The generalized Riemann hypothesis for Dirichlet L-functions

Volume XV · Real orbital integrals and matrix coefficients

26 November 2026 · Edition 700

Abstract

We estimate the archimedean factors in the orbital expansion. We compute matrix coefficients and height moments for the first-plane and higher-rank terms. For a rational flag with integral unmarked root ratios we bound the actual Bessel integral at its displaced archimedean frequency.

The guide in Volume I lists the sections and page counts for the series.

Contents

82 Real theta norms and projected coefficients . . . 3
82.1 Positive generators and the real scale of the theta tests . . . 3
82.2 An exact projected norm on the last compact pair . . . 6
82.3 Integral projection from the last noncompact real group . . . 10
82.4 Highest lines throughout the real theta correspondence . . . 13
82.5 An absolute upper bound for every real theta integral . . . 17
82.6 Totally positive generators in the comparison of real norms . . . 20
82.7 The last real projection and an integral coefficient . . . 21
82.8 Two consecutive integral evaluations . . . 23
82.9 An integral polynomial on every real pair . . . 24
82.10 An absolute estimate for the projected real integrals . . . 25
82.11 A compact conjugation average on a tempered column . . . 26
82.12 A tempered estimate with an auxiliary Hilbert space . . . 27
82.13 Absolute size of the last restriction on the period fields . . . 28
82.14 Vector evaluations and the signed theta row in the Hilbert norm . . . 30
83 Bounds for the first-plane integral . . . 32
83.1 Localizing the real frame integral . . . 32
83.2 The first vector on the common intervals . . . 33
83.3 Exponential decay for the first rational orbits . . . 34
83.4 Two bounds for the original real orbital integral . . . 35
83.5 First-plane parameters over the whole real field . . . 36
83.6 Squarefree first-plane parameters on the original intervals . . . 37
83.7 A sum over the actual algebraic integers . . . 39
83.8 A spherical matrix coefficient at arbitrary distance . . . 39
83.9 The factorial in the real first-plane estimate . . . 40
83.10 A weighted bound for the real orbital integral . . . 41
83.11 The weighted volume of the first-plane fiber . . . 42
83.12 The first flag coordinate of a tensor projection . . . 42
83.13 Separate real estimates for a field-valued multiplier . . . 44
83.14 The first-plane integral under a change of terminal frequency . . . 45
83.15 The real integral for one summand of the projected Gram form . . . 46
83.16 Integrating over the first isotropic line at infinity . . . 47
83.17 A full-flag estimate at small real parameter . . . 48
83.18 A height moment independent of the column dimension . . . 49
83.19 The real Haar measure in the first-plane integral . . . 49
83.20 The terminal minor and the size of a real unipotent fiber . . . 50
83.21 A negative test vector for the terminal minor . . . 51
84 Matrix coefficients and Hermite moments . . . 52
84.1 Integral vectors near a short isotropic line . . . 52

84.2 The second moment transverse to the stabilizer . . . 53
84.3 The norm of the initial Poincaré intertwiner . . . 55
84.4 The height at a large value of a cusp form . . . 56
84.5 A derivative bound for the lowest compact type . . . 57
84.6 A Sobolev bound for both terminal-period transfers . . . 59
84.7 A compactly supported operator fixing the incoming row . . . 60
84.8 An exponential moment for the initial Hermite expansion . . . 61
84.9 Uniform bounds for a discrete-series matrix coefficient . . . 62
84.10 Absolute Hermite coefficients of a real translate . . . 64
84.11 A matrix-norm moment for two smooth vectors . . . 65
84.12 A finite Hermite cutoff along the horizontal chain . . . 65
84.13 The real distribution of a terminal Bessel coefficient . . . 67
84.14 Consecutive frame lengths in the Bessel energy estimate . . . 69
84.15 The real Bessel norm and the formal degree . . . 71
85 Transverse pairs and higher-rank blocks . . . 73
85.1 The large-height contribution for every single reversed block . . . 73
85.2 The full contribution up to a bound on the absolute norm . . . 74
85.3 The real orbital integral for every transverse pair . . . 75
85.4 The archimedean integral for a whole isotropic block . . . 77
85.5 Minors controlling the entries between blocks . . . 78
85.6 A spherical integral for the two terminal vectors . . . 78
85.7 A Gram integral with one column on a sphere . . . 79
85.8 Integration of the inverse Gram determinant . . . 79
85.9 The mixed real orbital integral with two terminal lines . . . 80
85.10 Two distinct negative terminal lines . . . 81
85.11 Higher decay at the physical terminal frequency . . . 82
85.12 The real orbital estimate with a factorial for each exchanged vector . . . 83
85.13 The sum over all single reversed blocks . . . 84
85.14 The real integral for all blocks in a composition . . . 85
85.15 The terminal determinant for a reversed block . . . 86
85.16 The Gram bound for distinct negative vectors . . . 87
85.17 The orbit measure with a fixed positive vector . . . 88
85.18 The actual Bessel integral at the displaced real flag . . . 89
86 References . . . 91

82 Real theta norms and projected coefficients

82.1 Positive generators and the real scale of the theta tests

The choices of the additive character and of the skew form in Lemma 60.18 involve totally positive generators of two ideals of $F = E^+$. Their norms are prescribed by those ideals. The real theta kernels use every conjugate of each generator. We bound these conjugates separately, and then compute how a change of scale enters the real norm of the monomials in Theorem 60.21.

Lemma 82.1 (A bound using positive cyclotomic units)**.** *Suppose F is totally real, $F \subset \mathbb{Q}(\zeta_N)$, and $f = [F : \mathbb{Q}]$. Put $B_F = (f-1)\varphi(N)\log(2N)$. For each totally positive $u \in F^\times$ there is a totally positive unit $v \in \mathcal{O}_F^\times$ such that*

$$\max_{\tau : F \hookrightarrow \mathbb{R}} \left| \log \tau(uv) - \frac{1}{f} \log N_{F/\mathbb{Q}}(u) \right| \le B_F.$$

Proof. Let $G = \mathrm{Gal}(F/\mathbb{Q})$, labelling its real embeddings by a fixed inclusion and the elements of G. We construct a collection of positive units whose logarithm vectors span the hyperplane of coordinate sum zero. If $f = 1$ this space is zero and $v = 1$ works.

Take a nontrivial character of G. Its primitive Dirichlet character χ has conductor $h \mid N$ and is even because complex conjugation fixes F. The subfield fixed by the kernel of this character lies in both F and $\mathbb{Q}(\zeta_h)$. Write $F_h = F \cap \mathbb{Q}(\zeta_h)$, choose a prime to h with $\chi(a) \neq 1$, and put

$$y_a = \frac{1-\zeta_h^a}{1-\zeta_h}, \qquad v_a = N_{\mathbb{Q}(\zeta_h)/F_h}(y_a \overline{y_a}) \in F_h^\times.$$

A geometric sum shows that y_a is integral. If $ab \equiv 1 \bmod h$, the corresponding sum in powers of ζ_h^a is y_a^{-1}, hence y_a is a unit. All conjugates of v_a are positive. Each $|1-\zeta_h^j|$ with $(j,h) = 1$ is between $2\sin(\pi/h)$ and two. The lower bound is at least $4/h$ for $h \ge 2$. It follows that each coordinate of the log vector of v_a has absolute value at most $2\varphi(h)\log(2h)$.

This vector has nonzero coefficient in the χ-direction. Indeed the primitivity and even parity of χ give

$$\sum_{j \bmod h}^{*} \overline{\chi(j)} \log|1-\zeta_h^j| = -\mathfrak{g}(\bar\chi) L(1,\chi), \qquad \mathfrak{g}(\bar\chi) = \sum_{j \bmod h}^{*} \overline{\chi(j)} \zeta_h^j.$$

To see this, expand $-\log|1-s\zeta_h^j|$ in the cosine series at $0 < s < 1$. Evenness identifies the two exponential sums for each cosine. The primitive Gauss sum identity gives $\chi(n)\mathfrak{g}(\bar\chi)$ for the coefficient at s^n/n, also when $(n,h) > 1$ since both sides then vanish. Letting s tend to one gives the formula by Abel's theorem. The Gauss sum has absolute value $\sqrt{h}$, and Dirichlet's nonvanishing theorem gives $L(1,\chi) \neq 0$. Replacing j by aj and subtracting multiplies this nonzero coefficient by $\chi(a) - 1$. Taking the norm defining v_a sums log coordinates on the fibers above F_h, where the character is constant. The square of the absolute value contributes a factor two. Inflating from F_h to the embeddings of F contributes $[F : F_h]$. The coefficient thus remains nonzero.

Include all G-translates of the units just constructed. The complex span of their log vectors is a subrepresentation of the regular representation of the finite abelian group G. Each character occurs once there. Applying the finite character idempotents and using the preceding computation shows that the span contains every nontrivial character line. Unit norms give coordinate sum zero, so the real vectors span the real hyperplane as asserted. Choose $f-1$ independent vectors from this collection. Their supremum norms are at most $2\varphi(N)\log(2N)$, since $h \mid N$. Express the log vector of u minus its coordinate mean as a real linear combination

of this basis. Subtract the nearest integer from each coefficient by multiplying by the corresponding inverse unit power. Each remaining coefficient has absolute value at most one half. The triangle inequality proves the stated bound. □

Theorem 82.2 (Real and finite sizes for the same two generators)**.** *Keep the very primes of Proposition 60.23. There are totally positive elements β, γ with $(\beta)\mathfrak{d}_F = \mathfrak{p}$ and $(\gamma)\mathfrak{a}_0 = \mathfrak{q}$ such that, at every real embedding τ,*

$$\left|\log \tau\beta - \frac{1}{f}\log\frac{N\mathfrak{p}}{|D_F|}\right| \le B_F,$$
$$\left|\log \tau\gamma - \frac{1}{f}\log\frac{N\mathfrak{q}}{d_0^{f/2}}\right| \le B_F.$$

On the original intervals, $1 + \max_\tau(|\log \tau\beta| + |\log \tau\gamma|) = b^{o(1)}$, where $b = \exp(x^\epsilon)$. The two additional primes have logarithmic norms polynomial in x, and all original inert local data remain unramified.

Proof. The narrow ideal class equalities give initial positive generators. The two means of their logarithms have the displayed values since $N\mathfrak{d}_F = |D_F|$ and $\mathfrak{a}_0^2 = (d_0)$, the latter by extending to E and using $(\sqrt{-d_0})^2 = (d_0)$. Apply the lemma separately to the two generators. Unit multiplication preserves their ideals and real signs. For these fields, Section 6.6 gives $f \ll_\psi x^3$ and $\log(2N) \ll_{\psi,\epsilon} x^{\epsilon/5}\log^2(2x) + \log(2x)$. Thus $\log(1 + B_F) = o(x^\epsilon)$. The numerators of the two norm quotients have logarithms bounded in Proposition 60.23. For the denominators use $\log|D_F| \le f\log N$ and $d_0 \mid N$. These bounds are uniform on the chosen intervals and prove the assertion. □

Proposition 82.3 (The real scalar under dilation)**.** *Use one fixed compatible complex structure in each real oscillator, and give its Fock vacuum norm one. Keep the polynomials in these coordinates, the unit source vectors, and the Haar measures on the real groups fixed. Let $d_{j,\tau}$ be the degree of the monomial on edge $j \to j+1$ at the place τ in Theorem 60.21, and write $D_\tau = \sum_{j=0}^{m-1} d_{j,\tau}$, $\bar{D} = f^{-1}\sum_\tau D_\tau$. Let $R_\infty(\beta,\gamma)$ be the product of the absolute values of the real diagonal Rallis factors of these tests, with the scaled additive character and skew form. Use $R_\infty(1,1)$ for the same calculation at the real places with the original character and form. For $\lambda = \beta\gamma$ one has*

$$\frac{R_\infty(\beta,\gamma)}{R_\infty(1,1)} = \prod_\tau(\tau\lambda)^{-D_\tau}, \qquad N_{F/\mathbb{Q}}(\lambda) = \frac{N\mathfrak{p}\,N\mathfrak{q}}{|D_F|\,d_0^{f/2}}. \tag{82.1}$$

The absolute difference between the logarithm of this ratio and $-\bar{D}\log N_{F/\mathbb{Q}}(\lambda)$ is at most $2B_F\sum_\tau|D_\tau - \bar{D}|$ for the generators above. It is $o(b/x^A)$ for every fixed A whenever the sum of the degree deviations is bounded by a power of x.

Proof. The real symplectic pairing with its additive character acquires the positive scalar $s = \tau\lambda$, since the skew form is scaled by $\tau\gamma$ and the character by $\tau\beta$. A compatible complex structure for the initial pairing remains compatible for its positive multiple. In complex dimension l the Gaussian measure for its Fock inner product is proportional to $s^l\exp(-\pi sQ(z))\,dz$ for the initial positive quadratic form Q. The constant is chosen to give the vacuum norm one. The isometry to the initial Fock space sends a polynomial $P(z)$ to $P(z/\sqrt{s})$, by change of variables. The map $z \mapsto \sqrt{s}\,z$ is symplectic from the scaled space to the initial one and commutes with both real isometry groups. Its induced Fock isometry therefore intertwines their oscillators. Equivalently one may transport the Heisenberg translations and their multipliers

through this linear change. The characters defining the unitary splittings are unchanged. In the real oscillator the compact vacuum character depends on the signatures, the sign of the real additive character and these angular exponents. All agree after the positive dilation. Any difference between the two group splittings would be a character of the real unitary groups trivial on their maximal compacts. Such a character is trivial since the abelianization of a connected real unitary group is its determinant circle, already covered by the maximal compact. The metaplectic phase of an isometry has no effect on a diagonal matrix coefficient.

For homogeneous degree d, the image of the polynomial is exactly $s^{-d/2}P$. Its contraction gives the same target line. Its oscillator matrix coefficient, and hence its diagonal integral against the fixed source coefficient, is multiplied by s^{-d}. This reasoning applies at each edge with the successive unit lines, even though the fields used to define the real comparison alone need have no prescribed finite normalization at $(1,1)$. Multiplication gives the first identity. Multiplication of ideal norms gives the second. The error after subtracting the mean is

$$-\sum_{\tau}(D_\tau-\bar{D})\left(\log\tau\lambda-f^{-1}\log N_{F/\mathbb{Q}}(\lambda)\right).$$

Each factor in parentheses has absolute value at most $2B_F$ by the theorem. The stated asymptotic follows from $1+B_F=b^{o(1)}$. □

Corollary 82.4 (Conjugate choices with the same finite cost)**.** *With the real row and its polynomial coordinates fixed, the geometric mean of the ratios in* (82.1) *over the choices* $(\sigma\beta,\sigma\gamma)$, $\sigma\in\mathrm{Gal}(F/\mathbb{Q})$*, is exactly* $(N_{F/\mathbb{Q}}\lambda)^{-\bar{D}}$*. Some choice attains at most this value and some choice attains at least this value. All choices satisfy the same estimates on the two auxiliary prime norms and the conjugates of the generators.*

Proof. The ideals $\mathfrak{d}_F$ and $\mathfrak{a}_0$ are Galois invariant. For the latter this follows also from $\mathfrak{a}_0^2=(d_0)$ and unique factorization of fractional ideals. The conjugate primes are split in E since $E/\mathbb{Q}$ is abelian. They still lie above the same two distinct rational primes outside the original levels. For each fixed τ, the arithmetic mean over σ of $\log\tau\sigma\lambda$ equals $f^{-1}\log N_{F/\mathbb{Q}}(\lambda)$. Apply the first identity in (82.1), keeping D_τ at the fixed place, and exponentiate. The two inequalities follow from the elementary bounds on the mean of real numbers. □

Proposition 82.5 (Dependence on positive units at fixed prime ideals)**.** *Fix the ideals, the real monomials and the vacuum normalization of Proposition 82.3. The real normalizing factor is independent of the choice of positive generators if and only if* D_τ *is independent of* τ*. If the degrees are nonconstant, the possible positive real factors have infimum zero and are unbounded above even when both prime ideals themselves are fixed.*

Proof. For parallel degrees the right side of (82.1) is a power of a fixed ideal norm. For nonparallel degrees the vector $(D_\tau)_\tau$ has a nonzero projection onto the hyperplane of coordinate sum zero. The spanning assertion in Lemma 82.1 supplies a positive unit v with $c=\sum_\tau D_\tau\log\tau v\neq 0$. Replace β by βv^k, leaving γ fixed, for $k\in\mathbb{Z}$. All ideals and signs are preserved. By the dilation identity the real factor is multiplied by $\exp(-kc)$. Both conclusions follow by letting k run through positive and negative integers. The bounded representatives in Theorem 82.2 are a particular choice in this family of permissible generators. □

This controls the dependence of the real norm on the two generators in the vacuum normalization. The reference factors $R_\infty(1,1)$, the degrees for the designated arithmetic tensors, and the pairing with the required integral projection still enter the global period.

82.2 An exact projected norm on the last compact pair

At the last forward step the source and target at each real place are both compact. This permits an absolute calculation of the real factor in a fixed matrix frame, including a projection to a specified target line. For this section put $n = n_{m-1}$, $N' = n_m = n + t$, and write $a_T = -e_m + h_{m-1}$ and $a_S = -e_m + h_m$ at the real place in question. Write $p_1 > \cdots > p_n$ for the entries of P_{m-1} there. Use probability Haar measures on the two real groups. In orthonormal matrix coordinates $Z \in M_{N',n}(\mathbb{C})$ take the Gaussian $e^{-\pi \operatorname{tr}(Z^*Z)}dZ$, with each complex coordinate given ordinary plane Lebesgue measure. It has mass one. These are fixed coordinates for the reference real calculation. A further positive dilation has the exact effect of Proposition 82.3.

Lemma 82.6 (The unique degree at the last place)**.** *Put*

$$\lambda_i = p_i - \frac{n+1}{2} + i - a_T - \frac{N'}{2} \quad (1 \le i \le n), \qquad d_* = |\lambda| = \sum_i \lambda_i.$$

These entries form a partition. After the vacuum characters are removed, the joint compact type is $S_\lambda(\mathbb{C}^n) \boxtimes S_\lambda(\mathbb{C}^{N'})$, where S_λ denotes the Schur representation. It occurs once and only in degree d_* of the matrix polynomials. At a place τ with weight offset B_τ from equation 9.1 and the weight lists following that equation, the degree is

$$\begin{aligned} d_{*,\tau} = (k-1)B_\tau + \sum_{i=1}^{k-1} v_i - \frac{nN'}{2} \\ + \big((k-1)(2m-1) + m^2 + tm(m-1)\big)H_0. \end{aligned} \tag{82.2}$$

Here the lists are equivalently specified by $b_{i,\tau} = B_\tau + 2(i-1)H_0$ and $e_{i,\tau} = B_\tau + (2i-1)H_0$.

Proof. Both smaller signatures are zero by $s_j = \lfloor (m-j)/2 \rfloor$. The source highest weight is $(p_i - (n+1)/2 + i)_i$. In the forward convention the compact Fock vacuum has determinant exponents $a_T + N'/2$ on the source and $a_S + n/2$ on the target. These are the compact shifts of Ichino, Lemma 7.8, with both negative signature blocks empty. In particular both polynomial weights in the forward pair have the same partition. Equivalently this follows by expanding the symmetric algebra of the two defining spaces after removing the vacuum characters. Dualizing the entire realization gives the conjugate pair and the same dimensions and squared norms. Subtraction on the source gives the displayed λ_i. The entries are integral by the parity of the two ranks and splittings. Successive differences are nonnegative since $p_i - p_{i+1} \ge 1$. Moreover $\lambda_n = p_n - a_T - (t+1)/2 > 0$, since every entry of the centered source list exceeds $(t+1)/2$ by the spacings in Section 60.3.

For degree d, the symmetric power of the tensor of the two defining spaces is the space of simultaneous symmetric group invariants in the tensor of their d-fold powers. Schur–Weyl duality decomposes each tensor power into Schur modules times symmetric group irreducibles. The invariants have dimension one on each pair of equal symmetric group labels and zero on two different labels. Thus precisely the matching partitions of size d occur, with multiplicity one. On the target, padding λ by t zeros and adding $a_S + n/2$ also gives the highest weight computed from P_m in Theorem 60.6.

For the sum use $\sum_i ((n+1)/2 - i) = 0$. The centered entries on the source are $v_i + e_m$, $e_m - b_i$, and the strings of length t centered at $e_m - e_i$ for $i < m$. The sum of $e_m - b_i = (2m + 1 - 2i)H_0$ over $i \le m$ is $m^2 H_0$. The sums of the symmetric strings give $tm(m-1)H_0$. Inserting $e_m = B_\tau + (2m-1)H_0$ and subtracting $nN'/2$ proves (82.2). □

Write $D_n = \dim S_\lambda(\mathbb{C}^n)$ and $D_{N'} = \dim S_\lambda(\mathbb{C}^{N'})$. For unit source and target vectors u, v in the respective types, let e_v be the compact matrix-unit projector onto the line of v in each copy of the target type, zero on the other types. For a polynomial P the last projected local norm is

$$Z_v(P;u) = \int_{U(n)} \langle \omega(g) e_v P, e_v P \rangle \overline{\langle \sigma(g)u, u \rangle}\, dg.$$

The source and target actions in this formula include their vacuum twists. Equivariance of theta lifting identifies projection of the target form with applying e_v to the test. In particular selecting the line in this definition applies directly to the global norm formula of Section 60.8.

Theorem 82.7 (Monomials detecting a prescribed final line)**.** *Write $M = \binom{d_* + nN' - 1}{nN' - 1}$. For every pair of unit vectors u, v, and every coefficient-one monomial Z^α of degree d_*, set $E_\alpha = (\pi^{d_*}/\alpha!)^{1/2} Z^\alpha$, where $\alpha! = \prod_{i,j} \alpha_{ij}!$. Then*

$$\sum_{|\alpha| = d_*} Z_v(E_\alpha; u) = \frac{1}{D_n}, \qquad 0 \le Z_v(E_\alpha; u) \le \frac{1}{D_n}.$$

For some monomial of this degree one therefore has

$$\frac{1}{M D_n} \le \frac{\pi^{d_*}}{\alpha!} Z_v(Z^\alpha; u) \le \frac{1}{D_n}. \tag{82.3}$$

Over all real places, tests can be chosen with the sum of the absolute logarithms of these normalized local factors bounded by a fixed power of x, uniformly in the real unit lines on the intervals of Section 6.6.

Proof. For probability Haar measure, compact Schur orthogonality identifies integration against $\overline{\langle \sigma(g)u, u \rangle}$ with $1/D_n$ times the orthogonal matrix unit e_u in the oscillator. The two matrix units commute. On the polynomial space their product projects onto the one line $u \otimes v$ in the unique joint summand from Lemma 82.6. In particular

$$Z_v(P;u) = D_n^{-1} \| e_u e_v P \|^2.$$

A monomial has squared norm $\pi^{-|\alpha|}\alpha!$ by angular integration and the one-variable Gaussian moment. Distinct monomials are orthogonal. Hence the E_α at the displayed degree form an orthonormal basis of the whole degree space. Summing the diagonals computes the trace of a rank-one projection divided by D_n. Each diagonal is nonnegative and bounded by $1/D_n$, and their number is M, proving the inequalities.

On the given intervals the ranks and $f = [F : \mathbb{Q}]$ are polynomially bounded in x, while the logs of two plus the absolute weights are $O_{\psi,\epsilon}(x^\epsilon + \log(2x))$ by the row construction. The same bound holds for $\log(2 + d_* + nN')$, by the sum formula above. Since $\log M \le nN' \log(2 + d_* + nN')$, it is polynomial in x. Also $D_n \le M$ because the degree space contains the joint summand of dimension $D_n D_{N'}$. Summation of $\log(M D_n)$ over the f places proves the last assertion. □

Proposition 82.8 (Exact value for a product of leading minors)**.** *Take matching highest source and target lines in the polynomial realization, write $\lambda_{n+1} = 0$, and let $\Delta_i(Z)$ denote the determinant of the leading i-by-i submatrix. The polynomial with integer coefficients*

$$H_\lambda(Z) = \prod_{i=1}^{n} \Delta_i(Z)^{\lambda_i - \lambda_{i+1}}$$

has projected real norm

$$Z_v(H_\lambda;u)=\frac{\pi^{-d_*}}{D_n^2}\prod_{i=1}^{n}\frac{(\lambda_i+n-i)!}{(n-i)!}. \tag{82.4}$$

For a positive real scale s relative to these coordinates the right side is multiplied by s^{-d_*}.

Proof. The highest vector in the tensor product of $(\bigwedge^i\mathbb{C}^n)^{\otimes(\lambda_i-\lambda_{i+1})}$ is the product of the wedges $e_1\wedge\cdots\wedge e_i$ with those multiplicities. It is a unit for the induced unitary structure and spans the highest line in its irreducible Cartan component $S_\lambda(\mathbb{C}^n)$. On an n-by-n matrix X its diagonal coefficient for $S_\lambda(X)$ is the product of the leading minors above. As a polynomial on all N'-by-n matrices this is the joint highest line, using the same wedges in the first n target coordinates. This can also be seen by the unipotent invariances of each leading minor. Consequently e_ue_v fixes this polynomial with the matching lines. Its projected local norm is its squared Gaussian norm divided by D_n.

The Fischer squared norm of this leading-minor polynomial is computed in Lemma 70.4. In the Gaussian used there, Gram–Schmidt on the top square block separates the unitary part, with squared coefficient mean $1/D_n$, from independent gamma variables of shapes $n-i+1$. Their λ_i-th moments give the displayed factorial ratios. Passing from that Fischer norm to the present vacuum measure multiplies the squared norm by π^{-d_*}. Taking the additional factor $1/D_n$ from the source matrix unit now gives the formula. The dependence on s follows from Section 82.1. □

Theorem 82.9 (Integral evaluation in the last compact norm)**.** *Keep the two highest lines of Proposition 82.8. Let A be the integer ring of a number field L and let $Q\in A[Z]$ have column degrees λ_i. Assume that it is invariant under adding any multiple of an earlier column to a later column, as a polynomial identity. Let J have the identity matrix in its top n rows and zeros below. For every embedding $\iota:L\hookrightarrow\mathbb{C}$ and every positive scale s, one has the exact comparison*

$$\frac{Z_v^{(s)}(Q^\iota;u)}{Z_v^{(s)}(H_\lambda;u)}=|\iota(Q(J))|^2. \tag{82.5}$$

In particular it vanishes precisely when $Q(J)=0$. For $Q(J)\neq0$, taking the product over all embeddings gives

$$\sum_{\iota:L\hookrightarrow\mathbb{C}}\log\frac{Z_v^{(s_\iota)}(Q^\iota;u)}{Z_v^{(s_\iota)}(H_\lambda;u)}=2\sum_{\mathfrak{l}}\operatorname{ord}_{\mathfrak{l}}(Q(J))\log N\mathfrak{l}.$$

The scales $s_\iota>0$ may vary with the embedding. The same equalities hold for any sum of these polynomials with coefficients in A.

Proof. In the Cauchy decomposition of Lemma 82.6 the column conditions select the highest source line in the unique λ-summand. In this space the target weight $(\lambda_1,\dots,\lambda_n,0,\dots,0)$, before adding the vacuum character, is the one-dimensional highest weight space. Its orthogonal projector keeps exactly those row degrees, by integration over the diagonal compact torus. The resulting polynomial is $Q(J)H_\lambda$. Indeed the row-degree projection does not affect the value at J: the only monomial of those column degrees that can be nonzero there is $\prod_i Z_{ii}^{\lambda_i}$. The highest vector spans the line and takes the value one at J. This reproves in this compact pair the projection identity of Theorem 70.5, now in the local theta norm with its source factor. The identity $Z_v(P;u)=D_n^{-1}\|e_ue_vP\|^2$ gives (82.5). Both polynomials are homogeneous of the same degree, so their dilation factors cancel at each embedding separately. Finally $Q(J)$ is an algebraic integer. The product of its absolute values over all embeddings, counting conjugate complex embeddings separately, is the positive absolute norm of its principal ideal. Factoring

that ideal gives the stated sum. Linearity of evaluation applies before taking the square of the absolute value, including for sums with cancellations. □

Corollary 82.10 (Logarithmic size on the common interval)**.** *Use highest-line polynomials at the real places and any positive scales s_τ relative to the reference coordinates. For the projected local integrals of those coefficient-one polynomials one has*

$$\sum_\tau \log Z_{v_\tau}^{(s_\tau)}(H_{\lambda_\tau}; u_\tau) = (k-1)\sum_\tau B_\tau(\log B_\tau - 1 - \log\pi) - \sum_\tau d_{*,\tau}\log s_\tau + O_{\psi,\epsilon}(x^C)$$

for a fixed $C = C(\psi,\epsilon)$. The error is $o(b/x^A)$ for every fixed $A > 0$, uniformly on the intervals of Section 6.6.

Proof. In the reference norm, take the logarithm of (82.4). Replacing $\log((\lambda_i+n-i)!/(n-i)!)$ by $\lambda_i\log\lambda_i - \lambda_i$ costs $O(n\log(2+n+\lambda_1))$ per index, by the elementary Stirling estimate and the at most $n-i$ extra factors. We use $0\log 0 = 0$. The Weyl dimension product $D_n = \prod_{i<j}(\lambda_i - \lambda_j + j - i)/(j-i)$ has logarithm $O(n^2\log(2+n+\lambda_1))$. By the weight bounds in Theorem 82.7, the sum of all these errors over the embeddings is polynomial in x.

The $k-1$ centered entries $v_i + e_m$ are each $B_\tau + O(x^{C_1})$, as also are their shifted partition entries, for a fixed C_1. All the other centered entries are bounded by $2mH_0 + t$, and their partition entries are bounded by a power of x. These assertions use only the lists in Lemma 82.6 and the polynomial bounds for ranks, spacings and v_i. The three permitted sizes of B_τ all exceed every fixed power of x on the chosen intervals, as in the construction following equation 9.1. On an interval between B_τ and any of the first $k-1$ partition entries the derivative of $y\log y - y$ is $O(\log(2+B_\tau))$. Replacing each such entry by B_τ therefore costs only a power of x. The smaller entries, their number, and the number of places are polynomially bounded. The same replacement in $-d_{*,\tau}\log\pi$ costs only a power of x. Keep the scale term from Proposition 82.8 exact, sum, and use $b = \exp(x^\epsilon)$. □

In Theorem 60.21 the last source line is fixed by the previous tests. Theorem 82.7 therefore lets one prescribe a unit target line separately at every real place and retain a positive common constant after that projection. Indeed choose the last monomial for those two lines, apply the target matrix unit, and repeat the norm identity in Section 60.8 with the resulting projected local factors. The explicit finite factors and the bound Δ_x there are the same. The reference real norm and the factorial in (82.3) concern this last edge. The full arithmetic projection also acts on the finite classes, and its valuation and the normalizations on the preceding edges remain to be estimated.

The two highest lines in Proposition 82.8 may also be imposed in a complex chain if the penultimate real test can be chosen and projected. To see this, fix the unit source vector on that preceding edge. By Lemma 60.20 one real test gives a nonzero local contraction in the next compact representation. Translates of its contraction by the compact target span that irreducible representation. Thus a translate has nonzero projection to the desired highest line. Acting on the preceding test by the same translation and the compact matrix unit realizes that projection, by equivariance. The compact action preserves the homogeneous polynomial spaces in the Fock realization. The nonzero diagonal factor persists after this projection: one may use a reference cuspidal constituent, with all other diagonal factors nonzero, and project its nonzero pure archimedean tensor line by this same matrix unit. Its new norm is positive. The resulting fixed archimedean choices work for the full row as in Section 60.8 and allow the coefficient-one polynomial of (82.4) on the last edge. The compact averaging in the preceding test introduces a separate question about its arithmetic normalization.

82.3 Integral projection from the last noncompact real group

Fix one real place and take the preceding edge $j = m-2$. Put $n = n_j$, $p = n-1$, $L = n+t = n_{j+1}$. The groups on this edge are $G = U(p,1)$ and $H = U(L)$. The source parameter P_j has p plus entries $p_1 > \cdots > p_p$ followed, in the combined decreasing order, by its sole minus entry p_-. Put $a_T = -e_{m-1} + h_{m-2}$ and $a_S = -e_{m-1} + h_{m-1}$. Coordinates in the vacuum Fock norm are $Z \in M_{L,p}(\mathbb{C})$ and $W \in \mathbb{C}^L$, with independent Gaussian entries of density $e^{-\pi|z|^2}$ relative to plane measure. Haar measure on H is probability. Let d_σ denote the formal degree of the discrete series σ on G for the Haar measure used there.

Lemma 82.11 (The compact types on the penultimate edge)**.** *The highest weight of the lowest $U(p) \times U(1)$-type of σ, after removal of the compact vacuum characters, is $(\alpha_1, \dots, \alpha_p; -\beta)$, where*

$$\alpha_i = p_i - a_T - (p+L)/2 + i, \qquad \beta = H_0 - (t+1)/2 > 0.$$

The α_i form a partition, with $\alpha_p = \beta$. The corresponding compact target after its vacuum twist is removed has highest weight $(\alpha_1, \dots, \alpha_p, 0, \dots, 0, -\beta)$. The part of its oscillator isotypic space on the lowest source compact type lies in bidegree $(|\alpha|, \beta)$ in (Z, W).

Proof. At this edge the centered lists in Section 60.3 have p positive plus entries and precisely one negative minus entry $p_- - a_T = -H_0$. The least positive entry is $e_{m-1} - b_{m-1} = H_0$. The old strings have centers at least $2H_0$ and half-width $(t-1)/2 < H_0$. On this chamber the half-sum of the positive noncompact roots is $(1/2, \dots, 1/2; -p/2)$. The compact half-sum is $((p+1)/2 - i)_{1 \le i \le p}$ on the first block and zero on the last. The lowest compact type of the discrete series has weight $P_j + \rho_{\mathrm{nc}} - \rho_{\mathrm{c}}$, and occurs once. Remove the vacuum exponents $a_T + L/2$ and $a_T - L/2$ on the respective blocks. The plus entries become the given α_i, and the minus entry becomes $-H_0 + (L-p)/2 = -\beta$. The plus differences are nonnegative integers by regularity and parity of the parameter. The least is $H_0 - (L-p)/2 = \beta$.

The compact target vacuum exponent is $a_S + (p-1)/2$, from the same compact splitting weights as in Ichino [1, Lemma 7.8]. For the target parameter of Section 60.3, subtract its compact half-sum and this vacuum exponent. The old positive entries give α_i. The inserted string of length t gives zeros, and the last old entry gives $-\beta$. In the polynomial action of the compact groups, Z carries the defining target weights and the defining $U(p)$ weights. The W block carries dual target weights and the negative $U(1)$ weight. Thus the two source compact centers fix the two degrees of the lowest type, as claimed. □

Write $H(\alpha) = \|H_\alpha\|_{\mathrm{F}}^2$ for the hook product in Lemma 70.4. In the following test $H_\alpha(Z)$ uses the leading minors in the first p target rows. Let e_L be the last coordinate vector, $J = \binom{1_p}{0}$, $d_- = |\alpha| + \beta$, and set

$$P_0(Z, W) = H_\alpha(Z)\, W_L^\beta.$$

For a unit compact target vector v let e_v be its compact matrix unit in the oscillator. Define $Z_v(Q; u)$ as in Section 82.2 using this edge's groups and the integral over G.

Theorem 82.12 (A projected local norm with noncompact source)**.** *Let u span the highest line of the lowest compact source type, and v the compact target highest line, with unit vectors on both. Suppose that the polynomial $Q(Z, W)$ has column degrees α_i in Z, is invariant under upper unitriangular column operations in Z, and has degree β homogeneously in W. For a positive real dilation s relative to the reference frame one has*

$$Z_v^{(s)}(Q; u) = |Q(J, e_L)|^2\, (\pi s)^{-d_-}\, \frac{H(\alpha)\, \beta!}{d_\sigma}. \tag{82.6}$$

The normalized ratio to the value on P_0 is therefore the squared absolute value of an integral evaluation whenever the coefficients of Q are algebraic integers.

Proof. The compact member theorem for the oscillator realizes the isotypic space of the indicated irreducible H-type as the Hilbert tensor product of that type and its irreducible unitary theta partner on G. This is the compact-member decomposition used in Ichino, Section 7.4, in the proof of Lemma 7.7 for the compact factors. Its hypotheses apply to $H = U(L)$. The nonzero pair already identified in Theorem 60.6 determines this partner as σ by local Howe duality. On this tensor product the action of G is on the first factor alone. For a test projected by e_v, write the resulting vector as $z \otimes v$. Square integrability of the matrix coefficients of the discrete series and Cauchy–Schwarz give convergence of the defining local integral. The formal-degree orthogonality identity gives

$$Z_v(Q;u) = d_\sigma^{-1} |\langle z, u\rangle|^2.$$

This uses the invariant Hilbert norms, so any unitary identification of the compact isotypic summand gives the same value.

To determine the target projection, separate the target diagonal torus weights before the vacuum character is added. In a monomial let z_i be the total number of Z-entries from row i, and w_i the exponent of W_i. The target weight is $(z_i - w_i)_i$. For the required highest weight its last entry is $-\beta$, and $\sum w_i = \beta$. This forces $w_L = \beta$, all the other $w_i = 0$, and $z_L = 0$. The remaining weights force row degrees $(\alpha_1, \dots, \alpha_p, 0, \dots, 0)$ in Z. Extracting these row weights preserves the column identities. By Theorem 70.5, or by evaluating the highest row component at J, the extracted polynomial is exactly $Q(J, e_L)P_0$. The polynomial P_0 is highest on the target. Its two factors are highest vectors in the Schur module and the symmetric power of the dual defining representation, and their highest-weight tensor lies in the Cartan component. Therefore the weight projection just computed already lies in the specified irreducible highest line on the target. The compact matrix unit e_v kills the other weights and fixes this component.

On the source the polynomial P_0 has the highest weight of the lowest compact type from the lemma and is annihilated by its compact raising operators, by the minor and column identities. This type occurs with multiplicity one in σ. Thus the vector z for P_0 lies on the line of u and has norm $\|P_0\|$. Its two factors use independent Gaussian variables, so $\|P_0\|^2 = \pi^{-d_-} H(\alpha)\beta!$ by Lemma 70.4 and the one-variable Gaussian moment. Apply the formal-degree identity and then the dilation of Section 82.1. This proves (82.6), including zero evaluations. □

Corollary 82.13 (The last two integral evaluations)**.** *Choose the compact highest vector of Theorem 82.12 as the input to the last edge. For a polynomial Q on the preceding edge and Q' on the last edge satisfying their respective column conditions, the normalized product of the two projected local norms is*

$$|Q(J, e_L)\, Q'(J')|^2,$$

where J' is the top identity frame of Section 82.2. The normalizing denominator uses P_0 and H_λ on the respective edges with the same local scales. If the product of evaluations is nonzero and the coefficients are in the integers of a number field D, the sum of the logarithms of these ratios over all its embeddings equals

$$2 \sum_{\mathfrak{l}} v_{\mathfrak{l}}(Q(J, e_L) Q'(J')) \log N\mathfrak{l}.$$

Proof. The two intermediate lines agree because the compact representation and its positive-root ordering are the same on both sides. A change of its vacuum determinant character shifts every weight equally and preserves its highest line. Apply (82.6) first, take the intermediate

vector to have unit norm, and apply (82.5) next. The homogeneity factors cancel edge by edge. For nonzero evaluations, factor the principal ideal of the product exactly as in Theorem 82.9. □

Proposition 82.14 (A quantitative choice for arbitrary compact target line)**.** *Allow any unit u in the lowest compact source type and any unit v in the compact target. With*

$$M_- = \binom{|\alpha| + pL - 1}{pL - 1}\binom{\beta + L - 1}{L - 1},$$

the sum of $Z_v(P; u)$ over all reference unit monomials of bidegree $(|\alpha|, \beta)$ equals $1/d_\sigma$. For at least one such monomial,

$$(M_- d_\sigma)^{-1} \le Z_v(P; u) \le d_\sigma^{-1}.$$

Here $\log M_-$ is bounded by a power of x. For any choice of real Haar measure, the logarithmic ratio of d_σ to the formal degree for the regular parameter $\rho_i = (n + 1)/2 - i$ in the same chamber is also bounded by a power of x, uniformly after summation over real places. In the Haar measure giving the maximal compact probability in the integration formula and inducing $(1 - \|z\|^2)^{-(p+1)} d^{2p} z$ on the unit ball, that reference degree is $p!/\pi^p$.

Proof. Translate P_0 by $(U(p) \times U(1)) \times H$. Its translates span the tensor product of the lowest source compact type and the entire target representation, by irreducibility of both finite-dimensional factors. They stay in the stated bidegree. In the oscillator the vector $u \otimes v$ consequently lies in that bidegree. After compact target projection, formal-degree orthogonality from the preceding proof shows that $d_\sigma Z_v(P; u)$ is the squared projection of P to the single unit line $u \otimes v$. Summation in its homogeneous space is the trace one, and the number of monomials is M_-. The size of this binomial product follows from the same interval bounds on ranks and log weights as in Theorem 82.7.

In Harish-Chandra's formal degree formula for a discrete series the Haar-dependent constant and the root-length factors are the same for two parameters of this group and chamber. Their ratio is thus $\prod_{i<k}(P_i - P_k)/(k - i)$, where the complete parameter is in decreasing order. The reference parameter ρ is regular and integral after the usual half-sum shift and defines a discrete series in this chamber. Each ratio is at least one by the integral gaps. Taking logarithms bounds their sum by $O(n^2 \log(2 + \max_i |P_i| + n))$. The family bounds and the polynomial number of embeddings prove the estimate. For the formal-degree product see also *Wehrl Inequalities for Matrix Coefficients of Holomorphic Discrete Series*, Section 4.1, equations (12)–(13). A compact central factor changes only the common Haar constant.

For the reference constant, realize the scalar Bergman representation on the unit ball in $\mathbb{C}^p$ with ordinary Lebesgue measure, or its complex conjugate to match the chamber above. The group acts by change of variables with the holomorphic Jacobian, so its action is unitary. The constant functions have multiplicity one as a compact type. Their translates are the Bergman kernels at all ball points, up to nonzero scalars. These span a dense space by the reproducing property. An invariant orthogonal projection acts by zero or one on this compact line and then on all its translates, proving irreducibility. In the conjugate model the line has weight $(1, \dots, 1; -p) = 2\rho_{\mathrm{nc}}$: for $\mathrm{diag}(A, d)$ the inverse ball map is $z \mapsto dA^{-1}z$, whose holomorphic Jacobian is $d^p / \det A$. This is the scalar lowest compact type of the reference parameter ρ.

For a unit constant vector the squared absolute diagonal coefficient at g is $(1 - \|g0\|^2)^{p+1}$. In fact its coefficient is evaluation at zero of the Jacobian multiplier, up to the normalization of the constant, since the ball average of a holomorphic function is its value at zero. For a block matrix $g = \left(\begin{smallmatrix} A & b \\ c & d \end{smallmatrix}\right)$ the modulus of the Jacobian there is $|d|^{-(p+1)}$, by the fractional linear ball action and $|\det g| = 1$. The relation $|d|^2 - \|b\|^2 = 1$ gives $1 - \|g0\|^2 = |d|^{-2}$, with the same modulus for

the inverse. Integrating the squared coefficient in the stated invariant measure leaves exactly the Euclidean volume $\pi^p/p!$ of the ball. In particular this representation is square integrable, and its lowest type identifies it with the reference discrete series. Formal-degree orthogonality now gives $d_\rho = p!/\pi^p$. □

Corollary 82.15 (Uniform two-step main term)**.** *For each place τ let $Z_{-,\tau}$ and $Z_{+,\tau}$ be the projected integrals of P_0 and H_{λ_τ}, respectively, with reference scales $s_{-,\tau}, s_{+,\tau} > 0$. Let $d_{\rho,\tau}$ be the reference formal degree from Proposition 82.14, in the same Haar measure as the source integral on the first of these edges. Uniformly on the intervals of Section 6.6,*

$$\sum_{\tau} \log(d_{\rho,\tau} Z_{-,\tau} Z_{+,\tau}) = 2(k-1) \sum_{\tau} B_\tau (\log B_\tau - 1 - \log \pi) \\ - \sum_{\tau} \left(d_{-,\tau} \log s_{-,\tau} + d_{*,\tau} \log s_{+,\tau} \right) + O_{\psi,\epsilon}(x^C).$$

Proof. On the penultimate edge the only centered source entries containing B_τ are the $k-1$ entries $v_i + e_{m-1}$. They are $B_\tau + O(x^{C_1})$ for a fixed C_1 and eventually exceed all the remaining entries. The shifts in Lemma 82.11, as well as the other partition entries and β, are bounded by powers of x. In $\log H(\alpha)$ the factorials give $\sum_i (\alpha_i \log \alpha_i - \alpha_i)$ with polynomial error, and the Weyl differences in the denominator have a polynomial total logarithm. This is the Stirling argument of Corollary 82.10. The same argument replaces the $k-1$ large terms by those for B_τ. The extra $\log(\beta!)$ is polynomial. The ratio of formal degrees is covered by Proposition 82.14. Apply (82.6), keep its scale term exact, and add Corollary 82.10 for the last edge. All errors remain polynomial after summation over real places. □

To insert these tests in a complex chain one may prescribe the penultimate source line by compact projection at the prior edge. For fixed nonzero prior input the images of all polynomial tests under contraction span the target module, as in Corollary 60.7. On a reference cuspidal constituent choose a prior test with nonzero norm. Its translates in the real target span a dense invariant subspace of that irreducible unitary representation, so some translate has nonzero projection to the prescribed line of the multiplicity-one lowest compact type. Realize that projection by its compact matrix unit, as in Section 82.2. Continuity and density permit approximation by polynomial tests followed by this matrix unit. A term of some fixed homogeneous degree still has nonzero projected norm by linearity, and compact averaging preserves that degree. This fixes the preceding data for the full row. The last two choices then work with the previously determined finite tests. To pass to the full arithmetic period one still needs the earlier integral tensors, their finite class sums and the normalization of the earlier noncompact pairings.

82.4 Highest lines throughout the real theta correspondence

We determine a single polynomial with integer coefficients on every real pair in Section 60.3. In each intermediate representation we use the highest line of its lowest compact type for both the incoming and the outgoing map. We first determine the least polynomial degree for the actual discrete series.

Fix $0 \le j < m$, a real embedding, and the splitting conventions of Section 60.3. Write $n = n_j$, $t = t_{\mathrm{r}}$, $h = (t+1)/2$, $A = k + j(t+1)$, $y = s_{j+1}$, $z = s_j$. The source signature is $(p, q) = (A + y, z)$ and the target signature is $(P, Q) = (A + t + z, y)$. Use ℓ_i for the entries of the source parameter minus a_T in combined decreasing order. The first A entries are positive,

all of plus sign. Thereafter the signs are alternately minus and plus. Define three lists by

$$\begin{aligned} a_i &= \ell_i + i - A - h && (1 \le i \le A),\\ b_v &= \ell_{A+2v} - h && (1 \le v \le y),\\ c_u &= \ell_{A+2u-1} + h && (1 \le u \le z). \end{aligned}$$

Set $\alpha = (a_1, \dots, a_A)$, $\beta = (-b_y, \dots, -b_1)$ and $\gamma = (-c_z, \dots, -c_1)$. They are partitions, with the empty partition permitted. In fact the integral gaps in ℓ_i are at least one, the last positive entry is at least H_0, and the negative entries are at most $-H_0$, where $H_0 > h$.

Lemma 82.16 (The two compact highest weights)**.** *After removing the Fock vacuum characters, the highest weights of the lowest compact types in σ_j, σ_{j+1} are respectively*

$$(a, b\,;\, c), \qquad (a, 0^t, c\,;\, b).$$

The two compact types each have multiplicity one in their respective discrete series.

Proof. For a discrete series with parameter λ the highest weight of the lowest compact type is $\lambda + \rho_{\mathrm{nc}} - \rho_{\mathrm{c}} = \lambda + \rho_{\mathrm{tot}} - 2\rho_{\mathrm{c}}$, and that type has multiplicity one. Here positive roots use the order of the decreasing combined parameter. At combined position i in rank n we have $\rho_{\mathrm{tot},i} = (n+1)/2 - i$. At position v of a compact block of size d, $2\rho_{\mathrm{c},i} = d + 1 - 2v$. On the source subtract the vacuum $a_T \pm (P-Q)/2$, with the sign of the compact block. The first A entries have $v = i$. For the next entries the positions within their blocks are u at $A + 2u - 1$ and $A + v$ at $A + 2v$. Substitution, using $p - q + P - Q = 2A + t$, gives a_i, c_u, b_v, respectively.

On the target, Section 60.3 inserts the t entries $h - l$ $(1 \le l \le t)$ between the positive and negative lists, and reverses the compact signs of the old negative entries. Remove its vacuum $a_S \pm (p-q)/2$. A middle entry at combined and compact position $A + l$ becomes $h - l + A + l - A - h = 0$. An old negative entry is now at combined position $A + t + 2u - 1$ and plus position $A + t + u$, or at combined position $A + t + 2v$ and minus position v. The same half-sum formulas give c_u and b_v there. The old positive entries still give a_i. The Fock vacuum conventions and the compact harmonic correspondence for these signs are in Ichino [1, Lemma 7.8]. □

To describe the polynomial, a defining variable for a unitary block has torus weight e_i, while a dual variable has weight $-e_i$. In Fock coordinates write X, Y, W, V for the four matrices whose types on the source and target compact factors are as follows. Their row indices always belong to the target.

	size	source factor and action	target factor and action
X	$P \times p$	$U(p)$ defining	$U(P)$ defining
Y	$Q \times p$	$U(p)$ dual	$U(Q)$ dual
W	$P \times q$	$U(q)$ dual	$U(P)$ dual
V	$Q \times q$	$U(q)$ defining	$U(Q)$ defining

These are the four tensor products of compact defining or dual spaces in the polynomial polarization, with the constant vacuum weight removed. Let X_0 be the first A rows and columns of X. Let Y_0 use the last y rows and columns of Y, and W_0 the last z rows and columns of W. Order the indices in each dual square in decreasing order on both axes. For a partition ν of length d write

$$H_\nu(M) = \prod_{l=1}^{d} \det(M_{[1,l],[1,l]})^{\nu_l - \nu_{l+1}}, \qquad \nu_{d+1} = 0.$$

An empty square contributes one. Put

$$\mathcal{P}_j = H_\alpha(X_0)H_\beta(Y_0)H_\gamma(W_0), \qquad D_j = |\alpha| + |\beta| + |\gamma|. \tag{82.7}$$

Let $\mathcal{J}_j$ have the identity in each of the three indicated squares and zero elsewhere, including all of V.

Theorem 82.17 (An explicit polynomial surviving every real quotient)**.** *The least Fock degree among the compact types of σ_j on this pair is D_j. In any nonzero local theta quotient onto $\sigma_j \boxtimes \sigma_{j+1}$, the image of the integer polynomial $\mathcal{P}_j$ spans the tensor product of the highest lines of their lowest compact types. In particular the successive contractions of the $\mathcal{P}_j$'s on these lines are nonzero through the entire real chain.*

Proof. Leading minors are highest vectors for the two standard block actions. On a dual block its last coordinates in reverse order play the same role, since a compact raising operator on a dual basis sends a coordinate to a multiple of a later one. Consequently $\mathcal{P}_j$ is highest for both compact groups with the weights in Lemma 82.16. This also follows by taking the top exterior powers of the nested initial or terminal flags. The polynomial has degree D_j and $\mathcal{P}_j(\mathcal{J}_j) = 1$. For either compact group each variable contributes just one signed coordinate to its torus weight. Thus any monomial of a given vacuum-stripped weight has degree at least the sum of the absolute values of its coordinates. For each of our two compact types this sum is D_j. In particular both types have least degree D_j in the Fock space.

It remains to compare other compact types inside the source discrete series. In its combined ordering let $\delta_i = 1$ on the first A coordinates and $\delta_i = -1$ on the rest. This vector is dominant within each compact block. It pairs nonnegatively with every positive noncompact root $e_i - e_l$ because $i < l$. Let C denote the vector of the two constant source vacuum weights, and μ the lowest compact highest weight before removing C. Then $\langle \delta, \mu - C\rangle = D_j$. If ν is the highest weight of any compact type in the source discrete series, the Blattner multiplicity formula gives a compact Weyl permutation w and nonnegative integers m_ξ with

$$w(\nu + \rho_{\mathrm{c}}) = \mu + \rho_{\mathrm{c}} + \sum_{\xi > 0,\ \text{noncompact}} m_\xi \xi.$$

Indeed at least one summand of the signed partition formula must be nonzero for a type of positive multiplicity. Since both $\nu + \rho_{\mathrm{c}}$ and δ are compact dominant, sorting the coordinates within each block gives $\langle \delta, \nu + \rho_{\mathrm{c}}\rangle \geq \langle \delta, w(\nu + \rho_{\mathrm{c}})\rangle$. It follows that

$$\|\nu - C\|_1 \geq \langle \delta, \nu - C\rangle \geq \langle \delta, \mu - C\rangle = D_j.$$

The compact case is included, with no noncompact roots. This proves the degree assertion for the discrete series. For the multiplicity formula see Hecht and Schmid, *A proof of Blattner's conjecture*, and its $U(p,q)$ convention in Baldoni and Vergne, Section 2.2.

We explain why this polynomial survives the quotient. Use Howe's harmonic generation theorem and his least-degree lemma for a quotient of the polynomial oscillator. The space of a given source compact type in the oscillator is generated, under the target enveloping algebra, by its polynomials of least degree. Since μ occurs in the quotient, that degree-D_j subspace has nonzero image. The least-degree lemma identifies D_j also with the minimum degree of any target compact type in the quotient. Thus a target compact type contributing to the image of this subspace has least degree exactly D_j. It belongs there to the joint harmonics, since polynomials of least degree for either compact type are harmonic for it. In the joint harmonics a given source type has one target partner and their compact tensor occurs once. The polynomial $\mathcal{P}_j$, of least degree for both its types, is a nonzero joint harmonic highest

vector. It therefore lies in this unique compact tensor. The quotient map is nonzero there and injective on it by irreducibility. Multiplicity one in the two discrete series identifies the image of $\mathcal{P}_j$ with the asserted lines. The occurrence of the discrete-series quotient itself is Theorem 60.6. Its successive compact highest lines agree because the lowest compact type in an intermediate discrete series and its positive compact roots do not depend on the edge. These arguments use Howe, *Transcending classical invariant theory*, (3.9)–(3.10) and Lemma 4.1, and the explicit harmonic correspondence of Ichino, Lemma 7.8. □

Theorem 82.18 (Integral evaluation on all the matched lines)**.** *Let F be homogeneous of degree D_j and highest for the source compact group of the same weight as $\mathcal{P}_j$. Project it to the target compact highest line by its compact matrix unit. The resulting polynomial is exactly*

$$F(\mathcal{J}_j)\mathcal{P}_j.$$

In particular this matrix unit on the indicated source lattice has integral image and coefficient one on $\mathcal{P}_j$. The ratio of the squared norms of the contracted, projected theta vectors for F and $\mathcal{P}_j$, using the same nonzero intertwiner, source unit vector on the highest line, Haar measures and real scale, is $|F(\mathcal{J}_j)|^2$.

Proof. First take the target torus weight of $\mathcal{P}_j$. On the source, equality between its weight norm and the degree forces every used coordinate to have the sign of its column weight. For a zero weight the coordinate cannot occur at all. Otherwise opposite signs in one column would increase the degree by at least two. The identical argument on the target fixes the allowed rows. Thus in the chosen target weight only the three squares X_0, Y_0, W_0 can occur. The row and column degrees are α, β, γ on the successive squares in their specified orders. Source and target torus actions commute. The extracted polynomial consequently retains the column highest-vector identities for the independent changes within each of the three column sets.

For a square and any partition ν, the polynomials of its given highest column weight form one Schur module for its rows, by the Cauchy decomposition of the polynomial ring. The row weight equal to ν is the one-dimensional highest weight space in this module. Its generator H_ν is one at the identity, by the minor formula. Reversing the order in the two dual squares gives exactly the same assertion there. Taking the tensor product for the three independent squares shows that the extracted polynomial is a scalar multiple of $\mathcal{P}_j$. Evaluation at $\mathcal{J}_j$ reads this scalar from F itself. Indeed each monomial nonzero at that frame has its rows matched to its columns on each square and hence has the target weight just extracted. The result is already highest of the specified target compact type. Its matrix unit fixes this vector and kills all the other target weights. The projection formula follows. Applying the quotient of Theorem 82.17 and contracting on the source highest line gives the norm ratio. A common dilation of the coordinates on this space of homogeneous degree preserves the ratio. □

Corollary 82.19 (The product over a real chain)**.** *For tests $F_{j,\tau}$ as in Theorem 82.18 on the fields and intervals of Section 6.6, the ratio of the full products of these projected real squared norms, relative to the polynomials $\mathcal{P}_{j,\tau}$, equals*

$$\left| \prod_{\tau} \prod_{j=0}^{m-1} F_{j,\tau}(\mathcal{J}_j) \right|^2 .$$

For a cuspidal row present in Theorem 60.19, the same ratio holds for the projected real factors in the global Rallis norm at the prescribed highest lines, with a nonzero reference factor on each edge. If all coefficients lie in $\mathcal{O}_L$ and the displayed product of evaluations is $u \neq 0$, summation of the logarithm of the ratio over all embeddings of L, acting on these polynomial coefficients, gives $2\sum_{\mathfrak{l}} v_{\mathfrak{l}}(u) \log N\mathfrak{l}$.

Proof. Take unit vectors on the successive lines. On each line an intertwiner contributes one scalar, with conjugation where the Hermitian contraction requires it. Squared absolute values multiply. This proves the first assertion by the two theorems. For an occurring cuspidal representation use one nonzero global lift from Section 60.7. Fix its other local arguments so that its local bilinear map at the chosen real place is nonzero. Equivariance and the local irreducible theta correspondence identify this map on oscillator polynomials with the small theta quotient followed by contraction, up to a nonzero multiplicity vector in the other factors. Its application to $\mathcal{P}_j$ and the prescribed source vector is nonzero by Theorem 82.17. The global norm identity with the other arguments fixed gives a nonzero diagonal real factor for this choice. The equality of the projected polynomials in Theorem 82.18 then gives the same ratio in that integral. The argument applies at every step by the successive nonzero cuspidal lifts of Section 60.7. Finally the frames have entries zero or one, so u belongs to $\mathcal{O}_L$. The product of $|\iota u|$ over all embeddings $\iota : L \hookrightarrow \mathbb{C}$ is the absolute norm of its principal ideal. Factoring this ideal gives the asserted sum. □

Proposition 82.20 (The reference Fock norms)**.** *In unit-vacuum Gaussian coordinates with dilation $s > 0$ as in Section 82.1 the squared norm of $\mathcal{P}_j$ in the Fock space is*

$$\|\mathcal{P}_j\|_s^2 = (\pi s)^{-D_j} H(\alpha) H(\beta) H(\gamma),$$

where $H(\nu)$ is the hook product of Lemma 70.4. Summed over all edges and real places on the selected intervals,

$$\sum_{\tau,j} \log \|\mathcal{P}_{j,\tau}\|_{s_{j,\tau}}^2 = m(k-1) \sum_{\tau} B_\tau (\log B_\tau - 1 - \log \pi) - \sum_{\tau,j} D_{j,\tau} \log s_{j,\tau} + O_{\psi,\epsilon}(x^C).$$

Here B_τ is the weight offset of Section 82.2.

Proof. The Gaussian entries in the three squares are disjoint. Their moments therefore multiply. Lemma 70.4 gives their Fischer norms. The homogeneous change of scale gives the displayed power. Of the parts of α exactly the original $k-1$ entries from the v_i are $B_\tau + O(x^{C_1})$, by Section 60.3 and the bounds on the spacings. Every other entry of the three partitions, and the number of entries, are bounded by a fixed power of x, uniformly in j, τ. The factorial expression for each hook product in Lemma 70.4 and Stirling's inequality give an error polynomial in x after replacing each large part by B_τ in $u \log u - u$. Indeed $\log(2 + B_\tau)$ is polynomially bounded on the intervals, and this also bounds each logarithm of a Weyl difference up to a polynomial error. The number of summands $m[E^+ : \mathbb{Q}]$ is polynomially bounded. Keeping each dilation term exact gives the formula. □

The factor converting this last Fock norm into the projected theta norm on a pair with two noncompact groups is a separate absolute normalization. Sections 82.2 and 82.3 calculate it for the two terminal edges. The integer evaluations above determine the dependence on a change of test throughout the chain. The absolute normalized global period also involves the finite class sum and the projection at the original finite level.

82.5 An absolute upper bound for every real theta integral

At every real place keep the parameters, highest lines and unit-vacuum Fock norm of Section 82.4. Write $G = U(p, q)$, $n = p + q = n_j$, $K = U(p) \times U(q)$, and use an arbitrary fixed Haar measure on G. For a unit vector u on the prescribed source line let d_σ be the formal degree of its discrete series. The following bound uses the actual integral, prior to cancellation against a reference integral.

Lemma 82.21 (Integrability of the matrix unit)**.** *Every K-finite matrix coefficient of each source σ_j in the chain belongs to $L^1(G)$. If*

$$f_u(g) = d_\sigma \, \overline{\langle \sigma_j(g)u, u\rangle}$$

with inner products linear in the first argument, then $f_u = f_u^*$ and $f_u * f_u = f_u$ in $L^1(G)$. For any unitary representation ω of G its integrated operator $\omega(f_u)$ is an orthogonal projection.

Proof. The group has compact center. When $q > 0$, its derived group is $\mathrm{SU}(p, q)$, and the product with the central circle covers G with finite kernel. Absolute values of the coefficients are invariant under the unitary central character. Thus one may apply the semisimple integrability theorem on the derived group. Restriction of this irreducible representation to the derived group stays irreducible, since every element of G is the product of a derived-group element and a central element.

For a discrete series with parameter λ, a sufficient criterion for integrability of its compact-finite coefficients is that every noncompact root ξ satisfies

$$|(\lambda, \xi)| > \frac{1}{4} \sum_{\zeta \text{ root}} |(\zeta, \xi)|.$$

This is the L^1 consequence of D. Miličić, *Asymptotic behaviour of matrix coefficients of the discrete series*, pp. 84–85. Use the Euclidean pairing on the trace-zero hyperplane. In type A_{n-1}, for $\xi = e_i - e_l$, two roots pair with absolute value two, and $4(n-2)$ pair with absolute value one. The right side is $n - 1$. Subtracting a common scalar from the coordinates of the parameter preserves every pairing.

On our source, all the positive centered entries are plus, and the first negative entry is at most $-H_0$. The absolute values on the positive side are at least H_0. The negative tail is $-H_0, -3H_0, -5H_0, \ldots$ and its signs alternate. Hence any two entries of opposite compact sign are separated by at least $2H_0$. The noncompact roots are exactly the differences between such entries. The chosen spacing satisfies $H_0 \geq k + m(t_{\mathrm{r}} + 3) > n_m$, since $n_m = k + m - 1 + mt_{\mathrm{r}}$. This verifies the strict inequality for all n_j. For $q = 0$ integrability follows from compactness.

Unitarity and unimodularity give $f_u^*(g) = \overline{f_u(g^{-1})} = f_u(g)$. Schur orthogonality of the square-integrable representation with formal degree d_σ gives

$$\int_G \langle \sigma_j(h)u, u\rangle \langle \sigma_j(h^{-1}g)u, u\rangle \, dh = d_\sigma^{-1} \langle \sigma_j(g)u, u\rangle.$$

The integral exists by Cauchy–Schwarz on its two square-integrable coefficients. Conjugation and multiplication by d_σ^2 give the idempotent identity. An integrated unitary representation is a star representation of $L^1(G)$. The image of a self-adjoint idempotent is an orthogonal projection, including when its image is zero. □

Theorem 82.22 (Absolute real Rallis bound in the polynomial norm)**.** *Let ϕ be a Fock test, e a compact target matrix unit projecting orthogonally onto the specified target highest line in each copy of its compact type, and ω the oscillator. Write $Z_e(\phi; u)$ for the local integral of the coefficient of $e\phi$ against the conjugate source coefficient at u. Then*

$$d_\sigma Z_e(\phi; u) = \|\omega(f_u) e\phi\|^2 \leq \|e\phi\|^2.$$

In particular for every F of Theorem 82.18, at dilation $s > 0$,

$$0 \leq Z_e^{(s)}(F; u) \leq \frac{|F(\mathcal{J}_j)|^2}{d_\sigma} (\pi s)^{-D_j} H(\alpha) H(\beta) H(\gamma). \tag{82.8}$$

For $F = \mathcal{P}_j$ the ratio to the upper bound is the squared norm of the projection of $\mathcal{P}_j / \|\mathcal{P}_j\|$ by $\omega(f_u)$. It belongs to $(0, 1]$ on a surviving cuspidal row, and is one at the last two real edges.

Proof. The oscillator coefficient is bounded by $\|e\phi\|^2$. Lemma 82.21 makes its integral against f_u absolutely convergent. The definition of the integrated operator gives the inner product $\langle\omega(f_u)e\phi, e\phi\rangle$, which is the squared norm of its orthogonal projection. The identity $eF = F(\mathcal{J}_j)\mathcal{P}_j$ in Theorem 82.18 and the three Gaussian moments in Proposition 82.20 now give (82.8). The homogeneous dilation multiplies both squared norms by the same power s^{-D_j}. Nonvanishing on an occurring cuspidal row follows from Corollary 82.19. The exact evaluations of Sections 82.2 and 82.3 give equality on their reference tests. Equivalently in those two cases the compact target isotypic summand is a tensor product with the source discrete series, and the reference polynomial lies on the single source line. □

Proposition 82.23 (A common Haar normalization in all signatures)**.** *Give K probability in the compact disintegration and take on G/K the matrix ball $\mathcal{D}_{p,q} = \{Z \in M_{p,q}(\mathbb{C}) : I_q - Z^*Z > 0\}$ with quotient measure*

$$\det(I_q - Z^*Z)^{-n} d^{2pq}Z.$$

For any source parameter $\lambda_1 > \cdots > \lambda_n$ in combined order, its formal degree in this measure is

$$d_\sigma = \pi^{-pq} \prod_{l=1}^{q} \frac{\Gamma(p+l)}{\Gamma(l)} \prod_{i<k} \frac{\lambda_i - \lambda_k}{k-i}. \tag{82.9}$$

In particular $\sum_{\tau,j} |\log d_{\sigma_{j,\tau}}|$ is bounded by a fixed power of x on the common intervals.

Proof. We compute the constant at the infinitesimal character of $\rho_i = (n+1)/2 - i$ in the holomorphic chamber with all plus entries first. For $q > 0$ consider the scalar Bergman space of square-integrable holomorphic functions on the ball with Lebesgue measure. The action by the inverse fractional linear transformation with its holomorphic Jacobian is unitary by change of variables. The constants form one compact line, with multiplicity one as the central circle rotating all the entries separates degree zero from positive degrees. Evaluation at zero is a constant multiple of integration, by that circle action. Translates of the unit constant therefore represent evaluations at all points of the ball up to nonzero scalars. Their span is dense, since a function orthogonal to all of them vanishes at every point. An invariant orthogonal projection acts on the constant line by zero or one, hence has the same action on its dense span of translates. This proves irreducibility.

Write $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. At zero the fractional linear action $Z \mapsto (AZ + B)(CZ + D)^{-1}$ has differential $X \mapsto (A - BD^{-1}C)XD^{-1}$. Its Jacobian is $(\det g)^q (\det D)^{-n}$. Since $|\det g| = 1$ and $D^*D - B^*B = I_q$, its modulus squared is $\det(I_q - (g0)^*(g0))^n$. The inverse matrix has bottom right block D^*, so its Jacobian at zero has the same modulus. More generally

$$I_q - (gZ)^*(gZ) = (CZ + D)^{-*}(I_q - Z^*Z)(CZ + D)^{-1}.$$

The Jacobian at Z has modulus $|\det(CZ + D)|^{-n}$, by the same differential calculation with $A - (gZ)C$ in the left factor. Indeed the Schur complement of $CZ + D$ in $g \left(\begin{smallmatrix} I & Z \\ 0 & I \end{smallmatrix}\right)$ is $A - (gZ)C$. This proves invariance of the stated quotient measure. The squared diagonal coefficient of the unit constant is the squared Jacobian at zero by evaluation. Its integral in quotient measure is therefore the ordinary Euclidean volume of the ball.

For $a \geq 0$ integrate $\det(I - Z^*Z)^a$ over the ball with Lebesgue measure. Write $Z = (X, v)$ and put $v = (I - XX^*)^{1/2}w$. The block determinant is $\det(I - X^*X)(1 - \|w\|^2)$, by the Schur complement, and the Jacobian on the last complex column is $\det(I - XX^*) = \det(I - X^*X)$. Thus integrating the column multiplies the integral for $q - 1$ columns and exponent $a + 1$ by $\pi^p\Gamma(a+1)/\Gamma(a+p+1)$. This last factor is the radial beta integral on the unit ball in $\mathbb{C}^p$. Starting with $a = 0$ gives for the volume $\pi^{pq} \prod_{l=1}^{q} \Gamma(l)/\Gamma(p+l)$. The representation is square integrable.

In its conjugate model the constant has compact weight $(q,\dots,q\,;-p,\dots,-p) = 2\rho_{\mathrm{nc}}$, as follows by taking the Jacobian of $Z \mapsto A^{-1}ZD$ for $(A,D) \in K$. It is the scalar lowest type of the holomorphic or conjugate-holomorphic discrete series with parameter ρ in the indicated chamber. Schur orthogonality gives reciprocal ball volume as its degree. For $q = 0$ the reference is the trivial representation of a compact group of mass one.

Harish-Chandra's formal degree formula for a fixed compact Cartan gives a Haar-dependent constant times the absolute product of the positive-root pairings of the parameter. The constant is independent of the discrete series chamber for that Cartan. Its value is consequently fixed by the reference just computed. Taking the ratio to ρ gives the last product of (82.9), with absolute differences taken in combined decreasing order. This also gives the usual Weyl dimension formula for $q = 0$. For the general degree formula see Harish-Chandra, *Discrete series for semisimple Lie groups II*, Acta Math. **116** (1966), 1–111, and the proof by Atiyah and Schmid, *A geometric construction of the discrete series for semisimple Lie groups*.

Finally the gaps in λ are integral and at least one, so each Weyl ratio is at least one. Its logarithm is bounded by $O(\log(2 + n + \max|\lambda_i|))$. The logarithms of the gamma ratios cost $O(n^2 \log(2+n))$ in total. The ranks, m, the number of embeddings and the logarithms of the largest weights are all polynomially bounded in x on the intervals, giving the assertion. □

Corollary 82.24 (An absolute bound along the full chain)**.** *Use the measures of Proposition 82.23. On a surviving cuspidal row, put* $R_\infty^\circ = \prod_{\tau,j} Z_e^{(s_{j,\tau})}(\mathcal{P}_{j,\tau}; u_{j,\tau})$*. Uniformly on the fields and intervals of Section 6.6,*

$$\log R_\infty^\circ \le m(k-1)\sum_\tau B_\tau(\log B_\tau - 1 - \log\pi) - \sum_{\tau,j} D_{j,\tau}\log s_{j,\tau} + O_{\psi,\epsilon}(x^C).$$

For other real Haar measures the same bound holds for the sum of $\log(d_{\sigma_{j,\tau}} Z_e^{(s_{j,\tau})})$*. For changed tests of the stated degree and highest source weight one adds the exact logarithm in Corollary 82.19 when the evaluation product is nonzero.*

Proof. Multiply (82.8) for the reference polynomials and use Proposition 82.20 to sum the Gaussian logarithms. In ball measure the formal-degree logarithms are absorbed by the uniform error just proved. The changed tests use exactly the same orthogonal projections and Haar measures. Their relative factor is therefore the one in Theorem 82.18. □

The absolute upper bound permits independent local loss factors $\|\omega(f_u)\mathcal{P}_j\|^2/\|\mathcal{P}_j\|^2$. A lower bound for their full product and control of the finite class sum in the designated global period still require further arguments.

82.6 Totally positive generators in the comparison of real norms

In Theorem 60.30, the additive character is scaled by $\beta \in F^\times$ and the skew form by $\gamma \in F^\times$, with $F = E^+$. Both are totally positive. Fix the two split prime ideals of that theorem. Their relations to the absolute different $\mathfrak{d}_F$ and the ideal $\mathfrak{a}_0$ below $(\sqrt{-d_0})$ are

$$(\beta)\mathfrak{d}_F = \mathfrak{p}, \qquad (\gamma)\mathfrak{a}_0 = \mathfrak{q}.$$

Theorem 82.25 (Individual real scales on the period fields)**.** *Put* $f = [F:\mathbb{Q}]$*, take the given cyclotomic multiple* N *with* $E \subset \mathbb{Q}(\zeta_N)$*, and write* $B_F = (f-1)\varphi(N)\log(2N)$*. The generators can be chosen with*

$$\max_\tau \left|\log\tau u - f^{-1}\log N_{F/\mathbb{Q}}u\right| \le B_F \quad (u = \beta,\gamma), \qquad 1 + \max_\tau(|\log\tau\beta| + |\log\tau\gamma|) = b^{o(1)}$$

on the same intervals as Theorem 13.11. At the real places give the Fock vacuum norm one, and keep the polynomial coordinates and source vectors fixed. Write D_τ for the sum of the chosen monomial degrees through all edges at τ and $\bar{D} = f^{-1}\sum_\tau D_\tau$. If R_∞ is the product of the absolute diagonal real factors, then

$$\frac{R_\infty(\beta,\gamma)}{R_\infty(1,1)} = \prod_\tau \left(\tau(\beta\gamma)\right)^{-D_\tau}, \qquad N_{F/\mathbb{Q}}(\beta\gamma) = \frac{N\mathfrak{p}\, N\mathfrak{q}}{|D_F|\, d_0^{f/2}}.$$

The logarithm of the real-factor ratio differs from $-\bar{D}\log N_{F/\mathbb{Q}}(\beta\gamma)$ by at most $2B_F\sum_\tau |D_\tau - \bar{D}|$. The geometric mean of the ratios over all simultaneous conjugates of the two generators, with the real tests fixed, is exactly $(N_{F/\mathbb{Q}}(\beta\gamma))^{-\bar{D}}$.

Proof. For every nontrivial character of the totally real abelian field F, take its primitive even Dirichlet character χ of conductor $h \mid N$. The squared absolute values of $(1-\zeta_h^a)/(1-\zeta_h)$ are totally positive units in the real cyclotomic field. Norming to $F \cap \mathbb{Q}(\zeta_h)$ gives units whose log coordinates are at most $2\varphi(N)\log(2N)$ in absolute value. Taking $\chi(a) \neq 1$ gives a nonzero coefficient in the χ-direction because the χ-Fourier coefficient of $\log|1-\zeta_h^j|$ is a nonzero Gauss sum times $L(1,\chi)$. The character idempotents now show that the log vectors of these units and their conjugates span the sum-zero hyperplane. Rounding coordinates in an independent subset gives the bound B_F on the distance of the logs of a positive generator from their mean after multiplication by a positive unit. Use this for both generators. Their norms and the bound on $\log N$ in Section 6.6 give the uniform assertion.

At a real embedding set $s = \tau(\beta\gamma) > 0$. In the vacuum normalization the dilation from the scaled Fock space to the original one takes $P(z)$ to $P(z/\sqrt{s})$. It intertwines both unitary groups since scalar dilation commutes with their actions and respects the splittings. A homogeneous degree-d test therefore contributes s^{-d} to the ratio of the diagonal integrals. Multiply over the edges and embeddings. Subtract the mean of the log vector of the product of the generators to obtain the bound. Averaging over the Galois group makes every fixed coordinate equal to this mean, giving the geometric-mean assertion. Detailed proofs are in Section 82.1. □

In particular the factor from the generators for parallel degrees is determined by their ideal norms. If the degrees are nonparallel, fixed ideals alone allow arbitrarily large and arbitrarily small positive real factors. Indeed there is a positive unit whose logarithm has nonzero scalar product with the degree vector, by the spanning argument. Multiplying one generator by its integer powers gives both limits. The calculation is in the fixed vacuum norm. The scale of the original real integrals, the designated rational tensor and the valuations of its projected period remain necessary for the desired prime estimate.

82.7 The last real projection and an integral coefficient

At the last forward step both groups are compact at infinity. Put $n = n_{m-1}$, $N' = n_m = n+t$. For a fixed real place write $p_1 > \cdots > p_n$ for P_{m-1} in Theorem 60.26, and let $a_T = -e_m + h_{m-1}$. In a reference orthonormal complex matrix frame use the Fock inner product with density $e^{-\pi\,\mathrm{tr}(Z^*Z)}$ and probability Haar measure on $U(n)$ and $U(N')$.

Theorem 82.26 (A projected last theta norm in integral coordinates)**.** *The source and target, after removal of the compact vacuum characters, have the same partition*

$$\lambda_i = p_i - (n+1)/2 + i - a_T - N'/2 \quad (1 \le i \le n)$$

with zeros appended on the target. Write $d = |\lambda|$ and $D_n = \dim S_\lambda(\mathbb{C}^n)$. Fix real-place unit lines u, v in these two complex representations, and write $Z_v(P;u)$ for the real local Rallis

integral of a test after projection to the target line. Among the reference unit monomials of degree d there is one with

$$(MD_n)^{-1} \le Z_v(P;u) \le D_n^{-1}, \qquad M = \binom{d+nN'-1}{nN'-1}.$$

The sum of the logarithms of MD_n over the real places is bounded by a power of x on the common intervals. In particular the composed complex operator in Theorem 60.30 can still detect every surviving column on its real input line after fixing any one target line at each real place and choosing the last tests accordingly.

Take now the matching highest lines and let $H_\lambda = \prod_i \Delta_i^{\lambda_i-\lambda_{i+1}}$ in the leading minors, with $\lambda_{n+1} = 0$. For a polynomial Q with algebraic integer coefficients, column degrees λ_i, and invariance under upper unitriangular column operations, write $J = \binom{1_n}{0}$. At any coefficient embedding, with any scale $s > 0$ relative to the reference real frame, the projected squared norm is

$$Z_v^{(s)}(Q;u) = |Q(J)|^2 \, \frac{(\pi s)^{-d}}{D_n^2} \prod_{i=1}^{n} \frac{(\lambda_i + n - i)!}{(n-i)!}. \tag{82.10}$$

Thus for $Q(J) \neq 0$, the sum of the logarithms of the ratios $Z_v^{(s)}(Q;u)/Z_v^{(s)}(H_\lambda;u)$ over all coefficient embeddings is exactly $2\sum_{\mathfrak{l}} \operatorname{ord}_{\mathfrak{l}}(Q(J)) \log N\mathfrak{l}$.

Proof. The compact vacuum shifts are $a_T + N'/2$ on the source and $-e_m + h_m + n/2$ on the target. Subtraction from the highest weights of Theorem 60.26 gives the stated partition. By Schur–Weyl duality the symmetric algebra of the matrix coordinates contains exactly one joint summand of this pair of types, in degree d. Schur orthogonality expresses Z_v as $1/D_n$ times the squared norm of the projection to $u \otimes v$ in that summand. The normalized degree-d monomials are an orthonormal basis of size M. The sum of their projected squared norms is the trace one. Since $D_n \le M$ and $\log M \le nN' \log(2 + d + nN')$, the interval weight and rank bounds give the logarithmic estimate. Apply it with the incoming line from the preceding tests at each real place. Projecting the test by the target matrix unit projects the automorphic output by equivariance. This gives positive local factors common to the full fixed row, as required in the proof of Theorem 60.30.

The column identities put Q in the source highest line. Its component on the target highest line is $Q(J)H_\lambda$, since the row degrees of the diagonal monomial are the highest weight and $H_\lambda(J) = 1$. The Fischer squared norm of H_λ is $D_n^{-1}\prod_i(\lambda_i + n - i)!/(n-i)!$, by the Gaussian Gram–Schmidt computation of Lemma 70.4. The reference vacuum introduces π^{-d}, the source matrix unit contributes the other $1/D_n$, and Theorem 82.25 gives the factor s^{-d}. The principal ideal of $Q(J)$ now gives the final identity by its norm. The complete local computation, with its realization inside the complex chain, is in Section 82.2. □

For the coefficient-one highest vector the reference calculation also gives a uniform asymptotic at the required precision. Write B_τ for the offset following (9.1) and d_τ for the last degree there. Then

$$\sum_{\tau} \log Z_{v_\tau}^{(s_\tau)}(H_{\lambda_\tau};u_\tau) = (k-1)\sum_{\tau} B_\tau(\log B_\tau - 1 - \log\pi) - \sum_{\tau} d_\tau \log s_\tau + O_{\psi,\varepsilon}(x^C).$$

Indeed the first $k-1$ parts of λ_τ differ from B_τ by a power of x, and all other parts are polynomially bounded in x. Stirling's estimate in (82.10) and the Weyl dimension formula give the display, as proved in Corollary 82.10. It evaluates the final real projection for a specified

polynomial lattice. The global period requires the polynomial supplied by the preceding lifts and their sum over finite classes to survive in the designated arithmetic component, with the required height bound. Those comparisons and the remaining strict orders in (12.15) still enter (W).

82.8 Two consecutive integral evaluations

One step earlier the source is $U(p,1)$ and the target is compact. Write $n = n_{m-2}$, $p = n-1$, $L = n + t_{\mathrm{r}}$ and $a_T = -e_{m-1} + h_{m-2}$. At the real place, let $p_1 > \cdots > p_p$ be the plus entries of P_{m-2}. Put

$$\alpha_i = p_i - a_T - (p+L)/2 + i, \qquad \beta = H_0 - (t_{\mathrm{r}}+1)/2, \qquad d_- = |\alpha| + \beta.$$

Here α is a partition and $\alpha_p = \beta > 0$. Use coordinates $(Z,W) \in M_{L,p}(\mathbb{C}) \times \mathbb{C}^L$ in the unit-vacuum Gaussian frame of Theorem 82.26. Set $J = \binom{1^p}{0}$ and let e_L be the last coordinate vector in $\mathbb{C}^L$.

Theorem 82.27 (The two terminal real integrals)**.** *Take the highest line of the lowest compact source type and the highest line in the compact intermediate representation. Suppose $Q(Z,W)$ has column degrees α_i in Z, is invariant under its upper unitriangular column operations, and is homogeneous of degree β in W. For the indicated unit lines u, v, the projected local integral on this edge is*

$$Z_v^{(s)}(Q;u) = |Q(J,e_L)|^2 (\pi s)^{-d_-} H(\alpha)\beta!/d_\sigma.$$

Here d_σ is the formal degree of the source discrete series in the chosen real Haar measure, $H(\alpha) = \prod_i (\alpha_i + p - i)! / \prod_{i<j}(\alpha_i - \alpha_j + j - i)$, and $s > 0$ is the scale relative to the reference frame. On taking v as the input to Theorem 82.26, the ratio of the two-edge product to the same product for the coefficient-one highest polynomials equals

$$|Q(J,e_L)\, Q'(J')|^2,$$

where Q' is the last test and J' its identity frame. For algebraic integer coefficients a nonzero product of evaluations therefore gives the exact logarithmic norm identity of Theorem 82.26 with this product inside each valuation.

Proof. The sole negative centered entry of the source is $-H_0$, and its least positive entry is H_0. Adding the half-sum of noncompact positive roots and subtracting the compact half-sum gives the lowest compact type of this discrete series. Removing the vacuum characters gives precisely $(\alpha; -\beta)$. The compact target after its vacuum shift has highest weight $(\alpha_1, \dots, \alpha_p, 0, \dots, 0, -\beta)$ by Theorem 60.26. In a monomial its target weights are the Z-row counts minus the W-exponents. The last target weight $-\beta$ forces all W-variables to be W_L. The others force the highest row degrees in Z. The surviving polynomial is consequently $Q(J,e_L) H_\alpha(Z) W_L^\beta$, by the highest-line calculation in Theorem 82.26. It lies on the chosen source and target lines.

Since the target is compact, its isotypic summand of the oscillator realizes the Hilbert tensor product of the target type and its irreducible unitary source partner. The source is a discrete series, so formal-degree orthogonality evaluates the projected integral as the squared norm of this line component divided by d_σ. The two polynomial factors use independent Gaussian variables, with squared Fischer norms $H(\alpha)$ and $\beta!$. The dilation and the original Gaussian give $(\pi s)^{-d_-}$. The intermediate group has the same highest line on both edges, so multiplication by (82.10) proves the second identity. Section 82.3 also proves that a unit monomial can detect any lowest compact input line and any compact target line on this edge. □

For the coefficient-one highest polynomials, the logarithms on these two edges have the same leading contribution $(k-1)\sum_\tau B_\tau(\log B_\tau - 1 - \log\pi)$ per edge, after extracting the exact dilation terms and the reference formal degree on $U(p,1)$. That degree is $p!/\pi^p$ for probability compact integration and ball measure $(1-\|z\|^2)^{-(p+1)}d^{2p}z$. The total error is bounded by a fixed power of x, by Corollary 82.15. The arithmetic input reaching these two edges, its finite class sums and its nonvanishing at the prescribed level remain to be controlled. Theorem 13.12 bounds the cost of the finite test places on the same intervals. The remaining global strict orders are those in (12.15).

82.9 An integral polynomial on every real pair

Fix j, a real place τ, and use the forward theta quotient of Theorem 60.26. Subtract the target splitting exponent a_T from the ordered source parameter. Its entries ℓ_i consist of $A = k + j(t_{\mathrm{r}}+1)$ positive plus entries followed by alternating negative entries of signs minus, plus, starting with minus. Put $y = s_{j+1}$, $z = s_j$, $h = (t_{\mathrm{r}}+1)/2$, and set

$$a_i = \ell_i + i - A - h, \quad b_v = \ell_{A+2v} - h, \quad c_u = \ell_{A+2u-1} + h.$$

The three partitions used below are $\alpha = a$, the reversed list $\beta = (-b_y, \dots, -b_1)$, and the reversed list $\gamma = (-c_z, \dots, -c_1)$. Write $D = |\alpha| + |\beta| + |\gamma|$.

Theorem 82.28 (The matched highest lines on the full real chain)**.** *At every step one can use the highest lines of the lowest compact types of its two discrete series. In the unit-vacuum polynomial model, a product $\mathcal{P}_{j,\tau}$ of three polynomials in minors, with integer coefficients, maps nontrivially to these lines. Its degree D is the least Fock degree of any compact type of the source discrete series on this pair. For homogeneous degree-D tests $F_{j,\tau}$ of the same highest source compact weight, the product of the squared norms of the projected real theta vectors, divided by the product for $\mathcal{P}_{j,\tau}$, is*

$$\left| \prod_{\tau} \prod_{j=0}^{m-1} F_{j,\tau}(\mathcal{J}_{j,\tau}) \right|^2, \tag{82.11}$$

where the frames $\mathcal{J}_{j,\tau}$ have entries zero and one. This also gives the ratios of the projected real Rallis factors on each surviving cuspidal row of Theorem 60.29.

Proof. The stripped lowest compact weights are $(a, b\,;\, c)$ at the source and $(a, 0^{t_{\mathrm{r}}}, c\,;\, b)$ at the target. This follows by adding the half-sum of positive roots and subtracting twice the compact half-sum at each slot, then removing its compact vacuum character. In the defining source-plus and target-plus matrix take the first square of size A. Take the terminal square of size y in the dual source-plus and target-minus matrix and the terminal square of size z in the dual source-minus and target-plus matrix. Reverse both index orders in a dual square. For a partition ν the product of the leading l-minors with exponents $\nu_l - \nu_{l+1}$ is highest for its two block actions. Use this product on the three squares with partitions α, β, γ to obtain $\mathcal{P}_{j,\tau}$. It is one on the frame with these squares the identity and every other coordinate zero.

Here is the point that ensures survival when both groups are noncompact. Put $\delta_i = 1$ on the positive source entries and -1 on the negative tail. It is compact dominant and pairs nonnegatively with every positive noncompact root. Let C be the vector of compact vacuum weights and let μ be the actual lowest compact highest weight. For any compact highest weight ν occurring in the source, the Blattner formula places a compact Weyl translate of $\nu + \rho_{\mathrm{c}}$ in $\mu + \rho_{\mathrm{c}}$ plus the cone of positive noncompact roots. Pairing with δ and sorting within the compact blocks gives

$$\|\nu - C\|_1 \geq \langle \delta, \nu - C\rangle \geq \langle \delta, \mu - C\rangle = D.$$

Each oscillator variable contributes just one signed weight on the source, so every monomial of this weight has degree at least that norm. Both chosen compact types attain degree D. Howe's least-degree lemma for the theta quotient now preserves their joint harmonic copy. Concretely, the full source isotypic space is generated from its least-degree part by the target enveloping algebra. That part must survive somewhere. Every target type in the quotient has degree at least D by the same lemma, so its surviving part at degree D is joint harmonic. There is exactly one compact tensor of joint harmonics of the indicated source type. Its highest vector is $\mathcal{P}_{j,\tau}$.

For any test in the assertion, take the indicated target torus weight. The absolute weights on both sides sum to the total degree. This forces every entry used into the same three squares, with the specified row and column degrees. In each square the source highest identities and the highest row weight pick out the one-dimensional line of the corresponding minor polynomial by the Cauchy decomposition. Its coefficient is evaluation at the identity on the square. Thus the target highest projection gives precisely $F_{j,\tau}(\mathcal{J}_{j,\tau})\mathcal{P}_{j,\tau}$. The intermediate highest lines agree on consecutive edges, so contraction with unit vectors and multiplication gives (82.11). On an occurring cuspidal row the global norm formula and the nonzero local quotient identify the same relative factors. The degree calculation, compact weights, and this application of the norm formula are proved in Section 82.4. □

For integral coefficients, a nonzero product of evaluations in (82.11) has the principal-ideal norm formula of Theorem 82.27 after summation over coefficient embeddings. The Fock norm of its reference polynomial is $(\pi s)^{-D}H(\alpha)H(\beta)H(\gamma)$ at scale s. The ratio in (82.11) cancels the absolute real theta normalizations. Determining those on earlier noncompact pairs, controlling the designated global coefficient at the original finite level, and bounding the strict orders in (12.15) are still needed in the common-interval proof.

82.10 An absolute estimate for the projected real integrals

Let $\sigma = \sigma_{j,\tau}$ and let u be a unit vector on the source highest line in Theorem 82.28. Write d_σ for its formal degree relative to the chosen real Haar measure. We use $Z_e^{(s)}$ for the actual real Rallis integral after orthogonal projection to the target compact highest line, at dilation s. Normalize the Fock vacuum to have norm one.

Theorem 82.29 (Upper bound for the absolute real product)**.** *On every real pair, for a test F satisfying the weight and degree hypotheses of Theorem 82.28,*

$$0 \le d_\sigma Z_e^{(s)}(F;u) \le |F(\mathcal{J}_{j,\tau})|^2(\pi s)^{-D}H(\alpha)H(\beta)H(\gamma).$$

On a surviving cuspidal row use the reference polynomials $\mathcal{P}_{j,\tau}$ on all pairs and let R_∞° be the product of their projected real integrals. Give the compact factors probability in the disintegration of $U(p,q)$ and take the usual matrix ball with measure $\det(I - Z^*Z)^{-(p+q)}d^{2pq}Z$ on its quotient. Then $R_\infty^\circ > 0$ and, on the same fields and intervals,

$$\log R_\infty^\circ \le m(k-1)\sum_\tau B_\tau(\log B_\tau - 1 - \log\pi) - \sum_{\tau,j} D_{j,\tau}\log s_{j,\tau} + O_{\psi,\varepsilon}(x^C).$$

Proof. For a noncompact source, two entries of opposite compact sign differ by at least $2H_0$. Indeed all positive entries are plus, and the negative tail consists of odd negative multiples of H_0 with alternating signs. The discrete-series integrability test of Miličić requires that for every noncompact root ξ, $|(\lambda,\xi)| > \frac{1}{4}\sum_\zeta |(\zeta,\xi)|$, as a sufficient condition. The sum is over

the full root system. In type A_{n-1} the right side is $n-1$, and the spacing H_0 exceeds the largest rank n_m. Therefore the coefficient $f_u(g) = d_\sigma \overline{\langle \sigma(g)u, u\rangle}$ is integrable. This also holds at a compact source. Schur orthogonality gives $f_u * f_u = f_u = f_u^*$. Consequently its integrated oscillator operator is an orthogonal projection and $d_\sigma Z_e(F;u) = \|\omega(f_u)eF\|^2 \le \|eF\|^2$. The exact formula for eF and its Gaussian norm in Theorem 82.28 give the local inequality.

In the specified ball measure the formal degree equals

$$\pi^{-pq} \prod_{l=1}^{q} \frac{\Gamma(p+l)}{\Gamma(l)} \prod_{i<l} \frac{\lambda_i - \lambda_l}{l-i}, \qquad \lambda_1 > \cdots > \lambda_{p+q}.$$

To obtain the constant, integrate the squared coefficient of the unit constant in the scalar Bergman discrete series. Its Jacobian factor cancels the invariant density, leaving ball volume $\pi^{pq} \prod_{l=1}^{q} \Gamma(l)/\Gamma(p+l)$. Harish-Chandra's formal-degree formula gives the Weyl ratio to that reference parameter. All ranks and logarithms of the largest weights are polynomially bounded in x. Thus $\sum_{\tau,j} |\log d_\sigma|$ is polynomially bounded as well. On every edge the reference consists of three minor polynomials with Gaussian norms given above. Exactly $k-1$ parts are $B_\tau + O(x^{C_1})$, while all other parts are polynomially bounded. Stirling's inequality in the hook products gives the claimed upper bound after summing. Positivity on an occurring cuspidal row follows from Theorem 82.28. The ball integral, all measure constants and the root estimate are given in Section 82.5. □

For each reference test the ratio to its local upper bound is the squared norm of the orthogonal projection of a unit vector. The two final ratios are exactly one. The earlier ratios and the finite class sum enter the lower arithmetic comparison for the designated global period. The bound for the original strict global classes in (12.15) is also still needed for the common interval.

82.11 A compact conjugation average on a tempered column

Let L be a nonarchimedean local field, O its integer ring and P the size of its residue field. Put $G = \mathrm{GL}_k(L)$, $C = \mathrm{GL}_k(O)$. Use probability Haar measure on C.

Lemma 82.30 (A bound independent of the compact type)**.** *Let π be a tempered unitary representation of G. Let $a \in G$ have elementary divisors $(-d_-, 0, \dots, 0, d_+)$ with $d_\pm \ge 0$, and put $D = d_- + d_+$. For every measurable $|\phi| \le 1$ on C,*

$$\left\| \int_C \phi(h)\pi(hah^{-1})\, dh \right\| \le k^D P^{-(k-1)D/2}.$$

Proof. Let λ be the left regular representation and e_C its orthogonal projection onto left C-invariants. For $u, v \in L^2(G)$, take absolute values inside the integral of the regular matrix coefficient and set $x = hy$ in its integral over G. Cauchy–Schwarz in h bounds the answer by

$$\int_G U(a^{-1}y)V(y)\, dy, \qquad U(y) = \left(\int_C |u(hy)|^2\, dh \right)^{1/2},$$

with V defined using v. Both functions are left C-invariant and have norms $\|u\|, \|v\|$. Thus $\|e_C \lambda(a) e_C\|$ bounds the operator in the regular representation. Let ω be the compactly supported measure obtained by pushing $\phi(h)\, dh$ forward under conjugation of a. For vectors in π fixed by a compact open subgroup J, the matrix coefficient of $\pi(\omega)$ is unchanged on replacing ω by $e_J * \omega * e_J$, with e_J normalized Haar measure. This convolution has compact support and a locally constant density. Temperedness gives weak containment in λ, hence the norm

inequality on that density. The two regular compact averages are contractions. Smooth vectors are dense, which proves the same bound for every pair of vectors in π.

For the spherical regular bound let $\Xi(g)$ be the coefficient of the normalized C-fixed vector in normalized induction of the trivial character of the diagonal torus from the upper triangular Borel. This unitary representation has a one-dimensional fixed space by Iwasawa decomposition. In its compact model Ξ is an integral of a strictly positive half-modulus, so $\Xi > 0$. It is bi-C-invariant and $\Xi(a^{-1}) = \Xi(a)$ by unitarity and reality. Inserting the rank-one fixed-space projection between two matrices gives $\int_C \Xi(ghg')\,dh = \Xi(g)\Xi(g')$. Consequently the positive convolution kernel $e_C\lambda(a)e_C$ and its transpose both send the function Ξ on the discrete set $C\backslash G$ to $\Xi(a)\Xi$. All cosets have the same Haar mass. In Schur's test, apply the weighted Cauchy–Schwarz inequality to each row with weights $\Xi(y)/\Xi(x)$. Summation by columns and the transpose identity then give operator norm at most $\Xi(a)$.

The spherical coefficient $\Xi(a)$ is the value at torus parameters all equal to one of the Satake transform of 1_{CaC}, divided by $\mathrm{vol}(CaC)$. The coefficientwise bound in the proof of Proposition 83.8, using the D one-step diagonals and their inverses, bounds this numerator by $(kP^{(k-1)/2})^D$. Its denominator there is at least $P^{(k-1)D}$. Division proves the estimate. □

82.12 A tempered estimate with an auxiliary Hilbert space

Let $G = \mathrm{GL}_k(L)$, $C = \mathrm{GL}_k(O_L)$ and P be as in Lemma 82.30. The first-plane orbital integral at the terminal prime involves an additional variable recording a covector. We shall use the following form of the compact estimate.

Lemma 82.31 (Compact invariance in a tensor product)**.** *Let π be tempered on G, let V be a Hilbert space carrying a continuous unitary action $h \mapsto R_h$ of C, and let S be a contraction on V. If $u, v \in \mathcal{H}_\pi \otimes V$ are fixed by $\pi(h) \otimes R_h$ for all $h \in C$, then*

$$|\langle(\pi(A) \otimes S)v, u\rangle| \le \Xi(A)\|u\|\|v\|.$$

Here Ξ is the positive spherical function of Lemma 82.30. In particular if the only nonzero exponents of A are $-d, d$, allowing $d = 0$, the right side is at most $k^{2d}P^{-(k-1)d}\|u\|\|v\|$.

Proof. Average the coefficient under simultaneous conjugation by $\pi(h) \otimes R_h$. Its value is unchanged, and the integrand operator becomes $\pi(hAh^{-1}) \otimes D(h)$ with $D(h) = R_h S R_h^{-1}$. Each $D(h)$ has norm at most one. In the left regular representation on $L^2(G; V)$, change the integration variable $x = hy$. For test functions v_0, u_0, the absolute matrix coefficient of the average is bounded by

$$\int_G \int_C \|v_0(hA^{-1}y)\|_V \, \|u_0(hy)\|_V \, dh\, dy.$$

Cauchy–Schwarz in h gives a regular matrix coefficient between the two scalar left C-invariant functions constructed in Lemma 82.30, with the pointwise Hilbert norm in place of absolute value. Their norms are the original norms. The regular bound in that lemma is $\Xi(A)$.

To transfer the estimate, first test on finite sums of simple tensors whose first factors are smooth. Fix a compact open J fixing these first factors and choose an orthonormal basis of the finite-dimensional span V_0 of all their second factors. Compress $D(h)$ to V_0. Each matrix entry of the average now involves a compact measure on G. Convolving on the left and right by probability Haar measure on J preserves the test coefficient and gives a compactly supported locally constant density in each matrix entry. Since π is tempered, its integrated action is a representation of the reduced group C^*-algebra. On a finite matrix of densities this representation is contractive in the regular matrix norm, by contractivity of the induced star representation of the matrix algebra. In the regular model the compression to V_0 and

the two J-averages are all contractions. The preceding regular bound thus applies to the test coefficient. Density gives the result for the averaged operator on the whole tensor product. The explicit bound for $\Xi(A)$ is the last estimate in Lemma 82.30. □

82.13 Absolute size of the last restriction on the period fields

Fix the parameters of (9.1) on the fields of Theorem 15.11. Write $b = \exp(x^{\varepsilon})$, $X' = \exp(x^{\varepsilon/3})$ and $d = t_{\mathrm{r}} + 1$. Use normalized Fischer forms with the highest-vector square equal to one in both endpoint modules. At each embedding write c_τ, κ_τ for the factors of the standard restriction from Section 74.13. The word and translate are implicit in these two numbers.

Theorem 82.32 (An absolute arithmetic bound for the restriction scalar)**.** *There is one positive integer D_x on each field E_x, valid for every word and translate at this x, with*

$$\log D_x \ll x^{C_0}(1 + b/X').$$

For each such choice D_x is divisible by the absolute numerator and the denominator in lowest terms of each number in the list

$$\prod_\tau c_\tau, \qquad \prod_\tau \kappa_\tau, \qquad \prod_\tau c_\tau \kappa_\tau, \qquad \prod_\tau c_\tau^2 \kappa_\tau.$$

In particular $\log D_x = o(b/x^A)$ for every fixed $A > 0$, uniformly as the left endpoint of the interval tends to infinity.

Proof. We first locate the large powers in one embedding. Denote the exponents in Section 74.13 by $a_i = \mu_i - \lambda_{i+1}$ and $b'_i = \lambda_i - \mu_i$. Continue to write b_j, e_j for the parameters of the repeated tail. Put $N = k + md = n + 1$. In the initial part of the lists of Corollary 105.2 one has, for $1 \le i < k$,

$$a_i = v_i - u_{i+1} - 1/2, \qquad b'_i = u_i - v_i - 1/2.$$

At index $i_j = k + (j-1)d$ the differences are

$$\begin{aligned} a_{i_j} &= e_j - b_j - d/2, \\ b'_{i_1} &= u_k + b_1 - 1/2, \qquad b'_{i_j} = b_j - e_{j-1} - d/2 \quad (j > 1). \end{aligned}$$

Every remaining difference vanishes, because the two entries compared there belong to the same constant block. Thus outside the three entries

$$A = a_{k-1}, \qquad B = b'_1, \qquad C = b'_k$$

all exponents are bounded by a fixed polynomial in x, uniformly in the word. Indeed $a_i = \Delta c_{i+1}$ for $i \le k-2$, and $b'_i = 4\Delta - 1 - \Delta c_i$ for $2 \le i < k$. Both tail parameter spacings equal H_0. The three exceptions are fixed across words at the given translate. Since $v_{k-1} = 0$, at a high place $A = \lceil xb/X' \rceil + (k-2)/2$, and elsewhere $A = 0$. At every place $B, C \ll xb + x^{C_1}$. In particular it is possible for C to have order b at an ordinary place.

Take the two products X_0, P_0 of Lemma 98.3 with exactly these exponents. The highest-vector calculation gives

$$\kappa_\tau = \frac{q(X_0)}{q(P_0)} \frac{q(X_w)}{q(X_0)} \frac{q(P_0)}{q(p_{l_w})}.$$

For the middle quotient form a box as in Theorem 98.2. Its lower product is X_0 and its upper exponents are the maxima of the exponents of X_w over all words. The three indicated large

powers have already been included, so the degree difference is polynomial in x. The logarithm of two plus the upper degree is also polynomial. The theorem gives a common integer of polynomial logarithm divisible by both reduced parts of the middle quotient for every word. For the last quotient use a box with lower product P_0. The exponent of D_i in p_{l_w} is $a_i + b'_i$. After removing the three specified powers, the same degree bound therefore holds. Lemma 98.3 now shows that both parts of every κ_τ divide a single integer of logarithm $O(x^{C_2}(1 + b/X'))$. At an ordinary place the first quotient is exactly one and a polynomial logarithm suffices.

For c_τ, the length of the interval of permitted integers at index i, after omitting μ_i, is at most $d_0 = \sum_i a_i$. This sum is at most $A + x^{C_3}$. All the factors in the product for c_τ have absolute value at most $n + \lambda_1 - \lambda_N$. Their logarithms are $O(\log(2 + x^C + xb))$. For a fixed reference word the logarithm of $|c_\tau|$ is therefore $O(x^{C_4}(1 + b/X'))$, by counting at most $n^2 d_0$ factors. For every other word, the proof of Theorem 74.29 bounds both parts of the ratio of these integers by divisors of one integer of polynomial logarithm, before combining it with the Fischer factors. Multiplying that integer by the absolute reference value gives one common multiple of all the $|c_\tau|$.

Take the square of each common multiple for $|c_\tau|$, multiply by the integer for κ_τ, and multiply over the embeddings and the translates. There are polynomially many of these pairs by Proposition 6.4. Addition of prime valuations proves the four divisibilities, with a single fixed exponent C_0 after increasing it to include the number of pairs. Finally both b and X' exceed every fixed power of x for fixed $\varepsilon > 0$. This proves the little-oh estimate. □

Corollary 82.33 (The contribution to norms in every exterior degree)**.** *Remove the explicitly evaluated normal-line factor Z of Corollary 105.2. All nonzero singular values of the remaining restriction have the same square $s_w = \prod_\tau c_\tau^2 \kappa_\tau$. For every fixed $A > 0$,*

$$\sup_w |\log s_w| + \sum_p \sup_w |v_p(s_w)| \log p = o(b/x^A)$$

with the suprema also over translates. The same bound applies to the logarithm of each nonzero singular value squared in an exterior power, divided by its positive exterior degree, and to its finite valuations with this division. At g_1, g_2, T these valuations vanish.

Proof. The tensor product identity in Theorem 74.28 gives the scalar on the product target. All nonzero singular values of an exterior power are products of the indicated number of nonzero singular values, by diagonalizing in unitary bases. Both reduced parts of s_w divide D_x. Hence $|\log s_w| \le \log D_x$ and $|v_p(s_w)| \le v_p(D_x)$. The prime sum is at most $\log D_x$, which proves the assertion by the theorem. The marked primes satisfy the cutoff in Theorem 105.1. □

Proposition 82.34 (The normal line in the integral frames at the marked primes)**.** *On each lifted definite class the last restriction, written in the integral coefficient frames of the large and small lattices, is the standard map of Theorem 74.28. On these full coefficient lattices it is a split surjection at every prime above g_1, g_2, T, including the places where Z has nonzero valuation in the original rational frames.*

Proof. Write M for the small unimodular Hermitian lattice. In the lift the lattice whose Hermitian dual is used to evaluate the coefficients is $M \oplus \pi \mathcal{O}_E e_N$, with the standard unit normal vector e_N. Indeed the terminal marking on the normal coordinate has kernel $\pi \mathcal{O}_E$. Reduction at each prime dividing π tests that coordinate times $\bar{\pi}$, a unit there, and these primes have product (π) since T splits completely. Dualizing the orthogonal sum gives $M \oplus z^{-1} \mathcal{O}_E e_N$, where $z = \bar{\pi}$. Pairing with πe_N tests exactly z times the last coordinate. This also describes

the lifting condition in `Development/AbelianVarietyTheory/Definitions/AutomorphicMembership/ExtUHInjective.lean`, line 488.

At a finite place take a basis matrix G' for M. The corresponding basis matrix of the lifted dual lattice is $G = \mathrm{diag}(G', z^{-1})$. For a column of polynomial covectors, pullback into a frame with basis columns G uses $\rho_\lambda(G^{\mathsf{t}})$. This follows first for linear covectors by evaluation on the basis columns, then for products, minors and the separate determinant lines by functoriality. Write L for the standard restriction at one embedding and $L_z = z^{-\delta} L$ in the rational frames, suppressing the embedding on z, G, G'. Block equivariance and the normal degree of Corollary 105.2 give

$$L\,\rho_\lambda(G^{\mathsf{t}}) = \rho_\mu((G')^{\mathsf{t}})\,L_z.$$

Explicitly, each monomial retained by extraction uses the bottom row $\sum b'_i$ times. The determinant power contributes the additional exponent λ_N. Their sum is $\delta = \sum_i \lambda_i - \sum_i \mu_i$. The remaining n-block commutes with extraction and intertwines under the smaller isotypic projection. This proves the displayed equality with its specified sign. For the testing slot the change is $\rho_\mu((G')^{-1})$, preserving the same bilinear value, since transpose matrices are adjoint in the Fischer pairing. Tensor over all embeddings above the coefficient place. In these frames the lattice of minor polynomials and the lattice of integral coefficients agree at the three primes by Theorem 100.6 and the hook bound. The integral sections of Theorem 74.28 therefore give the claimed splitting on each coefficient space. □

The proposition describes the map at an individual class. The signed theta column restricts the values that can occur jointly at all the lifted classes. The absolute estimate for that restricted table and its pairing with the incoming Bessel row is still required in Section 6.6.

82.14 Vector evaluations and the signed theta row in the Hilbert norm

Use the residue sets G_P and evaluations E_G of Theorem 77.22. In the initial Hilbert space a class has unit mass, and the coefficient metric is the same on all of its markings. Put $Q = T^{fk}$. The integral source has already been specified in Theorem 77.20.

Proposition 82.35 (The least squared singular value of simultaneous evaluation)**.** *Let q_P be the largest number of members of G_P on any one residue line. If $q_P < T - 1$ for every P, then $E_G Q^{-1/2} \mathcal{F}_0$ from the separately zero-sum Hilbert space onto the grid values is surjective. Its singular values, counted on the target, are at most one, and the smallest squared singular value equals*

$$\prod_{P \mid \pi} \left(1 - \frac{q_P}{T-1}\right).$$

In particular this is at least $1 - f\sqrt{T}/(T-1)$ for the same grids of size at most $\sqrt{T}$ per place as in Theorem 77.22.

Proof. The normalized transform is unitary onto the corresponding zero-sum vector space by Theorem 60.33. Consequently the evaluation map times its adjoint is the restriction to G of the orthogonal projection onto that space, with both row and column indices in G. At one place the projection subtracts the mean on each scalar orbit. If t members of G_P lie on one line, their principal block is $I_t - \mathbf{1}\mathbf{1}^*/(T-1)$. Its constant line has eigenvalue $1 - t/(T-1)$ and its orthogonal complement has eigenvalue one. Distinct lines have zero cross terms. The full matrix is the tensor product of these local matrices and the identity on the coefficient space. Thus its smallest eigenvalue is the displayed product and is positive. The last bound follows by multiplying the elementary inequalities $0 \le q_P/(T-1) \le 1$ and using $\prod(1-y_P) \ge 1 - \sum y_P$. □

We next use vector frequencies to measure the actual output. Fix the base tuple and any finite Hilbert space Z of projected tests at the small terminal classes. Use its dual Hilbert space as the range, and write $\mathcal{P} : S^{\circ} \to Z^{\vee}$ for evaluation of the signed theta column on those tests, in the arithmetic class frames of Theorem 60.32. For initial class a and final class b, group the paths by the tuple C of their composite residue matrices. Let $W_{ab}(C) : D_a \otimes \mathbb{C} \to Z^{\vee}$ be the sum of their coefficient maps with all the class masses and terminal contractions of that theorem. This definition includes the coupled choices on the last two edges. The two class frames are held fixed. For each tuple of nonzero initial vectors v set

$$H_a(v) = \sum_{b,C} \left(\prod_{P|\pi} \psi_P(- f_{b,P}(C_P v_P)) \right) W_{ab}(C),$$
$$R_a(v) = \prod_{P|\pi} (1 - A_P) H_a(v).$$

Here A_P averages over multiplication of v_P by all nonzero residue scalars, and $f_{b,P}$ is the specified terminal covector. In H_a one may sum over all paths before discarding zero pullbacks. The projection to R_a makes both conventions agree. For fixed b the character in this formula is the matrix Fourier character attached at each place to the rank-one map $-v_P \otimes f_{b,P}$, since its trace against C_P is $-f_{b,P}(C_P v_P)$.

Theorem 82.36 (The Gram operator of the prescribed signed arithmetic column)**.** *On the full ordinary initial table with separate sums zero,*

$$\mathcal{P}\mathcal{P}^{*} = Q^{-1} \sum_{a} \sum_{v} R_a(v) R_a(v)^{*}.$$

The vector sums are over tuples with all coordinates nonzero. In particular, restricting the inner sums to any specified sets of vectors gives a lower bound as positive Hermitian operators on $Z^{\vee}$. The formula uses only the centered matrix Fourier coefficients on the indicated rank-one frequencies. For one algebraic terminal test F there is also the following inclusion in its actual fractional ideal of values. For any such v, any $d_P \neq 1$ in $k_P^{\times}$ and any $y \in D_a$, the ideal contains

$$u^{-f} z_{\max} \sum_{B \subseteq S_T} (-1)^{|B|} H_a(v_B)(y)(F), \qquad (v_B)_P = \begin{cases} d_P v_P & P \in B, \\ v_P & P \notin B. \end{cases}$$

All scalars and coefficient lattices in the integral assertion are taken over the same enlarged coefficient field as in Theorem 77.20.

Proof. Insert Fourier inversion into each term of Theorem 60.32. For $g_a = \mathcal{F}_0 \phi_a$ the resulting vector in $Z^{\vee}$ is $Q^{-1} \sum_{a,v} H_a(v) g_a(v)$. Each g_a has separate sums zero. Its pairing with $A_P H_a$ therefore vanishes, by collecting all terms of the scalar orbit. Hence H_a can be replaced by R_a. A path for which $f_{b,P} C_P = 0$ at one place has trivial character there, and its entire contribution is killed by $1 - A_P$.

In the normalized vector coordinates $Q^{-1/2} g_a$ the operator has blocks $Q^{-1/2} R_a(v)$. Each family $R_a(v)^{*} z$, for $z \in Z^{\vee}$, is already in the separately zero-sum space. Consequently the adjoint computed in the full array takes its values in the actual complex source. The unit mass at the initial class and unitarity of the normalized transform give the displayed Gram operator by matrix multiplication. Each summand is positive, which proves the lower bound for arbitrary subsets.

For the integral assertion prescribe the entire vector table at a to be $Q u^{-f} z_{\max} y$ times the tensor product of the differences at $v_P, d_P v_P$. Set all other class tables to zero. Theorem 77.20

proves this is the transform of an integral ordinary input. Its evaluation by the inversion formula cancels Q and gives the asserted alternating sum. For an actual complex lower estimate one must still control the signed coefficient sums $H_a(v)$ on the prescribed tests and compare the resulting operator with the analytic Bessel row of Section 6.6. □

83 Bounds for the first-plane integral

83.1 Localizing the real frame integral

At a real place write $G = U(S)(\mathbb{R})$, $U = U_\ell(\mathbb{C})$ for the upper unitriangular matrix group, and $X = X_*$ for the full frame in Proposition 80.1. Write H for its pointwise stabilizer and $R = \{g : gX \in XU\}$. Use χ for the character of U on the first superdiagonal. In the second position of the mixed oscillator, $\rho(p_*)$ denotes evaluation, leaving a Schwartz vector on the core. The residual action of g after the first polarization is denoted by $\Omega(g)$, with its splitting and change-of-measure factors. Thus on a product $A(x)\rho$ the first action is $A(g^{-1}x)\Omega(g)\rho$.

Lemma 83.1 (Real localization at any flag length)**.** *There is a smooth local section s of $G \to R\backslash G$ through the identity. Given any smooth compactly supported a on a sufficiently small section domain, there is a smooth compactly supported function A on the full ambient matrix space, supported among matrices of full rank, such that*

$$\int_U \overline{\chi(u)}\,[\omega(s(y))(A \otimes \rho)](Xu, p_*)\,du = a(y)\,[\Omega(s(y))\rho](p_*).$$

The same integral is zero on cosets outside that domain. The test A works for every residual Schwartz function ρ. One may take $a \geq 0$ of integral one on the real quotient, with support in an arbitrarily small neighborhood of the identity coset.

Proof. The Gram matrix of X is zero or has just its last diagonal entry nonzero. Hence $u^*X^*h_SXu = X^*h_SX$ for every $u \in U$. Full frames of this Gram matrix form a single G-orbit by Witt extension, applied to the isometry defined by matching the columns. The orbit $\mathcal{O}$ is embedded in the open set of full-rank matrices. Indeed, at a frame x there choose a matrix Z with $x^*h_SZ = 1$. The differential of the Gram map applied to $ZD/2$ is D for any Hermitian matrix D. The regular-level-set theorem proves the assertion. The group action identifies $H\backslash G$ smoothly with this orbit. Indeed the orbit map of a Lie group has constant rank, and its image here is the entire embedded level set. In particular its differential is onto, since a smooth constant-rank map of smaller rank cannot cover this level set.

The action $x \mapsto xu$ is free and proper. For the latter, if x and xu lie in fixed compact subsets of the full-rank set, a left inverse of x can be chosen with uniformly bounded norm, for example $(x^*x)^{-1}x^*$ for a positive definite auxiliary metric. Thus u is bounded in the closed matrix group U. Witt extension gives surjectivity of the map $R \to U$ defined by $gX = Xu$. Its kernel is H. The group R is closed because XU is closed in the space of matrices, using a left inverse to X. The quotient by a closed Lie subgroup has local smooth sections, obtained by exponentiating a complement to its Lie algebra. On such a section domain, $(y, u) \mapsto s(y)^{-1}Xu$ is a smooth parametrization of the corresponding open part of $\mathcal{O}$. Its inverse reads y from the class of the frame modulo U, then recovers u with the left inverse.

Choose a nonnegative compactly supported smooth function α on U of Haar integral one. On this part of $\mathcal{O}$ prescribe

$$A(s(y)^{-1}Xu) = a(y)\chi(u)\alpha(u)$$

and extend by zero on the rest of the orbit. This is smooth and has compact support in the full-rank locus. Cover the support by finitely many submanifold charts. In each chart multiply the

prescribed tangential function by a transverse bump equal to one on the orbit. A partition of unity and extension by zero give a smooth compact ambient extension whose restriction to the entire orbit is as prescribed. Substitution in the integral proves the equality and the support claim. A smooth positive quotient density permits normalization of any nonzero nonnegative bump to integral one. Taking products gives the assertion at all real places simultaneously. The construction places no bound on derivatives as the neighborhoods shrink. □

83.2 The first vector on the common intervals

Use the normalized unit-frequency section W_1 corresponding to the original word and real row. Write $\mathbb{E}_1$ for integration on $B_\infty\backslash G_\infty$ against $\|W_1\|^2$ divided by its integral. With u' the Euclidean unit vector of the first isotropic line, put

$$D = 4s^2 + 3, \qquad \mathcal{P}_D = \mathbb{E}_1 \prod_{\tau} \max(1, \|g_\tau^{-1} u'\|)^{2D}.$$

Thus the power in this moment is $8s^2 + 6$.

Theorem 83.2 (The first-vector estimate for the prescribed orbit)**.** *For O_1 of Theorem 89.7, uniformly in all prescribed words and after the same terminal compression,*

$$\|O_1\|_{\mathrm{op}} \le \exp(-(16 - o(1))(r-1) f x^{\varepsilon})\, \mathcal{P}_D^{1/2}.$$

The moment is finite at every x on the common intervals. In particular a bound $\log \mathcal{P}_D = O(f s^3 x^{\varepsilon})$ there would give the same exponential decay for this orbit.

Proof. At one place let $l_1, \dots, l_s$ be the successive Euclidean lengths of the unit Witt vectors under g^{-1}, orthogonalizing each against the preceding ones. Set $l_{s+1} = Z(g) \ge 1$ and put $N = \max(1, \max_{1 \le i \le s} l_{i+1}/l_i)$. In the block representative of Lemma 83.21 these quantities are $|\lambda_i|^{-1}$ and $\|m_0^{-1} e\|$. One has $\|m_0\| \le 2Z(g)$. For instance, compact rotations of the positive coordinates put $m_0 e$ in one positive direction and the negative line. On that plane a hyperbolic rotation with parameter t sends e to this vector up to compact phases, with $\cosh t = a$ from that lemma. The remaining isometry fixes e and is Euclidean unitary. Thus $\|m_0\| = e^t \le 2a$ for $t \ge 0$. The diagonal-block formula now gives

$$\nu(g) \le 2 \max(1, l_1) N^s.$$

Indeed $l_i \le l_1 N^{i-1}$ for all i, including $i = s+1$, while $1/l_i \le N^{s-i+1}$ for $i \le s$.

The flag vectors of Theorem 84.17 have length $\sqrt{2}$ before translation. At unit frequency its local maximum M_τ therefore satisfies $N_\tau \le \max(1, \sqrt{2} M_\tau)$. Take $j = sD$ in that theorem. Its derivative argument applies with terminal vector e, since it computes the simple coefficient using exactly the vector of the chosen frequency. The isometry for frequency one is Corollary 84.21. On the local lowest type, integration of N_τ^{2j} is consequently a positive quadratic form bounded by $\{C r^2 (L_* + j + 1)\}^{2j}$ times the norm squared, with an absolute enlarged C. The same estimate holds for the product of these weights with this bound raised to f. In fact the restricted forms on the finite lowest types are positive operators, and their tensor product has norm at most the product of the norms. This covers arbitrary vectors of the product lowest type.

Apply Cauchy–Schwarz with the probability measure $\mathbb{E}_1$ to the products of the first lengths and of the ratios. It yields

$$\mathcal{M}_D(W_1) \le 2^{Df}\, \mathcal{P}_D^{1/2} \{C r^2 (L_* + sD + 1)\}^{sDf}.$$

Here $L_* \ll \sqrt{r}(2\exp(x^{\varepsilon}) + r)$ by the lowest-type bound in Theorem 65.24. The logarithmic cost outside $\mathcal{P}_D^{1/2}$ is $O(f s^3 x^{\varepsilon})$. Also $l_1 \le \nu(g)$, so the moment is finite by Proposition 64.20.

Insert the last estimate into Theorem 83.18 and multiply by the original finite mass from Theorem 89.4. The numerical prefactor in Theorem 89.7 has $A_\sharp$ in place of A_*. Its logarithm still has the stated upper limit, since $\log A_\sharp \le O(r+\log(2s))$ from its formula. The extra cost just obtained is $o((r-1)fx^{\varepsilon})$ because $m^5x^{\varepsilon}=o(k)$ on this family. This proves the result. □

The source formulates the positive first-length moment in `Formalism/SharpRestriction/CuspSpanNormBounds/BesselShellMomentBounds.lean`, line 1634. The calculation here gives the other length estimates needed for this orbit on the original real row. Theorem 83.3 below proves the uniform bound for this moment. The full projected period and the remaining strict cohomology sum in Section 6.6 are open.

83.3 Exponential decay for the first rational orbits

Use the fields and intervals of Section 6.6 and the original word, finite level and terminal frequency. Write $b=\exp(x^{\varepsilon})$, $f=[E^{+}:\mathbb{Q}]$, $r=k+2s+1$ and $D=4s^2+3$. For the unit-frequency section W_1, let $\mathcal{P}_D$ be the normalized first-vector moment in Theorem 83.2. Let $V=\prod_{\tau}\nu_{\tau}$ be the quotient height from Section 64.7, and let $\mathbb{E}_1$ denote the probability with density proportional to $\|W_1\|^2$. Set $c_0=g_1g_2\sqrt{-q}$, and let O_j denote the normalized orbital Gram operator for γ_{jc_0}. In the sum below j is a positive squarefree integer prime to $2qd_0g_1g_2T$, all of whose prime factors split completely in E. The integer one is included.

Theorem 83.3 (The unit-frequency moment and the first rational orbits)**.** *Uniformly over the prescribed words, for every integer $1\le A\le 4r^2$ one has $\log \mathbb{E}_1V^A\le CfsA\log b$. For the prescribed first-vector moment and orbit one has*

$$\log \mathcal{P}_D=O(fs^3\log b),\qquad \|O_1\|_{\mathrm{op}}\le \exp\bigl(-(16-o(1))(r-1)f\log b\bigr).$$

For every fixed $0<\alpha<16$, on the same intervals,

$$\sum_{j\le b^{\alpha}}\|O_j\|_{\mathrm{op}}\le \exp\bigl(-(16-\alpha-o(1))(r-1)f\log b\bigr)$$

with the specified restriction on j. Both operator estimates hold after compression to the original terminal character and the retained coordinates.

Proof. We prove the moment estimate at a single real place first. Put $l(g)=\|g^{-1}U_1\|$, where U_1 is the first Euclidean unit Witt vector. Start at the prescribed row of parameter λ with lowest compact type U. Follow any sequence of a lowerings with nonzero outgoing Bessel maps as in Proposition 67.6, reaching parameter μ. For each $q'\ge 0$ with $a+q'\le 4r^2$ we prove

$$d_{\mu}\int_{B\backslash G} l(g)^{2q'}\|\Lambda_{\mu}(\sigma_{\mu}(g)w)\|^2\,d\dot g\le \exp(Csq'\log b)\,\|w\|^2,$$

for every vector w whose compact types occur in $(F^{*})^{\otimes a}\otimes U$. These vectors are compact-finite by admissibility. At $q'=0$ this is the isometry in Theorem 84.20. The shifted model exists at every retained step and is normalized by the same Haar measures.

To pass from q' to $q'+1$, take a Euclidean orthonormal dual basis $(\zeta_d)_{d=1}^{r}$ in the Witt frame. For each d, tensor equivariance gives the vector identity

$$\zeta_d(g^{-1}U_1)\Lambda_{\mu}(\sigma_{\mu}(g)w)=\sum_{i}\beta_i\Lambda_{\mu-e_i}(\sigma_{\mu-e_i}(g)\,p_i(\zeta_d\otimes w))\,,$$

where the sum includes only the nonzero maps. Indeed the left side applies $\Lambda_{\mu}\mathrm{ev}$ to $g(\zeta_d\otimes w)$. Insert $1=\sum_i \iota_ip_i$, commute g through the inclusions and apply Proposition 67.6. The squared

absolute values of the scalar factors on the left sum over d to $l(g)^2$. Bound the squared norm of each right side by r times the sum of the squared norms of its terms. Multiply by $d_\mu l(g)^{2q'}$, sum over d, and integrate. The induction hypothesis applies to each projected vector by Lemma 73.5. The formal-degree and scalar factor is bounded by $(2\pi)^{2s}(1+2L)^2$ as in Proposition 67.6. For each d, the sum of the remaining squared norms is at most $\exp(C_1 s\log b)\|w\|^2$ by Lemma 73.5. The whole step costs at most

$$r^2(2\pi)^{2s}(1+2L)^2\exp(C_1 s\log b)\le \exp(Cs\log b).$$

This proves the induction with a single C, independent of the path or q'. In particular it bounds the moment of every even order up to $8r^2$ on every initial compact vector.

The positive form for $\max(1,l)^{2A}$ on U has operator norm at most $1+\exp(CsA\log b)$ in the normalized model, since $\max(1,l)^{2A}\le 1+l^{2A}$. The operator for the product of the local weights on the product compact type is their tensor product, as is seen by integrating pairs of simple tensors and using polarization. Taking the f-fold product of the bounds yields a bound $\exp(CfsA\log b)$ for the product first-vector moment of order $2A$, including arbitrary vectors of the product compact type. In particular $\log \mathcal{P}_D = O(fs^3\log b)$. For the height, Theorem 83.2 gives pointwise $\nu_\tau \le 2\max(1,l_\tau)N_\tau^s$, where N_τ is the maximum of one and the consecutive flag-length ratios. By the local derivative estimate of Theorem 84.17 and the tensor-product argument of Theorem 83.2,

$$\mathbb{E}_1\prod_\tau N_\tau^{2sA}\le\{Cr^2(L_*+sA+1)\}^{2sAf}.$$

Since sA is polynomial in x, Cauchy–Schwarz applied to the two product weights gives the claimed bound for $\mathbb{E}_1 V^A$. Theorem 83.2 now yields the first operator estimate because $s^3 = o(r-1)$ on these intervals.

For the sum, use the positive majorant which gave Theorem 83.2. Its finite factor at $j=1$ is the unsigned finite mass and its real factor bounds the integral of the product of the two section norms. By Theorem 83.18, replacing c_0 by jc_0 in the real majorant multiplies the pure-power factor by $j^{-(r-1)f}$, and every positive logarithmic factor weakly decreases. The moment just proved is independent of j. As calculated in the proof of Theorem 89.6, the unsigned finite mass at all added split primes increases, relative to its value at $j=1$, by at most $2j^{(2k+2s-2)f}$. That proof also bounds finite translations and orthogonal compressions by the same majorant for the signed operator. Consequently

$$\|O_j\|_{\mathrm{op}}\le 2\,j^{(k-2)f}\exp\bigl(-(16-o(1))(r-1)f\log b\bigr)$$

uniformly. There are at most b^α terms, and $(k-2)/(r-1)\to 1$ while $(r-1)f\to\infty$. This gives the claimed sum. □

Theorem 89.8 sums this family over all its split squarefree multipliers. Other rational orbits, the arithmetic comparison of the integral period, and the uniform strict cohomology bound on these fields still require proof.

83.4 Two bounds for the original real orbital integral

Let $R=r-1$, $c_0=Q\sqrt{-q}$ and $U_0=Q\sqrt{q}$. For an integer $j\ge 1$ write I_j^+ for the real orbital integral of the product of section norms for γ_{jc_0}, divided by its diagonal real norm, in the original terminal frequency and word. Use the unsigned mass $m_f(c_0)$ at the single base parameter as a common normalization for all these real quantities.

Lemma 83.4 (Two powers of the rational multiplier)**.** *There are positive numbers A_0, A_1, independent of j, such that*

$$m_f(c_0)I_j^+ \le \min\{A_0 j^{-Rf},\, A_1 j^{-2Rf}\}.$$

On the original intervals they may be chosen uniformly in all the prescribed words with

$$\log A_0 \le -(16-o(1))Rf\log b, \qquad \log A_1 \le CRfs\log b, \qquad b=\exp(x^{\varepsilon}).$$

Proof. Theorem 83.18 bounds the positive real integral at the physical terminal frequency by using the unit-frequency moment of the same word. Its pure-power factor at jc_0 is j^{-Rf} times that at c_0, and the positive logarithmic factors can only decrease for $j \ge 1$. Multiply the real majorant at c_0 by $m_f(c_0)$. Its logarithm satisfies the stated estimate for A_0 by the proofs of Theorems 83.2 and 83.3, which bound this positive majorant before taking the signed orbital pairing.

For the other power use the exact frequency substitution of Proposition 83.14. Its quotient Haar factor is T^{-Rf}, and the parameter at each real place has modulus jU_0/T at unit frequency. The proof of Theorem 83.10 applies at this frequency and to this tuple of real parameters. Indeed its fiber calculation in Theorem 83.16 uses only a nonzero imaginary scalar at each place, and its Cauchy–Schwarz argument uses only left invariance of the section norm. It gives

$$A_1 = m_f(c_0)C_E\,T^{Rf}(2/U_0)^{2Rf}\left(\prod_{\tau} C_{r-2,h_\tau}\right)\mathbb{E}_1\prod_{\tau} l(g_\tau)^{2R},$$
$$l(g_\tau)=\|g_\tau^{-1}U_1\|.$$

Here $C_E \le 1$ by Proposition 83.19, and $C_{r-2,h_\tau} \le \pi^{r-1}/(r-2)!$ by Theorem 83.16. The length $l(g_\tau)$ is bounded by the minimum coset norm $\nu_\tau(g_\tau)$ because B fixes the Euclidean unit vector U_1. Since $2R \le 4r^2$, the height estimate of Theorem 83.3 bounds the logarithm of the expectation by $O(fsR\log b)$. Finally $\log m_f(c_0) = O(fr\log x)$, by Theorem 89.4 and the conductor bound, and $\log T = O(s\log b)$ by Theorem 71.17. As $U_0 \ge 1$, the asserted upper bound follows by multiplication. It remains valid after taking the largest majorant among the finitely many words for that field and interval point. □

83.5 First-plane parameters over the whole real field

Fix $E, F = E^+, x$ and the word from Section 6.6. A good split place in this paragraph is a finite place of F split in E/F, away from $2qd_0QT$, at which the ordinary row and column levels are the full lattice stabilizers. Let $\mathcal{A}$ be the set of all nonzero $a \in \mathcal{O}_F$ such that (a) is squarefree and supported at these places. All generators are included, in particular every unit of F. Write $O(a)$ for the normalized ordinary orbital Gram operator at $\gamma_{Q\sqrt{-q}\,a}$. This parameter is skew under conjugation of E/F, as required in Proposition 89.15.

Theorem 83.5 (The sum over integral generators with squarefree split support)**.** *On the original power intervals one has*

$$\sum_{a\in\mathcal{A}} \|O(a)\|_{\mathrm{op}} \le \exp\bigl(-(16-o(1))(r-1)fx^{\varepsilon}\bigr).$$

This holds uniformly in the prescribed words and lowest-type unit vectors, with the original terminal frequency, including after projection to its terminal character and retained coordinates.

Proof. Put $R = r-1,\ h = s-1,\ B_x = 4(k+s+1)^2$, and let $\omega_F((a))$ count the distinct prime ideals dividing (a). At $v \mid (a)$, put $P = Nv$. In the first split component the parameter $Q\sqrt{-q}\,a$

has valuation one. The computation in Theorem 79.1 works over the local integer ring $\mathcal{O}_v$ by counting vectors, covectors and scalars over its residue field and lifting them modulo a single uniformizer. Proposition 77.11 then gives precisely a_P identity terms and u_P terms per neighbor in the notation of Proposition 78.10. Each phase has modulus at most one after spherical averaging. The Satake bound of Proposition 79.2 applies at this split place with residue size P. The proof of Proposition 78.10 therefore bounds this local operator by $B_x P^R$, also when the residue degree over the rational prime exceeds one. At all other finite places multiplication by a is by a unit. The unchanged valuation tests give at most the original unsigned mass there, independently of that unit. With $c_0 = Q\sqrt{-q}$ the finite bound is consequently

$$m_f(c_0)\, B_x^{\omega_F((a))}\, |N_{F/\mathbb{Q}} a|^R.$$

Combine this with the signed real factorization of Proposition 83.13 by the same unfolding as in Theorem 89.8. Put $t_\tau = |\tau a|$. The product of t_τ^{-R} cancels the finite norm power exactly. For $t_\tau < 1$ use the first real estimate and retain its logarithmic factor. For $t_\tau \ge 1$ use both real estimates. After removing t_τ^{-R}, their minimum is at most $a_0(a_1/a_0)^\theta t_\tau^{-\theta R}$ for any $0 \le \theta \le 1$. Since $a_1/a_0 \ge 1$, multiplying over the real places gives

$$\begin{aligned}\|O(a)\|_{\mathrm{op}} \le m_f(c_0) C_E\, a_0^{(1-\theta)f} a_1^{\theta f} B_x^{\omega_F((a))} H(a)^{-\theta R} \\ \cdot \prod_\tau (1 + 2\log^+(1/t_\tau))^h.\end{aligned}$$

Here $H(a) = \prod_\tau \max(1, t_\tau)$. Every prime ideal has norm at least two, so $2^{\omega_F((a))} \le |N_{F/\mathbb{Q}} a| \le H(a)$. The logarithm of the last product is at most $2hL_-(a) \le 2h \log H(a)$, by Lemma 83.7. Choose

$$\theta = \frac{\log B_x / \log 2 + 2h + 3}{R}.$$

This lies in $(0,1)$ for sufficiently large x. The remaining height factor is at most $H(a)^{-3}$, which Lemma 83.7 sums over the entire ring, including all choices of generator, at cost 8^f.

Finally $\theta s = O((s \log r + s^2)/R) = o(1)$ by the original rank inequalities. Also $C_E \le 1$ and $\log m_f(c_0) = O(fr \log x) = o(Rf \log b)$. The two bounds for $\log a_i$ in Proposition 83.13 now yield the claimed exponent. Orthogonal compression preserves every norm upper bound. Distinct parameters here label distinct rational terms, since their first minors are distinct as in Proposition 89.15. □

Theorem 89.13 gives the first-plane estimate at all finite supports and valuations.

83.6 Squarefree first-plane parameters on the original intervals

On the fields and levels of Section 6.6 put $\mathcal{O}_E^- = \{\xi \in \mathcal{O}_E : \bar{\xi} = -\xi\}$. For $0 \ne \xi \in \mathcal{O}_E^-$ there is a unique integral ideal $\mathfrak{b}(\xi)$ of $F = E^+$ such that $\xi\mathcal{O}_E = \mathfrak{b}(\xi)\mathcal{O}_E$. Indeed E/F is unramified at finite places by Theorem 72.9, and at a split place its two valuations on ξ agree. These valuations specify the exponents of the ideal in F. Let $\mathcal{S}$ consist of the nonzero skew integers for which $\mathfrak{b}(\xi)$ is squarefree and supported away from QT at places where the row and column levels are the full lattice stabilizers. Both split and inert primes are allowed, as are different generators and mixed prime supports. Write O_ξ for the normalized orbital Gram operator of the first-plane element $\gamma_{Q\xi}$.

Theorem 83.6 (The contribution of all good squarefree skew parameters)**.** *Uniformly on the original power intervals and prescribed words, with $R = r - 1$, $f = [F : \mathbb{Q}]$, one has*

$$\sum_{\xi \in \mathcal{S}} \|O_\xi\|_{\mathrm{op}} \le \exp\bigl(-(16 - o(1))Rfx^\varepsilon\bigr).$$

The same estimate holds on the original terminal-character projection and the retained coordinates.

Proof. At $v \mid \mathfrak{b}(\xi)$, write $P = Nv$. The signed finite operator has norm at most $B_x P^R$, with $B_x = 4(k+s+1)^2$. At an inert place this is Proposition 79.3. At a split place apply the full-level residue count of Theorem 79.1 and the spherical compression of Proposition 78.10. These use a uniformizer as the first component of $Q\xi$ and the actual residue cardinality P, independent of its residue characteristic or degree. The character enters with modulus at most one in the compression. At every other place the valuation is the minimal one of Theorem 77.10, including above Q and at the terminal graph. The relative column is the identity there, with unsigned mass one. The finite operator is thus bounded in norm by

$$B_x^{\omega(\mathfrak{b})} (N\mathfrak{b})^R, \qquad \mathfrak{b} = \mathfrak{b}(\xi).$$

Here ω counts prime ideal factors. The product decomposition follows by unfolding as in Theorem 89.8, since every $\gamma_{Q\xi}$ commutes with the column and has the same stabilizer J by Proposition 89.15. Its first minor $Q\xi$ distinguishes the rational terms.

Choose one extension to E of each real embedding τ of F, and put $t_\tau = |\tau\xi|$. The ordinary absolute ideal norm gives

$$N\mathfrak{b} = \prod_\tau t_\tau, \qquad H(\xi) = \prod_\tau \max(1, t_\tau).$$

Indeed $N_{E/\mathbb{Q}}(\xi) = \prod_\tau t_\tau^2 = (N\mathfrak{b})^2$. The real factor is C_E times the product of the signed local factors of Proposition 83.13 with real multipliers $\tau(\xi/\sqrt{-q})$. At a single place its two bounds therefore take the form

$$a_0' t_\tau^{-R} (1 + 2\log^+(1/t_\tau))^h, \qquad a_1' t_\tau^{-2R}, \qquad h = s - 1.$$

One may take $a_0' = a_0 q^{R/2}(1+\log q)^h$ and increase $a_1' = a_1 q^R$ to $\max(a_0', a_1 q^R)$ if necessary. This uses $1 + 2\log^+(\sqrt{q}/t) \le (1+\log q)(1+2\log^+(1/t))$. Since $\log q = O(\log x)$ by Theorem 12.15, these constants still satisfy $\log a_0' \le -(16 - o(1))R \log b$ and $\log a_1' \le CRs \log b$ for $b = \exp(x^\varepsilon)$.

The factor $\prod_\tau t_\tau^{-R}$ cancels the entire finite norm power. For $t_\tau < 1$ use the first local estimate. For $t_\tau \ge 1$ use the geometric mean of the two estimates with exponents $1-\theta, \theta$. The inequalities

$$2^{\omega(\mathfrak{b})} \le N\mathfrak{b} \le H(\xi), \qquad \sum_\tau \log^+(1/t_\tau) \le \log H(\xi)$$

follow from the norm identity and $N\mathfrak{b} \ge 1$. The logarithmic factors together are at most $H(\xi)^{2h}$. Taking $\theta = (\log B_x / \log 2 + 2h + 3)/R$ therefore gives

$$\|O_\xi\|_{\mathrm{op}} \le C_E (a_0')^{(1-\theta)f} (a_1')^{\theta f} H(\xi)^{-3}.$$

We sum over all skew integers directly. For our fixed complex embeddings their imaginary parts are f real coordinates. Two distinct skew integers in the same product of half-open unit intervals would have a nonzero skew difference with absolute norm strictly below one. Thus each such product contains at most one. In the box indexed by integers $j_\tau = \lfloor \log_2 \max(1, t_\tau) \rfloor$ there are at most $6^f 2^{\sum j_\tau}$ points, by the interval count in Lemma 83.7, and $H(\xi) \ge 2^{\sum j_\tau}$. Summing gives at most 8^f for the sum of $H(\xi)^{-3}$ over all nonzero skew integers. In particular no choice of generator for an ideal in F occurs in this bound. Finally $C_E \le 1$ and $\theta s = o(1)$ by the rank inequalities in Theorem 83.5. This proves the stated exponent. Orthogonal compression decreases the norm. □

Theorem 89.12 treats repeated prime factors and arbitrary valuations at the marked primes for the full skew family.

83.7 A sum over the actual algebraic integers

For a totally real field F of degree f and $a \in \mathcal{O}_F \setminus \{0\}$, put

$$H(a) = \prod_{\tau : F \hookrightarrow \mathbb{R}} \max(1, |\tau a|), \qquad L_-(a) = \sum_{\tau} \max(0, -\log|\tau a|).$$

The height distinguishes the different archimedean sizes of generators of the same ideal. These sizes occur in the real orbital integrals of Section 6.6.

Lemma 83.7 (A uniform height sum including all unit multiples)**.** *One has $L_-(a) \le \log H(a)$. For any $\delta > 1$,*

$$\sum_{a \in \mathcal{O}_F \setminus \{0\}} H(a)^{-\delta} \le \left(\frac{6}{1 - 2^{1-\delta}} \right)^f .$$

In particular the sum at $\delta = 3$ is at most 8^f.

Proof. The difference $\log H(a) - L_-(a)$ equals $\log |N_{F/\mathbb{Q}} a|$, which is nonnegative because the norm is a nonzero rational integer. For the sum, assign a to the tuple $t_\tau = \lfloor \log_2 \max(1, |\tau a|) \rfloor$. In this tuple's box one has $|\tau a| < 2^{t_\tau + 1}$. Cover the corresponding interval on each real line by at most $2^{t_\tau + 2} + 2 \le 6\, 2^{t_\tau}$ half-open unit intervals. Two distinct algebraic integers cannot belong to one product of such intervals. Their difference would have all its real absolute values strictly less than one, contradicting the norm inequality. Thus the box contains at most $6^f 2^{\sum_\tau t_\tau}$ algebraic integers, while $H(a) \ge 2^{\sum_\tau t_\tau}$. Sum the resulting f geometric series over the nonnegative integers t_τ to obtain the assertion. □

83.8 A spherical matrix coefficient at arbitrary distance

At a full-level place v of the prescribed automorphic column write K_H for the lattice stabilizer and $P = Nv$. For a translation by a write $M(a)$ for its compression to the K_H-fixed vectors. Translate on the same prescribed real type as in Proposition 79.2. Normalize K_H to have volume one.

Proposition 83.8 (Spherical compression for a pair of elementary divisors)**.** *For a split column position with exponents $(-d_-, 0, \dots, 0, d_+)$, $d_\pm \ge 0$, one has*

$$\|M(a)\| \le k^{d_- + d_+} P^{-(k/2-1)(d_- + d_+)}.$$

For an inert column with the same exponents and $d_- = d_+ = d$, one has $\|M(a)\| \le (k+1)^d P^{-(k-2)d}$.

Proof. In the split case elementary divisors give the displayed diagonal representative a. In the inert case the hyperspecial Cartan decomposition gives representatives in a hyperbolic basis, with diagonal exponents $(\nu_1, \dots, \nu_{k/2}, -\nu_{k/2}, \dots, -\nu_1)$. Their elementary divisors as matrices over E_w determine them up to signed permutations, so the pair in question is the d-th power of the representative in Proposition 79.3. Haines and Rostami, Cartan decomposition, Section 1.

Let $\phi_a = 1_{K_H a K_H}$, and let S be its normalized Satake transform in the upper triangular Borel. Dividing the eigenvalue of ϕ_a by its degree $\mathrm{vol}(K_H a K_H)$ gives the eigenvalue of $M(a)$, by averaging over both compact factors. The transform of a nonnegative function has nonnegative coefficients by its unipotent integral. It preserves convolution as multiplication of Laurent polynomials. Thus if $a = a_1 \cdots a_j$ is a product of diagonal elements, the coefficients of $S\phi_a$ are at most those of $\prod_i S\phi_{a_i}$. Indeed $\phi_{a_1} * \phi_{a_2} \ge \phi_{a_1 a_2}$, since at $a_1 a_2$ one may integrate over $a_1 K_H$, which has volume one. Bi-invariance and induction give the assertion for the product.

At a split place use d_+ index-P translations and d_- inverse translations. Their Satake polynomials are $P^{(k-1)/2}\sum_i X_i$ and $P^{(k-1)/2}\sum_i X_i^{-1}$. At the absolute values of the parameters of any prescribed constituent each is at most $kP^{k/2}$, by Proposition 79.2. Consequently the numerator of the normalized eigenvalue is bounded by $(kP^{k/2})^{d_-+d_+}$. At an inert place use d copies of the neighbor translation. Its polynomial in Proposition 79.3 has nonnegative coefficients and is at most $(k+1)P^k$ at the absolute values of the prescribed parameters. This gives $((k+1)P^k)^d$ for the corresponding numerator.

For the denominator let δ be the Borel modulus. At the displayed diagonal a, chosen with the negative exponent first, the coefficient of the corresponding monomial in $S\phi_a$ is at least $\delta(a)^{1/2}$ because the integral contains all integral upper unipotents. The coefficient at the opposite Weyl translate is the same. Evaluation on the trivial representation multiplies this last coefficient by $\delta(a)^{1/2}$, since the inducing character is $\delta^{-1/2}$ and the longest Weyl element inverts the modulus. All contributions there are nonnegative. The degree, which is the trivial eigenvalue, is therefore at least $\delta(a)$. In the split case the upper triangular entries give $\delta(a) = P^{(k-1)(d_-+d_+)}$. In the inert case $\delta(a) = P^{2(k-1)d}$ by the first-line calculation of Proposition 79.3. Dividing the respective numerator bounds by these powers proves the claim on each spherical constituent and hence on their orthogonal sum. □

83.9 The factorial in the real first-plane estimate

For $0 \neq \xi \in \mathcal{O}_E^-$ put $t_\tau = |\tau\xi|$ and $H(\xi) = \prod_\tau \max(1, t_\tau)$. Write $d = k+1$ and $h = s-1$. The first-plane estimate of Theorem 83.18 includes the volume of a complex ball of dimension d. Its factorial is needed when marked primes consume the full factor Q^{Rf}.

Lemma 83.9 (A uniform bound with the factorial retained)**.** *With $\mathfrak{m}$ and O_ξ as in Proposition 77.3 and Theorem 83.6, there is a constant C independent of the interval point, word and ξ such that*

$$\|O_\xi\|_{\mathrm{op}} \le (N\mathfrak{m})^{R-1} H(\xi)^{-3} Q^{-Rf} (d!)^{-f} \exp(CRf).$$

Proof. At the original frequency the first estimate in Theorem 83.18, applied locally as in Proposition 83.13, bounds the signed factor by

$$Q^{-R} L\, t_\tau^{-R} (1 + 2\log^+(1/t_\tau))^h, \quad L = 2^R A_\sharp \exp(Cs(4s^2+3)\log b)(1 + 2\log^+(2T/Q))^h.$$

This follows by separating the positive logarithm of $2T/(Qt_\tau)$ into its constant part and its t_τ part as in that proposition. The explicit constants in Proposition 83.11 give

$$A_\sharp = (2s^2+1)(2+3(s+1))\, 2^{1+2h} (4\sqrt{2})^d \pi^{r-2}/d!.$$

The rank formulas in Theorem 6.29 give $s^3 \log b = o(k)$, and Theorem 71.17 gives $\log(1 + 2\log^+(2T/Q)) = O(\log x)$. Consequently $\log L \le -\log(d!) + CR$, and we may use $L = \exp(CR)/d!$ as a common larger constant.

Lemma 85.11 gives a second bound $Q^{-R} U t_\tau^{-3R}$ with $U = \exp(C' Rs \log b)$. Here the proof of that lemma supplies Q^{-3R}, so Q^{-R} can be taken outside both bounds. Increase C' if necessary to ensure $U \ge L$. The product of the signed real factors has just the additional field constant $C_E \le 1$ by Proposition 83.13. Remove t_τ^{-R} from each factor, cancelling $(N\mathfrak{g}\, N\mathfrak{m})^R$ in Proposition 77.3 by the norm identity of Theorem 83.6. Leave $(N\mathfrak{m})^{R-1}$ aside. The remaining finite factor is at most $H(\xi)^{2s+1+\log C_k/\log 2}$. All logarithmic factors cost at most $H(\xi)^{2h}$, since $\prod t_\tau \ge 1$ bounds the sum of the negative logarithms by $\log H(\xi)$.

For $t_\tau \ge 1$ interpolate the two real estimates with exponent θ on the second. At $t_\tau < 1$ use the first. Take $2R\theta = 2s + 1 + \log C_k/\log 2 + 2h + 3$. The extra power $H(\xi)^{-2R\theta}$ then leaves

exactly $H(\xi)^{-3}$ in the majorant. The remaining per-place constant after taking out Q^{-R} is at most $L^{1-\theta}U^{\theta}$. Its logarithm is at most

$$-\log(d!) + CR + \theta\{\log(d!) + C'Rs\log b\}.$$

Indeed $0 < \theta < 1$ on the specified large intervals. By the ranks in Theorem 6.29, $\log k = O(\log x) = o(s)$, $s^3 \log b = o(k)$, and $\log(d!) \le d\log d$. Thus $\theta\log(d!) = O((s+\log k)\log k) = o(R)$ and $\theta Rs\log b = O((s+\log k)s\log b) = o(R)$. Multiplication over the f places proves the estimate. □

83.10 A weighted bound for the real orbital integral

Use $D = 2r-2$ and the measures of Proposition 83.19. For a nonzero real Bessel section of the original word, put

$$M_D(W) = \frac{\int_{B_\infty\backslash G_\infty} \|W(g)\|^2 \prod_\tau R(g_\tau)^D \, d\dot{g}}{\int_{B_\infty\backslash G_\infty} \|W(g)\|^2 \, d\dot{g}}.$$

All norms R use the same unit first vector as in Theorem 83.16.

Theorem 83.10 (A common moment for the first-plane real integrals)**.** *For every rational skew parameter $c \neq 0$ of Proposition 89.15, the real factor of Theorem 83.15 satisfies*

$$|\mathcal{I}_\infty(\gamma_c; W)| \le C_E \, M_D(W) \prod_\tau \left(\frac{2}{|\tau c|}\right)^D C_{n,h_\tau}.$$

The right side also bounds the integral of the product of the two real norms divided by the same diagonal normalization. The moment is finite for the original lowest-type sections and is independent of the rational parameter c.

Proof. The change of opposite vector made in Theorem 83.16 preserves the orbital integral and the integral of norms. Indeed left multiplication by the corresponding central unipotent normalizes J_∞ and preserves its Haar measure, so it preserves the quotient measure. It acts on W on the left by the trivial character. We may therefore use the representatives γ' of that theorem simultaneously at all real places.

Write $R_1(g) = \prod_\tau R(g_\tau)$ and $R_2(g) = R_1((\gamma')^{-1}g)$, using the product element γ'. They are positive and invariant on the left under J_∞. Apply Cauchy–Schwarz to the product of norms on $J_\infty\backslash G_\infty$ with weight $(R_1/R_2)^D$ in the first squared integral and its inverse in the second. In the first integral disintegrate over $B_\infty\backslash G_\infty$. The fiber $J_\infty\backslash B_\infty$ is U_∞ in coordinates Jn with the measure of Proposition 83.19. The norm $\|W\|$ and R_1 are constant along that fiber. The integral of R_2^{-D} is exactly the product in the claimed bound, times C_E, by Theorem 83.16. In the second squared integral left translate g by γ'. This preserves quotient measure because γ' centralizes J_∞. The resulting first fiber integral has parameter $-c$ and hence has exactly the same value. The geometric mean of these two equal expressions proves the bound.

For the sections in question the calculation is integrable. For every $b' \in B(F_\tau)$, $R(g_\tau) \le \|(b'g_\tau)^{-1}\|_{\mathrm{op}} = \|b'g_\tau\|_{\mathrm{op}}$, since b' fixes u' and inverse and original operator norms agree in signed coordinates. Taking the infimum bounds R_1 by the height V of Proposition 64.20. That proposition gives boundedness of W and integrability of $\|W\|V^L$, where $L = 8(f+1)(r+1)^3 > D$. Thus the positive moment in the statement is finite. All disintegrations with nonnegative integrands also follow directly from Tonelli's theorem. □

83.11 The weighted volume of the first-plane fiber

Set $h = s-1,\ d = k+1,\ n = d+2h$, and use $\delta = |C|$ and $y = \gamma(C)^{-1}n(w,z)g$. Here g is the representative from Lemma 83.21. Let

$$K = 2s^2, \qquad B_* = (2+3(s+1))\,2^{\,1+2h}(4\sqrt{2})^d\,\pi^n/d!.$$

Proposition 83.11 (Weighted sections of the unipotent fiber)**.** *For every $R \geq 1$, with $H = R\nu(g)$, one has*

$$\int_{\nu(y)\leq R}\left(\frac{\mathcal{A}(g)}{\mathcal{A}(y)}\right)^d dw\,dt \leq B_*\,Z(g)^2H^K\,\delta^{-(n+1)}(1+2\log^+(1/\delta))^h.$$

Proof. The exterior product defining $\mathcal{A}$ has unit norm before translation. On any coset attaining the minimum this gives $\nu^{-s-1} \leq \mathcal{A} \leq \nu^{s+1}$, using the equal norms of a unitary matrix and its inverse. The ratio $\theta = \mathcal{A}(y)/\mathcal{A}(g)$ therefore belongs to $[H^{-s-1}, H^{s+1}]$. Divide this into its intersections with $(2^{j-1}, 2^j]$, $j \in \mathbb{Z}$. There are at most $2+2(s+1)\log H/\log 2 \leq (2+3(s+1))H$ nonempty pieces.

On such a piece Lemma 83.21 bounds the squared radius in the coordinates $x-\eta v_+/a, \eta/a$ by $2\sqrt{2}\,2^j/\delta$. The inverse change of coordinates has real Jacobian a^2. Multiplying the ball volume by the upper bound $2^{(1-j)d}$ on θ^{-d} gives

$$a^2\,\frac{\pi^d}{d!}(4\sqrt{2}/\delta)^d.$$

This is independent of j. On the same set $n^{-1}\gamma(C)b$ has norm at most H for some $b \in B$, by the argument of Theorem 83.17. For the h complex pairs the bound of Theorem 83.20 gives $\pi^{2h}(2H^{s+1}/\delta)^{2h}(1+2\log^+(1/\delta))^h$. The real t-coordinate ranges over an interval of length at most $2H/\delta$. All these containing sets may be multiplied for an upper bound even though the original conditions couple the variables. Multiply by the number of pieces. The total exponent of H is $1+2h(s+1)+1 = 2s^2$. Finally $a^2 \leq Z(g)^2$. □

83.12 The first flag coordinate of a tensor projection

At one real place use the Euclidean orthonormal basis, in order,

$$U_1,\dots,U_s,\quad w_1,\dots,w_k,e,\quad V_s,\dots,V_1,\qquad U_j = u_j/\sqrt{2},\quad V_j = \sqrt{2}\,v_j.$$

Here the w_a form an orthonormal basis of the positive column and u_j, v_j are the Witt vectors of Section 99.16. The row has rank $r = k+2s+1$. Write S_μ for the smooth discrete series with decreasing integral Harish–Chandra parameter μ and the signs of (65.4). Write F^* for the dual of its group's defining representation. In this paragraph adjacent gaps of μ exceed two. Let P_i be the projection in $F^* \otimes S_\mu$ onto the summand with parameter $\mu - e_i$, where e_i in a parameter denotes a coordinate unit vector. The coherent character and splitting argument of Lemma 64.19 applies with the dual defining representation, whose weights are the $-e_i$. The canonical smooth direct sum follows from the Casselman–Wallach globalization theorem. Thus the projections intertwine the group actions on smooth vectors. See also Bernstein and Krötz, *Smooth Fréchet Globalizations of Harish–Chandra Modules*.

Put $\omega = h(U_1,\cdot)$, the last dual basis vector, and write ev for evaluation at U_1 in the dual factor. Both are fixed by B. For the unit-frequency character of Section 99.16, let $\Lambda : S_\mu \longrightarrow E_u$ be a continuous Bessel map to the compact column of parameter $u = (u_a)_{a=1}^k$.

Proposition 83.12 (A quantum minor on the prescribed Bessel model)**.** *There is a nonzero constant η, common to all i, μ, u at these ranks, with $|\eta| = (2\pi)^{2s}/2$, such that*

$$\Lambda \operatorname{ev} P_i(\omega \otimes v) = \eta \, \frac{\prod_{a=1}^{k}(u_a + 1/2 - \mu_i)}{\prod_{j \neq i}(\mu_j - \mu_i)} \, \Lambda(v) \qquad (v \in S_\mu).$$

Proof. Denote the complexified Lie algebra matrix units and their actions by E_{ij}. For two ordered lists I, J of length t, set $T(z)_{ij} = z\delta_{ij} + E_{ij}$ and define

$$M_{I,J}(z) = \sum_{\pi \in \mathfrak{S}_t} \operatorname{sgn}(\pi) \prod_{a=1}^{t} T(z - a + 1)_{i_{\pi(a)}, j_a}.$$

Factors act by composition in their printed order. We shall also use the expansion that permutes the columns and has factors in row order with arguments $z-t+1, \dots, z$. Here is a verification. In t copies of the defining space let P^0_{ab} interchange copies a, b, and let A_t be the signed permutation sum divided by $t!$. Use $T_a(z)$ for the matrix $T(z)$ in copy a. The matrix-unit commutator gives

$$R_{ab}(u - v) T_a(u) T_b(v) = T_b(v) T_a(u) R_{ab}(u - v), \qquad R_{ab}(w) = 1 - P^0_{ab}/w.$$

Indeed its entries are $(u - v)[T_{ij}(u), T_{kl}(v)] = T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)$. In lexicographic order of the pairs $a < b$, the product of the $R_{ab}(b - a)$ equals $t! A_t$. To see this, move the factors involving t to the right without changing their relative order, commuting only disjoint transpositions. After the induction hypothesis the remaining calculation is

$$A_{t-1} \prod_{i<t} \left(1 - \frac{(i\, t)}{t - i}\right) = A_{t-1} \left(1 - \sum_{i<t} (i\, t)\right) = t A_t.$$

In the expansion, a nonempty set of transpositions with least index i lies in the left coset of $(i\, t)$ under $\mathfrak{S}_{t-1}$, as its inverse sends t to i. Its signed coefficient after antisymmetrizing contributes to that coset a total of $-\prod_{i<j<t}(1 + 1/(t-j))/(t-i) = -1$. The last equality in the display is the signed coset decomposition. The exchange relation now reverses all t factors, moving the antisymmetrizer from the left to the right. Taking entries gives alternation in both lists. On the right the factors occur in decreasing slot order. Reversing both index lists, which leaves their combined sign unchanged, gives the asserted row expansion.

For the full ordered list $J_0 = (1, \dots, r)$, put $D_r(z) = M_{J_0, J_0}(z)$. It is central. In fact commuting E_{ab} past the factors replaces each occurrence of row b by row a, and subtracts the replacements of column a by column b, by the matrix-unit relations. Repetitions vanish, and for $a = b$ the terms cancel. On a highest weight vector of weight λ its value is $\prod_j (z - j + 1 + \lambda_j)$. In a nonidentity permutation the last moved column has a strictly smaller row index and its upper matrix unit kills the highest weight vector after the diagonal factors to its right. The Harish–Chandra isomorphism with shift $(r + 1 - 2j)/2$ thus gives

$$D_r(Y + (r-1)/2) = \prod_j (Y + \mu_j)$$

on the entire discrete series. The same calculation applies to the compact k-column with parameter u.

The split Casimir C on $F^* \otimes S_\mu$ sends a coordinate vector $x = (x_j)$ to $(-\sum_j E_{ij} x_j)_i$. This follows from $E_{ab}\zeta = -\zeta \circ E_{ab}$ on the dual. Its eigenvalue on summand j is $-\mu_j - (r-1)/2$. Indeed twice C is the difference between the tensor Casimir and the two separate Casimirs, for

the trace form. The row eigenvalue is $\sum \mu_i^2 - \sum((r+1-2i)/2)^2$, lowering coordinate j changes it by $1-2\mu_j$, and the dual defining eigenvalue is r.

Let $A(z)_{ij} = (-1)^{i+j} M_{J_0\setminus j, J_0\setminus i}(z)$, with the inherited order. Expanding the last column gives $A(z)T(z-r+1) = D_r(z)I$. For an off-diagonal entry the appended column repeats one already present. At $z = Y + (r-1)/2$, the matrix on the left multiplying vectors first is scalar $Y + \mu_j$ on summand j. Consequently $A(z)$ there equals $\prod_{l\neq j}(Y+\mu_l)$. This is a polynomial identity, so substitution $Y = -\mu_i$ gives $A(z) = \prod_{l\neq i}(\mu_l - \mu_i)P_i$.

It remains to compute the $(1,r)$ entry after applying Λ. In the complexified unipotent radical all upper matrix units outside the middle square act through scalars under Λ. The only nonzero scalars belong to the successive arrows in the ordered list $U_1, \dots, U_s, e, V_s, \dots, V_1$, where an arrow from column b to row a means E_{ab} for two consecutive entries a, b of this list. For the two arrows touching e the absolute value is $2\pi/\sqrt{2}$, and for every other arrow it is 2π. Indeed the real differential is $4\pi i \operatorname{Re} d$, with $d(X) = \sum_{j<s} X_{U_j,U_{j+1}} + X_{U_s,e}/\sqrt{2}$. Its complex-linear extension is $2\pi i(d(X) + \overline{d(-JX^*J)})$, where J interchanges the two Witt strings, fixes the positive column and negates e. This also proves vanishing for every other upper entry in the radical. The column matrix units act through the differentiated compact column representation by equivariance.

The minor for the $(1,r)$ entry omits row V_1 and column U_1. Remove the rows $U_1, \dots, U_s$ in this order, using the row expansion. At each step the row index is absent from the remaining columns, and every column to be tested lies strictly to its right. Thus Λ on the leftmost factor kills all terms except the column at the next arrow. Deleting this row and that column leaves the parameter z unchanged in the row expansion. Now take the columns $V_1, \dots, V_s$ in that order, moving the chosen column to the front by alternation. Its index is absent from the remaining rows, which lie strictly to its left. In the column expansion only the row at the adjoining arrow survives. Each deletion changes z to $z-1$. The remaining square is precisely the positive k-column at $z - s = Y + k/2$, of value $\prod_a (Y + 1/2 + u_a)$ under Λ. All permutation signs and arrow scalars, including the cofactor sign, form the same constant η for each i. Their product has the asserted absolute value. Evaluation at U_1 selects row one, and insertion of ω selects column r. This proves the formula. □

The quantum minor used here also appears in `Development/AbelianVarietyTheory/Proofs/JetHarmonics/ProjectorEntryValueUnderLinearFunctional.lean`, line 24.

83.13 Separate real estimates for a field-valued multiplier

Let $c_0 = Q\sqrt{-q}$, $U_0 = Q\sqrt{q}$, $R = r-1 = k+2s$ and $h = s-1$. For each real place τ choose any Haar density on $B_\tau\backslash G_\tau$. On $J_\tau\backslash G_\tau$ use the compatible density with ordinary $dw\,dt$ on the first unipotent fiber as in Proposition 83.19. For a real $z \neq 0$ write $I_\tau(z)$ for the signed real orbital integral at parameter $(\tau c_0)z$ and the original frequency ϖ_τ, divided by its diagonal norm. The quotient of these densities in particular specifies the normalization of this number.

Proposition 83.13 (The local factors for independent real multipliers)**.** *On the lowest compact type the signed normalized real factor for a tuple (z_τ) is $C_E \prod_\tau I_\tau(z_\tau)$, for every nonzero row vector. There are positive $a_0 \le a_1$, the same for every place and word, such that for $t = |z| > 0$,*

$$|I_\tau(z)| \le a_0 t^{-R}(1 + 2\log^+(1/t))^h,$$
$$|I_\tau(z)| \le a_1 t^{-2R},$$
$$\log a_0 \le -(16 - o(1))R\log b, \qquad \log a_1 \le CRs\log b.$$

Here $C_E \le 1$ is the field constant of Proposition 83.19. Neither the signs nor the relative sizes of the different z_τ are restricted.

Proof. Polarize the numerator in the two input vectors of a fixed real word, giving a sesquilinear form on its finite-dimensional lowest compact type. The integrals for two different vectors converge absolutely by the weighted Cauchy–Schwarz proof of Theorem 83.18, since each squared integral uses the height moment of its own input. The denominator similarly gives a positive definite form by Corollary 84.21. The numerator and denominator are invariant under simultaneous right translation by the maximal compact group. Indeed one changes the quotient variable by right multiplication. Schur's lemma now makes the numerator a scalar multiple of the denominator on the irreducible compact type, even if the scalar is not real.

The argument also applies to the product compact group and the irreducible external tensor product of its lowest types. One may therefore evaluate the global scalar on any nonzero pure tensor. On this vector the integrands factor, the signed numerator being absolutely integrable. The common scalar in the quotient of the two product measures comes from the unipotent fiber $J_\infty\backslash B_\infty$. Its density is $C_E\prod_\tau dw_\tau\,dt_\tau$ by Proposition 83.19. This proves the product formula, independently of the local choices of base density.

Work now at a single place. The local estimates in the proof of Theorem 83.3, before the forms at different places are tensored, give $\mathbb{E}_{1,\tau}\nu_\tau^A \le \exp(CsA\log b)$ for $1 \le A \le 4r^2$. With $D = 4s^2+3$, use the local positive integral bound of Theorem 83.18 and the exact frequency change of Proposition 83.14. The result is at most

$$A_\sharp \exp(CsD\log b)(2/U_0)^R t^{-R}(1+2\log^+(2T/(U_0t)))^h.$$

The last parenthesis is at most $(1+2\log^+(2T/U_0))(1+2\log^+(1/t))$. Use the factor independent of t to define the first constant. Its logarithm is at most $-(16-o(1))R\log b$ since $\log Q = (16+o(1))\log b$, $\log A_\sharp = O(r+\log(2s))$, $\log(1+2\log^+(2T/U_0)) = O(\log x)$ and $s^3 = o(R)$.

For the second estimate use the same frequency substitution and the local calculation of Theorem 83.16 with power $2R$. As in Lemma 83.4, this gives the constant

$$C_{r-2,h_\tau}\,T^R(2/U_0)^{2R}\,\mathbb{E}_{1,\tau}l(g_\tau)^{2R}$$

times t^{-2R}. Use $l \le \nu_\tau$, $C_{r-2,h_\tau} \le \pi^R/(r-2)!$, and $\log T = O(s\log b)$ to bound its logarithm by $CRs\log b$. Uniform constants may be taken in both calculations. Increasing the second constant to $\exp(CRs\log b)$ ensures $a_1 \ge a_0$ on the specified sufficiently distant intervals. □

83.14 The first-plane integral under a change of terminal frequency

Keep the original finite level and all its characters. Here the change of variables takes place at infinity. For unit terminal frequency and a tuple $c = (c_\tau)$ of nonzero purely imaginary parameters, use $\mathcal{I}(c;W_1)$ for the quotient of the two integrals in Theorem 83.15, using $\gamma(C_\tau)$ of Theorem 83.20 with $iC_\tau = c_\tau/2$. This definition also applies to the terminal-frequency section W_ϖ. Write $\mathcal{I}^+$ for the same quotient with the product of norms in the numerator. The finite factors do not occur in either definition.

Proposition 83.14 (Exact frequency change for an individual real orbit)**.** *Use corresponding sections W_ϖ, W_1 from Proposition 84.13, and write $f = [F:\mathbb{Q}]$. For $n = r-2$, the real factor of the original rational orbit is*

$$\mathcal{I}_\infty(\gamma_c;W_\varpi) = T^{-(n+1)f}\,\mathcal{I}((\tau c/T)_\tau;W_1).$$

The identical scaling holds for $\mathcal{I}^+$.

Proof. The central unipotent change of opposite vector in Theorem 83.16 commutes with J and acts on each section by its trivial character. Apply this change in the numerator to pass

from the rational vector to the opposite vector of Euclidean length $1/\sqrt{2}$. At a place with terminal parameter $\beta = \varpi_\tau$, let a multiply each unit first flag vector of a Witt pair by β and its partner by $\bar{\beta}^{-1}$. It fixes the column and e. Conjugation by a sends $\gamma(C_\tau)$ to $\gamma(C_\tau/|\beta|^2)$. It normalizes B and its subgroup J on the middle space. Let j_B, j_J be its absolute Haar moduli there.

For either subgroup $L = B, J$, substitution on its left quotient gives $\int_{L\backslash G} F(ag)\, d\dot{g} = j_L \int_{L\backslash G} F(g)\, d\dot{g}$. For instance take $F(g) = \int_L F_0(lg)\, dl$, change l to ala^{-1} inside the integral, and use left invariance of Haar on G. Local quotient sections give the formula for all integrable functions, and then for nonnegative functions by monotone convergence. Since $B = J \ltimes U$, j_B/j_J is the modulus on U. Write a_M for the middle block of a. On U the coordinates transform as

$$w \longmapsto \bar{\beta}\, a_M w, \qquad t \longmapsto |\beta|^2 t.$$

Here $|\det_{\mathbb{C}} a_M| = 1$. Thus $j_B/j_J = |\beta|^{2n+2}$. The scalar $j_B^{-1/2}$ on each section in the numerator contributes j_B^{-1}, whereas the substitution on $J\backslash G$ contributes j_J. The denominator is preserved by the isometry on $B\backslash G$. Multiply the ratios over the real places and use $|\varpi_\tau|^2 = T$. The same argument works before the real vectors are contracted. □

83.15 The real integral for one summand of the projected Gram form

Fix one original word ρ, its unit lowest-type row vector and the corresponding V_ρ-valued Bessel section W of Theorem 64.22. Use the right-invariant seed of equation (70.16) at K^0. Its column inner product is normalized so that the diagonal orbital term is the identity. Put $\gamma = \gamma_{Q\sqrt{-q}}$ and let J be its stabilizer from Proposition 89.15.

Choose Haar measures as follows. In U, the radical of the first-line stabilizer, give every U_L of Theorem 77.10 finite local volume one. Normalize the remaining real Haar factor so that $[U]$ has volume one. Use this product measure with that on J to measure $B = J \ltimes U$, and take compatible quotient measures everywhere. Both $[J]$ and $[B]$ can then be given volume one: their volume ratio is the volume of $[U]$, by integration first on the normal unipotent group. Write $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$ and use the analogous notation for its subgroups. With inner product linear in the first entry, set

$$\mathcal{I}_\infty(\gamma; W) = \frac{\displaystyle\int_{J_\infty\backslash G_\infty} \langle W(t), W(\gamma^{-1}t)\rangle\, d\dot{t}}{\displaystyle\int_{B_\infty\backslash G_\infty} \|W(t)\|^2\, d\dot{t}}.$$

The numerator integrand is well defined since γ commutes with J and the two J_∞-actions on the sections are the same and unitary.

Theorem 83.15 (One rational term in the projected cuspidal Gram form)**.** *The numerator is absolutely convergent and the denominator is positive. The mixed orbital Gram term at γ for ordinary finite columns ℓ, ℓ' is*

$$\mathcal{I}_\infty(\gamma; W)\, \langle \ell, F_q^{\mathrm{col}} \ell' \rangle.$$

In particular its contribution to the normalized signed trace on any nonzero terminal space of Theorem 66.1, including the retained-coordinate version, is $\operatorname{Re}\mathcal{I}_\infty(\gamma; W)$ *times a real number between* $\prod_{v|q} L_{b_v}$ *and* $\prod_{v|q} U_{b_v}$.

Proof. In the orbital integral unfolded in Theorem 64.22 integrate first on $[J] = J(F)\backslash J(\mathbb{A}_F)$ and then on $J(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. The absolute majorant in that theorem applies to the norms of the

sections before contraction by any column covector. For the fixed rational orbit this majorant has finite integral. It is constant in absolute value under the common left J-action, and has a finite right coset of positive measure on which both finite supports occur, for example the identity coset. The integral of the product of the two real norms on $J_\infty\backslash G_\infty$ is therefore finite. The denominator is the squared norm of the nonzero real section. These facts justify all the following disintegrations.

The unipotent radical of J has the same Bessel character on the two sides. Its factors cancel in the product. The integral on $[J]$ therefore reduces to the common column argument on $[H]$, with its Haar probability. On each finite cell write $t_f = n_f$, modulo the left J-action and the right level, as in Theorem 77.10. If the second column is represented by h', the finite covectors in the integrand are evaluated at h_f and $h_f h'$. The local phases are one by Theorem 89.5. On the real factors both vectors are translated by the same $\rho(h_\infty)$, since the relative element commutes with H. Integrate also over right translations of h_∞ by the compact real group H_∞, with probability measure. This preserves the integral over $[H]$. Schur orthogonality on the irreducible word contracts the real vectors to $\langle W(t), W(\gamma^{-1}t)\rangle/\dim\rho$; the remaining pairing is that of the two column covectors at h_f and $h_f h'$.

For the identity orbital term the same calculation has $\|W(t)\|^2/\dim\rho$ on $B_\infty\backslash G_\infty$ and the identity on the finite columns. The choices of measure give the same probability on the common column argument in both calculations. Dividing by the diagonal normalization cancels the dimension and leaves the displayed ratio of real integrals. Each ratio of finite stabilizer measures is exactly the one counted in Theorem 77.10. Summing the translated column pairings accordingly gives F_q^{col} by Theorem 89.5. Finally average over an orthonormal basis in the chosen projected space. Theorem 79.6 bounds each diagonal matrix element between the two stated products. □

Thus this orbit has an explicitly positive finite multiplier in the projected trace. Its sign in that trace is determined by the real integral. Controlling the full trace requires estimates for the real integrals and the sum over rational orbits, with the arithmetic normalization of Section 6.6.

83.16 Integrating over the first isotropic line at infinity

At one real place put $R(g) = \|g^{-1}u'\|$, in the frame of Proposition 83.19. The Bessel group fixes u', so this function descends to its left quotient. Let $n = p_0 + q_0 = r - 2$, with p_0, q_0 the signatures on M, and write $h(w, w) = |w_+|^2 - |w_-|^2$. For $D = 2n + 2$ set

$$C_{n,h} = \int_{\mathbb{C}^n\times\mathbb{R}} (1 + |w|^2 + t^2 + h(w, w)^2/4)^{-D/2}\, dw\, dt.$$

Let γ' be the first-plane transvection formed with the opposite vector $v'/\sqrt{2}$ and parameter $c \in i\mathbb{R} \setminus \{0\}$. It is conjugate to the original γ_c by an element of the center of U at this place. That center has trivial Bessel character since $s \geq 2$, and it commutes with J.

Theorem 83.16 (The real integral on a first-line fiber)**.** *With the unscaled coordinate measure of the preceding display, for every $g \in G(F_\tau)$ one has*

$$\int_{\mathbb{C}^n\times\mathbb{R}} R\big((\gamma')^{-1}n(w, z)g\big)^{-D}\, dw\, dt = (|c|/2)^{-D}C_{n,h}.$$

The same identity holds if c is replaced by $-c$, and

$$2\,4^{-D/2}\frac{\pi^n}{n!} \leq C_{n,h} \leq \frac{\pi^{n+1}}{n!}.$$

Proof. Decompose $g = n_0 a m k_0$, where k_0 preserves the signed Euclidean metric, $m \in U(M)$, and a multiplies u', v' by $\lambda, \bar{\lambda}^{-1}$. To see the decomposition, the maximal compact group acts transitively on isotropic lines: the positive and negative parts of a unit isotropic vector have equal lengths and can be rotated independently. After removing a right compact element, g thus belongs to the parabolic stabilizing the first line. Its diagonal blocks have the indicated Levi form, and its remaining factor is the displayed radical. Right translation of the integration variable absorbs n_0 by invariance of the unipotent measure. Now put $R = |\lambda|^{-1}$.

The first vector under γ' goes to $u' + (c/2)v'$. Applying $n(w,z)^{-1}$ gives $(1 + (c/2)\bar{z})u' - (c/2)w + (c/2)v'$. Write $c/2 = iC$ with real $C \neq 0$. Remove k_0 by Euclidean norm invariance and make the changes of variables $w \mapsto m^{-1}w$, $t \mapsto t + 1/C$. They have Jacobian one, since a unitary matrix has complex determinant of modulus one. The squared norm after applying g^{-1}, in the resulting variables, is

$$C^2\{R^{-2} + |w|^2 + R^2(t^2 + h(w,w)^2/4)\}.$$

Substitute $w = w_1/R$, $t = t_1/R^2$. The Jacobian is R^{-2n-2}, and taking the power $-D/2$ supplies the factor R^D. These powers cancel at $D = 2n + 2$.

On $|w| \le 1$, $|t| \le 1$ the denominator before taking the power is at most four. This gives the lower bound. For the upper bound omit the nonnegative term $h(w,w)^2/4$. The integral of $(1 + |y|^2)^{-n-1}$ in $2n + 1$ real variables is $\pi^{n+1}/n!$: insert the gamma integral for the power, integrate the Gaussian in y, and use $\Gamma(1/2) = \sqrt{\pi}$. This proves finiteness and the stated bound. □

83.17 A full-flag estimate at small real parameter

Write $\nu_\tau(g) = \inf_{b \in B_\tau} \|bg\|_{\mathrm{op}}$ and $V(g) = \prod_\tau \nu_\tau(g_\tau)$, as in Proposition 64.20. For a nonzero section put

$$\mathcal{M}_a(W) = \frac{\int_{B_\infty \backslash G_\infty} \|W(g)\|^2 V(g)^a \, d\dot{g}}{\int_{B_\infty \backslash G_\infty} \|W(g)\|^2 \, d\dot{g}}.$$

In the notation of Theorem 83.20 let $A_* = (K_* + 1)2^{n+1}\pi^n/d!$, and keep the real Haar constant C_E of Proposition 83.19.

Theorem 83.17 (A bound uniform down to zero real parameter)**.** *Let $iC_\tau = c_\tau/2$, $\delta_\tau = |C_\tau| > 0$. For the lowest-type sections of the prescribed word,*

$$|\mathcal{I}(c;W)| \;\le\; \mathcal{I}^+(c;W) \;\le\; C_E A_*^f \, \mathcal{M}_{2K_*+1}(W) \prod_\tau \delta_\tau^{-(n+1)} (1 + 2\log^+(1/\delta_\tau))^{s-1}.$$

The moment is finite, and it is common to all c.

Proof. Start at one real place, suppressing τ. The matrix norm is at least one on G and is unchanged by inversion. The minimum in ν is attained. Indeed a bounded sequence in Bg has a convergent subsequence in the ambient matrix space. The relation defining the unitary group survives the limit and implies invertibility, and the subgroup B is closed. In a fiber $J\backslash B = U$ we may therefore use a representative with $\|g\| = \nu(g)$. Changing this representative only translates U or conjugates it by J, preserving measure. The latter group commutes with $\gamma(C)$.

For $y = \gamma(C)^{-1} n(w,z) g$ with $\nu(y) \le R$, choose $b_0 \in B$ with $\|b_0 y\| \le R$. Then $n(w,z)^{-1}\gamma(C) b_0^{-1} = g(b_0 y)^{-1}$ has norm at most $R\nu(g)$. Apply Theorem 83.20 with exactly this bound. For $p = K_* + 1$, integrate the resulting volume estimate against $pR^{-p-1}\,dR$ over $[1, \infty)$. Because $\nu(y) \ge 1$, Tonelli gives

$$\int_U \nu(y)^{-p} \, dw \, dt \le A_* \, \nu(g)^{K_*} \delta^{-(n+1)} (1 + 2\log^+(1/\delta))^{s-1}.$$

Here $p\int_1^\infty R^{K_*-p-1}\,dR = p$. The same estimate holds with $-C$.

Return to all places and put $V_1 = V(g)$, $V_2 = V(\gamma^{-1}g)$. In the integral of the product of norms on $J_\infty\backslash G_\infty$ apply Cauchy–Schwarz with weight $(V_1/V_2)^p$ in the first squared integral and its reciprocal in the second. In the first, integrate on $J_\infty\backslash B_\infty$ with product measure $C_E\prod_\tau dw_\tau\,dt_\tau$. The first section norm and V_1 descend to the base. The fiber estimates bound the remaining base weight by the stated product times $V(g)^{p+K_*}$. In the other squared integral translate by γ; this centralizes J_∞ and switches C to $-C$. The two upper bounds coincide. Divide by the diagonal squared norm to obtain the result. Finally $2K_*+1 \le 8(f+1)(r+1)^3$. Proposition 64.20 bounds the section and proves integrability of its norm times this power of V. □

83.18 A height moment independent of the column dimension

Use the positive integral $\mathcal{I}^+$ from Proposition 83.14, first at one common real frequency, with $\delta_\tau = |c_\tau|/2$. Put $D = 4s^2+3$, $p = 2s^2+1$, $A_\sharp = pB_*$, where B_* is from Proposition 83.11. For the product height V, write $\mathcal{M}_D(W)$ as in Theorem 83.17.

Theorem 83.18 (An orbital bound of order $4s^2+3$ in the height)**.** *For each nonzero lowest-type section one has*

$$\mathcal{I}^+(c;W) \le C_E A_\sharp^f \mathcal{M}_D(W)\prod_\tau \delta_\tau^{-(r-1)}(1+2\log^+(1/\delta_\tau))^{s-1}.$$

At terminal frequency ϖ, use W_1 in the moment and replace the product by $\prod_\tau(2/|\tau c|)^{r-1}(1+2\log^+(2T/|\tau c|))^{s-1}$ for the original orbit γ_c.

Proof. At one place integrate the estimate of Proposition 83.11 against $pR^{-p-1}\,dR$ for $R\ge 1$. As $p = K+1$, this bounds the integral over U of $(\mathcal{A}(g)/\mathcal{A}(y))^d\nu(y)^{-p}$ by $A_\sharp Z(g)^2\nu(g)^K$ times the local parameter factor. The choice of coset representative is harmless. Both $\mathcal{A}$ and ν descend to $B\backslash G$; a change by J preserves the fiber integral since it commutes with γ, and the other changes translate the unipotent variable.

Apply Cauchy–Schwarz on $J_\infty\backslash G_\infty$ as in Theorem 83.17, now with weight

$$\prod_\tau \frac{\mathcal{A}(g_\tau)^d\,\nu_\tau(g_\tau)^p}{\mathcal{A}((\gamma^{-1}g)_\tau)^d\,\nu_\tau((\gamma^{-1}g)_\tau)^p}.$$

After fiber integration, remove $C_E A_\sharp^f$ and the parameter factors. The remaining weight on the base is at most

$$\prod_\tau Z(g_\tau)^2\nu_\tau(g_\tau)^{p+K}.$$

Since $Z(g)\le\nu(g)$ by Lemma 83.21, this weight costs at most $V^{p+K+2} = V^D$. In the second squared integral the substitution by γ gives the same bound with $-C_\tau$. Normalize by the diagonal norm. The moment is finite by Proposition 64.20. The assertion in original frequency follows from Proposition 83.14, including its exact quotient-measure factor. □

83.19 The real Haar measure in the first-plane integral

Write $n = r-2$, $F = E^+$, $f = [F:\mathbb{Q}]$, and let Δ_E, Δ_F denote the absolute discriminants. In the signed Euclidean metric the first vector $u = e_+ + e_-$ has length $\sqrt{2}$. Put $u' = u/\sqrt{2}$, $v' = (e_+ - e_-)/\sqrt{2}$ at infinity. They have Euclidean length one and Hermitian pairing one. Use the signed orthonormal coordinates of the original space on the remaining block M.

Proposition 83.19 (Haar normalization on the first unipotent radical)**.** *Give the finite groups U_L volume one as in Theorem 83.15 and give $[U]$ volume one. Write the real unipotent in the u', M, v' frame as $n(w,z)$ of Theorem 77.10, with $z = -h(w,w)/2 + it$. With ordinary complex area and real length at each place, the resulting real Haar measure is*

$$C_E \prod_\tau dw_\tau \, dt_\tau, \qquad C_E = 2^{-f} \Delta_E^{-n/2} \Delta_F^{-1/2}.$$

Proof. Start in the rational integral frame u, M, v of Proposition 89.15. The quotient of U by its center is the additive E-space M; its center is the additive trace-zero F-line. Locally the integral subgroup maps onto L_M. For $w \in L_M$ one must solve $z + \bar{z} = -h(w,w)$ integrally. In a split algebra one can prescribe either summand of the trace. At the other places two is a unit, by Proposition 100.7. The kernel is the trace-zero integer lattice. Rational surjectivity follows by taking $z = -h(w,w)/2$. Product measure in w and the imaginary part of z is Haar at infinity, since translations have triangular Jacobian with diagonal one. Integration over the central quotient and then the additive quotient of M multiplies the two covolumes. Indeed their rational fundamental sets lift to a fundamental set for U, by first reducing w and then its central coordinate.

The additive covolume of M, with its finite lattices assigned volume one, is $(2^{-f}\sqrt{\Delta_E})^n$. For the standard integer lattice this is the discriminant formula in complex area. The change to the terminal graph lattice in the middle block acts on the two split halves by a matrix and its inverse dual, hence changes the product of additive volumes by one. The other finite lattices in this block are the original ones.

Write $E^- = F\sqrt{-d_0}$. The valuations of d_0 in F are even because it is a square up to sign in E and E/F is unramified at every finite place. Thus there is an integral ideal $\mathfrak{a}$ of F with $\mathfrak{a}^2 = d_0 \mathcal{O}_F$. Testing valuations gives $\mathcal{O}_E \cap E^- = \sqrt{-d_0}\,\mathfrak{a}^{-1}$. In imaginary-part coordinates its covolume is $d_0^{f/2}\sqrt{\Delta_F}/N\mathfrak{a} = \sqrt{\Delta_F}$. The normalization in rational coordinates is the reciprocal of the product of the two covolumes.

The opposite vector v of the integral frame differs from $v'/\sqrt{2}$ by a purely imaginary multiple of u. The upper unipotent changing one to the other is central in U, so this replacement leaves its coordinates unchanged. Rescaling the first vector to u' and the opposite vector to v' then rescales w by $\sqrt{2}$ and z by two. Their real Jacobian is 2^{n+1} at each place. Dividing the rational-coordinate normalization by $2^{(n+1)f}$ gives the displayed value. □

83.20 The terminal minor and the size of a real unipotent fiber

At a real place use the Euclidean unit Witt pair u', v' of Proposition 83.19. In their perpendicular use Euclidean unit Witt pairs p_i, q_i, $1 \le i \le h = s-1$, for the remaining isotropic flag, the negative unit terminal vector e, and the positive column of dimension k. These vectors form a Euclidean orthonormal basis, with Hermitian pairings $h_V(p_i, q_i) = 1$. We use h_V conjugate linear in its first argument. Set $d = k+1$ and $n = 2h + d = r - 2$. Write

$$w = \sum_{i=1}^{h} (a_i p_i + b_i q_i) + x + \eta e, \qquad z = -h_V(w,w)/2 + it.$$

Here x belongs to the positive column and $t \in \mathbb{R}$. Use ordinary real length and complex area. For a real $C \neq 0$, let $\gamma(C)$ fix the middle block and v', and send u' to $u' + iCv'$. Put $\delta = |C|$, $\log^+ a = \max(0, \log a)$, and $K_* = 1 + (s+1)n$.

Theorem 83.20 (The full decorated flag on a real fiber)**.** *For $H \ge 1$, the set of (w,t) for which $\|n(w,z)^{-1}\gamma(C)b\|_{\mathrm{op}} \le H$ for some $b \in B$ has volume at most*

$$\frac{2^{n+1}\pi^n}{d!}\, H^{K_*} \delta^{-(n+1)} (1 + 2\log^+(1/\delta))^h.$$

Proof. Put $g = n(w,z)^{-1}\gamma(C)$. In the block order u', M, v', the first column of g is $(A, -iCw, iC)$ with $A = 1 + iC\bar{z}$. For $p = p_i$ or $p = e$ one has $gp = p + h_V(w,p)u'$. Every $b \in B$ adds earlier vectors to later ones in the list $u', p_1, \dots, p_h, e$, with all diagonal entries one. Consequently it fixes every initial exterior product. For the first $j+1$ columns, $0 \le j \le h$, take the coefficient of that product along the same Euclidean unit exterior vector. It is

$$D_j = A + iC \sum_{i=1}^{j} \bar{b}_i a_i, \qquad |D_j| \le H^{j+1}.$$

Indeed the indicated submatrix has lower right block I_j, first row $(A, \bar{b}_1, \dots, \bar{b}_j)$ and lower first column $-iC(a_1, \dots, a_j)^{\mathrm{t}}$. The exterior norm of gb in this degree is at most H^{j+1}. Appending e changes the coefficient to $D_e = D_h - iC|\eta|^2$, with $|D_e| \le H^{s+1}$. Since

$$h_V(w,w) = 2\,\mathrm{Re} \sum_i \bar{b}_i a_i + \|x\|^2 - |\eta|^2, \qquad \mathrm{Im}\, D_e = -\frac{C}{2}(\|x\|^2 + |\eta|^2),$$

the full list bounds the squared length in d complex coordinates by $P = 2H^{s+1}/\delta$. The volume of this ball is $\pi^d P^d / d!$, by polar integration.

For each i, $D_i - D_{i-1} = iC\bar{b}_i a_i$ bounds $|a_i b_i|$ by P. The first column gives $|a_i|, |b_i| \le H/\delta = L_0$. One pair with these three bounds has volume at most $\pi^2 P^2[1 + 2\log^+(L_0^2/P)]$. If $L_0^2 \le P$ the two disks suffice. Otherwise integrate $\pi \min(L_0^2, P^2/\rho^2)$ against $2\pi\rho\, d\rho$ from zero to L_0, splitting at P/L_0. As $s \ge 2$, $L_0^2/P \le 1/\delta$. Finally, $\mathrm{Re}\, A = 1 + Ct$ puts t in an interval of length $2H/\delta$. Multiplication of these volume bounds gives the claim. In particular, indefinite cross terms of the middle space do not cost an extra power of $1/\delta$ in each positive column coordinate. □

83.21 A negative test vector for the terminal minor

Use the real coordinates of Theorem 83.20. Let P be the exterior product of the s Euclidean unit vectors in the isotropic flag, in order. Put

$$\alpha(g) = \|g^{-1}P\|, \qquad \mathcal{A}(g) = \|g^{-1}(P \wedge e)\|, \qquad Z(g) = \mathcal{A}(g)/\alpha(g).$$

Each norm is calculated in its exterior power. All three functions are invariant under B on the left. In particular $Z(g)$ is the distance from $g^{-1}e$ to the span of the transformed flag.

Lemma 83.21 (An ellipsoid from a negative unit vector)**.** *In every B-coset there is a representative g of norm $\nu(g)$, a positive-column vector v_+, and $a \ge 1$ with $\|v_+\|^2 = a^2 - 1$, $Z(g)^2 = 2a^2 - 1$, such that for every w, t, $y = \gamma(C)^{-1} n(w,z) g$ satisfies*

$$\|x - \eta v_+/a\|^2 + |\eta|^2/a^2 \le \frac{2\sqrt{2}}{|C|}\,\frac{\mathcal{A}(y)}{\mathcal{A}(g)}.$$

Here x, η are the positive and terminal coordinates of w.

Proof. The maximal compact group is transitive on the isotropic flags of the given dimensions. Indeed the projections of an isotropic subspace into the positive and negative signed spaces are injective and give an isometry between their images. Orthonormal bases adapted to the projected flags extend to orthonormal bases of the two signed spaces. Removing a right compact factor thus puts a group element in the flag parabolic. Its diagonal blocks on the two isotropic strings are scalars λ_i and $\bar{\lambda}_i^{-1}$ on their successive quotients. The middle block is $m_0 \in U(k,1)$. Removing the upper unipotent factor on the left gives a representative consisting of these

diagonal blocks followed by the right compact factor. This is allowed because that unipotent radical lies in B. On multiplying on the left by B, the projections onto the diagonal blocks still have the same norms, since the possible extra middle factor is unitary for the Euclidean norm. The block diagonal representative therefore attains the minimum $\nu(g)$.

In this representative put $l_i = |\lambda_i|^{-1}$. Then $\alpha(g) = \prod_i l_i$, $Z(g) = \|m_0^{-1} e\|$. Multiply $m_0 e$ by a phase to write it as $v_* = v_+ + ae$ with $a > 0$. Its Hermitian norm is -1. The last diagonal entry of m_0^{-1} is the conjugate of that of m_0, by the unitary relation for the signed form. Thus $Z(g)^2 = 2a^2 - 1$, and $\|m_0^{-1} v_*\| = 1$.

Let P^- be the ordered product of the s unit opposite Witt vectors. Pair $P^- \wedge v_*$ with $n^{-1}\gamma(C)(P \wedge e)$ by the determinant of Hermitian pairings. Divide this determinant by $h_V(v_*, e) = -a$ and call the result D_*. Using the first column and the other flag columns from Theorem 83.20 gives

$$D_* = D_h - iC|\eta|^2 + iC\bar{\eta}\, h_V(v_+, x)/a.$$

Indeed the row for v_* has first entry $-iCh_V(v_*, w)$, middle entries zero and final entry $-a$. Expansion using the identity block on the p_i gives the formula. Consequently $-2\operatorname{Im} D_*/C$ equals the left side of the asserted inequality, by completing the square.

To bound D_*, apply g^{-1} to both exterior vectors in the Hermitian pairing. The first then has Euclidean norm $1/\alpha(g)$, using $\|m_0^{-1} v_*\| = 1$ and the opposite scalings. The second has norm $\mathcal{A}(y)$. A signed-unitary Hermitian pairing on an exterior power is bounded by the product of Euclidean norms, since its sign matrix is Euclidean unitary. It follows that $|D_*| \le \mathcal{A}(y)/(a\alpha(g))$. Use $Z(g)/a \le \sqrt{2}$ to conclude. □

84 Matrix coefficients and Hermite moments

84.1 Integral vectors near a short isotropic line

Use the initial unitary group $H = U(Y)$ of Theorem 89.1 and its self-dual lattice $\Lambda = \Lambda_{Y,g}$ at g. Let $r = \dim_E Y$, $f = [E^+ : \mathbb{Q}]$, $d = 2fr$, and fix a cyclotomic modulus M for E. Write $D_E = |\operatorname{disc} E|$, $\operatorname{rd}(E) = D_E^{1/(2f)}$, and $\|v\|^2 = \sum_\tau |v|_{\tau,g}^2$. Regard the lattice in this Euclidean real space by its diagonal embedding. Let $q_* \ge 1$ be an upper bound for the dimension of a rational totally isotropic subspace of Y. Let λ_1 be the least nonzero Euclidean length in Λ.

Lemma 84.1 (Short spanning vectors for a rational flag and its perpendicular)**.** *If $b_g < 1$, then $\lambda_1 \le 2\sqrt{f \operatorname{rd}(E)}\, b_g^{1/f}$. If $\lambda_1 < 1/2$, there are a nonzero totally isotropic E-subspace $W \subset Y$ and $0 < a < a' \le 1/2$ with*

$$a/a' \le (2\lambda_1)^{1/q_*}.$$

Vectors in Λ of length at most a span W over E, and every integral vector outside W has length at least a'. The real space W_∞ has a spanning set of integral vectors of lengths at most Ma. If $dMa \le a'$, the real space $(W^{\perp_h})_\infty$ also has a spanning set of integral vectors of lengths at most Md^d/a'.

Proof. Choose a line attaining $b_g < 1$, which exists by the finiteness argument before Theorem 89.17. For a generator w let $\mathfrak{c} = \mathfrak{c}_g(w)$. The scalars taking w into Λ form $\mathfrak{c}^{-1}$. Its covolume in the product of complex embedding planes is $2^{-f} D_E^{1/2}/N\mathfrak{c}$. Consider the product of disks in these planes with radii $2\sqrt{\operatorname{rd}(E)}\, b_g^{1/f}/|w|_{\tau,g}$. Its volume is $(4\pi)^f D_E^{1/2}/N\mathfrak{c}$, since $\prod_\tau |w|_{\tau,g} = b_g\sqrt{N\mathfrak{c}}$. Minkowski's convex body theorem supplies a nonzero scalar of $\mathfrak{c}^{-1}$ in this product. Multiplication of w by that scalar proves the first bound.

List the distinct lengths below $1/2$ at which the E-span of the integral vectors up to that length increases. The first is λ_1. Every vector in this list's spans is generated by vectors whose

lines have height less than one, since their content is integral and their norm at every real block is less than $1/2$. All these spans are totally isotropic by Theorem 89.1. There are consequently at most q_* lengths in the list. Adjoin $1/2$ at the end. The product of each length divided by its successor is $2\lambda_1 < 1$. One ratio a/a' is at most $(2\lambda_1)^{1/q_*}$. Take W to be the span through a.

There are $[E : \mathbb{Q}]$ linearly independent integers of E bounded at every embedding by M. Indeed the relative traces to E of the powers of ζ_M span E over $\mathbb{Q}$, since the powers span the cyclotomic field and trace is onto. Each trace is a sum of at most M roots of unity. Multiplying an E-basis of the short vectors by these integers gives a real basis of W_∞ of lengths at most Ma. Rounding coefficients in this basis approximates any point of W_∞ within $dMa/2$ by an integral vector of W.

Let L be the orthogonal Euclidean projection of Λ onto $W_\infty^{\perp_{\mathrm{eucl}}}$. This is a lattice because W_∞ is spanned by lattice vectors. A basis of the saturated subgroup $\Lambda \cap W_\infty$ extends to a basis of Λ. Lift a nonzero vector of L to Λ, and subtract the integral approximation to its component in W_∞. The resulting vector is outside W and has length at least a'. If $dMa \le a'$, the triangle inequality shows that the projected vector has length at least $a'/2$.

For a Euclidean lattice L of rank j, let $\lambda_i(L)$ be its successive minima for the unit ball. One has $\lambda_1(L)\lambda_j(L^*) \le j^j$. Indeed $\lambda_i(L)\lambda_{j+1-i}(L^*) \ge 1$, since the indicated independent vectors on the two sides cannot all pair to zero, and their pairings are integers. Minkowski's second theorem bounds the product of all these inequalities' left sides by $4^j/\operatorname{vol}(B_j)^2$. The unit ball contains the cube of side $2/\sqrt{j}$, giving j^j. Thus L^* has independent spanning vectors of lengths at most $2d^d/a'$.

In the adapted frames let J be the orthogonal signature operator, so that half the trace of h on rational vectors is the Euclidean pairing with J in one slot. Self-duality over $\mathcal{O}_E$ and the definition of the codifferent $\mathfrak{D}_E^{-1}$ give $\Lambda^* = 2J(\mathfrak{D}_E^{-1}\Lambda)$. Indeed the trace dual condition against all multiples in $\mathcal{O}_E$ says precisely that the Hermitian pairings lie in the codifferent. The Euclidean dual of L is $\Lambda^* \cap W_\infty^{\perp_{\mathrm{eucl}}}$. Consequently applying $J/2$ to its spanning vectors gives vectors of $\mathfrak{D}_E^{-1}\Lambda$ perpendicular to W for h. Here orthogonality to the whole realification of an E-subspace tests both real and imaginary parts in every block.

Finally $M\mathfrak{D}_E^{-1} \subset \mathcal{O}_E$. The different of $\mathbb{Q}(\zeta_M)$ is generated by $\Phi_M'(\zeta_M)$, since $\mathbb{Z}[\zeta_M]$ is its full integer ring. Differentiation of $T^M - 1$ shows that this generator divides M. Transitivity of the different and contraction of extended ideals give $M\mathcal{O}_E \subset \mathfrak{D}_E$. Multiplying the preceding vectors by M gives the asserted integral spanning set of $(W^{\perp_h})_\infty$. □

84.2 The second moment transverse to the stabilizer

Use the real groups, parameters and Haar measures of Theorems 84.9 and 59.7. All inner products in this argument are linear in their first variable. Use Euclidean length in signed orthonormal coordinates and $W(g) = \|g\|_{\mathrm{op}}$. Endow $H\backslash G$ with the quotient of the chosen Haar measures.

Theorem 84.2 (Detection of the discrete summand in a bounded region)**.** *Suppose the isometric embedding $j : \sigma \to \pi|_H$ has multiplicity one. For all unit vectors v_0, w in the respective lowest compact types there is an absolute constant C such that*

$$d_\pi \int_G |g^{-1}z|^2 |\langle \pi(g)v_0, jw\rangle|^2 \, dg \le T_0^{Cn}.$$

Consequently, with an absolute $A \ge 1$, there exists g_0 with $W(g_0) \le T_0^{An}$ and $\|P\pi(g_0)v_0\| \ge T_0^{-An^3}$.

Proof. Write V_i for the smooth tensor summand with parameter $\lambda - e_i$, and π_i for its representation. Its invariant signed form restricts to $\epsilon_i\langle\ ,\ \rangle_i$ with $\epsilon_i \in \{1,-1\}$, in the normalization of Theorem 84.9. The functional $\ell_i(x) = \langle x_z, jw\rangle$ is defined on V_i. We first prove that it extends to the Hilbert completion with squared norm $|\gamma_i|$. This point is needed because w is smooth under H, whereas our squared integral is over G.

Choose a smooth vector a of G with $c = \langle jw, a\rangle \neq 0$, using density. Put $y_i = P_i I_z a$ and $p_w(h) = d_\sigma \overline{\langle \sigma(h)w, w\rangle}$. This coefficient is integrable by the decay and volume bounds of Theorem 84.9. Formal degree orthogonality gives a self-adjoint convolution idempotent. The argument of Theorem 59.15 identifies its operator on $\pi|_H$ with the orthogonal projection onto $\mathbb{C}jw$, by multiplicity one. Its operator Q_i on the Hilbert space of $\pi_i|_H$ is also an orthogonal projection. No multiplicity assumption for that restriction is needed. If δ_z is the sign of the coordinate z, then for smooth $x \in V_i$,

$$\begin{aligned}\epsilon_i\langle x, Q_i y_i\rangle_i &= \int_H p_w(h)[(\pi \otimes (\mathbb{C}^n)^*)(h)x, I_z a]\, dh \\ &= \delta_z \langle \pi|_H(p_w)x_z, a\rangle = \delta_z c\, \ell_i(x).\end{aligned}$$

In the first line orthogonality of the tensor summands permits replacing y_i by $I_z a$ in the signed pairing. In the second line the stabilizer fixes the last coordinate and its covector. Both integrals are absolutely convergent by unitarity in the respective Hilbert spaces. Taking $x = y_i$ and using Theorem 59.7 gives $\epsilon_i \|Q_i y_i\|_i^2 = \delta_z \gamma_i |c|^2$. The preceding identity now represents ℓ_i by a Hilbert vector of squared norm $|\gamma_i|$, also when this quantity vanishes.

For the coordinate covector e^l, take the last component of the translated tensor $v_0 \otimes e^l$. The dual defining action gives

$$(g^{-1})_{lz}\langle \pi(g)v_0, jw\rangle = \sum_i \ell_i\big(\pi_i(g)P_i(v_0 \otimes e^l)\big).$$

Cauchy–Schwarz for this sum of n terms and Schur orthogonality in each summand bound the squared integral times d_π by $n\sum_i (d_\pi/d_{\pi_i})|\gamma_i|\,\|P_i(v_0 \otimes e^l)\|_i^2$. The degree ratio is at most 2^{n-1} by the Vandermonde argument in Proposition 84.11. The projector estimate on the lowest compact type in Theorem 84.9 gives a squared norm at most T_0^{Cn}, and Theorem 59.7 bounds $|\gamma_i|$ by $(2B+1)^{n-1}$. Summing over l proves the transverse moment estimate.

The unweighted squared integral is $1/d_\pi$. Take $R^2 = 2T_0^{Cn}$ for a constant valid in that estimate. At least half the integral is supported on $|g^{-1}z| \le R$. This region is the inverse image of a set $\mathcal{B}_R$ in $H\backslash G$. On a fiber Hg formal degree orthogonality for σ integrates the unweighted square to $\|P\pi(g)v_0\|^2/d_\sigma$. It remains to bound the quotient volume. Each coset in $\mathcal{B}_R$ has a representative with $W(g) \le 2R$. Indeed a compact rotation and a boost in one signed plane carry z to $g^{-1}z$, as in the proof following Theorem 67.3. The norm of that isometry and of its inverse is at most $2|g^{-1}z|$. In a definite space a rotation suffices. For every such representative, $U_{1/4}g \subset \{W \le 3R\}$, where the small ball is taken in H. Quotient integration and the volume bounds in Theorem 84.9 give

$$\frac{d_\pi}{d_\sigma}\,\mathrm{vol}(\mathcal{B}_R) \le \frac{d_\pi\,\mathrm{vol}_G\{W \le 3R\}}{d_\sigma\,\mathrm{vol}_H(U_{1/4})} \le T_0^{Cn^3}.$$

Here $(3R)^{4n^2} \le T_0^{Cn^3}$. Integrating the fiber identity over $\mathcal{B}_R$ now shows that at some coset $\|P\pi(g)v_0\|^2 \ge (4T_0^{Cn^3})^{-1}$, since a smaller value throughout would contradict the half-mass lower bound. This norm is invariant under changing representatives on the left by H. Choose the representative of norm at most $2R$ and enlarge the absolute constant in the two asserted estimates. □

84.3 The norm of the initial Poincaré intertwiner

Fix a pure unit v_0 in the lowest compact type of the physical core discrete series π. Write $\ell_a(F) = \mathcal{B}(R(a)F, u)$ on $H(v_0)$, where $a \in G_\infty$ and u is the prescribed unit lowest-type row in Theorem 59.10. Let d_π and d_σ be the product real formal degrees of the core and row respectively. Use product Haar measures, counting measure on rational points, and keep the original $K_f = K(\mathfrak{m})$ and $J = \iota^{-1}K_f$.

Proposition 84.3 (A translated first restriction with an absolute lower bound)**.** *There is $a \in G_\infty$ satisfying $W(a_\tau) \le T_0^{Cn^3}$ for all τ such that*

$$\|\ell_a\|_* \;\ge\; \frac{\mathrm{vol}(J)}{d_\sigma}\left(\frac{d_\pi}{\mathrm{vol}(K_f)}\right)^{1/2} \|u\|_\infty \,\exp[-P_{\psi,\varepsilon}(x)].$$

Here a polynomial depending only on the fixed family and ε works uniformly on the original intervals, for every such row.

Proof. Choose g with $|u(g)| > \|u\|_\infty/2$, and put $c = \iota(g)$. The Poincaré series in Theorem 59.10 is $Y_b(b)$ in the notation of Proposition 59.8. We record the size of the compact averaging used to form b. The transverse estimate in Theorem 84.2 starts at any prescribed local lowest unit in the conjugate core, so its conjugate in the physical core can be chosen to be the local factor of v_0. The direction selection in Theorem 59.6 translates it by an element of norm at most $T_0^{Cn^3}$. Compact translates have norm one. Finally the compact projection in Theorem 59.15 integrates against a matrix unit in the lowest row type S. For unit $e, w \in S$, its density is $(\dim S)\overline{\langle\sigma(k)e, w\rangle}$ in a choice of coefficient convention and has L^1 norm at most $\sqrt{\dim S}$ by compact orthogonality. The mixed matrix unit in the proof of Theorem 59.6 permits an arbitrary prescribed lowest row vector. Complex conjugation preserves these bounds. Thus in the physical core one has

$$b = \int \pi(a)v_0 \, d\nu(a), \qquad \|\nu\|_{\mathrm{TV}} \le T_0^{Cfn^2}, \qquad \sup_{a\in\mathrm{supp}\,\nu} W(a_\tau) \le T_0^{Cn^3}$$

with compact support. For the variation bound we used $\log\dim S \ll fn^2 \log T_0$, as follows from the compact Weyl dimension formula in the proof of Theorem 59.10.

We bound the norm of $Y_b(v_0)$. In this calculation use the coefficient convention of Proposition 59.8 and put $f_0(t) = \langle\pi(t)v_0, b\rangle$. This Poincaré series has just one contributing finite double coset. Its squared norm is the level volume times the integral of the squared periodization of f_0 on $\Gamma_c\backslash G_\infty$. Indeed at finite representative c_f the contributing rational points are exactly those defining Γ_c; translating the real variable by c_∞^{-1} gives the stated quotient. Haar measure on the finite right coset contributes $\mathrm{vol}(K_f)$, with rational stabilizers counted in the quotient by Γ_c. Opening the square and unfolding one of the two sums gives

$$\|Y_b(v_0)\|_2^2 = \mathrm{vol}(K_f) \sum_{\gamma\in\Gamma_c} \int_{G_\infty} f_0(t)\overline{f_0(\gamma t)}\, dt.$$

For absolute convergence let $L = 8n^2 + 2n + 9$ be the decay exponent of Theorem 84.9. Since v_0 is a pure lowest tensor and ν has the displayed support and variation, that theorem and submultiplicativity give $|f_0(t)| \le T_0^{Cfn^5} w(t)^{-L}$. Now $w(t)w(\gamma t) \ge w(\gamma)$. For $p = 4n^2 + 2$ the product of absolute integrands is bounded by $T_0^{Cfn^5} w(\gamma)^{-p} w(t)^{-(L-p)}$. The last weight is integrable and the sum of the preceding weights is finite by Lemma 89.3. This justifies unfolding.

Formal-degree orthogonality gives the absolute value of each integral as $d_\pi^{-1}|\langle\pi(\gamma)b, b\rangle|$. Apply the same decay estimate on both compact translates in the two instances of b. It gives

$|\langle \pi(\gamma)b, b\rangle| \le T_0^{Cfn^5} w(\gamma)^{-L}$, again using $W(a_\tau) \le T_0^{Cn^3}$ at each end. The sum bound of Lemma 89.3 therefore yields

$$\|Y_b(v_0)\|_2 \le \left(\frac{\mathrm{vol}(K_f)}{d_\pi}\right)^{1/2} \exp[P_1(x)]$$

for a polynomial P_1 uniform at the chosen points.

By Theorem 59.1, Y_b extends to a bounded intertwiner and $Y_b(v_0)$ belongs to $H(v_0)$. The compact integral of its translates by ν is exactly $Y_b(b)$. Its pairing with u is the integral of $\ell_a(Y_b(v_0))$ against ν. To see this directly, the compact translates of the smooth bounded cusp form are uniformly bounded on the automorphic quotient and u is integrable on $[H]$. Fubini gives the assertion. Theorem 59.10 bounds this last pairing below in modulus by $\mathrm{vol}(J) d_\sigma^{-1} \|u\|_\infty \exp[-Cfn^5 \log b]$. It follows that $Y_b(v_0) \neq 0$. The continuous function of a has a maximum in modulus on the compact support, at least the pairing modulus divided by $\|\nu\|_{\mathrm{TV}}$. Divide by $\|Y_b(v_0)\|_2$ and combine the displayed estimates to obtain the assertion. $\square$

84.4 The height at a large value of a cusp form

Use the lattice, ranks and modulus M of Lemma 84.1. Let $m_0 \ge 1$ be a rational integer. On $H = U(Y)$ let $K_H(m_0)$ be the stabilizer of the original finite lattice acting trivially modulo m_0. For a tuple of real Lie algebra elements $A = (A_\tau)_\tau$ put $\|A\|_1 = \sum_\tau \|A_\tau\|_{\mathrm{HS}}$, where the Hilbert–Schmidt norms use the adapted frames at the identity. A cusp form has integral zero along the unipotent radical of each proper rational parabolic subgroup, at every right translate.

Theorem 84.4 (A uniform height bound from cuspidality)**.** *Let u be a nonzero bounded smooth cuspidal automorphic function on H, right invariant under $K_H(m_0)$, and put $U = \|u\|_\infty$. Suppose $|u(s\exp A) - u(s)| \le L\|A\|_1$ for all adelic s and all real A. Write $\kappa = m_0 M^2 (2d)^{d+5}$, where $d = 2f \dim_E Y$. At any point with $|u(g)| > U/2$,*

$$b_g \ \ge \left(\frac{[4\kappa(1 + L/U)]^{-q_*}}{4\sqrt{f\,\mathrm{rd}(E)}}\right)^f.$$

The bound uses the original field, lattice and level of the form.

Proof. Set $D_0 = [4\kappa(1 + L/U)]^{-1}$. The case $b_g = 1$ is immediate. For $b_g < 1$, Lemma 84.1 shows it suffices to prove $2\lambda_1 \ge D_0^{q_*}$. Suppose otherwise, and put $\theta = (2\lambda_1)^{1/q_*} < D_0$. The lemma gives an isotropic subspace W with $a/a' \le \theta$, so $dMa \le a'$. Let N be the unipotent radical of the proper parabolic stabilizing W in H. We will show that every point of $N(\mathbb{A}_F)g$, up to left rational translation and the right finite level, can be reached from g by two real exponentials of total norm at most $\kappa\theta$.

Choose a rational Witt decomposition $Y = W \oplus D \oplus W'$, with W' isotropic and dual to W under h. Denote by $\mathfrak{z}$ the central Lie subspace of maps supported in the block $W' \to W$ and skew adjoint there. The elements of N act as the identity on W, $W^{\perp_h}/W$ and $Y/W^{\perp_h}$. In this order of the three blocks their matrices are upper unitriangular. In the Lie algebra the $W' \to D$ block is determined by the negative adjoint of the $D \to W$ block. The quotient by $\mathfrak{z}$ is therefore the additive space of the latter blocks, and any product of three strictly upper block matrices is zero.

For integral $w \in W$ and $v \in W^{\perp_h}$ define

$$X_{w,v}(s) = wh(v, s) - vh(w, s).$$

This is a rational, integral, skew-adjoint endomorphism in the Lie algebra of N. Its square is $-h(v, v)wh(w, \cdot)$. Use for w the spanning vectors of length at most Ma and for v those of

length at most Md^d/a' from the lemma. Their operators span the quotient over $\mathbb{R}$. Indeed on D the first block is $wh(v,\cdot)$ and these rank-one maps span all complex matrices in each real block. The spanning sets cover the realifications, so real bilinearity also isolates the individual places. Taking both vectors from the short spanning set in W gives operators spanning $\mathfrak{z}_\infty$. In bases of W and W' paired by h, these blocks are $wv^* - vw^*$, which span the skew-Hermitian matrices. Choosing coordinate vectors and their imaginary multiples gives the diagonal and both real directions of every off-diagonal pair. Select a basis among the images for each of the two spaces. Each selection has at most d^2 members.

For each chosen X, use the element $t \in \mathcal{O}_E$ with $t + \bar{t} = 1$ from Theorem 89.1 to form

$$\gamma_X = 1 + m_0 X + t m_0^2 X^2.$$

Since $X^* = -X$ and $X^3 = 0$, multiplication by the adjoint gives the identity. Both γ_X and its adjoint preserve Λ and act trivially modulo $m_0\Lambda$. Hence $\gamma_X \in N(F)$ and its finite component belongs to $g_f K_H(m_0) g_f^{-1}$. Its logarithm is $m_0 X + (t - 1/2) m_0^2 X^2$. The second term belongs to $\mathfrak{z}$, as it is skew adjoint and supported in the last block. For a centrally supported generator $X^2 = 0$.

Take any $v \in N(\mathbb{A}_F)$. Left multiplication by a rational point first puts its finite component inside $g_f K_H(m_0) g_f^{-1}$. Indeed $N(F)$ is dense in $N(\mathbb{A}_{F,f})$. Exponential and logarithm here are inverse polynomials of degree at most two on the strictly upper blocks. They identify the rational Lie algebra and its finite adeles with the corresponding unipotent groups, preserving integral coordinates outside finitely many primes. Density follows from finite adelic approximation in the additive F-vector space. Further left multiplication by integer powers of the γ_X preserves this finite condition. Modulo $\exp \mathfrak{z}_\infty$ the real group is additive. Round the coordinates in the chosen basis of the elements $m_0 X$ in that quotient. The resulting real component has the form $\exp(A_0)\exp(Z_0)$, where A_0 is a sum of the corresponding $m_0 X$'s with coefficients in $[-1/2, 1/2]$ and $Z_0 \in \mathfrak{z}_\infty$. Now round in the central basis, using the integral powers of the central generators. This also puts every coefficient of Z_0 in $[-1/2, 1/2]$ in that basis. This argument includes the case of a zero-dimensional quotient.

Conjugate A_0, Z_0 by g_∞^{-1} and call the results A, Z. The norm of each conjugated $X_{w,v}$ is bounded by $2\|w\|\|v\|$ for the Euclidean lengths at g, since the Hilbert–Schmidt norm of a rank-one map is the product of the two lengths and Cauchy–Schwarz applies in the sum over places. Thus

$$\|A\|_1 + \|Z\|_1 \le d^2 m_0 M^2 (d^d a/a' + a^2) \le \kappa\theta.$$

Here $a^2 \le a/a'$ as $a, a' \le 1/2$. Left automorphy and right finite invariance now give $u(vg) = u(g \exp A \exp Z)$. The Lipschitz bound at two successive points yields $|u(vg) - u(g)| \le L\kappa\theta$. The quotient $N(F)\backslash N(\mathbb{A}_F)$ is compact, as follows from its central filtration by additive adelic quotients. Integrating against Haar probability and using the vanishing constant term gives $|u(g)| \le L\kappa\theta < U/2$, a contradiction. Combining the resulting lower bound for λ_1 with the first inequality of Lemma 84.1 proves the assertion. □

84.5 A derivative bound for the lowest compact type

Give $U(p,q)$, $r = p + q$, the left invariant metric determined at the identity by the Hilbert–Schmidt inner product on its real Lie algebra. In an automorphic row, fixing all coordinates except this real factor gives a function on the whole Lie group with the same bound in supremum norm. Thus an estimate for this metric can be used at every finite level.

Proposition 84.5 (A uniform derivative estimate for a bounded row)**.** *Suppose a bounded smooth automorphic row u at every real place belongs to a discrete series with Harish-Chandra*

parameters of absolute value at most $B \geq 1$, and to its lowest compact type. Put $U = \|u\|_\infty$ and $r = \dim_E Y$. Then at every adelic point and for every real Lie algebra tuple A,

$$|u(g \exp A) - u(g)| \leq 40\sqrt{r}\,(B + 2r)U \sum_\tau \|A_\tau\|_{\mathrm{HS}}.$$

The same result holds for any single compact type at each place whose highest-weight coordinates have absolute value at most $B + r$.

Proof. Write Δ for the Laplace–Beltrami operator, with negative spectrum. In a left invariant orthonormal frame it is the sum of squares, because the group is unimodular. In the Cartan decomposition let X_i be an orthonormal basis of the compact Lie algebra and Y_j one of its perpendicular. The trace form $\mathrm{Re}\,\mathrm{tr}(AB)$ has signs minus and plus on these two spaces. Its Casimir is therefore $\Omega = -\sum_i X_i^2 + \sum_j Y_j^2$, and the compact Casimir is $\Omega_K = -\sum_i X_i^2$. Thus $-\Delta = 2\Omega_K - \Omega$. Fix all other arguments of the row. Its function on this group satisfies $-\Delta u = \lambda u$, where $\lambda \geq 0$ since the sum of negatives of squares acts positively in the unitary series.

In the diagonal coordinates for this trace form, the Casimir on infinitesimal character ν is $\sum_i \nu_i^2 - \sum_i \rho_i^2$, with ρ half the sum of positive roots. This follows from the Harish-Chandra isomorphism for the quadratic Casimir. On the compact type of highest weight μ one uses $\nu = \mu + \rho_K$, separately on $U(p)$ and $U(q)$. For the lowest compact type, the discrete-series lowest-weight formula is $\mu = \nu_G + \rho_{\mathrm{nc}} - \rho_K$, in the chamber of its Harish-Chandra parameter ν_G. Each coordinate of $\rho_{\mathrm{nc}} - \rho_K$ has absolute value at most $(r-1)/2$, as each of the other coordinates contributes a single signed half. Thus $|\mu_i| \leq B + r$. The same bound is part of the hypothesis for the other types. Since every coordinate of a half-sum is bounded by $r/2$, the two Casimir formulas give $0 \leq \lambda \leq 10r(B + 2r)^2$.

We give the metric estimate including its rank dependence. For left invariant vectors the bracket has norm at most twice the product of their norms, by matrix multiplication. The Koszul formula therefore gives $\|\nabla_X Y\| \leq 3\|X\|\|Y\|$. Substituting in the definition of curvature gives $\|R(X,Y)Z\| \leq 24\|X\|\|Y\|\|Z\|$. Taking a trace in dimension r^2 yields $\mathrm{Ric} \geq -K$ with $K = 24r^2$. This invariant metric is complete.

Let P_t be its heat semigroup with generator Δ. The local reverse Poincaré inequality under $\mathrm{Ric} \geq -K$ reads, for a bounded real function a,

$$|\nabla P_t a|^2 \leq \frac{K}{1 - e^{-2Kt}}\,(P_t(a^2) - (P_t a)^2).$$

For clarity, the semigroup gradient inequality obtained from Bochner's formula is $P_s|\nabla v|^2 \geq e^{-2Ks}|\nabla P_s v|^2$. Differentiate $P_s((P_{t-s}a)^2)$, apply the gradient inequality to $v = P_{t-s}a$, and integrate $0 < s < t$. This gives the displayed formula for compactly supported smooth data. On a complete manifold with Ricci curvature bounded below the heat semigroup is conservative. Smooth compact cutoffs of a bounded smooth function converge under P_t locally with first derivatives for $t > 0$, by bounded convergence for the heat kernel and interior parabolic estimates. This gives the same formula here. This semigroup interpolation is also given in Kopfer and Sturm, Section 2.1.

For a bounded real eigenfunction of $-\Delta$ of eigenvalue λ one has $P_t a = e^{-t\lambda}a$. For example, apply Itô's formula to $e^{s\lambda}a(B_s)$ for the nonexplosive diffusion with generator Δ. The stopped local martingale is bounded on every finite time interval, so passage to the limit preserves its expectation. Take $t = (1 + \lambda + K)^{-1}$. Convexity of e^{-2s} on $[0,1]$ gives $1 - e^{-2Kt} \geq (1 - e^{-2})Kt$. Hence $\|\nabla a\|_\infty \leq 3\sqrt{1 + \lambda + K}\,\|a\|_\infty$. For a complex row take the real part after multiplication by a unit scalar chosen to make its derivative in a given direction real. The bound for λ gives

$3\sqrt{1+\lambda+K} \le 40\sqrt{r}(B+2r)$. Integrate along the exponential path at one place and vary the places successively. This proves the assertion. □

84.6 A Sobolev bound for both terminal-period transfers

Fix an edge from i to $i+1$ in the construction of Theorem 67.5, with $i \ge 2$. Fix the input cusp forms B and A on the horizontal source and the next core, respectively. Keep a single allowed original finite test and, on a Jacobi edge, its distinguished line test. Write $P(\phi)$ for the outgoing complex period as a functional of the real horizontal test in the original polarization. In real position coordinates at all infinite places together let d be the dimension, h_ν the normalized product Hermite functions, and $\mathcal{N}$ the number operator with eigenvalue $|\nu| = \sum \nu_j$.

Proposition 84.6 (An explicit order for the period functional)**.** *With $s = 2d+2$, for every real Schwartz test,*

$$|P(\phi)| \le M\|(1+\mathcal{N})^s \phi\|_2.$$

One may take M to be the finite lattice sums and integrals of the absolute input forms given below. The order s is independent of their weights, the finite levels and all previous choices of Hermite degrees. For a prescribed finite list of marked terms one may use the maximum of their constants.

Proof. Use the defining integral before unfolding, as in Propositions 66.4 and 62.6. Its variables are g on the horizontal source, h on the core, and n on the target flag radical. Apart from unit characters its integrand is the product of $B(g)A(h)$, the horizontal theta at (g, nh), and, on a Jacobi edge, the conjugate line theta. Write $Q(n,h)$ for the absolute value of the last factor, and put $Q = 1$ on a Bessel edge. The flag quotient has a compact fundamental set. For the two reductive quotients take finitely many Siegel sets and representatives of the finite components at the given compact open levels. The finite arguments of the horizontal oscillator then lie in a finite union of compact sets. On such sets its action on a fixed finite test takes only finitely many values, because the test has an open stabilizer. Subdivide the integration cover accordingly into pieces $\mathcal{D}_\lambda$. On a piece the translated finite test is $\phi_{f,\lambda}$. Put

$$H_\lambda = \sum_{y \text{ rational position}} |\phi_{f,\lambda}(y)|(1+|y_\infty|)^{-(d+1)}.$$

It is finite. In fixed rational coordinates the finite support lies inside a common inverse denominator times the integral lattice. The number of points of that real lattice in a ball of radius $R \ge 1$ is $O_\lambda(R^d)$, by packing disjoint balls. Summing over dyadic shells proves convergence.

Let $S(g, nh)$ be the real symplectic matrix acting on the horizontal oscillator and set $W = 1 + \|S(g,nh)\|_{\mathrm{op}} + \|S(g,nh)^{-1}\|_{\mathrm{op}}$. There is a constant c_d in normalized real oscillator coordinates for which the following choice follows from the estimate below.

$$M = c_d \sum_\lambda H_\lambda \int_{\mathcal{D}_\lambda} |B(g)A(h)|\, Q(n,h) W^{2d+2}\, dg\, dh\, dn.$$

Fixed coordinate changes can equivalently be included in the matrix norms and in H_λ. The integrals are finite. On the compact unipotent set the matrix norms have polynomial growth on the two Siegel sets. The fixed line theta has polynomial growth there by its lattice sum, as in Proposition 62.6. The two smooth compact-finite cusp forms have rapid decrease in the respective unbounded chamber variables. These bounds also justify using an overlapping integration cover to majorize the absolute period.

It remains to prove the real estimate leading to this formula. Put $a = \lfloor d/2 \rfloor + 1$. By Fourier inversion and Cauchy–Schwarz, the sup norm of a Schwartz function is bounded by

a dimension-dependent constant times the sum of its L^2 derivative norms through order a. Indeed $(1+|\xi|^2)^{-a}$ is integrable on $\mathbb{R}^d$. Apply this estimate to $x^\gamma f$ for $|\gamma| \le d+1$. Expanding $(1+\sum |x_k|)^{d+1}$ bounds $\sup_x (1+|x|)^{d+1}|f(x)|$ by a fixed sum of L^2 norms of words of length at most $d+1+a \le 2d+2$ in coordinates and derivatives.

Take $f = \omega(S)\phi$. Moving a word through $\omega(S)$ replaces each of its generators by a linear combination of the $2d$ generators, with coefficients bounded by a fixed multiple of W. This follows by differentiating the covariance of the Schrödinger representation. Each resulting word is a fixed combination of words in creation and annihilation operators. A word of length u sends h_ν to a single shifted basis element times a scalar of modulus at most $(1+|\nu|+u)^{u/2}$, or to zero. Distinct surviving indices remain distinct under that fixed shift. Parseval thus bounds its norm on ϕ by a dimension-dependent constant times $\|(1+\mathcal{N})^s\phi\|_2$ for $u \le s$. Combining the finitely many estimates defines c_d and gives the real supremum bound with the additional factor W^s. The horizontal rational theta sum on each piece is at most H_λ times this supremum. Integration proves the proposition. □

84.7 A compactly supported operator fixing the incoming row

Let G be a product of the real unitary groups of a horizontal row, K a maximal compact subgroup and $g = \dim_{\mathbb{R}} G$. On its Lie algebra take the invariant real trace form on each matrix factor. It is negative on $\mathfrak{k}$, including the center, and positive on the noncompact Cartan complement $\mathfrak{p}$. Reversing the sign on $\mathfrak{k}$ gives a positive inner product. Use its left-invariant, right-K-invariant metric and its Haar volume. For a different fixed Haar measure rescale convolution densities accordingly. Let Δ be the nonnegative Laplacian. For an orthonormal basis adapted to the two summands the associated left-invariant vector fields give

$$\Delta = -\sum_j X_j^2 = -\Omega_G + 2C_K, \qquad C_K = -\sum_{X_j \in \mathfrak{k}} X_j^2.$$

Indeed unimodularity makes each first-order divergence zero. Here Ω_G is the Casimir for the signed trace form.

Proposition 84.7 (Compact convolution fixing finitely many types)**.** *Let a unitary representation have a fixed infinitesimal character on its smooth vectors, or be a Hilbert sum of copies with that character. For a finite nonempty list $\mathcal{T}$ of its K-types write $\lambda_\tau = -\Omega_G(\pi) + 2C_K(\tau)$. Put $M = \sum_{\tau\in\mathcal{T}} \dim \tau$, $H = 1 + \max_{\tau\in\mathcal{T}} \sqrt{\lambda_\tau}$. There is a smooth b supported in the fixed compact set $\overline{B(1,1)}K$ such that $\pi(b)$ is the identity on the listed types. For each integer $l \ge 0$,*

$$\|b\|_{C^l} \le C_{G,l} M H^{2(l+2g+2)}.$$

Proof. On a smooth vector v of type τ, $-\sum d\pi(X_j)^2 v = \lambda_\tau v$. Pairing with v in the unitary inner product proves $\lambda_\tau \ge 0$. Choose a fixed even nonnegative smooth function of integral one supported in $(-1,1)$. Its cosine transform $m(z)$ is real, even and rapidly decreasing, and $m(0) = 1$. Choose $0 < c \le 1$ such that $m(z) \ge 1/2$ for $|z| \le c$. Let $R = c/H$ and $k_R = m(R\sqrt{\Delta})\delta_1$. The metric is complete. By the finite propagation theorem for the wave equation the kernel here is supported in $\overline{B(1,R)}$. Indeed the multiplier is the integral of the cosine wave operators for times of absolute value at most R. The kernel is real by this formula. Cheeger, Gromov and Taylor, Section 1.

We detail its derivative estimate. Local elliptic regularity on nested coordinate balls about the identity and Sobolev point evaluation give $|u(1)| \le C\|(1+\Delta)^{g+1}u\|_2$ for smooth compactly supported u. The same inequality on the operator domain follows by local regularity there, bounding every lower power in L^2 by the highest shifted power. Hence $(1+\Delta)^{-(g+1)}\delta_1 \in L^2$, by

duality. On the fixed compact ball of radius two, the same interior estimates followed by Sobolev embedding bound the C^l norm of a function supported in the unit ball by $C_l\|(1+\Delta)^{l+g+1}u\|_2$. By the spectral theorem, with $j = l + 2g + 2$,

$$\|(1+\Delta)^{l+g+1}k_R\|_2 \le \|(1+\Delta)^{-(g+1)}\delta_1\|_2 \sup_{z\ge 0}(1+z^2)^j|m(Rz)| \le C_l R^{-2j}.$$

This also proves smoothness by taking all l.

For a smooth v of type τ and any smooth dual Hilbert vector w, $F(y) = \langle \pi(y)v, w\rangle$, linear in v, satisfies $\Delta F = \lambda_\tau F$. The wave solution with initial data $(F, 0)$ is locally $\cos(t\sqrt{\lambda_\tau})F$. For its value at the identity through time R, one may first cut off F to compact support with the cutoff equal to one near the closed radius-R ball. Uniqueness and finite propagation give the stated solution at that point. Integrating in time and using the real self-adjoint kernel gives $\int_G k_R(y)F(y)\,dy = m(R\sqrt{\lambda_\tau})F(1)$. It follows that $\pi(k_R)$ multiplies the listed type by this positive scalar. Approximation extends the conclusion to its full Hilbert isotypic space, since smooth vectors projected onto a compact type are dense there.

With dk of mass one on K, take the measure

$$d\mu(k) = \sum_{\tau\in\mathcal{T}} \frac{(\dim\tau)\overline{\chi_\tau(k)}}{m(R\sqrt{\lambda_\tau})}\,dk.$$

Schur orthogonality gives the inverse scalar on each listed type. It also gives $\int_K |\chi_\tau|\,dk \le 1$ by Cauchy–Schwarz, so $\|\mu\| \le 2M$. The right convolution $b = k_R * \mu$ has the required support and fixes the listed types. Right translations by K have uniformly bounded derivatives of every fixed order in a left-invariant frame. The estimate for k_R and $R = c/H$ give the asserted bound. □

84.8 An exponential moment for the initial Hermite expansion

On the first even core oscillator at place τ, let $D = n(n+d)$ be the real position dimension, L_τ the energy bound for ϕ_τ, and $W(a_\tau) \ge 1$ the matrix norm of the real translate in Lemma 84.10. Put

$$S_\tau = (1+16D^3)W(a_\tau)^2, \qquad \eta_\tau = \frac{1}{4\mathrm{e}S_\tau(L_\tau+3)}, \qquad \eta = \min_\tau \eta_\tau.$$

Here e is the base of natural logarithms. The indices below run over all product Hermites for the orthogonal sum of the bottom row and its positive line at every place.

Lemma 84.8 (Exponential coefficient bound after one real translation)**.** *Let c_α be the coefficients of the single Schwartz test $\omega(a)\phi^0$ of Lemma 59.5. With total degree $|\alpha| = \sum_\tau |\alpha_\tau|$,*

$$\sum_\alpha |c_\alpha| \exp(\eta|\alpha|) \;\le\; S, \qquad S = \prod_\tau 2(1-\exp(-2\eta_\tau))^{-D/2}\|\phi_\tau\|_2.$$

In this bound each $W(a_\tau)$ may be replaced by an upper bound fixed in advance, using the resulting smaller η_τ throughout.

Proof. Fix a place, and write $y = \omega(a_\tau)\phi^0_\tau$. For every integer $j \ge 1$,

$$\|(1+\mathcal{N})^j y\|_2 \le [S_\tau(L_\tau + 2j + 1)]^j\|\phi_\tau\|_2.$$

Indeed, in normalized creation and annihilation operators, the conjugate of each generator by $\omega(a_\tau)$ has sum of absolute linear coefficients at most $4DW(a_\tau)$. This is the covariance computation of Lemma 84.10. Expand $(1 + \sum_l a_l^* a_l)^j$ after substitution. The sum of the

absolute word coefficients is at most S_τ^j. The words have length at most $2j$. Each step changes the energy by at most one, with operator norm at most the square root of one plus the current energy bound. Their norms on ϕ_τ^0 are therefore bounded by $(L_\tau + 2j + 1)^j \|\phi_\tau\|_2$.

For $j \geq 1$, $L_\tau + 2j + 1 \leq (L_\tau + 3)j$ and $j! \geq (j/\mathrm{e})^j$. Consequently the series with terms $(2\eta_\tau)^j (1+\mathcal{N})^j y/j!$ converges in L^2 with sum of norms at most $2\|\phi_\tau\|_2$, including the constant term. Each Hermite coordinate of the sum is that of y times $\exp(2\eta_\tau(1 + |\alpha_\tau|))$, by the scalar exponential series. Cauchy–Schwarz now gives

$$\sum_{\alpha_\tau} e^{\eta_\tau |\alpha_\tau|} |\langle h_{\alpha_\tau}, y\rangle| \leq (1 - e^{-2\eta_\tau})^{-D/2}\, 2\|\phi_\tau\|_2.$$

The squared numerical factor before the 2 is the product of D geometric series. The archimedean test is a pure tensor, so its absolute coefficients multiply. Multiplying these local inequalities and using $\eta \leq \eta_\tau$ proves the lemma. □

84.9 Uniform bounds for a discrete-series matrix coefficient

Fix $n \geq 2$, $B \geq 1$, and put $T_0 = 10^4 n^4 (B+1)$, $D = 4n^2 + 2$, $s = 2n + 5$, $L = s + 2D$. For $U(p,q)$ of rank $m = p + q \leq n$ write $W(h) = \|h\|_{\mathrm{op}}$ in signed orthonormal coordinates and $U_\delta = \{h : \|h - 1\|_{\mathrm{HS}} \leq \delta\}$. Here $W(h^{-1}) = W(h) \geq 1$. For a discrete series ρ let d_ρ be its formal degree with respect to the indicated Haar measure. Constants denoted by C below are absolute and may increase between inequalities.

Theorem 84.9 (Power decay and Haar bounds with uniform constants)**.** *Suppose the Harish-Chandra coordinates of ρ have absolute value at most B and consecutive gaps greater than $24n^2 + 2$. For unit vectors a, b in the lowest compact type one has*

$$|\langle \rho(h) a, b\rangle| \leq T_0^{Cn^3} W(h)^{-L}.$$

For $S \geq 1$ the same parameters satisfy

$$d_\rho \,\mathrm{vol}\{W \leq S\} \leq T_0^{Cn^2} S^{4m^2},$$
$$d_\rho \int W(h)^{-D}\, dh \leq T_0^{Cn^2}, \qquad d_\rho \,\mathrm{vol}(U_{1/4}) \geq T_0^{-Cn^3}.$$

Proof. We track the constants in translation by the dual defining representation $V = (\mathbb{C}^m)^*$. Write e^l for its coordinate covectors, E_{ab} for complex matrix units, and λ for the current Harish-Chandra parameter. For one translation the coherent character identity used in Lemma 64.19 gives the discrete series with parameters $\lambda - e_i$, each once. All shifts here are in the same regular chamber. Their infinitesimal characters are distinct. Consequently the center splits the finite-length tensor module into these irreducible constituents. On smooth globalizations this is also a direct sum, by the Casselman–Wallach globalization theorem. We use this observation for translated smooth vectors as well as compact-finite vectors. After at most $L < 24n^2$ shifts a difference of two parameter entries has changed by at most L, so each translation used below satisfies the same chamber condition.

On this tensor use the invariant Hermitian form $[\ ,\ ]$ given by the Hilbert form of ρ times the signed form on V. The difference of the tensor Casimir and the two factor Casimirs, divided by two, is the split Casimir A. For the trace form its matrix on vectors with m components is $A_{ab} = -d\rho(E_{ab})$, since the dual action of E_{ba} is $-E_{ab}$. The Casimir eigenvalue on parameter ν is $\sum \nu_i^2 - \sum \rho_{G,i}^2$, with ρ_G the half-sum of positive roots. On the dual defining representation its value is m (equivalently $\sum E_{ab}E_{ba}$ acts by m). Thus on summand i the eigenvalue of A is $\xi_i = -\lambda_i - (m-1)/2$, and its projector is

$$P_i = \prod_{j \neq i} \frac{A - \xi_j}{\xi_i - \xi_j}.$$

The denominators have absolute value at least one. The split Casimir is self-adjoint for [,], since each Casimir is a signed sum of squares of infinitesimal generators of an isometric action. The real distinct eigenvalues give orthogonal summands. On each, the nonzero restricted form is a real multiple of the unitary form. Uniqueness follows by Schur's lemma on the irreducible Harish-Chandra module and then by density on smooth vectors. It is nonzero by orthogonality and nondegeneracy of the full form. Normalize the Hilbert norm on this summand so that its square is $|[x,x]|$. In particular

$$\|P_i(a\otimes e^l)\|_i^2 = |[P_i(a\otimes e^l), a\otimes e^l]|.$$

The absolute value on the right is at most $\|a\|$ times the ordinary product-space Hilbert norm of $P_i(a\otimes e^l)$ in the parent representation. This estimate avoids any choice of norm for a translation intertwiner.

Here are bounds for the differential operators in this expression. In any intermediate unitary constituent the coordinates of its parameter are at most $B+L$ in absolute value. The compact highest-weight coordinates meeting our vectors are at most $B+n+L$ in absolute value. Initially this follows from the lowest-type formula in Proposition 84.5. Tensoring with a defining dual changes the compact bound by at most one, since all weights of a unitary compact type lie in the convex hull of the Weyl orbit of its highest weight. The projections preserve compact types. Applying one Lie algebra element increases the bound by at most one as well, by the compact action on the matrix units in the adjoint representation. On the sum of the resulting types after at most n derivatives, the Casimir identity $-\Delta = 2\Omega_K - \Omega$ in Proposition 84.5 gives

$$\sum_X \|d\rho(X)a\|^2 \le 20n(B+4n+2L)^2\|a\|^2$$

for a real orthonormal Lie algebra basis X. The Casimir calculation applies on each compact isotypic space and these spaces are orthogonal. A complex matrix splits as $X+iY$ in the real form, with both norms at most its Hilbert–Schmidt norm, since real conjugation $Z\mapsto -JZ^*J$ is isometric. Successive applications show that any word of length $j\le n$ in matrices of Hilbert–Schmidt norm at most one acts on the starting vector with norm at most $T_0^{2j}\|a\|$.

More generally put $J_m(a)$ equal to the maximum of the Hilbert norms of the matrix-unit words of length at most $m-1$ applied to a, including a itself. Expanding the projector gives

$$\|P_i(a\otimes e^l)\|_i^2 \le T_0^{Cn}\|a\|J_m(a).$$

Indeed each multiplication by $A-\xi_j$ contributes at most $m+1$ choices to a matrix entry, one derivative or a scalar of size at most $B+L+n$. Taking the product-space norm adds at most $\sqrt{m}$. This proof of the last bound works for every smooth a. For our compact types the preceding word estimate bounds its right side by $T_0^{Cn}\|a\|^2$.

To obtain decay, multiply the coefficient by an entry $(h^{-1})_{la}$. Up to a sign this gives the coefficient for the signed form between the tensor vectors with covectors e^l, e^a. Decomposing them gives a sum of at most m signed unitary coefficients, with each pair of vector norms growing in product by at most T_0^{Cn}. Repeat L times using the same indices. The number and the norms of all terms together are bounded by $(mT_0^{Cn})^L$. At least one entry of h^{-1} has absolute value at least $W(h)/m$. Since $L=O(n^2)$, this proves the first assertion.

For a lowest unit vector a and $h\in U_\delta$, take the matrix logarithm at the identity. For $\delta\le 1/4$ it has Hilbert–Schmidt norm at most 2δ, by its power series, and lies in the real Lie algebra by uniqueness of the logarithm near one and $Jh^*J=h^{-1}$. With $\delta=(8T_0^2)^{-1}$, integrating the first derivative along this exponential gives $|\langle\rho(h)a,a\rangle|\ge 1/2$. Formal degree orthogonality therefore gives $d_\rho\,\mathrm{vol}(U_\delta)\le 4$. For any $0<\delta\le 1/4$ the ball $W\le S$ is covered

by at most $(1+2\sqrt{m}S^2/\delta)^{2m^2}$ left translates of U_δ. Take a maximal δ/S-separated subset in the ambient Hilbert–Schmidt norm and compare balls of half that radius in the real $2m^2$-dimensional matrix space. A center g has $W(g^{-1}) \le S$, so distances at most δ/S give the claimed translate. The small choice of δ proves the volume bound. On shells $2^j \le W < 2^{j+1}$ their weighted contributions to the integral of W^{-D} decrease by a factor at least 2^{D-4m^2} in the resulting geometric majorant. This gives the second estimate. Finally use the same cover with $\delta = 1/4$ and the first assertion for $a = b$ to obtain $1/d_\rho = \int |\langle \rho(h)a, a\rangle|^2 dh \le T_0^{Cn^3}\,\mathrm{vol}(U_{1/4})$, by summing the shells with exponent $2L > 4m^2+1$. This proves the lower bound. □

84.10 Absolute Hermite coefficients of a real translate

Consider the first even core edge C_0, C_1, of ranks $n, n+d$ with $d = n+4$. At a real place its symplectic space is the underlying real space of a complex matrix space of dimension $D = n(n+d)$. The Schrödinger position space has real dimension D. Use signed orthonormal Hermitian bases on both groups. Choose the corresponding positive oscillator structure respecting $C_0 = Y \perp Ez$, and the orthonormal product Hermite basis $(h_\alpha)_{\alpha\in\mathbb{N}^D}$ in these coordinates. Write $\mathcal{N}h_\alpha = |\alpha|h_\alpha$ for the number operator and put

$$s = \lfloor D/2 \rfloor + 1, \qquad A(D,L) = \big[(1+16D^3)(L+2s+1)\big]^s.$$

Here the energy of a polynomial Fock vector is its largest nonzero Hermite degree.

Lemma 84.10 (A summable bound on the first real oscillator)**.** *Let ϕ have energy at most L and let $G = U(C_0)_\tau$. For $W(g) = \|g\|_{\mathrm{op}}$ in the defining signed coordinates,*

$$\sum_\alpha |\langle h_\alpha, \omega(g)\phi\rangle| \le \sqrt{\zeta(2)}\,\|(1+\mathcal{N})^s\omega(g)\phi\|_2 \le \sqrt{\zeta(2)}\,A(D,L)W(g)^{2s}\|\phi\|_2.$$

For the bottom core discrete series π, every compact-finite unit vector u has finite weighted coefficient integral

$$M_\tau(u) = d_\pi \int_G |\langle u, \pi(g)u\rangle| W(g)^{2s}\,dg.$$

Here d_π is the formal degree in the chosen measure. If u belongs to the lowest compact type, then $M_\tau(u) \le T_0^{Cn^3}$, with T_0 of Theorem 84.9 and an absolute C.

Proof. For degree j there are at most $(1+j)^{D-1}$ indices, since the first $D-1$ entries determine the last. As $2s \ge D+1$, the sum of $(1+|\alpha|)^{-2s}$ is at most $\zeta(2)$. Cauchy–Schwarz with these weights proves the first inequality.

Use the usual creation and annihilation operators a_i^*, a_i in the Hermite basis, so $\mathcal{N} = \sum_i a_i^* a_i$. Conjugation by $\omega(g)$ substitutes the real linear action of g or its inverse on the Heisenberg generators. This follows by differentiating the defining covariance of the Weil representation on each one-parameter Heisenberg subgroup. In the Hilbert–Schmidt metric on the matrix space, composition on the C_0 factor has norm at most $W(g)$, since $\|g^{-1}\|_{\mathrm{op}} = W(g)$. The common positive scalar in the symplectic form scales the two orthonormal Heisenberg coordinates together. In the normalized complex combinations defining a_i, a_i^* each coefficient of the substitution is therefore bounded by $2W(g)$. Circle factors in a group splitting disappear in conjugation.

Write $b_i = \omega(g)^{-1} a_i \omega(g)$. Each has sum of absolute linear coefficients at most $4DW(g)$. Expanding $(1+\sum_i b_i^* b_i)^s$ gives words of length at most $2s$ in a_i, a_i^*, with sum of absolute coefficients at most $(1+16D^3)^s W(g)^{2s}$, since $W(g) \ge 1$. On the space of energies at most j each individual operator has norm at most $\sqrt{j+1}$, by its action on the Hermite basis. The norm of each word on ϕ is at most $(L+2s+1)^s\|\phi\|_2$. Unitarity and the triangle inequality prove the second bound.

The core gaps exceed $24n^2+2$ in the proof of Theorem 59.10. Since $2s \le 2n^2+4n+2$, the integer $2s+4n^2+1$ is below those gaps for $n \ge 2$. Use this exponent for the coefficient in Lemma 64.19 and integrate the remaining power $W^{-(4n^2+1)}$ by its Haar bound. This proves finiteness for every compact-finite u. For a unit vector in the lowest type, Theorem 84.9 gives decay with exponent $8n^2+2n+9$ and prefactor $T_0^{Cn^3}$. After subtracting $2s$ the exponent is still at least $4n^2+2$. The formal-degree Haar bound in the same theorem therefore gives the stated uniform estimate. □

84.11 A matrix-norm moment for two smooth vectors

Keep the notation T_0, W, J_m, d_ρ of Theorem 84.9. The moment below permits one vector to be the projection of a smooth vector from a larger unitary representation. Such a projection can have infinitely many compact types.

Proposition 84.11 (A second moment for the full real matrix norm)**.** *Under the parameter hypotheses of Theorem 84.9, for all smooth vectors a, b in ρ,*

$$d_\rho \int_{U(p,q)} \|h\|_{\mathrm{HS}}^2 |\langle \rho(h)a, b\rangle|^2 \, dh \le T_0^{Cn} J_m(a)\|a\| J_m(b)\|b\|.$$

The constant is absolute, including for compact signatures.

Proof. For each pair of indices l, k, use the smooth decomposition and the signed invariant form of Theorem 84.9. Apart from the sign of the k-th coordinate,

$$(h^{-1})_{lk}\langle \rho(h)a, b\rangle = \sum_i \epsilon_i \langle \rho_i(h) P_i(a \otimes e^l), P_i(b \otimes e^k)\rangle_i,$$

where the sign on a summand is ϵ_i. The right side is an identity of smooth coefficients in the corresponding globalization. Write d_i for the formal degree of summand i. The formal degree formula on this fixed group is a common positive Haar factor times the absolute Vandermonde product of the Harish-Chandra entries. On lowering λ_i by one, each of its $m-1$ Vandermonde factors changes by a ratio at least $1/2$, because each original difference has absolute value at least two. Thus $d_\rho/d_i \le 2^{m-1}$. The Cauchy–Schwarz inequality for the sum over i, followed by formal degree orthogonality in each ρ_i, bounds the integral of the absolute square by

$$m \sum_i d_i^{-1} \|P_i(a \otimes e^l)\|_i^2 \|P_i(b \otimes e^k)\|_i^2.$$

Use the smooth-vector projector estimate of Theorem 84.9 and sum over l, k. The factor $m^4 2^{m-1}$ is absorbed by T_0^{Cn}. Since $h^{-1} = Jh^*J$, the sum of the squared matrix entries of h^{-1} is exactly $\|h\|_{\mathrm{HS}}^2$. □

84.12 A finite Hermite cutoff along the horizontal chain

Keep one of the fields in Theorem 67.5 and the same compact-finite input forms on an edge $i \ge 2$. Use the real section through the identity of Lemma 83.1. In local coordinates $y = 0$ there write $J(y)$ for the scalar fiber integral before insertion of the first-position cutoff, as in Theorems 66.16 and 66.14. Then $J(0)$ is the nonzero preceding period on an odd-index edge and the preceding period times $\|q\|_2^2$ on an even-index edge. Write a for the real dimension of the section. Choose a closed coordinate ball of radius $0 < \rho_0 \le 1$ inside it. Put

$$L = \sup_{|y| \le \rho_0} \|\nabla J(y)\|, \qquad \rho = \min\{\rho_0, \ |J(0)|/[2(1+L)]\}.$$

Let c_f, v_f be the given finite base scalar and orbit measure on this edge, and T the number of original marked terms in its finite partition. Put $\eta = c_f v_f |J(0)|/(2T)$. These three finite quantities are those in the corresponding transfer theorem.

Theorem 84.12 (Degree and size of an individual real test)**.** *Let d, s, M be as in Proposition 84.6, with the maximum taken over the marked terms of this edge. There are finite positive chart and test constants C_0, C_{s+1} described below. Set*

$$K = \max\left\{1, \left\lceil \frac{2MC_{s+1}}{\eta\, \rho^{\,a/2+2(s+1)}} \right\rceil \right\}.$$

One original marked term and one normalized product Hermite test h_ν of total degree $|\nu| \le K$ satisfy

$$|P_{i+1}(h_\nu)| \ge \frac{\eta\, \rho^{a/2}}{2C_0 \binom{K+d}{d}^{1/2}}.$$

This estimate can be applied successively from the nonzero second restriction through the prescribed raw terminal Bessel period, retaining the same terminal integral coefficient.

Proof. The function J is continuously differentiable on the closed coordinate ball. A derivative acts on the smoothly translated row by a first Lie algebra derivative, or on the translated residual oscillator test. On a compact section these operations use bounded combinations of differentiated compact-finite cusp forms and bounded Schwartz seminorms. The differentiated cusp forms still have rapid decrease. The integrable chamber majorants in the proofs of Propositions 66.4 and 62.6 therefore apply after differentiation, giving the assertion by dominated convergence. In particular L is finite. Throughout the ball of radius ρ the difference $|J(y) - J(0)|$ is at most $|J(0)|/2$.

We record the real seminorms in the localization. Choose once a nonnegative smooth product bump of integral one supported inside the open unit ball in the section coordinates. Dilate it by ρ and divide by the smooth positive quotient-density factor, giving it quotient integral one. Write Φ_ρ for its ambient horizontal Schwartz test. In the orbit chart (y, u) used in Lemma 83.1, multiply the section bump by the fixed upper-group bump and the prescribed character in u. Use upper support inside a fixed compact coordinate patch. Extend to a tubular chart by a fixed transverse bump of value one at zero. The core, line and opposite-position factors in the remaining coordinates are fixed Schwartz functions. One may shrink ρ_0 at the outset to fit the patch and use a finite family of such charts. All coordinate maps and their required derivatives are bounded on the supports in the frame coordinates.

For every fixed integer $j \ge 0$ this construction gives

$$\|(1 + \mathcal{N})^j \Phi_\rho\|_2 \le C_j\, \rho^{-a/2-2j}.$$

Indeed the section bump and its derivatives of order v cost $O(\rho^{-a-v})$, by the chain rule and the fixed density factor. The volume of the section support is $O(\rho^a)$. The operator $(1 + \mathcal{N})^j$ is a sum of polynomial-coefficient differential words of length at most $2j$. The other frame variables stay on fixed compact supports and the weighted residual derivatives are square integrable. Taking L^2 norms gives the indicated power. Equivalently C_j can be taken as the sum of the resulting coordinate derivative and Jacobian bounds times the weighted norms of the fixed factors. A fixed metaplectic change to the original polarization replaces each word by a combination of words of the same length, by the covariance used in Proposition 84.6. Include its matrix coefficients in C_j. These constants are independent of the chosen radius.

The two orbital transfer formulas now give modulus at least $c_f v_f |J(0)|/2$ before their finite partition, by integrating against this nonnegative bump. Some individual marked term has

modulus at least η, since the partition coefficients have modulus one. Fix it. For the spectral projection E_K onto all Hermite degrees at most K, Parseval gives

$$|P_{i+1}((1-E_K)\Phi_\rho)| \le \frac{M}{K+1}\|(1+\mathcal{N})^{s+1}\Phi_\rho\|_2 \le \eta/2.$$

Hence the period on $E_K\Phi_\rho$ has modulus at least $\eta/2$. There are $\binom{K+d}{d}$ multi-indices in this range, by the count of weak compositions with a slack coordinate. Cauchy–Schwarz bounds the sum of the absolute Hermite coefficients by the square root of this number times $\|\Phi_\rho\|_2$. This proves the stated individual bound.

The selected output is compact-finite and cuspidal by the first-occurrence statement of Theorem 61.1, so it is an admissible input to the next edge. For the final raw character, apply the rational involution of Lemma 66.10 to the smooth final test *before* its Hermite projection. It preserves the absolute period, takes a marked coset to a marked coset, and at infinity is a fixed metaplectic action. Include this action in C_j and take M for the raw period. The same estimates now select a single test in the original terminal coordinates. This proves the last assertion by finite induction, starting with Theorem 62.4. □

The bounds use the input forms through $M, L, J(0)$ and the chosen charts and residual tests through C_j. Their growth along the varying chain, and the passage from this complex period to the integral projection of Corollary 74.7, remain to be controlled.

84.13 The real distribution of a terminal Bessel coefficient

Keep the unitary space of Section 99.16, with $r = k+2s+1$, and set $B = N \rtimes U(W_0)$. At each real place, sections for the frequency β transform by χ_β on N and by the column representation on $U(W_0)$. We use the invariant quotient measure on $B_\infty\backslash G_\infty$. The next formulas apply to nonnegative integrals, including infinite values.

Proposition 84.13 (Change of the terminal frequency and the height)**.** *At a single real place let a_β multiply the u_i by $\beta \in \mathbb{C}^\times$, the v_i by $\bar\beta^{-1}$, and fix the middle space pointwise. Put $J_\beta = |\beta|^{2s(r-s)}$. The operation*

$$W(g) \longmapsto J_\beta^{-1/2} W(a_\beta g)$$

is a right-equivariant isometry from square-integrable sections of frequency one to those of frequency β. For $\beta = \varpi_\tau$ simultaneously at all real places, write U_ϖ for its product. For every nonnegative Borel function b and every real seed section W,

$$\int b(H(g))\|(U_\varpi W)(g)\|^2\, d\dot g = \int b(\sqrt{T}H(g))\|W(g)\|^2\, d\dot g. \tag{84.1}$$

Here both sides use the same height of Section 64.3.

Proof. Conjugation by a_β fixes the first $s-1$ simple coordinates of N and multiplies its last coordinate by β. It centralizes $U(W_0)$. Thus left translation of the argument gives the indicated character. It commutes with every right translation. To compute the measure, use the decomposition $L \oplus W \oplus L^\vee$. In the Lie algebra of B, the compact block and the strictly upper triangular block on L are unchanged. The block from W to L has complex dimension $s(k+1)$ and is multiplied by β. Its opposite partner is determined by skew adjointness. The block from $L^\vee$ to L consists of skew-Hermitian s-by-s matrices of real dimension s^2, and is multiplied by $|\beta|^2$. Therefore the absolute determinant on the Lie algebra of B is $|\beta|^{2s(k+1)+2s^2} = J_\beta$.

Both groups are unimodular. For a compactly supported scalar function v on G, put $v_B(g) = \int_B v(bg)\, db$. In $v_B(a_\beta g)$ change $b = a_\beta b' a_\beta^{-1}$. Quotient integration and left invariance of Haar

measure on G give $\int v_B(a_\beta g)\,d\dot g = J_\beta \int v_B(g)\,d\dot g$. Local sections and a partition of unity give the same formula for every nonnegative function on the quotient. Finally, $(a_\beta g)^{-1}u_i = \beta^{-1}g^{-1}u_i$ for every i, so $h_\tau(a_\beta g) = |\beta|^{-1}h_\tau(g)$. All $|\varpi_\tau| = \sqrt{T}$. Combining these formulas and the scalar normalization proves (84.1). Replacing β by its inverse gives the inverse isometry. □

Two other quotient functions will locate a Bessel coefficient with large frequency. At each place use Euclidean distance to a complex subspace and put

$$A_\tau(g) = \operatorname{dist}(g_\tau^{-1}u_s, g_\tau^{-1}L_{s-1}), \qquad Z_\tau(g) = \operatorname{dist}(g_\tau^{-1}e, g_\tau^{-1}L).$$

Here $L_0 = 0$. Left multiplication by B_∞ preserves the corresponding affine spaces, so the functions descend. One has $0 < A_\tau \le h_\tau$ and $Z_\tau \ge 1$, the latter because $e + L$ has Hermitian norm -1. Define

$$H_{\mathrm{end}}(g) = \Big(\prod_\tau A_\tau(g)Z_\tau(g)\Big)^{1/f}.$$

Lemma 84.14 (Terminal derivatives on a region of the real quotient)**.** *Let W be a smooth section of frequency ϖ. At each place choose a real Frobenius-orthonormal basis of $\mathfrak{u}(k{+}s, s{+}1)$. For $j \ge 1$, let $\mathcal{I}_j$ consist of words I with j right derivatives at each place in these basis directions, in a fixed order of the places and in every order within each place. On any measurable region Ω contained in $H_{\mathrm{end}} \le U$,*

$$\|\mathbf{1}_\Omega W\|_2 \le \Big(\frac{U}{2\pi\sqrt{T}}\Big)^{fj}\left(\sum_{I\in\mathcal{I}_j}\|\mathbf{1}_\Omega D_I W\|_2^2\right)^{1/2}. \tag{84.2}$$

The inequality also holds in the seed norm, with all its finite class coordinates and the same fixed frequency.

Proof. Fix g and a real place. Choose $u' \in u_s{+}L_{s-1}$ and $e' \in e{+}L$ that realize the two displayed distances after applying g_τ^{-1}. Orthogonal projection gives the minimizing representatives. For a scalar c of modulus one define

$$Y(w) = c\,u'h(e', w) - \bar c\,e'h(u', w).$$

This is skew adjoint. It kills L, sends $L^\perp$ to L, and sends the whole space into $L^\perp$. It belongs to the Lie algebra of N since it acts as zero on every graded piece of the completed flag. Its first $s-1$ simple coordinates are zero, and the u_s coefficient of $Y(e)$ is $-c$. Choose the phase so that $-c\varpi_\tau = \sqrt{T}$. Then $\chi_{\varpi,\tau}(\exp(tY)) = \exp(4\pi i\sqrt{T}\,t)$ for real t, by the trace from $\mathbb{C}$ to $\mathbb{R}$.

Put $X = g_\tau^{-1}Yg_\tau$. The Frobenius norm of a rank-one map $vh(w,\cdot)$ is $\|v\|\,\|w\|$, since the signature matrix is Euclidean unitary. It follows that $\|X\|_{\mathrm{F}} \le 2A_\tau(g)Z_\tau(g)$. Freeze this X while differentiating at the given g. The identity $g_\tau\exp(tX) = \exp(tY)g_\tau$ gives the factor $(4\pi i\sqrt{T})^j$ on taking j derivatives. Apply this at all places, whose actions commute. Express the resulting product of derivatives in the fixed bases. The Euclidean norm of its coefficient vector is $\prod_\tau \|X_\tau\|_{\mathrm{F}}^j$, by expansion of the tensor powers. Cauchy–Schwarz gives the pointwise version of (84.2); integration gives the result. The coefficients are treated as fixed at each evaluation point, so this argument introduces no derivatives of the minimizing representatives. Summing the squared inequalities with the finite class inner product gives the last assertion. □

Theorem 84.15 (Projection on the automorphic quotient)**.** *Use a single terminal frequency ϖ and any finite set of the selected words and columns. Let $\mathcal{H}_\sigma$ be the subspace of automorphic L^2 in the isotypic space of a prescribed real discrete series σ and in its specified compact type. It is the image of the two commuting orthogonal isotypic projections. Let $L_* \ge 1$ bound the compact*

Lie norms and the square roots of the absolute Casimir scalars in Lemma 99.25 on this type. For $R > 0$ put $B(R) = 1$ when $R \le R_0$ and $B(R) = (1 + 16(R^2/Q_0)^f)^{rs}$ when $R > R_0$. For every seed supported in $H < R$, $H_{\mathrm{end}} \le U$, and every integer $j \ge 1$,

$$\| \operatorname{proj}_{\mathcal{H}_\sigma} P\Phi\| \le B(R) \left(\frac{Cr^2(L_* + fj + 1)U}{\sqrt{T}} \right)^{fj} \|\Phi\|_{\mathrm{seed}} \tag{84.3}$$

with an absolute C.

Proof. Take first a smooth vector F in a finite sum of the given copies. The functional $\Phi \mapsto \langle P\Phi, F\rangle$ on compact test seeds has a locally square-integrable representing section W_F in the seed inner product. Indeed restriction to any bounded height is bounded by Theorem 64.11. The Riesz representatives agree on overlaps by testing on sections there. On $H < R_0$ the summation map is isometric. Together these bounds give $\|\mathbf{1}_{H<R} W_F\|_{\mathrm{seed}} \le B(R)\|F\|$. Use complex inner products linear in the first argument, so that W_F has the same covariance as the test seeds.

For a compact test seed and real a, ordinary summation commutes with $R(a)$. Changing variables in the pairing therefore gives $R_B(a) W_F = W_{R_G(a)F}$ as distributions. Differentiation gives $D_I W_F = W_{D_I F}$. In particular all these weak derivatives are locally square-integrable, and on $H < R$ their norms are at most $B(R)\|D_I F\|$. One may use smooth representatives in Lemma 84.14: in a local bundle chart the right Lie directions span the tangent space, so all coordinate derivatives of each order are linear combinations with smooth coefficients of such words of at most that order. The local Sobolev embedding theorem in each finite order gives a smooth representative. The fixed finite class and word coordinates carry unitary fibre metrics and satisfy the same argument.

There are r^{2fj} words in $\mathcal{I}_j$, since the real dimension at each place is r^2. By Lemma 99.25, each $\|D_I F\|$ is at most $\{Cr(L_* + fj + 1)\}^{fj}\|F\|$. In (84.2) use $\Omega = \{H < R, H_{\mathrm{end}} \le U\}$ and the preceding derivative bounds. This gives $\|\mathbf{1}_\Omega W_F\|_{\mathrm{seed}}$ bounded by the factor in (84.3) times $\|F\|$. Pairing against Φ gives the assertion for such test vectors F. Finite sums from copies of σ in this compact type are dense in $\mathcal{H}_\sigma$, by the definition of the isotypic subspace and the bounded compact-type projector. Within each copy compact-finite vectors of the discrete series are smooth. The constant is uniform on these copies. Taking the supremum over their unit sphere gives (84.3). □

84.14 Consecutive frame lengths in the Bessel energy estimate

At a real place τ use the flag of Section 84.13 at g_τ. In this section set

$$v_i = g_\tau^{-1} u_i \ (1 \le i \le s), \qquad v_{s+1} = g_\tau^{-1}(\varpi_\tau e), \qquad V_i = \langle v_1, \dots, v_i\rangle, \qquad V_0 = 0.$$

Define $a_{i,\tau} = \operatorname{dist}(v_i, V_{i-1}) > 0$ for $1 \le i \le s+1$. These are functions of the real seed coset. The first s are its Gram–Schmidt lengths. The last is $\sqrt{T} Z_\tau \ge \sqrt{T}$. With the indices starting at zero in the archive, these are exactly the functions in `Formalism/SharpRestriction/CuspSpanNormBounds/RelevantZeroGramTerms.lean`, line 299. Put

$$M_\tau(g) = \max_{1 \le i \le s} \frac{a_{i+1,\tau}}{a_{i,\tau}}, \qquad M(g) = \left(\prod_\tau M_\tau(g) \right)^{1/f}.$$

Lemma 84.16 (A derivative for each consecutive pair)**.** *For each g_τ and each $i \le s$ there is a real unitary Lie direction X_i with $\|X_i\|_{\mathrm{F}} \le 2a_{i,\tau}/a_{i+1,\tau}$ such that every smooth Bessel section of frequency ϖ satisfies*

$$W(g_\tau \exp(tX_i)) = \exp(4\pi i t)\, W(g_\tau) \qquad (t \in \mathbb{R}).$$

Here the other arguments of the section are held fixed.

Proof. Write J for the signature matrix, so $h(y,w) = y^{*}Jw$, $J^{*} = J$, and $J^{2} = 1$. Let P_{V} denote Euclidean orthogonal projection. At this point and this index set

$$u = (1 - P_{V_{i-1}})v_i, \qquad t_0 = (1 - P_{V_i})v_{i+1}, \qquad w = Jt_0/\|t_0\|^2.$$

Thus $u \in v_i + V_{i-1}$, $\|u\| = a_{i,\tau}$, $h(w,V_i) = 0$, $h(w,v_{i+1}) = 1$, and $\|w\| = a_{i+1,\tau}^{-1}$. Define

$$X_i y = uh(w,y) - wh(u,y).$$

It is skew adjoint and has Frobenius norm at most $2\|u\|\|w\|$. Since $u \in V_s$ and V_s is isotropic, this map kills V_i, sends V_j to V_i for $i < j \le s$, and sends $V_s^{\perp_h}$ to V_s. By skew adjointness it also lowers the dual flag. Thus $g_\tau X_i g_\tau^{-1}$ lies in the Bessel unipotent Lie algebra. If $i < s$, the coefficient of v_i in $X_i v_{i+1}$ modulo V_{i-1} is one. The other simple coordinates, including the terminal one, are zero: in the later steps the image lies in the earlier V_i. If $i = s$, all the ordinary simple coordinates vanish and $X_s(\varpi_\tau g_\tau^{-1} e) = u$. The terminal coordinate relative to e is then ϖ_τ^{-1}, which is multiplied by ϖ_τ in the actual character. In both cases the character on the exponential of the conjugate is $\exp(4\pi i t)$. Left covariance proves the identity. The factor $a_{i+1,\tau}^{-1}$ in the norm comes from the vector representing the dual functional on the next quotient line; this calculation allows Z_τ to be arbitrarily large. □

Theorem 84.17 (Joint energy of the consecutive ratios)**.** *Let W be a smooth real section, possibly with finite class coordinates. For every $j \ge 1$, pointwise and hence on every measurable region Ω,*

$$(2\pi)^{fj}\|\mathbf{1}_\Omega M^{fj} W\|_2 \;\le\; \left(\sum_{I \in \mathcal{I}_j} \|\mathbf{1}_\Omega D_I W\|_2^2\right)^{1/2}. \tag{84.4}$$

Here $\mathcal{I}_j$ uses the fixed Frobenius bases in Lemma 84.14. In particular at a single place, in an isometrically realized square-integrable real Bessel model, a compact-type vector W with the derivative parameter L_* of Lemma 99.25 satisfies

$$\int M_\tau^{2j}\|W\|^2\, d\dot{g} \le \{Cr^2(L_* + j + 1)\}^{2j}\,\|W\|_2^2.$$

Proof. At a given point choose an index attaining the maximum at each real place. Freeze the resulting X_i and take j derivatives there, at all the places. Lemma 84.16 gives a factor $(4\pi i)^{fj}$. The coefficient vector of the resulting words in the fixed bases has Euclidean norm at most $\prod_\tau (2/M_\tau)^j = 2^{fj}/M^{fj}$. Pointwise Cauchy–Schwarz gives precisely the inequality, with the norms replaced by the pointwise fibre norms. The right hand side uses the same list of words at every point, including when the maximizing index changes. Squaring and integrating proves (84.4). For the one-place assertion there are r^{2j} words. The hypothesis on the realization transfers the Hilbert derivative estimate of Lemma 99.25 to each of them. This proves the moment bound. The consecutive ratios are the ones entering the energy estimate in `Formalism/SharpRestriction/CuspSpanNormBounds/BesselShellMomentBounds.lean`, line 1642. □

Corollary 84.18 (Uniformity in the terminal direction)**.** *In Theorem 84.15, suppose the seed is supported in $H < R$, $M \ge V$, where $V > 0$. Then*

$$\|\operatorname{proj}_{\mathcal{H}_\sigma} P\Phi\| \le B(R)\left(\frac{Cr^2(L_* + fj + 1)}{V}\right)^{fj}\|\Phi\|_{\mathrm{seed}}.$$

In fact at every point, if $h_i = (\prod_\tau a_{i,\tau})^{1/f}$ *and* $Z = (\prod_\tau Z_\tau)^{1/f}$*, one has*

$$M^{s-i+1} h_i \;\geq\; \sqrt{T}\, Z \qquad (1 \leq i \leq s).$$

In particular $M \geq \sqrt{T}\, Z/h_s \geq \sqrt{T}/H$*.*

Proof. For a unit test vector F in the required discrete series and compact type use its local representative W_F from Theorem 84.15. On $H < R$, the L^2 norm of $D_I W_F$ is bounded by $B(R)\|D_I F\|$. Equation (84.4) on the support region and the word count from that theorem give the bound on its adjoint. Duality gives the stated estimate. For the geometric assertion, multiply the consecutive ratios from i through s at one place. Their product is $a_{s+1,\tau}/a_{i,\tau}$; each is at most M_τ. Take the geometric mean over the places and use $Z_\tau \geq 1$ and $h_s \leq H$. □

84.15 The real Bessel norm and the formal degree

Fix one real place. Write $G = U(V)$, $C = U(W_0)$, and $B = N \rtimes C$ as in Section 84.13. The column space W_0 is definite, so C is compact. Let ρ be the prescribed irreducible representation of C on E_ρ and σ the discrete series of the row. Give C probability Haar measure. Choose Haar measures on N and G and use their quotient measures throughout. Write d_σ for the formal degree with respect to this choice on G. Inner products in this section are linear in the first argument.

Put $\mathcal{G} = G \times C$ and $\mathcal{H} = \{(nc, c) : n \in N,\ c \in C\}$. On $\mathcal{H}$ the character $\xi(nc, c) = \chi_\beta(n)$ is unitary. This is the real unitary Gan–Gross–Prasad Bessel pair. The larger space adds s hyperbolic planes and the line of e to the smaller space. Its character is obtained from the successive flag coordinates and the last pairing with e. The change to a nonzero terminal frequency β is the conjugation in Proposition 84.13. Changing the sign of both Hermitian forms, when needed for the sign convention on the line, preserves the unitary groups.

Lemma 84.19 (Averaging a Schwartz function over the period subgroup)**.** *Let* $\mathcal{C}(\mathcal{G})$ *be the Harish-Chandra Schwartz algebra. For* f *in this algebra set*

$$Qf(g) = \int_{\mathcal{H}} f(hg)\overline{\xi(h)}\, dh, \qquad f^*(g) = \overline{f(g^{-1})}.$$

Then $Qf \in L^2(\mathcal{H}\backslash\mathcal{G}, \xi)$*. With* $(f_1 * f_2)(h) = \int_{\mathcal{G}} f_1(y) f_2(y^{-1}h)\, dy$ *one has*

$$\langle Qf, Qf' \rangle = \int_{\mathcal{H}} (f * f'^*)(h)\overline{\xi(h)}\, dh. \tag{84.5}$$

Proof. For smooth compactly supported functions, expand the convolution as $\int_{\mathcal{G}} f(hg)\overline{f'(g)}\, dg$. Integrating in h first gives $Qf(g)$. Disintegrating dg along left $\mathcal{H}$-orbits and using $Qf(h_0 g) = \xi(h_0) Qf(g)$ gives the left side of (84.5). The integrations are absolutely convergent in this case by compact support.

For the passage to Schwartz functions we use the restriction estimate of Beuzart-Plessis for this Bessel pair. Lemma 6.5.1(ii) of [2] gives $\int_{\mathcal{H}} \Xi_{\mathcal{G}}(h)(1 + \ell(h))^{-a}\, dh < \infty$ for sufficiently large a, where ℓ is the logarithmic norm and $\Xi_{\mathcal{G}}$ is the elementary spherical function. Thus integration on $\mathcal{H}$ against $\overline{\xi}$ is continuous in a Schwartz seminorm. The same holds after any fixed right translation. The Harish-Chandra Schwartz algebra has continuous convolution and involution and contains C_c^∞ densely [2, Sections 1.5 and 2.6]. Approximate f by $f_j \in C_c^\infty$ there. Applying (84.5) to the difference $f_j - f_i$ shows that Qf_j is Cauchy in the quotient Hilbert space. The translated restriction estimate gives pointwise convergence to the displayed Qf. An almost everywhere convergent subsequence of the Hilbert space limit identifies the limits. Polarization and continuity now give (84.5). □

Theorem 84.20 (The normalized Bessel model of a discrete series)**.** *Suppose that the continuous Bessel functionals on the smooth representation $\pi = \sigma \boxtimes \rho^\vee$ form a line. There is an intertwining map on smooth row vectors $\Lambda : \sigma^\infty \to E_\rho$, unique up to a complex scalar of absolute value one under the normalization below, such that*

$$d_\sigma \int_{B\backslash G} \|\Lambda(\sigma(g)v)\|^2 \, d\dot{g} = \|v\|^2. \tag{84.6}$$

Its norm and phase convention up to this scalar are determined by the Haar measures and the matrix coefficient integral over $\mathcal{H}$.

Proof. For compact-finite smooth vectors a, b of π set $f_{a,b}(g) = \langle \pi(g)a, b\rangle$ and

$$E(a, b) = Qf_{a,b}(1).$$

The group $\mathcal{G}$ has compact center, and π is a discrete series. Its compact-finite matrix coefficients belong to $\mathcal{C}(\mathcal{G})$. Equivalently, in Harish-Chandra inversion the smoothing rank-one operators on this discrete representation are a Schwartz spectral component supported at its isolated atom. Inverting that component gives the coefficients, up to formal degree and inversion of the group argument [2, Theorem 2.6.1].

The continuous relative character on smooth vectors in [2, Sections 7.1–7.2] therefore agrees with the ordinary integral defining E on these vectors and on their translates. In the convention with $\pi(h)$ in the second argument, substitute h^{-1} for h. Theorem 7.2.1 there gives nonvanishing of this form because the continuous $(\mathcal{H}, \xi)$-functional is nonzero. The form is Hermitian by the same substitution. In each argument its covariance is that of the functional or its conjugate. Choosing a generator ℓ' of the line, we consequently have $E(a, b) = c\,\ell'(a)\overline{\ell'(b)}$ on smooth vectors with c real and nonzero. Indeed fixing the second argument first gives a multiple of ℓ', and Hermitian symmetry determines its dependence on that argument.

For compact-finite vectors v, u, Schur orthogonality on $\mathcal{G}$ gives

$$(f_{v,u} * f^*_{v,u})(h) = d_\pi^{-1}\|v\|^2 f_{u,u}(h).$$

To see the order of the entries, write the first factor in the integral as $\langle \pi(g)v, \pi(h)^{-1}u\rangle$ and the conjugate factor as $\overline{\langle \pi(g)v, u\rangle}$. Lemma 84.19 now gives $\|Qf_{v,u}\|^2 = d_\pi^{-1}\|v\|^2 E(u, u)$. Choose u with $\ell'(u) \neq 0$, possible by density of the compact-finite vectors in the smooth topology. Positivity forces $c > 0$. Put $\ell = \sqrt{c}\,\ell'$. At every point g one has

$$Qf_{v,u}(g) = E(\pi(g)v, u) = \ell(\pi(g)v)\overline{\ell(u)}.$$

Canceling $|\ell(u)|^2$ gives the square norm $d_\pi^{-1}\|v\|^2$ for the scalar function $g \mapsto \ell(\pi(g)v)$. Approximate an arbitrary smooth v by compact-finite vectors in its smooth topology. The identity gives convergence in L^2, and continuity of ℓ identifies the limit pointwise along a subsequence.

Define Λ by $\ell(v \otimes z) = z(\Lambda(v))$ for $z \in E_\rho^\vee$. It transforms by χ_β on N and by ρ on C. The quotient of the product pair identifies with $N\backslash G$ through $(g, c) \mapsto Nc^{-1}g$. With probability Haar measure on C, quotient integration first along N and then along C gives exactly the measures on $N\backslash G$ and $B\backslash G$ fixed above. For $v \otimes z$ the scalar square norm is thus

$$\int_{B\backslash G}\int_C |z(\rho(c)\Lambda(\sigma(g)v))|^2 \, dc\, d\dot{g} = \frac{\|z\|^2}{\dim \rho}\int_{B\backslash G} \|\Lambda(\sigma(g)v)\|^2 \, d\dot{g}.$$

The equality is compact Schur orthogonality. The product formal degree is $d_\pi = d_\sigma \dim \rho$, by the same orthogonality on the two factors. The scalar identity gives (84.6). The line of functionals and this norm fix the modulus of the scaling. □

Corollary 84.21 (Isometry at every real place of the period field)**.** *For every word on the row of Theorem 65.7, and for $\beta = 1$ or ϖ_τ, the map*

$$v \longmapsto \sqrt{d_{\sigma_\tau}}\,\Lambda_\tau(\sigma_\tau(\,\cdot\,)v)$$

extends to an isometric right-equivariant embedding in the local seed Hilbert space. Its tensor product over the real places retains this property. In Theorem 84.17 the local derivative moments therefore apply to these embeddings on the whole real quotient.

Proof. The cited packet calculation gives exactly the one-dimensional space required in Theorem 84.20, for each column and each nonzero terminal character. The formula commutes with right translations by the representation law and extends by density. Products of the local norm identities give the assertion on pure tensors, then on finite sums by polarization and on their completion. For a smooth input, differentiating the equivariance identity in the Hilbert norm transfers every derivative to the same isometric realization. This establishes the local norm condition at `Formalism/SharpRestriction/CuspSpanNormBounds/BesselShellMomentBounds.lean`, line 1625. □

85 Transverse pairs and higher-rank blocks

85.1 The large-height contribution for every single reversed block

For $2 \le j \le s$ let $\mathcal{B}_j$ consist of the relevant supported pairs of Proposition 78.7, counting each rational orbit once. Let $H(\xi) = \prod_\tau \max(1, |\tau\xi|)$, with the scalar in its reversed cross block written $c = Q\xi$.

Theorem 85.1 (A height cutoff uniform in the block length)**.** *For $H_0 \ge 1$ and $1 \le \ell \le j$, on the original fields and intervals,*

$$\sum_{\substack{w \in \mathcal{B}_j \\ H(\xi) \ge H_0}} \|O_w\|_{\mathrm{op}} \le \frac{\exp(CRj\ell fs \log b)}{Q^{(3\ell R - 2r)jf}\, H_0^{3j\ell R - 4j - 3}}.$$

In particular for every fixed $D > 0$ there is M_D such that the sum over all $2 \le j \le s$ with $\log H(\xi) \ge M_D fs \log b$ is at most $\exp(-DRfs \log b)$. These bounds hold after the original orthogonal projections. The remaining labels in these one-block families form a finite set at each point of the interval.

Proof. Take the power $A_1 = 3\ell R$ in Lemma 78.12 at a place with $|\tau\xi| < 1$, and $A_2 = 6\ell R$ at each other place. The product of all inverse powers of $|\tau c|$ is at most

$$(Q^f A_\xi)^{-3j\ell R}\, H(\xi)^{-3j\ell R}.$$

Here $Q \ge 1$ and $A_\xi = \prod_\tau |\tau\xi|$. Such independent local choices give a bound for the actual product orbital integral. Indeed the diagonal sesquilinear forms and the polarized signed forms commute with the real right maximal compact group on its given irreducible lowest type. They are scalar on that type, by Schur's lemma. Evaluate them on a decomposable vector, use product measures, and bound each signed factor by the local positive integral, as in Proposition 83.13. Alternatively the weighted Cauchy–Schwarz proof uses the product of the chosen local height weights. The field factor is at most one by Lemma 78.12. Multiply by Proposition 78.2. The remaining power of A_ξ is $-(3\ell R - 2r)j$. It is negative for the ranks in question, and $A_\xi \ge 1$ by integrality. This leaves the individual estimate

$$\exp(CRj\ell fs \log b)\, Q^{-(3\ell R - 2r)jf} H(\xi)^{-3j\ell R}.$$

For fixed ξ, the algebraic integer $\eta = T\xi^{j}\lambda$ has prescribed $\eta\bar{\eta} = T^{2}(\xi\bar{\xi})^{j}$. As in Theorem 89.13, its ideal divides the ideal of this product, the number of divisors is at most the absolute norm, and generators for a given ideal differ by a root of unity. Thus at most $8f^{2}T^{4f}H(\xi)^{4j}$ choices of λ occur. Each gives at most one label for the specified cross block. The factor $8f^{2}T^{4f}$ is absorbed in the exponential by Theorem 71.17. The sum of H^{-3} over all nonzero integers of E is at most 27^{f} by the disk count of Theorem 89.13. On $H \geq H_{0}$ take the remaining negative powers at H_{0}, yielding the first inequality.

For large x, $3jR-4j-3 \geq 2jR$ for every indicated j. Set $\ell = 1$, take $M_{D} > C+D+1$, and sum the resulting geometric series in j to obtain the second assertion. Orthogonal compression decreases the norms. Finally bounded H bounds every complex coordinate of ξ, so only finitely many integers and then finitely many relative scalars occur. □

Theorem 89.13 treats all heights at $j = 1$. The bounded-norm calculation below gives a second estimate when $j \geq 2$.

85.2 The full contribution up to a bound on the absolute norm

For each $2 \leq j \leq s$ let $\mathcal{B}_{j}$ be the one-block family of Theorem 85.1, with $c = Q\xi$. The absolute norm and the height of ξ can be very different on the prescribed varying fields. The following sum includes every unit in $\mathcal{O}_{E}$ allowed by the support conditions, at every block length.

Theorem 85.2 (Single reversed blocks of bounded absolute norm)**.** *Put $d = k+1$. Uniformly for $0 \leq \vartheta \leq 1$, one has*

$$\sum_{\substack{w\in\mathcal{B}_{j}\\ |N_{E/\mathbb{Q}}\xi|^{1/2}\leq Q^{\vartheta f}}} \|O_{w}\|_{\mathrm{op}} \;\leq\; (d!)^{-jf}\exp(Crjf)\,Q^{-(1-\vartheta)rjf}.$$

There is no restriction on the complex sizes of the conjugates of ξ, on the relative scalar, or on its valuations beyond the stated norm and support conditions. The assertion holds on the original fields, words, levels and intervals and after the prescribed orthogonal projections.

Proof. For a single label multiply the bounds of Theorem 85.12 at all real places. Their product bounds the real operator relative to the diagonal, since the positive norms dominate the polarized forms on the given irreducible lowest type, as in Theorem 85.1. The rational measure factor is at most one by Lemma 78.12. The product of $|\tau c|$ is $Q^{f}A_{\xi}$, where $A_{\xi} = |N_{E/\mathbb{Q}}\xi|^{1/2}$. Proposition 78.3 therefore gives

$$\|O_{w}\|_{\mathrm{op}} \leq (d!)^{-jf}\exp(Crjf)Q^{-(B_{j}-j^{2})f}A_{\xi}^{2rj-B_{j}}.$$

Here $B_{j}-rj = j^{2}(4s-j)-2js-j^{2}$. For $j \geq 2$ and $s \geq j$ this is at least j^{2}, since $s(4-2/j) \geq j+2$. It is also $O(j^{2}s) = o(rj)$ uniformly. Thus $2rj-B_{j} > 0$ for large x. Substitute $Q^{\vartheta f}$ for A_{ξ}. The remaining negative exponent of Q^{f} is at least $(1-\vartheta)rj$, since its excess is $(1+\vartheta)(B_{j}-rj)-j^{2} \geq 0$.

To sum over all heights take $H_{*} = \exp(Mfs\log b)$ with a fixed sufficiently large M. Below this height the disk estimate $\sum_{\xi\neq 0}H(\xi)^{-3} \leq 27^{f}$ from Theorem 89.13 gives at most $27^{f}H_{*}^{3}$ integers. For each there are at most $8f^{2}T^{4f}H_{*}^{4j}$ admissible scalars, by Theorem 85.1. The logarithm of the resulting count is $O_{M}(jfs\log b) = o(rjf)$.

For the heights $H(\xi) \geq H_{*}$ use Theorem 85.1 with $\ell = 1$. Since $3j(r-1)-4j-3 \geq 2j(r-1)$, choosing M greater than its absolute cost constant gives a tail at most $\exp(-jrfs\log b)$, even without the norm restriction. This is at most $(d!)^{-jf}Q^{-rjf}$ for sufficiently large x. Indeed $\log(d!) \leq d\log d$, $\log Q = O(\log b)$ by the marked-prime choice and $\log d = O(\log x) = o(\log b)$, while s tends to infinity. Combining the two sums proves the estimate. Orthogonal projections are contractions. □

Theorem 85.13 removes the norm restriction by estimating the finite column operator at every prime.

85.3 The real orbital integral for every transverse pair

Consider the remaining transverse positions on the fields and intervals of Section 6.6, so $\mathcal{G} > 0$. At one real place let $\mathcal{P}_\tau$ be the integral of the product of the two section norms divided by their diagonal norms, as in Theorem 85.9. On the common group use probability on its compact factor and ordinary additive measure on the entries of its internal unitriangular blocks in the normalized monomial frames. At the original frequency these are coordinates in the decorated frames with terminal vector $z = \varpi e$. Let J_i be the block endpoints for the extended pair, $i(p)$ the index of the block containing p, and $u_p = d - J_{i(p)}$. Use δ_p from Proposition 80.3 and R from Proposition 85.17.

Theorem 85.3 (A real bound including the last nonsingleton block)**.** *Put $\kappa = k - d$, $E_0 = \kappa + 2ds + s^2$, with $d = s + 1$. On sufficiently distant common intervals, for every pair of prescribed words and $\eta_i \geq 0$ with $\sum_i J_i \eta_i \leq 200d(d + \log_2(2(k+1)))$, one has*

$$\mathcal{P}_\tau \;\leq\; \frac{\exp(Ckd)\, R^{-2\kappa}\, \delta_d^{-E_0}}{(k!)^s\, T^{s(k+d)}} \prod_{p=1}^{d} \delta_p^{-2u_p} \prod_i \max(1, \delta_{J_i})^{-\eta_i}.$$

This holds also for $D_s = 0$. In that case $R = \sqrt{2}$ and $\delta_d = |\tau D_d|/(2T)^d$.

Proof. We start at unit frequency as in Proposition 80.3. With exterior products in the extended space set $\alpha_0(g) = \| \bigwedge^d (g^{-1} P_0) \|$. This is positive and left B-invariant, since the action on the frame is upper unitriangular. Let g minimize the operator norm in its original B-coset and put $N = \nu(g)$. Extending an original isometry by the identity on t preserves this norm, which is at least one. In the fiber above g write $y = w^{-1} b g$, $b \in J \backslash B$, using the representatives in Proposition 85.17. The columns of $y^{-1} P_0$ are

$$g^{-1} (Z_0^\dagger, -U, I_d)^{\mathrm{col}}\, A^\dagger C.$$

Indeed $b_0^{-1} b = \mathrm{diag}(A, h, A^{-*}) n(U, Z_0)$ and $b_0^{-1} w P_0 = P_- C$.

On $\nu(y) \leq H$, choose a minimizing left translate of y. The columns before g^{-1} in this formula then have operator norm at most $M = NH$ after right multiplication by an upper unitriangular matrix b_1. This uses the equality of the operator norms of an isometry and its inverse in the signed Euclidean coordinates. In particular $\| A^\dagger C b_1 \| \leq M$. Apply the minor identity of Lemma 85.5 to this matrix, permuting rows within each block and removing pivot phases. In column p the u_p entries strictly between blocks have moduli bounded by M^p/δ_p, and $\delta_p \leq M^p$. The operation relating these entries to the coordinates of A permutes and conjugates them and preserves volume. Thus the A-volume is at most $\pi^{\sum u_p} M^{2 \sum p u_p} \prod_p \delta_p^{-2u_p}$. All singular values of $A^\dagger C b_1$ are at most M and their product is δ_d. Its inverse consequently has norm at most M^{d-1}/δ_d. The column formula gives $\|U\|, \|Z_0\| \leq L = M^d/\delta_d$.

For the terminal determinant, factor the same g in the extended unitary group as

$$g = n(U_g, Z_g)\, \mathrm{diag}(A_g, h_g, A_g^{-*}) k_g$$

where k_g is Euclidean unitary. To obtain this factorization move $g^{-1}\,\mathrm{span}(P_0)$ onto the standard maximal isotropic space by a compact isometry. Its projections to the two signed spaces identify isometric subspaces, and their orthonormal bases extend to bases of the signed spaces. The remaining parabolic element has the displayed block form by the unitary equations. In

particular $\alpha_0(g) = |\det A_g|^{-1}$. Write $n(U, Z_0)n(U_g, Z_g) = n(U + U_g, Z')$. Its block relation gives $-Z' - Z'^{\dagger} = (U+U_g)^{\dagger}(U+U_g)$. In the formula for $y^{-1}P_0$, remove the left compact factor k_g^{-1}. The leading d-square is now $A_g^{-1}Z'^{\dagger}A^{\dagger}C$. Its determinant modulus is bounded by the full exterior norm $\alpha_0(y)$. Also, if $-Z' - Z'^{\dagger} = K \geq 0$, then $\det K \leq 2^d |\det Z'|$. For $K > 0$ conjugate $-Z'$ by $K^{-1/2}$. Its real part is $I/2$ and all its singular values are at least $1/2$ by Cauchy–Schwarz. For singular K the inequality is automatic. We have proved

$$\det((U + U_g)^{\dagger}(U + U_g)) \leq \frac{2^d}{\delta_d} \frac{\alpha_0(y)}{\alpha_0(g)}.$$

Integrate $(\alpha_0(g)/\alpha_0(y))^{\kappa}$ on this shell. At fixed A the last column of U has the sphere law of Proposition 85.17. The first s columns lie in balls of radius L, and the free skew entries have volume at most $\exp(Cs^2)L^{s^2}$ since the skew part has norm at most L. Enlarge these ranges independently and use Lemma 85.7, translated by U_g. The quotient density supplies 2^s. From $k!/(k-i-1)! \leq k^{i+1}$ and $d \log k = o(k)$ on these intervals, all its fixed numerical factors and $2^{d\kappa}$ are at most $\exp(Ckd)/(k!)^s$ after the factorials in the lemma are included. The shell integral is therefore bounded by

$$\frac{e^{Ckd}}{(k!)^s} R^{-2\kappa} \delta_d^{-E_0} \prod_p \delta_p^{-2u_p} M^{K_0}, \qquad K_0 = d(2ds + s^2) + 2\sum_p p u_p.$$

For endpoints with $\delta_{J_i} \geq 1$, multiply by $(M^{J_i}/\delta_{J_i})^{\eta_i} \geq 1$ on the shell. Write K_1 for the enlarged exponent of M.

Integrate the shell inequality against $pH^{-p-1}\,dH$ for $H \geq 1$ with $p = K_1 + 1$. By Tonelli this inserts $\nu(y)^{-p}$ and leaves pN^{K_1}. Apply Cauchy–Schwarz to the two original section norms on $J\backslash G$ with squared-integral weights

$$\Big(\alpha_0(g)/\alpha_0(w^{-1}g)\Big)^{\kappa} \Big(\nu(g)/\nu(w^{-1}g)\Big)^{p} \quad \text{and its reciprocal.}$$

In the first squared integral the section norm descends to $B\backslash G$ and the preceding calculation integrates its fiber. The remaining normalized moment is of order $2K_1 + 1 = O(d^3 + d \log k)$. The local estimate in Theorem 83.3, rounding the exponent upwards, costs $\exp(O(sd^3 \log b)) \leq \exp(Ckd)$ by Theorem 6.29. For the reciprocal integral translate by w and interchange the two frames. Their cross matrix becomes C^*, with the same pivot moduli in each reversed block and the same R. Common unipotent coordinates in the second frame are transported by $D \mapsto C^{-1}D^{-*}C$. Inside each block the pivot scalar cancels, and reversal, phase changes and unitriangular inversion preserve ordinary Haar measure. The compact common factor has unit mass in both frames. Thus the second disintegration has exactly the required normalization and uses the diagonal moment for the second word. This proves the estimate at unit frequency.

Finally conjugate from the physical frequency as in Proposition 83.14. On B the change scales the $(k+1)s$ rectangular complex entries by modulus $\sqrt{T}$, the s^2 real center coordinates by T, and the s-flag upper entries with modulus one. Its Haar modulus is $T^{s(k+d)}$. On J it carries the original monomial-frame coordinates to the unit ones with modulus one. Indeed all d vectors of the first extended decorated frame have acquired the same scalar $\sqrt{2}\,\tau\varpi$ relative to the unit frame, by Proposition 80.3. Thus the unitriangular coefficients in those frames agree entry by entry, and the compact probabilities are preserved. The change of normalized section integrals contributes the common modulus divided by the B-modulus, by the quotient integration formula in Proposition 83.14. This gives the displayed power of T. □

This estimates a real factor of the original orbital integral even when the isotropic cross determinant vanishes. For an integral projected period one must still combine the bound

with the finite operators and the adelic measure of the common $U(k-1)$. In particular the determinant entering the estimate is $|\tau\Delta|/(2T)^d$, and integrality of Δ/Q^d alone gives no uniform lower bound of one for this number.

85.4 The archimedean integral for a whole isotropic block

At one real place use Euclidean unit Witt vectors and write $l_j(g)$ for the Euclidean norm of the exterior product of the first j vectors after g^{-1}. In these coordinates let C_0 be the invertible cross block for an exchange as in Proposition 78.7. Use $W \in \mathbb{C}^{n\times j}$ and a skew-Hermitian j-square matrix V as coordinates on the unipotent radical. In its upper right block put $Z = -W^{\dagger}HW/2 + V$, where H is the signed Hermitian matrix on M_j and $\dagger$ is ordinary conjugate transpose. The three upper blocks in the unipotent are $-W^{\dagger}H, W, Z$. Give W ordinary complex area entry by entry and V imaginary diagonal length and complex area above the diagonal. These are Haar coordinates by the upper triangular multiplication law.

Theorem 85.4 (The matrix volume on a first-block fiber)**.** *For $A = 2\alpha$ with $\alpha > n+3j/2-1$, the integral of $l_j(w^{-1}u_0g)^{-A}$ over this radical equals*

$$|\det C_0|^{-A}\, l_j(g)^{A-2(n+j)}\, K_{n,j,A}.$$

Its constant satisfies

$$K_{n,j,2\alpha} \le \frac{\pi^{nj+j^2/2}}{2^{j(j-1)/2}} \prod_{i=1}^{j} \frac{\Gamma(\alpha - n - j/2 - i + 1)}{\Gamma(\alpha - i + 1)}.$$

Proof. The maximal compact group acts transitively on isotropic j-planes because their positive and negative projections are isometric copies with matched orthonormal bases. Thus one may write $g = u_1 \operatorname{diag}(D, m, D^{-*})k_0$, where m is unitary for H and k_0 is Euclidean unitary. Right translation in U absorbs u_1. The columns defining the transformed exterior vector then have blocks

$$(D^{-1}Z^{\dagger},\ -m^{-1}W,\ D^{\dagger})^{\mathrm{col}}C_0.$$

Here the notation means stacking the three blocks. The D-dependent change $W = mW_1D^{\dagger}$, $V = DV_1D^{\dagger}$ gives $Z = DZ_1D^{\dagger}$. The Jacobian is $|\det D|^{2(n+j)}$. Indeed the complex rectangular block gives the power $2n$, as $|\det m| = 1$. Congruence on skew-Hermitian matrices gives the power $2j$, by diagonalizing the positive part of D and multiplying the real diagonal and complex off-diagonal scalings. The common right matrix on the stacked columns is $D^{\dagger}C_0$. Since $l_j(g) = |\det D|^{-1}$, this proves the first identity with

$$K_{n,j,2\alpha} = \int \det(I + W^{\dagger}W + ZZ^{\dagger})^{-\alpha}\, dW\, dV.$$

For a positive Hermitian j-square matrix B use the analogous independent entry measure. Set $\Gamma_j(z) = \int_{B>0} e^{-\operatorname{tr} B}(\det B)^{z-j}\, dB$ for $z > j-1$. Integrating its last column by the Schur complement gives $\Gamma_j(z) = \pi^{j-1}\Gamma(z-j+1)\Gamma_{j-1}(z)$. In fact if B' is the preceding principal block and b the off-diagonal column, the new positive scalar is $B_{jj} - b^{\dagger}(B')^{-1}b$, and the Gaussian integral over b gives $\pi^{j-1}\det B'$. Congruence by a positive $S^{-1/2}$ consequently gives

$$(\det S)^{-\alpha} = \Gamma_j(\alpha)^{-1} \int_{B>0} e^{-\operatorname{tr}(BS)}(\det B)^{\alpha-j}\, dB.$$

Insert $S = I + W^{\dagger}W + ZZ^{\dagger}$. Tonelli's theorem applies to the positive integrals. At fixed B, W, the Gaussian in Z integrated over the translate of the skew-Hermitian subspace is at most its integral on that subspace through zero. To see this, orthogonally project the translation onto

the subspace for the real inner product associated to the positive quadratic form $\operatorname{tr}(BZZ^{\dagger})$. Translation inside the subspace preserves measure and the orthogonal component supplies a factor at most one.

Conjugate B unitarily to diagonal entries $b_i > 0$. The skew-Hermitian Gaussian gives $\pi^{j^2/2}(\det B)^{-1/2}\prod_{i<i'}(b_i+b_{i'})^{-1}$. By the arithmetic and geometric mean inequality this is at most $\pi^{j^2/2}2^{-j(j-1)/2}(\det B)^{-j/2}$. The W-Gaussian gives $\pi^{nj}(\det B)^{-n}$ by integration over its rows. The remaining integral is $\Gamma_j(\alpha-n-j/2)$, finite in the stated range. Taking the quotient of the two gamma integrals proves the bound. □

85.5 Minors controlling the entries between blocks

Lemma 85.5 (Quotients of minors for the internal unipotent)**.** *Use C, z from Proposition 78.9. After a signed permutation of rows, $z^{*}Cb_1$, for any upper unitriangular b_1, becomes LDb_1, where $D_{pp} = c_{i(p)}$ is diagonal and L is block lower unitriangular with identity diagonal blocks. Its strict off-block entries are, up to sign, the conjugates of the independent entries in z, each used once. If an index ℓ belongs to a later block than p, then*

$$L_{\ell p}G_p = \det(LDb_1)_{(1,\dots,p-1,\ell),(1,\dots,p)}.$$

Also $\det(LDb_1)_{[p],[p]} = G_p$.

Proof. Write $C = SD$ with S the block diagonal product of the signed reversals. Premultiplication by S^{-1} gives $L = S^{-1}z^{*}S$, of the stated shape. In the first p columns, upper triangularity of b_1 permits only rows $1,\dots,p$ of that matrix to enter. Its leading square has determinant one. The corresponding part of D has determinant G_p. On selecting the first $p-1$ rows and row ℓ of L, its final column is zero except for $L_{\ell p}$, and the preceding leading determinant is one. This proves the formula. Selecting the first p rows instead gives the last assertion. □

85.6 A spherical integral for the two terminal vectors

Lemma 85.6 (The average of a negative power of the terminal pairing)**.** *Let $a_g = |h(e, m_g e)| \geq 1$, $B_\tau = |\tau\beta| > 1$ and use the probability measure on h_0 in Proposition 85.10. For an integer $1 \leq p \leq k-1$,*

$$\int a_1^{-p}\,dh_0 \leq (a_g B_\tau)^{-p}\frac{\Gamma(k)\Gamma(k-p)}{\Gamma(k-p/2)^2} \leq e^{Ck}(a_g B_\tau)^{-p}.$$

Proof. After multiplying the two terminal vectors by phases, their negative coordinates are a_g, B_τ. Their positive parts have lengths $\sqrt{a_g^2-1}$ and $\sqrt{B_\tau^2-1}$. The second positive direction is uniform on the complex unit sphere. Consequently a_1 has the distribution of $a_g B_\tau|1-\rho z|$, with z the first coordinate of a uniform unit vector in $\mathbb{C}^k$ and

$$\rho = \sqrt{1-a_g^{-2}}\sqrt{1-B_\tau^{-2}} < 1.$$

This formula also applies to $a_g = 1$.

Write $(a)_m = a(a+1)\cdots(a+m-1)$, with $(a)_0 = 1$. Independent rotations give $\mathbb{E}[z^m\bar{z}^n] = 0$ for $m \neq n$ and $\mathbb{E}|z|^{2m} = m!/(k)_m$. For the latter formula take independent complex Gaussians with density $\pi^{-1}e^{-|u|^2}$. Their direction is uniform, independent of their radius. The numerator moment of the first Gaussian is $m!$ and the $2m$-th radial moment is $\Gamma(k+m)/\Gamma(k)$, by polar integration. With $\alpha = p/2$, the absolutely convergent binomial series for $(1-\rho z)^{-\alpha}$ and its conjugate now give

$$\mathbb{E}|1-\rho z|^{-p} = \sum_{m\geq 0}\frac{(\alpha)_m^2}{m!(k)_m}\rho^{2m} \leq \frac{B(\alpha, k-2\alpha)}{B(\alpha,k-\alpha)}.$$

For the inequality substitute the beta integral with density $x^{\alpha-1}(1-x)^{k-\alpha-1}/B(\alpha,k-\alpha)$ for the ratio $(\alpha)_m/(k)_m$, by taking its m-th moment. Sum the other binomial series with $\rho=1$ inside this positive integral. It gives the additional factor $(1-x)^{-\alpha}$, which is integrable since $k-2\alpha>0$. The resulting beta ratio is the first gamma quotient in the assertion. At $\rho=1$ the positive series increases termwise with p. Its largest value in this range is attained at $p=k-1$. Stirling's formula bounds $\Gamma(k)/\Gamma((k+1)/2)^2$ by e^{Ck}. □

85.7 A Gram integral with one column on a sphere

Lemma 85.7 (A spherical column and translated ball cutoffs)**.** *Let $k>d=s+1\geq 2$, $L,R>0$, and put $\kappa=k-d$. In a k-by-d complex matrix A, integrate each of the first s columns over a ball of radius L with arbitrary center, using ordinary volume. Integrate its last column over a sphere of radius R with arbitrary center, using uniform probability. Then*

$$\int \det(A^{\dagger}A)^{-\kappa}\,dA \leq R^{-2\kappa}\pi^{ks}L^{2ds}\prod_{i=1}^{s}\frac{1}{i!(k-i-1)!(d-i)}.$$

Proof. First fix the last column v. The event $v=0$ has sphere measure zero. Starting with v, add the other columns in their given order. Their successive perpendicular lengths and $\|v\|$ have squared product $\det(A^{\dagger}A)$ by Gram–Schmidt. The columns added so far are independent almost everywhere by Fubini, since every proper complex subspace has zero volume.

Integrate the added columns in reverse order. For the i-th added column its parallel component lies in complex dimension i, and its perpendicular component in dimension $k-i$. The original ball is contained in the product of the two projected balls of radius L. The parallel volume is $\pi^{i}L^{2i}/i!$. A translate of the perpendicular ball integrates $\|z\|^{-2\kappa}$ to at most its value on the centered ball. Indeed each superlevel set of the integrand is a centered ball and its intersection with a ball of fixed radius is largest when their centers coincide. Integrating the superlevel volumes proves the claim. Polar integration gives the other factor $\pi^{k-i}L^{2(d-i)}/((k-i-1)!(d-i))$. These bounds do not depend on the preceding columns.

It remains to integrate $\|v\|^{-2\kappa}$. In real dimension $n=2k$, the average of $\|u+R\omega\|^{2-n}$ over unit directions ω is at most R^{2-n}, for every u. Here is a direct proof. Away from $\|u\|=R$ the average is radial and harmonic by differentiation under the integral, since the radial Laplacian of r^{2-n} vanishes on $r>0$. Solutions of $f''+(n-1)f'/r=0$ have the form $a+br^{2-n}$. In the interior boundedness at zero and the value at zero give the constant R^{2-n}. In the exterior the asymptotic $r^{2-n}(1+o(1))$ at infinity gives r^{2-n}. On the intervening sphere Fatou's lemma applied from inside gives the asserted upper bound. Raise this nonnegative integrand to the power $\kappa/(k-1)\in(0,1)$. Jensen's inequality bounds the new average by $R^{-2\kappa}$. Multiplication of the factors proves the result. □

85.8 Integration of the inverse Gram determinant

Lemma 85.8 (A rectangular determinant integral with a column cutoff)**.** *Let $d\geq j\geq 1$ be integers and $L>0$. Give the d-by-j complex matrix Y ordinary entry measure, and restrict each column to the Euclidean ball of radius L. Then*

$$\int \det(Y^{\dagger}Y)^{-(d-j)}\,dY \leq \pi^{dj}L^{2j^2}\prod_{i=1}^{j}\frac{1}{(i-1)!(d-i)!(j-i+1)}.$$

Proof. Outside a set of measure zero all initial columns are independent. Indeed, having fixed $i-1<d$ independent columns, their complex span has zero $2d$-dimensional measure. Decompose the i-th column into components parallel and perpendicular to this span. Write

t_i for the length of the perpendicular component. Gram–Schmidt orthogonalization gives $\det(Y^{\dagger}Y)=\prod_i t_i^2$.

Integrate the last column first and continue in reverse order. At each step enlarge the ball to the product of the two balls of radius L for its orthogonal components. The parallel part has complex dimension $i-1$ and volume $\pi^{i-1}L^{2(i-1)}/(i-1)!$. For the other part put $v=d-i+1$. Polar integration gives

$$\int_{\|z\|\le L}\|z\|^{-2(d-j)}dz=\frac{\pi^{v}L^{2(j-i+1)}}{(v-1)!(j-i+1)}.$$

In fact the sphere at radius one has area $2\pi^{v}/(v-1)!$, as follows by integrating $e^{-\|z\|^2}$ in Cartesian and polar coordinates. The radial power is integrable since $j-i+1>0$. The product of these two factors does not depend on the preceding columns. Tonelli’s theorem and multiplication over i give the asserted bound. □

85.9 The mixed real orbital integral with two terminal lines

Retain the real quotient in Proposition 85.10. For lowest-type sections W_1,W_2 of prescribed words use the positive real quantity

$$\mathcal{P}_{\tau}(w)=\frac{\displaystyle\int_{J\backslash G}\|W_1(g)\|\,\|W_2(w^{-1}g)\|\,d\dot g}{\displaystyle\prod_{\ell=1}^{2}\left(\int_{B\backslash G}\|W_{\ell}(g)\|^2\,d\dot g\right)^{1/2}}.$$

Take nonzero sections, ordinary coordinate measures on the unipotents and probability measure on each compact middle group. The sections in the numerator and denominator have the same physical terminal frequency. Set $d_0=k+1$, $E_s=d_0-s+3s^2$, and $a_*=-D_{s+1}/Q^{s}=T\beta a_t$. The congruences in Proposition 88.11 imply that a_* is integral at every finite place.

Theorem 85.9 (Real decay in the separation of the negative lines)**.** *For $\eta_i,\eta_*\ge 0$ with $\sum_i J_i\eta_i+(s+1)\eta_*\le 200s(s+1+\log_2(2(k+1)))$, the local integral satisfies the bound*

$$\mathcal{P}_{\tau}(w)\le\frac{e^{Crs}}{(d_0!)^{s}}\,|\tau\beta|^{-(d_0-3s)}|\tau G_s|^{-E_s}\prod_{p=1}^{s}|\tau G_p|^{-2u_p}\cdot\max(1,|\tau a_*|)^{-\eta_*}\prod_{i=1}^{t}\max(1,|\tau a_i|)^{-\eta_i}.$$

The two words may coincide or be distinct. The constant is uniform in their choice, in the relative middle vector and in the prefix composition.

Proof. Work first at unit terminal frequency on $y=w^{-1}h_0\widehat{z}u_0g$, with g minimal. Use the notation of Theorem 85.14, now $j=s$ and $h=0$. On the shell $\nu(y)\le H_0$ write $M_0=NH_0$. The first s columns still have blocks $(Z^{\dagger},-W,I_s)^{\mathrm{col}}\Omega b_1$, since the rotations and m act only on the middle space. Thus all bounds there for the entries between blocks, and the norm bound $M_0^{s}/\delta_{[s]}$ for W,Z,V, continue to hold.

Set $b_1'=d_0-s$. In bounding $(\mathcal{A}(g)/\mathcal{A}(y))^{b_1'}$ on this shell, Lemma 85.16 contributes the factor

$$(2^{s})^{b_1'}\delta_{[s]}^{-b_1'}a_1^{2s}(Z_g/a_1)^{b_1'}$$

before the inverse Gram determinant integral. Its column radius has an additional factor N, since $\|Y\|\le 3N\|W\|$. By Lemma 85.8 the cost of this radius increase is N^{2s^2}. All surviving powers of $\delta_{[p]}$ and M_0 are the same as in Theorem 85.14.

Integrate the factors involving a_1, Z_g over the actual compact fiber. Put $p_1 = b_1' - 2s = d_0 - 3s$, which lies between 1 and $k-1$ on the large intervals. Since $Z_g = \sqrt{2a_g^2 - 1} \le \sqrt{2}\, a_g$, Lemma 85.6 gives

$$\int a_1^{2s} (Z_g/a_1)^{b_1'}\, dh_0 \le e^{Ck} a_g^{2s} |\tau\beta|^{-p_1} \le e^{Ck} N^{2s} |\tau\beta|^{-p_1}.$$

This is valid before contracting the column vectors or summing finite cells.

In addition to the endpoint tests in Theorem 85.14, the shell bounds $|\tau\beta|\delta_{[s]} \le M_0^{s+1}$. Indeed take the Hermitian pairing of $P \wedge e$ with its translated image. Its modulus for the exchange is this product by block diagonality. Left and right upper changes of decorated frame preserve the exterior vector and its pairing, as do the compact column rotations. Its Euclidean pairing bound on a representative of norm at most M_0 is M_0^{s+1}. Insert its inverse power η_* with the compensating factor $M_0^{(s+1)\eta_*}$ whenever $|\tau a_*| \ge 1$. Treat the other endpoint tests at the places $|\tau a_i| \ge 1$ as before.

The squared-integral weights in Cauchy–Schwarz are the ratio of the full exterior norms to power b_1' and the ratio of the heights to power one larger than the total shell exponent of H_0, and their reciprocals. Integrate the shell bounds as in Theorem 85.14. The extra moments of N above have exponent $2s^2 + 2s$. The endpoint tests cost at most the weighted sum in the hypothesis in each shell. All required base moments therefore have exponent $O(s^3 + s\log(k+1))$, within Theorem 83.3. Their logarithmic cost is $O(s^4 \log b) = o(rs)$. In the second squared integral translate by w. The inverse has the same absolute prefix products and its middle terminal coefficient is $\bar\beta$, by the unitary relation for m. It normalizes J by Proposition 85.10. The diagonal moment estimate applies to the second word with the same constants.

The terminal-frequency dilation fixes the middle space, so it commutes with m and the compact quotient. Scale $\delta_{[p]}$ to $|\tau G_p|/(2T)^p$ and use the quotient factor $T^{-s(n+s)}$ as in Theorem 85.14, with $n = k+1$. For the additional terminal test, $|\tau\beta|\,|\tau G_s|/(2T)^s = Q^s |\tau a_*|/(T(2T)^s)$. Its extra constant has logarithm $O((s+1)\eta_* \log(2T))$, within $O(rs)$. This proves the asserted estimate with the stated normalizations. □

The real estimate applies to the smaller common column when $D_s \neq 0$. Its finite orbital factors and the normalization of the adelic common $U(k-1)$ still enter the global period. When $D_s = 0$ the final reversed block contains an isotropic and a terminal vector, and the representative above does not cover it.

85.10 Two distinct negative terminal lines

Return to the remaining transverse pairs with common positive column of rank $k-1$. Here we consider those for which the isotropic cross determinant D_s is nonzero. Then $\mathcal{G} = |D_{s+1}|^2 - T^2|D_s|^2$ is totally positive, where $|z|^2 = z\bar z$. In the normal form of Lemma 88.10 the terminal entry is a singleton block, since both s and $s+1$ are block endpoints. Use the composition and products G_p of Proposition 78.9 for the isotropic prefix of length s. Its signed determinant convention gives $G_s = D_s$.

Proposition 85.10 (A rational representative and its compact real quotient)**.** *Write $\beta = -D_{s+1}/(TD_s)$. One can represent the pair by an exchange with prefix cross block C as in Proposition 78.9 and middle block $m \in U(W_0 \perp Ee)$, with $h(e,e) = -1$, such that*

$$me = v + \beta e, \qquad h(v,v) = \beta\bar\beta - 1, \qquad v \in W_0.$$

Here m fixes $v^\perp \cap W_0$ pointwise. The common stabilizer J is the product of the diagonal prefix unitriangular groups and $U(v^\perp \cap W_0)$, and the exchange normalizes J with modulus one.

At a real place the extra quotient in $J\backslash B$ is the unit sphere of the positive column, with its rotation-invariant probability measure when both compact groups have mass one.

Proof. After the two upper unitriangular changes of frame, the terminal vectors are perpendicular to both isotropic lists, since the last block of the cross matrix is a singleton. The two isotropic spaces pair perfectly and span a nondegenerate space of dimension $2s$. The map fixing the original isotropic vectors and the original terminal vector, and sending the second isotropic list to the standard opposite list multiplied by C, is an isometry on their span. Witt extension gives a rational isometry of the ambient space fixing the first decorated frame. The second terminal vector divided by ϖ is now a negative unit in $W_0 \perp Ee$. Its e-coordinate is β by the determinant ratio and $\varpi\bar{\varpi} = T$. The norm identity gives the asserted length of v. In particular $|\tau\beta| > 1$ at every real place.

Put $t = e - (v + \beta e)$ and $b_0 = h(t, e) = \bar{\beta} - 1$. This is nonzero. On the middle space take

$$m(z) = z - t\, h(t, z)/b_0.$$

Expanding the Hermitian pairing shows this is an isometry, because $h(t, t) = b_0 + \bar{b}_0$. It sends e to the required vector and fixes the indicated perpendicular. Using it as the middle block therefore realizes both decorated frames.

An element of J stabilizes the two opposite isotropic spaces and is block diagonal on their sum and its perpendicular. The prefix calculation is the one in Proposition 78.9. The middle part fixes e and $v + \beta e$, so it fixes v as well. This gives the stated group. The middle block commutes with that group, and the exchange acts by inverse adjoints and reversal inside the prefix blocks. These operations preserve Haar measure there. With $\widehat{z}, u_0$ as in that proposition, representatives on the real fiber can thus be taken in the order $h_0 \widehat{z} u_0$, with h_0 in the middle $U(k)$ modulo the left stabilizer $U(k-1)$. The pushforward by $h_0 \mapsto h_0^{-1} v/\|v\|$ is the unique invariant probability on the sphere. Conjugation by these compact matrices preserves the other Haar factors. □

85.11 Higher decay at the physical terminal frequency

At a real place τ, write $l(g) = \|g^{-1}u'\|$ for the length of the first Euclidean unit isotropic vector in Theorem 83.16. Use the same quotient Haar densities as in Proposition 83.13. Write $\mathcal{I}_{\tau,\xi}$ for the signed local numerator divided by its diagonal norm at parameter $Q\tau\xi$ and the original terminal character. Thus $\mathcal{I}_{\tau,\xi} = I_\tau(\tau(\xi/\sqrt{-q}))$ in the notation of that proposition.

Lemma 85.11 (A real estimate beyond twice the first-plane exponent)**.** *Let $R = r - 1 = n + 1$, $n = \dim_{\mathbb{C}} M$, and $t = |\tau\xi| > 0$. On the prescribed words,*

$$|\mathcal{I}_{\tau,\xi}| \le a_2\, t^{-3R}, \qquad \log a_2 \le CRs \log b.$$

The constant is uniform on the common intervals and across the real places. The same bound holds for the normalized local integral of the product of the two section norms.

Proof. First use unit terminal frequency and let $c_\tau \neq 0$ be the imaginary local orbital parameter. In the first unipotent coordinates of Theorem 83.16 put $L = l(g)$. For $A \ge 2R$ its change of variables gives

$$\int l((\gamma')^{-1} n(w, z) g)^{-A}\, dw\, dv = (|c_\tau|/2)^{-A} L^{A-2R} C_{n,\tau,A}.$$

Here v is the real central coordinate. In fact before dilation the squared length is $(|c_\tau|/2)^2\{L^{-2} + |w|^2 + L^2(v^2 + h_M(w, w)^2/4)\}$, after the unit-Jacobian middle isometry and central translation.

Substituting $w = y/L$, $v = u/L^2$ gives the Jacobian L^{-2n-2}. The remaining integral has integrand $(1 + |y|^2 + u^2 + h_M(y,y)^2/4)^{-A/2}$, so $C_{n,\tau,A} \le C_{n,\tau,2R} \le \pi^R/n!$. Changing the sign of c_τ has the same result.

On $J(F_\tau)\backslash G_\tau$, the quotient by the common stabilizer of the two flags, apply Cauchy–Schwarz as in Theorem 83.10, with weight $(l(g)/l((\gamma')^{-1}g))^A$ in the first squared integral and its reciprocal in the second. Disintegrating the first over $B_\tau\backslash G_\tau$ multiplies its diagonal density by $l(g)^A$ and by the fiber integral just computed. The product contains the power $l(g)^{2A-2R}$. For the second integral translation by γ' replaces c_τ by $-c_\tau$, because γ' centralizes the stabilizer. The two bounds therefore agree. The integrands after squaring are nonnegative and Tonelli's theorem applies.

Take $A = 3R$. The normalized moment required is $\mathbb{E}_{1,\tau} l(g)^{4R} \le \exp(CRs \log b)$ by the local proof of Theorem 83.3, since $4R \le 4r^2$. This also proves convergence. Now use Proposition 83.14 at the physical frequency. Its Jacobian contributes T^{-R}, and the unit-frequency parameter has modulus Qt/T. Thus one may take the uniform majorant for

$$(\pi^R/n!)\, T^{2R} (2/Q)^{3R} \mathbb{E}_{1,\tau} l(g)^{4R}$$

as a_2. Theorem 71.17 gives $\log T = O(s \log b)$, which proves the stated size bound. □

85.12 The real orbital estimate with a factorial for each exchanged vector

Put $d = k + 1,\ h = s - j,\ n = 2h + d$, and define

$$D_0 = d - j, \qquad K_j = j^2(4s - j), \qquad B_j = jD_0 + K_j.$$

Here K_j is an exponent of a local height. Keep the positive orbital integral divided by the diagonal norm, with the coordinate measures in Lemma 78.12.

Theorem 85.12 (The real factor at the original terminal frequency)**.** *For $2 \le j \le s$ its value at τ is at most*

$$(d!)^{-j} \exp(Crj)\, |\tau c|^{-B_j}.$$

The constant is uniform in the word, real place, block length, and point on the prescribed intervals.

Proof. Work first at unit terminal frequency, with $\delta = |c_\tau|/2$. Take g minimal in its Bessel coset, put $N = \nu(g)$, and use $y = w^{-1}u_0 g$. For $H_0 \ge 1$ consider the part of the radical on which $\nu(y) \le H_0$. There is $b_0 \in B$ with $\|u_0^{-1} w b_0\| \le N H_0$. Indeed choose a representative $b_0^{-1} y$ of norm at most H_0, take its inverse and multiply by g. For these unitary matrices the inverse has the same operator norm. On the first plane b_0 has upper unitriangular matrix b_1. The first j columns of $u_0^{-1} w b_0$ have blocks $Z^\dagger, -W, I_j$, each multiplied on the right by $C_0 b_1$. Put $L = NH_0/\delta$. The last block gives $\|b_1\| \le L$, hence $L \ge 1$ on the set in question. The product of the singular values of b_1 is one, so $\|b_1^{-1}\| \le L^{j-1}$. The first two blocks now give

$$\|Z\|,\ \|W\|,\ \|V\| \le L^j.$$

For the center we used $V = (Z - Z^\dagger)/2$. An empty set contributes zero throughout.

Put $U(g,y) = \mathcal{A}(g)/\mathcal{A}(y)$. Lemma 85.15 gives, almost everywhere,

$$U(g,y)^{D_0} \le (2^j \sqrt{2})^{D_0} \delta^{-jD_0} \det(Y^\dagger Y)^{-D_0}.$$

Changing the coordinates X, η to Y gives $dX\, d\eta = a^{2j} dY$ by triangularity. Its columns have length at most $3L^j$, since $\|v_+\|/a \le 1$. Apply Lemma 85.8 with this radius. The entries of

A, B lie in disks of radius L^j. The skew center lies in the product of its j real intervals and $j(j-1)/2$ complex disks of that radius. The degree in L^j from these factors and the integral in Y is

$$4hj + j^2 + 2j^2 = 4hj + 3j^2.$$

Thus the bound on the set $\nu(y) \le H_0$ is

$$\int U(g,y)^{D_0}\, du_0 \;\le\; (d!)^{-j} e^{Crj}\, a^{2j}\, \delta^{-B_j} (NH_0)^{K_j}.$$

In estimating the constants we used $d!/(d-i)! \le d^i$ for $1 \le i \le j$. The resulting logarithmic cost $O(j^2 \log d)$, and all powers of $2, 3, \pi$ from the balls and determinant, are $O(rj)$ on these intervals.

Take $p = K_j + 1$. Integrate the last bound against $pH_0^{-p-1}\, dH_0$ for $H_0 \ge 1$. Since $\nu(y) \ge 1$, Tonelli gives the same shell bound with weight $\nu(y)^{-p}$, with $H_0^{K_j}$ replaced by p. In the Cauchy–Schwarz inequality on $J\backslash G$ use $U(g, w^{-1}g)^{D_0} (\nu(g)/\nu(w^{-1}g))^p$ for the first squared section norm and the reciprocal for the second. Disintegrate over $B\backslash G$ as in Lemma 78.12. The preceding computation bounds the inner integral, and $a \le \nu(g)$. The remaining diagonal moment has exponent $2K_j + 2j + 1$ on $\nu(g)$. The reciprocal weight gives the same estimate on changing variables by w. Indeed w^{-1} is an exchange of the same form with the same singular values of its cross block, and w preserves the measure on J.

The moment exponent is at most $4r^2$ for large x. The local proof of Theorem 83.3 bounds its logarithmic cost by $O(j^2 s^2 \log b) = o(rj)$, since $j \le s$ and $s^3 \log b = o(k)$ by Theorem 6.29. This proves the unit-frequency bound. At the physical frequency replace δ by $|\tau c|/(2T)$ and multiply by $T^{-j(n+j)}$, as in Lemma 78.12. The exponent $B_j - j(n+j) = K_j - 2js$ on T is $O(j^2 s)$, and $\log T = O(s \log b)$ by Theorem 71.17. Finally $2^{B_j} \le e^{Crj}$ on the same intervals. All of these costs fit in the stated constant. □

85.13 The sum over all single reversed blocks

Theorem 85.13 (All coefficients and all heights for one reversed block)**.** *On the fields and intervals of Section 6.6, for every prescribed word and every $2 \le j \le s$,*

$$\sum_{w \in \mathcal{B}_j} \|O_w\|_{\mathrm{op}} \le ((k+1)!)^{-jf} \exp(Crjf).$$

Here $\mathcal{B}_j$ includes every relevant supported single reversed block of length $j < s$, and for $j = s$ every such block in the transverse full common column case. The estimate holds at the original level and terminal character. Together with Theorem 89.13 it gives a total at most $((k+1)!)^{-f} \exp(Crf)$ for these positions including $j = 1$. It remains valid after all prescribed orthogonal projections.

Proof. Put $d = k+1,\ R = r-1,\ H = H(\xi)$, and use the exponent $B_j = j(d-j) + j^2(4s-j)$ of Theorem 85.12. For $j \ge 2$, its bounds in the proof of Theorem 85.2 give $kj \le B_j \le 2rj$ for large x. At a real place put $t = |\tau\xi|$. For every $t > 0$ we have the two majorants

$$L_0 (Qt)^{-B_j}, \qquad U_0 (Qt)^{-G}, \qquad G = 6jR,$$

where $L_0 = (d!)^{-j} \exp(C_1 rj)$ by Theorem 85.12 and $U_0 = \exp(C_2 rjs \log b)$ by Lemma 78.12 with $\ell = 1$. Increase C_2 to have $U_0 \ge L_0$. They bound the positive integral with the same measures and terminal frequency. The product bounds polarized coefficients relative to the diagonal, with field factor at most one, by Theorem 85.1.

Use $L_0(Qt)^{-B_j}$ at the places $t < 1$. At all other places take the geometric interpolation with weight

$$\theta = \frac{\Delta_j + 4j + 3}{G - B_j}$$

on the second estimate. By Proposition 79.7 and $G - B_j \geq 3rj$ for sufficiently large r, $\theta = O((s + \log(k+1))/r) < 1$, and $\theta > 0$. Since $Q \geq 1$, the product is at most

$$L_0^f (U_0/L_0)^{\theta f} (Q^f A_\xi)^{-B_j} H^{-\Delta_j - 4j - 3}.$$

Multiply by Proposition 79.7. Because $B_j \geq kj$ and $1 \leq A_\xi \leq H$, the factors remaining from $Q^f A_\xi$ cost at most $Q^{f\Delta_j} H^{\Delta_j}$. The logarithm of Q^{Δ_j} is $O(rj)$, using $\log Q = O(\log b)$. For the interpolation constant, $\log(d!) \leq d \log d$ gives

$$\theta \log(U_0/L_0) = O\big(j(s + \log(k+1))(s \log b + \log d)\big) = o(rj).$$

Here $\log(k+1) = O(\log x) = o(s)$ and $s^3 \log b = o(k)$ on the common intervals. Thus each single label costs at most

$$(d!)^{-jf} \exp(Crjf)\, H^{-4j-3}.$$

For fixed ξ, the number of allowed λ is at most $8f^2 T^{4f} H^{4j}$ by Theorem 85.1, and each pair of parameters gives at most one label. The sum of H^{-3} over the nonzero integers of E is at most 27^f by Theorem 89.13. Since $\log T = O(s \log b)$, the extra constants fit into $\exp(Crjf)$. This proves the individual block-length bound. For large x, $(d!)^{-f} \exp(Crf)$ tends to zero since $r \asymp k$ and $\log(d!) \gg k \log k$. The geometric sum in $j \geq 2$, and the separate codimension-one result, give the asserted total. Compression by orthogonal projections decreases operator norms. □

For a prefix with several reversed blocks, Proposition 78.9 identifies the additional unipotent coordinates. Their real and finite contributions are compared in Theorem 89.14.

85.14 The real integral for all blocks in a composition

For a composition of j put $h = s - j$, $d = k + 1$ and $E_j = d - j + 4hj + 3j^2$. Keep the ordinary unipotent coordinate measures at a single real place τ. The integral below is the positive orbital integral relative to the same diagonal norm as in Theorem 85.12.

Theorem 85.14 (Decay in the terminal and intermediate prefix products)**.** *Let $2 \leq j \leq s$. For real $\eta_i \geq 0$ such that $\sum_i J_i \eta_i \leq 200j(s + 1 + \log_2(2(k+1)))$, the integral at the original frequency is bounded by*

$$\frac{\exp(Crj)}{(d!)^j} |\tau G_j|^{-E_j} \prod_{p=1}^{j} |\tau G_p|^{-2u_p} \prod_{i=1}^{t} \max(1, |\tau a_i|)^{-\eta_i}.$$

All constants are uniform in the composition and the field and interval parameters.

Proof. Work first with unit terminal frequency. Put $\delta_{[p]} = |\tau G_p|/2^p$. On the fiber above a minimal representative g, let $N = \nu(g)$ and $y = w^{-1} \widehat{z} u_0 g$. On $\nu(y) \leq H_0$ there exists an element of B whose upper unitriangular action b_1 on the j-plane gives the following bounds. With $\Omega = z^* \tau C/2$, the three blocks of the first j columns in the calculation of Theorem 85.12 are $(Z^\dagger, -W, I_j)^{\mathrm{col}} \Omega b_1$, of operator norm at most $N H_0$. Here one multiplies the inverse of a minimal representative of y on the left by g, and uses the last identity in Proposition 78.9. Put

$M_0 = NH_0$. Every p-minor of the last block is at most M_0^p, by the exterior-power norm. In the notation of Lemma 85.5 this implies

$$\delta_{[p]} \le M_0^p, \qquad |L_{\ell p}| \le M_0^p/\delta_{[p]} \quad (i(\ell) > i(p)).$$

The volume of all possible z is therefore at most $\pi^{\sum u_p} M_0^{2\sum p u_p} \prod_p \delta_{[p]}^{-2u_p}$, since there are u_p such entries in column p, each with ordinary complex area.

Next use $|\det(\Omega b_1)| = \delta_{[j]}$ and its norm bound to get $\|(\Omega b_1)^{-1}\| \le M_0^{j-1}/\delta_{[j]}$. The norms of W, Z, V are therefore at most $M_0^j/\delta_{[j]}$. Fix z. In the proof of Lemma 85.15 remove the right matrix Ω from the first j columns. This multiplies the pairing determinant by $\det \Omega$ and leaves the identical Schur complement and Hermitian part in that proof. Consequently the bound on $\det(Y^\dagger Y)$ uses $\delta_{[j]}^{-1}$ in place of δ^{-j}. Apply Lemma 85.8 with column radius $3M_0^j/\delta_{[j]}$ and use the entry bounds on the two residual rectangular blocks and the skew center. The combined degree in $M_0^j/\delta_{[j]}$ is $4hj + 3j^2$, as in Theorem 85.12. It follows that the integral of $(\mathcal{A}(g)/\mathcal{A}(y))^{d-j}$ on this shell, now over both z and u_0, is at most

$$\frac{e^{Crj} a^{2j}}{(d!)^j} M_0^K \, \delta_{[j]}^{-E_j} \prod_p \delta_{[p]}^{-2u_p}, \qquad K = j(4hj + 3j^2) + 2\sum_p p u_p.$$

Here $a \le N$ is the scalar of the negative test vector. All fixed numerical factors from the z-disks cost $e^{O(j^2)}$.

At this place let I consist of the endpoints with $|\tau a_i| \ge 1$. On the nonempty shell $M_0^{J_i}\delta_{[J_i]}^{-1} \ge 1$ for each of these endpoints. We may multiply the last majorant by $\prod_{i\in I}(M_0^{J_i}/\delta_{[J_i]})^{\eta_i}$. Put $K' = K + \sum_{i\in I} J_i\eta_i$ and $p_0 = K'+1$. Integrate the shell bound against $p_0 H_0^{-p_0-1}\, dH_0$ on $[1,\infty)$. As in Theorem 85.12, apply weighted Cauchy–Schwarz on the full $J\backslash G$, using the ratio of the two full terminal exterior norms to power $d-j$ and the ratio of the two quotient heights to power p_0. The second squared integral, on changing variables by w, uses w^{-1}, whose block coefficients are unit signs times $\bar{c}_i$. All the absolute prefix products in its shell bound are therefore the same. The remaining normalized base moment has exponent $2K'+2j+1 = O(j^2 s + j\log(k+1))$. For a nonintegral exponent take the next integer, as $\nu \ge 1$. This is less than $4r^2$ for large x, and Theorem 83.3 bounds its logarithmic cost by $O(j^2 s^2 \log b) = o(rj)$.

Finally replace $\delta_{[p]}$ by $|\tau G_p|/(2T)^p$. The frequency dilation acts trivially on z, since it is scalar on the prefix. Its factor on the quotient density is therefore $T^{-j(n+j)}$ as in Lemma 78.12. In the factors for $i \in I$ use $G_{J_i} = Q^{J_i} a_i$ and $Q \ge 1$. The exponent of T left over is at most

$$jE_j + 2\sum_p p u_p + \sum_{i\in I} J_i\eta_i - j(n+j) = O(j^2 s + j\log(k+1)).$$

Its logarithmic cost is $o(rj)$ because $\log T = O(s\log b)$ and $s^3 \log b = o(k)$. The powers of two cost $e^{O(rj)}$, proving the assertion. □

85.15 The terminal determinant for a reversed block

At a real place keep the Euclidean Witt basis used in Theorem 85.4. Write $h = s - j$, $d = k+1$ and $\delta = |c_\tau|/2$ at unit terminal frequency. The cross block is C_0, with C_0/δ Euclidean unitary. For the full isotropic exterior vector P and the negative unit terminal vector e, put

$$\alpha(g) = \|g^{-1}P\|, \qquad \mathcal{A}(g) = \|g^{-1}(P \wedge e)\|.$$

Both descend to the left Bessel quotient. In the h residual Witt pairs, the definite column and the terminal line, decompose the rectangular radical coordinate into row blocks

$$W = pA + qB + X + e\eta.$$

Here A, B are h by j, X is k by j and η is one by j. Denote the skew-Hermitian center coordinate by V and the upper right block by Z.

Lemma 85.15 (A Gram determinant in the positive and terminal coordinates)**.** *Choose the minimal representative g of its Bessel coset as in Lemma 83.21. There exist $a \geq 1$ and a positive-column vector v_+ of squared length $a^2 - 1$. For $y = w^{-1}u_0 g$ form the d-by-j matrix*

$$Y = \begin{pmatrix} X - v_+\eta/a \\ \eta/a \end{pmatrix}.$$

Then

$$\det(Y^\dagger Y) \leq 2^j \sqrt{2}\, \delta^{-j} \frac{\mathcal{A}(y)}{\mathcal{A}(g)}, \qquad a \leq \nu(g).$$

These conclusions hold for every residual scalar λ in Proposition 78.7.

Proof. Use the full-flag parabolic and remove its upper unipotent on the left, retaining a possible right compact factor. As shown in Lemma 83.21, the middle block is $m_0 \in U(k,1)$. A phase multiple of $m_0 e$ is $v_* = v_+ + ae$. We have $\mathcal{A}(g)/\alpha(g) = \sqrt{2a^2 - 1}$ and $\|g^{-1}(P^- \wedge v_*)\| = 1/\alpha(g)$, where P^- is the exterior product of all the opposite unit flag vectors. In particular $a \leq \|m_0\| \leq \nu(g)$.

Pair $P^- \wedge v_*$ with $u_0^{-1} w (P \wedge e)$ by the determinant of Hermitian pairings. In the first j columns remove the right factor C_0. Their blocks are now $Z^\dagger, -W, I_j$. Each residual column has a factor λ of modulus one. After removing them, the pairing matrix is

$$\begin{pmatrix} Z^\dagger & B^\dagger & -\eta^\dagger \\ -A & I_h & 0 \\ -(v_+^\dagger X - a\eta) & 0 & -a \end{pmatrix}.$$

This follows by applying u_0^{-1} to the residual flag vectors. Its top row block on a middle vector z is $W^\dagger H z$ for the middle Hermitian form H. The Schur complement of $\mathrm{diag}(I_h, -a)$ is

$$D = Z^\dagger + B^\dagger A - \eta^\dagger \eta + \eta^\dagger v_+^\dagger X/a.$$

Applying g^{-1} to both exterior vectors preserves their Hermitian pairing. Its modulus is at most the product of their Euclidean norms, since the sign matrix of the Hermitian form on the exterior power is Euclidean unitary. The factors removed from its determinant have modulus $a\delta^j$. Thus

$$|\det D| \leq \mathcal{A}(y)/(a\delta^j \alpha(g)).$$

Now $Z + Z^\dagger = -W^\dagger H W$. Substituting the four row blocks of W and using $\|v_+\|^2 = a^2 - 1$ gives $-D - D^\dagger = Y^\dagger Y$. For $S = Y^\dagger Y/2 > 0$, congruence by $S^{-1/2}$ changes $-D$ into the identity plus a skew-Hermitian matrix. Its determinant has modulus at least one, by diagonalizing the skew-Hermitian part. It follows that $2^{-j} \det(Y^\dagger Y) \leq |\det D|$, also when Y has deficient rank because the left side then vanishes. Finally $\sqrt{2a^2 - 1}/a \leq \sqrt{2}$, which proves the assertion. □

85.16 The Gram bound for distinct negative vectors

At unit terminal frequency write a minimal representative g as in Lemma 83.21, with middle block $m_g \in U(k,1)$. Put $v_* = m_g e$, $Z_g = \|m_g^{-1} e\|$, $N = \nu(g)$. For $y = w^{-1} h_0 \widehat{z} u_0 g$, set

$$e_1 = h_0^{-1} m e, \qquad a_1 = |h(v_*, e_1)|, \qquad \Omega = z^* \tau C/2.$$

Write W for the $(k+1)$-by-s rectangular block of u_0 and Z for its upper right block. As before $\mathcal{A}(g) = \|g^{-1}(P \wedge e)\|$ for the full isotropic exterior vector P.

Lemma 85.16 (Change of rectangular coordinates for the two terminal lines)**.** *There is a complex linear change from W to Y, at fixed g, h_0, with*

$$dW = a_1^{2s} dY, \qquad \|Y\| \le 3N\|W\|, \qquad \det(Y^\dagger Y) \le \frac{2^s Z_g}{a_1 |\det \Omega|} \frac{\mathcal{A}(y)}{\mathcal{A}(g)}.$$

Proof. Let $H = \mathrm{diag}(I_k, -1)$. The first s columns of $u_0^{-1}\widehat{z}^{-1}h_0^{-1}w$ are $(Z^\dagger, -W, I_s)^{\mathrm{col}}\Omega$. The column for e is $(W^\dagger H e_1, e_1, 0)^{\mathrm{col}}$. Pair their exterior product against $P^- \wedge v_*$, where P^- is the product of the opposite isotropic vectors. After removing Ω, the terminal diagonal entry is $h(v_*, e_1)$, whose modulus a_1 is at least one. In fact $m_g^{-1}e_1$ is a negative unit and the modulus of its e-coordinate is a_1. The Schur complement is

$$D = Z^\dagger + W^\dagger H e_1 \, h(v_*, W)/h(v_*, e_1).$$

The transformed test exterior vector has norm $1/\alpha(g)$, with $\alpha(g) = \|g^{-1}P\|$, and $\mathcal{A}(g)/\alpha(g) = Z_g$, as in Lemma 85.15. Bounding the pairing by the Euclidean norms therefore gives

$$|\det D| \le \frac{\mathcal{A}(y)}{a_1 |\det \Omega| \alpha(g)}.$$

Write $m_g^{-1}W = (X', \eta')^{\mathrm{col}}$ and take a phase multiple of $m_g^{-1}e_1$ with positive final coordinate, say $v_1 + a_1 e$. The phase cancels in the quotient defining D. Then $\|v_1\|^2 = a_1^2 - 1$. Since $Z + Z^\dagger = -W^\dagger H W$, substitution and completion of the square give

$$-D - D^\dagger = Y^\dagger Y, \qquad Y = \begin{pmatrix} X' - v_1\eta'/a_1 \\ \eta'/a_1 \end{pmatrix}.$$

Indeed the cross terms are $-X'^\dagger v_1 \eta'/a_1$ and its adjoint, and the coefficient on $\eta'^\dagger \eta'$ is one. Thus $2^{-s}\det(Y^\dagger Y) \le |\det D|$, as in Lemma 85.15. The two matrix changes have absolute complex determinants one and a_1^{-1} on each column, respectively. Finally $\|m_g^{-1}\| \le N$ and $\|v_1\|/a_1 \le 1$. This gives the Jacobian, the norm bound and the claimed determinant inequality. □

85.17 The orbit measure with a fixed positive vector

Use the unit-frequency coordinates of Proposition 80.3 at one real place. Their Hermitian form on the three blocks has identity off-diagonal blocks pairing the two d-spaces and identity on the positive k-space. Let $\widetilde{B}$ stabilize the decorated flag of P_0 in the extended group, and B be its subgroup fixing t. The latter is precisely the original Bessel subgroup. Write $\gamma_0 = 1/\sqrt{2}$. Then $t = (\gamma_0 \mathbf{e}_d, 0, \gamma_0 \mathbf{e}_d)$. The common stabilizer of the original pair is denoted by J.

Proposition 85.17 (Coordinates and Haar measure on the original fiber)**.** *Use unit mass on compact groups and ordinary additive measures on the entries of the common unitriangular groups in the monomial frames. There are coordinates on $J\backslash B$ consisting of a block upper unitriangular matrix A with identity diagonal blocks, s arbitrary vectors in $\mathbb{C}^k$, a skew-Hermitian s-square, and one point on a sphere of radius*

$$R = \sqrt{2\,\tau\mathcal{G}}/|\tau\Delta|.$$

Their quotient measure is 2^s times ordinary additive measures and sphere probability, for the original Euclidean Witt measures on B. When $D_s = 0$ the radius is $\sqrt{2}$.

Proof. Every element of $\widetilde{B}$ has a unique expression

$$\mathrm{diag}(A, h, A^{-*})\, n(U, Z_0), \qquad n(U, Z_0) = \begin{pmatrix} I & -U^\dagger & Z_0 \\ 0 & I & U \\ 0 & 0 & I \end{pmatrix}, \qquad Z_0 + Z_0^\dagger = -U^\dagger U.$$

Here initially A is any upper unitriangular d-square, and $h \in U(k)$. This follows from flag preservation and the three block equations for the Hermitian form. At t the condition that the displayed matrix fix t gives $U_d = 0$, $Z_0 \mathbf{e}_d = A^{-1}\mathbf{e}_d - \mathbf{e}_d$. For any A, h these equations can be solved, for example by taking $U = 0$ and completing the last column to a skew-Hermitian matrix. Thus B realizes every upper unitriangular change of its extended frame. Left and right changes within B allow the two flags to have cross matrix C from Proposition 80.3. Denote the resulting unit-frequency relative isometry by w.

There is $b_0 \in \widetilde{B}$ fixing the first frame pointwise and taking $P_- C$ to wP_0. Indeed these second frames have the same cross matrix with P_0, and both are totally isotropic. The resulting isometry of the two nondegenerate spans extends over their positive perpendiculars. Write $t' = b_0^{-1} t = (a, m, \gamma_0 \mathbf{e}_d)$. Pairing t' with P_0 gives its last block as displayed. Pairing it with $P_- C$ gives $C^* a = \gamma_0 \mathbf{e}_d$, since w fixes t. In particular a lies on the first coordinate line of the last reversed block. Its positive middle part satisfies $\|m\|^2 = \tau \mathcal{G} / |\tau \Delta|^2$ by Proposition 80.3.

Left multiplication by b_0^{-1} sends B onto the elements of $\widetilde{B}$ taking t to t'. In the displayed coordinates their defining equations are

$$U_d = h^{-1} m / \gamma_0, \qquad Z_0 \mathbf{e}_d = A^{-1} a / \gamma_0 - \mathbf{e}_d.$$

The last diagonal entry has the required real part because $2\gamma_0 \operatorname{Re} a_d + \|m\|^2 = 1$. Consequently the first s columns of U are free, as is the leading s-square of $(Z_0 - Z_0^{\dagger})/2$. These data determine every remaining entry of Z_0 by its Hermitian-part equation.

In these two opposite frames the common stabilizer consists of block diagonal matrices $\operatorname{diag}(D, h_0, D^{-*})$. Indeed preserving both isotropic spaces eliminates both off-diagonal radicals. The conditions that D and $C^{-1} D^{-*} C$ be upper unitriangular allow just the entries within individual diagonal blocks of D, as in Proposition 78.9. They fix a and the last dual coordinate vector. Fixing t' thus requires exactly $h_0 m = m$. In particular h_0 runs over $U(k-1)$. Taking the left quotient removes the diagonal blocks from A and replaces h by $h^{-1} m / \gamma_0$ on its sphere.

For the density, use product measure on the free entries just described, on the upper entries of A, and on h. Left multiplication by a fixed element of $\widetilde{B}$ transports this product between its two fibers whenever the target vector changes. It left translates A, h. For fixed A, h it translates U by a fixed matrix, by conjugating the left unipotent through $\operatorname{diag}(A, h, A^{-*})$. For fixed A, h, U it translates the free skew entries as well, by the multiplication law for $n(U, Z_0)$. All three changes have Jacobian one. In particular the product at $t' = t$ is Haar on B. To compare it with the original Witt normalization it suffices to compare at the identity. The ks positive rectangular entries, the s-square skew center, the compact factor, and the upper entries among the first s indices agree to first order. If η_i is the negative middle coordinate in original rectangular column i, the image of $p_d = (t + e)/\sqrt{2}$ has first-order coefficient $\bar{\eta}_i / \sqrt{2}$ on p_i. Thus $d^2 A_{id} = d^2 \eta_i / 2$. This proves the factor 2^s. The block decomposition of A has product Haar measure by triangular multiplication. Dividing by the measures on the common groups gives the stated quotient. All these calculations are on the fiber of the original group fixing t. □

85.18 The actual Bessel integral at the displaced real flag

Take the integral ratios of Theorem 9.21 and the period in Proposition 64.41. Denote the ordinary integral with probability Haar measures on $[N]$ and $[H]$ by $P_{\chi_*}(A, \phi)$. A scalar attached to the period normalization multiplies this integral. Fix a real place τ and its chosen complex embedding σ. For the one-parameter root of Lemma 99.20, in the original signed coordinate plane, write

$$\chi(n_\tau(t)) = \exp(4\pi i \operatorname{Re}(\alpha_\tau t)), \qquad Q_\tau = |\alpha_\tau \, \sigma(c_s)|.$$

The coefficient α_τ is nonzero since this is the simple root adjacent to the marked middle line. This definition retains the scalar in the given additive character.

Theorem 85.18 (Decay on the fixed row type at the new archimedean frequency)**.** *Use a smooth square-integrable row A with the level, compact type and Casimir bounds of Theorems 99.24 and 99.26. Let r be the rank, $d=[F:\mathbb{Q}]r^2$, I the index of its finite invariance subgroup in the standard lattice level, B_0 the height bound in the restriction estimate and L the parameter for the derivatives in Lemma 99.25. For every integer $k\ge 0$,*

$$|P_{\chi_*}(A,\phi)|\le \sqrt{I}\,\exp\{Cd\log(2dB_0)\}\frac{\{Cr(L+k+d+1)\}^{k+d}}{(4\pi Q_\tau)^k}\|A\|_2\,\|\phi\|_2.$$

The constant C is absolute. On the common intervals take the finite level and real parameter bounds of Theorem 65.2. The rational diagonal in Theorem 9.21 can also be chosen with $Q_\tau\ge T$. With this choice, uniformly for sufficiently large x and all the indicated rows and columns,

$$|P_{\chi_*}(A,\phi)|\le \exp(-cQ_\tau/r)\|A\|_2\,\|\phi\|_2$$

with $c>0$ absolute. The logarithmic upper bound for $2gW$ in that theorem increases by at most $O(1+\log T+\log^+(1/|\alpha_\tau|))$.

Proof. In the three-dimensional coordinate block u_s,e,v_s the diagonal scales u_s by c_s, fixes e and scales v_s by $\bar c_s^{-1}$. Conjugating the matrix $n_\tau(t)$ gives $n_\tau(\sigma(c_s)t)$, including the quadratic term in its upper corner. Put $\xi=\overline{\alpha_\tau\sigma(c_s)}/Q_\tau$. Along $b\mapsto n_\tau(b\xi)$, $b\in\mathbb{R}$, the conjugated character is $\exp(4\pi ibQ_\tau)$. This root commutes with the column group. Right translation in $[N]$ and differentiation k times therefore give

$$P_{\chi_*}(X_\xi^k A,\phi)=(4\pi iQ_\tau)^k P_{\chi_*}(A,\phi).$$

Differentiation is valid on the compact period domains, where the smooth row and all the derivatives in use are bounded. Cauchy–Schwarz on the two probability spaces bounds the absolute value on the left by $\|\phi\|_2\sup_{n,h}|(X_\xi^kA)(nh)|$. Theorem 99.24 bounds that supremum by the prefactor $\sqrt{I}\exp\{Cd\log(2dB_0)\}$ times the largest L^2 norm of derivatives through order d of X_ξ^kA. The generator X_ξ has Frobenius norm two, as in Theorem 99.26, since $|\xi|=1$. Thus the words have length at most $k+d$ and Lemma 99.25 bounds each norm by $\{Cr(L+k+d+1)\}^{k+d}\|A\|_2$. This proves the first inequality for the given global automorphic integral.

In the construction of Theorem 9.21, keep the same congruence class of the rational integer a modulo J. Add $2gT/|\alpha_\tau|$ to the lower bound there for a, and take the first representative above the new bound. The logarithmic increase asserted in the statement follows from $a\le J$ plus the new lower bound. The same estimate of the conjugates of θ_s still gives $|\sigma(c_s)|\ge W/2=a/(2g)$. This proves $Q_\tau\ge T$ while retaining all the finite conditions.

Fix an absolute $\kappa>0$ small enough and set $k=\lfloor \kappa Q_\tau/r\rfloor$. The level and weight hypotheses give $L\le\exp(C'x^\varepsilon)$, polynomial bounds on r,d, and a polynomial bound for $\log I+\log B_0$. Lemma 65.1 gives $\log T\ge(16-o(1))mx^\varepsilon$ with $m\to\infty$. Thus $Cr(L+k+d+1)/(4\pi Q_\tau)\le 1/2$ for large x, uniformly for $Q_\tau\ge T$. The k powers contribute 2^{-k}. The remaining logarithm, apart from the input norms, is at most

$$\tfrac12\log I+Cd\log(2dB_0)+O(d\log(2Q_\tau)).$$

This is $o(Q_\tau/r)$ uniformly in the same range. Indeed the first two terms grow polynomially and $\log(2Q)/Q$ decreases for $Q\ge T$ for large T. Absorbing the multiplier proves the second inequality. □

For example, suppose a nonzero algebraic period p in a number field J_0 has conjugates $a_\nu P_{\chi_*,\nu}(A_\nu,\phi_\nu)$, all satisfying this estimate with $Q_{\tau,\nu}\ge T$ and common rank r. Its denominator ideal $\mathfrak{d}(p)$ then satisfies

$$\frac{\log N\mathfrak{d}(p)+\sum_\nu \log(|a_\nu|\,\|A_\nu\|_2\,\|\phi_\nu\|_2)}{[J_0:\mathbb{Q}]}\ \ge\ cT/r.$$

Indeed $|N(p)|\ge (N\mathfrak{d}(p))^{-1}$ because its numerator ideal is integral. Multiplying the analytic bounds over the embeddings proves the assertion. Applying it to the designated theta column would require identifying that algebraic period and its conjugates and determining the scalars a_ν.

86 References

[1] A. Ichino, *Theta lifting for discrete series representations of real unitary groups*, 2020. Preprint.

[2] R. Beuzart-Plessis, *A local trace formula for the Gan–Gross–Prasad conjecture for unitary groups: the archimedean case*, Astérisque **418** (2020). Sections 1.5, 2.6 and 7.2, Lemma 6.5.1(ii), and Theorem 7.2.1.