cuspidal row. In the basis expansion only two classes in the same terminal scalar orbit can contribute. Each such pair has retained initial column. The cutoff applies before any averaging over second flags.

For a full-rank pair let C be a rational positive complement to the common radical as in Theorem 64.3. Projection gives two rational isometries $\pi_0, \pi_1 : C \longrightarrow P$, transporting the second column to the first model by γ^{-1}. Thus $\iota = \pi_1 \pi_0^{-1}$ identifies the two actions of $U(C)$. By Theorem 76.2 the real cell operator for an identical word is a scalar times $\rho(\iota)$; the scalar absorbs the phase of the unitary identification. For different words Schur's lemma gives zero.

To take the trace, choose one finite class representative h at the base of the common scalar orbit. Choose its other representatives by right multiplication by scalar unit lifts at the terminal

places. These act only on the finite coordinate. For each side of the cell the finite Levi coordinate can therefore be written $q_{i,f} h t_i k_i$, with $q_i \in U(P)(F)$, t_i such a lift and k_i in the original column level. This is exactly equality in the corresponding double coset. The rational factor is unique for fixed representative by Lemma 70.23. The class covector is transported by $\rho(q_i)^{-1}$, as in the construction preceding (70.15). The finite phases are common to the whole real basis. In these coordinates the remaining unitary map is $\rho(q_1^{-1}\iota q_0)$, or its adjoint for the opposite convention. Summing its diagonal entries proves the absolute-value formula with $g = q_1^{-1}\iota q_0$. □

In the words of (9.1) the shifted weight gaps are at least 3Δ, since interior entries descend by 4Δ before a change of either bit and the endpoint gaps satisfy the same bound. For fixed $A, B > 0$, one may choose polynomial spacing $\Delta \geq 2e^A k x^B$ in Theorem 9.13 before selecting T. Every noncentral full-rank cell admitting a denominator with $(N_{E/\mathbb{Q}}\mathfrak{a})^{1/(2f)} \leq x^B$ then contributes at most $e^{-Af(k-1)}|a|$ to this normalized real trace. This leaves the central terms, large denominators, and the total size of the scalar amplitudes to be estimated for the prescribed period.

Proposition 88.8 (An integral denominator from the two isotropic flags)**.** *On any original full-rank cell in the preceding trace let X_0, X_1 be the isotropic subspaces, $I = X_0 \cap X_1$, and Λ the common row lattice. Put $U_i = (\Lambda \cap X_i)/(\Lambda \cap I)$. With the convention that h is linear in its second slot, consider the map*

$$D : U_0 \longrightarrow U_1^{\sharp}, \qquad x \longmapsto (y \longmapsto h(y,x)/Q), \qquad \mathfrak{d} = \operatorname{Ann}_{\mathcal{O}_E}(U_1^{\sharp}/DU_0).$$

Here $U_1^{\sharp}$ is the module of conjugate-linear integral functionals. This is a map of lattices of equal rank with finite cokernel. Its ideal $\mathfrak{d}$ is a denominator for g in Proposition 88.6. If $I = 0$ and $s = d - 1$, then

$$N_{E/\mathbb{Q}}\mathfrak{d} \;\leq\; |N_{E/\mathbb{Q}}(D_s/Q^s)|.$$

Thus the determinant or, more sharply, the largest local elementary divisors of this integral map test the denominator in the original trace.

Proof. Write $Y_i = X_i + Ez_i$ and $A = (Y_0 + Y_1)^{\perp}$. By Theorem 64.3 the full-rank case gives a common negative vector $z_* \in Y_0 \cap Y_1$ and $Y_i = X_i + Ez_*$. The kernel of the pairing between X_0 and X_1 is I on either side. Indeed, an $x \in X_0$ perpendicular to X_1 is also perpendicular to z_*. Thus it lies in $Y_1^{\perp}$, whose isotropic vectors all belong to X_1 by positivity of the column quotient. Interchanging the indices gives the assertion on the other side. The resulting pairing on the quotients is perfect over E.

At every place above Q, the primitive integral representatives of the two isotropic flags have the same reduction up to an upper unitriangular change. In particular each vector of $\Lambda_v \cap X_1$ is congruent to one of $\Lambda_v \cap X_0$ modulo $Q\Lambda_v$. The original level imposes this congruence in both split factors, as in Proposition 88.11. Hence $h(y,v) \in Q\mathcal{O}_E$ for $y \in \Lambda \cap X_1$ and $v \in \Lambda \cap Y_0^{\perp}$; away from Q this just uses integrality. Since every such v is orthogonal to I, $y \mapsto h(y,v)/Q$ defines an element of $U_1^{\sharp}$. This includes $v \in \Lambda \cap X_0$, proving that D is integral. The rational nondegeneracy just proved makes its cokernel finite.

Fix $\beta \in \mathfrak{d}$. For every such v there exists $u \in \Lambda \cap X_0$ with $h(y, \beta v - u) = 0$ for every $y \in X_1$, by the definition of the annihilator and by clearing Q. Now $\beta v - u$ lies in $Y_0^{\perp}$, is perpendicular to X_1, and is perpendicular to z_*. It belongs to $\Lambda \cap A$. The projections of A/I onto both rational column spaces are isometries, so its image in the second column is ι of its image in the first. In the first the image is precisely $\beta[v]$. Consequently $\mathfrak{d}\,\iota L_0 \subset L_1$ for the two projected integral column lattices. In the common class coordinates of Corollary 88.7, these lattices are $q_0 M, q_1 M$

in their two models. Indeed the terminal scalar changes preserve the defining hyperplanes, and hence their lattices by Lemma 87.10. This gives $\mathfrak{d}\, gM \subset M$.

Over a local discrete valuation ring the cokernel is a sum of cyclic modules by diagonal reduction. Its annihilator has the largest elementary-divisor exponent, bounded by their sum, the valuation of the determinant in integral bases. If $I = 0$, the primitive ordered flag representatives give such bases at each finite prime. Their allowed upper unitriangular changes leave the leading cross minor unchanged. The determinant of D is therefore $\overline{D_s}/Q^s$ in these local bases, up to a unit. Multiply the local bounds to obtain the stated norm inequality. The zero-rank case gives the unit ideal. □

88.5 The relative scalar on the common isotropic part

Let X_b be the span of the $s = d-1$ isotropic vectors of the b-th decorated frame, for $b = 0, 1$, and $Y_b = X_b + Ez_b$ with $h(z_b, z_b) = -T$. Write $X_{b,i}$ for its prefix of length i. Suppose $I = X_0 \cap X_1$ has dimension $e > 0$ and put $t = s - e$. The exceptional rank in a signed first moment can range from $t = 0$ to $t = s - 1$.

Theorem 88.9 (Classification data with a common isotropic subspace)**.** *For any relevant pair there are allowed changes of the two representatives such that their isotropic lists and final vectors have the form*

$$\begin{aligned} &u_{0,1}, \dots, u_{0,t}, \quad v_{t+1}, \dots, v_s, \qquad z_*, \\ &u_{1,1}, \dots, u_{1,t}, \quad \lambda v_{t+1}, \dots, \lambda v_s, \quad \lambda z_*. \end{aligned}$$

Here the v_i form a basis of I, $\lambda\bar{\lambda} = 1$, and the cross matrix on the first t positions is a direct sum of reversed blocks of the kind in Lemma 88.10. The lengths, block scalars, and λ determine the rational double coset. The only nonzero leading cross minors are the block endpoints in the first t positions.

Proof. Theorem 64.7 gives the same filtration on I from both flags, with one jump at each of $t+1, \dots, s$. It also supplies a common negative vector. The projections from $Y_0 \cap Y_1$ to Y_b/X_b have the same kernel I: a vector of the intersection lying in one isotropic subspace has norm zero and hence lies in the other, since Y_b/X_b is negative. Choose z_* inducing the prescribed final vector for the first flag. The second one is induced by λz_* for a unique λ. Equality of the negative norms gives $\lambda\bar{\lambda} = 1$.

At a jump i, take $v \in I$ with leading coefficients a_0, a_1 in the two lists. In the proof of (64.4), if $i < s$, the common successor has precisely these same coefficients at position $i+1$. Iterate to position s. At that last position (64.4) says $a_0 z_* - a_1 \lambda z_* \in X_0 + X_1$. But z_* is perpendicular to this sum and has nonzero norm. Hence $a_0 = \lambda a_1$ at every jump. At each i choose v_i with first coefficient one. Changing each list by adding earlier vectors then gives the two suffixes in the statement. These vectors are rational, since the flags and filtrations are defined over E.

The kernel of the cross pairing on X_0, X_1 is exactly I on each side, again by Theorem 64.7. Both prefixes of length t complement I, so their pairing matrix is invertible. Reduce it to monomial form by upper unitriangular changes of the prefixes. Their sum is a nondegenerate subspace perpendicular to I and to z_*. Each inversion test of Proposition 87.40 acts on this sum and can be extended by the identity on its perpendicular. It then fixes the common suffix and the two terminal representatives, and is in both unipotent radicals. The two character functionals on this element are exactly the ones for the prefixes, with the coefficients at all later simple positions zero. The same adjacent-descent computation therefore gives consecutive reversed blocks. For $j > t$ the first j rows include a nonzero vector of I, which pairs to zero with the entire second frame. This proves the minor assertion, including at d.

If the data agree for two pairs, map the vectors of the two prefixes, the first suffix and z_* to their counterparts. Together these vectors form a basis of their span. The Gram matrices agree: the prefixes pair by the specified blocks, the negative vector has norm $-T$, and I is the radical of this span. Witt's extension theorem extends the isometry of the subspaces to the ambient space. Equivalently one can first match the nondegenerate prefixes and negative line, and then split off and match hyperbolic partners of the common isotropic basis in their perpendiculars. The same map also carries the second suffix and final vector because λ agrees. The resulting pairs are in one rational orbit. □

88.6 Counting rational orbits in the terminal period

Let E/F be the CM extension of Section 99.16, $f = [F : \mathbb{Q}]$, and choose one embedding of E above each real place τ of F. We write τx for its value on x. Put $d = s + 1 \geq 2$. Write $Y = (u_1, \dots, u_s, z)$ for the ordered frame, with $z = \varpi e$ and $h(z, z) = -T$. The two factors in the unfolded period use the same T. At the finite places we use the level K^0 of Theorem 64.1. Set $Q = g_1 g_2$. The congruence at Q will enter the count separately at every nonzero leading minor.

A decorated flag is the class of Y under right upper unitriangular changes of basis. Thus it specifies each successive flag vector modulo its predecessors, including the last vector. Its stabilizer in $G = U(h)$ is $B = NH$. Indeed an element stabilizing the class acts trivially on each one-dimensional isotropic quotient. On the middle Hermitian quotient it fixes the image of z, so its action there belongs to $H = U(W_0)$. The kernel of these graded actions is N. For $\gamma \in G(F)$, let $Y' = \gamma Y$, and write

$$Z = h(Y, Y'), \qquad D_0 = 1, \qquad D_j = \det(Z_{ab})_{1 \leq a,b \leq j}, \qquad \mathcal{G} = D_d \overline{D_d} - T^2 D_{d-1} \overline{D_{d-1}}.$$

Independent changes of representative of the two flags send Z to $U^* Z V$, where U, V are upper unitriangular. Each D_j is unchanged, since its leading square transforms by the corresponding leading unitriangular matrices. A pair is *relevant* if its two characters agree on the common real unipotent group in Lemma 76.1. A pair is *supported at the finite places* if $\gamma_f \in B_f K^0 B_f$. Both properties descend to the rational double coset.

Lemma 88.10 (Normal form of a relevant pair)**.** *Suppose $\mathcal{G} \neq 0$. The normal form of Z under the two unitriangular changes determines its rational double coset. For a relevant pair the normal form is block diagonal. Each block of length h has successive antidiagonal entries $c, -c, c, \dots, (-1)^{h-1} c$, with $c \in E^\times$, and all its other entries vanish. The nonzero leading minors have indices precisely the successive block endpoints. For a composition $h_1 + \cdots + h_l = d$ with endpoints $j_\nu = h_1 + \cdots + h_\nu$, a prescribed tuple of endpoint minors occurs for at most $\prod_{\nu=1}^{l} h_\nu$ rational double cosets.*

Proof. The determinant of the concatenated Gram matrix is $(-1)^d \mathcal{G}$ by (64.2). Its $2d$ vectors are therefore independent and have nondegenerate span. By Theorem 64.3 the orthogonal complement is positive definite of rank $k - 1$, and $\mathcal{G}$ is totally positive. In particular $D_d \neq 0$.

For an invertible Z clear its first row using its leftmost nonzero entry. Add multiples of this pivot column to later columns and multiples of the first row to later rows to clear the pivot column. Repeat on the remaining rows and columns in their original order. This gives a monomial matrix M with entries $a_i = M_{i,w(i)} \neq 0$. Such a form is unique. The ranks of all northwest rectangles are invariant and determine the location of each nonzero entry by taking successive differences. If $M' = LMV$ has the same permutation, with L lower and V upper unitriangular, the i-th diagonal entry of $M' V^{-1} M^{-1} = L$ is $a'_i / a_i = 1$.

Two pairs with the same normal form admit representatives with the same concatenated Gram matrix. Both diagonal Gram blocks are $-TE_{dd}$, where E_{ij} denotes a matrix unit. The linear map carrying the first concatenated basis to the second is an isometry on their spans. Witt’s extension theorem for Hermitian spaces extends it over E to $G(F)$. In this nondegenerate case one can see the extension directly. Diagonalize the first span and match its orthogonal basis vectors in turn. In the perpendicular to the vectors already matched take vectors v, v' of the same nonzero norm a. For some $y \in \{v', -v'\}$ the number $b = h(v - y, v)$ is nonzero, since the two choices give numbers with sum $2a$. Put $w = v - y$. The map $t \mapsto t - wh(w, t)/b$ sends v to y and preserves h, as $b + \bar{b} = h(w, w)$. If $y = -v'$, compose with minus the identity on this perpendicular. Extend by the identity on the vectors already matched. Induction gives the required isometry. Thus the two ordered pairs of decorated flags are in the same $G(F)$-orbit, which is equivalent to equality in $B(F)\backslash G(F)/B(F)$.

On a frame an element of N acts by an upper unitriangular matrix U, and its character at a real place is the additive character of the trace of $\sum_{i<d} U_{i,i+1}$. The coefficient in the last position uses $z = \varpi e$, as in Section 99.16. Conjugation by an upper unitriangular matrix preserves the first superdiagonal, so the formula applies to the representatives used to obtain M.

Take an inversion $i < j$, $w(i) > w(j)$, and work at one real place. For any $x \in \mathbb{C}$ put

$$U_x = 1 + xE_{ij}, \qquad V_x = 1 - \bar{x}(a_i/a_j)E_{w(j),w(i)}.$$

Here the a_i are evaluated at the chosen place. Multiplication gives $U_x^{*} M V_x = M$. On the two independent spans these upper unitriangular maps preserve their self pairings. Together they define a unipotent isometry, extended by the identity on the orthogonal complement. Its projection to each of the two compact column groups is unipotent and unitary, hence the identity. It is therefore in both unipotent radicals. Equality of the characters, applied to all real multiples of x, gives

$$\mathbf{1}_{j=i+1}\Re(x) = \mathbf{1}_{w(i)=w(j)+1}\Re\big(-\bar{x}\,a_i/a_j\big).$$

An adjacent descent must decrease by one. At each such descent $x = 1$ and $x = \sqrt{-1}$ give $a_{i+1} = -a_i$. The equality also holds in E because the embedding is injective. Each maximal descending run of w is a consecutive interval of values in reverse order. Since two successive runs are disjoint intervals joined by an ascent, their value intervals are in increasing order. They partition $\{1, \dots, d\}$, proving the asserted block form.

Within a partially included reversed block the first row of the partial block is zero. A whole block has determinant c^h, since the reversal sign and the alternating row signs cancel. Consequently $D_{j_\nu}/D_{j_{\nu-1}} = c_\nu^{h_\nu}$, with $j_0 = 0$. Each equation has at most h_ν roots in the field E. The uniqueness just proved gives the claimed multiplicity. □

Proposition 88.11 (Congruences of the leading minors)**.** *Every rational pair supported at the finite places satisfies*

$$D_j \in Q^j \mathcal{O}_E \quad (1 \le j < d), \qquad D_d + T D_{d-1} \in Q^d \mathcal{O}_E.$$

All D_j are algebraic integers. If $\mathcal{G} \neq 0$, then $|\tau D_d| > T|\tau D_{d-1}|$ for every τ.

Proof. Write the finite translate as $b_1 k b_2$ with $k \in K^0$. Each exterior product of the first j frame vectors is fixed by B_f. Pairing it with its translate gives the same minor as for k. The frame is integral in the local Hermitian lattices and k preserves them, including the two mutually dual factors at T, by Theorem 64.1. Thus every minor is integral at all finite places.

Fix either prime $g \mid Q$. At each place above g the reduction of kY is Y times an upper unitriangular matrix in both split factors. The two actions are described in the proof of Theorem 64.4. Since the self pairing of the frame is $-TE_{dd}$, it follows that every entry of $h(Y, kY) + TE_{dd}$ is in g times the local integer ring. A leading j-square for $j < d$ therefore has determinant in g^j. The full determinant after adding TE_{dd} is $D_d + TD_{d-1}$ by expansion in that entry. It lies in g^d. The two rational primes are distinct and split completely, giving the asserted global inclusions. The real inequality follows from total positivity of $\mathcal{G}$. □

For a tuple of minors with nonzero endpoints indexed by I, put

$$w(D) = \prod_{j \in I} \prod_{\tau} (1 + |\tau D_j|^2)^2.$$

The source uses constant antidiagonal entries on a reversed block. Multiplication of the cross matrix on the left by $\operatorname{diag}(1, -1, 1, -1, \ldots)$ changes the alternating convention of Lemma 88.10 into this convention. It multiplies D_j by $(-1)^{j(j-1)/2}$ and preserves $w(D)$. For constant blocks the successive endpoint ratios are $(-1)^{h(h-1)/2} c^h$. Either convention therefore gives the same bound for the number of choices of block scalars.

Theorem 88.12 (Sum over all integral block data)**.** *Over any CM field E of degree $2f$, sum $w(D)^{-1}$ over all compositions of $d \geq 1$ and all nonzero block scalars in E for which the endpoint minors belong to $\mathcal{O}_E$. In either sign convention this sum is bounded by*

$$L^f (L^f + 2)^{d-1} \leq 8^{fd}, \qquad L = 81/16.$$

In particular the bound 64^{fd} in the source holds for every pair of positive terminal norms in its definition of integral block points.

Proof. For $\delta \in \mathcal{O}_E$ set $p(\delta) = \prod_{\tau} (1 + |\tau\delta|^2)^{-2}$. The centers $(\tau\delta)_\tau$ of distinct algebraic integers have sup distance at least one. Indeed the square of the product of the coordinate distances is the absolute norm of a nonzero algebraic integer, hence at least one. The open polydiscs of radius $1/2$ around the centers are disjoint.

In one coordinate average $(1 + |y|^2)^{-2}$ over the disc of radius $1/2$ about z with uniform area probability. The average of $|y|^2$ is $|z|^2 + 1/8$, by rotational symmetry and radial integration. Convexity of $t \mapsto (1 + t)^{-2}$ gives a lower bound $(1 + |z|^2 + 1/8)^{-2}$, which is at least $(8/9)^2 (1 + |z|^2)^{-2}$. The disc area is $\pi/4$. The integral of the same density on $\mathbb{C}$ is π, by polar coordinates. Multiply over the f coordinates and use disjointness on any finite set of centers. This gives $\sum_{\delta \in \mathcal{O}_E} p(\delta) \leq [4(9/8)^2]^f = L^f$ on passing to the supremum over finite sets.

For l endpoints the sum over independent integral values is at most L^{fl}. The successive scalar equations give multiplicity at most $\prod_{\nu} h_\nu \leq 2^{d-l}$, since $h \leq 2^{h-1}$ for $h \geq 1$. There are $\binom{d-1}{l-1}$ compositions of this length. The binomial theorem gives $L^f (L^f + 2)^{d-1}$, and $L^f + 2 \leq (L + 2)^f < 8^f$. An integral representative in the source's cell definition has integral leading minors because the lower and upper unitriangular factors preserve them. The further conditions on its Gram determinant restrict the set being summed. This proves the last assertion, uniformly in the two terminal norms. □

Theorem 88.13 (The sum at the prescribed finite level)**.** *Let $\mathcal{R}$ be the set of relevant rational double cosets supported at the finite places with $\mathcal{G} \neq 0$. For $0 \leq t < 2$ set*

$$M_t = \int_0^\infty \frac{4 r^{3+t}}{(1 + r^2)^3} \, dr.$$

Then

$$\sum_{\gamma\in\mathcal{R}} w(D(\gamma))^{-1} \le (9M_t+2)^{fd} Q^{-(2+t)fd}. \tag{88.1}$$

In particular the sum is at most $\bigl(128\log(2Q)/Q^4\bigr)^{fd}$.

Proof. Fix a composition with endpoint set I. For $j\in I$ set $P_j = Q^j$, with one modification. When the terminal block has length one put $P_d = Q^{d-1}\min(Q,T)$. For every endpoint tuple under consideration,

$$\prod_{\tau} |\tau D_j| \ge P_j^f.$$

For $j<d$ this follows by taking the norm of the nonzero integer D_j/Q^j. For a terminal block of length greater than one we have $D_{d-1}=0$ and the same argument applies at d. For a terminal singleton we have $d-1\in I$, and Proposition 88.11 gives $\prod_\tau |\tau D_d| > T^f Q^{(d-1)f}$. Also, given all preceding endpoint values, distinct choices of D_j differ by an element of $Q^j\mathcal{O}_E$. At d this uses the fixed residue $-TD_{d-1}$, with $D_{d-1}=0$ when that index is absent. In particular the product of the coordinate distances of two such choices is at least P_j^f.

Impose bounds $|\tau D_j|\le r_{j,\tau}$, where all radii are positive, and put $R_j = (\prod_\tau r_{j,\tau})^{1/f}$. There are no tuples unless $R_j\ge P_j$ at every endpoint. For one endpoint at a time divide each complex coordinate by its radius. The centers now lie in the unit polydisc and have sup distance at least P_j/R_j by the product bound on differences. The disjoint open polydiscs of radius $P_j/(2R_j)$ around them fit in the polydisc of radius $1+P_j/(2R_j)$. Comparison of volumes gives at most $(1+2R_j/P_j)^{2f}\le 9^f(R_j/P_j)^{2f}$ choices. Iterating conditionally on earlier values and allowing the scalar multiplicity bounds the number of rational labels in the box by

$$\Bigl(\prod_\nu h_\nu\Bigr)\prod_{j\in I} 9^f (R_j/P_j)^{2f}\mathbf{1}_{R_j\ge P_j}.$$

The identity $(1+u^2)^{-2} = \int_u^\infty 4r(1+r^2)^{-3}\,dr$ expresses the required weighted count as the integral of these box counts against the product of the radial densities. Tonelli's theorem applies since the summands are nonnegative. Use $\mathbf{1}_{R_j\ge P_j}\le (R_j/P_j)^{tf}$. For one endpoint the resulting integral is at most $(9M_t)^f P_j^{-(2+t)f}$. It converges for $0\le t<2$. For a terminal block of length greater than one, $\prod_{j\in I} P_j \ge Q^d$ comes from the last endpoint alone. For a terminal singleton the last two endpoints contribute at least $Q^{2(d-1)}\ge Q^d$, since $d\ge 2$ and $T\ge 1$. The binomial sum in Theorem 88.12, here with $L=9M_t$, now proves (88.1).

For the explicit form take $\eta = 1/\log(2Q)$, $t=2-\eta$. Since $Q\ge 2$, $0<\eta<1$. Splitting the integral at one and bounding its denominator below by 1 and r^6, respectively, gives $M_t\le 1+4/\eta$. Hence $9M_t+2\le 47\log(2Q)$, while $Q^\eta\le e$ and $47e<128$. □

Corollary 88.14 (The measure on the individual finite cells)**.** *Fix two different words and unit vectors in the corresponding input spaces of the untruncated periodization at K^0. Denote the scalar contribution of a finite cell c above γ by $J_{\gamma,c}$, including the finite coefficient and its volume, with the normalization of (76.1). If $\sum_c |J_{\gamma,c}|\le A/w(D(\gamma))$ for every $\gamma\in\mathcal{R}$, then the absolute value of this mixed Gram entry is at most $A\bigl(128\log(2Q)/Q^4\bigr)^{fd}$.*

Proof. The sum is absolutely convergent by Theorem 64.22. Lemma 76.1 removes the irrelevant terms and Corollary 64.8 removes the terms with zero determinant for distinct words. Apply the triangle inequality and Theorem 88.13 to the remaining labels. □

The source specifies its block points, weight, and summation in `Formalism/SharpRestricti on/CuspSpanNormBounds/Point.lean`, line 11, `Formalism/SharpRestriction/CuspSpanNormBou`

`nds/LeadMinorK.lean`, line 150, and `Formalism/SharpRestriction/CuspSpanNormBounds/OneP laceEstimates.lean`, line 35. Its bound 64^{fd} occurs in `Formalism/SharpRestriction/CuspS panNormBounds/BesselShellMomentBounds.lean`, line 265. Theorem 88.12 proves this estimate from the scalar equations and the area integral. Theorem 88.13 gives the further saving from the prescribed congruences. In the mixed Gram entry the constant A includes the sum of the absolute orbital integrals over the individual finite cells of one rational orbit. Controlling this quantity uniformly and controlling the singular terms for a word paired with itself remain necessary for an estimate of the whole Gram matrix by this argument.

88.6.1 Integral cross matrices within a rational orbit

Put $O = \mathcal{O}_E$. Write $U^-(S)$ and $U^+(S)$ for the lower and upper unitriangular groups over a ring S. A rational label specifies a double orbit under these groups over E. Counting integral flags requires the smaller equivalence relation over O. The following bound accounts for all the integral classes above the label.

Theorem 88.15 (Integral classes of a marked cross matrix)**.** *Let M be the monomial normal form of a relevant nondegenerate label in Lemma 88.10, and write $a_i = M_{i,w(i)}$ for its nonzero entries. Put*

$$b_0 = 1, \qquad b_j = Q^{-j} \prod_{i=1}^{j} a_i \quad (1 \le j < d), \qquad u_j = \#\{i > j : w(i) > w(j)\}.$$

Consider the integral matrices in $U^-(E)MU^+(E)$ which satisfy $Z \equiv -TE_{dd} \bmod QO$, where E_{dd} has its only nonzero entry, one, at (d,d). If such a matrix exists, all b_j are algebraic integers. With $h_j = |N_{E/\mathbb{Q}} b_j|$, the number of $U^-(O), U^+(O)$ double classes is at most

$$C(M) = \prod_{j=1}^{d-1} h_j^{\,d-j+u_j}.$$

If $t = d-1$, the absolute index of the image in O^t of the top t rows of any such Z is at most $Q^{2ft} h_t$, where $[E:\mathbb{Q}] = 2f$.

Proof. Both unitriangular groups fix E_{dd} on their respective sides. Thus $Z \mapsto Z' = (Z + TE_{dd})/Q$ carries the specified matrices bijectively to the integral matrices of one rational orbit and respects integral equivalence in both directions. Its monomial form M' has the same permutation. If $w(d) = d$, its last entry is $(a_d + T)/Q$, and the others are a_i/Q. The last entry is nonzero because $\mathcal{G} > 0$ at every real place, so $|D_d| > T|D_{d-1}|$ there. If $w(d) \neq d$, row $w^{-1}(d) < d$ of M has its sole nonzero entry in column d. A rational lower unitriangular row operation removes the added entry at (d,d), giving $M' = M/Q$.

Permute the columns of Z' into the order $w(1), \dots, w(d)$, and call the resulting matrix B. It has a factorization $B = L\Delta U$, where L is lower unitriangular, Δ has diagonal $M'_{i,w(i)}$, and U is upper unitriangular with $U_{ij} = 0$ when $i < j$ and $w(i) > w(j)$. To see this, write $B = L_0 \Delta V$ using the given rational orbit. The matrix V is unitriangular in the order defined by w, so an off-diagonal entry V_{ij} can be nonzero only for $w(i) < w(j)$. Eliminate its entries below the ordinary diagonal by adding multiples of earlier rows, clearing each pivot column before proceeding. A nonzero entry to be cleared in position (i,j) has $w(i) < w(j)$. Every nonzero entry of the pivot row is in a column k with $w(j) \le w(k)$, so the row operation preserves the allowed positions and the diagonal ones. This gives $V = L_1 U$. Then $L = L_0 \Delta L_1 \Delta^{-1}$ has the required form.

For $j < d$, the leading j-minor of B is b_j. It is nonzero and integral. For $i > j$, the minor with rows $1, \dots, j-1, i$ and columns $1, \dots, j$ is $b_j L_{ij}$. For $i > j$, the minor with rows $1, \dots, j$ and columns $1, \dots, j-1, i$ is $b_j U_{ji}$. These identities follow by taking determinants in the displayed triangular factorization. Consequently the entries below column pivot j of L and to the right of row pivot j of U lie in $b_j^{-1}O$.

Choose representatives of $b_j^{-1}O/O$, with zero representing zero. There are h_j choices. In each row i of L, reduce its entries in columns $j < i$ in decreasing order of j by adding integral multiples of row j. The later columns already fixed in that row are unchanged. Every entry in column j stays in $b_j^{-1}O$, since the operations are integral. In each column i of U use the corresponding column operations with $j < i$ in decreasing order. A nonzero coefficient is used only for $w(j) < w(i)$, so in the original column order the operation is an allowed integral upper unitriangular change. These operations preserve the zero pattern just proved, as well as the denominators within each row. The two reductions act on opposite sides of $L\Delta U$. They give a list of representatives of length at most the displayed product, with $d - j$ positions on the left and u_j on the right for index j.

For $t > 0$, the columns $w(1), \dots, w(t)$ of the first t rows of Z have determinant $Q^t b_t$ in that order, since adding TE_{dd} leaves these rows unchanged. Their image is contained in the image of all the columns. Taking the index gives $|N_{E/\mathbb{Q}}(Q^t b_t)| = Q^{2ft} h_t$. For $t = 0$ the index and bound are both one, with $h_0 = 1$. □

The bound uses only the block scalars of the rational label. It has a simpler expression in terms of the endpoints. For each proper endpoint put $H_{j_\nu} = |N_{E/\mathbb{Q}}(D_{j_\nu}/Q^{j_\nu})|$, and put $H_0 = 1$ and $H_d = |N_{E/\mathbb{Q}}((D_d + TD_{d-1})/Q^d)|$. Every H_{j_ν} is a positive integer for a represented label. At successive endpoints the magnitudes of the prefix pivot products of M' are precisely these norms. Inside a block of length greater than one the pivot magnitudes are constant. The logarithms of their prefix product norms therefore interpolate linearly between the two endpoint logarithms. It follows that each h_j is at most $\prod_\nu H_{j_\nu}$. Since $\sum_{j<d}(d - j + u_j) \le d(d-1)$, the number of classes is at most $(\prod_\nu H_{j_\nu})^{d(d-1)}$. The finer product in the theorem can be used separately for each label.

88.7 The rational orbits at the prescribed period level

Fix x in one of the intervals of Theorem 13.15, and one of its translates. In the period over E_x/F, $F = E_x^+$, let $f = [F : \mathbb{Q}]$, $d = s + 1 = m_0/2$, and $Q = g_1 g_2$. The Gram matrix in Theorem 88.2 is the sum on the original level K^0 at the terminal frequency ϖ. For two distinct words, the nonzero terms in the off-diagonal block are supported on relevant rational double cosets with nonzero Gram determinant. Write $\mathcal{R}$ for all such rational labels allowed by the finite support, including any whose orbital integral is zero. We count a label once even when it supports several finite cells.

Theorem 88.16 (Weighted count on the full period fields)**.** *For $\gamma \in \mathcal{R}$ let I_γ index the nonzero leading minors D_j of the cross matrix of the two ordered frames. Put $w_\gamma = \prod_{j \in I_\gamma, \tau} (1 + |\tau D_j|^2)^2$, where τ runs through one complex embedding above each real place of F. At every x and every translate as above,*

$$\sum_{\gamma \in \mathcal{R}} w_\gamma^{-1} \le \Big(\frac{128 \log(2Q)}{Q^4} \Big)^{fd} \le \exp\!\big(-(64 - o(1)) f d\, x^{\varepsilon}\big). \tag{88.2}$$

The second estimate is uniform on the intervals $[V, V^{3/2}]$ as V tends to infinity. The subset for which $D_{d-1} \neq 0$ satisfies the stronger bound

$$\sum_{\substack{\gamma \in \mathcal{R} \\ D_{d-1} \neq 0}} w_\gamma^{-1} \;\leq\; \exp\bigl(-(192 - o(1)) f d\, x^{\varepsilon}\bigr).$$

Proof. For $Y = (u_1, \dots, u_s, \varpi e)$ write $Z = h(Y, \gamma Y)$. Changing the two ordered representatives multiplies Z by a lower and an upper unitriangular matrix. Gaussian elimination leaves one nonzero entry in each row and column. Agreement of the two additive characters on each common elementary unipotent forces each descending run of the permutation to reverse a consecutive block, with successive entries $c, -c, c, \dots$. A block of length h multiplies the preceding nonzero leading minor by c^h. Each tuple of nonzero minors therefore has at most the product of the block lengths as its multiplicity. The two auxiliary level primes force $Q^j \mid D_j$ for $j < d$ and $Q^d \mid D_d + T D_{d-1}$. The standard separation of distinct algebraic integers by the product of the distances at their complex embeddings now bounds the number of tuples in any product of discs. Integrating the resulting count against $4r(1 + r^2)^{-3}\, dr$ in every coordinate gives the first inequality in (88.2). The character calculation, congruences, volume bound and integral are proved in Section 88.6, Lemma 88.10 through Theorem 88.13. Since $b = \exp(x^{\varepsilon})$ and $(kb)^8 \leq g_i \leq 2(kb)^8$ by Proposition 72.6, with k of polynomial size, $\log Q = (16 + o(1)) x^{\varepsilon}$ uniformly. This proves the second inequality.

Here is the additional saving when the last block has length one, the case $D_{d-1} \neq 0$. Write R_j for the geometric mean of the disc radii at endpoint j. For $j = d - 1$, a box with a point must have $R_{d-1} \geq Q^{d-1}$. At d, the positivity of $|\tau D_d|^2 - T^2 |\tau D_{d-1}|^2$ gives $R_d \geq T Q^{d-1}$, and distinct last minors with the preceding data fixed differ by an element of $Q^d \mathcal{O}_{E_x}$. Here $T \geq Q$ for large V, since Theorem 9.13 gives $\log T \geq (16 - o(1)) m_0 x^{\varepsilon}$ and $m_0 \to \infty$ uniformly. The count for the last endpoint is at most

$$9^f (R_d / Q^d)^{2f}\, \mathbf{1}_{R_d \geq T Q^{d-1}}.$$

For $0 < t < 2$ bound the indicator by $(R_d / (T Q^{d-1}))^{tf}$. Writing $M_t = \int_0^\infty 4 r^{3+t} (1 + r^2)^{-3}\, dr$, the integral at d is at most $(9 M_t)^f Q^{-2df} (T Q^{d-1})^{-tf}$. The endpoint $d - 1$ supplies $(9 M_t)^f Q^{-(2+t)(d-1)f}$. Use the same integral at the remaining endpoints, discard their further powers of Q^{-1}, and sum the multiplicities over compositions as in Section 88.6. This gives

$$(9 M_t + 2)^{f d} Q^{-(2+t)(2d-1)f} (Q/T)^{tf}.$$

Taking $t = 2 - 1/\log(2Q)$ gives $9 M_t + 2 \leq 47 \log(2Q)$. The factor in $Q^{-(2+t)(2d-1)f}$ saves $(128 - o(1)) f d\, x^{\varepsilon}$ in the logarithm. Since $m_0 = 2d$ and $d \to \infty$, the last factor saves at least $(64 - o(1)) f d\, x^{\varepsilon}$. The positive logarithm from the first factor is $o(f d\, x^{\varepsilon})$, proving the assertion. □

Corollary 88.17 (A sufficient bound for a mixed Gram entry)**.** *Consider two distinct words and unit input vectors at K^0. Let $J_{\gamma,c}$ be the scalar orbital integral of finite cell c above γ, with its finite coefficient and volume. Suppose for some $\delta > 0$ that at all sufficiently large x their absolute sums satisfy*

$$\sum_{c} |J_{\gamma,c}| \leq w_\gamma^{-1} \exp\bigl((64 - \delta) f d\, x^{\varepsilon}\bigr) \qquad (\gamma \in \mathcal{R}).$$

Then the absolute value of this mixed entry is at most $\exp(-(\delta - o(1)) f d\, x^{\varepsilon})$.

Proof. Theorem 88.2 gives absolute convergence and vanishing of the zero-determinant mixed cells. Averaging over the common unipotent group removes the irrelevant cells. Sum the displayed inequality over the remaining rational labels and use (88.2). □

Theorem 88.16 and the cohomology estimates in Theorem 13.15 use exactly the same E_x at each point of the interval. The hypothesis of Corollary 88.17 concerns orbital integrals with their actual volumes and normalizations. Its verification, the terms for equal words, and the integral lower bound for the projected Bessel coefficient $p_{\lambda,H}\mathcal{F}A$ of Corollary 74.7 remain open in this argument. On the cohomological side the total order of the surviving groups in (13.12) also remains to be bounded.

88.8 A common real scale and the change of finite Fourier measure

Take the fields of Theorem 12.15, with its polynomial modulus. Write $F = E_x^+$, $E = E_x$, $f = [F : \mathbb{Q}]$, and denote the fixed conductor of K_ψ by F_ψ. Keep the two split prime ideals $\mathfrak{p}, \mathfrak{q}$ and positive generators β, γ of Proposition 60.23 and Theorem 82.2. The rational ramified prime q and the split ideal $\mathfrak{q}$ lie over different rational primes.

Theorem 88.18 (An exact real scale with the original skew form)**.** *Let $\mathfrak{d}_F$ be the absolute different and let $\mathfrak{a}_0\mathcal{O}_E = (\sqrt{-d_0})$, as in Lemma 60.12. The ideal $\mathfrak{b} = \mathfrak{d}_F\mathfrak{a}_0^{-1}$ is integral and supported on places split in E/F. Set $\beta' = (d_0\gamma)^{-1}$. Then*

$$(\beta')\mathfrak{d}_F = \mathfrak{b}\mathfrak{q}^{-1}, \qquad (\gamma)\mathfrak{a}_0 = \mathfrak{q}, \qquad \beta'\gamma = 1/d_0.$$

The trace character scaled by β' has conductor $\mathcal{O}_v$ at every inert finite place. At every real place keep the vacuum, coordinates, and homogeneous tests of Proposition 82.3. For every tuple of degrees covered by that proposition,

$$R_\infty(\beta', \gamma) = d_0^{\sum_\tau D_\tau} R_\infty(1, 1).$$

For the two original conductor ideals, every choice of their generators has nonrational product $\beta\gamma$.

Proof. At d_0 the only ramified quadratic layer in $K_\psi C_J(\sqrt{-q}, \sqrt{-d_0})$ comes from $\sqrt{-d_0}$. The other factors have conductor prime to d_0, and adjoining this square root to an unramified completion has ramification index two. At q the same argument uses $\sqrt{-q}$. Inertia at either prime fixes the other imaginary quadratic field, on which complex conjugation acts nontrivially. Its intersection with $\mathrm{Gal}(E/F)$ is trivial. Thus the absolute ramification index of F is two at both primes. Its different exponent is one there. Indeed, over the maximal unramified subfield of the completion, a ramified quadratic extension of odd residue characteristic is generated by a root of $X^2 - a$ for a of valuation one. The powers of that uniformizer form an integral basis, and the derivative has valuation one.

Squaring the defining ideal of $\mathfrak{a}_0$ and descending gives $\mathfrak{a}_0^2 = (d_0)$. Its valuation is therefore one above d_0 and zero elsewhere. This proves integrality of $\mathfrak{b}$ and removes all places above d_0 from its support. Every remaining rational prime in the support divides $2F_\psi q$, by the conductor of the compositum. At an odd divisor p of $F_\psi q$, $-d_0 \equiv 1 \pmod p$, by the congruences in Theorem 5.6, on every sufficiently distant interval. Hensel's lemma gives a square root in $\mathbb{Q}_p$. At two, $-d_0 \equiv 1 \pmod 8$ likewise makes it a square in $\mathbb{Q}_2$. Since $E = F(\sqrt{-d_0})$, every place above these primes splits in E/F.

The displayed ideal identities follow by multiplication, using $(d_0) = \mathfrak{a}_0^2$. The local annihilator of $\mathcal{O}_v$ is the inverse of $(\beta')\mathfrak{d}_{F,v}$, hence equals $\mathcal{O}_v$ at an inert place. The skew scalar is still $\gamma\sqrt{-d_0}$ with the prescribed sign. At a real embedding the positive factor of the symplectic pairing is $\tau(\beta'\gamma) = 1/d_0$. The Fock isometry of Proposition 82.3 multiplies a degree-d vector by $d_0^{d/2}$ and its diagonal coefficient by d_0^d. Multiplying over the edges and the real places gives the formula.

Finally take a place $v \mid q$ and normalize v to have value group $\mathbb{Z}$. The original ideals $\mathfrak{p}, \mathfrak{q}$ were chosen away from q. Their defining equalities give $v(\beta\gamma) = -v(\mathfrak{d}_F\mathfrak{a}_0) = -1$, whereas $v(t) = 2v_q(t)$ for every rational $t \neq 0$. □

Proposition 88.19 (Finite Fourier measures and dual lattice indices)**.** *Put $u = \beta'/\beta$. At a finite place v fix a nonsingular bilinear pairing of an n-dimensional F_v-space with its dual. Use self-dual measures in dual bases, keeping the same coordinate conventions for both additive characters. For every Schwartz function h, and every full lattice L in the domain, one has*

$$\widehat{h}_{\beta'}(y) = |u|_v^{n/2}\widehat{h}_{\beta}(uy), \qquad L_{\beta'}^{\perp} = u^{-1}L_{\beta}^{\perp}.$$

The support of the valuations of u consists of split places. For all edges of the theta chain together, the sum of the logarithms of the mutual lattice indices and of the absolute logarithmic changes of Haar measure is $O_{\psi,\varepsilon}(x^{C})$ for a fixed C, uniformly in the lattices and in x on the selected intervals.

Proof. With the original measure the new Fourier kernel is the old kernel at uy. Applying the Fourier transform twice with this substitution introduces $|u|_v^{-n}$. Fourier inversion thus gives the positive measure multiplier $|u|_v^{n/2}$ in dual coordinates. The two annihilators follow by testing the character on L. For a lattice $M = L_{\beta}^{\perp}$, one of $M, u^{-1}M$ contains the other, and the product of their indices over their intersection equals $(Nv)^{n|v(u)|}$. Here Nv is the residue field cardinality.

The two conductor equalities give $(u) = \mathfrak{b}/(\mathfrak{p}\mathfrak{q})$. The numerator and denominator have disjoint support, by the choice of the two primes. Consequently

$$\sum_{v<\infty} |v(u)| \log Nv = \log \frac{|D_F|}{d_0^{f/2}} + \log N\mathfrak{p} + \log N\mathfrak{q}.$$

Indeed $N\mathfrak{d}_F = |D_F|$ and $N\mathfrak{a}_0 = d_0^{f/2}$. The conductor–discriminant formula gives $|D_F| \leq N_x^f$, since each of the f characters of the abelian field F has conductor dividing N_x. The polynomial bound on N_x, the degree bound in Theorem 12.15, and Proposition 60.23 give a polynomial bound for the displayed sum. A Schrödinger coordinate space for an edge between ranks n_1, n_2 has dimension n_1n_2, half the F_v-dimension of the tensor space. By Section 60.3 the largest rank is $n_m = k + m - 1 + mt_{\mathrm{r}}$, so the sum of the coordinate dimensions over the m edges is at most mn_m^2. Theorem 6.29 gives $t_{\mathrm{r}} = O(x^{400})$ and polynomial bounds for k, m. Multiplication by three halves times this bound accounts for both the lattice indices and the Haar multipliers. □

The Fourier identity evaluates the original transform at uy. For a prescribed terminal frequency y, the integral period still requires a lower bound at that transported argument and control of the signed projected sum. Theorem 72.16 gives an exact comparison when the Schwartz functions and splittings are transported together.

88.9 Rational orbits meeting the retained columns

Use Y, Q, T, d, f from Corollary 87.39. For a relevant rational label γ, let D_j be its leading cross minors. Its weight is $w_\gamma = \prod_{j:D_j\neq 0}\prod_{\tau}(1+|\tau D_j|^2)^2$, where one complex embedding above each real place is used. Let $\mathcal{R}_Y$ consist of the labels with $\mathcal{G} \neq 0$ for which some original finite cell has retained initial column.

Theorem 88.20 (Weighted count of the supported rational labels)**.** *For $0 \leq u < 2$, with $M_u = \int_0^\infty 4r^{3+u}(1+r^2)^{-3}\,dr$, one has*

$$\sum_{\gamma\in\mathcal{R}_Y} w_\gamma^{-1} \leq (9M_u + 2)^{fd} Q^{-(2+u)fd} Y^{-u/2}.$$

On the common intervals this is at most $\exp(-(98-o(1))fd\,x^{\varepsilon})$. *On labels with a single reversed block of length* d *it is at most* $\exp(-(132-o(1))fd\,x^{\varepsilon})$.

Proof. The nonzero leading minors have as indices the endpoints J of a composition of d, including d. At the nonterminal endpoints $D_j \in Q^j\mathcal{O}_E$. At d the inclusion is $D_d + TD_{d-1} \in Q^d\mathcal{O}_E$ by Proposition 88.11. Thus, with the preceding values fixed, the product of the complex distances between two distinct values of D_j is at least Q^{jf}. The product of the magnitudes of each nonzero D_j, $j < d$, has the same lower bound. At the terminal endpoint,

$$\prod_{\tau} |\tau D_d| \;\geq\; |N_{F/\mathbb{Q}}(\, D_d\bar{D}_d - T^2 D_{d-1}\bar{D}_{d-1}\,)|^{1/2} > Q^{df}\sqrt{Y}.$$

Here both the total positivity and the cutoff refer to the very same cell. Since $Y > 1$ eventually, all endpoints satisfy the ordinary lower bound as well.

In a product of discs of coordinate radii $r_{j,\tau}$, put $R_j = (\prod_\tau r_{j,\tau})^{1/f}$. Dividing each coordinate by its radius puts the centers in a unit polydisc, separated in sup norm by at least Q^j/R_j. Packing polydiscs of half that radius gives at most $9^f(R_j/Q^j)^{2f}$ centers whenever the box is nonempty. Bound the support indicator by $(R_j/Q^j)^{uf}$ at $j < d$, and by $(R_d/Q^d)^{uf}Y^{-u/2}$ at d. These inequalities apply successively with all preceding values fixed. Integrate each radius against $4r(1+r^2)^{-3}\,dr$. Its tail from $|\tau D_j|$ equals $(1+|\tau D_j|^2)^{-2}$, so Tonelli gives a factor $(9M_u)^f Q^{-(2+u)jf}$ at each endpoint and a single additional factor $Y^{-u/2}$. For block lengths h_i, the number of scalar tuples with the given minors is at most $\prod_i h_i \leq 2^{d-|J|}$, since each successive scalar satisfies an equation of degree h_i. Sum over the subsets of $\{1,\dots,d-1\}$ of earlier endpoints, retaining at least the power of Q^{-1} from d. The binomial formula gives the first bound.

For one block write $D_d = \pm c^d$. All earlier minors vanish when $d > 1$; we are on intervals with $d \to \infty$. The terminal congruence implies $c/Q \in \mathcal{O}_E$, by the valuation at every finite prime. The nonzero centers c are separated in the product of embedding distances by Q^f. In a box of geometric mean radius R_0, the preceding terminal inequality requires $R_0 > QY^{1/(2df)}$. Packing as above, and using the indicator with exponent af, bounds this subsum by

$$(9M_{d,a})^f Q^{-(2+a)f} Y^{-a/(2d)}, \qquad M_{d,a} = \int_0^\infty \frac{4d\,r^{2d+1+a}}{(1+r^{2d})^3}\,dr.$$

Indeed the new radial weight is $(1+r^{2d})^{-2}$; its negative derivative supplies the density. For $0 \leq a < 4d-2$, splitting at one gives $M_{d,a} \leq 2 + 4d/(4d-2-a)$.

Take $u = 2 - 1/\log(2Q)$ and $a = 4d-2-1/\log(2Q)$. The bound $M_u \leq 1 + 4/(2-u)$ follows by the same split. We have $\log Q = (16+o(1))x^{\varepsilon}$ and $\log Y = (34+o(1))fd\,x^{\varepsilon}$ by $m = 2d$ and Theorem 71.9. All logarithms of the moment prefactors are $o(fd\,x^{\varepsilon})$ after their indicated powers. The two exponents in the statement follow. □

For a matrix entry supported on these columns, an upper bound of the form $\sum_b |J_{\gamma,b}| \leq w_\gamma^{-1}\exp((98-\eta)fd\,x^{\varepsilon})$, with fixed $\eta > 0$, would therefore bound the entire nondegenerate part by $\exp(-(\eta - o(1))fd\,x^{\varepsilon})$. Here b runs through the actual finite cells and $J_{\gamma,b}$ includes the archimedean integral. The theorem controls the sum of label weights. Bounding these integral amplitudes remains necessary.

88.10 Counting the last block scalar

Retain the columns of Theorem 87.38, with their hyperplane threshold Y. For a single reversed block, Theorem 88.20 counted the block scalar before raising it to the block length. The same calculation works with preceding blocks once their contribution to the denominator of the last scalar is included.

Theorem 88.21 (A bound uniform in every terminal block length)**.** *On the same intervals, for all the supported nondegenerate labels meeting the retained columns,*

$$\sum_{\gamma\in\mathcal{R}_Y} w_\gamma^{-1} \le \exp\bigl(-(132-o(1))fd\,x^{\varepsilon}\bigr).$$

Proof. Let $h\ge 2$ be the length of the last reversed block and set $j=d-h$. For fixed preceding data put $A=D_j$, taking $D_0=1$. Then $A=Q^j a$ for a nonzero $a\in\mathcal{O}_E$. The final minor is $D_d=\pm Ac^h$. Since $D_{d-1}=0$, Proposition 88.11 gives $a(c/Q)^h\in\mathcal{O}_E$. At every prime $\mathfrak{p}$ this says $v_{\mathfrak{p}}(c/Q)\ge -\lfloor v_{\mathfrak{p}}(a)/h\rfloor$. Consequently c/Q ranges in a fractional ideal $\mathfrak{l}$ with $N\mathfrak{l}\ge |N_{E/\mathbb{Q}}a|^{-1/h}$.

In the complex embedding τ count the possible scalars in the variable $v_\tau=|\tau A|^{1/h}\tau c$. For distinct choices of c the difference divided by Q belongs to $\mathfrak{l}$, and the norm of a nonzero element of this ideal is at least $N\mathfrak{l}$. Thus the product of the coordinate distances is at least

$$|N_{E/\mathbb{Q}}A|^{1/(2h)}Q^f(N\mathfrak{l})^{1/2}\ge Q^{df/h}.$$

The dependence on the earlier value a cancels. Also $|v_\tau|^h=|\tau D_d|$, so that in any box of these variables with geometric mean radius R_0 a retained contribution requires $R_0>Q^{d/h}Y^{1/(2hf)}$ by Corollary 87.39. The count in that box is at most $9^f(R_0/Q^{d/h})^{2f}$ when it is nonempty, by the polydisc packing argument of Theorem 88.13. For $0\le b_0<4h-2$, bound the support indicator by $(R_0/Q^{d/h})^{b_0 f}Y^{-b_0/(2h)}$. Integrating the negative radial derivative of $(1+r^{2h})^{-2}$ gives the following bound for the sum of the final weight at the fixed prefix:

$$(9M_{h,b_0})^f Q^{-(2+b_0)df/h}Y^{-b_0/(2h)},\qquad M_{h,b_0}\le 2+\frac{4h}{4h-2-b_0}.$$

As in Theorem 88.20, the moment is the integral of $4h\,r^{2h+1+b_0}(1+r^{2h})^{-3}$. Counting c itself already includes all possible last-block roots.

Write $\eta=1/\log(2Q)$. Sum the earlier block scalars with their original weights. All their endpoint minors D_i are nonzero multiples of Q^i. Packing at each earlier endpoint, inserting the exponent $2-\eta$ and summing over compositions as in Theorem 88.13, bounds that sum by

$$(9M_{2-\eta}+2)^{fj}Q^{-(4-\eta)jf}.$$

For $j=0$ the empty prefix contributes one. In this estimate the power of Q^{-1} comes from the final prefix endpoint j; the root multiplicities $\prod h_i$ of the prefix are included in the binomial sum. It applies to any subset of prefixes because all summands are nonnegative.

Choose $b_0=4h-2-\eta$. Uniformly over $2\le h\le d$, the logarithms of the two moment prefactors together are $o(fd\,x^{\varepsilon})$: the first is $O(f\log(d\log(2Q)))$ and the second $O(fd\log\log(2Q))$. The saving from the last endpoint alone, divided by $fd\,x^{\varepsilon}$, is $132-34/h+o(1)$, since $\log Q=(16+o(1))x^{\varepsilon}$ and $\log Y=(34+o(1))fd\,x^{\varepsilon}$. For $h\ge\sqrt{d}$ this suffices. For $2\le h<\sqrt{d}$ the prefix contributes a further $64j/d+o(1)=64-o(1)$. The sum over the at most d lengths only costs $\log d$ in the logarithm. When $h=1$ the terminal congruence is shifted. The terminal-singleton estimate of Theorem 88.16 instead saves at least $192-o(1)$ on the same scale. This completes the proof. □

Accordingly the amplitude hypothesis following Theorem 88.20 suffices with $132-\eta_0$ in place of $98-\eta_0$, for any fixed $\eta_0>0$.

89 Rational orbits on the first plane

89.1 Short rational translates at the initial period

Keep the field E/F and the bottom decomposition $X = Y \perp Ez$ of Proposition 73.3. Write h for its Hermitian form, linear in the second variable, and $\eta = h(z,z) \in \{1,-1\}$. Write $G = U(X)$, $H = U(Y)$, and $\iota : H \to G$ for extension by the identity on z. Fix $g \in H(\mathbb{A}_F)$. The finite lattice and the real norms at this point are

$$\Lambda_g = X(E) \cap \iota(g_f)\widehat{\Lambda}_X = \Lambda_{Y,g} \oplus \mathcal{O}_E z, \qquad |y + bz|_{\tau,g}^2 = |y|_{\tau,g}^2 + |\tau b|^2.$$

Here $X(E)$ denotes the rational vectors of the E-space X. At each real place of F choose a frame in which h has diagonal entries ± 1 and use the Euclidean norm after applying $\iota(g_\tau)^{-1}$ in the adapted frame. Thus $|\tau h(v,w)| \le |v|_{\tau,g}|w|_{\tau,g}$. For a nonzero rational vector put

$$\mathfrak{c}_g(v) = \{\lambda(v) : \lambda \in \mathrm{Hom}_{\mathcal{O}_E}(\Lambda_g, \mathcal{O}_E)\}, \qquad H_g(Ev) = \frac{\prod_\tau |v|_{\tau,g}}{\sqrt{N\mathfrak{c}_g(v)}}.$$

Each integral linear form is extended E-linearly, and N denotes absolute ideal norm. The product runs over one embedding in each conjugate pair. Scaling v scales the numerator and denominator by the same positive number. Let W_g be the E-span of the lines of height strictly less than one.

For the original nonzero integral modulus $\mathfrak{m}$, set $\mathfrak{b} = \mathfrak{m} \cap \bar{\mathfrak{m}}$. Write $K(\mathfrak{m})$ for the unitary stabilizer of $\widehat{\Lambda}_X$ acting trivially modulo $\mathfrak{m}\widehat{\Lambda}_X$. A class $\gamma H(F)$ in $G(F)/H(F)$ meets the finite test if $\iota(g_f)^{-1}\gamma\iota(h_f) \in K(\mathfrak{m})$ for some $h_f \in H(\mathbb{A}_{F,f})$. In the grid, $\mathfrak{m} = \prod_{v \in S_0} \mathfrak{w}_v^{e_v}$, with one prime $\mathfrak{w}_v$ above each chosen split place v of F. Hence $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ are coprime, $\mathfrak{b} = \mathfrak{m}\bar{\mathfrak{m}}$, and $\sqrt{N\mathfrak{b}} = N\mathfrak{m}$. The choice of the primes and modulus is recorded in `Development/AbelianVarietyTheory/Definitions/Divisors/BasePlaceNotMemChiGridBadPlaces.lean`, line 43–82.

Theorem 89.1 (All short displacements at the original modulus)**.** *W_g is a totally isotropic subspace of Y. Every class meeting the finite test has displacement $w = \gamma z - z$ in $\mathfrak{b}\Lambda_g$. The nonidentity classes meeting the test and satisfying $\prod_\tau |w|_{\tau,g} < \sqrt{N\mathfrak{b}}$ correspond bijectively, by their displacements, to the vectors*

$$0 \neq w \in \mathfrak{b}\Lambda_{Y,g} \cap W_g, \qquad \prod_\tau |w|_{\tau,g} < \sqrt{N\mathfrak{b}}.$$

These assertions hold at every g, including on the dyadic components of the stated fields.

Proof. In local lattice bases the content ideal is generated by the coordinates of v. Thus $v \in \mathfrak{a}\Lambda_g$ exactly when $\mathfrak{c}_g(v) \subset \mathfrak{a}$, for any fractional ideal $\mathfrak{a}$. Integrality of the Hermitian form on this lattice gives $h(v,w) \in \overline{\mathfrak{c}_g(v)}\mathfrak{c}_g(w)$. When $h(v,w) \neq 0$, taking norms of the principal ideal and using the real Cauchy–Schwarz inequalities yields $H_g(Ev)H_g(Ew) \ge 1$. Taking $v = w$ and taking two distinct lines shows that all the short lines are pairwise orthogonal and isotropic. If $v = y + bz$ with $b \neq 0$, the integral projection onto the coefficient of z gives $b \in \mathfrak{c}_g(v)$. Then $\prod_\tau |v|_{\tau,g} \ge |N_{E/\mathbb{Q}} b|^{1/2} \ge \sqrt{N\mathfrak{c}_g(v)}$. This places W_g in Y.

The lattice $\widehat{\Lambda}_X$ is Hermitian self-dual. If $k \in K(\mathfrak{m})$, then $k^{-1} - 1$ also takes this lattice into $\mathfrak{m}\widehat{\Lambda}_X$. Taking its Hermitian adjoint shows that $k - 1$ takes it into $\bar{\mathfrak{m}}\widehat{\Lambda}_X$: duality gives this directly by pairing against each integral vector. Hence $K(\mathfrak{m}) = K(\mathfrak{b})$. Since $\iota(h_f)z = z$, the finite support condition implies $w \in \mathfrak{b}\Lambda_g$. If $w \neq 0$ satisfies the strict product bound in the statement, the content inclusion gives $H_g(Ew) < 1$. Distinct classes yield distinct vectors because the rational stabilizer of z is exactly $H(F)$.

For the converse the field contains $t = (1 + \sqrt{-q})/2 \in \mathcal{O}_E$ with $t + \bar{t} = 1$, since the selected prime is 3 mod 4. Given such a w, set

$$N_w(v) = \eta^{-1}\big(wh(z,v) - zh(w,v)\big), \qquad T_w = 1 + N_w + tN_w^2.$$

Orthogonality and isotropy give $N_w(z) = w$, $N_w(w) = 0$, $N_w^* = -N_w$, and $N_w^3 = 0$, since its image lies in $Ew + Ez$. Consequently $T_w^* T_w = 1 + (t + \bar{t} - 1)N_w^2 = 1$, and $T_w z = z + w$. Since $w \in \mathfrak{b}\Lambda_g$ and $\bar{\mathfrak{b}} = \mathfrak{b}$, both N_w and $T_w - 1$ carry Λ_g into $\mathfrak{b}\Lambda_g$. The same holds for $T_w^* - 1 = -N_w + \bar{t}N_w^2$. Hence $\iota(g_f)^{-1} T_w \iota(g_f) \in K(\mathfrak{b})$, using $\widehat{\Lambda_g} = \iota(g_f)\widehat{\Lambda}_X$ by the local description of a lattice. This gives the required rational class at exactly the original test. The integral element t works at the primes above two as well. □

89.2 Short vectors in every terminal Hermitian lattice

Let $N = n_m$ be the rank of the terminal even core, $f = [F : \mathbb{Q}]$, $D = 2fN$, and D_E the absolute field discriminant. Use the definite Hermitian lattices Λ_c of Proposition 73.4 and their given level $\mathfrak{m}$. Set

$$R = 2D^{D/2}|D_E|^{N/2}, \qquad B = (\sqrt{N}\,R)^{Nf}, \qquad H_* = (1 + 2R^2)^{2fN^2} R^{4fND}\,\mathrm{Nm}(\mathfrak{m})^{N^2}.$$

Lemma 89.2 (Uniform bases over the field and a class count)**.** *For each class c there are $x_1, \dots, x_N \in \Lambda_c$ forming an E-basis. In the Hermitian metrics at every selected complex embedding the column map from the standard orthonormal basis to these vectors has all singular values in $[B^{-1}, B]$. Contraction with the same vectors gives integral linear functionals $\xi_1, \dots, \xi_N$ in the dual lattice. Their column map into the Hermitian dual has the same bounds. The number of terminal level classes satisfies $h_m \le H_*$.*

Proof. Write $\Lambda = \Lambda_c$ and h for its definite Hermitian form. Give its underlying $\mathbb{Z}$-lattice the positive bilinear form $\mathrm{Tr}_{E/\mathbb{Q}}\, h$. Its dual for this form is $\mathfrak{D}_{E/\mathbb{Q}}^{-1}\Lambda$. Indeed trace integrality against all multiples of an element of Λ is precisely the condition that its Hermitian pairing take values in the inverse different. Hermitian unimodularity gives the formula after localization at each ideal. The index of Λ in the trace dual is $|D_E|^N$, since Λ has local rank N. The Euclidean covolume is therefore $|D_E|^{N/2}$. Each nonzero vector has squared trace length a positive integer and hence length at least one.

The Euclidean unit ball contains the cube of side $2/\sqrt{D}$. Minkowski's second theorem bounds the product of the D successive minima by $D^{D/2}|D_E|^{N/2}$. The preceding lower bound for the first minimum gives D independent vectors of length at most R. They span the rational space, so we can choose an E-basis $x_1, \dots, x_N$ among them. Write X_τ for their column map at τ. Each column has norm at most R since its squared norm contributes to the trace norm. The Gram determinant $\Delta = \det(h(x_a, x_b))$ is a nonzero totally positive integer of F. At each real embedding its value is $|\det X_\tau|^2 \le R^{2N}$ by Hadamard's inequality. Its norm to $\mathbb{Q}$ is an integer at least one. Thus $|\det X_\tau| \ge R^{-N(f-1)}$ at each embedding. All singular values are at most $\sqrt{N}R$. Dividing the determinant by the product of the largest $N - 1$ bounds the least singular value below by B^{-1}. The Hermitian form identifies each x_a with a linear functional ξ_a by placing x_a in the conjugate-linear slot. Integrality and the equality of singular values follow from this anti-isometry.

For the count, every class at maximal level is determined by the isometry type of its lattice in the rational Hermitian space. Conversely a rational isometry carrying one lattice to another relates their representatives by the adelic stabilizer. In the basis just found every Gram entry belongs to $\mathcal{O}_E$ and has modulus at most R^2 at all embeddings, by Cauchy–Schwarz. Two distinct algebraic integers differ by at least one in maximum modulus over the embeddings, by

the norm of their difference. In $\mathbb{C}^f$ with maximum complex modulus, the open balls of radius $1/2$ about these integers are disjoint and lie in the ball of radius $R^2+1/2$. Comparing real volumes bounds their number by $(1+2R^2)^{2f}$. Taking all N^2 entries gives the first factor in H_*.

For a fixed Gram matrix use $M=\sum_a \mathcal{O}_E x_a$ as coordinate lattice. Then $M\subseteq \Lambda\subseteq M^{\#}$, where the sharp denotes its Hermitian dual. In this basis the determinant index gives $|M^{\#}/M|=|\operatorname{Nm}_{E/\mathbb{Q}}\Delta|\le R^{4fN}$. The image Λ/M is generated as an abelian group by at most D elements since Λ has $\mathbb{Z}$-rank D. There are at most $|M^{\#}/M|^D$ such subgroups, by counting ordered generating lists padded with zero. Each preimage defines at most one lattice for this fixed matrix. Finally, the number of level classes over one maximal-level class is at most the level index. At a marked split place this index is $|\mathrm{GL}_N(\mathcal{O}_v/\varpi_v^{a_v})|\le (Nv)^{a_v N^2}$, and elsewhere it is one. Multiplying these bounds proves $h_m\le H_*$. □

89.3 Separation of the arithmetic returns at the chosen point

Let $G=U(C_0)$, $H=U(Y)$, $K_f=K(\mathfrak{m})$, and $g\in H(\mathbb{A}_F)$ be a point of Theorem 67.3. Put $c=\iota(g)$. The real subgroup associated with its level component, in coordinates centered at c_∞, is

$$\Gamma_c=\{c_\infty^{-1}\gamma_\infty c_\infty:\gamma\in G(F),\ c_f^{-1}\gamma_f c_f\in K_f\}.$$

Write $p=4n^2+2$, $d_{\mathbb{R}}=2fn$, $w(h)=\prod_\tau W(h_\tau)$, where W is the defining operator norm. With the line height b_g of Theorem 89.17, define

$$\lambda=b_g^{1/f},\qquad R_g=2d_{\mathbb{R}}^{d_{\mathbb{R}}/2}|D_E|^{n/2}\lambda^{-(d_{\mathbb{R}}-1)},\qquad \rho=\min\{1/4,\lambda/(16R_g)\}.$$

Lemma 89.3 (A uniform sum over all arithmetic returns)**.** *Every nonidentity member of Γ_c satisfies $\max_\tau\|\gamma_\tau-I\|_{\mathrm{op}}\ge\lambda/R_g$. For the product real group one has*

$$\sum_{\gamma\in\Gamma_c} w(\gamma)^{-p}\le 2^{fp}\,T_0^{Cfn^3}(1+4/\rho)^{2fn^2}$$

with T_0 as in Theorem 59.10. At every point above half the supremum of its prescribed lowest-type row, the logarithm of this bound is at most a polynomial in x, uniform in the row, on the same original intervals.

Proof. Use the underlying integral lattice $\Lambda=C_0\cap c_f\widehat{\Lambda}_{C_0}$, using the diagonal embedding of the Hermitian E-space. At infinity use the positive Euclidean metrics in the frames translated by c_∞ and sum their squared norms. Every nonzero $v=y+tz$ in Λ has product of its local norms at least b_g. Indeed the lattice is $\Lambda_{Y,g}\oplus\mathcal{O}_E z$, and if $t\neq0$ that product is at least $|N_{E/\mathbb{Q}}t|^{1/2}\ge1$. If $t=0$ and $h(y,y)\neq0$, the integral value $h(y,y)$ has nonzero integer norm in F, and $|\tau h(y,y)|\le|y|_{\tau,g}^2$ everywhere. In the isotropic case the product is $H_g(Ey)\sqrt{N\mathfrak{c}_g(y)}$, with integral content, and $H_g(Ey)\ge b_g$. This uses the original g and its level lattice. In particular every nonzero Euclidean length is at least λ.

The absolute determinant of the trace form on this $\mathbb{Z}$-lattice is $|D_E|^n$. Its trace dual is $\mathfrak{D}_E^{-1}\Lambda$ by Hermitian unimodularity, with this index over Λ. In the positive metric the trace form is twice the Euclidean form composed with the orthogonal signature involution. The Euclidean covolume is therefore $2^{-fn}|D_E|^{n/2}$. Applying Minkowski's second theorem in dimension $d_{\mathbb{R}}$, as in Lemma 89.2, gives $d_{\mathbb{R}}$ independent integral vectors of length at most R_g, using the lower bound for all preceding minima. A nonidentity arithmetic return moves at least one of these vectors. Its displacement is a nonzero integral vector, since the finite return belongs to the stabilizer of the level lattice. Comparing its length to that of the starting vector gives the stated separation.

Let U_ρ be the product of the local operator-norm balls of radius ρ about I in the groups. The sets γU_ρ are disjoint, since the distance to I of a local member of $U_\rho U_\rho^{-1}$ is at most $2\rho/(1-\rho) < \lambda/R_g$. At one place cover the group ball $\|h - I\|_{\mathrm{HS}} \le 1/4$ by translates of $U_{\rho,\tau}$. A maximal $\rho/4$-separated set in the Hilbert–Schmidt norm does this, since the inverses of its centers have operator norm at most $4/3$. The disjoint Euclidean balls of radius $\rho/8$ about the centers lie inside the ball of radius $1/4 + \rho/8$. Thus at most $(1 + 4/\rho)^{2n^2}$ translates suffice. Write d_π for the product formal degree of the core series. Applying the lower volume estimate of Theorem 84.9 at each place gives

$$d_\pi \operatorname{vol}(U_\rho) \ge T_0^{-Cfn^3} (1 + 4/\rho)^{-2fn^2}.$$

For $t \in \gamma U_\rho$ one has $w(t)^{-p} \ge 2^{-fp} w(\gamma)^{-p}$. Integrating on the disjoint translates and using the upper integral estimate of the same theorem, $d_\pi \int w(t)^{-p}\, dt \le T_0^{Cfn^2}$, proves the sum bound after increasing C.

At a point above half the supremum, $\log(1/b_g) \ll f^2 n^5 \log b$ by the proof of Theorem 67.3. Also $\log T_0 \ll \log b = x^{\varepsilon}$, f, n are polynomially bounded, and $\log |D_E| \le 2f \log M$ for the cyclotomic modulus of Lemma 72.22. Its logarithm is $o(\log b)$. The definitions of λ, R_g, ρ now give the claimed polynomial bound for the logarithm. □

89.4 The finite mass of a nontransverse orbit on the common interval

For a first-plane orbit let $m_f(\gamma) = \prod_{v<\infty} m_v(\gamma)$ with the quotient measures of Theorem 77.10. Equivalently this is the mass in the finite adelic orbital quotient divided by its identity counterpart. Indeed the groups and their test sets are restricted products, the local stabilizers are compact open, and after division all but finitely many factors are one. This measure is taken before pairing the column coordinates, inserting the unipotent characters, or multiplying by the real orbital integral. Write $f = [F : \mathbb{Q}]$.

Theorem 89.4 (A finite mass at the original cuspidal period level)**.** *On each interval of Theorem 12.15 and for every sufficiently large x in it, use its very primes q, d_0, Q, T and set $c = Q\sqrt{-q}$. For $b_v = \mathrm{N}v$ the entire finite factor above γ_c is*

$$m_f(\gamma_c) = \prod_{v \mid q} b_v^{2s-2} \left[b_v^{2k} + (b_v - 3 + 2s(b_v - 1)^2) b_v^k + 1 \right].$$

The rational label has weight $w_{\gamma_c}^{-1} = (1 + Q^2 q)^{-2f}$ in Theorem 87.43. Uniformly on these intervals,

$$\log m_f(\gamma_c) = (k + s - 1) f \log q + O(f s\, q^{2-k}), \qquad \frac{\log(m_f(\gamma_c)/w_{\gamma_c})}{fk \log q} \longrightarrow 1.$$

More generally, for any c of Proposition 89.15 whose ideal beyond $Q\mathcal{O}_E$ is the extension of a squarefree product of split places outside QT, the exact product runs over those places in place of $v \mid q$.

Proof. The two primes of Q split completely in E and differ from q, d_0, T. Thus c has valuation one there and is a unit elsewhere except above q. Theorem 77.10 gives local mass one at all these minimal valuations, including at two and at the terminal graph. At q, $-d_0 \equiv 1 \bmod q$ and q is odd. The simple roots of $X^2 + d_0$ lift to $\mathbb{Q}_q$. Since $E = F(\sqrt{-d_0})$, every place of F above q splits in E.

The fields K_ψ, C_J are unramified at q by their conductors. Adjoining the square root of $-q$ to their local compositum has ramification index two by the Eisenstein polynomial $X^2 + q$.

The other quadratic field is already split there. This proves that each completion of F at q has ramification index two over $\mathbb{Q}_q$ and that $\sqrt{-q}$ has valuation one in either split component. The row level there is the full lattice stabilizer and T is a unit. Apply Theorem 79.1 at every such place. The same two local theorems give the general squarefree assertion. Proposition 89.15 gives the single minor; each of its archimedean squared moduli equals Q^2q.

At a place over q the bracket divided by b_v^{2k} is $1 + O(sb_v^{2-k})$, with an absolute constant, by direct division. Its logarithm satisfies the same bound for large k, since $s = O(\sqrt{x})$ and the original column ranks satisfy $m^5x^{\varepsilon} = o(k)$ as in the dimension estimate for the chosen words. The local degree formula gives $\sum_{v|q} \log b_v = (f/2)\log q$. These identities prove the asymptotic for $\log m_f$. Finally $\log w_{\gamma_c} = 2f\log(1 + Q^2q)$ and $\log Q = (16 + o(1))x^{\varepsilon}$ on the same intervals. Since $s = o(k)$ and $x^{\varepsilon} = o(k\log q)$, division by $fk\log q$ gives the limit. □

The positive finite factor can therefore exceed the weight used to count rational labels even for a single orbit. To estimate its contribution to the signed trace of Theorem 66.1, the two column coordinates and their phases must be contracted with the real kernel. The signed trace and the normalized integral theta pairing still require estimates for this contraction and for the other orbits.

89.5 The additive phases and the operator on the column

Use $c = Q\sqrt{-q}$ in Theorem 89.4 and the finite Bessel seed of equation (70.16) with the original character χ_{ϖ}. At $v \mid q$ let A_v denote the sum of the translations to all neighbors of the first split column lattice. This is the unnormalized spherical Hecke operator supported on the double coset with exponents $(1, 0, \dots, 0, -1)$. Use the normalization in which the identity orbital factor acts by the identity on the column.

Theorem 89.5 (The finite operator with its Bessel phases)**.** *For the right-invariant seed at level K^0 the finite operator for γ_c is the product over $v \mid q$ of*

$$F_v = a_b\,1 + u_b A_v, \qquad b = \mathrm{N}v,$$

with a_b, u_b of Proposition 77.11. All the local additive phases in this formula are one.

Proof. At $v \mid q$, $F_v/\mathbb{Q}_q$ has ramification index two, and q is odd. Its different has exponent one. For example, in either split completion the extension is obtained by adjoining $\sqrt{-q}$ to an unramified field. The Eisenstein generator in this totally ramified quadratic extension gives an integral basis $1, \sqrt{-q}$: in $a + b'\sqrt{-q}$ the two terms have different valuation parities. Its different is therefore generated by $2\sqrt{-q}$, also by direct trace duality in that basis. The absolute trace character is trivial on $c^{-1}O$, the inverse different at v. This holds in both split factors.

Take each first-plane representative $n(l, H, z)$ from Theorem 79.1. Its simple flag entries have denominators dividing c on both sides. The first finite phase is therefore one. To compute the phase at $g^{-1} = \gamma_c^{-1}n(l, H, z)$, take the integral primitive representatives constructed in that same proof. If $i = \infty$ the primal list needs no changes. Otherwise every change uses the fixed combination $Au + cv$ of its earlier integral columns with A a unit. The coefficients added to a later column have denominators dividing c. The dual changes likewise have denominators dividing c, using D a unit when $j < \infty$. Integral lifts of the coordinates of X, Y only multiply these coefficients by integers. Denote these upper unitriangular changes by V in the two split factors. By Lemma 76.30 an element $k_0 \in K_v^0$ carries the reference decorated list to g times the reference list changed by V. Thus $k_0^{-1}g \in B(F_v)$ acts on the list by V^{-1}. For a unitriangular matrix, inversion negates its first superdiagonal. The simple entries for $(k_0^{-1}g)^{-1}$, which is the left factor of g^{-1}, are consequently in $c^{-1}O$ too. Passing between e and ϖe uses only units here.

The trace character of their sum is one. These computations use the conductor of precisely the character occurring in the period.

At a minimal place Theorem 77.10 has the representative 1 and $\gamma_c^{-1} \in K_v^0$. At $v \mid q$ the first column is unchanged and its second column is the lattice just calculated. In integrating a pair of column vectors over their common column argument, a left and a right integral change of the relative position leave the pairing unchanged by Haar invariance and the level condition. The stabilizer weights count precisely the right U_L-cosets by Theorem 77.10. Proposition 77.11 now gives the identity and neighbor coefficients with their stated multiplicities. Different places act by commuting translations. The restricted product has just the nonidentity factors displayed. □

For a determinant character of the row level that is trivial on all its finite units the same formula applies directly. At a split full level, any unit character through the determinant can also be removed locally by twisting the translation action by its inverse, extended to give a uniformizer value one. Relative column representatives here can be chosen with determinant one, since both column lattices have equal determinant valuation. These observations leave the formula for the ordinary seed independent of choices of bases or of uniformizer.

89.6 An explicit tail for rational first-plane orbits

Keep the original fields, intervals, word, finite level and terminal frequency. Let $\mathcal{T}$ consist of the positive squarefree integers coprime to $2qd_0QT$ all of whose prime divisors split completely and are unramified in E. There are infinitely many such primes by Chebotarev. Put $c_0 = Q\sqrt{-q}$, $c_j = jc_0$, and let O_j be the orbital Gram operator for the ordinary seed at γ_{c_j}, normalized by its diagonal term as in Theorem 83.15. Define the positive real number

$$B_x = m_f(\gamma_{c_0})\, C_E\, M_D(W) \prod_{\tau} (2/|\tau c_0|)^D C_{n,h_\tau}.$$

Theorem 89.6 (A summable tail on the prescribed column)**.** *For every integer $Y \ge 1$, every sufficiently distant common interval, and $A = (2s+2)f$, one has*

$$\sum_{\substack{j \in \mathcal{T} \\ j > Y}} \|O_j\|_{\mathrm{op}} \le 2B_x \, \frac{Y^{1-A}}{A-1}.$$

The bound applies also after compression to a terminal character or to the retained coordinates. For the actual real-frequency sections, with $m = 2(s+1)$, its positive prefactor satisfies

$$\liminf_{x \to \infty} \frac{\log B_x}{(2r-2) f m x^{\varepsilon}} \;\ge\; 8$$

along these intervals, uniformly in the prescribed words.

Proof. The distinct j give distinct labels since their first minors are distinct. At the places above q, multiplying c_0 by a unit leaves the unsigned finite mass unchanged by Theorem 79.1. Above a prime $p \mid j$ every local factor has valuation one, residue size p, and is at full split level. There are f such places in F. At the other places use the minimal-valuation calculation in Theorem 77.10. Thus the extra unsigned mass is

$$\prod_{p \mid j} \left(p^{\,2k+2s-2} [1 + O(s p^{2-k})] \right)^f .$$

The constant is absolute by the polynomial in Theorem 79.1. For $k \ge 5$, $\sum_{a\ge 2} a^{2-k} \le 2^{2-k} + 2^{3-k}/(k-3)$ by integral comparison. Since fs is polynomial in x and k grows faster than m^5, the product of all the bracketed errors, even over all rational primes, is at most two on sufficiently distant intervals.

Disintegrate the orbital pairing as in Theorem 83.15. At an extra prime the local phase can have modulus one without being trivial. On a fixed representative both finite factors are unitary translations of column arguments, with scalar phases. The integral of their pairing over the common $[H]$ is at most the product of the column norms, by Cauchy–Schwarz and Haar invariance. This remains true if a representative is not spherical before averaging. Summing the finite weights therefore costs at most the unsigned mass. The bound for the real integral of norms in Theorem 83.10 gains j^{-Df}. Combining the estimates gives $\|O_j\|_{\mathrm{op}} \le 2B_x j^{-A}$, since $D = 2r-2 = 2k+4s$. Summing by integral comparison proves the first assertion. Orthogonal projections have norm at most one.

To measure the prefactor use the simultaneous concentration of Theorem 65.24, writing $b = \exp(x^{\varepsilon})$. On a set of relative $\|W\|^2$-mass at least $1 - \exp(-c_1 b r^2)$, with $c_1 > 0$ absolute, every $R(g_\tau) = a_{1,\tau}/\sqrt{2}$ is at least $\sqrt{T/2}(br^4)^{-s}$. Thus $\log M_D(W)/(Df)$ is at least $\frac{1}{2}\log(T/2) - s\log(br^4) + o(1)$. By Theorem 71.17, $\log T = (17+o(1))mx^{\varepsilon}$ and $\log Q = (16+o(1))x^{\varepsilon}$. Also $\log q = O(\log x)$ by Theorem 12.15. The lower bound of Theorem 83.16 gives $\log C_{n,h} \ge -O(n\log(2n))$. The discriminants have logarithms $O(f\log x)$ by the polynomial cyclotomic conductor and the conductor-discriminant formula. Consequently the loss from C_E and the local constants is $O(Df\log x)$. Finally the unsigned mass is at least one, $\log|\tau c_0| = \log Q + \frac{1}{2}\log q$, and $s/m \to 1/2$. Division by $Dfmx^{\varepsilon}$ gives the claimed lower limit. □

The lower limit concerns the positive prefactor in the tail bound. The signed contribution at $j=1$ is the real integral of Theorem 83.15 times its finite operator. Its possible cancellation and that of the other rational orbits still enter the period estimate in Section 6.6.

89.7 The first orbit at the original terminal prime

Keep the original fields, power intervals, level, word and column of Theorem 83.15. Use $c_0 = Q\sqrt{-q}$, $U_0 = Q\sqrt{q}$, and write O_1 for the orbital Gram operator at γ_{c_0}, normalized by the diagonal. Let $M_* = \mathcal{M}_{2K_*+1}(W_1)$, the full-flag height moment of the unit-frequency real section from Theorem 83.17. Put

$$H_x = m_f(\gamma_{c_0})\, C_E A_*^f (2/U_0)^{(r-1)f}(1 + 2\log^+(2T/U_0))^{(s-1)f}.$$

Theorem 89.7 (The critical power of the terminal prime)**.** *On these same intervals, including after projection to the prescribed terminal character and retained coordinates,*

$$\|O_1\|_{\mathrm{op}} \le H_x M_*, \qquad \limsup_{x\to\infty} \frac{\log H_x}{(r-1)fx^{\varepsilon}} \le -16.$$

Both inequalities are uniform in the words in question. The remaining moment M_* uses the actual word at that point of the interval.

Proof. By Theorem 83.15 the norm is at most the absolute real factor times the norm of the finite operator. The latter is bounded by the unsigned mass, since every translation entering the finite pairing has norm one before spherical averaging. Orthogonal compression preserves this bound. Apply Proposition 83.14 and then Theorem 83.17 at $\delta_\tau = U_0/(2T)$. The resulting product is

$$T^{-(r-1)f}\big(2T/U_0\big)^{(r-1)f} = (2/U_0)^{(r-1)f}.$$

This leaves precisely $H_x M_*$.

For the limiting upper bound, $\log m_f = (k+s-1)f\log q + O(fsq^{2-k})$ by Theorem 89.4, and $\log q = O(\log x)$ by Theorem 12.15. The discriminant formula of Proposition 83.19 gives $C_E \le 1$. Since $d! \ge 1$, $\log A_* \le O(r+\log(2K_*))$. The terminal-prime estimate of Theorem 71.17 gives $\log(1+2\log^+(2T/U_0)) = O(\log x)$. All these logarithmic contributions are $o((r-1)fx^\varepsilon)$. Finally $\log Q = (16+o(1))x^\varepsilon$, as follows from the two level primes between $(k\exp(x^\varepsilon))^8$ and $2(k\exp(x^\varepsilon))^8$, with k polynomial in x. Substitution proves the upper limit. □

The estimate controls the first term by a positive majorant whose explicit factor decays on the original intervals. A uniform upper bound for M_* at the required scale would turn it into a numerical bound for this term. The signed sum over all terms and the comparison with the integral theta lattice in Section 6.6 are still open.

89.8 The sum over all split squarefree first-plane multiples

Fix a field and a point of the intervals in Section 6.6, one of the prescribed words and a unit vector of the lowest row type. Put $c_0 = g_1 g_2\sqrt{-q}$, $r = k+2s+1$, $R = r-1$ and $b = \exp(x^\varepsilon)$. Let $\mathcal{T}$ be the positive squarefree integers prime to $2qd_0g_1g_2T$ whose prime factors split completely in E, including 1. For $j \in \mathcal{T}$, take the ordinary orbital Gram operator O_j at γ_{jc_0} with the diagonal normalization of Theorem 83.15.

Theorem 89.8 (Exponential decay for the entire split squarefree family)**.** *On the original fields, intervals and finite level,*

$$\sum_{j\in\mathcal{T}} \|O_j\|_{\mathrm{op}} \le \exp\bigl(-(16-o(1))(r-1)fx^\varepsilon\bigr), \qquad f = [E^+:\mathbb{Q}].$$

The estimate is uniform in the word and the unit vector. It holds also after compression to the original terminal character or the retained coordinates. In particular the normalized signed trace of this whole family on any such nonzero space has absolute value at most the same bound.

Proof. The relative element γ_{jc_0} commutes with the same column group and the same stabilizer J as γ_{c_0}, by Proposition 89.15. Thus the unfolding of Theorem 83.15 applies with each of the finite multipliers in Proposition 78.10. More explicitly, for each finite unipotent coset tuple one integrates over the common column quotient. Averaging over its right compact real column group contracts the real pair by Schur orthogonality, while the finite covectors are paired after their relative Hecke translation. The factor $1/\dim\rho$ cancels against the diagonal. The finite additive phases depend only on the finite tuple and remain in the finite operator. The modulus of the common real factor is bounded by I_j^+. All disintegrations are legitimate for each j by the absolute convergence of Theorem 64.22 or, for the positive real integral, by Lemma 83.4.

Multiplying Proposition 78.10 with Lemma 83.4 and cancelling $m_f(c_0)$ gives

$$\|O_j\|_{\mathrm{op}} \le H_x^{f\omega(j)} \min\{A_0, A_1 j^{-Rf}\}.$$

Put $L_x = \log H_x/\log 2$, so $H_x^{\omega(j)} \le j^{L_x}$ by $2^{\omega(j)} \le j$. Since $L_x = O(\log r)$ and $R = r-1 \to \infty$, the number $\theta = (L_x+2)/R$ lies in $(0,1)$ for large x. Bound the minimum by the product of its first and second entries raised to $1-\theta$ and θ, respectively. This yields

$$\|O_j\|_{\mathrm{op}} \le A_0^{1-\theta} A_1^{\theta} j^{-2f}.$$

Summing over every positive integer on the right costs at most $1 + \int_1^\infty t^{-2f}\,dt \le 2$. By Lemma 83.4 the logarithm of the product of constants is bounded by

$$-(1-\theta)(16-o(1))Rf\log b + C\theta Rfs\log b.$$

Here $\theta = o(1)$ and $\theta s = o(1)$ because $s \log r = o(r)$. Indeed the common construction has $m = 2s + 2$, $m^5 x^{\varepsilon} = o(k)$, and r polynomial in x. This proves the asserted exponent. Compression by an orthogonal projection decreases each operator norm. The absolute value of a normalized trace is at most the operator norm, so absolute convergence and the same bound also hold for its signed sum. □

This sum includes arbitrarily large rational multipliers. The projected period in Section 6.6 also contains other first-plane parameters and other relative flag positions. The estimates for the full period and for the strict cohomology on these intervals remain open.

89.9 All first-plane terms prime to the marked level

For each x in the power intervals of Section 6.6 use the skew coefficients $\xi \in \mathcal{O}_E^-$ and the descended ideal $\mathfrak{b}(\xi)$ defined before Theorem 83.6. Denote by $\mathcal{S}_{\mathrm{full}}$ all nonzero ξ whose ideal is supported away from QT at places of full row and column lattice level. The prime exponents of this ideal are arbitrary. As before O_ξ is the normalized orbital operator at $\gamma_{Q\xi}$.

Theorem 89.9 (The full-level first-plane sum with unrestricted valuations)**.** *For $f = [E^+ : \mathbb{Q}]$, $R = r - 1$, on the original fields, intervals and words one has*

$$\sum_{\xi \in \mathcal{S}_{\mathrm{full}}} \|O_\xi\|_{\mathrm{op}} \le \exp\bigl(-(16 - o(1))Rfx^{\varepsilon}\bigr).$$

The estimate holds on the terminal-character projection and the retained coordinates at the same level.

Proof. Let $C_k = 4(k+1)^2$, $h = s - 1$, and let $\Omega(\mathfrak{b})$ count prime factors with multiplicity. Proposition 77.2 at each divisor, and the minimal-valuation result of Theorem 77.10 at all the remaining places, bound the finite operator by

$$C_k^{\Omega(\mathfrak{b})} (N\mathfrak{b})^{R+2s+1}.$$

These are the local factors of the original rational orbit by the unfolding in Theorem 83.6. For $t_\tau = |\tau\xi|$ let $H(\xi) = \prod_\tau \max(1, t_\tau)$. Then $\prod_\tau t_\tau = N\mathfrak{b}$, and $2^{\Omega(\mathfrak{b})} \le N\mathfrak{b} \le H(\xi)$.

Use the two signed real bounds with constants a_0', a_1' in the proof of Theorem 83.6. Their common power $\prod_\tau t_\tau^{-R}$ cancels $(N\mathfrak{b})^R$, leaving at most $H(\xi)^{2s+1}$ from the finite norm power and $H(\xi)^{\log C_k / \log 2}$ from the multiplicities. The product of logarithmic factors is at most $H(\xi)^{2h}$. For $t_\tau \ge 1$ interpolate with the higher real bound with exponent

$$\theta = \frac{\log C_k / \log 2 + 2s + 1 + 2h + 3}{R}.$$

At $t_\tau < 1$ use the first bound alone. Since $a_1' \ge a_0'$, this gives

$$\|O_\xi\|_{\mathrm{op}} \le C_E (a_0')^{(1-\theta)f} (a_1')^{\theta f} H(\xi)^{-3}.$$

The sum of the last height power over all nonzero skew integers is at most 8^f by the skew-lattice count in Theorem 83.6. Here $\theta s = O((s \log k + s^2)/R) = o(1)$. In particular $0 < \theta < 1$ on the sufficiently distant intervals. Using $C_E \le 1$, $\log a_0' \le -(16 - o(1))Rx^{\varepsilon}$, and $\log a_1' \le CRsx^{\varepsilon}$ proves the stated exponent. The norm bound passes to orthogonal compressions. □

The remaining first-plane coefficients can have extra valuation at the marked level. The full projected period also contains other relative flag positions. The integral lower comparison and the uniform strict cohomology bound on these fields and intervals still require proofs.

89.10 A finite window for the marked first-plane contribution

At each point of the common intervals consider every nonzero skew integer ξ and its orbit $\gamma_{Q\xi}$. Let $\mathfrak{m}$ be the part of the descended ideal of ξ above g_1g_2T. Let $H = H(\xi)$ be the product of the larger of 1 and $|\tau\xi|$ over the real places of F, using one extension of each place to E.

Theorem 89.10 (The entire first-plane sum and its marked range)**.** *Uniformly in the prescribed words on the original intervals,*

$$\sum_{0\neq\xi\in\mathcal{O}_E^-} \|O_\xi\|_{\mathrm{op}} \le \exp(CRfs\log b).$$

For every fixed $0<\delta<1$,

$$\sum_{N\mathfrak{m}\le(Qk^{1-\delta})^f} \|O_\xi\|_{\mathrm{op}} \le \exp\bigl(-(\delta-o(1))kf\log k\bigr).$$

For every $A>0$ there is a constant M_A for which

$$\sum_{\log H\ge M_Afs\log b} \|O_\xi\|_{\mathrm{op}} \le \exp(-ARfs\log b).$$

All three conclusions hold after restriction to the original terminal-character projection and retained coordinates.

Proof. For all $y\ge 1$ the height summation of Theorem 83.6 and Lemma 83.9 give

$$\sum_{N\mathfrak{m}\le y} \|O_\xi\|_{\mathrm{op}} \le y^{R-1}Q^{-Rf}\{(k+1)!\}^{-f}\exp(CRf).$$

The 8^f from summing H^{-3} is included in the exponential. Insert $y=(Qk^{1-\delta})^f$. The factors involving Q give Q^{-f}. The integral of $\log u$ from 1 to $k+1$ gives $\log((k+1)!)\ge k\log k-O(k)$. Since $R=k+2s$ and $s=o(k)$, the second assertion follows.

For an unrestricted parameter use the higher bound of Lemma 85.11 at every place with $t_\tau\ge 1$, and the first real bound of Lemma 83.9 at $t_\tau<1$, where $t_\tau=|\tau\xi|$. The per-place constants are at most $\exp(CRs\log b)$. After removing the common product of t_τ^{-R} the large places give H^{-2R} and the small-place logarithms cost at most H^{2h}, $h=s-1$. For the remaining finite factors note that $2s+1+\log C_k/\log 2\le R-1$ for large x. Consequently Proposition 77.3 gives

$$C_k^{\Omega(\mathfrak{g})}(N\mathfrak{g})^{2s+1}(N\mathfrak{m})^{R-1}\le(N\mathfrak{g}\,N\mathfrak{m})^{R-1}\le H^{R-1}.$$

The field factor satisfies $C_E\le 1$. The unfolding gives

$$\|O_\xi\|_{\mathrm{op}}\le\exp(CRfs\log b)H^{-R-1+2h}.$$

Since $R=k+2s$, the last power is exactly H^{-k-3}. Summing H^{-3} costs at most 8^f, proving the unrestricted bound. Above a height cutoff H_0, the remaining factor is at most H_0^{-k}. Taking $\log H_0=M_Afs\log b$, using $R/k\to 1$, and choosing $M_A>2(C+A+1)$ proves the third assertion. An orthogonal compression only decreases the norms. □

For example, $\xi=Q\sqrt{-q}$ has $N\mathfrak{m}=Q^f$. The second assertion includes all its multiples by $a\in\mathcal{O}_F$ which are units above QT, with no size restriction at good primes. The part of the first-plane sum outside the two decay estimates is finite for each x, since $H<H_0$ bounds every embedding and the skew integers form a discrete lattice there. For instance $\xi=Q^2\sqrt{-q}$ belongs to this range. Its marked norm is Q^{2f} and its logarithmic height is $f(2\log Q+\frac12\log q)$, which is below $M_Afs\log b$ for large x. Theorem 89.12 bounds the entire skew family at these marked valuations.

89.11 Arbitrary powers at both unitriangular primes

For a nonzero skew integer $\xi \in \mathcal{O}_E^-$ use the descended ideal $\mathfrak{b}$ and the normalized operator O_ξ at $c = Q\xi$ of Theorem 83.6. Let $\mathfrak{t}$ be the part of $\mathfrak{b}$ supported over T. Define

$$D_Q(\xi) = \prod_{\substack{v \mid Q \\ v(\xi) > 0}} Nv.$$

In particular $D_Q(\xi) \le Q^f$, where $f = [F : \mathbb{Q}]$.

Theorem 89.11 (Decay with unrestricted valuations above $g_1 g_2$)**.** *For $y \ge 1$, uniformly on the fields, words and intervals of Section 6.6,*

$$\sum_{N\mathfrak{t} \le y} \|O_\xi\|_{\mathrm{op}} \le y^{R-1} Q^{-(R-k)f} \{(k+1)!\}^{-f} \exp(CRf), \qquad R = k + 2s.$$

In particular for fixed $0 < \delta < 1$, the entire sum with $N\mathfrak{t} \le k^{(1-\delta)f}$ is at most $\exp(-(\delta - o(1))kf \log k)$ as $x \to \infty$. There is no restriction on the valuations above either unitriangular prime. Both inequalities hold on the original terminal projection and on the retained columns.

Proof. Write $\mathfrak{a} = \mathfrak{b}/\mathfrak{t}$ and $C_k' = 4(k+1)^4$. Multiplication of Propositions 77.2, 77.3 and 77.7 bounds the finite factor by

$$D_Q(\xi)^k (C_k')^{\Omega(\mathfrak{a})} (N\mathfrak{a})^{R+2s+1} (N\mathfrak{t})^{2R-1}.$$

Each active place above Q contributes its extra $(Nv)^k$ exactly once, independently of valuation. Remove $\prod_\tau |\tau\xi|^{-R} = (N\mathfrak{b})^{-R}$ from the real estimates. The remaining good and unitriangular powers, apart from $D_Q(\xi)^k$, cost at most $H^{2s+1+\log C_k'/\log 2}$, with $H = \prod_\tau \max(1, |\tau\xi|)$. The terminal factor left over is $(N\mathfrak{t})^{R-1}$.

At a place where $|\tau\xi| \ge 1$, interpolate the two real estimates in Lemma 83.9 with weight $\theta = (2s + 1 + \log C_k'/\log 2 + 2h + 3)/(2R)$ on the higher exponent, where $h = s - 1$. At the other places use the lower exponent. Their logarithmic terms cost at most H^{2h} by $\prod_\tau |\tau\xi| \ge 1$. This leaves H^{-3} in the bound. Since $\log C_k' = O(\log k)$, the estimates for $\theta \log((k+1)!)$ and $\theta Rs \log b$ in that lemma still give $o(R)$. With the same diagonal and field normalizations we obtain

$$\|O_\xi\|_{\mathrm{op}} \le D_Q(\xi)^k (N\mathfrak{t})^{R-1} H^{-3} Q^{-Rf} \{(k+1)!\}^{-f} \exp(CRf).$$

Use $D_Q(\xi) \le Q^f$, since the two rational primes split completely. The sum of H^{-3} is at most 8^f by Theorem 83.6. This proves the first bound. Taking $y = k^{(1-\delta)f}$, use $\log((k+1)!) \ge k \log k - O(k)$, $s = o(k)$ and $R \ge k$ to obtain the stated decay. Orthogonal projections do not increase the operator norm. □

For example every $Q^j \sqrt{-q}$, $j \ge 0$, has $N\mathfrak{t} = 1$, as does its product with any nonzero F-integer prime to T. All these terms are included in one convergent sum. A large terminal part can still occur below the height cutoff of Theorem 89.10. The next theorem bounds its contribution at every terminal valuation.

89.12 Exponential decay for the whole first-plane family

Let $F = E^+$, $f = [F : \mathbb{Q}]$, $R = k + 2s$ and $Q = g_1 g_2$. Use the original finite row level, terminal character and real frequencies of Section 6.6. For every nonzero skew integer ξ let O_ξ be the orbital Gram operator of $\gamma_{Q\xi}$ divided by the diagonal normalization of Theorem 83.6.

Theorem 89.12 (The full first-plane contribution on the common intervals)**.** *Uniformly in the prescribed words and interval points,*

$$\sum_{0\neq\xi\in\mathcal{O}_E^-} \|O_\xi\|_{\mathrm{op}} \le Q^{-(R-k)f}\{(k+1)!\}^{-f}\exp(CRf).$$

In particular the entire sum is at most $\exp(-(1-o(1))kf\log k)$*. The same estimates hold for the sum after compression to the specified terminal character and retained coordinates. On any nonzero such space they also bound the absolute normalized trace of the sum.*

Proof. Let $\mathfrak{b}$ be the descended ideal of ξ, $C_k' = 4(k+1)^4$ and $D_Q = \prod_{v\mid Q,\ v(\xi)>0} Nv$. Away from QT use Proposition 77.2. Above Q use Proposition 77.7, and above T use Proposition 78.11. Since $k+4s \le R+2s+1$, their product in the common-column unfolding is bounded by

$$D_Q^k (C_k')^{\Omega(\mathfrak{b})} (N\mathfrak{b})^{R+2s+1}.$$

This estimate includes every valuation at every finite place. The two real estimates at the physical frequency, the field constant, and the height sum in the proof of Theorem 89.11 apply to the same parameters. Explicitly, the factor $\prod_\tau |\tau\xi|^{-R}$ cancels $(N\mathfrak{b})^R$. With $H = \prod_\tau \max(1,|\tau\xi|)$, the remaining finite powers other than D_Q^k cost at most $H^{2s+1+\log C_k'/\log 2}$. Take the interpolation weight $(2s+1+\log C_k'/\log 2 + 2(s-1)+3)/(2R)$ at the places $|\tau\xi|\ge 1$. The other places cost at most $H^{2(s-1)}$ in logarithmic factors. The result of Lemma 83.9 with these powers is

$$\|O_\xi\|_{\mathrm{op}} \le D_Q^k H^{-3} Q^{-Rf}\{(k+1)!\}^{-f}\exp(CRf).$$

The two unitriangular primes split completely, so $D_Q \le Q^f$. Theorem 83.6 bounds the sum of the height weights by 8^f. Absorb this in the exponential and use $\log((k+1)!) \ge k\log k - O(k)$ and $R/k \to 1$. The stated estimate also implies absolute convergence in operator norm. Orthogonal compression is contractive, and normalized trace on a finite-dimensional space has norm one. This proves the last assertions. □

This exhausts the parameters of Theorem 83.6, including simultaneous divisibility by both unitriangular primes and the terminal prime. The next theorem allows arbitrary relative scalars in the same intersection dimension.

89.13 All pairs whose isotropic spaces meet in codimension one

Let $\mathcal{S}_1$ denote the relevant, finitely supported rational labels in the ordinary one-word expansion for which $\dim(X_0\cap X_1) = s-1$, counted once each as in Theorem 87.43. Write O_γ for an individual orbital Gram operator in the identity normalization on the original column.

Theorem 89.13 (The entire codimension-one contribution)**.** *On the common fields and intervals of Section 6.6, uniformly in the prescribed word,*

$$\sum_{\gamma\in\mathcal{S}_1} \|O_\gamma\|_{\mathrm{op}} \le Q^{-(R-k)f}\{(k+1)!\}^{-f}\exp(CRf), \qquad R = k+2s.$$

The sum includes every allowable relative scalar. In particular it is at most $\exp(-(1-o(1))kf\log k)$*. The same bounds apply after the terminal-character or retained-column projections and to the absolute normalized trace on each nonzero projected space.*

Proof. For $\gamma = w_{Q\xi,\lambda}$ use $H(\xi) = \prod_\tau \max(1,|\tau\xi|)$, using one embedding of E above each embedding of F. The product of the real absolute values is A_ξ in Proposition 79.5. In particular

$1 \le A_\xi \le H(\xi)$. Factoring $\prod_\tau |\tau\xi|^{-R} = A_\xi^{-R}$ out of Proposition 89.16 cancels the first R powers of A_ξ in the finite estimate. Besides $D_Q(\xi)^k$, the remaining finite cost is at most $H^{4s+1+2\log C_k/\log 2}$.

At a place with $|\tau\xi| \ge 1$ interpolate the two real estimates with weight θ on the second, where

$$2R\theta = 4s + 1 + 2\log C_k/\log 2 + 2(s-1) + 7.$$

At smaller places use the first estimate. The product of their logarithmic terms costs at most $H^{2(s-1)}$, because the sum of the negative logarithms of $|\tau\xi|$ is bounded by $\log H$. Thus H^{-7} remains. As in Lemma 83.9, $\theta = O((s+\log k)/R)$, $\theta\log((k+1)!) = o(R)$ and $\theta Rs\log b = o(R)$ on the prescribed intervals. The field factor is at most one. We have obtained the individual bound

$$\|O_\gamma\|_{\mathrm{op}} \le D_Q(\xi)^k H(\xi)^{-7} Q^{-Rf}\{(k+1)!\}^{-f}\exp(CRf).$$

For a fixed ξ, put $\eta = \xi\lambda$. It is integral and $\eta\bar\eta = \xi\bar\xi$. Hence (η) divides $(\xi\bar\xi)$. The number of divisors of an integral ideal is at most its absolute norm, because $e+1 \le 2^e$ for each prime exponent e. Here the norm is A_ξ^4. Two possible generators of the same ideal differ by a unit of absolute value one at all embeddings. Such a unit is a root of unity by Kronecker's theorem. There are at most $8f^2$ roots of unity in E, since a primitive root of order N has degree $\varphi(N) \ge \sqrt{N/2}$ and $[E:\mathbb{Q}] = 2f$. The inequality follows by multiplying $\varphi(p^e) \ge p^{e/2}$ for odd prime powers and $\varphi(2^e) \ge 2^{(e-1)/2}$ for $e \ge 1$. Thus there are at most $8f^2H(\xi)^4$ possible λ's. For each pair (ξ,λ) there is at most one label by Theorem 88.9.

For completeness, the sum of H^{-3} over all $\xi \in \mathcal{O}_E \setminus \{0\}$ is at most 27^f. For coordinate radii $r_\tau \ge 1$, center a product of open complex disks of radius $1/2$ at each integer in the box $|\tau\xi| \le r_\tau$. Distinct products are disjoint, since otherwise the difference of the integers would have absolute norm strictly less than one. Comparison with the product of disks of radii $r_\tau + 1/2$ bounds the count by $\prod_\tau (1+2r_\tau)^2 \le 9^f \prod_\tau r_\tau^2$. Integrate this count against $\prod_\tau 3r_\tau^{-4}\,dr_\tau$ on $[1,\infty)^f$. The contribution of each point is H^{-3}, and the majorant integrates to 27^f.

Now $D_Q(\xi) \le Q^f$. Sum the individual bound using the two counts and absorb $8f^2 27^f$ into $\exp(CRf)$. The factorial estimate and the assertions concerning projections and traces follow as in Theorem 89.12. □

In this ordinary one-word expansion, the case $X_0 = X_1$ has only the identity label by Proposition 72.20. Theorem 89.14 below treats every other proper nonzero intersection dimension and the transverse pairs with full common column.

89.14 The full sum for all common isotropic flags

For $1 \le j < s$ let $\mathcal{F}_j$ be the relevant, supported single-word rational labels with $\dim(X_0 \cap X_1) = s - j$. Let $\mathcal{F}_s$ denote the transverse labels with full common column. Each rational label is counted once in its ordinary diagonal normalization.

Theorem 89.14 (All proper intersections and the transverse full column)**.** *On the fields and intervals of Section 6.6, for every prescribed word and every $1 \le j \le s$,*

$$\sum_{w \in \mathcal{F}_j} \|O_w\|_{\mathrm{op}} \le ((k+1)!)^{-jf}\exp(Crjf).$$

The sum over all j is at most $((k+1)!)^{-f}\exp(Crf)$. There is no restriction on the prefix composition, coefficients or individual heights. These estimates hold at the original levels and after the prescribed orthogonal projections.

Proof. For $j = 1$ use Theorem 89.13. For $j \ge 2$ take a composition as in Proposition 78.9 and let $H_i = \prod_\tau \max(1, |\tau a_i|)$. Set $\eta_i = 3$ for $i < t$ and $\eta_t = \delta_k + 7$, with δ_k as in Proposition 78.4. The power constraint in Theorem 85.14 holds because $\sum_i J_i \le j^2$ and $j \le s$. Multiply its bounds over the real places. The factors from the intermediate products cancel their entire finite cost in Proposition 78.4, since

$$\prod_\tau |\tau G_p|^{-2u_p} = |N_{E/\mathbb{Q}} G_p|^{-u_p}.$$

The remaining rational measure factor is at most $2^{f\sum_p u_p}$ by Proposition 78.9 and the whole-plane measure calculation of Lemma 78.12. The common compact column contributes the same Schur dimension factor as on the diagonal, since w centralizes its group and the characters agree on J for each relevant pair. The lowest-type product and polarization argument of Theorem 85.1 therefore bounds the ordinary operator by these products.

Now put $A = |N_{E/\mathbb{Q}} G_j|^{1/2} = Q^{jf} |N_{E/\mathbb{Q}} a_t|^{1/2}$. We have $1 \le A \le Q^{jf} H_t$, and $E_j = d - j + 4hj + 3j^2 \ge k$. Consequently the remaining norm power $A^{k+\delta_k - E_j}$ costs at most $Q^{jf\delta_k} H_t^{\delta_k}$. The contribution of $Q^{jf\delta_k}$ and the measure cost is $\exp(O(rjf))$, since $\log Q = O(\log b)$ and $(s + \log(k+1)) \log b = o(r)$. Thus each label is bounded by

$$((k+1)!)^{-jf} \exp(Crjf)\, H_t^{-7} \prod_{i<t} H_i^{-3}.$$

For given nonzero integers a_i, the successive block coefficients solve $(c_i/Q)^{j_i} = a_i / a_{i-1}$, with $a_0 = 1$ and with the signs fixed by our definition of G_p. Each equation has at most j_i roots in E, giving at most 2^j choices altogether. The integer $T a_t \lambda$ has prescribed relative norm $T^2 a_t \bar{a}_t$, so the ideal divisor and root-of-unity count in Theorem 85.1 gives at most $8 f^2 T^{4f} H_t^4$ possible scalars. The parameters determine a rational label by Proposition 78.9. It remains to sum $\prod_i H_i^{-3}$ over the tuples. The disk count of Theorem 89.13 gives at most 27^{ft}. There are 2^{j-1} compositions, corresponding to subsets of the $j-1$ possible cuts. Since $t \le j$ and $\log T = O(s \log b) = o(r)$, all these counting factors are absorbed by $\exp(Crjf)$. This proves the first assertion. Summing the geometric majorant, as in Theorem 85.13, proves the second, and orthogonal projections are contractions. □

When $X_0 = X_1$, Proposition 72.20 leaves only the identity at the given level. Thus the remaining one-word contribution outside the cases above comes from transverse pairs with smaller common column. For pairs of distinct words the same positions have zero real operator by Theorem 76.2. Indeed the representative in Proposition 78.9 centralizes the whole definite column of rank k, and the restrictions of the two real types to that column are inequivalent. The mixed real integrals converge by the same weighted Cauchy–Schwarz estimate as above, applying the diagonal moment bound to each word separately. The complete projected period and the strict cohomology estimate of Section 6.6 remain open.

89.15 An integral plane for the first relative orbit

Take the field $E = E_x$ of Theorem 12.15, put $F = E^+$, and keep the row G, the group $B = N \rtimes H$ and the original level K^0 of Theorem 64.1. Write $Q = g_1 g_2$ for the product of the two split level primes. Assume $s = m/2 - 1 \ge 2$. Put $u = e_+ + e_-$ in the first signed coordinate plane, with $h(e_\pm, e_\pm) = \pm 1$. Let M be the perpendicular of that plane. The subsequent flag vectors, the terminal vector ϖe, and the entire definite column W_0 belong to M. Write J for their Bessel group in $U(M)$, using the flag with u deleted. Embed J by the identity on the first plane.

Proposition 89.15 (A first-plane orbit integral also at two)**.** *There is an integral isotropic partner v of u over E such that u, v are a basis of the signed plane lattice. For every $0 \neq c \in Q\mathcal{O}_E$ with $\bar{c} = -c$, put*

$$\gamma_c y = y + c\, v h(v, y).$$

This is a rational isometry in K^0 at every finite place. It represents a relevant orbit with $t = 1$, relative scalar one, and $B \cap \gamma_c B \gamma_c^{-1} = J$. Its sole nonzero leading minor is $D_1 = c$. On every oriented column one finite representative gives the same column on the two sides.

Proof. Let $\alpha = (1 + \sqrt{-d_0})/2$. It satisfies the integral monic equation $X^2 - X + (1 + d_0)/4 = 0$, and $\alpha + \bar{\alpha} = 1$. Take $v = e_+ - \alpha u$. Then $h(u, v) = 1$ and $h(v, v) = 1 - \alpha - \bar{\alpha} = 0$. The determinant of u, v in the signed basis is -1. The identity $\bar{c} = -c$ cancels the two linear terms in the pairing of two images under γ_c. Its quadratic term vanishes by isotropy of v. Its inverse is γ_{-c}. The difference from the identity and the corresponding difference for the inverse have entries in $Q\mathcal{O}_E$. This gives the prescribed reduction at g_1, g_2. The graph defining the level above T changes only the middle lattice in M. Elsewhere the integral basis computation applies to the full stabilizer.

The two first vectors are u and $u + cv$. An element of the common stabilizer fixes both, hence fixes their plane pointwise. On M the two remaining decorations are identical and their stabilizer is precisely J. Their simple characters coincide there. The coordinate of u on the image of the second flag vector is zero there for both flags. This proves relevance. The common isotropic space is spanned by $u_2, \dots, u_s$, on which the two flags have the same decoration. The cross matrix pairs the first two first vectors to c, all its other isotropic coordinates to zero, and the two terminal vectors to $-T$. Since $s \geq 2$, this proves the minor assertion. Finally γ_c commutes with H and lies in the finite level. At a representative $h_f \in H(\mathbb{A}_{F,f})$ of the column, h_f and $\gamma_c^{-1} h_f = h_f \gamma_c^{-1}$ therefore give the same finite right coset. □

89.16 A representative for every pair with one moving line

Use the ordinary orbital expansion for one prescribed word on the fields of Section 6.6. Write X_0, X_1 for the two s-dimensional isotropic spaces. Suppose $\dim(X_0 \cap X_1) = s - 1 > 0$. Let u, v_1 be the first integral hyperbolic pair in Proposition 89.15, and M its perpendicular. For $c \in E^\times$ and $\lambda\bar{\lambda} = 1$ define a rational isometry by

$$w_{c,\lambda} u = c v_1, \qquad w_{c,\lambda} v_1 = u/\bar{c}, \qquad w_{c,\lambda}|_M = \lambda I_M.$$

The first two images form a hyperbolic pair and the last map preserves the form on M.

Proposition 89.16 (Rational parameters and their archimedean estimates)**.** *Every relevant supported label with this intersection has a representative $w_{c,\lambda}$ with $c = Q\xi$, $0 \neq \xi \in \mathcal{O}_E$ and $\xi\lambda \in \mathcal{O}_E$. Its common stabilizer is the group J in Proposition 89.15. For $t = |\tau\xi| > 0$ the two real majorants used in Lemma 83.9 apply to its local normalized coefficient. With $h = s - 1$, these are*

$$Q^{-R} \frac{\exp(CR)}{(k+1)!}\, t^{-R} (1 + 2\log^+(1/t))^h, \qquad Q^{-R} \exp(C' R s \log b)\, t^{-3R}.$$

The field factor for their product is the same $C_E \leq 1$.

Proof. Theorem 88.9 gives a length-one prefix with scalar $c = h(u, u_1)$ and the scalar λ on the common suffix and terminal vector. These data determine the rational double coset. The displayed isometry has those data. Proposition 72.20 gives the two assertions of integrality. The stabilizer of both decorations fixes u and cv_1, so it fixes their nondegenerate plane pointwise.

Its restriction to M must preserve the remaining decoration, whose stabilizer is J. Conversely the scalar λI_M commutes with J, proving equality and commutation with the relative isometry. The common-column unfolding of Theorem 89.8 therefore has the same rational quotient and the same Haar constants.

At a real place use the Euclidean unit Witt pair u', v' of Theorem 83.16. Passing from the original opposite vector to $v'/\sqrt{2}$ conjugates by a central first unipotent. Such elements commute with J and their Bessel character is trivial since $s \geq 2$. In this unit pair the exchange on the plane has lower left entry $C = \tau c/2$ and upper right entry $1/\bar{C}$. Write $\delta = |C|$. Multiplication by the unit scalar $C/(i\delta)$ on the plane changes the exchange with lower entry $i\delta$ into the exchange with entry C. On M the additional scalar $\tau\lambda$ also has absolute value one. Their block diagonal product is Euclidean unitary, normalizes B, commutes with J, and preserves each of the functions $l, \nu, \mathcal{A}$ on the left. For l and $\mathcal{A}$ this follows from the unit scalar on each vector of their defining exterior products. For ν it follows by taking the infimum over B after using the normalizer property. This block diagonal product commutes with the plane exchange.

Write $n(z)$ for a central first unipotent with purely imaginary upper plane entry z. Multiplying the exchange with lower entry $i\delta$ on the left and right by $n(1/(i\delta))$ gives the transvection with lower entry $i\delta$ and diagonal entries one. In a first unipotent fiber integral, such changes are absorbed by left invariance of the heights and translation of the unipotent variable. Consequently the fiber bounds for l^{-3R} in Lemma 85.11 and for the $(\mathcal{A}, \nu)$ weight in Theorem 83.18 hold with this δ. The reciprocal Cauchy–Schwarz integrals use the inverse exchange, which has the same absolute parameter and the inverse unit scalar on M. Only the heights in these fiber bounds are transported. The original squared section norms in the base integrals are bounded by the moments of Theorem 83.3.

The change from the terminal frequency ϖ_τ to unit frequency conjugates the exchange to one with lower entry C/T and the reciprocal conjugate upper entry. Its modulus on the quotient is T^{-R}, by the computation in Proposition 83.14 for $J \subset B$. The resulting estimates, including the factorial in the low estimate, are exactly the two bounds displayed. Schur orthogonality and product Haar measure give the field factor as in Proposition 83.13. □

89.17 A uniform sum over the transverse rational classes

At g let $\mathcal{D}_g$ be the set of displacements meeting the finite test in Theorem 89.1. It includes zero. Group these vectors by their images in X/W_g. Set $n = \dim_E X$, $f = [E^+ : \mathbb{Q}]$, and

$$b_g = \min\bigl(1, \inf_{\ell} H_g(\ell)\bigr), \qquad D_g(w) = \prod_{\tau} |z + w|_{\tau,g}^{-(2n+5)}.$$

The infimum can be taken over the isotropic rational lines of Y, with value $+\infty$ for an empty set. For every fixed g, one has $b_g > 0$. To see this, rescale a generator of each line to put its content in a fixed finite list of ideals representing the ideal classes. Multiplication by a unit keeps this content. The logarithms of the absolute values of the units form a lattice in the hyperplane of coordinate sum zero, by Dirichlet's unit theorem. Hence when $H_g(\ell) \leq 1$ we can rescale the generator further by a unit to give uniformly bounded norm in each real block, with bounds depending on E, g. The rational vectors of each of the indicated contents lie in the corresponding fractional ideal times Λ_g. Their real images form a subset of a lattice, so only finitely many lines occur. All their heights are positive.

Theorem 89.17 (Absolute mass for distinct classes modulo the short lines)**.** *Write* $C_n =$

$3^{4n+5}2^{2n+5}$, *so that* $\log C_n < 6n+9$. *On the actual fields and at every point* g,

$$\sum_{\substack{C\in\mathcal{D}_g \bmod W_g\\ C\neq 0}} \sup_{w\in C\cap\mathcal{D}_g} D_g(w) \le C_n^f\,(\sqrt{N\mathfrak{b}})^{-(2n+4)},$$

$$\sum_{0\neq w\in\mathcal{D}_g} D_g(w) \le C_n^f\,(b_g\sqrt{N\mathfrak{b}})^{-(2n+4)}.$$

In the first sum C *denotes a coset in* X *represented in* $\mathcal{D}_g$. *The first bound is independent of the lengths of the short isotropic vectors at* g.

Proof. We first give the packing estimate. Let $\mathcal{V}$ be a set in $\prod_{\tau=1}^{f}\mathbb{C}^n$ containing zero, with $\prod_\tau \|v_\tau - v'_\tau\| \ge R > 0$ for distinct vectors. For each nonzero v assign the integers $k_\tau \ge 0$ given by $2^{k_\tau} \le 1+\|v_\tau\| < 2^{k_\tau+1}$, and put $K=\sum_\tau k_\tau$, $B_\tau = 2^{k_\tau+1}$. In an occupied box $R \le \prod_\tau B_\tau = 2^{K+f}$ by separation from zero. Set $p=(R/\prod_\tau B_\tau)^{1/f}\le 1$. Products of open balls of radii $pB_\tau/2$ around the points in this box are disjoint, since an intersection would violate the product separation. They are contained in the product of the balls of radii $(1+p/2)B_\tau$ about zero. Comparing volumes in real dimension $2n$ per block bounds the number of points by

$$(1+2/p)^{2nf} \le 3^{2nf}\,2^{2n(K+f)}R^{-2n}.$$

Each point has weight $\prod_\tau(3/(1+\|v_\tau\|))^{2n+5} \le 3^{(2n+5)f}2^{-(2n+5)K}$. Multiplying and using $R^4 \le 2^{4(K+f)}$ bounds the weight in the box by $3^{(4n+5)f}2^{(2n+4)f}R^{-(2n+4)}2^{-K}$. Summing 2^{-K} over the f nonnegative indices gives 2^f. This proves the stated constant for any finite set of points and then for the full sum of nonnegative terms. Finally $3^4 2^2 < e^6$ and $3^5 2^5 < e^9$.

For an admissible displacement w the vector $z+w$ has Hermitian length $\eta=\pm1$. In each adapted Euclidean norm $|z+w|_{\tau,g}\ge 1$ and $|z|_{\tau,g}=1$. The triangle inequality gives $1+|w|_{\tau,g}\le 3|z+w|_{\tau,g}$, so $D_g(w)$ is at most the weight just summed. Every difference of two displacements lies in $\mathfrak{b}\Lambda_g$. If the difference is outside W_g, its line has height at least one, and the product of its real norms is at least $\sqrt{N\mathfrak{b}}$. Choose one point from each of finitely many nonzero cosets and adjoin zero. The packing estimate applies independently of these choices. Taking suprema over the choices and then over finite sets of cosets proves the first inequality. For the complete set every nonzero difference has height at least b_g, by Theorem 89.1 and the definition of b_g. Applying the same packing estimate with this smaller separation proves the second inequality. □

89.18 The integral orbit and its measure

Take the notation of Proposition 62.5. Write $G=U(S)$, $H=U(C)$, $u=\dim_E S$, and let K_f be the given level on G. At an unmarked place it is the stabilizer of Λ_S, and at a marked place it is its principal congruence subgroup of depth a_v.

Lemma 89.18 (The finite orbit cut out by the position test)**.** *For* $y\in S(\mathbb{A}_{F,f})$ *of Hermitian length* -1,

$$y\in z+\mathrm{j}\widehat{L} \quad\Longleftrightarrow\quad y\in K_f z.$$

Consequently the support on $H(\mathbb{A}_{F,f})\backslash G(\mathbb{A}_{F,f})$, *under* $g\mapsto g^{-1}z$, *is the image of* K_f. *Give both lattice stabilizers at each place volume one, before imposing any congruence, and take the quotient measure. The measure of this support is*

$$c_f = \prod_{v\in S_0}\frac{(Nv)^{-a_v(2u-1)}}{1-(Nv)^{-u}} \;\ge\; (N\mathfrak{m})^{-(2u-1)}.$$

Proof. The lattices $L = \mathfrak{l}^{-1}\Lambda_S$ and Λ_S have the same stabilizer, since locally their ratio is a scalar on the entire space. The same holds for congruences, which are conditions on the endomorphism $g - 1$. A compact translate of z lies in the required coset because $z \in L$ and the action reduces to the identity at every marked place.

At an unmarked nonsplit place the residue characteristic is odd. If y in the self-dual lattice has length -1, its line splits off integrally by orthogonal projection. Its orthogonal complement is unimodular and has the same rank as the complement of z. Every unimodular Hermitian lattice over this unramified quadratic extension admits an orthonormal basis. Indeed the residue form has a vector of nonzero norm by polarization and nondegeneracy. Split off a lift of unit norm value, rescaling that value to one by surjectivity of the norm on units. The last surjectivity follows from the residue norm and Hensel lifting on principal units. Induction gives the basis. An integral isometry of the two complements, extended by $z \mapsto y$, now gives the requisite compact element.

At a split v, the two halves of L_v are an integral module and its dual. Write $z = (t, \lambda)$ and $y = (t', \lambda')$, so that $\lambda(t) = \lambda'(t') = -1$. The action on pairs is $(t, \lambda) \mapsto (At, \lambda A^{-1})$. Take an integral basis $t, e_2, \dots, e_u$ with the last vectors in the kernel of λ; such a basis exists since $\lambda(t)$ is a unit. At an unmarked place one can take an analogous basis for y and map the former basis to it. At a marked place the coset condition gives $t' \equiv t,\ \lambda' \equiv \lambda \bmod \pi^{a_v}$. Define

$$At = t', \qquad Ae_j = e_j + \lambda'(e_j)t' \quad (2 \le j \le u).$$

Then $A \equiv I \bmod \pi^{a_v}$, so it is invertible, and $\lambda' A = \lambda$ on the chosen basis. This proves the claim at every split place, including the unmarked dyadic ones. Taking the product of the local compact elements proves the finite adelic assertion. The stabilizer in G of z is H because the line is nondegenerate. This identifies the stated support on the quotient.

Integration on the image of K_f gives $\operatorname{vol}_G(K_f)/\operatorname{vol}_H(K_f \cap H)$. The intersection is the same level in dimension $u - 1$, by the orthogonal splitting of L. The unmarked quotient volumes are one. At a marked place, reduction of $\mathrm{GL}_d(\mathcal{O}_v)$ modulo π^{a_v} gives index $(Nv)^{a_v d^2} \prod_{k=1}^{d}(1 - (Nv)^{-k})$. Taking the ratio for $u - 1, u$ gives the claimed formula. □

90 The mean terminal trace

90.1 Counting the flags fixed by a residue comparison

The upper unitriangular group $U_k(\mathbb{F}_q)$ stabilizes a complete flag with a specified nonzero vector on each one-dimensional quotient. We compute the probability $P_k(A; q)$ of Proposition 97.1 by forgetting these quotient vectors.

Lemma 90.1 (Fixed complete flags of a unipotent matrix)**.** *If A is not unipotent then $P_k(A; q) = 0$. Otherwise let λ be the partition of k given by the Jordan blocks of $A - 1$. Write $b_i = \lambda'_i$ for the number of blocks of length at least i, put $\ell = k - b_1$, and write $P(\lambda) = P_k(A; q)$. With $P(\varnothing) = 1$ one has*

$$P(\lambda) = \sum_{i : b_i > b_{i+1}} \frac{q^{b_i} - q^{b_{i+1}}}{q^k - 1} P(\lambda - \square_i), \qquad P(\lambda) \le k^{\ell} q^{-(k^2 - \sum_i b_i^2)/2}.$$

Here $\lambda - \square_i$ shortens a length-i block by one. Uniformly for $q \ge 2$ and sufficiently large k, $P(\lambda) \le q^{-k\ell/3}$.

Proof. Every matrix conjugate into U_k is unipotent. Conversely if A is unipotent and preserves a complete flag, it acts as one on every successive quotient. All decorations of that flag are

then fixed. Each ordinary complete flag has the same number of decorations, so it suffices to choose a uniform ordinary flag and test its invariance.

Put $N = A-1$. The first line L must lie in $\ker N$. For $i \ge 1$ the nested spaces $\ker N \cap \operatorname{im} N^{i-1}$ have dimensions b_i, as is seen on each Jordan block. The fraction of all first lines in the i-th space and outside the next is the coefficient in the formula. For such a line the image rank of N^a on the quotient by L is $\operatorname{rank} N^a - \mathbf{1}_{a<i}$ for every $a \ge 0$. The list of ranks determines the Jordan partition and gives precisely $\lambda - \square_i$. The rest of a uniform flag with first line L is uniform on the quotient. This proves the recursion over every finite field.

Write $c(\lambda) = (k^2 - \sum_i b_i^2)/2$. Removing the indicated box decreases c by $k - b_i$. Each coefficient in the recursion is at most $q^{-(k-b_i)}$, because $1 - q^{-(b_i - b_{i+1})} \le 1 - q^{-k}$. On any sequence of removals the product of these upper bounds telescopes to $q^{-c(\lambda)}$. To bound the number of sequences, fill each removed box with the number of boxes present just before that removal. These fillings increase along rows and columns and recover the entire sequence. There are ℓ boxes outside the first column. Recording their entries in fixed spatial order gives at most k^{ℓ} possibilities. The remaining entries in the first column must increase, so the record determines the whole filling.

Finally $\sum_i b_i^2 \le b_1 \sum_i b_i = b_1 k$, whence $c(\lambda) \ge k\ell/2$. For all sufficiently large k, $k \le 2^{k/6} \le q^{k/6}$. This proves the last bound, including the identity, for which $\ell = 0$ and $P = 1$. □

90.2 The trace when the two maps agree on the whole perpendicular

Keep the word, prescribed terminal space and retained fraction $\vartheta \ge 1/2$ of Proposition 90.4. The rank in the next result is the rank of $p_1 - \beta p_0$ on the common rational perpendicular.

Theorem 90.2 (The full rank-zero comparison contribution)**.** *The sum of the absolute normalized cell traces over all matching, relevant smaller-column transverse positions with $\mathcal{G} > 0$ and comparison rank zero is at most*

$$(k!)^{-df} \exp(Cfkd)$$

on the common fields and intervals. All endpoint heights and all values of the saturation index occur in this estimate.

Proof. Fix a label, set $R = A_-/Q^{df}$, and put $t_0 = k - 2d - 2$. This is nonnegative on the common intervals. On every cell being counted $I \ge R$ by Theorem 90.9. The index is constant on all refinements of a coarse row cell. Theorem 87.5 with $m = k - 2d$ bounds its t_0-th moment in the normalized compatible finite measure by $\exp(Cf)$. Hence the fraction with $I \ge R$ is at most $\exp(Cf)\min(1, R^{-t_0})$, by Markov's inequality and the trivial bound one.

In Proposition 90.4 the estimate before summing the finite covolumes has a uniform factor depending only on the label. We may therefore multiply the bound of Theorem 78.5 by this fraction when restricting to the present cells. Give just the terminal endpoint power $\theta_t = k - t_0 = 2d + 2$. Its weighted cost $L = d(2d+2)$ satisfies $L \le r^2/2$, and the extra exponent $CfdL \log b$ is $O(fkd)$ on the original intervals. In the product bound the remaining factors obey

$$Q^{-dfk} A_-^{k} \min(1, R^{-t_0}) H_t^{-(k-t_0)} \le 1.$$

For $R \le 1$ use $H_t \ge 1$. For $R > 1$ use $R \le A_- \le H_t$. The bound for this label is now $(k!)^{-df} \exp(Cfkd) \prod_i H_i^{-3}$. Summing by the integral endpoint count in Theorem 90.6 costs at most $4^d 27^{fd}$, which is absorbed in the stated exponential. □

90.3 An integral column for the ordinary series

On a good orbit of Theorem 99.5 fix a representative with positive column lattice L and a prescribed terminal character λ. For an archimedean covector ℓ, let $b(\ell)$ be the column supported on the orbit with coefficients obtained by transporting ℓ and multiplying by $\overline{\lambda(z)}$ at the scalar $z \in Z$, in the convention of Theorem 88.4. Its squared class norm is $c_{\rho,K^0}|Z|\|\ell\|^2$.

Use the integral covector Schur model of Proposition 62.7. If the polynomial degree after subtracting the determinant power j_τ is e_τ at τ, put $e_s = \sum_\tau e_\tau,\ a_s = \sum_\tau (e_\tau + k|j_\tau|)$. Here the highest weights refer to the representation whose dual supplies the covectors. Write $d_\rho = \dim \rho$. For $D = 2fk$ define

$$R_s = 2D^{D/2}|\Delta_E|^{k/2}T^f, \qquad B_s = (\sqrt{k}R_s)^{kf}.$$

Proposition 90.3 (A literal integral character column on a good orbit)**.** *There are k^{e_s} integral covectors on the chosen class whose frame operator on the complex covector space lies between $B_s^{-2a_s}$ and $B_s^{2a_s}$ times the identity. For at least one of these covectors,*

$$\|\Theta S b(\ell)\|^2 \geq c_{\rho,K^0}|Z|\,\alpha e^{-\Delta_x}\frac{d_\rho B_s^{-2a_s}}{k^{e_s}}M_{J,\mathcal{O}}.$$

Each coefficient of $b(\ell)$ belongs to the original integral covector model after adjoining the character values.

Proof. The lattice L has Hermitian determinant T^f by Lemma 87.10. Its trace lattice has rank D and Euclidean covolume $|\Delta_E|^{k/2}T^f$. This follows from a free sublattice by the trace discriminant identity and passage through the index, as in Lemma 89.2. Every nonzero vector has trace norm at least one, by positivity and integrality. The unit ball contains the cube of side $2/\sqrt{D}$. Minkowski's second theorem therefore gives D rationally independent lattice vectors of length at most R_s. They span the space over E, so choose k that form an E-basis. At every complex embedding their column matrix has largest singular value at most $\sqrt{k}R_s$. The product of the squared moduli of its determinant over the f embeddings is the norm of a nonzero integral Gram determinant and is at least one. Hadamard's inequality at the other embeddings, and the upper bound for the other $k-1$ singular values at the chosen one, give least singular value at least B_s^{-1}. The same vectors yield integral functionals by Hermitian contraction, with the same singular values.

Take all ordered tensors of these functionals and restrict them to the integral Young image. Include the determinant factor by the exterior product of the chosen vectors or functionals according to sign, exactly as on the dual side of Proposition 62.7. The tensor and exterior singular-value inequalities give the two claimed frame bounds. In representatives obtained by terminal scalar translation, the lattice L is the same because the covector kernels are fixed. The transport of the coefficients introduces only the indicated unitary level characters. These have root-of-unity values on the finite markings. Thus the orbit sum is integral in its stated model.

The map $i : \ell \mapsto b(\ell)/(c_{\rho,K^0}|Z|)^{1/2}$ is an isometry onto the orbit character space. The first moment is $d_\rho^{-1}\|Si\|_{\mathrm{HS}}^2$ by its trace identity. The matching complex norm theorem 66.1 implies $\|\Theta Si\|_{\mathrm{HS}}^2 \geq \alpha e^{-\Delta_x} d_\rho M_{J,\mathcal{O}}$ by summing over an orthonormal basis. Taking the trace of $(\Theta Si)^*\Theta Si$ against the lower frame bound gives a lower bound $B_s^{-2a_s}\|\Theta Si\|_{\mathrm{HS}}^2$ for the sum over the integral list. Some member attains at least its average. Undoing the normalization of i proves the assertion. □

90.4 The finite mass in the normalized terminal trace

Work on a field and interval of Section 6.6 with $d = s + 1$, $f = [F : \mathbb{Q}]$, $h = k - 1$, and $Q = g_1 g_2$. Fix one prescribed word. Let J_0 be a prescribed nontrivial terminal character space or the simultaneous zero-sum space of Theorem 88.4. One may retain all its finite class coordinates or a union of complete scalar orbits of fraction $\vartheta \geq 1/2$ in the full set of finite classes. Theorem 87.38 gives such a set on the common intervals. In this section P_0 denotes the projection onto this possibly restricted space. Write each ordinary orbital operator O_γ as the sum of its finite-cell terms $O_{\gamma,c}$. For a supported relevant transverse pair with $\mathcal{G} > 0$ set

$$B_\gamma = \frac{\sum_c |\operatorname{tr}(P_0 O_{\gamma,c})|}{\operatorname{tr} P_0}.$$

Take the original level and the real group measures of Theorem 85.3. The real sections have diagonal norm one. Write $a = \sum_\nu \binom{h_\nu}{2}$ for the block unipotent dimension over E, and $\mathcal{P}_\tau$ for the positive, locally normalized orbital integral there.

Proposition 90.4 (Cancellation of the fine level and of the word dimension)**.** *With the finite mass $\mathcal{M}_\gamma^{=}$ of Theorem 76.32, the projected trace satisfies*

$$B_\gamma \;\leq\; \frac{1}{\vartheta}\,\frac{\mathfrak{m}_{k-1}}{\mathfrak{m}_k}\,\frac{\mathcal{M}_\gamma^{=}/\mathfrak{m}_{k-1}}{(T^k-1)^f}\,\prod_\tau \mathcal{P}_\tau.$$

One can use the local factors for any unit pure tensor in the prescribed lowest type.

Proof. Give every compact real column group probability and each full finite row lattice stabilizer K_v^* volume one. At T this is the modified row lattice of Theorem 64.1. At Q write $I_\sigma = [K^* : K^0]$ and let I_π be the corresponding index on the definite column. There are

$$n_{\mathrm{cl}} = \mathfrak{m}_k I_\pi (T^k - 1)^f$$

fine column classes. Indeed each terminal covector has stabilizer of index $T^k - 1$ in its parent compact by transitivity on nonzero covectors. Multiplying the genus mass by the local indices gives the refined mass by orbit–stabilizer, and the fine rational stabilizers are trivial by Lemma 70.23. Each retained class carries the same $d_\rho = \dim \rho$ real coordinates.

In the coordinate formula of Theorem 88.4 two classes can contribute only on the same scalar orbit. For each such ordered pair the coefficient in the normalized trace has modulus at most $1/(\vartheta n_{\mathrm{cl}} d_\rho)$, with the basis elements on the two classes matched by unitary transport. On a cell, sum over the orthonormal real covectors before taking the absolute value. At each real integration point this sum is the inner product of the transported two real vectors, times a phase of modulus at most one. This follows from the completeness relation of an orthonormal basis. Its modulus is bounded by the product of the two norms, including when the common compact group is only $U(k-1)$.

The unit class-coordinate inner product includes c_{ρ,K^0} from Theorem 70.24. The sum of the covolume weights of all fine cells is $c_E^a (I_\pi/I_\sigma)\mathcal{M}_\gamma^{=}$, by the refinement and mass computations of Theorem 76.41, with $c_E = 2^{-f}\sqrt{|\Delta_E|}$. The integral of the two real norms over the common unipotent quotient equals the integral over the quotient by the whole common group, since the norms are invariant under the compact factor, which has probability one. On a pure tensor this factors over the real places. The trace of each signed cell, polarized in its lowest-type row vectors, is invariant under simultaneous right translation by the product maximal compact. Haar invariance gives this directly in the convergent integral. Schur's lemma on the irreducible product type therefore gives the same signed trace on every unit row vector.

The identity constant has $d_\rho c_{\rho,K^0} = v_N/(I_\sigma \operatorname{vol}_f K_N^*)$, where $K_N^* = N_f \cap K^*$. Indeed the integral unipotent levels agree at Q by Lemma 76.19. Substitution leaves exactly the proposed right side times $c_E^a \operatorname{vol}_f(K_N^*)/v_N$.

This last scalar is at most one. For detail, separate N into the radical of the full s-plane and the upper unitriangular group on its s coordinates. With ordinary Euclidean Witt coordinates at infinity, the covolume of the radical is the one in Lemma 78.12 at $j = s,\ n = k+1$. It has $s(k+1)$ complex rectangular variables, $\binom{s}{2}$ complex skew entries and s trace-zero diagonal entries. The additional upper group has $\binom{s}{2}$ integral E-coordinates, with additive covolume c_E each. The integral upper matrices lift on the two isotropic strings and have determinant one on the radical, so the covolumes multiply. The middle graph at T preserves the unipotent volume by Theorem 64.2. This gives

$$\frac{v_N}{\operatorname{vol}_f K_N^*} = 2^{fs(k+d)} c_E^{s(k+s)} |\Delta_F|^{s/2}.$$

Here the power of two comes from rescaling the integral Witt pairs to Euclidean length one, as in Lemma 78.12. Since $c_E \ge 2^{-f}$, $|\Delta_F| \ge 1$, and $a \le s(k+s)$, the quotient of c_E^a by this covolume is at most $2^{-f(s+a)}$. This proves the proposition. □

90.5 Counting single flags with a prescribed comparison rank

Fix one integral cross class Z with $\mathcal{G} > 0$ on the original row. Equivalence of terminal flags uses integral upper changes of frame and the row group N_Λ of Section 87.6. For a fixed row and initial flag count each terminal flag class once if it has a representative admitting a scalar-orbit comparison of rank h from Theorem 80.4. Use the normalized average on the initial terminal column genus. The bound below also applies when an arbitrary row over each column is chosen, or is averaged with a probability measure.

Theorem 90.5 (A mean bound for one integral cross class)**.** *Put $c = h+1,\ j = k-c$, and suppose $1 \le j < k$. With $n_T = T^f,\ \nu = 4 + \log_2(2f)$, set*

$$Y = 2^{fh} \prod_{\tau} (|\tau D_d| + T|\tau D_s|), \qquad A_c(Z) = 9^{cf} \max(1, G/n_T)^c J_Z^{c+s},$$

where J_Z is the index of the maximal-minor ideal in Theorem 87.21. For $Y \ge 1$ the mean number of the stated flag classes is at most

$$2\,\frac{\mathfrak{m}_j \mathfrak{m}_c}{\mathfrak{m}_k}\,(Y/n_T)^k n_T^{c+2s}\, Y^{\nu+1+s} A_c(Z).$$

For $Y < 1$ the count is zero.

Proof. If a representative admits the comparison, let $M = p_0(\Lambda \cap \ker(p_1 - \beta p_0))$ in the initial positive column lattice L. This has rank j, lifts integrally to $\Lambda \cap C$, and has determinant at most Y by Theorem 90.9. The existence of a sublattice with these properties suffices for the upper bound. In particular the determinant of an integral Hermitian lattice is at least one, which gives the zero assertion.

Let $M_* = (EM) \cap L$, $D = \delta(M_*)$ and $a = [M_* : M]$. Thus $Da \le Y$. The determinant D is an integer by the local discriminant formula used in Theorem 87.11. For fixed M_* there are at most $a^{j+\lambda}$ submodules of index a, where $\lambda = \log_2(2f)$, by Lemma 87.22. For each one Theorem 87.21 bounds the number of flags lifting it by $(n_T^2 a)^s A_c(Z)$. These two bounds hold in every row over the given column. The resulting upper bound is a function of L alone. By Theorem 87.11 the mean number of possible M_* at determinant D is at most

$$\frac{\mathfrak{m}_j \mathfrak{m}_c}{\mathfrak{m}_k} n_T^{c-k} D^{k+\nu}.$$

The normalized measure on any of the original full markings gives the same mean for a function of L by Lemma 87.10. The formula concerns the full mean, so restricting the count to specified initial coordinates can only reduce its unnormalized sum.

After removing the constant factors we must bound $\sum_{a\le Y} a^{j+\lambda+s}\sum_{D\le Y/a} D^{k+\nu}$. The inner sum is at most $(Y/a)^{k+\nu+1}$. Since $j=k-c$ and $\nu=4+\lambda$, the remaining sum of powers of a is bounded by

$$\sum_{a\le Y} a^{s-c-5} \le Y^s \sum_{a\ge 1} a^{-c-5} \le 2Y^s.$$

Restoring the factors proves the result. □

90.6 Two ranges in the original terminal trace

Keep the original word, terminal space, fields and intervals of Theorem 78.5. For $\mathcal{G}>0$ write $H_*=\max_i H_i^{1/j_i}$, using the heights of its normalized integral endpoints. Recall that A_- measures the product of the differences $|\tau D_d|-T|\tau D_s|$, divided by Q^{df} and truncated below at one.

Theorem 90.6 (The small terminal differences and the large endpoint heights)**.** *There is a constant M such that, for every fixed $0<\epsilon_0<1$, the sum of B_γ over all relevant supported transverse pairs of smaller common column satisfying at least one of*

$$A_- \le (Qk^{\,1-\epsilon_0})^{df}, \qquad H_* \ge b^{Mfd}$$

is at most $\exp(-(\epsilon_0-o(1))fkd\log k)$. Here $o(1)\to 0$ on the common intervals, and the estimate is uniform in the prescribed character and word. On each field at each point of an interval the complement is a finite set of tuples of algebraic integers.

Proof. For any fixed composition the original block data have multiplicity at most $\prod_i h_i \le 2^d$ over the tuple of the a_i, by Lemma 88.10. Indeed the proper minors are $Q^{j_i}a_i$, and the last original minor is $Q^d a_t - TD_s$, so every original endpoint is recovered. There are at most 2^{d-1} compositions. For $H(z)=\prod_\tau \max(1,|\tau z|)$, the sum of $H(z)^{-3}$ over the nonzero integers of E is at most 27^f, as in Theorem 89.13. One can see this by packing disjoint polydiscs of radius $1/2$ at the integer points in the complex embedding. Distinct centers have sup distance at least one by the norm inequality. A box with radii $r_\tau\ge 1$ thus contains at most $9^f\prod r_\tau^2$ points. Integrating against $\prod_\tau 3r_\tau^{-4}dr_\tau$ gives the stated bound. Therefore the sum of $\prod_i H_i^{-3}$ over all labels is at most $4^d 27^{fd}$.

For the first range use Theorem 78.5 with all $\theta_i=0$. The factor $Q^{-dfk}A_-^k$ is at most $k^{(1-\epsilon_0)dfk}$. Since $\log(k!)\ge k\log k-k$, the factorial and the preceding count give the required exponent.

For the second range take an index i attaining H_*. In the same theorem put power k at the terminal endpoint and additional power kd/j_i at endpoint i. Add the two powers if the indices agree. Their weighted sum is $2kd\le r^2/2$ on sufficiently distant intervals. The first power absorbs A_-^k by $A_-\le H_t$. The other contributes H_*^{-kd}. Take M larger than twice the constant of the term $CfdL\log b$ in the product theorem. The resulting negative power of b absorbs that entire term. Summation as above gives at most $(k!)^{-df}\exp(Cfkd)Q^{-dfk}$ on this range. The sum of the two upper bounds proves the assertion for their union.

Outside the two ranges, each a_i is integral and every one of its complex embeddings has modulus at most $H_i<b^{Mfdj_i}$. The integral embedding is a lattice and hence meets the resulting compact box in finitely many points. The block multiplicity already proved gives the last assertion. □

Combining this result with Theorem 89.14 in the first moment of Theorem 88.4 leaves only the smaller-column transverse terms with both a larger terminal difference and bounded endpoint heights. The theorem gives a uniform contribution outside that range. Controlling the signed trace inside it and comparing the resulting period with the designated integral pairing are still required.

90.7 A rank estimate in the ordinary terminal trace

Use the normalized terminal trace and the retained coordinate space of Proposition 90.4. For a fine transverse cell c with its two columns in one scalar orbit take the unique comparison of Proposition 97.1. Define

$$\ell_v(c) = \operatorname{rank}_{\mathbb{F}_{q_v}}(A_{\alpha,v} - 1), \qquad S(c) = \sum_{v \mid Q} \ell_v(c) \log q_v.$$

These numbers are independent of the coordinates on the coarse rational cell. The other cells have zero trace by the character projector formula in Theorem 88.4.

Theorem 90.7 (The terminal difference and the comparison ranks)**.** *Fix $0 < \epsilon_0 < 1$. On the fields and intervals of Section 6.6, sum the absolute normalized cell traces over all relevant smaller-column transverse labels with $\mathcal{G} > 0$. Restrict the sum to matching cells satisfying*

$$A_- \le (Qk^{\,1-\epsilon_0})^{df} \exp(S(c)/3), \quad \text{or} \quad H_* \ge b^{Mfd}.$$

Here A_-, H_*, M are as in Theorem 90.6. The sum is at most $\exp(-(\epsilon_0 - o(1))fkd \log k)$. In the complementary range the height tuples form a finite set on each field at each parameter value and

$$S(c) < 3 \log\!\left(\frac{A_-}{(Qk^{\,1-\epsilon_0})^{df}} \right) \le 3Mfd^2 \log b.$$

Proof. First fix a label and impose $S(c) \ge S_0$ with $S_0 \ge 0$. For a contributing comparison each marked reduction is unipotent. Lemma 90.1 bounds its proportion of all common refinements by $\exp(-kS_0/3)$. The number of coarse rational comparisons above one cell is at most $\exp(Cfk \log^2(2fk))$. Proposition 97.1 therefore gives the product of these two bounds for the proportion of the finite measure above each compatible coarse cell. The same upper bound holds if we retain only the chosen class coordinates.

The proof of Proposition 90.4 bounds the absolute normalized trace of each retained cell by its finite covolume times a bound independent of the cell for this label. We may thus insert the preceding factor before summing the covolumes. Every subsequent estimate in Theorem 78.5 concerns the label alone. Putting all its free endpoint powers equal to zero gives the bound

$$\frac{\exp(Cfkd - kS_0/3)}{(k!)^{df}}\, Q^{-dfk} A_-^k \prod_i H_i^{-3}.$$

We used $\log^2(2fk) = O(d)$, since f, k are polynomially bounded in x and $d \asymp x^{1/2-\varepsilon/16}$.

For the first range put $V = (Qk^{1-\epsilon_0})^{df}$ and take $S_0 = 3\max(0, \log(A_-/V))$. Thus

$$Q^{-dfk} A_-^k \exp(-kS_0/3) \le k^{(1-\epsilon_0)dfk}.$$

The sum of $\prod_i H_i^{-3}$ over the labels is at most $4^d 27^{fd}$ by the lattice count in Theorem 90.6. The factorial bound $\log(k!) \ge k \log k - k$ proves the desired exponent. On the second range Theorem 90.6 already bounds the sum of the absolute traces of all cells. Adding the two bounds proves the assertion for the union. Outside the union the first inequality is reversed and $A_- \le H_t \le H_*^d < b^{Mfd^2}$. These inequalities give the displayed bounds. Finiteness follows from the integrality of the endpoints as in Theorem 90.6. □

90.8 The first moment for the prescribed ordinary Bessel series

Work at the original terminal frequency, field, interval and fine level of Section 6.6. For each prescribed word take a unit pure real vector in its stipulated lowest type. Let S be its normalized ordinary Bessel periodization, J a specified character space with all local terminal characters nontrivial, or the simultaneous zero-sum space. Allow the retained classes of Corollary 88.7 of fraction $\vartheta \geq 1/2$. Write ξ for a random unit signed column from the first-moment formula on J, and $M_J = \mathbb{E}\|S\xi\|^2$.

Theorem 90.8 (The ordinary first moment on each prescribed terminal space)**.** *Uniformly in the prescribed word and terminal space,*

$$|M_J - 1| \leq \exp\bigl(-(1 - o(1))fk \log k\bigr).$$

For the matching complex theta operator in Theorem 66.1, with its norm factor $\alpha = C_x P(\mathbf{a}) > 0$, some unit signed column satisfies

$$\|\Theta S\xi\|^2 \geq \alpha e^{-\Delta_x} \left(1 - \exp\bigl(-(1 - o(1))fk \log k\bigr)\right) > 0$$

on sufficiently distant common intervals.

Proof. The ordinary Gram formula of Theorem 88.4 and Corollary 88.7 expresses $M_J - 1$ as the real part of the sum of the nonidentity orbital traces divided by $\dim J$. For coincident isotropic spaces only the identity contributes at this level by Proposition 72.20. Theorem 89.14 bounds the sum of operator norms for proper nonzero intersections and for transverse pairs with full common column by $((k+1)!)^{-f} \exp(Crf)$. This also bounds their absolute normalized traces. The remaining labels are transverse with smaller common column and $\mathcal{G} > 0$, as in Theorem 64.3.

Use Theorem 90.7 with $\epsilon_0 = 1/2$. Its two ranges together cost at most $\exp(-(\frac{1}{2} - o(1))fkd \log k)$. Outside them, $P_- = \prod_\tau (|\tau D_d| - T|\tau D_s|) > 1$ and $H_* < b^{Mfd}$. Theorem 90.2 already bounds all comparison ranks equal to zero. For every other contributing cell Theorem 80.4 gives $1 \leq h \leq C_0 d^2$ with one constant. We can enlarge the sum to all cells of these ranks at the same labels.

For such a label multiply Proposition 104.5 over the real places with all free endpoint powers zero. With $\kappa = k - d$, $E_0 = \kappa + 2ds + s^2$ it gives

$$\prod_\tau \mathcal{P}_\tau \leq \frac{\exp(Cfkd)}{(k!)^{sf}}\, G^{-\kappa} P_-^{2\kappa - E_0} n_T^k.$$

Indeed the between-block partial pivot products at any index with $u_p > 0$ are integral after multiplication by $(2T)^p$ in modulus. More explicitly, their product over all places, after this rescaling at each one, is the product of the moduli of the nonzero integral intermediate pivot product ω_p of Lemma 97.2. The final phase leaves those pivots unchanged. Its normalized inverse powers cost at most $(2T)^{2f \sum_p p u_p}$. The exponent of T at one place becomes $dE_0 + 2\sum_p p u_p - s(k+d) = k + O(d^3)$. The extra factor costs $\exp(O(fd^4 \log b))$ because $\log T = O(d \log b)$. The powers of two cost at most $\exp(Cfkd)$. This proves the displayed estimate.

Now apply Proposition 90.10. Its n_T^{-k} cancels n_T^k. The remaining Gram power is $G^{k-\kappa} = G^d$, whose logarithm is $O(fd^3 \log b)$. The exponent of P_- is $2\kappa - E_0 - k < 0$, so this factor is at most one. The error $\exp(Cfd^6 \log b)$ fits in $\exp(Cfkd)$ since $d^5 \log b = o(k)$ by the parameter sizes in Theorem 6.29. It remains to bound the genus factor times 2^{fhk}.

Put $c = h + 1$, $j = k - c$, so $c, j \geq 2$ and $c = O(d^2) = o(k)$. Theorem 87.33 gives $\mathfrak{m}_j \mathfrak{m}_c / \mathfrak{m}_k \leq \mathfrak{m}_2 (C/(kR_F))^{fjc}$, with root discriminant $R_F \geq 1$ and $\mathfrak{m}_2 \leq b^{Cf}$. Uniformly on this

range $jc = k(1+h) - c^2 \ge k(1+h/2)$ for large x, because $c^2/(kh) = o(1)$. For sufficiently large k, the decay $(k/C)^{-fkh/2}$ absorbs 2^{fhk}. Consequently

$$2^{fhk}\frac{\mathfrak{m}_j\mathfrak{m}_c}{\mathfrak{m}_k} \le (k!)^{-f}\exp(Cfk).$$

This leaves $(k!)^{-df}\exp(Cfkd)$ for each fixed rank and label.

The number of labels in the remaining range is at most $4^d 27^{fd}\exp(Cfd^3\log b)$. In fact $\sum_\gamma \prod_i H_i^{-3} \le 4^d 27^{fd}$ by Theorem 90.6, and $H_i \le b^{Mfd^2}$ for at most d endpoints. The number of ranks is at most k. These counts leave the same bound $(k!)^{-df}\exp(Cfkd)$ for the entire intervening sum of absolute cell traces. Add the two height ranges and the full-column estimate above. As $d \to \infty$, $r = k+2s+1 \sim k$, and $\log(k!) \ge k\log k - k$, the stated error for M_J follows.

For the specified whole cuspidal row the ordinary image already belongs to the projected space. The two-sided norm estimate in Theorem 66.1 now gives $\mathbb{E}\|\Theta S\xi\|^2 \ge \alpha e^{-\Delta_x} M_J$. The signed column ensemble is finite, so a member attains at least this mean. This proves the second assertion. □

The vector supplied by the first moment depends on the word and on the terminal character. The arithmetic comparison in Corollary 74.7 asks for the specified integral row and its distinguished pairing with controlled cost. The strict cohomology sum in Section 6.6 and this integral comparison are still open.

90.9 The determinant of the integral agreement lattice

Use the spaces of Lemma 104.6. In the original row Λ let $K = \Lambda \cap C_0$, the integral lattice on which the two column maps agree. Put $G = |N_{F/\mathbb{Q}}\mathcal{G}|$.

Theorem 90.9 (A determinant cutoff from an integral comparison)**.** *For every matching transverse cell of positive Gram determinant, with comparison of rank h as in Theorem 80.4,*

$$\delta(K) \le \frac{2^{fh}G}{\prod_\tau(|\tau D_d| - T|\tau D_s|)} = 2^{fh}\prod_\tau(|\tau D_d| + T|\tau D_s|).$$

In particular, if $h = 0$, then $A_- \le Q^{df} I$, where I is the index of Theorem 87.32.

Proof. The original row is self-dual. Put $B' = \Lambda \cap W'$. Orthogonal projection of Λ onto W' gives $(B')^\vee$. Locally every integral functional on B' extends to the row since B' is primitive, and self-duality represents the extended functional. The same reasoning applies to K in C_0. The quotient $\Lambda/(B' \oplus K)$ identifies by projection with each discriminant module. Indeed subtraction of a projected element of B', or of K on the other side, gives the kernel assertion. Thus $\delta(B') = \delta(K)$, using the positive square root of the discriminant index also on the nondegenerate indefinite lattice B'.

Take primitive integral frame bases for $U_i = \Lambda \cap Y_i$ as in Lemma 87.20. Changing either frame by an upper unitriangular matrix preserves its full cross determinant and its leading s-minor. Choose a full integral lattice M_1 in C_1 contained in Λ, and set $U' = U_0 \oplus U_1 \oplus M_1$, $b_0 = [B' : U']$. The Gram determinant of the first two frames has absolute norm from F equal to G. The third summand is orthogonal to them. Passing to the over-lattice divides the determinant by its additive index. It follows that

$$b_0 = \frac{G\,\delta(M_1)}{\delta(K)}.$$

The orthogonal projections of $\Lambda\cap Y_i^{\perp}$ onto W' are exactly $D_i=(B')^{\vee}\cap S_i$. In fact every lift of a vector of D_i by the surjection just proved differs from it by a vector of C_0, itself orthogonal to Y_i. For integral lifts v,w on the two sides, the row pairing is integral and so is the column pairing since $\beta L_0=L_1$. The difference $\mathcal{H}(v,w)$ depends only on the projections to W'. Hence $\mathcal{H}(D_0,D_1)\subset\mathcal{O}_E$.

In the coordinates of Lemma 104.6 the lattice $(U')^{\vee}\cap S_i$ is $\mathcal{O}_E^{d}\oplus M_1^{\vee}$. The index of D_i in it equals b_0. To see this, restrict the functionals on B' and on U' to U_i. Both maps onto its module dual are surjective because U_i is primitive in each lattice and the quotient is projective over the Dedekind ring. Their kernels are the two lattices in S_i. Consequently their index is the full dual index $[(U')^{\vee}:(B')^{\vee}]=b_0$. With the standard positive metric on the d coordinates and the given one on C_1, each D_i therefore has normalized Euclidean covolume $b_0/\delta(M_1)=G/\delta(K)$. Here the normalized covolume of a module of rank a uses the factor c_E^{-a} of Theorem 80.4. In this convention a Hermitian dual has reciprocal covolume.

The nondegenerate integral pairing maps D_1 into the Hermitian dual of D_0 by the matrices N_τ. Its image has positive integral index, equal by covolumes to

$$\left(\frac{G}{\delta(K)}\right)^{2}\prod_{\tau}|\det N_\tau|^{2}.$$

Taking square roots and applying Lemma 104.6 gives the first inequality. The identity follows by factoring $\tau\mathcal{G}$ at each place. If $h=0$, then $K=\Lambda\cap C$, whose determinant is $G/(Q^{2df}I)$ by Theorem 87.32. Substitution gives $\prod_\tau(|\tau D_d|-T|\tau D_s|)\le Q^{2df}I$. Since $Q^{df}I\ge 1$, this proves the assertion for the truncated A_- as well. □

90.10 From the mean flag count to the terminal trace

For a supported smaller-column transverse label put

$$P_-=\prod_{\tau}(|\tau D_d|-T|\tau D_s|),\qquad n_T=T^{f}.$$

For one word and one retained terminal space as in Proposition 90.4, let $B_{\gamma,h}$ be the sum of absolute normalized traces of its matching fine cells of comparison rank h. The normalization divides by the dimension of the retained space.

Proposition 90.10 (The finite count with the original trace weights)**.** *Fix C_0. When $\mathcal{G}>0$, $H_*<b^{Mfd}$ and $0\le h\le C_0d^2$, put $c=h+1$, $j=k-c$. On sufficiently distant common intervals,*

$$B_{\gamma,h}\le\exp(Cfd^{6}\log b)\,\frac{\mathfrak{m}_j\mathfrak{m}_c}{\mathfrak{m}_k}\left(\frac{2^{fh}G}{P_-n_T}\right)^{k}\prod_{\tau}\mathcal{P}_\tau.$$

The $\mathcal{P}_\tau$ are the real factors of Proposition 104.5. The constant is uniform in the label, the word and the retained fraction $\vartheta\ge 1/2$.

Proof. We first count fine rational cells above one initial column coordinate y. The initial finite support is B_fK^0/K^0. Its quotient by $B(F)$ is the fine column class space. Indeed $B=N\rtimes H$, the induced level projects onto K_H by Theorem 64.1, and $N(F)$ is dense in N_f by successive additive approximation in its unipotent coordinates. We may take a finite representative t_y in $H_f\subset B_f$. In the row lattice Λ it specifies, the rational stabilizer of the coset t_yK^0 in $B(F)$ is exactly N_Λ of Section 87.6. In fact its column component lies in the trivial fine rational column stabilizer. Within $N(F)$, integrality is sufficient. At the marked primes the whole integral unipotent radical reduces into the row level by Lemma 76.19, and t_y normalizes the rational

radical. Elsewhere the row level is the lattice stabilizer, using the original graph at T. This proves the asserted stabilizer equality.

Write a row coset above y as bt_yK^0 with $b \in B(F)$. If it supports the second decoration in the relative position γ, transport this decoration back to t_y by b^{-1}. This maps the fine rational cells above y injectively to the terminal flag classes used in Theorem 90.5. Here is the injectivity with the stabilizers included. If two transported flags agree after a change $n \in N_\Lambda$, then $b_2nb_1^{-1}$ belongs to both $B(F)$ and $\gamma B(F)\gamma^{-1}$, since it stabilizes the second decoration as a rational flag. It identifies the original row cosets because n fixes t_yK^0. Integral upper changes of frame represent exactly the same decorated rational flag among its primitive integral representatives, as in Lemma 87.20. The image of a rank-h matching cell has a representative with the integral lift of determinant at most $Y = 2^{fh}G/P_-$ used in Theorem 90.5.

In the trace estimate before summation in Proposition 90.4, the bound for one fine cell is

$$\frac{\operatorname{vol}_f K^0}{\vartheta n_{\mathrm{cl}}\, d_\rho c_{\rho,K^0}}\ \operatorname{vol}(\Gamma_D\backslash D_\infty)\prod_\tau \mathcal{P}_\tau,$$

where n_{cl} is the full number of fine initial column classes. This follows by contracting the real covectors in the normalized projector and integrating over the common unipotent quotient before taking the product of real norms. With the coarse row compact of volume one, the factor $\operatorname{vol}_f K^0/(d_\rho c_{\rho,K^0})$ equals $\operatorname{vol}_f K_N^*/v_N$ by the same proposition. There it was proved that this times $c_E^{a_0}$ is at most one. Proposition 78.6 therefore bounds the weight of a cell by $\exp(Cfd^6\log b)\prod_\tau \mathcal{P}_\tau/(\vartheta n_{\mathrm{cl}})$.

Summing over y now gives the uniform average of the flag counts on all fine initial coordinates. The pointwise bound by sublattice counts in Theorem 90.5 depends only on the column lattice. Its average over these coordinates is its genus mean, by Lemma 87.10. Discarded coordinates can be included in an upper estimate. There are at most $\exp(Cfd^6\log b)$ integral cross classes by Lemma 97.2. For each such class apply Theorem 90.5, using $J_Z \le G$ from Section 87.6. If $Y < 1$ the count vanishes. Otherwise

$$\log Y = O(fd^2\log b),\quad \log G = O(fd^2\log b),\quad \log n_T = O(fd\log b).$$

For the first inequality use $Y = 2^{fh}\prod_\tau(|\tau D_d| + T|\tau D_s|)$ and the endpoint height bounds of Lemma 97.2. Since $c = O(d^2)$ and $\nu = O(\log x) = O(d)$, the logarithm of $n_T^{c+2s}Y^{\nu+1+s}A_c(Z)$ is $O(fd^4\log b)$. The factors for the cross classes, the covolume and $\vartheta^{-1} \le 2$ all fit in the displayed exponential. The remaining factors are precisely those in the proposition. □

91 References

[1] W. T. Gan, J. P. Hanke and J.-K. Yu, *On an exact mass formula of Shimura*, Duke Math. J. **107** (2001), 101–133. Section 4, formulas (4.2)–(4.3) and Propositions 4.4–4.5.