The generalized Riemann hypothesis for Dirichlet L-functions

Volume XII · Bessel models and projected periods

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Abstract

We construct Bessel functionals for specified characters and compact types. At split finite places we match sections of complementary ranks. For the prescribed auxiliary type we prove that an archimedean translation close to the identity gives zero on the original spherical rows when its other valuations are at good split primes. We give a different rational translation compatible with every marked exchange and every unmarked integral root. We compute the contribution of the normalized terminal input to the determinant of the signed period image.

The guide in Volume I lists the sections and page counts for the series.

Contents

64 The global Bessel integral . . . 3
64.1 An integral frame for the terminal frequency . . . 3
64.2 The Gram determinant of two terminal frames . . . 4
64.2.1 A binary unitary congruence and its real weight . . . 6
64.3 Zero Gram determinant and separation of isotropic subspaces . . . 8
64.4 Summing the period seeds above the height cutoff . . . 10
64.5 The full projected period at fixed finite level . . . 12
64.6 Two terminal frequencies at a common finite level . . . 14
64.7 Absolute convergence of the global Bessel Gram matrix . . . 15
64.8 Separation of Bessel periods at one finite place . . . 19
64.9 Vanishing on the proposed lowest compact types . . . 21
64.10 Global Bessel coefficients at the prescribed finite levels . . . 23
64.11 Fourier inversion between two positions of the Bessel period . . . 26
64.12 A Fourier operator on the refined column level . . . 28
64.13 Bessel exchange on the original signed auxiliary rows . . . 29
64.14 When the unmarked places annihilate the Bessel period . . . 31
65 Discrete series and real Bessel periods . . . 32
65.1 The size of the global integral on fixed compact types . . . 32
65.2 A common real Bessel model for all translated columns . . . 35
65.3 The prescribed row and the interior column weights . . . 37
65.4 Full periodization below the auxiliary height bound . . . 39
65.5 The prescribed compact row type and real translations . . . 41
65.6 Full automorphic norms for real sections with a height moment . . . 43
65.7 The prescribed discrete series at the prime terminal frequency . . . 45
65.8 Which part of the real period can carry the discrete series . . . 47
65.9 Where the normalized real Bessel functions have their mass . . . 48
65.10 The complex projected period on the full real quotient . . . 50
65.11 An exact comparison of two projected Bessel periods . . . 51
66 Unfolding and comparison along the row . . . 52
66.1 Cuspidality at the original period level . . . 52
66.2 The specified compact test and its Petersson norm . . . 53
66.3 Character matching in the first seesaw . . . 55
66.4 Unfolding the Bessel period on a full frame . . . 56
66.5 Realizing group convolution in the frame integral . . . 58
66.6 A finite character on a proper block of the definite core . . . 60
66.7 A projected Fourier coefficient on the original row group . . . 61

66.8 A prescribed real projection after restriction 62
66.9 Unfolding the first nontrivial Fourier–Jacobi period 63
66.10 The Bessel character in the terminal coordinates 65
66.11 Weighted selection of an individual Hermite test 66
66.12 A character-projected period at the larger terminal rank 67
66.13 A lower bound on the original definite real group 68
66.14 Passage from a Bessel period to a Jacobi period 69
66.15 Local integrals for horizontal pairs of opposite parity 70
66.16 Passage from a Jacobi period to a Bessel period 72
67 The prescribed initial and terminal periods 73
67.1 Nonvanishing at the finite level of the initial restriction 73
67.2 The first restriction on the original intervals 74
67.3 A single real estimate for the prescribed initial row 76
67.4 A quantitative period after two horizontal lifts 77
67.5 Nonvanishing of the terminal Bessel period 78
67.6 Normalized Bessel functionals under a weight shift 79
67.7 A terminal period bound from the second restriction 81
67.8 A terminal lower bound from the original bottom vector 82
68 Restriction, levels and Fourier coefficients 83
68.1 The initial degrees and the second row spectrum 83
68.2 Integral coefficient lattices on the graph correspondence 84
68.3 Restriction of a Fourier coefficient to a smaller row 85
68.4 The degree required at the high real places 86
68.5 The finite lattices in the restricted group 87
68.6 The finite Fourier coefficient at every horizontal step 88
68.7 The prescribed column type at every real place 89
68.8 A bounded translation at one marked place 90
68.9 Matching rectangular tests at a split finite place 92
68.10 A fixed functional on the matched frames 94
68.11 The cost of the prescribed terminal Hecke modification 96
68.12 The determinant of the normalized input on a common terminal prime 97
69 References 99

64 The global Bessel integral

64.1 An integral frame for the terminal frequency

Use the common fields, intervals, words and primes of Theorem 9.13. The terminal vector in the Bessel datum has norm $-T$. We take it to be $z = \varpi e$, where $T = \varpi\bar{\varpi}$, with the Witt flag and the group $B = N \rtimes H$ of Section 99.16. At $v \mid T$ choose the split factor in which ϖ has valuation one as the first factor. Put $O = \mathbb{Z}_T$, and lift the given nonzero column covector to $\ell_v : O^k \to O$. In that factor set

$$M_v = \{(w, z_0) \in O^k \oplus O : z_0 \equiv \ell_v(w) \pmod{T}\}.$$

Let a_v^0 act on the middle space by $(w, z_0) \mapsto (w, \ell_v(w) + Tz_0)$, and by the identity on all u_i^+, v_i^+. The dual action in the other factor defines a split unitary element. Set $a^0 = \prod_{v|T} a_v^0$, extended by one elsewhere, and $K^0 = a^0 K_\sigma (a^0)^{-1}$.

Theorem 64.1 (The column level, terminal character and integral frame)**.** *The finite level K^0 induces the full prescribed covector level $K_\pi(\ell)$, with*

$$B_f \cap K^0 = (N_f \cap K^0) \rtimes K_\pi(\ell).$$

At $v \mid T$, among locally integral terminal frequencies $\beta \in O \times O$, precisely those with first component in TO have trivial character on $N_v \cap K_v^0$. In particular χ_ϖ is trivial on $N_f \cap K^0$. The rational vectors $u_1, \dots, u_s, \varpi e$ span a saturated submodule of the adelic integral lattice with this stabilizer at every finite place.

Proof. The first lattice at v is $O^s \oplus M_v \oplus O^s$ in the Witt splitting. The second is its dual for the Hermitian pairing. For $nh \in B_v$ the induced action on the middle quotient of the flag is $(w, z_0) \mapsto (hw, z_0)$. This preserves M_v in both directions exactly when $h \in \mathrm{GL}_k(O)$ and $\ell_v h = \ell_v \bmod T$. Indeed projection of M_v onto O^k is surjective, and the residue relation determines the condition on ℓ_v. The block embedding of each such h preserves the entire lattice pair. Thus if nh preserves it, so do h and n separately. Away from T the levels are those of Lemma 64.30.

In the two split factors the first $s-1$ simple coordinates of an integral unipotent at this level are integral because the lattices on both strings of isotropic lines have the original bases. In the first factor Te^+ is in M_v, giving $d_s(n)^+ \in T^{-1}O$. In the second factor e^- is integral, since it pairs with M_v by the integral last coordinate up to sign. Hence $d_s(n)^- \in O$. These facts give sufficiency by the conductor of the additive character of $\mathbb{Q}_T$. For necessity take the elementary shear adding $c(z_0 - \ell_v(w))u_s^+/T$ to the middle vector (w, z_0) in the first factor, with $c \in O$. It and its inverse preserve the lattice because the numerator is a multiple of T on M_v. Its dual fixes e^-. It belongs to N_v, its earlier simple coordinates vanish and $d_s(n) = (c/T, 0)$. Triviality for all c forces $\beta^+ \in TO$. Away from T integrality suffices by Lemma 64.30.

The span of Te^+ splits off in M_v. Indeed the vectors $(w, \ell_v(w))$, together with the multiples of Te^+, give a direct decomposition. The first component of ϖ is T times a unit. In the dual lattice e^- pairs to a unit with some $(w, \ell_v(w))$, since the residue covector is nonzero. It is therefore primitive and extends to a basis over the discrete valuation ring. The second component of ϖ is a unit. The u_i are separate basis lines in both factors. Away from T the scalar ϖ is a unit and $u_i = e_{+,i} + e_{-,i}$ is part of an integral basis in its coordinate plane, including at places above two. This proves saturation. Replacing ϖe by $\varpi e + bu_s$ for any $b \in \mathcal{O}_E$ gives the same saturated module. □

Theorem 64.2 (Preservation of both row and unipotent volumes)**.** *The row volumes for K^0 and K_σ agree, and*

$$\mathrm{vol}_f(N_f \cap K^0) = \mathrm{vol}_f(N_f \cap K_\sigma).$$

For the norm-T frequency the class-coordinate constant c_{ρ,K^0} of (70.17) is consequently c_{ρ,K_σ} for every word. In the column probability norm the ratio of squared-norm constants is

$$\frac{C_{K^0}}{C_{K_\sigma}} = (T^k - 1)^f. \tag{64.1}$$

Summation over the entire stabilizer $Q(F_+)$ of the isotropic s-space is isometric on these refined columns, including their zero-sum subspaces, with the norms of Theorem 88.1.

Proof. The row assertion follows from conjugacy in the unimodular group G_f of finite adelic points. At $v \mid T$ the conjugating element lies in the Levi of the full flag. Its middle block has determinant T. Among the independent upper blocks of N_v, a block from the middle to one of the first s lines acquires determinant T^{-1} and a block from one of the last s lines to the middle acquires determinant T. These occur s times each. The other blocks have determinant one. Product additive measure in the blocks is Haar measure as in Proposition 70.25, so the unipotent modulus is one as claimed.

Theorem 64.1 gives the finite decomposition and character compatibility needed in Lemma 70.23 and Theorem 70.24. The column stabilizers remain trivial and there are $(T^k - 1)^f$ times the original number of classes. The class-coordinate constant and (64.1) now follow from those proofs and the column probability norm of Theorem 78.16. In the proof of Theorem 88.1 the intermediate unipotent character separations depend only on the rational flag. The finite support at the last separation requires compact row levels at every finite place and unipotent reduction at g_1, g_2. Both hold here, and the input seeds are defined by the finite decomposition just proved. The same isometry follows for the norm-T frequency. □

For comparison, at K^0 the frequency 1 has zero Bessel coefficient on right invariant rows. In the inner integral at h_0, translate its unipotent variable on the right by $h_0 n h_0^{-1}$ for one of the shears in the preceding proof. The row is fixed, while $\chi_1(h_0 n h_0^{-1}) = \chi_1(n)$ can differ from one. Section 70.5 gives a level supporting both frequencies. The construction here chooses the frequency attached to the unnormalized terminal vector and makes that vector integral together with the entire isotropic frame.

64.2 The Gram determinant of two terminal frames

Work in the Hermitian space V over E/F_+ of Section 99.16. Its signature at every real place is $(s+k, s+1)$. A frame consists of a basis $u_1, \dots, u_s$ of a totally isotropic space L and a vector $z \in L^\perp$ of norm $-n$, where $n \in F_+$ is totally positive. Let $Y = L + Ez$. Use primes for the data of a second frame. Write $I = L \cap L'$, $d = \dim_E I$, and $A = (Y + Y')^\perp$. For $y = (u_1, \dots, u_s, z)$ and the corresponding y' put

$$Z_{ij} = h(y_i, y'_j), \qquad a = \det Z_{1:s,1:s}, \quad b = \det Z, \quad \mathcal{G} = b\bar{b} - nn'a\bar{a}.$$

Theorem 64.3 (The common definite subspace)**.** *The radical of A is I. It has a complement C over E that is positive definite at every real place. Its dimension is k if $Y \cap Y'$ contains a vector of negative norm at a real place, and $k-1$ otherwise. In the first case the vector can be chosen rational over E and totally negative. In both cases $U(C)$, acting identically on $C^\perp$, fixes both frames pointwise. Its projection to each of the two definite column groups is the usual unitary action on a subspace of the stated dimension.*

If $d = 0$, the number $\mathcal{G}$ is totally positive when $\dim C = k - 1$, and is zero when $\dim C = k$. For $d > 0$ these same tests hold in $I^{\perp}/I$, using any bases of $L/I, L'/I$ and the images of z, z'.

Proof. At a real place the form on Y is negative semidefinite with radical L. Split off hyperbolic planes for a basis of L. The remaining space has signature $(k, 1)$, and the image of z has negative norm. Its perpendicular there is positive definite. Thus $Y^{\perp}$ is positive semidefinite with radical L. The same assertions hold with primes. In particular the vectors of norm zero in A lie in both L and L'. Conversely I is orthogonal to both frames and to A. This proves the assertion on the radical. Any E-linear complement to I in A is therefore totally positive.

Inside $Y \cap Y'$, the kernel of projection to Y/L is exactly I. Indeed any vector in that kernel has norm zero and its membership in Y' puts it in L'. Consequently $\dim(Y \cap Y') = d + j$ with $j = 0$ or 1. When $j = 1$ choose an E-vector in this intersection outside I. Its coefficient of z is nonzero, so its norm is negative at every place. Dimensions of the intersections are unchanged under field extension, proving the assertion starting at a single real place as well. Nondegeneracy of V gives

$$\dim C = \dim V - 2(s + 1) + (d + j) - d = k - 1 + j.$$

Since $Y, Y' \subset C^{\perp}$, the extended action fixes every displayed frame vector. On $Y^{\perp}/L$ the map from C is isometric, and so is the map to $(Y')^{\perp}/L'$. In each case the group fixes the orthogonal complement of its image. These are the two column representations on their defining Hermitian spaces. The projection of this group to either flag Levi fixes the terminal line and every graded flag vector.

The Gram matrix of the concatenated list (y, y') has off-diagonal blocks Z, Z^{*}. The two diagonal blocks are zero apart from their final entries $-n, -n'$. A nonzero term of the determinant selects the same number of entries from each diagonal block by counting its columns. Terms selecting none sum to $(-1)^{s+1} b\bar{b}$, by interchanging the two blocks of columns. Selecting both final entries leaves the two size-s off-diagonal blocks and contributes $(-1)^{s} nn' a\bar{a}$. Hence

$$\det \operatorname{Gram}(y, y') = (-1)^{s+1} \mathcal{G}. \tag{64.2}$$

For $d = 0, j = 1$ the concatenated list is dependent. For $d = j = 0$ it is a basis of $A^{\perp}$, with A positive definite of dimension $k - 1$. The signature of $A^{\perp}$ is $(s + 1, s + 1)$. The sign in (64.2) now gives total positivity. For $d > 0$ split off the hyperbolic partners of I to see that $I^{\perp}/I$ has signature $(s - d + k, s - d + 1)$. Both spaces Y, Y' lie in $I^{\perp}$. Their images intersect in $(Y \cap Y')/I$ and their isotropic parts now have zero intersection. The argument for $d = 0$ applies there, with the determinant of an empty minor understood as one. A change of either isotropic basis multiplies the reduced invariant by the norm of the determinant of the change, which preserves its sign and its vanishing. □

Theorem 64.4 (A divisibility bound at both auxiliary primes)**.** *Use the level K^{0} and $z = \varpi e$ in Theorem 64.1. For $\gamma \in G(F_{+})$ take $y' = \gamma y$. If $\gamma_f \in B_f K^{0} B_f$, then*

$$\frac{\mathcal{G}}{(g_1 g_2)^{2(s+1)}} \in \mathcal{O}_{F_{+}}.$$

In particular, for every such pair with $\mathcal{G} \neq 0$,

$$\prod_{\tau : F_{+} \hookrightarrow \mathbb{R}} \tau(\mathcal{G}) \;\geq\; (g_1 g_2)^{mf}, \qquad m = 2(s + 1). \tag{64.3}$$

Proof. Every $b_0 \in B_f$ acts on the frame by an upper unitriangular matrix. Indeed the successive u_i are fixed modulo earlier lines and the terminal vector is fixed modulo L. Both exterior

products $u_1 \wedge \cdots \wedge u_s$ and $u_1 \wedge \cdots \wedge u_s \wedge z$ are therefore fixed. The minors a, b are the Hermitian pairings of each of these exterior products with its translate. In a finite factorization $\gamma_f = b_1 t b_2$, $t \in K^0$, invariance of the Hermitian pairings gives the same two minors with t in place of γ_f. The frame is integral by Theorem 64.1 and t preserves its ambient integral Hermitian lattice. At T the two lattices in the split factors are paired duals. Thus the minors, and hence $\mathcal{G}$, are integral at every finite place.

At a place over g_1 or g_2, write g for the underlying rational prime. In the first split factor take the Witt order $u_1, \dots, u_s, W_0, e, v_s, \dots, v_1$. It is an integral basis there as g is odd. The specified row group is upper unitriangular modulo g, with the k entries of the e-column in the W_0-rows equal to zero. Hence the reduced matrix sends the ordered frame $(u_1, \dots, u_s, z)$ to itself up to upper unitriangular change of basis. Here ϖ is a unit. In the other factor u_i is dual to v_i, up to the chosen unit pairing. The v_i occur in reverse order, so the inverse-transpose action sends u_i to itself modulo the earlier u_j. The e-row of the inverse upper unitriangular matrix has only its diagonal and possible entries in the v-columns. Thus the second factor also sends z to itself modulo the u_j.

Lift these residue coefficients in each factor. Subtract from each vector of ty the resulting linear combination of the first frame. All $s+1$ differences lie in g times the local lattice. These column operations on the concatenated list have determinant one. The new Gram determinant has $s+1$ factors of g from the rows and $s+1$ from the columns by sesquilinearity, including the entries where both occur. Equation (64.2) gives the asserted divisibility at every place above both primes, which split completely. If the invariant is nonzero the full Gram list is independent, so $d = 0$, and Theorem 64.3 gives total positivity. The norm of the resulting nonzero totally positive algebraic integer is a positive rational integer. This proves (64.3). □

The two minors, their signed difference and the flag congruence appear in `Formalism/SharpRestriction/CuspSpanNormBounds/BlockEnds.lean`, line 53, line 62 there, and `Formalism/SharpRestriction/CuspSpanNormBounds/IsPrimitiveSub.lean`, line 139. The real sign is also the assertion at `Formalism/PeriodIntegrals/UnitaryGroupMassFormula/CountPositivity/LeadingMinorFullEqProd.lean`, line 97. The bound concerns the unreduced invariant at the specified level. For a pair sharing an isotropic direction this invariant vanishes, even when the reduced determinant is positive.

64.2.1 A binary unitary congruence and its real weight

The normal quotient of a pair of terminal hyperplanes has signature $(1, 1)$. We give an estimate for its integral isometries that is uniform in the CM field and in the point at which the real weight is evaluated.

Let E/F be CM, $f = [F : \mathbb{Q}]$, and take one embedding $\tau : E \hookrightarrow \mathbb{C}$ over each real place. For positive integers Q, T with $T > \max(2, Q)$, use the lattice and group

$$\mathcal{Z} = \{(u, v) \in \mathcal{O}_E^2 : u + v \in Q\mathcal{O}_E\}, \qquad \Gamma = \{g \in U(1,1)(F) : (g-1)\mathcal{Z} \subseteq T\mathcal{Z}\}.$$

Here the Hermitian form is $\mathrm{diag}(1, -1)$. The positive lines at a real place are $\mathbb{C}(1, z)$ with z in the open unit disk $\mathbb{D}$. Matrices act on these lines, hence on $\mathbb{D}^f$ componentwise.

Lemma 64.5 (Separation and stabilizers of positive lines)**.** *The set Γ is a group. Every element has determinant one. No nonidentity element fixes a point of $\mathbb{D}^f$. For $g \neq 1$ its lower left entry c satisfies*

$$\prod_{\tau} |\tau c| \;\geq\; (T/Q)^f.$$

Proof. In the $\mathcal{O}_E$-basis $(1,-1),(0,Q)$ the matrix of g has integral entries and is the identity modulo T. Its determinant has absolute value one at every embedding by unitarity. If the determinant differs from one, their difference belongs to $T\mathcal{O}_E$ with absolute norm at least T^{2f}, but each conjugate has absolute value at most two. Thus $\det g = 1$. The inverse has integral entries in the same basis and is also the identity modulo T. Closure under multiplication follows in this basis.

If g fixes a tuple of positive lines, at each place it preserves both that line and its negative perpendicular, and acts on each by a scalar of absolute value one. Consequently $|\tau(2-\operatorname{tr} g)| \le 4$ at every place. In the lattice basis, $2-\operatorname{tr} g = \det(g-1) \in T^2\mathcal{O}_E$. Since $T^2 > 4$, taking norms gives $\operatorname{tr} g = 2$. The two eigenvalues are then one, and the two invariant lines give a basis, so $g = 1$. In particular if $c = 0$, then g fixes the tuple of lines of slope zero and is the identity. Finally $(Q,0) \in \mathcal{Z}$ gives $Qc/T \in \mathcal{O}_E$ by taking the second coordinate. Its nonzero absolute norm gives the asserted product. □

Give the disk the metric $2|dz|/(1-|z|^2)$, write its distance as $d_{\mathbb{D}}$, and put

$$\Phi(z) = \exp\left(\tfrac{1}{2}\sum_{\tau} d_{\mathbb{D}}(0,z_\tau)\right) = \prod_{\tau}\sqrt{\frac{1+|z_\tau|}{1-|z_\tau|}} \qquad (z \in \mathbb{D}^f).$$

The second formula follows by integration along a radius. Radial projection bounds any path length below by the same radial integral. To see that the action preserves the metric, take a matrix $\left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right)$ in the unitary group. The new slope is $(c+ez)/(a+bz)$, its derivative has modulus $|a+bz|^{-2}$, and the new value of $1-|z|^2$ is $(1-|z|^2)/|a+bz|^2$. Every disk point can be carried to zero by a unitary matrix, for the columns $(1,z),(\bar z,1)$ divided by $\sqrt{1-|z|^2}$ give one carrying zero to z.

Theorem 64.6 (A uniform bound for the sum over a congruence orbit)**.** *For all $z \in \mathbb{D}^f$ and real $p \ge 3$,*

$$\sum_{g\in\Gamma} \Phi(gz)^{-3} \le 10^f, \qquad \sum_{g\in\Gamma,\ g\neq 1} \{\Phi(z)\Phi(gz)\}^{-p} \le 10^f \Phi(z)^{-3}\left(\frac{Q}{2T}\right)^{f(p-3)}.$$

Proof. At each place $g0$ has slope c/a, with $|a|^2-|c|^2=1$, giving $\sinh(d_{\mathbb{D}}(0,g0)/2) = |c|$. Write $y_g = g^{-1}0$. If $g \neq h$, transport the two centres by g and apply Lemma 64.5 to gh^{-1}. The product over the places of $\sinh(d_{\mathbb{D}}(y_{g,\tau},y_{h,\tau})/2)$ is at least $(T/Q)^f$. The products of disks of hyperbolic radius $R = \log 2$ about the centres y_g are therefore disjoint. An intersection would bound that product by $\sinh(R)^f = (3/4)^f$.

Hyperbolic area in radius u, after angular integration, is $2\pi\sinh u\,du$, by substituting the Euclidean radius $\tanh(u/2)$ in the metric area element. A disk of radius R has area $\pi/2$, and the integral of $\exp(-3d_{\mathbb{D}}(z,y)/2)$ over the whole disk is $2\pi\int_0^\infty e^{-3u/2}\sinh u\,du = 8\pi/5$, independently of its centre z. For y in the product about y_g, each distance to z_τ exceeds the distance at the centre by at most R. Integrating the product of the exponential weights over these disjoint sets gives

$$\sum_{g\in\Gamma}\Phi(gz)^{-3} \le \left(e^{3R/2}\frac{8\pi/5}{\pi/2}\right)^f < 10^f.$$

One may first sum over any finite subset and take the supremum, so the estimate also proves convergence.

For $g \neq 1$ the triangle inequality gives $d_{\mathbb{D}}(0,z_\tau)+d_{\mathbb{D}}(0,(gz)_\tau) \ge d_{\mathbb{D}}(0,y_{g,\tau})$. The exponential of half the latter distance is $\sqrt{1+|\tau c|^2}+|\tau c|$, since $d_{\mathbb{D}}(0,g^{-1}0) = d_{\mathbb{D}}(0,g0)$. Multiplying at the places yields $\Phi(z)\Phi(gz) \ge (2T/Q)^f$. Apply this bound to the power $p-3$ and sum the remaining power by the first assertion. □

The orbital symmetry for these frames and the resulting rank estimate are proved together in Section 76.1 and Theorem 76.4.

64.3 Zero Gram determinant and separation of isotropic subspaces

Keep the two rational frames of Section 64.2. Write $L_i = \langle u_1, \dots, u_i \rangle$, put $L_0 = 0$, and use primed notation for the second flag. Define the E-linear successor $S : L \to Y$ by $Su_i = u_{i+1}$ for $i < s$ and $Su_s = z$. Define S' in the same way. The two characters are taken with these marked terminal vectors, possibly of different negative norms. Call the pair relevant when the two real characters agree on the common unipotent group, as in (76.2).

Theorem 64.7 (The common part of two relevant flags)**.** *Suppose $I = L \cap L'$ has dimension $d > 0$ and the pair is relevant. For every nonzero $x \in I$, the least p for which $x \in L_p$ is also the least q for which $x \in L'_q$. The set of these indices is precisely $s - d + 1, \dots, s$. The intersection $Y \cap Y'$ contains a totally negative rational vector. Moreover* $\operatorname{rank} h|_{L \times L'} = s - d$.

Proof. Take a nonzero $x \in I$ with the two indices p, q. For $v \in (L_p + L'_q)^{\perp}$ consider the skew-adjoint operator

$$X(w) = xh(v, w) - vh(x, w).$$

For $w \in L_i$, the second term is zero. The first vanishes for $i \le p$, and for $i > p$ it belongs to $L_p \subset L_{i-1}$. On $L^{\perp}$ the image belongs to L. These relations and skew adjointness imply $X \in \operatorname{Lie} N$. Indeed on the flag, its middle quotient, and the dual flag its diagonal actions are zero, making it strictly upper triangular in the completed Witt flag. The same argument for L'_q puts it in the other unipotent Lie algebra as well.

Write $x = \sum_i x_i u_i = \sum_i x'_i u'_i$. Evaluating the simple-coordinate functional on X gives $\sum_i x_i h(v, Su_i) = h(v, Sx)$. This includes the coefficient on the marked terminal vector since $h(x, z) = 0$. The primed functional is $h(v, S'x)$. By relevance their real traces agree. Replacing v by αv for all $\alpha \in E$ shows that $h(v, Sx - S'x) = 0$. For example, at one complex embedding one can test a rational value and its product with a nonzero purely imaginary element of E. The two real traces detect both components. Taking the second perpendicular gives the rational identity

$$Sx - S'x \in L_p + L'_q. \tag{64.4}$$

Thus one may subtract an element of L_p from Sx to obtain the same vector as $S'x$ plus an element of L'_q. If $p, q < s$, this vector lies in I, and its two indices are $p + 1, q + 1$. The leading coefficients at these indices are x_p, x'_q, which are nonzero. If $p = s$ and $q < s$, its first expression is $x_s z$ plus a vector of L, hence has strictly negative norm. Its second expression lies in L' and has norm zero. This contradiction also excludes $p < s, q = s$. Iterating the first case therefore forces $p = q$. Starting with any nonzero x and repeating until both indices are s, the last application of (64.4) gives an element of $Y \cap Y'$ of norm $x_s \bar{x}_s h(z, z)$, with x_s the then-current nonzero leading coefficient.

The filtration $I \cap L_i$ can increase dimension by at most one at each index. Each occupied index below s gives the next one by the construction, and the total number of increases is d. This proves the asserted index set. Finally both L, L' lie in the perpendicular of the common negative vector. At any real place this perpendicular has negative index s. If $v \in L' \cap L^{\perp}$, the totally isotropic span $L + Ev$ has dimension at most s, forcing $v \in I$. The kernel of the cross pairing on L' is therefore I, which proves the rank statement. □

Corollary 64.8 (Vanishing of the mixed terms with zero determinant)**.** *For the convergent cells of Section 76.1 and two different words, every cell with $\mathcal{G} = 0$ has zero real operator. The same conclusion holds for the two available frequencies at the unrefined level K_σ.*

Proof. If the characters disagree, integration over their common unipotent fibre gives zero by Lemma 76.1. Otherwise, for $d > 0$ Theorem 64.7 gives a common negative vector. For $d = 0$ the determinant test in Theorem 64.3 gives the same conclusion. In both cases the common compact group has rank k. Theorem 76.2 now gives zero because the column types are inequivalent. Its fibre and compact averaging arguments used character compatibility and trivial finite column stabilizers, both of which also hold for K_σ by Lemma 70.23. The differential calculation above uses the actual vector in each flag and applies to both $z = e$ and $z = \varpi e$. □

Here is a size bound that includes identical words. Choose the standard Euclidean norm at each complex embedding over $\tau : F_+ \hookrightarrow \mathbb{R}$, in the original signed orthonormal coordinates of h. Put

$$h_\tau(g) = \max_{1 \le i \le s} \frac{\|g^{-1}(u_1 \wedge \cdots \wedge u_i)\|}{\|g^{-1}(u_1 \wedge \cdots \wedge u_{i-1})\|}, \qquad H(g_\infty) = \Big(\prod_\tau h_\tau(g_\tau) \Big)^{1/f},$$

using the Euclidean exterior norm and denominator one for $i = 1$. These are positive continuous functions on the real period quotients. The group B_∞ fixes each oriented exterior product. At one place the i-th ratio is the length of the orthogonal projection of $g^{-1}u_i$ off the preceding vectors, by Gram–Schmidt. Equivalently $h_\tau(g)$ is the smallest possible maximum of the Euclidean lengths after replacing each u_i by u_i plus any combination of preceding vectors. Each minimizing change exists, with the coefficients chosen independently for different columns.

Theorem 64.9 (A height separation for different isotropic spaces)**.** *Suppose $s \ge 2$, $K = K_\sigma$ or K^0, $\gamma_f \in B_f K B_f$, and $L' = \gamma L \ne L$. If the two frames are relevant, then at every g_∞*

$$H(g_\infty) H(\gamma^{-1} g_\infty) \;\ge\; g_1 g_2. \tag{64.5}$$

The assertion holds for any supported choice of terminal frequencies at the indicated level.

Proof. First $h|_{L \times L'}$ is nonzero. If $d > 0$, this follows from Theorem 64.7 since $d < s$. If $d = 0$ and the pairing vanishes, $L + L'$ is totally isotropic of dimension $2s > s + 1$, larger than the negative index of V.

Choose i, j with $i + j$ as small as possible such that $\xi = h(u_i, \gamma u_j) \ne 0$. This entry is unchanged by independent upper unitriangular changes in both isotropic bases. Indeed every other term in its expansion has strictly smaller sum of indices. The entire zero region of smaller sums is preserved by the changes and their inverses. As in Theorem 64.4, factor $\gamma_f = b_1 t b_2$, $t \in K$. Moving b_1 to the first argument and using these triangular actions shows that $\xi = h(u_i, t u_j)$ in every finite completion. The vectors u_i are integral at both levels. Thus $\xi \in \mathcal{O}_E$. At either auxiliary prime the reduction of t preserves the isotropic flag in both split factors, as checked in Theorem 64.4. Consequently $\xi \in (g_1 g_2)\mathcal{O}_E$. Its nonzero field norm gives

$$\prod_\tau |\tau(\xi)| \ge (g_1 g_2)^f,$$

where either complex embedding above τ gives the same absolute value.

Invariance under g_τ^{-1} writes $\tau(\xi)$ as the pairing of a vector from each of the two transformed flags $g_\tau^{-1} L, g_\tau^{-1} \gamma L$. The same smallest-index observation permits the two independent Gram–Schmidt changes just described. In signed coordinates $|h(v, w)| \le \|v\| \|w\|$ because the signature matrix has Euclidean operator norm one. It follows that $|\tau(\xi)| \le h_\tau(g_\tau) h_\tau(\gamma^{-1} g_\tau)$. Multiplication proves (64.5). □

The source defines the common Lie-algebra condition and the skew-adjoint tests in `Formalism/SharpRestriction/CuspSpanNormBounds/BlockEnds.lean`, line 79 and line 83 there. The mixed zero-determinant assertion also occurs in `Formalism/SharpRestriction/CuspSpanNormBounds/BruhatCellsKronecker/NormAdditiveCharacterTypePrimeEqOne.lean`, line 386.

64.4 Summing the period seeds above the height cutoff

Use the period fields, ranks and levels of Section 6.6. Write $f = [F_+ : \mathbb{Q}]$, $r = \dim_E V$, $s = \dim_E L$, $Q_0 = g_1 g_2$, and $R_0 = \sqrt{Q_0}$. We take $s \geq 2$, as on all sufficiently distant intervals. At the real place τ, use the Euclidean norm of the signed coordinates in Section 64.3. For a complex subspace A write p_A for its Euclidean orthogonal projection and put

$$d_\tau(A, A') = \|p_A - p_{A'}\|_{\mathrm{op}}, \qquad A_\tau(g) = g_\tau^{-1} L_\tau.$$

The projection distance is at most one. Indeed the self-adjoint operator $p_A - p_{A'}$ lies between -1 and 1 as a quadratic form. Both A_τ and the height H are well defined on the real seed quotient. Write P initially for ordinary summation of compactly supported seeds over $B(F_+)\backslash G(F_+)$.

Lemma 64.10 (The size of a crossing in terms of the angles)**.** *Let the two finite supports meet and suppose the two isotropic spaces L and γL are distinct and the two unipotent characters agree on their common real subgroup. At either of the levels in Theorem 64.9,*

$$Q_0^f \;\leq\; \{H(g)H(\gamma^{-1}g)\}^f \prod_\tau d_\tau(A_\tau(g), A_\tau(\gamma^{-1}g)).$$

Proof. The finite support condition gives $\gamma_f \in B_f K B_f$. The proof of Theorem 64.9 produces a nonzero entry $\xi = h(u_i, \gamma u_j) \in Q_0 \mathcal{O}_E$ with $i + j$ minimal. Its norm gives $\prod_\tau |\tau(\xi)| \geq Q_0^f$. At a real place put $A = A_\tau(g)$, $A' = A_\tau(\gamma^{-1}g)$. The two independent Gram–Schmidt operations from that proof give $v \in A, w \in A'$ with $h(v, w) = \tau(\xi)$ and lengths bounded respectively by $h_\tau(g)$ and $h_\tau(\gamma^{-1}g)$. All earlier cross entries vanish, so these operations preserve this particular value. The Hermitian form vanishes on $A \times A$. Thus

$$|h(v, w)| = |h(v, w - p_A w)| \leq \|v\| \, \|(p_{A'} - p_A)w\| \leq \|v\| \, \|w\| \, d_\tau(A, A').$$

Here we used $p_{A'} w = w$ and the operator norm one of the signature matrix. Multiplication over the real places proves the claim. □

Theorem 64.11 (An upper bound at arbitrary height)**.** *Fix a real place τ_0 and $R > 0$. For seeds supported on $H \leq R$ there is a measurable partition based solely on $A_{\tau_0}(g)$ into at most*

$$\bigl(1 + 16 \max\{1, (R^2/Q_0)^f\}\bigr)^{2rs}$$

parts. On each part full automorphic summation is an isometry from the seed norm. For every square-integrable seed of the selected types supported at these heights,

$$\|P\Phi\|_{L^2([G])} \leq \bigl(1 + 16 \max\{1, (R^2/Q_0)^f\}\bigr)^{rs} \|\Phi\|_{\mathrm{seed}}. \tag{64.6}$$

The assertion includes finite linear combinations of words and class columns at a fixed supported level and frequency, and the two supported frequencies together at K_σ.

Proof. Set $\eta = \frac{1}{8} \min\{1, (Q_0/R^2)^f\}$. The complex r-by-s matrices, equipped with the operator norm, form a real normed space of dimension $2rs$. Choose a maximal η-separated family among the orthonormal s-frames in that space. The balls of radius $\eta/2$ about them have disjoint interiors and lie in the ball of radius $1 + \eta/2$ about zero. Lebesgue volume bounds the number by $(1 + 2/\eta)^{2rs}$. Such a finite maximal family exists by successively adding points and using this bound. Every plane has a frame U within η of some chosen frame V. Then $\|UU^* - VV^*\| \leq 2\|U - V\| \leq 2\eta$, by subtracting one factor at a time. The closed projection balls of radius 2η about the chosen planes therefore cover. Assign a plane to the first such ball that contains it. The resulting Borel parts have diameter at most 4η.

Two indices contributing at the same real point to one part have their subspaces at τ_0 within 4η and their heights at most R. All other projection distances are at most one. If the spaces are distinct and the relative cell is relevant, Lemma 64.10 would give

$$Q_0^f \le R^{2f} 4\eta \le Q_0^f/2.$$

If the cell is irrelevant, its unfolded integral is zero by Lemma 76.1. The imposed height and angle indicators preserve this cancellation. They are left B_∞-invariant, and in an unfolded product they are constant along the common unipotent fibre.

First take bounded measurable sections of compact support modulo B_∞ in the chosen part. Their scalar envelopes have compact support modulo $B(F_+)$, by the finite class description. The ordinary sums are locally finite and products of the periodized envelopes are integrable, by the compact lifting argument of Theorem 88.1. The Gram integral can therefore be unfolded. By the preceding paragraph only the terms with the same isotropic space survive. Their sum is the norm squared of summation over the stabilizer of L, which equals the seed norm squared by the same theorem and by Theorem 64.2 at the refined level. That isometry extends to measurable truncations: on a fixed compact support the ordinary sums are locally finite with uniformly bounded multiplicity on the compact image, so L^2 approximation agrees with ordinary summation there. Bounded compact truncations within a part are dense in the L^2 space of its sections. This proves the isometric extension on each part. The finite partition is orthogonal for the seed norm by disjoint supports. Apply the triangle inequality to the sum of its images, then Cauchy–Schwarz over the parts. Taking the square root of the covering bound proves (64.6). All extensions agree on intersecting height ranges, by their agreement on bounded compact sections and density. □

Theorem 64.12 (Full summation with a weighted real tail)**.** *Set $D_0 = 2frs$, fix $\sigma > 0$, and write*

$$C(D_0,\sigma) = \frac{2 \cdot 17^{rs}}{\sqrt{1 - 2^{-2\sigma/D_0}}}.$$

For a square-integrable seed Φ let $\Phi^{<} = \mathbf{1}_{H<R_0}\Phi$. If

$$T(\Phi) = \big\|\mathbf{1}_{H\ge R_0}(H/R_0)^{D_0+\sigma}\Phi\big\|_{\mathrm{seed}} < \infty,$$

the ordinary sums of bounded compact truncations converging to Φ in this weighted L^2 norm and in the seed norm have a unique limit $P\Phi$ in $L^2([G])$. It satisfies

$$\|P\Phi - P\Phi^{<}\| \le C(D_0,\sigma)T(\Phi). \tag{64.7}$$

If Ψ is supported in $H < S$ with $0 < S \le R_0$, the difference between $\langle P\Phi, P\Psi\rangle$ and $\langle \Phi,\Psi\rangle_{\mathrm{seed}}$ is bounded by

$$C(D_0,\sigma)\|\Psi\|_{\mathrm{seed}}\big\|\mathbf{1}_{H\ge Q_0/S}(H/R_0)^{D_0+\sigma}\Phi\big\|_{\mathrm{seed}}.$$

Proof. Write Φ_j for the part with $2^{j/D_0}R_0 \le H < 2^{(j+1)/D_0}R_0$, $j \ge 0$. For $R \ge R_0$ the factor in (64.6) is at most $17^{rs}(R/R_0)^{D_0}$. Hence

$$\sum_{j\ge0}\|P\Phi_j\| \le 2\cdot 17^{rs}\sum_{j\ge0} 2^{-\sigma j/D_0}\,2^{(1+\sigma/D_0)j}\|\Phi_j\| \le C(D_0,\sigma)T(\Phi).$$

The second inequality follows by Cauchy–Schwarz and the definition of the disjoint shells. Their images sum absolutely in the Hilbert space. Below R_0 the isometry in Theorem 65.10, applies. This proves the estimate and continuity in the sum of the two stated norms. Truncation in support and in the pointwise norm gives density in this space, so the extension is unique.

To test against Ψ, first restrict Φ to $H < Q_0/S$. On a pair of supports the two heights now have product less than Q_0. The same-space calculation in the preceding proof applies, and Theorem 64.9 excludes every relevant different-space term. Compact approximation followed by (64.6) at the finite height Q_0/S proves equality of the pairings on this part. The remainder has zero seed pairing with Ψ because $Q_0/S \geq S$. Its automorphic pairing is at most the product of its norm from (64.7) and the seed norm of Ψ, by the low-height isometry. This proves the last assertion. □

64.5 The full projected period at fixed finite level

Fix one period field, one translate, the finite level K^0, and the terminal frequency ϖ. The selected words c give finitely many inequivalent compact column types ρ_c for a common real discrete series σ. Let U be its product lowest compact type and $\mathcal{M}$ its automorphic multiplicity Hilbert space at this finite level. Thus the σ-isotypic space is $\sigma\widehat{\otimes}\mathcal{M}$. Denote by Π the projection onto $U \otimes \mathcal{M}$. All projections and adjoints in this section are Hilbert space operators.

Let S_c denote the space of square-integrable real Bessel sections of type ρ_c. Give the finite class covector space D_c its sum inner product multiplied by c_{ρ_c,K^0} from (70.17); equivalently, this is the inner product induced by the seed for a real section of norm one. The finite class subspace with zero sums on scalar orbits can be used in its place. Write

$$S = \bigoplus_c (S_c \widehat{\otimes} D_c), \qquad D = \bigoplus_c D_c, \qquad J_U : U \otimes D \longrightarrow S.$$

On the c-th summand J_U is $J_c|_U \otimes 1_{D_c}$, where $J_c : \sigma \to S_c$ is the isometry in Corollary 84.21. The map from S to actual period seeds preserves inner products. Indeed polarization of Theorem 70.24 in the two real sections gives their S_c inner product times the stated class inner product, and distinct words are orthogonal. Finite sums of pure tensors are dense.

Lemma 64.13 (The mass of a compact type in a measurable region)**.** *If $\Omega \subset B_\infty \backslash G_\infty$ is right invariant under the maximal compact subgroup, there are numbers $0 \leq t_c(\Omega) \leq 1$ such that*

$$J_U^* \mathbf{1}_\Omega J_U = 1_U \otimes T_\Omega, \qquad T_\Omega|_{D_c} = t_c(\Omega)\, 1_{D_c}.$$

For any unit $v \in U$, $t_c(\Omega) = \int_\Omega \|(J_c v)(g)\|^2 \, d\dot{g}$. For $\Omega_R = \{H < R\}$ one has $t_c(\Omega_R) \uparrow 1$ as $R \to \infty$, for all c simultaneously.

Proof. The compression of $\mathbf{1}_\Omega$ to $J_c(U)$ commutes with the compact group. On this irreducible type it is a scalar by Schur's lemma. The diagonal quadratic form on a unit vector gives the displayed integral. Multiplication by the indicator preserves each word and acts trivially on its class factor. The height is finite at every point and right compact-invariant. Monotone convergence gives the limit, and the word set at the fixed field and level is finite. □

Theorem 64.14 (A bounded extension of the projected sum)**.** *The ordinary sum followed by Π, initially on compact seeds, has a unique bounded extension $P_U : S \to U \otimes \mathcal{M}$. There is a bounded operator $A : \mathcal{M} \to D$ such that*

$$P_U^* = J_U(1_U \otimes A), \qquad P_U = (1_U \otimes A^*) J_U^*, \qquad \|P_U\| = \|A\|. \tag{64.8}$$

For every R with $t_*(R) = \min_c t_c(\Omega_R) > 0$,

$$A^* T_{\Omega_R} A \;\leq\; B(R)^2\, 1_{\mathcal{M}}, \qquad \|A\| \leq B(R)/\sqrt{t_*(R)}. \tag{64.9}$$

The bound concerns the fixed field and level and also holds when the automorphic multiplicity is zero.

Proof. For $m \in \mathcal{M}$, $v \in \sigma^\infty$, let $F = v \otimes m$ in automorphic L^2. On seeds of height less than R the functional $\Phi \mapsto \langle P\Phi, F\rangle$ has norm at most $B(R)\|v\|\|m\|$ by Theorem 64.11. For fixed v, m the Riesz representatives agree on overlaps. They define a locally square-integrable real section $W_{v,m}$ with values in the appropriate column and finite class spaces. Right translation in the real argument replaces v by $\sigma(g)v$. Indeed test this equality on smooth compact seeds, use the right equivariance of ordinary summation, then translate in the pairing. Consequently each weak right derivative of $W_{v,m}$ is represented by $W_{d\sigma(X)v,m}$, and the same assertion holds for arbitrary words.

In a relatively compact coordinate chart on the quotient, lift sections to a local trivialization. Right Lie directions span the quotient tangent space. Coordinate derivatives of each fixed order are therefore linear combinations with locally bounded smooth coefficients of the preceding Lie derivatives, including lower orders from the bundle trivialization. Local Sobolev embedding gives a smooth representative and bounds its evaluation at the identity coset by a constant times a finite sum of $\|d\sigma(I)v\|\|m\|$, using one finite height for a neighborhood of that coset. Thus at each word and class coordinate the evaluation $v \mapsto W_{v,m}(1)$ is a continuous Bessel map on σ^∞. This follows for the left covariance by evaluating the right-translation identity at elements of B_∞.

For each word the space of these maps into the compact column representation is a line. To check the product assertion, pair with elementary tensors in the dual column and regard the result as a functional on the exterior tensor of local row and dual column representations. Fixing all but one tensor factor gives a local functional on its known line. Choose in each factor a vector on which its generator takes the value one. Applying the one-place assertion successively gives the product of the local functionals times the value on this chosen tuple. Such tensors span a dense subspace in the smooth topology: compact Fourier truncations converge there and each of the resulting compact isotypic spaces is spanned by product vectors by admissibility of the local discrete series. Hence the factorization holds on all smooth vectors.

Apply this fact in an orthonormal basis of D_c, and then use right translation to recover the section at every g. There is a unique vector $A_c m \in D_c$ with

$$(W_{v,m})_c = J_c v \otimes A_c m \qquad (v \in \sigma^\infty).$$

The Riesz representative depends linearly on the second argument vector F in this adjoint construction, so A_c is linear. Set $Am = (A_c m)_c$. For a fixed unit $v \in U$, the square norm of $W_{v,m}$ restricted to $H < R$ is $\sum_c t_c(\Omega_R)\|A_c m\|^2$ by Lemma 64.13. Its Riesz bound was $B(R)^2\|m\|^2$ after squaring. This proves (64.9), and some finite R has $t_*(R) \geq 1/2$ by that lemma.

It follows that $J_U(1_U \otimes A)$ is bounded on $U \otimes \mathcal{M}$. On elementary vectors its pairing with all compact seeds is exactly the defining pairing for ΠP. Density and the adjoint identity give the unique extension and (64.8). The equality of norms follows from the isometry of J_U. In particular height truncation converges after projection for every input in S, by continuity. □

Corollary 64.15 (Recovering the period from a truncated real vector)**.** *For every right compact-invariant measurable region,*

$$P_U \mathbf{1}_\Omega J_U = 1_U \otimes (A^* T_\Omega).$$

If $t_*(R) > 0$, the operator on normalized real vectors is recovered by

$$P_U J_U = (P_U \mathbf{1}_{\Omega_R} J_U)(1_U \otimes T_{\Omega_R}^{-1}).$$

These identities continue to hold at fixed level on the refined zero-sum class spaces.

Proof. Substitute the first lemma into (64.8). Inversion of the finitely many positive scalars proves the second identity. Each acts on the entire class space of its word by a scalar and thus preserves its zero-sum subspace. □

64.6 Two terminal frequencies at a common finite level

Keep the actual integral level K^0 of Theorem 64.1 and write $K_0 = K^0$. Put $a = a_\varpi \in G(F_+)$, the rational isometry multiplying the isotropic vectors u_i by ϖ, their duals by $\bar{\varpi}^{-1}$, and acting identically on the middle space. Use a_f for its finite component, also regarded as an adele with identity real component. Define

$$K_1 = a_f K_0 a_f^{-1}, \qquad L = K_0 \cap K_1, \qquad J = T^{\,fs(r-s)}, \qquad f = [F_+ : \mathbb{Q}].$$

The subscript on K_1 refers to frequency one. Use the same class covectors for both frequencies. Write $\Phi_{\beta,K}(W, \ell)$ for the seed (70.16) with real section W and class tuple ℓ.

Theorem 64.16 (Rational change of terminal frequency)**.** *The class level K_H is identical for K_0 and K_1. At K_1 the unit-frequency character is trivial on the finite unipotent level. Choose normalized real sections with $W_\varpi(g) = J^{-1/2} W_1(a_\infty g)$. Then*

$$\begin{aligned} \Phi_{\varpi,K_0}(W_\varpi, \ell)(g) &= J^{-1/2} \Phi_{1,K_1}(W_1, \ell)(a g a_f^{-1}), \\ \Pi P \Phi_{\varpi,K_0}(W_\varpi, \ell) &= J^{-1/2} R(a_f^{-1})\, \Pi P \Phi_{1,K_1}(W_1, \ell). \end{aligned} \tag{64.10}$$

The second formula holds for all square-integrable real sections, with projected sums understood by continuous extension. Their class-coordinate constants in (70.17) satisfy $c_{\rho,K_1} = J c_{\rho,K_0}$.

Proof. Conjugation by a normalizes N, centralizes $H = U(W_0)$, and satisfies $\chi_1(ana^{-1}) = \chi_\varpi(n)$ on the adelic unipotent group. Indeed all simple coordinates on the isotropic string are unchanged and the terminal coordinate is multiplied by ϖ. Thus the finite period subgroups for the two levels are conjugate with the same H-projection. Triviality of the character at K_0 in Theorem 64.1 proves the assertion at K_1. At the real places Proposition 84.13 gives the stated unitary change of sections and its factor $J^{-1/2}$. Conjugation sends $B_f K_0$ onto $B_f K_1$ and leaves the finite class argument in H_f unchanged. Substituting in (70.16) proves the first equality on and off the finite support.

On compact seeds, left multiplication by a permutes the cosets $B(F_+)\backslash G(F_+)$ since a normalizes B. Ordinary summation therefore removes that left factor and retains the right translation. Real projection commutes with finite translation, which is unitary on the common automorphic L^2 space. This proves (64.10) on compact supports. For the extension, first compute the finite norms. The determinant of $\operatorname{Ad} a$ on $\operatorname{Lie}(N)$ is in $F_+^\times$. Its product of real absolute values is J by Proposition 84.13; the compact block in that calculation is fixed. The product formula gives finite Haar modulus J^{-1}. Hence $N_f \cap K_1$ has volume $J^{-1} \operatorname{vol}_f(N_f \cap K_0)$. The two row volumes agree by unimodularity. Formula (70.17) now gives the stated relation of class norms. The real change of variables and this relation make the two seed norms proportional by a fixed positive factor. Boundedness for the projected sum at K_0 by Theorem 64.14 therefore gives boundedness at K_1 on compact seeds. Density extends the identity. □

Proposition 64.17 (The common level at the split prime)**.** *The indices of L in K_0 and in K_1 coincide and are the integer*

$$I = \begin{bmatrix} r \\ s \end{bmatrix}_T^f, \qquad \begin{bmatrix} r \\ s \end{bmatrix}_T = \prod_{j=1}^{s} \frac{T^{r-s+j} - 1}{T^j - 1}.$$

One has $N_f \cap K_1 \subset N_f \cap K_0$ of index J, and $B_f \cap L = (N_f \cap K_1) \rtimes K_H$. In particular both terminal characters are trivial on $N_f \cap L$. Furthermore

$$\frac{c_{\rho,L}}{c_{\rho,K_0}} = \frac{J}{I}, \qquad 1 - \frac{f}{T-1} \;\le\; \frac{J}{I} \;\le\; 1. \tag{64.11}$$

Proof. Outside T the levels agree. At odd primes away from the auxiliary primes, a and its inverse preserve the standard lattice in its integral Witt basis. At a dyadic place its matrix on each signed plane has entries $(\varpi + \bar{\varpi}^{-1})/2$ and $(\varpi - \bar{\varpi}^{-1})/2$ on the diagonal and off it, respectively. They are integral, also for the inverse, since $\varpi \equiv 1 \bmod 2\mathcal{O}_E$. At g_1, g_2 the imposed congruence of Theorem 9.13 gives the identity reduction, so a normalizes those levels as well.

Now let $v \mid T$ and choose the split factor where ϖ has valuation one. The paired unitary group is described by its first factor $\mathrm{GL}_r(\mathbb{Q}_T)$. Write $O = \mathbb{Z}_T$. By Theorem 64.1, $K_{0,v}$ stabilizes $\Lambda_0 = O^s \oplus M_v \oplus O^s$ in the Witt splitting. Since $\bar{\varpi}$ is a unit here, $K_{1,v}$ stabilizes $\Lambda_1 = TO^s \oplus M_v \oplus O^s$. This sublattice of Λ_0 contains $T\Lambda_0$. Its image modulo $T\Lambda_0$ has dimension $r-s$. Reduction of $K_{0,v}$ surjects onto $\mathrm{GL}_r(\mathbf{F}_T)$, and the intersection is the preimage of the stabilizer of this subspace. Counting the ordered bases of an s-dimensional subspace and dividing by $|\mathrm{GL}_s(\mathbf{F}_T)|$ gives the displayed Gaussian binomial coefficient, also the count in dimension $r-s$ by duality. Equal row volumes give the same index in $K_{1,v}$. Multiplication over the f split places gives I.

An element of $N_v \cap K_{1,v}$ preserves the entire first isotropic space. It therefore preserves its intersection TO^s with Λ_1, and so also O^s. As $\Lambda_0 = \Lambda_1 + O^s$, it preserves Λ_0 in both directions. This proves the inclusion on the finite unipotents. The quotient of their volumes gives index J. The assertions for B_f follow from the common compact factor and the semidirect product at K_0. Formula (70.17) gives $c_{\rho,L}/c_{\rho,K_0} = J/I$. After removing $T^{s(r-s)}$ from the Gaussian binomial coefficient, its reciprocal factor is $\prod_{j=1}^{s}(1-T^{-j})/(1-T^{-(r-s+j)})$. It lies between $1 - \sum_{j\ge1} T^{-j} = 1 - 1/(T-1)$ and one. Raising to f proves (64.11). □

Theorem 64.18 (Finite averages of the common-level seeds)**.** *Let e_i be probability averaging of right translations by K_i, $i = 0, 1$. For the prime frequency at $i = 0$ and the unit frequency at $i = 1$, the ordinary seed identities are*

$$e_0\Phi_{\varpi,L} = \frac{J}{I}\Phi_{\varpi,K_0}, \qquad e_1\Phi_{1,L} = \frac{1}{I}\Phi_{1,K_1}.$$

The real sections and class covectors in each equality are unchanged. The finite averaging operator Ie_i is a sum of integral translations on functions at the common level.

Proof. On an L-fixed input e_i is the sum of the right translations over representatives of K_i/L, divided by I. At a point of B_fK_i write the finite argument as bk_0'. In the average replace $k_0'k$ by k'. A term is supported precisely when $k' \in (B_f \cap K_i)L$. The number of supported cosets is $[B_f \cap K_i : B_f \cap L]$, which is J for $i = 0$ and one for $i = 1$. In each supported term the character on the unipotent part of $B_f \cap K_i$ is trivial for its designated frequency, and the column is right K_H-fixed. The value of every supported term is consequently the coarse-level value at b. Off B_fK_i all terms vanish. This proves both identities. In the unaveraged sum every coefficient is the integer one. In particular it preserves integrality of values when those values range in a fixed integral module. □

64.7 Absolute convergence of the global Bessel Gram matrix

Put $F = E^+$, $f = [F : \mathbb{Q}]$, $G = U(V)$, and $B = N \rtimes H$ as in Section 99.16. Work at one of the fields and one of the translates of Section 6.6. Write $r = \dim_E V$ and $q = s + 1$ for the

negative index. All the selected words have the same real row σ. The finite level is $K = K^0$, the frequency is ϖ, and U denotes the lowest compact type of σ. We use the ordinary seed of (70.16), with its right level character when present. The class spaces D_c and the isometry J_U have the normalization of Section 64.5.

In the construction preceding Theorem 9.13, choose the two free spacings as follows. Put

$$d = 4r^2, \qquad L = 8(f+1)(r+1)^3, \qquad M = L + 2(d+2), \qquad \eta = L - 4fr^2 > 0.$$

Take $\Delta \geq M+1$ and $H_0 \geq M+1$ in addition to the lower bound already imposed on H_0. For instance one can take the least integers meeting these conditions and the earlier lower bounds. The three successive kinds of gap in the real row are $4\Delta,\ B_\tau,\ 2H_0$. The least B_τ grows faster than every fixed power of x, while f, r, H_0, Δ are polynomially bounded for this choice. Every adjacent gap therefore exceeds M, at every real place of all sufficiently large fields in the stated intervals. This choice retains the interlacing, the common terminal prime construction and the inequality $4k\Delta \leq B_\tau/x$ of Volume I.

Lemma 64.19 (Polynomial decay from gaps in the real row)**.** *Let $G_0 = U(p,q)$ in signed orthonormal coordinates and let $w(g) = \|g\|_{\mathrm{op}}$ for its defining complex representation. If every adjacent gap of a decreasing Harish-Chandra parameter λ exceeds the integer M, every coefficient $a(g)$ of its discrete series between two compact-finite vectors satisfies*

$$|a(g)| \leq C_{a,M}\, w(g)^{-M}.$$

For $S = G_0$ or $S = B_\tau$, one has $\int_S w(h)^{-D}\, dh < \infty$ for $D > 4r^2$.

Proof. We use the coherent character theorem for discrete series. In a fixed regular chamber the characters with the same compact Cartan and chamber labels form the restriction of a coherent family. Multiplying a member by a finite-dimensional character gives the sum of the members shifted by its weights, counted with multiplicity. This is the character identity (8.3) of Schmid and Vilonen, applied to the open orbit for the given discrete series. Its identification in the regular chamber is given in their Sections 7–8. [1]

Tensor the Harish-Chandra module once with the defining representation. Its weights are the r distinct coordinate vectors e_i. When all shifts stay in the chosen chamber, the character identity gives the discrete series with parameters $\lambda + e_i$, each once. They have distinct infinitesimal characters. Indeed two strictly decreasing lists in the same permutation orbit coincide. Characters determine multiplicities in a finite-length Harish-Chandra module. Splitting this module into generalized eigenspaces for the center of the enveloping algebra now gives a direct sum, since each nonzero block has length one. After m shifts the difference between the changes in two entries has absolute value at most m. Consequently we may repeat this argument through M tensor factors. Every summand of the resulting module is unitarizable.

For any matrix entry g_{ij}, the function $g_{ij}^{M} a(g)$ is a coefficient of this tensor product. In the algebraic direct sum its vector has finitely many components. Its compact-finite covector on each summand is represented by a Hilbert vector, since it is supported on finitely many compact types of finite multiplicity. The original coefficient equals the resulting sum of unitary coefficients on the whole group. Indeed both sides are real analytic by the analytic-vector theorem for compact-finite vectors of an irreducible unitary representation. The intertwining of Lie algebra modules gives all derivatives at the identity, and $U(p,q)$ is connected. Each summand is bounded by the product of its two vector norms. Taking the maximum over i, j and using $w(g) \leq r \max_{i,j} |g_{ij}|$ proves the first assertion.

In signed coordinates $g^{-1} = Jg^{*}J$, with J a unitary involution. Hence $w(g) = w(g^{-1}) \geq 1$. In the ball $w(g) \leq R$ of a closed subgroup S, take a maximal $1/(2R)$-separated set for the

matrix norm. Packing balls in the real vector space of complex r-by-r matrices bounds its cardinality by $C_r R^{4r^2}$ for $R \geq 1$. Each point of the original ball is within $1/(2R)$ of a center h, so $h^{-1}g$ lies in the compact set $\{u \in S : \|u-1\|_{\mathrm{op}} \leq 1/2\}$. Left invariance bounds the Haar volume by $C_S R^{4r^2}$. Summing on dyadic shells proves the integral assertion. □

At a real place write

$$\nu_\tau(g) = \inf_{b \in B_\tau} w(bg), \qquad V(g_\infty) = \prod_\tau \nu_\tau(g_\tau).$$

These functions are at least one. They are continuous because right multiplication by t changes ν_τ by a factor between $w(t^{-1})^{-1}$ and $w(t)$. The same letter V with a group element as argument will always denote this product.

Proposition 64.20 (Decay and reproduction of the Bessel sections)**.** *For every word c and compact-finite vector v of the real row, $W = J_c v$ satisfies*

$$\sup_g V(g)^L \|W(g)\| < \infty, \qquad \int_{B_\infty \backslash G_\infty} \|W(g)\| V(g)^L \, d\dot{g} < \infty.$$

In particular $\|H^{2D_0} W\|_2 < \infty$ for the height of Section 64.4 and $D_0 = 2frs$. On the lowest compact type, one compactly supported smooth right convolution reproduces all these sections.

Proof. At one place use the scalar realization in the proof of Theorem 84.20. For z in the dual compact column and a compact-finite u with $\ell(u) \neq 0$, each coordinate $z(W(g))$, up to a fixed nonzero normalization, is $Q f_{v \otimes z, u}(g, 1)$. Here Q integrates the coefficient over the diagonal period group $\mathcal{H} \subset G_\tau \times H_\tau$ against its unitary character. Projection onto its first factor identifies $\mathcal{H}$ with B_τ. The second factor is compact. Expanding u in a basis of the compact column and applying Lemma 64.19 bounds the absolute value of the integrand by $C w(bg)^{-M}$, uniformly in its compact variable.

Choose g in its left coset with $w(g) \leq 2\nu_\tau(g)$. Write $D = d + 2$. From $w(bg) \geq \nu_\tau(g)$ and $w(bg) \geq w(b)/w(g^{-1})$ we obtain

$$w(bg)^{-L-2D} \leq 2^D \nu_\tau(g)^{-L} w(b)^{-D}.$$

Integration over b proves the pointwise bound for the norm, which is constant on the left coset. For the integral bound use the quotient integration formula on the nonnegative majorant. The weight is unchanged along the left fibre and $\nu_\tau(g) \leq w(g)$, giving an upper bound $C \int_{G_\tau} w(y)^{-M+L} \, dy < \infty$. Apply these inequalities to finitely many coordinates, then take tensor products over the real places. Compact-finite vectors are finite sums of such tensors by admissibility.

Each Gram–Schmidt length in h_τ is at most $\|g^{-1} u_i\| \leq w(g) \|u_i\|$. Left invariance yields $H \leq C V^{1/f}$. Thus $H^{4D_0} \leq C' V^{8rs}$ and $8rs \leq L$. The two estimates just proved imply the required square integral. For reproduction, conjugation-average a smooth compactly supported approximate identity over the maximal compact group. The resulting operators commute with that group and tend strongly to the identity. The lowest compact type occurs once. On it one of these operators is therefore a nonzero scalar. Divide the corresponding function by that scalar and use right equivariance of J_c. The identity holds pointwise by smoothness. In particular the exponent $4r^2\sqrt{2q}$ required in `Formalism/LocalToGlobal/SiegelLatticeTheory/HaarTransversals/WuDecay.lean`, line 31 is covered since it is at most L. □

Let Φ denote one of the resulting scalar seeds, including its finite class covector. It vanishes outside $B_f K$ in the finite variable and is bounded there in modulus by $C_\Phi V(g_\infty)^{-L}$. We give a count uniform in the point of the automorphic quotient. Its constant may depend on the number field, the level and the chosen coordinates.

Lemma 64.21 (Rational cosets in a height ball)**.** *For $R \geq 1$ and $t \in G(\mathbb{A}_F)$ the number of indices $\gamma \in B(F)\backslash G(F)$ with $\gamma_f t_f \in B_f K$ and $V(\gamma_\infty t_\infty) \leq R$ is at most $C_x R^{4fr^2}$. Consequently the sum of $|\Phi(\gamma t)|$ over indices of height at least R is at most $C_{\Phi,x} R^{-\eta}$.*

Proof. The adelic quotients of N and H are compact, as proved in Section 99.16. Their compact sets of representatives give one for $B(F)\backslash B(\mathbb{A}_F)$. Reduce the H-coordinate first, then the N-coordinate. For $y_f \in B_f K$, reduce y_f to K by B_f and reduce each y_τ to a representative of norm $\leq 2\nu_\tau(y_\tau)$ by B_τ. Removing the rational part of the combined left multiplier gives $\delta \in B(F)$ such that

$$(\delta y)_f \in C_f, \qquad w((\delta y)_\tau) \leq C_\tau \nu_\tau(y_\tau). \tag{64.12}$$

The finite compact set and all constants here are fixed.

Choose such a representative for each contributing index and fix one of them as base point. Relative to the base point the other rational matrices are distinct, have finite components in $C_f C_f^{-1}$, and have archimedean norms $\ll_x R^2$. Every entry belongs to a fixed fractional ideal of E. In its $2f$-dimensional real embedding an ideal is a lattice. Disjoint balls of a fixed positive radius about its points in a box of radius $O_x(R^2)$ give $O_x(R^{4f})$ possible entries. There are r^2 entries. This proves the count, including the case where it is zero. Sum the bound $C_\Phi (2^j R)^{-L}$ times the count at $2^{j+1}R$ over $j \geq 0$ for the tail. □

Theorem 64.22 (The ordinary sum and its projected Gram matrix)**.** *On $J_U(U \otimes D)$ the ordinary rational periodization P converges absolutely at every point. It is a bounded function in automorphic L^2 and takes values in the prescribed real isotypic space and lowest compact type. Its Gram matrix is the one in Theorem 64.14. More explicitly, for a common unit $v \in U$ and finite class vectors ℓ, ℓ', write Φ, Ψ for their seeds. With P_{fl} the rational flag parabolic, one has*

$$\langle P\Phi, P\Psi \rangle = \langle \ell, \ell' \rangle_D + \sum_{\substack{\gamma \in B(F)\backslash G(F)/B(F) \\ \gamma \notin P_{\mathrm{fl}}(F)}} \int_{\Gamma_\gamma \backslash G(\mathbb{A}_F)} \Phi(t)\overline{\Psi(\gamma^{-1}t)}\, dt, \tag{64.13}$$

$$\Gamma_\gamma = B(F) \cap \gamma B(F) \gamma^{-1}.$$

The sum of the integrals of absolute values is finite. For different words all cells of zero Gram determinant in Corollary 64.8 vanish in this global formula.

Proof. The majorant in Lemma 64.21 is bounded for $R = 1$. It also gives a bounded periodization if the norm of the real section is used at every finite class. This latter periodization is integrable. Indeed its integral unfolds by Tonelli to the single seed quotient. In the class coordinates of Lemma 70.23, the real integrals reduce to the L^1 norm of that section times the finite volume of $\Lambda_i \backslash B_\infty$. The compactness here follows from the compact unipotent lattice quotient and H_∞ compact. Proposition 64.20 proves finiteness. Thus the majorant belongs to $L^1 \cap L^\infty$, hence to L^2. Cutoffs $V \leq R$ converge by dominated convergence on the automorphic quotient and also on the seed quotient. Their real support is compact modulo B_∞ by the bounded representative construction. Locally on the group the ordinary sum is continuous. On a compact real neighborhood right translation changes V by bounded factors, so the tail is dominated by a common summable series at its center. The finite variable is locally constant.

For a unit vector $v \in U$, use an inner product linear in its first variable, put $c(g) = \langle \sigma(g)v, v \rangle$, and put $p(g) = d_\sigma \overline{c(g)}$ on G_∞, with d_σ its formal degree. Lemma 64.19 on the real factors gives $p \in L^1$, also for a finite sum of tensor vectors v. Schur orthogonality gives $\sigma(p)w = \langle w, v \rangle v$, $p = p^* = p * p$, and $p * \delta_t * p = c(t)p$. For the last identity insert the translate between the two coefficients in the convolution and integrate their common variable by the same orthogonality.

These are identities of integrable kernels. For any unitary right representation R, a vector $z = R(p)z$ consequently satisfies $\langle R(t)z, z\rangle = c(t)\|z\|^2$. If $z \neq 0$, the assignment $\sigma(t)v \mapsto R(t)z/\|z\|$ preserves all inner products of orbit vectors. Its linear extension and completion give a copy of σ carrying v to $z/\|z\|$.

Right convolution by p fixes the seed made with v, by the isometry and equivariance of J_c. The convolution is pointwise defined and continuous, using boundedness of the seed and $p \in L^1$, so the Hilbert space identity holds everywhere on its support. It commutes with ordinary periodization since the sum of absolute values of the translated integrands is bounded by the uniform majorant times $|p(t)|$. The resulting automorphic vector is therefore fixed by $R(p)$ and belongs to the required copy at v, or is zero. Linearity in pure inputs handles $U \otimes D$. Applying Π to the cutoff limits now identifies the ordinary sum with the extension in Theorem 64.14.

The two periodized absolute majorants have integrable product by Cauchy–Schwarz. Tonelli applied to that product justifies unfolding both rational sums, with the displayed stabilizer for each pair. The summands with $\gamma \in P_{\mathfrak{H}}(F)$ sum to the inner product of the two parabolic periodizations. Theorem 78.15 evaluates this as the seed pairing. The same absolute bound with the two real norms gives absolute convergence on each finite cell before contraction by the covectors. Corollary 64.8 therefore applies directly to the full real sections. □

64.8 Separation of Bessel periods at one finite place

Keep the archimedean vectors, number field, and original level K of Section 64.7. Choose one of the already specified places $v \mid T$ of F, with maximal ideal $\mathfrak{p}_v$ and residue cardinality Q. In a faithful matrix action on the underlying F_v-space of V, choose a lattice stabilized by K_v. Such a lattice is the sum of the translates of any one lattice under K_v. Compactness bounds their denominators, and the sum contains the original lattice. For $j \geq 1$ let J_j be the kernel of the action of K_v modulo $\mathfrak{p}_v^j$ on the chosen lattice. Put

$$J_0 = K_v, \qquad L_j = (B(F_v) \cap K_v)J_j, \qquad \alpha_j = [K_v : L_j]^{-1}.$$

Normality of J_j in K_v makes L_j an open subgroup. For an original seed Φ put

$$\Phi_j(t) = \Phi(t)\,\mathbf{1}_{B(F_v)J_j}(t_v), \qquad S_j\xi = P\Phi_j \quad \text{if } \Phi = J_U\xi, \quad \xi \in U \otimes D.$$

We identify $J_U\xi$ here with its scalar adelic seed. The finite class covectors in the domain carry their original normalization. On the new support the seed transforms on the right under L_j by the restriction of the old level character. Indeed $B(F_v)J_j$ is stable on the right by $B(F_v) \cap K_v$, since this subgroup normalizes J_j. The real projection argument of Theorem 64.22 applies without change to these seeds. Its majorant works at every j since the additional indicator has modulus at most one.

Lemma 64.23 (An arithmetic lower bound for overlapping supports)**.** *Let h_i be any of the original finite class representatives in H_f. Suppose that $\gamma \in G(F) \setminus B(F)$, $\gamma_f h_i \in B_f K$, and $\gamma_v h_{i,v} \in B(F_v)J_j$. For every $g \in G_\infty$,*

$$Q^j \leq C_x V(g)\, V(\gamma_\infty g).$$

The constant is independent of j, γ, g and of the class index i at these fixed data.

Proof. Apply (64.12) to $y = (g, h_i)$ and to γy. Let $\delta_0, \delta_1 \in B(F)$ be the multipliers, and set $a = \delta_1\gamma\delta_0^{-1}$. It lies outside $B(F)$. At the finite places $a_f \in C_f C_f^{-1}$, and at each real place

$$w(a_\tau) \leq C_\tau^2 \nu_\tau(\gamma_\tau g_\tau)\nu_\tau(g_\tau).$$

At v write $c_0 = \delta_{0,v} h_{i,v}$, $c_1 = \delta_{1,v}\gamma_v h_{i,v}$. Both lie in the fixed compact set C_v. We have $c_0 \in B(F_v)$ and $c_1 = bk$ with $b \in B(F_v)$, $k \in J_j$. As $k \in K_v$, the matrices $b = c_1 k^{-1}$ range in a fixed compact set. In the chosen lattice coordinates

$$a_v - bc_0^{-1} = b(k-1)c_0^{-1}$$

has entries in $\mathfrak{p}_v^{j-c}$ for a fixed integer c.

We record the equations cutting out B inside G. In the Witt frame of Section 99.16, with $A_i = \sum_{h \le i} Eu_h$ and $A_0 = 0$, they are

$$(a-1)u_i \in A_{i-1} \quad (1 \le i \le s), \qquad (a-1)e \in A_s.$$

The first conditions force an isometry to stabilize each flag space and to act as the identity on its successive lines. The dual lines then have identity action as well. The last condition fixes e in $A_s^{\perp}/A_s = W_0 \perp Ee$. The induced isometry of this middle quotient lies in $U(W_0)$. Removing that Levi factor gives precisely an element of N. The reverse inclusion follows from the same description. Expanding the conditions in an F-basis of E gives finitely many affine linear equations in F-matrix coordinates.

One of their values $z \in F$ at a is nonzero. All possible values belong to a fixed fractional ideal by the finite bound on a_f. The congruence above gives $\mathrm{ord}_v(z) \ge j - O_x(1)$, since every equation vanishes on bc_0^{-1} and its linear coefficients are fixed. Choose a positive rational integer clearing the fractional ideal. The absolute integral norm of its product with z is divisible by $Q^{\max(0, j - O_x(1))}$. At each infinite place $|\tau z| \le C'_\tau w(a_\tau)$, since the equations are affine linear and $w(a_\tau) \ge 1$. The two bounds for the same field norm give the assertion. □

Theorem 64.24 (The projected Gram matrix at increasing depth)**.** *Use the same Haar measure on the automorphic quotient at every level. On the entire original space $U \otimes D$, including the subspace of finite columns of zero sum on each scalar orbit,*

$$\|\alpha_j^{-1} S_j^* S_j - 1\|_{\mathrm{op}} \le C_x Q^{-j\eta}, \qquad \eta = L - 4fr^2 > 0.$$

Consequently, at each fixed field and choice of parameters, S_j is injective for all sufficiently large j.

Proof. Take two original seeds Φ, Ψ. After unfolding the first rational sum in $\langle P\Phi_j, P\Psi_j \rangle$, reduce its finite variable to the original class cells $h_i K$ as in Lemma 70.23. On $h_i K$ the first indicator restricts the local right coordinate to L_j. In fact $B(F_v)J_j \cap K_v = (B(F_v) \cap K_v)J_j$. The two factors have the same unitary right character on $K^{(v)}L_j$, so it cancels in their product. The original stabilizer Λ_i acts on this restricted finite cell since its conjugate at v lies in $B(F_v) \cap K_v$. A real fundamental region for Λ_i and the indicated finite cell therefore give, with the ordinary quotient measures,

$$\alpha_j^{-1} \langle P\Phi_j, P\Psi_j \rangle = \mathrm{vol}_f(K) \sum_i \int_{\Lambda_i \backslash G_\infty} \Phi(g, h_i) \overline{P\Psi_j(g, h_i)}\, dg.$$

The bounded integrable majorants of Theorem 64.22 justify the unfolding. The identity coset in $P\Psi_j(g, h_i)$ contributes $\Psi(g, h_i)$ because $h_{i,v} \in B(F_v)$. Hence its integral is exactly $\langle \Phi, \Psi \rangle_{\mathrm{seed}}$.

Write $t = V(g)$. Each remaining nonzero term of $P\Psi_j(g, h_i)$ has height at least $Q^j/(C_x t)$ by Lemma 64.23. The tail estimate of Lemma 64.21 now gives

$$|P\Psi_j(g, h_i) - \Psi(g, h_i)| \le C_{\Psi, x}\, Q^{-j\eta} t^{\eta}.$$

When the lower cutoff is less than one, the bound at height one gives the same inequality after enlarging the constant. The integral of $|\Phi(g,h_i)|V(g)^{\eta}$ on the real class quotient is finite. Disintegrate over $\Lambda_i\backslash B_\infty$, which is compact, and apply Proposition 64.20 with $\eta \le L$. Combining these estimates proves the asserted decay for each matrix entry of the difference from the original seed pairing. The seed map J_U is isometric. Summing the squared bounds over an orthonormal basis of its finite-dimensional domain gives the operator bound with a fixed constant. When the bound is less than one, every nonzero input has positive squared image norm. This proves injectivity for all those j. □

Proposition 64.25 (Averaging back to the prescribed level)**.** *Let e_K be right averaging at K_v against the conjugate of the old level character, with probability measure. On the same inputs,*

$$e_K S_j = \alpha_j S_0, \qquad \alpha_j^{-1} S_j^*(1-e_K)S_j = \alpha_j^{-1}S_j^*S_j - \alpha_j S_0^*S_0.$$

Proof. For $t_v \in B(F_v)K_v$ write $t_v = bk_0$. The indicator in $\Phi_j(tk)$, averaged over $k \in K_v$, has integral α_j because k_0k must lie in L_j. The original seed contributes its level character on the right, canceled by the conjugate character in the integral. Off $B(F_v)K_v$ every right translate is zero. Thus averaging the seed gives $\alpha_j\Phi$. Absolute summability allows the compact average to commute with periodization. This proves the first equality. Since $e_K = e_K^* = e_K^2$, taking the Gram matrix of e_KS_j gives the second equality. □

The original period matrix is $S_0^*S_0$. The estimate in Theorem 64.24 gives a small error for large j at fixed x. Its constant includes the real coefficient norms and the covolumes of the chosen fields. The required bound for the matrix at $j = 0$ on the original intervals, and its pairing with the designated integral theta rows, remain open.

64.9 Vanishing on the proposed lowest compact types

Consider the Bessel integral (99.7) on the fields of Section 99.16. At a real place write $P = k + s$, $Q = s + 1$ and $r = P + Q$, where $s = m/2 - 1 \ge 1$ and $k \ge 2$ is even. The group $H_\tau = U(k)$ acts on the positive subspace W_0. After ordering the positive coordinates with W_0 first, it is the upper left block in $U(P)$ and acts trivially on the remaining coordinates of $K_\tau = U(P) \times U(Q)$. We use the right actions on both spaces of automorphic forms and the conjugation of the column in (99.7).

Lemma 64.26 (A necessary compact branching condition)**.** *Suppose that at one real place the row belongs to a compact type of highest weight $(A;B)$, with $A_1 \ge \cdots \ge A_P$, and the column belongs to the type of highest weight $\nu_1 \ge \cdots \ge \nu_k$. A nonzero value of the integral requires*

$$A_i \ge \nu_i \ge A_{i+s} \qquad (1 \le i \le k). \tag{64.14}$$

These inequalities are necessary and sufficient for the occurrence of the column in the restriction of this compact row type.

Proof. For $y \in H_\tau$, simultaneous right translation of the two inputs changes h_0 to h_0y in (99.7). Right invariance of the measure gives $\mathcal{P}_\beta(R(y)F, R(y)\phi) = \mathcal{P}_\beta(F,\phi)$. On compact types the resulting sesquilinear form pairs only equivalent irreducible summands, by Schur's lemma. Multiplicity spaces and the $U(Q)$-factor carry the trivial H_τ-action.

For the restriction $U(n)$ to $U(n-1)$, a weight b occurs in a row c exactly when $c_i \ge b_i \ge c_{i+1}$. One proof uses the tableau character after a common determinant twist makes c a partition. The boxes containing the letter n form a horizontal strip. Removing them leaves precisely the shapes b satisfying the inequalities. Conversely all tableaux of each such shape b extend by

filling the strip with n. Setting the last variable equal to one gives the restriction character. Undoing the twist subtracts the same integer from all entries on both sides. This is also the Gelfand–Tsetlin branching rule [2, Theorem 2.1].

Iterating the inequalities gives (64.14). Conversely, for $s \geq 1$ fill the next row of length $k+s-1$ by $\gamma_i = \max(A_{i+1}, \nu_i)$ for $i \leq k$ and $\gamma_i = A_{i+1}$ for $i > k$, taking only indices in the row. It is decreasing in the weak sense and $A_i \geq \gamma_i \geq A_{i+1}$. For $1 \leq i \leq k$, one has $\gamma_i \geq \nu_i$ and $\gamma_{i+s-1} \leq \nu_i$ because $A_{i+s} \leq \nu_i$ and any entry of ν used at that index is at most ν_i. For $s = 1$ this row equals ν. Induction supplies an entire branching pattern. □

Let $\lambda_1 > \cdots > \lambda_r$ be an ordered Harish-Chandra parameter. A chamber of the discrete-series packet assigns P entries to the positive subspace, in positions $j_1 < \cdots < j_P$. In the lowest compact type the shift of the parameter is $\rho_{\mathrm{n}} - \rho_{\mathrm{c}}$, the difference of half the sums of the positive noncompact and compact roots. At position j_i the half-sum of all positive roots is $(r+1)/2 - j_i$, and the half-sum of the compact roots is $(P+1)/2 - i$. Subtracting twice the latter gives the first equality below. Subtracting the compact column's own half-sum gives the second.

$$A_i = \lambda_{j_i} - k/2 + 2i - j_i, \qquad \nu_i = u_i - (k+1)/2 + i. \tag{64.15}$$

Here u denotes the column Harish-Chandra parameter. These are the conventions of `Formalism/PeriodIntegrals/UpqRep.lean`, line 231. The entries of A are dominant. In fact $\lambda_{j_i} - \lambda_{j_{i+1}} \geq j_{i+1} - j_i$ by regular integrality, so $A_i - A_{i+1} \geq 2(j_{i+1} - j_i) - 2 \geq 0$.

Theorem 64.27 (Vanishing for the explicit terminal weights)**.** *Use the lists printed just before Theorem 72.8. The global integral vanishes for every lowest compact row type in the packet and the specified column. This holds for every terminal prime, every $\beta \in E^{\times}$, and all smooth finite components. It already holds at a place with the low column weight.*

Proof. Write $v_i = 2(k-1-i)$ for $1 \leq i < k$. The row is obtained by adjoining $-j(t_{\mathrm{r}}+4)$ for $1 \leq j \leq m$. The column has $u_i = v_i + 1/2$ for $i < k$, $u_k = -1/2$, with an additional a_τ in u_1, equal to a at a high place and zero otherwise. Since $j_1 \geq 1$, $j_k \geq k$, monotonicity of λ in (64.15) gives

$$\begin{aligned} A_1 &\leq v_1 - k/2 + 1, & \nu_1 &= v_1 - k/2 + 1 + a_\tau, \\ A_k &\leq -(t_{\mathrm{r}}+4) + k/2, & \nu_k &= k/2 - 1. \end{aligned}$$

The last inequality required by $A_k \geq \nu_k$ thus fails by at least $t_{\mathrm{r}} + 3 > 0$. Lemma 64.26 applies in every chamber. Finite translations and finite averages commute with H_τ, so the same vanishing holds after either operation. □

Proposition 64.28 (Order of a derivative that could contribute)**.** *Let f be the number of real places and h the number with $a_\tau = a$. Starting with the lowest compact types of Theorem 64.27, a differential operator of total order less than*

$$ha + f(t_{\mathrm{r}} + 3) \tag{64.16}$$

has zero period against the specified column. Furthermore, operators generated solely by the terminal one-parameter subgroups $n(t)$ of Lemma 99.20 have zero period in every order.

Proof. At one place the compact span of derivatives of order at most d is a quotient of a subspace of the sum of $\mathfrak{gl}_r(\mathbb{C})^{\otimes j} \otimes V_{(A;B)}$, $0 \leq j \leq d$. Indeed composition of derived actions intertwines the compact action on the input with conjugation on each Lie algebra factor. As a $U(P)$-module the latter factor consists of copies of the standard module V, its dual, $V \otimes V^*$, and trivial modules, by the four blocks in an r-by-r matrix.

Tensoring a dominant weight c with V increases precisely one coordinate by one in each constituent. Tensoring with V^* decreases precisely one. For instance, in the alternant numerator of the Weyl character formula multiply by $\sum_i z_i$ or by $\sum_i z_i^{-1}$. Distributing yields the sum of alternants with one exponent moved by one. A repeated exponent gives zero and the surviving sequences are still strictly decreasing. The denominator is unchanged. Consequently each Lie algebra factor adds at most one to the total of all upward coordinate moves, also in the $V \otimes V^*$ block. Every resulting highest weight A' therefore satisfies $\sum_i \max(A'_i - A_i, 0) \le d$. For the column to occur we need $A'_i \ge \nu_i$ for $i \le k$. The distinct indices $1, k$ alone require at least $a_\tau + t_{\mathrm{r}} + 3$ upward moves. Expanding a global differential operator in words supported at individual places and summing these local lower bounds proves (64.16).

Finally each $n(t)$ acts only on the three-space spanned by u_s, e, v_s and fixes W_0 pointwise. Its derivatives commute with H_τ, so applying any of them preserves the list of H_τ-types available in the row. Theorem 64.27 again gives zero. □

The exponential bound for the analytic integral in Theorem 65.2, holds in particular for a zero matrix. Its Corollary 65.3 requires a nonzero algebraic coefficient. The explicit weights above cannot supply that coefficient on these lowest compact types. The scalar in the identification in `Formalism/SharpRestriction/GlobalTheoremInputs.lean`, line 65 cannot turn the zero pairing on the specified spaces into a nonzero one.

64.10 Global Bessel coefficients at the prescribed finite levels

Let E/F_+, G, $H = U(W_0)$, N and χ_β be as in Section 99.16. Put $B = NH$ and write $p_H : B \to H$ for the quotient with kernel N. Use the row and column levels K_σ, K_π of Proposition 72.6. At the finite places away from g_1, g_2 use the respective standard lattice stabilizers. In particular the original stabilizers are used at the terminal prime T. We take probability measure on $[H]$ and $[N]$ as in (99.7). All assertions below use the shortened period field and the intervals of Proposition 9.9.

Proposition 64.29 (Column multiplicities and their inner products)**.** *Let $M_H = \mathfrak{m}_k I_\pi$, where I_π is the index of K_π in the full finite stabilizer and $\mathfrak{m}_k$ is the mass of the standard definite genus. Then M_H is the number of finite level classes. If ρ is the irreducible unitary representation of H_∞ of a specified column weight, put $D_\rho = \dim \rho = \prod_\tau W_k(u_\tau)$. Its multiplicity in $L^2([H])^{K_\pi}$ is $M_H D_\rho$, the quantity $d_{u,*}$ of Theorem 72.8.*

Choose representatives $y_1, \dots, y_{M_H}$ in $H(\mathbb{A}_{F_+,f})$. On the class y_i, and zero outside that class, set $\phi_{i,a,b}(h_\infty, y_i) = \langle e_a, \rho(h_\infty) e_b \rangle$ for an orthonormal basis of the representation, using an inner product linear in the second entry. These $M_H D_\rho^2$ functions are orthogonal and each has squared norm $1/(M_H D_\rho)$.

Proof. At the full level a class corresponds to a lattice L in the genus, with finite stabilizer $\mathrm{Aut}(L)$. Finiteness follows by taking its integral action on the underlying lattice with its positive definite trace form. Refinements of this class are the orbits of $\mathrm{Aut}(L)$ on the I_π cosets of the smaller level. All the refined stabilizers are trivial, as in the proof of Corollary 70.10. Indeed each stabilizer element has unipotent reduction on the underlying $2fk$-dimensional rational lattice at the split prime g_1. The principal congruence kernel in $\mathrm{GL}_{2fk}(\mathbb{Z}_{g_1})$ is torsion free by the binomial formula at the odd prime g_1. Thus an element of finite order here has g_1-power order. An element of order g_1 would require a rational cyclotomic factor of degree $g_1 - 1 > 2fk$ in its characteristic polynomial. Hence the action on refinements is free. It contributes $I_\pi / |\mathrm{Aut}(L)|$ classes. Summing proves the formula for M_H.

Each refined class in $[H]/K_\pi$ is one copy of H_∞. They have equal measure because they have the same finite level volume and trivial stabilizers. Each copy therefore has mass $1/M_H$.

The right regular representation on one such copy contains ρ with multiplicity D_ρ by the compact group decomposition into matrix coefficients. Schur orthogonality gives their stated inner products. In particular the multiplicity space can be described by one covector in $\rho^\vee$ for each class, by evaluation at $(1, y_i)$. □

Lemma 64.30 (The period domain at finite level)**.** *Let $K_B = B(\mathbb{A}_{F_+,f}) \cap K_\sigma$. The natural map*

$$Y = B(F_+)\backslash B(\mathbb{A}_{F_+})/K_B \longrightarrow X = G(F_+)\backslash G(\mathbb{A}_{F_+})/K_\sigma$$

is a smooth embedding with compact image. Moreover $p_H(K_B) = K_\pi$, with K_π embedded in K_σ, and χ_β extended trivially on H is trivial on K_B for every $\beta \in \mathcal{O}_E$.

Proof. Suppose representatives b_1, b_2 have the same image. Then $b_2 = \gamma b_1 k_f$ with $\gamma \in G(F_+)$ and $k_f \in K_\sigma$. The real component of γ belongs to B_∞. Since B is a closed algebraic subgroup over F_+, its polynomial equations evaluated at the rational element γ already vanish over F_+. Thus $\gamma \in B(F_+)$, and $k_f \in K_B$. This proves injectivity. Both quotients are locally discrete group quotients of their real Lie groups by left translation. To see discreteness, at a fixed finite level the allowed rational matrices lie in a fixed fractional lattice in all their real embeddings. The local map is the immersion of B_∞ in G_∞. The quotient for B is compact. Indeed $[N]$ is compact by its additive filtration, and $[H]$ is compact by definiteness. Compact sets of representatives for these two quotients give a compact set of representatives for $[B]$, by first moving the H-component rationally and then the N-component. An injective immersion of a compact manifold into a Hausdorff manifold is an embedding.

The action on the middle quotient projects $b = nh \in K_B$ to the W_0-matrix of h. It preserves the standard lattice there. Indeed projection to the W_0-coordinates and inclusion of those coordinates both preserve the original lattice, and the same argument applies to b^{-1}. At g_1, g_2 this block is upper unitriangular modulo the prime, so $h \in K_\pi$. Conversely the block embedding of K_π has identity on the other graded spaces. It satisfies all row level conditions, including the zero entries in the column of e. Thus $K_B = (K_\sigma \cap N(\mathbb{A}_{F_+,f})) \rtimes K_\pi$. For n in this intersection, nu_{i+1} and ne are integral at each finite place. Their relevant u_i-coefficients are integral even at two, since they are the coefficients of the original positive basis vectors in the respective sums of the u_j. Traces of integral elements to $\mathbb{Q}_p$ are integral. The standard finite additive character is therefore trivial on every term defining $\chi_\beta(n)$ for $\beta \in \mathcal{O}_E$. Invariance under H-conjugation extends this character to B as asserted. □

For a row function F write its Bessel coefficient as

$$\mathcal{F}_\beta F(h) = \int_{[N]} F(nh)\overline{\chi_\beta(n)}\, dn.$$

It is a function on $[H]/K_\pi$, by the character invariance and the inclusion of finite levels in Lemma 64.30. Let $\mathcal{C}$ be a sum of the full isotypic spaces of any finite set of column types in $L^2([H])^{K_\pi}$. It is finite dimensional and smooth, by Proposition 64.29.

Theorem 64.31 (Simultaneous prescribed global coefficients)**.** *For any finite set J of distinct nonzero elements of $\mathcal{O}_E$ there is an H_∞-equivariant linear map*

$$S : \bigoplus_{\beta \in J} \mathcal{C} \longrightarrow C_c^\infty(X) \quad \text{with} \quad \mathcal{F}_\alpha S((\phi_\beta)_\beta) = \phi_\alpha \quad (\alpha \in J).$$

Its image can be chosen finite under K_∞. For every $\eta > 0$ it can also be chosen with L^2 operator norm at most η, using the direct sum norm on the input and the fixed quotient measure on X.

Proof. First prescribe the restriction of the desired function to Y by the formula

$$(nh) \longmapsto \sum_{\beta \in J} \chi_\beta(n)\phi_\beta(h).$$

It descends on the left by rational invariance of each factor and on the right by Lemma 64.30. Distinct χ_α, χ_β are orthogonal on $[N]$. Indeed their ratio is nontrivial on the last one-parameter subgroup of Lemma 99.20, as is seen by varying its real argument at one place. Right invariance of the probability measure forces the integral of this ratio to vanish. Consequently the prescribed restriction already has all the stated coefficients.

Choose an H_∞-invariant Riemannian metric on X by averaging a metric over this compact right action. In a tubular neighborhood of the compact embedded Y, pull its prescribed function back by normal projection and multiply by a smooth function of squared normal distance which is one on Y and zero near the boundary of the neighborhood. Extension by zero gives a compactly supported smooth function. The neighborhood, projection and cutoff can be chosen invariant under H_∞; the construction is linear and equivariant. Denote the resulting map by S_0. Its norm can be made arbitrarily small by shrinking the tube. In detail, the prescribed functions on Y are uniformly bounded on the unit ball of their finite dimensional domain, and Y has positive codimension in X, since NH is a proper Lie subgroup of smaller dimension than G. Smooth quotient measure gives Y measure zero. The volumes of tubes of decreasing radius tend to zero by compactness.

To obtain compact-finite functions, apply to S_0 finite sums P_j of the right K_∞-isotypic orthogonal projections, exhausting its irreducible types. These commute with H_∞. On smooth functions they converge uniformly on compact sets to the identity. For example, on the compact group apply the Peter–Weyl decomposition to $k \mapsto F(xk)$. Partial sums converge in every compact Sobolev norm by the compact Laplacian. With cutoff eigenvalue j the norm of the omitted tail in order t is at most $(1+j)^{-1}$ times the norm in order $t+2$. The latter norms are bounded uniformly for x on a compact set because all its products with K_∞ form a compact set. Sobolev embedding then gives uniform convergence there. The supports after projection lie in the compact K_∞-orbit of the original support. Thus on the finite dimensional source the joint coefficient of $P_j S_0$ converges in operator norm to the identity. The coefficient maps and $P_j S_0$ are H_∞-equivariant and the finite column level is K_π. Since $\mathcal{C}$ contains the entire isotypic spaces of the input types, the range lies in $\bigoplus_{\beta \in J} \mathcal{C}$. For some j this endomorphism A_j satisfies $\|A_j - 1\| < 1/2$. Taking $S = P_j S_0 A_j^{-1}$ gives exactly the required identity. Since $\|P_j\| \le 1$ on $L^2(X)$ and $\|A_j^{-1}\| \le 2$, starting with $\|S_0\| < \eta/2$ proves the norm assertion. □

Corollary 64.32 (The two terminal frequencies on the same quotient)**.** *Take $J = \{1, \varpi\}$ for the chosen terminal generator $\varpi\bar{\varpi} = T$, and include the prescribed column types in $\mathcal{C}$. For any orthonormal basis (ϕ_i) of $\mathcal{C}$, there are smooth compact-finite row functions $F_{\beta,i}$ at the given levels with*

$$\mathcal{P}_\alpha(F_{\beta,i}, \phi_j) = \mathbf{1}_{\alpha=\beta}\mathbf{1}_{i=j} \qquad (\alpha, \beta \in J).$$

Proof. Apply Theorem 64.31 to the vector with ϕ_i in component β and all other components zero, then pair the resulting coefficient with ϕ_j on $[H]$. □

The functions in this construction have a finite set of compact types that can depend on the chosen tube. Their construction imposes neither a particular infinitesimal character nor the condition of lying in an iterated theta lift. In fact the lowest-type component for the coded source row gives zero in every entry by Corollary 65.8. To transfer the displayed global matrix to the required arithmetic row one must control its spectral projection and its values on the integral theta lattice.

64.11 Fourier inversion between two positions of the Bessel period

Fix a word u and its field from Theorem 9.13. Write A for the definite space of the column, $a = \dim_E A$, $H = U(A)$, and G for the row group. Write s for the length of its isotropic flag. Fix the marked vector $z = z(u)$ and the simple-root character χ of N. For a column ϕ and a smooth automorphic row F, put

$$L(F) = c_H \int_{[H]} \overline{\phi(h)} \int_{[N]} F(nh)\overline{\chi(n)}\, dn\, dh, \qquad (R(g)F)(t) = F(tg).$$

Here c_H is the scalar in the given pairing, $[N]$ has mass one, and H acts identically on the orthogonal complement of A. The two quotients have compact sets of representatives by Lemma 64.30. Thus these integrals exist for each fixed translate. For $n_0 \in N(\mathbb{A})$ one has $L(R(n_0)F) = \chi(n_0)L(F)$. Indeed hn_0h^{-1} lies in N and has the same simple-root coordinates as n_0. Right translation by this element in the inner quotient preserves its measure and contributes $\chi(n_0)$.

At position one the split factors are those over the chosen terminal generator $\pi_T(u)$ and the conjugate marked generators $\bar{\pi}_{g_j}$. Denote these chosen halves by S_u. At $v \in S_u$ let $k = F_{+,v}$, O its valuation ring, π a uniformizer and $q = |O/\pi O|$. Identifying the unitary group with the general linear group on this half determines the action on the other half by inverse adjoint. Use coordinates $e_1, \dots, e_s, A, z, f_1, \dots, f_s$, with e_i, f_i dual. Set $d_i = e_{i+1}$ for $i < s$ and $d_s = z$. We use d_i^* for the corresponding coordinate functional. For $x \in A_k$, $y \in A_k^*$ and $i \le j$, put

$$X_i(x) = 1 + x d_i^*, \qquad Y_i(y) = 1 + e_i y, \qquad Z_{ij}(t) = 1 + e_i t d_j^*.$$

The covectors on A_k are extended by zero on the other summands. The exchange group at index i fixes all basis columns except d_i, and its displacement on d_i can be any vector of A_k. It is therefore exactly $X_i(A_k)$. At position one these are the indices $1 \le i \le s$. At the symmetric position there are no exchanges. All the X_i commute.

Lemma 64.33 (The character on the split flag)**.** *The elements $Y_i(y)$ and $Z_{ij}(t)$ belong to $N(k)$. Their χ-values are one except for $\chi(Z_{ii}(t)) = \psi_i(t)$, a nontrivial additive character of k. With the character $\psi_v(\mathrm{Tr}_{E_v/k}(\theta \sum_l h(nd_l, f_l)))$, linearity of h in its first entry gives $\psi_i(t) = \psi_v(\theta_v^+ t)$, where $+$ denotes the chosen half. For $P = \prod_{j \ge i} X_j(x_j)$ one has*

$$PY_i(y)P^{-1} = Y_i(y) \prod_{j \ge i} Z_{ij}(-y(x_j)).$$

Proof. Each indicated element sends an isotropic flag vector to itself plus earlier flag vectors and acts trivially on the middle quotient. On the opposite half the added map in the inverse adjoint has input the coordinate dual to e_i, so it can move only f_i. It fixes all the columns used to compute the character. On the chosen half only $Z_{ii}(t)$ contributes to the sum of adjacent coordinates, and its contribution is t. The split trace adds the two components and $\theta_v^+ \neq 0$. Finally $Pe_i = e_i$, $P^{-1} = 1 - \sum_{j \ge i} x_j d_j^*$, and $yP^{-1} = y - \sum_{j \ge i} y(x_j) d_j^*$. Multiplying the rank-one matrices gives the asserted identity. □

Fix the original additive Haar measure μ_i on each X_i. Write $L_{>i}$ for integration of $L(R(\prod_{j>i} X_j(x_j))F)$ over the indicated variables at this place and $L_{\ge i}$ for the result after also integrating x_i. Exchanges at other places may already be present. These definitions include the functional $L_{>s} = L$ when there are no other exchanges.

Theorem 64.34 (Finite Fourier inversion for a specified row)**.** *All these exchange integrals converge absolutely. Suppose that F is fixed by $X_i(U)$ and $Y_i(V)$ for full O-lattices $U \subset A_k$, $V \subset$*

A_k^*. Write $\perp$ for annihilators under $(x,y)\mapsto\psi_i(y(x))$. Choose U,V small enough that $U\subset V^\perp$. Put $D=V^\perp/U$, $D'=U^\perp/V$ and $b(x)=L_{>i}(R(X_i(x))F)$. For every $y\in U^\perp$,

$$L_{\geq i}(R(Y_i(y))F)=\mu_i(U)\sum_{x\in D}b(x)\,\overline{\psi_i(y(x))},$$

$$L_{>i}(F)=\frac{1}{|D|\mu_i(U)}\sum_{y\in D'}L_{\geq i}(R(Y_i(y))F).$$

The first formula at $y=0$ evaluates the exchanged functional on the original row. If $U=V^\perp$, its value is $\mu_i(U)L_{>i}(F)$.

Proof. First consider the joint integrand before any exchange. Fix a basis of A_k. Smoothness gives an integer $m\geq 1$ such that the row is fixed by the principal congruence subgroup of depth m in these coordinates. The character ψ_i is the same for every i at this place. Let c be the integer for which the annihilator of O under this character is π^cO. For a tail P beginning at i, the lemma and N-equivariance show that $L(R(P)F)$ vanishes unless $x_i\in\pi^{c-m}O^a$. Indeed it must be unchanged by right multiplication by every $Y_i(y)$ with $y\in\pi^mO^{a*}$. The change on moving this matrix to the left of P is $\overline{\psi_i(y(x_i))}$.

To apply the same argument after $x_1,\dots,x_{i-1}$ have been bounded, move their commuting product onto F. Its open stabilizer implies that the translated row takes only finitely many values on this compact set. A single deeper congruence subgroup fixes them all. The tail argument then bounds x_i uniformly on this set. Starting at the first index proves compact support in all variables. Finite products of places satisfy the same argument since their groups commute. The joint integrand is locally constant. It is therefore absolutely integrable, and Fubini's theorem applies.

Conjugating $Y_i(y)$ through the factors X_j, $j>i$, produces only Z_{ij} with $j>i$, on which $\chi=1$. The subgroup Z_{ii} commutes with these factors. Thus $L_{>i}$ is Y_i-invariant and transforms by ψ_i under Z_{ii}. Applying the lemma with the additional factor $X_i(x)$ gives the multiplier $\overline{\psi_i(y(x))}$ in the integrand for $R(Y_i(y))F$. Taking $y\in V$ shows that b vanishes off $V^\perp$. It is constant on cosets of U by right invariance. Integration on these cosets proves the first formula. The pairing identifies D' with the full character group of D. Indeed in a basis diagonalizing the inclusion $U\subset V^\perp$, each pair of cyclic O-module quotients has the same order and has zero annihilator on either side. Summing the characters over D' therefore selects $x=0$ with multiplicity $|D|$, which proves the second formula. □

In particular both directions are integral away from the residue characteristic after removing the displayed Haar scalar. More precisely, work in a coefficient ring containing the finite character values. After division by $\mu_i(U)$, the forward matrix has entries in this ring and its inverse is the conjugate transpose divided by $|D|$, a power of the residue characteristic. This assertion concerns the full vector of Fourier values. For one value at $y=0$ the formula retains the individual signs.

Proposition 64.35 (The comparison on the two prescribed translates)**.** *Let Φ and W be the given finite-dimensional Hilbert spaces of columns and rows for a word with $s>0$. Take the two specified finite translates g_{sym},g_1. Inserting the given column and row forms, denote the symmetric pairing on the first translate by $\langle\phi,SF\rangle$ and the position-one pairing on the second by $\langle\phi,JF\rangle$, with the inner product linear in its second argument. Use the conjugate column space if the original parametrization uses conjugated forms. For a basis $F_1,\dots,F_n$ of W, the two pairings satisfy*

$$\langle\phi,SF\rangle=\kappa\langle U^s\phi,JF\rangle\quad(\phi\in\Phi,\ F\in W)$$

for some unitary U on Φ and some $\kappa \neq 0$ if and only if a single $t > 0$ satisfies

$$\langle SF_j, SF_l\rangle = t\langle JF_j, JF_l\rangle \qquad (1 \le j, l \le n).$$

Every entry on the right is a finite sum of products of symmetric periods on the second translate. A positive scalar multiplying U can be absorbed into κ.

Proof. The two linear maps to Φ are given by finite-dimensional Riesz representation. For an orthonormal column basis (h_k) the k-th coordinate of JF_j is the finite exchange sum of Theorem 77.14 for the column h_k and the row $R(g_1)F_j$. Thus its Gram entries are obtained by summing products of these coordinates with the first factor conjugated. The coordinates of SF_j use $R(g_{\mathrm{sym}})F_j$. If the pairing identity holds, $S = \kappa(U^s)^*J$, which gives $t = |\kappa|^2$. Conversely the Gram identities extend to every pair of rows by sesquilinearity. The map $JF \mapsto t^{-1/2}SF$ is then well defined and preserves the inner product. Extend it to a unitary V on Φ by matching orthonormal bases of the two orthogonal complements. Diagonalizing V^* and taking an s-th root of each eigenvalue gives a unitary U with $U^s = V^*$. Taking $\kappa = \sqrt{t}$ proves the assertion. □

The finite Fourier identities determine the entries to be compared. Their proportionality on the specified automorphic row space remains to be proved.

Corollary 64.36 (A translate annihilating the entire row space)**.** *Suppose $a, s \ge 1$ and fix a split finite place and a finite translate g_0. There is a compact lattice $C \subset k^a$ such that for every $x \notin C$, every column ϕ and every $F \in W$,*

$$L_\phi(R(X_i(x)g_0)F) = 0.$$

If the position-one pairing on $R(g_1)W$ is nonzero, the proportionality in Proposition 64.35 fails for each of these choices $g_{\mathrm{sym}} = X_i(x)g_0$.

Proof. Open stabilizers of the translates of a finite basis of W have open intersection. In particular $Y_i(V)$ fixes all of $R(g_0)W$ for a single full lattice V. The commutator formula, with just the factor $X_i(x)$, forces the symmetric coefficient to vanish outside $C = V^{\perp}$. This uses only the character on N and is uniform in ϕ. The symmetric Gram matrix is zero there. The nonzero position-one pairing has a strictly positive diagonal entry in some row basis. □

The supplied source records both translates as data. In its Tied condition, the exponential-twist and archimedean identities involve g_1, and the root-of-unity identity involves the character factors. All three are preserved when g_{sym} changes. The Hecke condition called adm5 depends only on the underlying arithmetic datum. Consequently, for any packet satisfying these conditions and having a nonzero position-one pairing, they continue to hold for choices of g_{sym} excluded by the corollary. The comparison for a selected packet must use a property of its particular g_{sym}. The two definitions occur in `ZetaProductPairs.lean`, lines 197–205 and 422–427.

64.12 A Fourier operator on the refined column level

Let $k = \mathbb{Q}_T$ at a split terminal place and take $\pi = \varpi$ on the chosen half. Write $O = \mathbb{Z}_T$, $q = T$, $K = \mathrm{GL}_a(O)$, $V = O^a/\pi O^a$. In V choose the last coordinate vector v_0, and in its dual choose the last coordinate covector ℓ_0. An integral change of basis makes any prescribed primitive covector take this form. Write P for the stabilizer of ℓ_0 in K and P' for the stabilizer of v_0. Use the twisted unitary column action $\rho = \widetilde{R}_H$ of Section 59.20. Its restriction to the finite scalar

residues on P- or P'-invariants is well defined because lifts differ by an element congruent to one. A superscript $\circ$ on either fixed space means that the sum of these scalar actions is zero on the vector.

Let $\psi_0(\bar{t}) = \psi(t/\pi)$ for the simple additive character of conductor O. For $0 \neq \eta \in V^*$ choose $t_\eta \in K$ with $\ell_0 t_\eta^{-1} = \eta$ modulo π. For $v \neq 0$ choose $k_v \in K$ with $k_v v_0 = v$ modulo π. Set $h = \mathrm{diag}(1, \dots, 1, \pi^{-1})$. We use

$$B\phi = \sum_{\eta \neq 0} \psi_0(\eta(v_0))\rho(t_\eta)\phi,$$
$$C\xi = \sum_{v \neq 0} \overline{\psi_0(\ell_0(v))}\rho(k_v)\xi,$$
$$D\phi = \rho(h^{-1})B\phi.$$

Theorem 64.37 (An invertible operator on the scalar zero-sum part)**.** *For every unitary $H(k)$-representation, D maps $(\mathcal{H}^P)^\circ$ onto itself and*

$$\|D\phi\|^2 = q^a\|\phi\|^2, \qquad D^{-1} = q^{-a}C\rho(h).$$

Both D and $q^a D^{-1}$ are finite sums of group actions with coefficients in $\mathbb{Z}[\psi_0(\mathbb{F}_q)]$. They preserve each scalar character subspace wherever that character is defined.

Proof. The choices in B differ by right multiplication by P and those in C by right multiplication by P'. Reindexing shows that $B\phi$ is P'-fixed, $C\xi$ is P-fixed, and

$$\rho(k_v)B\phi = \sum_{\eta \neq 0} \psi_0(\eta(v))\rho(t_\eta)\phi.$$

The analogous identity for $\rho(t_\eta)C\xi$ has coefficients $\overline{\psi_0(\eta(v))}$. The action of each scalar unit commutes with every group element in the definitions. If ϕ has zero scalar sum, then $\sum_{\eta \neq 0} \rho(t_\eta)\phi = 0$. Indeed the summands for a single line in V^* are one translate of the scalar sum, by the stabilizer description. The same argument applies on vector lines to ξ. Summing the displayed identity against $\overline{\psi_0(\ell_0(v))}$ and including $v = 0$, whose term is zero, gives $CB\phi = q^a\phi$ by finite character orthogonality. The second identity gives $BC\xi = q^a\xi$ on the corresponding vector space.

Put $N = q^a - 1$. In the first identity sum the squared norms over all $v \in V$, setting its value at zero equal to zero. The left side is $N\|B\phi\|^2$ by unitarity. On the right, orthogonality leaves $q^a \sum_{\eta \neq 0} \|\rho(t_\eta)\phi\|^2 = q^a N\|\phi\|^2$. Lastly $h^{-1}P'h = P$. In the matrix of a member of P', the entries above the last diagonal entry in the last column are divisible by π and the last diagonal entry is one modulo π. Conjugating by h^{-1} divides those entries by π and multiplies the off-diagonal entries of the last row by π, giving exactly P. The action of h^{-1} is unitary with inverse $\rho(h)$. This proves the theorem. □

On automorphic columns the action is by the specified finite translations. It commutes with projection at a different place. At the chosen place the theorem applies inside any closed subspace stable under its full local group. In particular it acts on the actual cuspidal column space of the specified real type with all the other finite levels fixed.

64.13 Bessel exchange on the original signed auxiliary rows

Keep the signed auxiliary row projectors, the terminal graph and the separately zero-sum columns on one field of Section 6.6. Choose d_* from Theorem 9.18. For the ordinary global character χ of the unipotent group N, write

$$\chi_*(n) = \chi(d_* n d_*^{-1}).$$

It is a character of $N(F)\backslash N(\mathbb{A}_F)$ since d_* is rational and normalizes N. The column group H preserves it. Use superscripts on $\mathcal{B}, \mathcal{J}$ to specify the character.

Theorem 64.38 (A rational character compatible with the original projector)**.** *For each original signed row A and every indicated column ϕ,*

$$\mathcal{J}_{\phi}^{\chi_*}\big(\widetilde{R}(g_T)A\big) = C_T \prod_{v\mid g_1 g_2} \prod_{i=1}^{s} \mu_{i,v}(\pi_v O_v^a)\, \mathcal{B}_{D_T^s\phi}^{\chi_*}(A).$$

Both sides use A at the original signed auxiliary levels. The terminal scalar and column operator are those of Theorem 59.24. At either auxiliary place the character on the ordinary period agrees with λ_Γ throughout $N \cap \Gamma$. For any row and column one also has the exact global identity

$$\mathcal{B}_{\phi}^{\chi_*}(A) = \mathcal{B}_{\phi}^{\chi}(R(d_*)A),$$

where $R(d_*)$ is the full ordinary adelic translation, including all archimedean places. No Haar scalar occurs in this second identity.

Proof. In an auxiliary chosen half, conjugation by d_* multiplies the j-th string entry by the ratio of its two consecutive eigenvalues. This ratio differs from the one in Theorem 77.17 by a unit congruent to one modulo π_v and has valuation -1. On O_v both ratios give the same additive character, because their difference lies in O_v and the original conductor ideal is O_v. The product formula on the string entries of Lemma 77.16 proves the compact character agreement. The annihilator of O_v for each new simple character is now $\pi_v O_v$. The original signed row is fixed by $Y_i(O_v^{a*})$ and $X_i(\pi_v O_v^a)$. These lattices are exact annihilators of each other for the commutator pairing with χ_*. The finite Fourier inversion in Theorem 64.34 therefore has only its zero coset and contributes $\mu_{i,v}(\pi_v O_v^a)$ for each integration on this row.

At T every new simple-root ratio is one modulo π_v. In particular the annihilator of O_v is still O_v and the characters on $\pi_v^{-1}O_v/O_v$ are exactly the old ones. In the single-step proof of Lemma 59.23 the only finite character entries are $\psi_i(\pi_v^{-1}y)$ for $y \in O_v$. Its zero term, the transfer D_v and each Haar factor are therefore identical. The conjugation of the off-diagonal string entries still has trivial character, since a diagonal cannot change their root positions. This verifies the commutator calculation with the new character separately at each simple root. The compact-support argument of Theorem 64.34 applies here because at each step it uses only the annihilator of the current nontrivial simple character and an open stabilizer of the smooth translated row. It bounds the support at the finitely many marked places successively. Thus one may perform all terminal steps and all auxiliary integrations in the indicated order, proving the first identity.

In the ordinary integral over $[H]$ and $[N]$, one has $A(nhd_*) = A(d_*^{-1}nd_*h)$ by commutation with H and left invariance under $G(F)$. Make the change $n' = d_*^{-1}nd_*$. Conjugation preserves $N(F)$, and the mass-one invariant measure on its adelic quotient is preserved. Equivalently the local Jacobians multiply to one by the product formula on its rational Lie algebra. The character becomes $\chi(d_*n'd_*^{-1})$. The column and its measure have stayed fixed. This proves the second identity for the original Haar normalization. □

In the original Bessel normalization the rational translation on the right includes its components away from the auxiliary primes. The height bound in Theorem 9.18 controls the valuations and the archimedean sizes of the rational flag coordinates. Its lower bound forces an unmarked finite valuation on each isotropic coordinate. Measuring the translations on the prescribed lattices requires accounting for their integral frames. Identifying the resulting signed value with the independent integral theta column in `Formalism/SharpRestriction/Divisibilit`

y/ZetaProductPairs.lean, line 107 still requires evaluation with those components and its prescribed endpoint projector. Theorem 74.26 gives exact vanishing for the attempted comparison which simply projects a matched auxiliary translate back to its original signed type.

64.14 When the unmarked places annihilate the Bessel period

Use the original auxiliary character, row level and terminal frame of Theorem 64.38. Write $g = g_1 g_2$ and $\rho_j = c_j/c_{j+1}$ with $c_{s+1} = 1$. At a good unmarked split place the row has the full lattice level. The two flag halves are integral and the absolute-trace character on every simple string entry has conductor O_v. These properties hold at all but finitely many completely split places. In fact the trace multiplier and the coefficients of the original frame and its inverse are units or integral as required outside finitely many primes. The standard finite additive character of $\mathbb{Q}_p$ has conductor $\mathbb{Z}_p$. Whenever prescribing the support of the new valuations, we may include the exceptions in the unmarked congruences in Theorem 9.20 or Theorem 9.21.

Lemma 64.39 (The integral root condition on a spherical row)**.** *At a good unmarked split place, $\chi_*(n) = \chi(d_* n d_*^{-1})$ is trivial on $N \cap K_v$ precisely when $v_{\mathfrak{p}}(\rho_j) \geq 0$ at both primes $\mathfrak{p}$ of E above v, for every $1 \leq j \leq s$. A negative valuation on any one root implies $\mathcal{B}_\phi^{\chi_*}(A) = 0$ for every global column ϕ and every original row A fixed by K_v.*

Proof. The string is $e_1, \dots, e_s, z, f_s, \dots, f_1$ on the chosen half. The simple entries have multipliers $\rho_1, \dots, \rho_s$ followed by $\bar{\rho}_s, \dots, \bar{\rho}_1$ under conjugation. The original additive coefficients are units. The character on N is their product on these simple entries, by the split trace calculation of Lemmas 64.33 and 77.16. This last calculation of the trace uses only the flag and the inverse adjoint on the second half. Every element of $N \cap K_v$ has integral entries there, and each individual simple string root subgroup $Z(O_v)$ lies in K_v. The conductor therefore gives the equivalence.

If a multiplier has negative valuation, choose $u \in Z(O_v)$ with $\chi_*(u) \neq 1$. The root involves only the string and fixes the column space, so u commutes with H. In the ordinary integral, right translation of the unipotent variable by u is a measure-preserving change of variables on $[N]$. It multiplies the value on replacing A by $R(u)A$ by $\chi_*(u)$. The row is fixed by u. Hence its period is zero for every column. □

Theorem 64.40 (A lower bound for the finite denominators and exact vanishing)**.** *Suppose a rational diagonal matches the auxiliary residues and is the identity in the split flag coordinates modulo T^b. Set*

$$D_j^- = \sum_{\mathfrak{p} \nmid gT} \max(0, -v_{\mathfrak{p}}(\rho_j)) \log N\mathfrak{p}.$$

For every simple ratio one has the global inequality

$$\exp(D_j^-/n) \left(\prod_{\sigma : E \hookrightarrow \mathbb{C}} |\sigma(\rho_j) - 1| \right)^{1/n} \geq T^b/g.$$

In particular take a flag from Theorem 9.20 whose unmarked nonzero valuations occur at good split places and for which $|\sigma(c_i) - 1| \leq \delta < 1/2$ everywhere. For $s \geq 1$ one has $\mathcal{B}_\phi^{\chi_*}(A) = 0$ for every original row A fixed at these good places and every column ϕ. Both sides of the exchange identity of Theorem 64.38 then vanish on its separately zero-sum columns.

Proof. The element $y = g(\rho_j - 1)/T^b$ is nonzero because $v_{\mathfrak{p}}(\rho_j) = -1$ at every $\mathfrak{p} \mid g$. There g has valuation one and T is a unit, so y is a unit. At the primes over T the full congruence makes y integral. At any other prime where ρ_j has negative valuation the ultrametric inequality

gives $v_{\mathfrak{p}}(\rho_j - 1) = v_{\mathfrak{p}}(\rho_j)$. At all remaining primes y is integral. Thus $\log|N_{E/\mathbb{Q}}(y)| \geq -D_j^-$. Evaluating that same norm at the archimedean embeddings gives the displayed inequality.

For the last positive root $\rho_s = c_s$. Its archimedean geometric mean in the formula is at most δ, so $D_s^- \geq n\log(T^b/(g\delta)) > 0$. Indeed the chosen terminal prime satisfies $T \equiv 1 \bmod g$ and $T > g$ by Theorem 7.11. At least one unmarked prime of E has negative valuation on this ratio. On the assumed support it lies at a good split place where the row is spherical. Lemma 64.39 forces the ordinary period to be zero. The exchange formula expresses its first-position value as an explicit multiple of this period evaluated at another column, so that value is zero as well. □

Proposition 64.41 (Exchange with integral ratios at every compensating place)**.** *Use the diagonal of Theorem 9.21, with the unmarked exceptions included in M. The rational character χ_* agrees with the prescribed signed auxiliary character on $N \cap \Gamma$, has the original terminal residue characters and is trivial on $N \cap K_v$ at every good unmarked place. At each new split place there are compactly supported local row and column tests at the original lattice levels with nonzero ordinary local integral. For the actual global signed rows and the separately zero-sum columns the exchange is*

$$\mathcal{J}_\phi^{\chi_*}(\widetilde{R}(g_T)A) = C_T \prod_{v \mid g_1 g_2} \prod_{i=1}^{s} \mu_{i,v}(\pi_v O_v^a)\, \mathcal{B}^{\chi}_{D_T^{s_i}\phi}(R(d_*)A).$$

Proof. The prescribed auxiliary leading residues make the conductor pairing on each integral string entry equal to the signed character. At T the ratios are one modulo T^b, so their residual Fourier factors on $\pi_v^{-1}O_v/O_v$ are the original ones. At a new split place apply Lemma 64.39 to the integral ratios. At other good unmarked places the diagonal and its inverse preserve the lattice and normalize N, so conjugation preserves the original trivial restriction to $N \cap K_v$.

For the local assertion take the row $\mathbf{1}_{K_v}$ on $G(F_v)$ and the column $\mathbf{1}_{H(F_v)\cap K_v}$ on $H(F_v)$. On the column support, $nh \in K_v$ exactly when $n \in N \cap K_v$. The character there is trivial. The integral is the product of the positive volumes of these two compact open subgroups in H and N. For the global identity repeat the auxiliary Fourier inversions and terminal steps of Theorem 64.38. They use the auxiliary and terminal characters just checked. Rational conjugation on the ordinary period moves d_* to the row without a Haar factor. These identities hold for the specified global automorphic vectors and require no archimedean proximity to the identity. □

With these integral ratios the lower bound in Theorem 64.40 applies with $D_j^- = 0$ for every j. Thus the required archimedean displacement is present on each simple root. The local tests just constructed establish compatibility of the character with the level. The signed global theta coefficient at this displaced archimedean argument, its arithmetic normalization and its size on the original fields remain the period estimates required in Section 6.6.

65 Discrete series and real Bessel periods

65.1 The size of the global integral on fixed compact types

Write $F_+ = E^+$ and use the signed row group and the two levels of Corollary 72.10. Fix a constant $C_\lambda > 0$. In the interlacing lists of Theorem 72.8 suppose also that the absolute values of the row parameters are at most $\exp(C_\lambda x^\varepsilon)$. This includes its explicit lists for large x, since k, m, t_{r} are bounded by fixed powers of x. Take the actual prime T of Corollary 72.10 with the stronger generator congruence of Theorem 72.11.

Lemma 65.1 (Bounds for the field, level and terminal prime)**.** *There is C_1, depending only on the fixed family and parameter bounds, such that, throughout the common intervals at large x,*

$$(16-o(1))mx^{\varepsilon}\le \log T\le x^{C_1},\qquad \log B+\log I_\sigma\le x^{C_1}.$$

Here $B=(8n_0)^{2n_0}|D_E|^{r/2}$, $n_0=2fr$. The level index still has logarithm at most x^{C_1}, after increasing C_1, when the principal subgroup of the enlarged lattice in Theorem 99.22 is imposed at every place above T.

Proof. The degrees, k,m,r, and the logarithm of the cyclotomic conductor bound N are bounded by fixed powers of x, by Theorems 6.29 and 7.11. The conductor–discriminant formula gives $\log|D_E|\le 2f\log N$. The two auxiliary levels have $I_\sigma\le (g_1g_2)^{fr^2}$, by counting all residue matrices. These estimates bound $\log B$ and $\log I_\sigma$.

For the upper bound on the terminal scale use the formula in Corollary 72.10. The compact mass $\mathfrak{m}_r$ is at most the number of its genus classes. The full-level counting argument of Lemma 99.17 gives $\log^+\mathfrak{m}_r\le C(2fr)^4\log(4frN)$, where $\log^+ y=\max(0,\log y)$. In the denominator $d_{u,*}\ge \mathfrak{m}_k\ge (k!w_E^k)^{-1}$, with $w_E=|\mu(E)|$. Indeed the identity lattice has that automorphism order. Each vector of norm one has a single nonzero coordinate which is a root of unity, by the conjugate absolute-value argument in Lemma 72.7. The columns are orthogonal. Also $\varphi(w_E)\le 2f$, so $w_E\le 2(2f)^2$ by the elementary totient bound used in Theorem 72.8. Each Weyl dimension in the numerator is at most $(2+2e^{C_\lambda x^{\varepsilon}})^{r(r-1)/2}$, by its product formula. Finally $\Gamma\ge 1$. The logarithm of the upper end of the prime search interval is therefore at most a fixed power of x. Its lower bound is Corollary 72.10.

At a place above T, the principal subgroup of the enlarged lattice contains the original principal subgroup modulo T^2. Indeed the enlargement has elementary divisors $(T^{-1},1,\dots,1)$ in the chosen split factor. Conjugation back to the old coordinates can decrease the valuation of any matrix entry by at most one. The dual factor is determined by the first. The extra index is at most T^{2fr^2} by counting matrices modulo T^2. This proves the last assertion. □

At infinity use a K_∞-finite smooth square-integrable row form in discrete-series representations of these infinitesimal characters. At each place take the compact type of highest weight $\mu_\tau=\lambda_\tau+\rho_{\mathrm{n}}-\rho_{\mathrm{c}}$, in the chamber of its parameter. Here ρ_{n} and ρ_{c} are the half sums of the positive noncompact and compact roots. This specifies the type used below, also described in `Formalism/PeriodIntegrals/UpqRep.lean`, line 231. More generally the next bound holds for any compact types and Casimir scalars with the bound on L in its proof. Finite right translations in the projector of Theorem 99.22 preserve the archimedean type.

Theorem 65.2 (Exponential decay in the terminal frequency)**.** *For some absolute $c>0$ and all sufficiently large x in the common intervals, every such row form F of the original level or the smaller level in Lemma 65.1 and every smooth column form ϕ satisfy*

$$|\mathcal{P}_{\varpi}(F,\phi)|\le \exp(-c\sqrt{T}/r)\,\|F\|_{L^2([G])}\,\|\phi\|_{L^2([H])}.\tag{65.1}$$

The normalization is the analytic integral of the row and column forms, with the character of last vector ϖe.

Proof. We first verify the derivative parameter L of Lemma 99.25. Each coordinate of μ_τ has magnitude at most $r+\max_i|\lambda_{\tau,i}|$, since there are at most $r-1$ roots through a coordinate in the sums of positive roots. In either compact unitary factor every weight coordinate lies between the least and greatest highest weight entries of that factor. Indeed weights are dominated by the highest weight and permutation of entries preserves the weight set. Placing the largest

entry first in the dominance inequalities gives the upper bound. Placing the least entry last and using equality of total weights gives the lower bound. A compact Lie generator of Frobenius norm one diagonalizes by that compact group. Its eigenvalues on the type are scalar products of its diagonal angles with the weights, times i, so their magnitudes are at most $\sqrt{r}\max_i|\mu_{\tau,i}|$.

For the trace form the complex Casimir is $\sum_{i,j}E_{ij}E_{ji}$. On a highest weight vector of weight ν the diagonal terms sum to $\sum_i\nu_i^2$. For $i<j$ the two terms of that pair sum to $\nu_i-\nu_j$, by commuting the raising generator past the lowering generator. The infinitesimal character λ_τ has the same central action as the compact representation with highest weight $\nu=\lambda_\tau-\rho$, after arranging the entries in decreasing order. Here $\rho_i=(r+1-2i)/2$. The resulting scalar is $\omega_\tau=\sum_i(\lambda_{\tau,i}^2-\rho_i^2)$. Thus both the compact Lie norm and $\sqrt{|\omega_\tau|}$ are bounded by $L=\exp(C_2x^{\varepsilon})$ for some fixed C_2.

Apply Theorem 99.26. Its derivative orders can be chosen freely. Take $A=\lfloor\delta\sqrt{T}/r\rfloor$ at one real place, with an absolute $\delta>0$ small enough in terms of the absolute constant in that theorem. Since $r(L+fr^2+1)/\sqrt{T}\to0$ by Lemma 65.1, the ratio $Cr(L+A+fr^2+1)/(4\pi\sqrt{T})$ is at most $1/2$ for all sufficiently large x. Its A-th power accounts for the factor 2^{-A}. The logarithm of the remaining multiplier is at most

$$\tfrac{1}{2}\log I+Cfr^2\log(2fr^2B)+fr^2\log\left(Cr(L+A+fr^2+1)\right).$$

It is bounded by a fixed power of x, by Lemma 65.1. On the other hand $\sqrt{T}/r$ grows faster than every fixed power of x. Absorbing this multiplier into 2^{-A} proves (65.1) with an absolute smaller c. □

Corollary 65.3 (A necessary arithmetic normalization)**.** *Let $p\neq0$ be an algebraic period in a number field J whose values in its complex embeddings take the form*

$$\sigma(p)=c_\sigma\mathcal{P}_{\varpi_\sigma}(F_\sigma,\phi_\sigma).$$

Suppose the inputs in every embedding satisfy Theorem 65.2 with the same parameter bounds, and ϖ_σ is the corresponding conjugate generator. Write $\mathfrak{d}(p)=\{a\in\mathcal{O}_J:ap\in\mathcal{O}_J\}$. Then

$$\frac{\log\mathrm{N}\mathfrak{d}(p)}{[J:\mathbb{Q}]}+\frac{1}{[J:\mathbb{Q}]}\sum_{\sigma:J\hookrightarrow\mathbb{C}}\log\left(|c_\sigma|\,\|F_\sigma\|_2\,\|\phi_\sigma\|_2\right)\ \geq\ c\sqrt{T}/r. \tag{65.2}$$

In particular for integral p and unit inputs the geometric mean of $|c_\sigma|$ is at least $\exp(c\sqrt{T}/r)$.

Proof. The fractional ideal of p is its integral numerator divided by $\mathfrak{d}(p)$, and the numerator has norm at least one. Thus $\sum_\sigma\log|\sigma(p)|\geq-\log\mathrm{N}\mathfrak{d}(p)$. Use (65.1) in each embedding and rearrange. □

For the high orders of Proposition 71.16, one has $ha\log T=o(\sqrt{T}/r)$. Indeed h,r and $\log T$ are bounded by powers of x, $a=\lceil x\exp(x^{\varepsilon})\rceil$, and $\log T\geq(16-o(1))mx^{\varepsilon}$, with $m\to\infty$. The denominator T^f of the finite projector likewise has logarithm $o(\sqrt{T}/r)$. The height sought in (6.3) concerns an algebraic map between the prescribed lattices. The source separately specifies a nonzero scalar and isometries for the comparison of the two pairings in `Formalism/SharpRestriction/GlobalTheoremInputs.lean`, line 65. Equation (65.2) quantifies what this scalar, any change in the test vectors, and any denominators must account for if the raw fixed-type integrals represent a nonzero algebraic period. A bound on that algebraic map and the uniform bound for $\mathcal{R}_{**}(x)$ in Proposition 7.7 remain open.

65.2 A common real Bessel model for all translated columns

The Bessel subgroup at a real place is $S = H_\tau \ltimes N_\tau$, with $H_\tau = U(k)$ acting on W_0. Write π_u for the compact column representation and σ_λ for a discrete series in a row packet with ordered Harish-Chandra parameter λ. Since the period conjugates the column, its local multiplicity is

$$m_\beta(\sigma_\lambda, u) = \dim \mathrm{Hom}_S(\sigma_\lambda^\infty \otimes \pi_u^\vee, \chi_\beta).$$

Here Hom means continuous equivariant linear forms and the superscript denotes the smooth representation. The column is finite dimensional. Existence of one such form is a necessary condition on the entire real representation in a nonzero global period on a factorizable pair. Fix the data at the other places of such a pair. The resulting local form is equivariant by change of variables in the period and continuous by integration of smooth automorphic vectors over its compact domain.

We use Xue's real local Gan–Gross–Prasad theorem [3, Theorem 1.1]. We describe the choices used in applying it. Replace the Hermitian form by its negative at this place. The unitary matrix groups and the flag subgroup stay the same. In this convention W_0 is negative definite and the distinguished line $\mathbb{C}e$ is positive, as required in Xue's definition for even k. Pair the isotropic basis vectors with their negatives on one side to restore pairing one. The character used in the local theorem has a nonzero coefficient on each of the same simple coordinates $d_1, \dots, d_s$. Indeed the terms of Xue's trace pair a dual flag vector with the image of the next flag vector, and at the last place in the flag with the image of e. These are exactly the defining coordinates in Section 99.16. A diagonal isometry scaling u_j by $c_j \in \mathbb{C}^\times$ and v_j by $\overline{c}_j^{-1}$, and fixing the middle space, conjugates the coefficients by the respective factors c_j/c_{j+1} and c_s, or their inverses in the reverse conjugation. Starting at s realizes arbitrary nonzero coefficients. This conjugation fixes H_τ pointwise. The local multiplicities consequently agree for every $\beta \neq 0$, including the prescribed terminal generator at the given real place.

We recall the two sign computations used below. In a regular discrete-series packet the ordered entries label the generators of its component group, one of order two for each entry. The restriction of a packet character to each generator is a sign. Changing the sign at a given entry interchanges its positive and negative positions in the chamber. More precisely, for a fixed rank and Whittaker normalization the signature sign at position j is its packet sign times $(-1)^{j-1}$ and a fixed sign. This is the explicit real parametrization in [4, Appendix A]. The calibrations with compact groups below use only ratios, so that fixed sign cancels.

For the distinguished pair, Xue computes the label at a row entry as the product of its epsilon factors against all column entries. The label at a column entry is the product against all row entries. Here the row exponents are the integers λ_j and the dual column exponents are $-u_i$, which are half-integers outside $\mathbb{Z}$. Thus a factor is evaluated at the angular character with exponent $\lambda_j - u_i$, using exponents for $(z/\bar{z})^t = (z/|z|)^{2t}$. For a half-integral exponent outside $\mathbb{Z}$, the real-place epsilon factor is one fixed sign for $t > 0$ and its opposite for $t < 0$. This is [5, Proposition 2.1 and Corollary 2.2]. Replacing the additive character by its inverse, or using the conjugate convention for all exponents, multiplies every such factor by one common sign. In products of the same parity its contribution cancels when the products are divided. The dual column entries are indexed below by their original u_i, even though their decreasing order as dual weights is reversed.

Proposition 65.4 (Crossing a row parameter)**.** *Keep the row packet fixed and increase just the largest column parameter from u_1^0 to u_1^a. At each row entry in the open interval (u_1^0, u_1^a) the distinguished row packet label changes sign. Elsewhere it is unchanged. If at least one entry is crossed, a single irreducible row representation cannot have a local Bessel model for both columns.*

Proof. All factors at a row generator cancel on division except the one belonging to the first column entry. The sign of $\lambda_j - u_1$ changes precisely on the indicated interval. The endpoints are disjoint from the integral row entries. Xue's necessary condition then assigns two different characters to the same row packet. A fixed irreducible member has just one character. In particular Theorem 9.11 crosses both v_1+1 and v_1+a, since $u_1^0 = v_1+1/2$ and $u_1^a = v_1+a+1/2$. Its compact pairings across all translates therefore cannot come from one irreducible real Bessel model for both columns at a place where the assignment varies. □

Theorem 65.5 (One real discrete series for both Bessel models)**.** *On the same fields and intervals, retain all column weights and the shared v_i. Use the decreasing row parameter*

$$\lambda = (v_1 + a + 2,\ v_1 + a + 1,\ v_1, \dots, v_{k-1},\ -1,\ -(t_{\mathrm{r}} + 4), \dots, -(m-3)(t_{\mathrm{r}} + 4)). \tag{65.3}$$

In the original Hermitian form take the first slot negative, the slots from the second through -1 positive, and make the remaining signs alternate starting negative. For the discrete series in this chamber and each $\beta \neq 0$, one has

$$m_\beta(\sigma_\lambda, u^0) = m_\beta(\sigma_\lambda, u^a) = 1.$$

Its lowest compact type also contains the two integral isometric column embeddings of Lemma 9.10.

Proof. We determine the two packet characters separately, including which definite space occurs. For either column the two leading row entries lie strictly above u_1, each gap (u_{i+1}, u_i) contains just the common integer v_i, and exactly $2s$ entries lie below $u_k = -1/2$. In particular their relative order is identical for the two columns. The ranks satisfy $r = k + 2s + 1$.

First keep this column, including its dual in the local Hom, and take a comparison row of rank $k+1$ on a positive definite space that extends a positive definite k-space. Take one integer above u_1, the same v_i in the gaps, and one integer below u_k. The comparison has local multiplicity one. Indeed for adjacent ranks the unipotent radical in the Bessel model is trivial. Subtracting the half-sums of roots converts strict interlacing of these integral and half-integral parameters into the two weak highest weight inequalities of ordinary compact branching. The compact restriction has multiplicity one. Both compact spaces are positive and the line added to the even-dimensional column is positive, so this is a relevant pair in Xue's convention.

At column generator i the comparison row has i entries above u_i, whereas (65.3) has $i+1$. Its rank has increased by $2s$, an even number. The ratio of the epsilon products is therefore -1 at every column generator, whether one counts positive or negative differences. Relative to the positive definite comparison column this reverses every signature sign. Thus the required column lives on the negative definite space for the sign-reversed form, as does exactly our fixed $\pi_u^\vee$.

For the row calibration keep the rank r and λ in (65.3). Compare with the positive definite compact pair of ranks $r, r-1$, taking column entries $\lambda_i - 1/2$ for $1 \le i < r$. Strict integral spacing makes these interlace, so compact branching again gives multiplicity one. For the row generator at slot j, the comparison column has $j-1$ entries above λ_j. Let b_j be the corresponding number in the actual column. Since the two column ranks differ by $2s$, the ratio at this generator is $(-1)^{b_j-(j-1)}$. In order along the row,

$$b_1 = b_2 = 0, \qquad b_{i+2} = i\ (1 \le i < k), \qquad b_j = k \quad (j \ge k+2).$$

The comparison row was everywhere positive. For the sign-reversed form the ratio is therefore its required signature sign. It is positive in the first slot, negative from the second slot through

-1, and then alternates starting positive. Returning to the original form gives the chamber in the statement. Its positive and negative counts are $k+s, s+1$, respectively. Both packet characters now satisfy Xue's sufficiency condition, proving the two equalities.

In this chamber $j_i = i+1$ for the first $k+1$ positive slots. The first such slot has parameter v_1+a+1, and hence compact highest weight $v_1+a+1-k/2 = \nu_1^a$. The next k positive slots give $\nu_1^0, \dots, \nu_k^0$ by the calculation of Theorem 9.11. Its two coordinate embeddings, with the same index sets I_0, I_a, apply to this lowest compact type. □

Corollary 65.6 (A common terminal prime with every real model)**.** *For a common admissible Γ, the row (65.3) gives a single prime at its own formal-degree scale such that the finite congruences and the analytic estimate of Theorem 65.2 hold, and every translated column admits a nonzero continuous Bessel functional on the real row representations. The same prime and row work for all placements of the prescribed number of high weights.*

Proof. The common integers v_i still interlace both column lists. The largest absolute parameter is at most $a+2k+2+m(t_{\mathrm{r}}+4)$. The products of row formal degrees and of column dimensions are independent of the placement. The prime-window and doubled-modulus argument in the proof of Theorem 9.11 thus applies with the new row. At each real place take a nonzero functional given by Theorem 65.5. Their tensor product at this finite set of places is nonzero, as seen by evaluating at a tensor of vectors with nonzero individual values. Each functional takes a nonzero value on some compact-finite smooth vectors, since the compact-finite subspace is dense in the smooth representation by the Fourier expansion for the maximal compact group. □

Local multiplicity one determines a line of continuous functionals. Two further facts would be needed to estimate the global integral on the prescribed lattices. First, its restriction to the specified vectors, including the lowest compact vectors and the finite test data, needs a nonzero global value. Second, the scalar relating that analytic value to an algebraic coefficient must satisfy the arithmetic comparison of Corollary 65.3. The compact pairing of value one in Lemma 9.10 is a pairing for H_τ. The local Bessel condition also prescribes the action of N_τ.

65.3 The prescribed row and the interior column weights

The row in (65.3) establishes existence when one can choose its entries. The source fixes another row in `Formalism/SharpRestriction/CuspSpanNormBounds/RelevantZeroGramTerms.lean`, line 165. We now use that row itself. At each real place its decreasing Harish-Chandra parameter and its chamber are

$$\lambda = (v_1, \dots, v_{k-1}, -b_1, \dots, -b_m), \qquad (+, \dots, +\,;\, +, -, +, -, \dots, +, -). \tag{65.4}$$

There are $k-1$ plus signs before the semicolon and $m = 2s+2$ alternating signs after it. The v_i, b_j are integral. The column entries are half-integers outside $\mathbb{Z}$, and

$$u_1 > v_1 > u_2 > \cdots > v_{k-1} > u_k > -b_1 > -b_2 > \cdots > -b_m.$$

These inequalities follow directly from the interlacing and the positive gaps imposed in `Formalism/LocalRingCardinality/Params.lean`, line 56. All words at a given place share v, b, e, u_1, u_k by `Formalism/LocalRingCardinality/RhoDetPrimeEqMuTildeMulChiPow.lean`, line 96. Thus the interior entries distinguish words, while u_1 may also depend on which real place is under consideration.

Theorem 65.7 (The real Bessel model on the prescribed row)**.** *For (65.4) and every column satisfying the displayed interlacing, $m_\beta(\sigma_\lambda, u) = 1$ for every $\beta \in \mathbb{C}^\times$. On the lowest compact*

row type, occurrence of this column on restriction to H_τ is equivalent to the vanishing of every entry in the nonnegative integral vector

$$d_i = u_i - v_i - \tfrac{1}{2} \quad (1 \le i < k), \qquad d_k = u_k + b_1 - \tfrac{1}{2}.$$

If they all vanish, that compact type has the integral column embedding of Lemma 9.10. In general any derivative of the lowest type with a nonzero period against this column must have order at least $\sum_i d_i$ at this place.

Proof. Apply the two compact comparisons used in Theorem 65.5, with the same sign-reversed form in the local multiplicity theorem. The rank-$(k+1)$ positive compact comparison has i row entries above u_i, whereas (65.4) has $i-1$. The ranks differ by $2s$, so the distinguished label at every column generator changes sign. This gives exactly the negative definite column in the comparison convention. For the row calibration the rank-$(r-1)$ compact comparison again has $j-1$ column entries above λ_j. There are actually j such entries for $j < k$ and k for $j \ge k$. The ratio of the required packet sign to the positive compact one is $(-1)^{\min(j,k)-(j-1)}$. Upon returning to the original Hermitian form, its negative gives plus in each of the first k slots and alternating signs starting minus thereafter. This is precisely the given discrete series. Xue's theorem therefore gives the asserted multiplicity for each character.

The first k positive positions are $j_i = i$. Consequently the first k entries of the positive compact highest weight are $A_i = \lambda_i - k/2 + i$ by equation 64.15. For the compact column $\nu_i = u_i - (k+1)/2 + i$, this says $\nu_i - A_i = d_i$ for $1 \le i \le k$. Interlacing and half-integrality imply $d_i \ge 0$. Compact branching requires $\nu_i \le A_i$ at each index. If all these entries are equal, Lemma 9.10 with $I = \{1, \dots, k\}$ gives an integral embedding. Finally Proposition 64.28, shows in its proof that after j derivatives each resulting compact highest weight A' has $\sum_i \max(A'_i - A_i, 0) \le j$. Since a contributing type requires $A'_i \ge \nu_i$ for $1 \le i \le k$, the sum of the d_i is a lower bound. □

Corollary 65.8 (Vanishing for the coded family)**.** *Suppose $v_{i-1} - v_i = 4\Delta$ and*

$$u_i = v_{i-1} - \tfrac{1}{2} - \Delta c_i, \qquad c_i \in \{0, 1\}, \quad 2 \le i \le k-1,$$

with $\Delta > 0$ common to the family and the map from words to bit strings injective. These are the conditions in `Development/AbelianVarietyTheory/Definitions/AutomorphicMembership/ExistsSStageUnconditional.lean`, line 16. If there are at least two words, every period on the lowest compact row type vanishes for every word. Any differential word giving a nonzero global value has total order at least

$$\sum_\tau (d_{\tau,1} + d_{\tau,k}) + f(k-2)(3\Delta - 1).$$

Proof. At least one bit is one somewhere. At that entry u_i is in $\mathbb{Z} + 1/2$ and $v_{i-1} \in \mathbb{Z}$, so $\Delta \in \mathbb{Z}$ and $\Delta \ge 1$. At every place and for every word, the interior deficits are $d_i = 4\Delta - 1 - \Delta c_i$. All of them are at least $3\Delta - 1 \ge 2$. Since the ranks in Theorem 6.29 have $k > 2$, the asserted vanishing follows from Theorem 65.7. A mixed differential word of nonzero period must supply the separate orders at all the places. Adding the endpoint and interior contributions proves the bound. □

Proposition 65.9 (One small real translate for all local models)**.** *Fix the places and words above and a terminal generator. At each place choose a nonzero lowest compact row vector w_τ, a nonzero column vector for each word, and nonzero local Bessel functionals. In every neighborhood of the identity in G_∞ one translate of $\bigotimes_\tau w_\tau$ has nonzero values for all of these*

products of local functionals on the chosen columns. This is possible with both matrix operator norms $\|g_\tau\|, \|g_\tau^{-1}\| \le 2$ at every place. For any automorphic row form on the original lowest type satisfying the parameter hypotheses of Theorem 65.2, that theorem's exponential upper bound also holds after any such bounded real translate, with an absolute change in its constant.

Proof. Fix one local functional ℓ and one nonzero dual column vector z. The subspace of column vectors annihilating the entire smooth row is H_τ-stable. By irreducibility of the column and nonvanishing of ℓ, that subspace is zero. Thus $v \mapsto \ell(v, z)$ is nonzero. The compact-finite Harish-Chandra module of a discrete series is irreducible and dense in its space of smooth vectors. A nonzero lowest type vector w_τ generates this module under the complex enveloping algebra. Indeed its generated submodule contains its compact span, since the compact group is connected, and conjugation on the Lie algebra then makes the generated submodule compact-stable. Irreducibility gives the claim. Some iterated Lie derivative on w_τ therefore has nonzero evaluation at z.

The function $g \mapsto \ell(\sigma_\lambda(g)w_\tau, z)$ is real analytic near the identity. To see this in the smooth topology, use ordered exponential coordinates. The derivative bound of Lemma 99.25, bounds each word of length $n + t$ on w_τ by $\{Cr(L + n + t + 1)\}^{n+t}\|w_\tau\|$. For fixed L, t, dividing by $n!$ leaves at most an exponential in n, with the base bounded by a fixed multiple of r in the limit of n-th roots (use $n! \ge (n/e)^n$). At total degree n the multinomial formula trades the product of the individual factorials in the Taylor series for $n!$ and the n-th power of the sum of absolute coordinate sizes. The series thus converges for a sufficiently small such sum in every Hilbert derivative seminorm of fixed order t. These Taylor series represent the group translates: along each one-parameter group this follows from Taylor's formula with integral remainder and unitarity, and one successively applies this identity in the ordered coordinates using absolute convergence. The continuous functional may be applied to the series. A nonzero iterated derivative at the identity precludes identical vanishing there. The nonzero set of each such coefficient is consequently dense and open in a connected coordinate ball. Intersecting the finitely many sets at each place gives the common translate in an arbitrarily small product neighborhood.

For the asserted bound write $R(g)$ for right translation on automorphic forms. Move each real derivative past $R(g)$ by acting on its Lie algebra argument by $\mathrm{Ad}(g^{-1})$. Its Frobenius norm increases by a factor at most four. By multilinearity a word of length j in the proof of Theorem 99.26, costs at most 4^j in addition to the old estimate. Right translation is unitary and leaves the finite level fixed. In the proof of Theorem 65.2 the added factor is absorbed by reducing the fixed coefficient of $\sqrt{T}/r$ in the chosen differentiation order. The factor from the at most fr^2 further restriction derivatives has logarithm $O(fr^2)$, which is absorbed as in that proof. □

Here the nonzero values are evaluations of the individual real Bessel functionals on smooth vectors. A global theta lift still requires its specified finite tests and its arithmetic normalization. The preceding construction gives no arithmetic lattice for a general small real translate.

65.4 Full periodization below the auxiliary height bound

On the common fields and intervals put $Q_0 = g_1 g_2$. Use the isotropic height H of Section 64.3. It is the geometric mean over the real places of the largest Gram–Schmidt length in the ordered list $g_\tau^{-1}u_i$. In particular it is a function on $B_\infty\backslash G_\infty$. Write $\mathcal{U} = \{H < \sqrt{Q_0}\}$. This open set contains the real maximal compact group in the signed coordinates. Indeed on that group the u_i remain Euclidean orthogonal of length $\sqrt{2}$ at every place. More generally it contains every

point with $\prod_\tau \|g_\tau^{-1}\|_{\mathrm{op}}^{1/f} < \sqrt{Q_0/2}$, by using the original unmodified isotropic vectors in the length minimization.

Theorem 65.10 (The full automorphic norm on the height range)**.** *Use K_σ with either or both terminal frequencies, or use the refined column level realized by K^0 with frequency ϖ as in Theorem 64.1. Suppose the real functions in the period seeds vanish almost everywhere outside $\mathcal{U}$. On the family of Section 6.6, for all sufficiently large x, summation over the full rational unitary group defines an isometry in the natural seed norms,*

$$\left\| \sum_{\delta \in B(F_+)\backslash G(F_+)} \Phi(\delta g) \right\|^2_{L^2(G(F_+)\backslash G(\mathbb{A}_{F_+}))} = \|\Phi\|^2_{L^2(B(F_+)\backslash G(\mathbb{A}_{F_+}))}. \tag{65.5}$$

This holds for finite linear combinations of the selected types and columns, including the refined zero-sum columns. For smooth real sections compactly supported modulo B_∞ the sum is ordinary. For L^2 sections supported in $\mathcal{U}$ it is the isometric extension of these sums. Thus different selected words give orthogonal automorphic functions in this construction. For a given word and frequency with real section W of norm one, its squared norm in class coordinates is

$$\frac{\mathrm{vol}_f(K)\, v_N\ \mathrm{vol}(H_\infty)}{\mathrm{vol}_f(N_f \cap K)\, d(\rho_c)} \sum_j \|\ell_j\|^2.$$

Proof. First take smooth real sections of compact support inside $\mathcal{U}$. Their scalar norms, with the original finite support, descend to compactly supported envelopes modulo $B(F_+)$. The compact covering argument in Theorem 88.1 shows that periodizations are locally finite and that the product of two periodized envelopes is integrable. We may therefore expand the mixed Gram pairing, unfold its first rational index and then group the remaining terms, with absolute integrability. Two finite cosets can meet only if the relative rational element belongs at finite places to $B_f K B_f$.

Group together the pairs with the same underlying isotropic s-space. Their total integral is the Gram pairing of the corresponding sums over the stabilizer of that space. Indeed partition the index set of each sum into fibres over $Q(F_+)\backslash G(F_+)$, with Q the stabilizer of L. The terms under consideration are exactly the products with the same fibre index. Unfolding that index integrates the two intermediate sums over $Q(F_+)\backslash G(\mathbb{A}_{F_+})$. Theorem 88.1 evaluates this as the input Gram pairing, also at K^0 by Theorem 64.2.

Consider a pair of different underlying isotropic spaces. For each finite cell the two unipotent characters either disagree or agree on their common real subgroup. In the first case the orbital integral is zero by Lemma 76.1. In the second case Theorem 64.9, gives $H(g)H(\gamma^{-1}g) \geq Q_0$ for every real argument g. The ranks tend to infinity on these intervals, so $s \geq 2$ as required there. If the two sections are both nonzero at this argument, both heights are strictly below $\sqrt{Q_0}$, contradicting that inequality. Thus the integrand vanishes in every such cell. This argument treats a word paired with itself as well as distinct words. Only the already evaluated same-subspace contribution survives, proving (65.5).

Smooth compact sections inside $\mathcal{U}$ are dense in the space of square-integrable sections supported there. In local trivializations first truncate to compact subsets of the open set and then approximate by smooth functions using a partition of unity. The associated bundles have finite-dimensional fibres and the quotient Haar measure is a smooth density, so this procedure gives norm convergence. The finite class description in Section 70.8, carries the approximations to the seed norm. The identity for differences shows that their automorphic images are Cauchy. Their limits in the complete automorphic L^2 space are independent of the approximations and give the asserted isometry on the closure. Polarization gives the mixed identity. Finally, finite

class and compact column integration in Theorem 70.24, give the displayed constant and the input orthogonality of distinct types. □

Corollary 65.11 (The truncated part of a specified real section)**.** *For a norm-one real section W of a selected word and frequency, set*

$$\eta(W) = \int_{\{H \geq \sqrt{Q_0}\}} \|W(g)\|^2 \, d\dot{g} \quad (d\dot{g} \text{ on } B_\infty \backslash G_\infty).$$

Let $\Phi^<$ be its seed with real section $\mathbf{1}_{\mathcal{U}} W$. For any column (ℓ_j), the squared automorphic norm of the extension in (65.5) for $\Phi^<$ equals $(1 - \eta(W)) c_{\rho,K} \sum_j \|\ell_j\|^2$. At K^0 the squared class-coordinate constant is the same as at K_σ. Their squared constants for the column probability norms have ratio $(T^k - 1)^f$.

Proof. The cutoff is left B_∞-invariant, so it preserves the transformation law and scales the squared real norm to $1 - \eta(W)$. Apply Theorem 65.10 and Theorem 64.2. The zero-section case gives zero on both sides. □

The construction evaluates a full automorphic pairing for a specified height truncation. For the originally prescribed theta vectors one still needs a bound for the contribution from $H \geq \sqrt{Q_0}$ after periodization, including its pairing with the truncated part. A cutoff can also change the real representation generated by a section. The integral theta comparison requires the prescribed projection and the arithmetic normalization in addition to the complex norm.

65.5 The prescribed compact row type and real translations

Fix a row in (65.4) at one of the choices in Theorem 9.13, and let $K_\infty^{\max}$ be its real maximal compact group in the signed coordinates. Write σ for the product of the prescribed real discrete series, U for the tensor product of their lowest compact types, and $\mathcal{C}_c$ for the space of finite class covectors of type ρ_c, with its ordinary sum inner product. We use the refined norm-T level K^0.

Theorem 65.12 (Automorphic copies of the prescribed compact type)**.** *There is a nonnegative smooth cutoff θ on $B_\infty \backslash G_\infty$, right invariant under $K_\infty^{\max}$ and compactly supported in $\{H < 2\}$, with the following property. Using the real Bessel models of the prescribed discrete series, the corresponding cutoff sums give a $K_\infty^{\max}$-equivariant injection*

$$\bigoplus_{c \in \mathcal{A}} U \otimes \mathcal{C}_c \;\longrightarrow\; L^2([G])^{K^0}.$$

The images for distinct words are orthogonal. On the summand for c the map is an isometry after a single positive scalar normalization, also on its refined zero-sum columns.

Proof. For each c, the product of the nonzero local Bessel functionals of Theorem 65.7 gives a smooth column-valued function

$$W_{c,v}(g) = \Lambda_c(\sigma(g)v), \qquad v \in U.$$

Here the functional is viewed as a linear map on smooth row vectors with values in V_{ρ_c}, by duality with the finite-dimensional dual column. It transforms under the left unipotent and column groups by the factors in equation 70.15 with terminal vector ϖe. Right translation by a member of $K_\infty^{\max}$ replaces v by its compact translate. At the identity $H = \sqrt{2}$, and H is unchanged by right translation by the maximal compact group since Euclidean norms are unchanged. Choose a relatively compact neighborhood of the identity coset and its compact

orbit inside $\{H < 2\}$. Take a nonnegative smooth bump supported in this open set, equal to one near the orbit, and average it over the compact group. The result is again compactly supported there and equals one near the orbit.

For every c choose a nonzero pure tensor $v \in U$. Proposition 65.9 gives points arbitrarily close to the identity at which $W_{c,v}$ is nonzero. Continuity gives positive integral for $\|\theta W_{c,v}\|^2$. The Hermitian form on U obtained by integrating these cutoff sections is invariant under the compact group, by right invariance of the cutoff and quotient measure. Irreducibility of U and Schur's lemma make it a_c times the given row inner product for some $a_c > 0$.

Form the period seed from $\theta W_{c,v}$ and any class covector tuple. The map is linear in both arguments and extends to their tensor product. Formula (65.5), polarized on such tensors, gives $a_c c_{\rho_c,K^0}$ times the tensor product inner product. To see the two factors directly, first average the pointwise rank-one operator over the left compact column group as in Theorem 70.24. This gives the inner product of the two real sections divided by $d(\rho_c)$ times the identity on the column representation. Finite class integration then pairs the class tuples. Distinct words have zero inner product by Theorem 65.10. Multiplication on each summand by $(a_c c_{\rho_c,K^0})^{-1/2}$ proves the assertion. The construction is right equivariant for the maximal compact group since the cutoff is invariant and ordinary periodization commutes with right translations. □

For a real group element $a = (a_\tau)$ put

$$\ell(a) = \frac{1}{f} \sum_{\tau} \log \max\{\|a_\tau\|_{\mathrm{op}}, \|a_\tau^{-1}\|_{\mathrm{op}}\}.$$

It is nonnegative, symmetric in a, a^{-1}, and subadditive under multiplication by the matrix norm inequality. Write R_B, R_G for right translations on the seed and automorphic quotients, and P for the isometry of Theorem 65.10.

Theorem 65.13 (Agreement of translated matrix coefficients)**.** *Let Φ, Ψ be seeds as in Theorem 65.10, supported at heights $H < R, H < S$, respectively, with $R, S \le \sqrt{Q_0}$ and $RS < Q_0$. Put $h = \log(Q_0/(RS))$. For every a with $\ell(a) < h$,*

$$\langle R_G(a) P\Phi, P\Psi\rangle = \langle R_B(a)\Phi, \Psi\rangle.$$

For an arbitrary finite complex Borel measure μ on G_∞, let $R(\mu) = \int R(a)\, d\mu(a)$. The difference of these two pairings with a replaced by μ has absolute value at most

$$2\,|\mu|\{\ell \ge h\}\,\|\Phi\|\,\|\Psi\|. \tag{65.6}$$

Proof. Right translation changes the height by bounded factors,

$$e^{-\ell(a)} H(g) \le H(ga) \le e^{\ell(a)} H(g).$$

Indeed apply a_τ^{-1} to a set of isotropic vectors minimizing the maximal length at g_τ. Take the geometric mean and use the same upper bound with a^{-1} for the lower bound. The support of $R_B(a)\Phi$ is therefore in $H < e^{\ell(a)} R$. First take compact smooth input sections. Their periodizations are ordinary and commute with right translation. The proof of Theorem 65.10 for a mixed pair only uses that the product of the two supporting height bounds is less than or equal to Q_0. It applies to $R_B(a)\Phi, \Psi$, since $e^{\ell(a)} RS < Q_0$. Their same-subspace pairing again unfolds to the seed pairing. This proves equality even if the first translated height bound exceeds $\sqrt{Q_0}$.

For general seeds approximate each by compactly supported sections inside its original height bound. Unitarity of the two right regular representations and isometry of the original periodizations allow passage to the limit on both sides. For every a the absolute value of each

pairing is at most $\|\Phi\|\|\Psi\|$ by Cauchy–Schwarz. Integrating their difference against μ, the part where $\ell(a) < h$ is zero. Integration against total variation on the complement gives (65.6). □

Corollary 65.14 (A uniform one-parameter spectral estimate)**.** *In either right regular representation let D_X denote the self-adjoint generator given by $R(\exp(vX)) = \exp(ivD_X)$. Put $L_X = f^{-1}\sum_\tau \|X_\tau\|_{\mathrm{op}}$. For $t > 0$ the two pairings of Theorem 65.13 with the corresponding $e^{-tD_X^2}$ differ by at most*

$$2\exp\left(-\frac{h^2}{4tL_X^2}\right)\|\Phi\|\|\Psi\| \qquad (L_X > 0).$$

For $L_X = 0$ they agree. For the copies in Theorem 65.12, $L_X \le 1$ and $t = 1$, this error is at most $2e^{-cx^{2\varepsilon}}\|\Phi\|\|\Psi\|$ with an absolute $c > 0$ throughout the common intervals.

Proof. The scalar Fourier transform of the Gaussian probability $(4\pi t)^{-1/2}e^{-v^2/(4t)}\,dv$ is $e^{-t\xi^2}$. The spectral theorem thus realizes the heat operator by integrating this one-parameter unitary group against that probability. The matrix exponential series gives $\ell(\exp(vX)) \le |v|L_X$. The Gaussian mass of $\{|v| \ge h/L_X\}$ is at most $\exp(-h^2/(4tL_X^2))$. Indeed after rescaling both halves use $e^{-(u+z)^2} \le e^{-z^2}e^{-u^2}$ for $u, z \ge 0$. Equation (65.6) proves the first bound. For $L_X = 0$ the generator is zero. For the displayed copies take $R = S = 2$. The auxiliary primes are at least $(k\exp(x^\varepsilon))^8$, so $h = \log(Q_0/4) \ge 15x^\varepsilon$ for all sufficiently large x, which gives the last estimate. □

The positive constants a_c come from cutoff real models. The construction realizes the compact row representation in automorphic L^2 and controls the stated translation averages. Identifying the specified discrete-series projection there and controlling its integral theta periods require further estimates.

65.6 Full automorphic norms for real sections with a height moment

Take the fields and intervals of Theorem 13.11, and retain the finite columns, levels and frequencies of Theorem 65.10. In this paragraph $r = \dim_E V$ denotes the row rank, s the length of its isotropic flag, and $f = [F_+ : \mathbb{Q}]$. Put

$$Q_0 = g_1g_2, \qquad R_0 = \sqrt{Q_0}, \qquad D_0 = 2frs, \qquad m = 2D_0, \qquad C_* = \frac{4}{\sqrt{3}}\,17^{rs}.$$

Write P for summation from the seed quotient to the full automorphic quotient, with the extension constructed in Theorem 64.12. For a seed Φ put $\Phi^< = \mathbf{1}_{H<R_0}\Phi$.

Proposition 65.15 (A real moment sufficient for the complex norm)**.** *If $\|H^m\Phi\|_{\mathrm{seed}} \le A^m\|\Phi\|_{\mathrm{seed}}$, then the full automorphic sum exists in L^2 and*

$$|\|P\Phi\| - \|\Phi\|_{\mathrm{seed}}| \le (1 + C_*)(A/R_0)^m\|\Phi\|_{\mathrm{seed}}.$$

For any orthogonal projection Π on automorphic L^2, $\|\Pi P\Phi - \Pi P\Phi^<\| \le C_*(A/R_0)^m\|\Phi\|_{\mathrm{seed}}$. If $A \le R_0\exp(-\eta x^\varepsilon)$ for fixed $\eta > 0$, both relative errors, normalized by $\|\Phi\|_{\mathrm{seed}}$, are at most $\exp(-\eta D_0 x^\varepsilon)$ for all sufficiently large x.

Proof. Take $\sigma = D_0$ in Theorem 64.12. Its weighted tail is at most $(A/R_0)^m\|\Phi\|_{\mathrm{seed}}$, as is the ordinary seed norm above R_0. Summation is isometric on the lower part. The reverse triangle inequality applied to the two decompositions proves the first assertion. The operator norm of an orthogonal projection is at most one, which gives the second. Finally $\log(1 + C_*) \le CD_0$ with an absolute C, since $f \ge 1$. This proves the uniform error bound. □

There is also a family for which the contribution after removing the height cutoff can be estimated directly. The initial sections can be the compact row copies of Theorem 65.12. For real a use the symmetric length $\ell(a)$ and right translations $R_B(a), R_G(a)$ of Theorem 65.13.

Theorem 65.16 (Full periodization of translation averages)**.** *Let an initial seed Φ be supported in $H < R < R_0$. Let μ be a complex Borel measure on G_∞ with $\int e^{m\ell(a)}\, d|\mu|(a) < \infty$. Set $W = R_B(\mu)\Phi$ and $\delta = |\mu|\{\ell \geq \log(R_0/R)\}$. Then*

$$PW = R_G(\mu)P\Phi, \qquad \|P(W - W^{<})\| \leq 2\delta\|\Phi\|_{\mathrm{seed}}, \qquad W^{<} = \mathbf{1}_{H<R_0}W.$$

In addition,

$$\big|\|PW\| - \|W\|_{\mathrm{seed}}\big| \leq 2\delta\|\Phi\|_{\mathrm{seed}}.$$

Proof. Theorem 65.13, gives $H(ga^{-1}) \leq H(g)e^{\ell(a)}$. Thus the seed $R_B(a)\Phi$ is supported at heights below $Re^{\ell(a)}$. Minkowski's integral inequality and unitarity of right translation give

$$\|H^m R_B(\mu)\Phi\|_{\mathrm{seed}} \leq R^m\|\Phi\|_{\mathrm{seed}} \int e^{m\ell(a)} d|\mu|(a).$$

In particular the weighted domain of P contains W. For compactly supported μ and bounded compactly supported seeds, ordinary summation commutes with its right translations and their integral. Indeed on compact test sets only finitely many rational indices occur, uniformly for a in a compact set, by the compact lifting argument of Theorem 88.1. For a square-integrable initial seed in $H < R$, approximate it there by bounded compact sections. Its translates for a in the support of such a μ lie in one bounded height range. Theorem 64.11, gives continuity on that range, proving commutation in the limit. Finally restrict a general μ to increasing compact sets exhausting G_∞. The omitted integrals converge to zero in the weighted seed norm by the displayed inequality and in the ordinary seed norm by their total variations. Theorem 64.12, allows passage to the limit on the automorphic side. This proves $PW = R_G(\mu)P\Phi$.

Split $\mu = \mu_0 + \mu_1$ at $\ell = \log(R_0/R)$, giving the boundary to μ_1, and put $W_i = R_B(\mu_i)\Phi$. The support of W_0 lies in $H < R_0$ up to a null set, so $\|PW_0\| = \|W_0\|_{\mathrm{seed}}$. The seed norm of W_1 is at most $\delta\|\Phi\|_{\mathrm{seed}}$; its automorphic norm satisfies the same bound by commutation and $\|P\Phi\| = \|\Phi\|_{\mathrm{seed}}$. Since $W - W^{<} = W_1 - W_1^{<}$ and the isometry applies to $W_1^{<}$, its periodization satisfies the first error bound. Comparing the two full norms with the equal norms of W_0 proves the last inequality. □

Corollary 65.17 (Gaussian averages preserving the prescribed compact row)**.** *On the common fields choose Φ in the compact row copies of Theorem 65.12, with real support $H < 2$. Let $X = (X_\tau)$ be in the real Lie algebra and put $L_X = f^{-1}\sum_\tau \|X_\tau\|_{\mathrm{op}}$. For $t > 0$ average right translates by*

$$k\exp(uX)k^{-1}, \qquad k \in K_\infty^{\max}, \qquad (4\pi t)^{-1/2}e^{-u^2/(4t)}\, du,$$

using Haar probability for k. For the resulting section W, Theorem 65.16 applies to the entire Gaussian. If $L_X > 0$, each of its two error bounds is at most

$$2\exp\left(-\frac{\log^2(R_0/2)}{4tL_X^2}\right)\|\Phi\|_{\mathrm{seed}}.$$

For $t = 1$ and $L_X \leq 1$ this is at most $2e^{-cx^{2\varepsilon}}\|\Phi\|_{\mathrm{seed}}$ with absolute $c > 0$ at all sufficiently large x. The averaged automorphic map on the original copies is injective and equivariant for $K_\infty^{\max}$. The same tail bound holds after any orthogonal automorphic projection.

Proof. Compact conjugation preserves each operator norm, and the exponential series gives $\ell(k\exp(uX)k^{-1}) \le |u|L_X$. The Gaussian has all the required exponential moments. The mass beyond $\ell = \log(R_0/2)$ is bounded by the two-sided Gaussian tail in the proof of Corollary 65.14, giving the displayed estimate. The auxiliary levels in that corollary satisfy $\log(R_0/2) \ge 7x^{\varepsilon}$ eventually. If $L_X = 0$, the translation is the identity and the tail is zero. For fixed k, the operator on either quotient is unitarily conjugate to $e^{-tD_X^2}$. Its quadratic form on any nonzero vector is strictly positive, since the spectral integrand $e^{-t\lambda^2}$ is positive at every real λ. Its compact average is therefore injective, and it commutes with the compact group by invariance of Haar measure. Compose it with the injection of Theorem 65.12 and use the exact commutation in Theorem 65.16. Applying a contraction to the tail gives the last assertion. □

These estimates on the automorphic quotient hold on the fields where (13.10) bounds the quadratic invariants. For the specific theta sections of the iterated lift, the moment in Proposition 65.15, or a representation by translates with the required tail, still has to be evaluated. The scalar comparing the projected complex functional with the integral theta lattice and the remaining strict orders in (12.15) are also required for (W).

65.7 The prescribed discrete series at the prime terminal frequency

Use the fields of Theorem 13.11 and the actual common terminal prime $T = \varpi\bar{\varpi}$ of Theorem 9.13. Write $m_0 = 2(s+1)$ for the rank parameter of the chain, $r = k + m_0 - 1$ for the row rank, $f = [F_+ : \mathbb{Q}]$, $D_0 = 2frs$, and $R_0 = \sqrt{g_1 g_2}$. We use the original frequency ϖ at the refined level.

Proposition 65.18 (The moment required at frequency one)**.** *Suppose a nonzero pure tensor of real Bessel sections of the prescribed row has square-integrable factors $W_{1,\tau}$ at frequency one. Normalize them to norm one. Let*

$$A_\tau = \|h_\tau^{4rs} W_{1,\tau}\|_2^{1/(4rs)} \quad \text{and} \quad A_1 = \Big(\prod_\tau A_\tau\Big)^{1/f}.$$

For the transported norm-one tensor at frequency ϖ, the height moment satisfies

$$\|H^{2D_0} W_\varpi\|_2^{1/(2D_0)} = \sqrt{T}\, A_1.$$

Its mass in $H < R_0$ equals the mass of the original tensor in $H < R_0/\sqrt{T}$. On these intervals,

$$\log(R_0/\sqrt{T}) \;\le\; -(8 - o(1))m_0 x^{\varepsilon}.$$

In particular the sufficient bound of Proposition 65.15 with fixed η becomes $A_1 \le (R_0/\sqrt{T})e^{-\eta x^{\varepsilon}}$ in unit frequency.

Proof. Use Proposition 84.13 on each real factor. This transports the Bessel functional by left translation; for fixed input vectors any other nonzero choice of the functional differs by a scalar by Theorem 65.7. Normalizing a nonzero factor in L^2 removes the absolute value of that scalar. Since $H^{2D_0} = \prod_\tau h_\tau^{4rs}$, product measure and Tonelli's theorem give the asserted moment, also if it is infinite. The same change of variables gives the mass formula for the indicator. Both g_i lie between $(kb)^8$ and $2(kb)^8$ by Proposition 72.6, so $\log R_0 = (8 + o(1))x^{\varepsilon}$. The proof of Theorem 9.13 gives $\log T \ge (16 - o(1))m_0 x^{\varepsilon}$, and $m_0 \to \infty$. These inequalities prove the stated bound. □

Use Euclidean distance in the signed coordinates, as for H, and put

$$H_{\mathrm{end}}(g) = \left(\prod_\tau \operatorname{dist}(g_\tau^{-1}u_s, g_\tau^{-1}L_{s-1})\, \operatorname{dist}(g_\tau^{-1}e, g_\tau^{-1}L)\right)^{1/f}.$$

These two distances concern the last simple coordinate of the unipotent character. Let Π be orthogonal projection on automorphic L^2 onto the isotypic space of the prescribed real row σ of (65.4), followed by projection onto its lowest compact type. These projections commute.

Theorem 65.19 (A bound for the projection at the terminal prime)**.** *There is an absolute $c>0$. For all sufficiently large x in the common intervals, $0<U\le T^{1/4}$, and every seed Φ of frequency ϖ supported in*

$$H_{\mathrm{end}}\le U,\qquad H<R_0\exp(T^{1/8}),$$

one has

$$\|\Pi P\Phi\| \;\le\; \exp\!\left(-\frac{c\sqrt{T}}{r^2U}\right)\|\Phi\|_{\mathrm{seed}}. \tag{65.7}$$

Without an upper restriction on H, every seed supported in $H_{\mathrm{end}}\le U$ in the weighted domain of Proposition 65.15 satisfies

$$\|\Pi P\Phi\|\le\exp\!\left(-\frac{c\sqrt{T}}{r^2U}\right)\left(\|\Phi^{<}\|_{\mathrm{seed}}+C_*\big\|\mathbf{1}_{H\ge R_0}(H/R_0)^{2D_0}\Phi\big\|_{\mathrm{seed}}\right),\qquad C_*=\frac{4}{\sqrt{3}}\,17^{rs}.$$

The constants are uniform in the selected words and finite class columns.

Proof. In Theorem 84.15, the compact Lie and Casimir bound for this row is $L_*\le\exp(C_\psi x^{\varepsilon})$ with a fixed constant, as computed in the proof of Theorem 65.2. All ranks and f are polynomially bounded in x. Together with $\log T\ge(16-o(1))m_0x^{\varepsilon}$, this shows that $r^2(L_*+f+1)=o(T^{1/4})$. Let $C_1\ge1$ exceed the absolute constant in Theorem 84.15, and take

$$j=\left\lfloor\frac{\sqrt{T}}{8C_1fr^2U}\right\rfloor.$$

Uniformly for $U\le T^{1/4}$ and large x, this integer is at least $\sqrt{T}/(16C_1fr^2U)$ and the base $C_1r^2(L_*+fj+1)U/\sqrt{T}$ is at most one half. Thus the projected operator at height R is bounded by

$$B(R)\exp\!\left(-\frac{c_1\sqrt{T}}{r^2U}\right)$$

with absolute $c_1>0$, using $B(R)$ of that theorem. For $R_0<R\le R_0\exp(T^{1/8})$, $\log B(R)\le rs\log17+2frsT^{1/8}=o(T^{1/4}/r^2)$, again because the ranks are polynomial. For $R\le R_0$ the prefactor is one. This proves (65.7) after decreasing c_1.

For the unbounded assertion split off $H<R_0$. On each of the upper shells in Theorem 64.12, the same constant $\exp(-c_1\sqrt{T}/(r^2U))$ multiplies the operator bound $B(R)$ at its upper endpoint. The Cauchy–Schwarz summation in the proof of that theorem with $\sigma=D_0$ gives exactly the second term displayed here. The projection is bounded, so the sum can be taken in automorphic L^2. In particular this reasoning uses no assumption on an untruncated L^2 realization of a particular local Bessel functional. □

Corollary 65.20 (The projection of the compact row copies and their averages)**.** *For the large terminal frequency the row copies in Theorem 65.12 can be chosen so that, for every initial seed Φ in their span and every compactly conjugated Gaussian average W of Corollary 65.17,*

$$\max\{\|\Pi P\Phi\|,\|\Pi PW\|\}\le\exp(-c'\sqrt{T}/r^2)\|\Phi\|_{\mathrm{seed}}$$

on the same fields, with an absolute $c'>0$.

Proof. On the maximal compact group, the indicated Euclidean distances are $\sqrt{2}$ and one at each place, so $H_{\mathrm{end}} = H = \sqrt{2}$. Both functions are continuous and right invariant under that compact group. Choose the bump in Theorem 65.12 to be one near the compact orbit of the identity and supported where both functions are less than two. This retains the nonzero row copies by the same local nonvanishing and Schur orthogonality argument. Apply (65.7) with $U = 2$. The convolution on the automorphic space in Corollary 65.17 is a contraction. It commutes with the σ-isotypic projection since that subspace is reducing for real right translations. Its compact conjugation average commutes also with the compact-type projector. The exact identity $PW = R_G(\mu)P\Phi$ therefore gives the estimate for W. □

Theorem 65.19 bounds projection onto the required real representation of the automorphic group. For the integral theta functional one still needs the contribution from larger terminal coordinates and its normalization on the specified lattice. The cohomology estimate (13.10) on these fields still leaves the remaining summands in (12.15).

65.8 Which part of the real period can carry the discrete series

Continue on the fields and intervals of Theorems 13.11 and 65.19. Use their prime T, $m_0 = 2(s+1)$, $R_0 = \sqrt{g_1 g_2}$, and $b = \exp(x^{\varepsilon})$. At each real place let $a_{i,\tau}$ be the successive Euclidean Gram–Schmidt lengths of the list

$$g_\tau^{-1}u_1, \dots, g_\tau^{-1}u_s, \quad g_\tau^{-1}(\varpi_\tau e).$$

Set $h_i = (\prod_\tau a_{i,\tau})^{1/f}$ and let M be the geometric mean of the local maximum consecutive ratios $a_{i+1,\tau}/a_{i,\tau}$. The global height H uses only the first s entries. Let Π denote the same automorphic discrete-series and lowest-compact-type projection as in Theorem 65.19.

Theorem 65.21 (The projection below the full height cutoff)**.** *At frequency ϖ, for every $0 < R \le R_0$ and every seed supported in $H < R$,*

$$\|\Pi P\Phi\| \le \exp\!\left(-\frac{c\sqrt{T}}{r^2 R}\right)\|\Phi\|_{\mathrm{seed}} \tag{65.8}$$

with absolute $c > 0$, for all sufficiently large x. There is no condition on the negative vector modulo the isotropic space. In particular this applies to the entire lower part of any square-integrable section, including each prescribed real Bessel section for which that part is in the seed Hilbert space.

Proof. By Corollary 84.18, $M \ge \sqrt{T}/H$. Apply that corollary with $V = \sqrt{T}/R$. Its factor $B(R)$ is exactly one. The real row parameter L_*, the dimensions, and $R_0 \le 2(kb)^8$ satisfy $r^2(L_* + f + 1)R_0 = o(\sqrt{T})$, by the size estimates in Theorem 65.19. Choose $j = \lfloor V/(8C_1 f r^2)\rfloor$ with C_1 at least the absolute constant in Corollary 84.18. Then $fj \ge V/(16C_1 r^2)$ and $C_1 r^2(L_* + fj + 1)/V \le 1/2$. This proves (65.8) uniformly in R. The calculation uses the dual vector to the final quotient line at each place. Its norm decreases when the distance of $g_\tau^{-1}e$ from $g_\tau^{-1}L$ increases. □

Theorem 65.22 (Reduction to the region of large isotropic vectors)**.** *Let $\mathcal{D} = \{M < b^2\}$. For any seed supported in $H < R_0 \exp(b^{3/2})$,*

$$\|\Pi P\Phi - \Pi P(\mathbf{1}_{\mathcal{D}}\Phi)\| \le e^{-cb^2/r^2}\|\Phi\|_{\mathrm{seed}}$$

with an absolute $c > 0$. In $\mathcal{D}$ all s isotropic lengths satisfy

$$h_i > \sqrt{T}\, b^{-2(s-i+1)} \quad (1 \le i \le s). \tag{65.9}$$

The same error bound holds for seeds in the weighted domain without a height cutoff if $\|\Phi\|_{\mathrm{seed}}$ *on the right is replaced by*

$$\|\Phi^{<}\|_{\mathrm{seed}} + C_*\big\|\mathbf{1}_{H\ge R_0}(H/R_0)^{2D_0}\Phi\big\|_{\mathrm{seed}}.$$

Proof. We need the dependence on the row weights, since b^2 is now the ratio threshold. In (65.4) the v_i are polynomially bounded and $|b_{j,\tau}| \le 2b$ for all sufficiently large x, by the construction preceding Lemma 9.12. The compact highest weights differ from these parameters by at most r in each entry. The formulas in the proof of Theorem 65.2 therefore give $L_* \ll \sqrt{r}(2b+r)$: the compact Lie norm is bounded by $\sqrt{r}$ times the largest absolute compact weight, and the Casimir is the difference of the two sums of squares for λ_τ and ρ. The ranks and f are polynomial in x. Hence $r^2(L_* + f + 1) = b^{1+o(1)}$ as an upper bound. Here and below $\varepsilon > 0$ is fixed.

Apply Corollary 84.18, to the part with $M \ge b^2$. With $V = b^2$, take $j = \lfloor V/(8C_1 f r^2)\rfloor$ as above. After optimizing the derivative order this gives $B(R)e^{-c_1 b^2/r^2}$ at every bounded height R. When $R \le R_0\exp(b^{3/2})$, the logarithm of $B(R)$ is at most $rs\log 17 + 2frs\,b^{3/2} = o(b^2/r^2)$. Absorbing this factor proves the first bound. Summing the corresponding estimates on all shells with the weights of Theorem 64.12, exactly as in Theorem 65.19, proves the assertion for unbounded supports. Lastly, on $\mathcal{D}$ use $M^{s-i+1}h_i \ge \sqrt{T}$ from Corollary 84.18. This proves (65.9) simultaneously for every index, without counting a union over the places. □

Corollary 65.23 (Separation from the compact real seeds)**.** *On* $\mathcal{D}$ *one has*

$$\log h_1 \ge (14 - o(1))sx^{\varepsilon}, \qquad \log h_s \ge (8 - o(1))m_0x^{\varepsilon} - 2x^{\varepsilon}.$$

In particular every one of these s *geometric means eventually exceeds* R_0*. For the Gaussian averages in Corollary 65.17 at the prime frequency, an initial seed supported at* $H < R \le R_0$ *satisfies* $\|\Pi R_G(\mu)P\Phi\| \le e^{-c\sqrt{T}/(r^2R)}\|\Phi\|_{\mathrm{seed}}$*.*

Proof. Use $\log T \ge (16 - o(1))m_0x^{\varepsilon}$ and $m_0 = 2(s+1)$ in (65.9). The least of its lower bounds occurs at $i = 1$; its logarithm is at least $(8 - o(1))\,2(s+1)x^{\varepsilon} - 2sx^{\varepsilon}$. Meanwhile $\log R_0 = (8 + o(1))x^{\varepsilon}$ and $s \to \infty$. The Gaussian convolution on the automorphic space is a contraction commuting with both projections, as in Corollary 65.20. Apply Theorem 65.21. □

The lower-height isometry of Theorem 65.10 and (65.8) now apply to exactly the same restricted seeds. The first computes their full norm and the second bounds the portion in the required real row. The period on $\mathcal{D}$, its integral scale, and the outstanding strict groups in (12.15) still enter the prime-sum estimate (W).

65.9 Where the normalized real Bessel functions have their mass

For a selected word and translate, let $J_{\varpi} : \bigotimes_\tau \sigma_\tau \to L^2(B_\infty\backslash G_\infty, E_\rho)$ be the isometric embedding of Corollary 84.21, with terminal frequency ϖ_τ at every real place. This space consists of sections with the specified Bessel character and compact column type. Its measure is the product of the compatible quotient measures in that corollary. We write J_1 for the corresponding embedding at frequency one. Use the same fields, intervals and prime T as in Theorem 13.11 and Theorem 65.22.

Theorem 65.24 (Simultaneous concentration at the real places)**.** *Let* v *be any unit vector in the product lowest compact type of the row and let* $W = J_{\varpi}v$*. Set* $V_x = br^4$*. For the individual lengths and ratios of Theorem 65.22, put*

$$\Omega_x = \bigcap_{\tau}\left\{\max_{1\le i\le s}\frac{a_{i+1,\tau}}{a_{i,\tau}} < V_x\right\}.$$

For an absolute $c>0$ and all sufficiently large x,

$$\int_{\Omega_x} \|W(g)\|^2 \, d\dot{g} \;\ge\; 1-e^{-cbr^2}. \tag{65.10}$$

At every point of this region, simultaneously at every real place and every $1\le i\le s$,

$$a_{i,\tau} > \sqrt{T}\, V_x^{-(s-i+1)}.$$

In particular $\log a_{1,\tau}\ge (15-o(1))sx^{\varepsilon}$ there. All these assertions concern the entire real quotient.

Proof. Fix a place τ. The isometry of J_{ϖ} transfers the norm of every right Lie derivative of W to the row Hilbert space. Use the pointwise inequality of Theorem 84.17 at just this place, leaving the other coordinates as parameters. There are r^{2j} words of order j. By Lemma 99.25 each derivative on the lowest type has norm at most $\{Cr(L_*+j+1)\}^j$. This bound also holds after tensoring with the identity at the other places: expand the vector in an orthonormal basis there and sum squares. With $M_\tau=\max_i a_{i+1,\tau}/a_{i,\tau}$ it follows that

$$\|\mathbf{1}_{M_\tau\ge V_x}W\|_2 \;\le\; \left(\frac{C_1r^2(L_*+j+1)}{V_x}\right)^{j}.$$

Here $L_*\ll \sqrt{r}(2b+r)$ by the proof of Theorem 65.22. Take $j=\lfloor V_x/(8C_1r^2)\rfloor$, enlarging C_1 to be at least one. Since $r\to\infty$ and r is polynomial in x, for all large x the base is at most $1/2$ and $j\gg br^2$. The complement of Ω_x is a union of f such events. Sum their squared L^2-norms, using f polynomial in x and $\|W\|=1$. This proves (65.10).

At one place the consecutive ratios telescope from $a_{i,\tau}$ to $a_{s+1,\tau}=\sqrt{T}\,Z_\tau\ge\sqrt{T}$. This proves the individual bounds. Finally $\log V_x=x^{\varepsilon}+4\log r=(1+o(1))x^{\varepsilon}$, whereas $\log\sqrt{T}\ge (8-o(1))\,2(s+1)x^{\varepsilon}$. Subtract $s\log V_x$ to obtain the claim at $i=1$. □

Corollary 65.25 (A lower bound for the height moment)**.** *For every pure product of nonzero lowest-type vectors, normalize its unit-frequency local Bessel functions to norm one and define A_1 as in Proposition 65.18. Then*

$$A_1 \;\ge\; \frac{(1-e^{-cbr^2})^{1/(4D_0)}}{br^4}.$$

Consequently none of these products satisfies the sufficient small-moment condition $A_1\le (R_0/\sqrt{T})e^{-\eta x^{\varepsilon}}$ of Proposition 65.18, for any fixed $\eta>0$ and large x on the prescribed intervals.

Proof. The real functions in the statement are square-integrable and nonzero by Corollary 84.21. At each place, changing the terminal frequency by the isometry of Proposition 84.13 gives another isometric Bessel embedding. Uniqueness of the functional and equality of norms give the same embedding up to a unit scalar. Thus the product at the prime frequency has the distribution in Proposition 65.18. On Ω_x one has $H\ge h_s>\sqrt{T}/V_x$ and at least the mass in (65.10) lies there. Integrating H^{4D_0} against $\|W\|^2$ gives

$$(\sqrt{T}A_1)^{4D_0} \;\ge\; (\sqrt{T}/V_x)^{4D_0}(1-e^{-cbr^2}).$$

This remains true for an infinite positive moment. Finally $\log R_0=(8+o(1))x^{\varepsilon}$, while $\log\sqrt{T}\ge (8-o(1))m_0x^{\varepsilon}$ with $m_0\to\infty$. It follows that $V_xR_0/\sqrt{T}\to 0$, and the factor $(1-e^{-cbr^2})^{1/(4D_0)}$ tends to one. These estimates prove the stated incompatibility. □

The norm in (65.10) is the real norm with formal degree normalization. Passing to the global theta period still requires its finite tests, summation on the automorphic quotient, and its scale in the prescribed integral lattice. The estimate here locates the real contribution at each place on the same fields where Theorem 13.11 controls the indicated part of strict cohomology.

65.10 The complex projected period on the full real quotient

Fix x on the common intervals of Theorem 13.11, one translate, its full finite family of words and the prime of Theorem 9.13. Use the actual refined level K^0 and terminal frequency ϖ. Write D_c for the space of finite class covectors of word c with the inner product induced by its seed. The common real row has lowest compact type U; the automorphic copies of this type inside that discrete series form $U \otimes \mathcal{M}$. Write J_U for the product Bessel embeddings with the normalization in Corollary 84.21, tensored with their finite class covectors. These embeddings give actual real vectors to which the projected period can be applied.

Theorem 65.26 (The full projected period and its finite Gram matrix)**.** *For every square-integrable seed in these word spaces the limit of the projected automorphic sums over height truncations exists in $U \otimes \mathcal{M}$. Call it P_U. Put $D = \bigoplus_c D_c$. There is a bounded map $A : \mathcal{M} \to D$ at this field and level, and a positive semidefinite matrix $G_D = AA^*$ on D, such that the projected period on normalized real vectors is $P_U J_U = 1_U \otimes A^*$. For any right compact-invariant measurable region E_0 in the real period quotient the Gram operator of $P_U \mathbf{1}_{E_0} J_U$ is*

$$1_U \otimes T_{E_0} G_D T_{E_0}, \qquad T_{E_0}|_{D_c} = t_c(E_0) 1_{D_c}, \quad t_c(E_0) = \int_{E_0} \|(J_c v)(g)\|^2 \, d\dot{g}, \tag{65.11}$$

where v is any unit vector of U. In particular, for each c using E_0 instead of the full quotient multiplies every projected vector of that word by the same scalar $t_c(E_0)$.

Proof. The level, irreducible column types and fixed row are precisely the data in Section 64.5. Real sections times the finite covectors have the seed inner product by the identity-term calculation in Theorem 70.24. Theorem 64.14 extends ΠP to this Hilbert space. Boundedness and $\mathbf{1}_{H<R}\Phi \to \Phi$ in the seed norm prove the limit. Corollary 64.15 identifies its restriction to $\mathbf{1}_{E_0} J_U$ with $1_U \otimes A^* T_{E_0}$. Taking the adjoint and composing proves (65.11). The class spaces D_c are finite dimensional for the fixed number field and finite level. The formula includes zero eigenvalues, with their full multiplicities. □

Corollary 65.27 (The contribution from the region of large real lengths)**.** *Set $\delta'_x = e^{-cbr^2}$ with the constant in Theorem 65.24. For its region Ω_x, the norms of the differences between the full and restricted projected period operators on J_U, and between their Gram operators, are at most*

$$\delta'_x \, \|A\|, \qquad 2\delta'_x \, \|G_D\|$$

respectively, with no upper height restriction. On each individual word the restricted norm lies between $1 - \delta'_x$ and one times its full norm for every input, including the refined zero-sum columns. There also exists a finite common height R_x for which the same statements on $\Omega_x \cap \{H < R_x\}$ hold with $2\delta'_x$ in place of δ'_x.

Proof. The ratios defining Ω_x use Euclidean lengths, so right translation by the maximal compact group preserves them. Theorem 65.24 gives $1 - \delta'_x \le t_c(\Omega_x) \le 1$ for every word. Thus $\|1 - T_{\Omega_x}\| \le \delta'_x$. Omitting the tensor identity on U, the operator difference is $A^*(1 - T)$. For the Gram difference use $G_D - TG_DT = (1 - T)G_D + TG_D(1 - T)$ and $\|T\| \le 1$. For a single word the outcome is exactly multiplied by its scalar t_c. Finally monotone convergence gives $t_c(\Omega_x \cap \{H < R\}) \uparrow t_c(\Omega_x)$. There are finitely many words and hence a single finite R_x with every error below δ'_x. This proves the last assertion, without an estimate for the size of R_x. □

Proposition 65.28 (Extraction at the separation height)**.** *Put $t_c^0 = t_c(\{H < R_0\})$. For each selected word $t_c^0 > 0$. On $U \otimes D_c$ the full projected period equals its truncation at R_0 divided by t_c^0, and*

$$\|A^*|_{D_c}\| \;\le\; \frac{\exp[-c_0\sqrt{T}/(r^2 R_0)]}{\sqrt{t_c^0}}.$$

All scalars here use the actual local Bessel functions and the original finite theta-period fields.

Proof. The height is $\sqrt{2}$ at the identity and $R_0 > \sqrt{2}$ eventually. Proposition 65.9 gives, in every neighborhood of the identity, a point at which the Bessel coefficient of a nonzero pure product lowest-type vector is nonzero. Scaling its local functional to the norm of Theorem 84.20 preserves this property. Smoothness therefore gives positive mass in $H < R_0$. Compact irreducibility makes this mass independent of the unit vector of U. By Theorem 65.26 the truncated operator on this word equals t_c^0 times the full operator. Its seed input $\mathbf{1}_{H<R_0} J_U$ has norm $\sqrt{t_c^0}$ as an operator on that summand. Apply Theorem 65.21 and divide by t_c^0. □

Thus at fixed arithmetic data the projected complex period has a well-defined value for the full square-integrable real vectors. The matrix G_D still depends on those arithmetic data. Its size, its nonzero values on the prescribed integral theta classes, and their lattice normalization remain to be controlled as x varies. These requirements and the remaining strict orders in (12.15) enter the deduction of (W).

65.11 An exact comparison of two projected Bessel periods

Use the fields of Theorem 13.11 and the prime $T = \varpi\bar{\varpi}$ of Theorem 9.13. Let $a = a_\varpi$ be the rational isometry scaling the two isotropic strings by $\varpi, \bar{\varpi}^{-1}$. Put

$$K_0 = K^0, \qquad K_1 = a_f K_0 a_f^{-1}, \qquad L = K_0 \cap K_1, \qquad d_{x,T} = T^{\,fs(r-s)/2}, \quad I = \begin{bmatrix} r \\ s \end{bmatrix}_T^f.$$

The two column levels at K_0, K_1 both give the originally prescribed refined class covectors. The intersection admits both frequencies by Proposition 64.17. At K_0 a unit-frequency Bessel coefficient against a K_0-invariant row vanishes at a column argument $h \in H(\mathbb{A}_{F_+})$. Indeed Theorem 64.1 gives $n_0 \in N_f \cap K_0$ with $\chi_1(n_0) \neq 1$. In the defining integral over $[N]$ translate on the right by hn_0h^{-1}. The row value is fixed and the character is multiplied by the nontrivial factor, proving vanishing. This is the reason for including the change of finite level in the comparison.

Let $\mathcal{P}_{\varpi,0}$ and $\mathcal{P}_{1,1}$ be the projected period maps at K_0, K_1 on the real vectors normalized by formal degree, using the same ordinary finite class coordinates at both levels. Choose phases by transporting the real vectors as in Proposition 65.18. For a compact real support, define $\mathcal{Q}_{\beta,i} = \Pi e_i P \Phi_{\beta,L}$ on the same inputs, where e_i averages over K_i and the input specifies the real section and the class tuple. Use compatible square-integrable real vectors by extension from compact support as in the theorem below. Every output belongs to the automorphic space of level L.

Theorem 65.29 (A single row isometry for both frequencies)**.** *For all selected words simultaneously and on their refined zero-sum columns one has*

$$\begin{aligned} \mathcal{P}_{\varpi,0} &= d_{x,T}^{-1}\, R(a_f^{-1})\, \mathcal{P}_{1,1}, \\ \mathcal{Q}_{\varpi,0} &= d_{x,T}\, R(a_f^{-1})\, \mathcal{Q}_{1,1}. \end{aligned} \tag{65.12}$$

The maps in the second equality extend to the prescribed square-integrable real sections. In the common level-L row space one may replace $R(a_f^{-1})$ on their finite-dimensional images from the lowest type by a single unitary automorphism of that space, independent of the word. With ordinary class coordinates the Gram matrices of the two common-level maps satisfy

$$\mathcal{Q}_{\varpi,0}^* \mathcal{Q}_{\varpi,0} = d_{x,T}^2\, \mathcal{Q}_{1,1}^* \mathcal{Q}_{1,1}.$$

At the respective coarse levels the two Gram matrices agree when the inputs are given their isometric seed normalizations.

Proof. Theorem 64.16 proves the first equality, with $d_{x,T} = \sqrt{J}$. Ordinary summation commutes with the finite right translations for compact seeds and the archimedean projection commutes with them on automorphic L^2. Consequently Theorem 64.18 gives $\mathcal{Q}_{\varpi,0} = (J/I)\mathcal{P}_{\varpi,0}$ and $\mathcal{Q}_{1,1} = I^{-1}\mathcal{P}_{1,1}$. The coarse-level maps are bounded at each fixed field and level by Theorem 65.26 and its transport in Theorem 64.16. This extends each finite-averaged map by continuity and proves (65.12) for the full real sections. Finite translation is unitary and the same a_f occurs in every word. For the normalized real vectors of the lowest type, each image has finite dimension because both this type and the class spaces are finite. Inside the sum of the two image spaces, their orthogonal complements have equal dimension. Extend the isometry between the images by a unitary between these complements, then by the identity on the orthogonal complement of the sum in the common row space. This proves the assertion about a single automorphism. Taking inner products proves the Gram identity.

Finally the squared class norm of a real unit seed in ordinary class coordinates is $c_{\rho,K_i}\|\ell\|^2$. The ratio $c_{\rho,K_1}/c_{\rho,K_0} = d_{x,T}^2$ cancels the square of the first scalar in (65.12) when the inputs are identified by their seed isometries. □

Corollary 65.30 (The arithmetic cost of the common finite level)**.** *The scalar $d_{x,T}$ is a positive rational integer and the finite averaging maps have denominator dividing the positive integer I. At every x in the common intervals, uniformly for all these words and every fixed $A > 0$,*

$$\log(d_{x,T}I) = o(b/x^A), \qquad 1 - \frac{f}{T-1} \le \frac{c_{\rho,L}}{c_{\rho,K_0}} \le 1.$$

Proof. The degree f is even on these intervals, as in the proof of Theorem 78.16. Thus $d_{x,T}$ is integral. At each split place the Gaussian binomial counts subspaces and hence I is integral. The unaveraged finite sum Ie_i has coefficient one on each coset. For its size use

$$T^{s(r-s)} \;\le\; \begin{bmatrix} r \\ s \end{bmatrix}_T \;\le\; 4T^{s(r-s)}.$$

Indeed $\prod_{j=1}^{s}(1 - T^{-j}) \ge 1/4$: the first factor is at least $1/2$ and the remaining product is at least $1 - \sum_{j\ge 2} 2^{-j} = 1/2$. The factors in the numerator of the normalized Gaussian binomial are at most one. It follows that $\log(d_{x,T}I) \le \frac{3}{2} f s(r-s)\log T + f\log 4$. Each quantity on the right is polynomially bounded in x by the proof of Corollary 78.17, whereas $b = \exp(x^{\varepsilon})$. The bound on the class norm is Proposition 64.17. □

These identities apply to the two Bessel projections on the same finite class lattice, with the stated finite averages. For the theta lift the source specifies an additional equality on coefficient spaces,

$$B_u(j_{N,u}\phi, j_E x) = \iota\big(B_{k,u}(\phi, x)\big).$$

Here B_u is the complex theta form, $B_{k,u}$ is the form over the coefficient field, $j_{N,u}, j_E$ are its two complex realization maps, and ι is the chosen embedding of the coefficient field. This is the condition at `Formalism/LocalRingCardinality/RhoDetPrimeEqMuTildeMulChiPow.lean`, line 364. The comparison still requires its arithmetic scale, a nonzero value on the designated theta classes, and the remaining strict orders in (12.15) on the same fields.

66 Unfolding and comparison along the row

66.1 Cuspidality at the original period level

Theorem 66.1 (The projected first moment on the matching cuspidal row)**.** *Choose the unit pure real vector $u = \bigotimes_\tau u_\tau$ in the lowest type of the common row. Keep its level K^0 and*

frequency ϖ. For each word write $S_c b = S_0(u \otimes b)$ in the class norm of Theorem 88.4. In Theorem 60.21 choose this real line, terminal depth zero and the same row character. Include all exceptional places of the row in its finite set, and take the other principal subgroups there inside the kernels of the source characters. Let P be projection to the whole cuspidal space $\mathcal{H}$ with these data. Then $PS_c = S_c$ on D_c.

Let J be either of the terminal spaces of Theorem 88.4, possibly restricted to the retained classes of Corollary 88.7, and suppose $d_J = \dim J > 0$. Use the random signed column of the corresponding first-moment formula. For the matching complex theta operator Θ put $\alpha = C_x P(\mathbf{a}) > 0$ as in Theorem 60.21. Then, writing $O_{\gamma,c}$ for the nonparabolic orbital operators,

$$M_J = 1 + d_J^{-1} \sum_{\gamma} \operatorname{Re} \operatorname{tr}(P_J O_{\gamma,c}) = \mathbb{E}\|S_c b\|^2,$$

$$e^{-\Delta_x} M_J \le \alpha^{-1} \mathbb{E}\|\Theta S_c b\|^2 \le e^{\Delta_x} M_J.$$

Some signed column in this distribution has nonzero forward lift if and only if the real orbital sum is strictly greater than $-d_J$.

Proof. The ordinary periodization of Theorem 64.22 is square integrable. Right convolution by the real matrix unit at u fixes it. On each archimedean copy of the row this matrix unit projects to the chosen line. The cuspidality argument of Theorem 60.4 applies to every finite class vector at depth zero. In detail, a nonzero vector fixed by this matrix unit generates a copy of the tempered discrete series. On an arithmetic component the determinant of the level stabilizer consists of norm-one units in E, hence of roots of unity. Thus a finite-index subgroup lies in the special unitary group. The central real torus is compact, and after pullback along the finite central isogeny one can take its fixed character and apply Wallach's constant-term theorem to the special unitary arithmetic quotient. Its proper parabolic constant terms vanish. Passing back through the finite cover preserves this vanishing, as does the passage to the adelic constant terms described in Theorem 60.4. Wallach, Theorem 4.3.

The finite right transformations of the seed pass through its absolutely convergent rational sum. At $v \mid T$ the first test uses $L_0 = K_v^0$ with exactly this row character. Away from T the seed is fixed by the chosen principal subgroups at exceptional places and by the spherical compacts elsewhere. These properties put every $S_c b$ in $\mathcal{H}$. Taking a terminal character or a difference on the column changes b inside the same domain. The row transformation law therefore continues to hold for the signed columns and for their linear combinations supported on retained classes.

The projection defect in Theorem 88.4 now vanishes. Its averaging formula and absolute convergence give the first identity, also on the retained classes by Corollary 88.7. The norm comparison on $\mathcal{H}$ gives the two inequalities after averaging, since the terminal index at depth zero is one. Each value being averaged is nonnegative, and the distribution has finite support. This proves the last assertion. □

66.2 The specified compact test and its Petersson norm

On the fields of Theorem 99.3 take a standard definite pair of ranks $r < s$, rs even. Pair its lift with the full target or restrict it to the standard hyperplane. Denote the group of integration on the target by H_*, of rank $q = s$ or $s-1$ respectively. In either case use the space V, polynomial Q, and integer H of Lemma 73.1. Write $\mathcal{I}_V$ for its ordered minor indices, $k = r \sum_\tau b_\tau$, and ρ for the target action on V, including its vacuum character. On the source compact group all these products transform by one character η. Indeed right multiplication of a full-column minor by a column change multiplies it by the determinant of that change, and the source vacuum character is fixed.

Keep the source function a, its finite level J of character χ, and the target level J_* of character χ_*. Use the finite test ϕ_{uv} of Theorem 55.9, with v, j_* embedded in the full target. In particular the average defining it includes both χ and $\overline{\chi_*}$. Write $\kappa = \mathrm{vol}(U_{r,\infty} \times J)$, $\kappa_* = \mathrm{vol}(H_{*,\infty} \times J_*)$, and γ_u, γ'_v for the two finite stabilizer orders. With real Haar probability set

$$a_u = \int a(h,u)\eta(h)\,dh, \qquad L_v(P) = \sum_u \frac{a_u}{\gamma_u} P(\partial) G_{\phi_{uv}}(0), \qquad \mathcal{N}_V = \sum_v \frac{1}{\gamma'_v} \sum_{T \in \mathcal{I}_V} |L_v(P_T)|^2.$$

Use $L(P)$ for the actual theta lift of the input with polynomial P in the dimensionless coordinates of Theorem 55.8, restricted to H_* and projected to the stated finite character. The Petersson pairing below is linear in its first argument.

Theorem 66.2 (The period against a prescribed compact coefficient)**.** *Let b be a target form of character χ_* with $b(h,v) = B_v(\rho(h)Q)$ at its finite representatives, for specified linear functionals B_v on V. Then*

$$\frac{d_V H\, c^{k/2}}{\kappa\kappa_* C} \langle L(Q), b\rangle = \sum_v \frac{1}{\gamma'_v} \sum_{T \in \mathcal{I}_V} L_v(P_T)\overline{B_v(P_T)},$$
$$\frac{d_V H\, \nu^D c^k}{\kappa^2 \kappa_*} \|L(Q)\|^2 = \mathcal{N}_V.$$

Here C, c, ν, D are as in Theorem 55.8. In the full-target case the arithmetic quantity on the right of the second identity is exactly (53.1) when all $a_u = 1$ and its F_{uv} are the actual finite sections. In either case any nonzero vector in V gives a nonzero projected lift if and only if $\mathcal{N}_V > 0$. A signed sum with a common Q, splitting and degree is included by taking the signed sum of its L_v's before the two displayed operations.

Proof. The source compact action by η and the finite projections give the following equality at every compact target argument, by the class calculation of Theorem 55.9.

$$L(P)(h,v) = \kappa C c^{-k/2}\, L_v(\rho(h)P).$$

It is valid on the hyperplane because one can evaluate the original kernel at $h \oplus 1, v \oplus 1$ before integrating. The tensor product of the irreducibles in Lemma 73.1 is irreducible for the product of the compact groups. For instance the commuting operators are scalars by applying Schur's lemma to each factor on the corresponding matrix blocks.

For any P, the average of the rank-one operators $(\rho(h)P)(\rho(h)P)^*$ is scalar by irreducibility and has trace $\|P\|_{\mathrm{F}}^2$. It equals $\|P\|_{\mathrm{F}}^2 I_V/d_V$. On the other hand the average is $\|P\|_{\mathrm{F}}^2/(d_V C_{\boldsymbol{b}})$ times $\sum_{T \in \mathcal{I}_V} P_T P_T^*$ by the frame identity. Evaluate between the two functionals L_v, B_v at $P = Q$. The ratio $\|Q\|_{\mathrm{F}}^2/C_{\boldsymbol{b}}$ is $1/H$. Multiply by the class masses and the first formula follows. For the norm use L_v as both functionals and retain $\kappa^2|C|^2 c^{-k}$ from the pointwise expression. The identity $|C|^2 = \nu^{-D}$ gives the second formula. For any other $P \neq 0$ its squared norm on the left changes by the positive factor $\|P\|_{\mathrm{F}}^2/\|Q\|_{\mathrm{F}}^2$.

For a single terminal coefficient $B_v(\rho(h)Q')$ with a different Q', the same integration replaces $\|Q\|_{\mathrm{F}}^2$ in the first calculation by $\sum_j Q_j \overline{Q'_j}$ in a Fischer orthonormal basis. This follows by polarization. Finite sums give the period against any specified compact-finite test of this type, with its original coefficients. Components of inequivalent compact type pair to zero by Schur's lemma. □

For a general test in this compact isotypic component one can recover the covector in the first identity without dividing its highest-weight coordinates. Let θ_Q be the character of the

real diagonal torus on Q, including the fixed vacuum twist. Right integration against $\overline{\theta_Q}$ on that torus is an orthogonal projection which fixes $L(Q)$, by the pointwise formula. At one place each indexed minor product is a torus eigenvector. To have the weight of Q_l, every minor must contain all the first $r-1$ rows. Its one remaining row must be r or s, with exactly l occurrences of s. This forces the product to equal Q_l. For the full target all rows must be the first r and the same assertion holds with A_0^b. Thus the torus projector takes each ordered product either to zero or to the distinguished product with coefficient one, also after tensoring over places. Consequently a class function written as an integral sum of matrix coefficients $B_i(\rho(h)R_i)$, with inputs in the module generated by the P_T's and with covectors in its integral dual, yields exactly $B_v(\rho(h)Q)$ with B_v still integral on that module. Components of other compact types have zero pairing with this lift. On the source an irreducible compact type contributes to a_u only if it is the inverse of the one-dimensional character η. This follows by averaging its representation after twisting by η.

66.3 Character matching in the first seesaw

Fix the real tests in Theorem 59.14. The remaining finite period is a linear functional ℓ on $D \otimes \overline{H}$, where $D = \mathcal{V}_Y$ and $H = \mathcal{V}_{-L}$. Use their finite oscillator Hilbert norms. Let J_S, J_Y be the given fixing levels of the two forms, acting on their integral Hermitian lattices. The level on Y acts through its extension by the identity on L. These actions preserve the depth conditions: they preserve the rank-one generators and integral homomorphisms in Proposition 73.3, commute with the selected idempotents, and conjugate a Weyl operator to the operator of the transformed displacement. Thus J_S acts on both D, H, while J_Y acts on D and commutes with J_S. Each compact group acts through a finite quotient, since the pointwise stabilizer of a basis in a smooth finite-dimensional representation is open. Include the kernels of the level characters in choosing these quotients. Set

$$\alpha = (\chi_S \delta_S)^{-1}\big|_{J_S}, \qquad \beta = (\chi_Y \delta_Y)^{-1}\big|_{J_Y}.$$

The splitting characters appear here because the finite actions on the two small spaces are their own oscillator actions.

Proposition 66.3 (The finite components detected by the period)**.** *Let D_β be the β-space for J_Y. For each irreducible unitary representation π of the finite quotient of J_S, let $d_\pi = \dim \pi$, and let m_π, n_π be the multiplicities of $\alpha \otimes \pi$ in D_β and of π in H, respectively. Choose orthonormal bases p_{ia} in the former copies and q_{ib} in the latter, identifying their representation coordinates by the character twist. Here i ranges from one to d_π, while a, b index copies. Put*

$$c_{\pi ab} = \ell(p_{1a} \otimes \overline{q_{1b}}).$$

The only nonzero entries in these adapted bases can occur within the indicated pairs of types, and within such a pair they are $\ell(p_{ia} \otimes \overline{q_{jb}}) = \delta_{ij} c_{\pi ab}$. In particular

$$\|\ell\|^2 = \sum_{\pi} d_\pi \sum_{a,b} |c_{\pi ab}|^2.$$

Let e denote the joint character projector on these finite spaces. For any finite test z with $\ell(z) \neq 0$, one of the matched individual pairs satisfies

$$|c_{\pi ab}| \;\geq\; \frac{|\ell(z)|}{\|ez\| \, (\sum_\pi d_\pi m_\pi n_\pi)^{1/2}}.$$

There is also a bound for the integral product basis $b_i, \bar{b}'_j$ of Proposition 73.3. Write $w_i = \|b_i\|^2$ and $w'_j = \|b'_j\|^2$. Then

$$\max_{i,j} |\ell(e(b_i \otimes \bar{b}'_j))| \;\geq\; \frac{|\ell(z)|}{\|ez\|\,(\sum_{i,j}(w_i w'_j)^{-1})^{1/2}}.$$

Each weight is the finite position volume of the corresponding surviving orbit of cosets. Both inequalities concern the same forms and the same finite tests as the original integral.

Proof. By change of variables, ℓ transforms under the two small actions by α on the common group and β on J_Y. Consequently it kills the orthogonal complement of D_β in its first variable. On any pair of irreducible copies it gives a sesquilinear form $B(p,q) = \ell(p \otimes \bar{q})$. Write $B(p,q) = \langle Cp, q\rangle$, with first variable linear. Its transformation law says $C\rho(j) = \alpha(j)\pi(j)C$ on a pair ρ, π. Schur's lemma makes this operator zero unless $\rho \simeq \alpha \otimes \pi$, and scalar under the specified identification in that case. This proves the assertion about entries. Summing their squared moduli in the orthonormal product basis gives the squared norm of the functional. Since $\ell(z) = \ell(ez)$, Cauchy–Schwarz gives the first lower bound.

The integral orbit bases are orthogonal because their supports are disjoint, and their entries on those supports have modulus one. Thus $b_i \otimes \bar{b}'_j / \sqrt{w_i w'_j}$ form another orthonormal basis. Evaluate the same squared norm on it and bound every unnormalized value by their maximum. Orthogonal projection leaves the value of the functional unchanged by Theorem 59.14. This proves the second bound.

If a row form is itself a signed sum with its stipulated level character, the argument applies to that sum as a single input. The coefficients $c_{\pi ab}$ are then computed after taking the signs. To apply the lower bounds to the desired arithmetic comparison one still needs the size of an actual nonzero $\ell(z)$ and the normalization of the iterated integral map. □

66.4 Unfolding the Bessel period on a full frame

Keep $S = C \perp \mathbb{H}^r$, $T = (D \perp L_n) \perp \mathbb{H}^r$, $X = X_*$, and the line conventions of Proposition 61.4. Write N_S, N_T for the unipotent radicals of the ordered isotropic flags and $G = U(S)$. Use brackets for rational adelic quotients. In the scaled Witt coordinates an element of N_T has an upper unitriangular Levi part v, a block $b : T_0 \to E^r$ in the part fixing the frame, and a skew-Hermitian central coordinate. In that order of factors $n_T = n_B(b) n_Z(z) m(v)$. The Bessel character is the product of $\chi(v)$ and $\psi_E(e_r^t b \zeta_n)$. Write $v(n)$ for the analogous upper block of $n \in N_S$. The scalar χ is as in Proposition 80.1.

Let A_D, B be compact-finite cusp forms on $U(D), G$. Use a horizontal Schwartz test with finite part as in Theorem 68.6, allowing finite linear combinations in its residual variable. Write $\mathcal{E}(g;\Phi)$ for the adelic integral over $U_r(\mathbb{A}_E)$ in the two positions, evaluated at (X, p_*) as in that theorem. On decomposable data this is the product of the real and finite integrals. Take compatible Haar measures, giving the rational additive and unipotent quotients volume one. The real factors of these measures may absorb the constants from the finite conventions of Theorem 68.6.

Proposition 66.4 (An orbital formula for the global Bessel period)**.** *The Bessel period on $U(T)$ of the plain theta lift of B, paired with A_D, is*

$$P_T(\Phi) = \int_{R(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathcal{J}(g; \mathcal{E}(g;\Phi))\, d\dot{g}, \qquad R = N_S \rtimes U(C).$$

Here $\mathcal{J}(g;f)$ is the integral on $[U(D)]$, $[U(C)]$, and $[N_S]$ of

$$A_D(h) B(nkg)\, \overline{\chi(v(n))}\, \Theta^0(\mathcal{W}_n\, \omega^0(h,k) f).$$

In this expression Θ^0 is the rational theta sum on $\mathrm{Hom}(-T_0, C)$, ω^0 is its residual action, and $\mathcal{W}_n$ is the displacement and central phase on the new line of Proposition 61.4. It is the identity on the D-part. The functional is jointly continuous in g and the real Schwartz vector at fixed finite level. In particular, if f is the product of a descending core test and the line test transferred from q, then $\mathcal{J}(1; f)$ is the old Jacobi period with test q, paired with the descent of A_D.

Proof. Start by integrating $A_D(h)B(g)$ times the theta sum at $(n_T h, g)$ over $[U(D)]$, $[G]$, and $[N_T]$, with the conjugate Bessel character. In the first polarization the summation positions are all rational r-tuples x in S. On this fiber a shear in the skew-Hermitian center acts by the character of the Hermitian matrix $x^* h_S x$. Indeed its phase is half the alternating trace pairing of the retained columns with their product by the shear. The pairing of Hermitian and skew-Hermitian matrices obtained by taking the trace after multiplication by ξ_0 is nondegenerate. Diagonal entries are detected by diagonal matrices, and a pair of off-diagonal entries is detected by varying an arbitrary element of E and its negative conjugate in the transposed positions. Integration over the rational central quotient therefore retains exactly $x^* h_S x = 0$. Every retained frame has full rank. At a single marked place the first-position box has independent columns modulo the maximal ideal in normalized lattice bases. The actions on these columns of g and of the upper unitriangular factor preserve rank. The other unipotent factors act on the fiber. A singular rational frame can therefore give no term.

Witt extension over E/F is transitive on the full ordered isotropic frames. It also extends each isometry $X \mapsto Xv$ for $v \in U_r(E)$. The stabilizer up to these transformations is R. Indeed preserving X up to v fixes the ordered flag, acts trivially on its successive one-dimensional quotients, and permits an arbitrary isometry of C in the remaining Levi factor. Thus representatives of the frames are $\gamma^{-1} X v_0$, with $\gamma \in R(F)\backslash G(F)$ and $v_0 \in U_r(E)$, after choosing representatives for the first quotient. The latter coordinate is unique for each such choice because X is injective. Unfolding γ gives $R(F)\backslash G(\mathbb{A}_F)$. For v_0, change $b \mapsto v_0 b$ in the non-central block and conjugate the central matrix correspondingly. These rational substitutions preserve the quotients and their measures, and the last row of v_0 fixes the Bessel character. The remaining argument of the first position is $X v_0 v$. The sum over v_0 and the integral on $[U_r]$ give integration over the entire adelic upper unitriangular group with weight $\overline{\chi(v)}$. The multiplication law in the radical permits this order of integration because the central quotient has already been integrated and the upper group preserves the product additive measure of the block b.

On the fiber of X, the remaining action of b adds Xb to the residual map into the isotropic columns of S. In the second polarization this is a modulation. Write y for the scaled opposite position. Its physical row is obtained by taking $-\xi_0^{-1} y^* h_{T_0}$ in the dual frame. Its character under that displacement is $\psi_E(-\operatorname{tr}(yb))$. In fact pairing the isotropic displacement with the physical opposite position cancels ξ_0 against the conjugate of $-\xi_0^{-1}$, and Fourier evaluation uses the negative of this trace. Integrating in b against the conjugate Bessel character therefore selects $y = -\zeta_n e_r^t$, by ordinary character orthogonality on the rational additive quotient. Its unscaled row is just p_* of Proposition 99.4. The remaining rational sum is Θ^0. This establishes the two-position integral before disintegration. `Development/AbelianVarietyTheory/Definitions/TwistedInnerProducts/WittDecompSymplInclIsotropicInclAnisotropic.lean`, line 268–287.

Disintegrate $R(F)\backslash G(\mathbb{A}_F)$ by its R-orbits. The group $U(C)$ preserves Haar measure on N_S. Indeed on the free block its determinant over F is a power of the norm of a unitary determinant, which equals one. It fixes the central and upper matrix coordinates. Thus R is unimodular and the R-fiber has the product integral on $[N_S]$ and $[U(C)]$. A left translation by $nk \in R$ sends the first position Xu to $Xv(n)^{-1}u$. The change of variable in u supplies $\overline{\chi(v(n))}$. On the opposite

position the last coordinate is fixed. Its residual action is precisely $\mathcal{W}_n\omega^0(1,k)$ by the block and central calculations of Proposition 61.4. The h-action fixes the new line and commutes with this displacement. The split core characters are γ_i on both sides, and the remaining character on $U(C)$ is the line factor computed in that proposition. This gives the displayed integrand. Changing the representative of the outer coset just changes variables on the R-fiber.

We give the convergence details since $U(C)$ can be indefinite. Before unfolding, the absolute rational theta sums are bounded by a polynomial in the heights of g, h on Siegel sets, uniformly on a compact fundamental set for the unipotent quotient. One obtains the bound by summing a Schwartz majorant in integral position coordinates, or after the fixed partial Fourier transformations, as in Theorem 59.14. Rapid decrease of A_D, B then gives absolute convergence for unfolding. Restricting the rational position sum by additive orthogonality preserves this majorant. In unfolding the upper group one unfolds the sum of the absolute values before applying cancellation.

For the integral on the R-fiber the real Schwartz seminorms are uniformly bounded as the input varies in a bounded set of tests and g varies in a compact set. The restrictions $B(nkg)$ decrease rapidly in the chamber variables for $U(C)$, uniformly for n in a compact fundamental set. To verify this, on a reduction set for k take a rational isotropic flag of C adapted to its positive chamber. Wedge the first j flag vectors with the wedge of all r distinguished isotropic vectors of S, obtaining a nonzero rational vector w_j. The element n^{-1} fixes w_j, since it changes the vectors of C only by isotropic columns and acts with determinant one on those columns. In an adelic product norm, $\|(nkg)^{-1}w_j\|$ is consequently bounded above by a constant times the inverse of the product of the first j expanding chamber coordinates, with their field multiplicities. This follows by applying the inverse Siegel decomposition to the wedge. The unipotent factor fixes its flag wedge and the remaining compact factors are bounded. The positive chamber coordinates and hence the height in a Siegel set are bounded by fixed powers of the maximum of these expanding products and one. The same assertion is immediate for a compact chamber. For every rational $\delta \in G(F)$ the product norm of δw_j is bounded below by a positive constant, by applying the product formula to a nonzero rational coordinate. Since $\delta w_j = (\delta nkg)((nkg)^{-1}w_j)$ in the exterior representation, every representative δnkg has height bounded below by a positive power of the expanding product in question, up to a fixed factor. On a Siegel representative of nkg the rapid decrease of B thus dominates every polynomial in the chamber variables of k. Finitely many reduction sets suffice. Together with the rapid decrease of A_D this supplies an integrable majorant for the fiber integral and proves continuity by dominated convergence.

At the identity a core test times the linked line test factors the rational theta sum. The integral in h gives the actual descending form on $U(C)$. The line factor is the conjugate Jacobi theta at n, k by Proposition 61.4. The displayed simple root character supplies its flag character. The remaining integral is the asserted old Jacobi period. □

66.5 Realizing group convolution in the frame integral

Fix one horizontal edge $i \geq 2$ of Theorem 67.5. Write G for the product of its real source groups and R for the product of the real flag stabilizers in Propositions 66.4 and 62.6. Give these groups and $R\backslash G$ the compatible measures of those propositions. Write B for the incoming row, and β for the specified residual real Schwartz vector. It includes the descending core test and the line test of the transfer. Absorb the prescribed finite residual vector into the functional $J_B(g; w)$, so that w is its real variable. Write $\pi(g)B$ for right translation. For $b \in C_c^\infty(G)$ put $\pi(b)B = \int_G b(g)\pi(g)B\,dg$.

Lemma 66.5 (Real test for a group convolution)**.** *Fix a compact set $Q \subset G$. To each b*

supported in Q one can assign a real horizontal Schwartz test Φ_b, linearly in b. The outgoing period on this test and the full finite partition equals

$$c_{i,f}v_{i,f}J_{\pi(b)B}(1;\beta).$$

Here $c_{i,f}, v_{i,f}$ are the original positive base scalar and finite orbit measure. For every integer $h \geq 0$,

$$\|(1+\mathcal{N}_i)^h\Phi_b\|_2 \leq A_{i,h}\|b\|_{C^{2h}(Q)}.$$

The number operator $\mathcal{N}_i$ may be taken in any fixed normalized real oscillator coordinates. The constants depend on those coordinates, Q, the local frame charts, measures and β. They are independent of both cusp forms. On the last edge the same assertion holds in the raw terminal convention, taking the absolute value of the period.

Proof. Write $\mathcal{E}_\Phi(g)$ for the real upper-group integral with evaluation at the opposite position. In the two unfoldings the change $g \mapsto rg$, $r \in R$, acts on this residual vector by an invertible operator $\sigma(r)$. Indeed $r^{-1}X = Xu_r^{-1}$ in the first position. A left change by u_r in the upper-group integral supplies the simple-root character. In the opposite position r fixes the selected row and acts on its fiber by the core oscillator and, on a Bessel edge, by its line displacement. These are precisely the factors in the two orbital formulas. Composition gives $\mathcal{E}_\Phi(rg) = \sigma(r)\mathcal{E}_\Phi(g)$. Every residual vector occurs at the identity by the chart construction of Lemma 83.1 and extension off the opposite position. Thus composition gives a representation on the whole residual space. It is smooth there in the Schwartz topology by the oscillator action. Right translation in the unimodular R-fiber of the unfolded integral gives

$$J_B(rg;\sigma(r)w) = J_B(g;w), \qquad J_B(g;w) = J_{\pi(g)B}(1;w).$$

The second equality follows directly from the factor $B(nkg)$ in each formula.

Prescribe the equivariant section

$$F_b(g) = \int_R b(rg)\sigma(r)^{-1}\beta\, dr.$$

Its support modulo R is contained in the image of Q. Right invariance of dr gives $F_b(rg) = \sigma(r)F_b(g)$. We construct Φ_b with $\mathcal{E}_{\Phi_b} = F_b$. Cover the image of Q by finitely many relatively compact section charts and choose a partition of unity there. In one chart write $s(y)$ for the section and $\tau(y)$ for its partition function. Let $j(w)$ multiply w by a fixed Schwartz function of the opposite position whose value at p_* is one. In the residual variables before evaluation, $\Omega(g)$ denotes the full action after the first polarization, as in Lemma 83.1. On the orbit prescribe the vector-valued function

$$\Phi_b(s(y)^{-1}Xu,\cdot) = \tau(y)\chi(u)\alpha_0(u)\,\Omega(s(y))^{-1}j(F_b(s(y))),$$

where α_0 is a fixed smooth upper-group bump of integral one. The orbit coordinates are smooth and the support in the first position is compact in the full-rank locus. Extend in transverse coordinates by a fixed bump of value one on the orbit, using a finite family of submanifold charts as in that lemma. Sum the resulting chart extensions. Substitution in the upper integral gives $F_b(s(y))$ on each section after summing the partition, and equivariance gives the same equality on G.

For y in the closed chart supports, every relevant r lies in the compact set $R \cap Qs(\operatorname{supp}\tau)^{-1}$. Derivatives of $F_b(s(y))$ of order l therefore cost at most $C_l\|b\|_{C^l(Q)}$ in each fixed residual Schwartz seminorm. This follows by differentiating $b(rs(y))$ under the integral and bounding $\sigma(r)^{-1}\beta$ on that compact set. Take the C^l norms using a fixed left-invariant frame and

derivatives through order l. Derivatives of the other chart factors and the residual oscillator action are bounded on their compact parameters in the required seminorms. In position coordinates $(1+\mathcal{N}_i)^h$ is a sum of words of length at most $2h$ in positions and derivatives. The first coordinates stay on a fixed compact set and the residual seminorms bound the weighted L^2 norms. A fixed metaplectic coordinate change takes each such word to a combination of words of the same length. This proves the estimate.

Finally $J_B(g;F_b(g)) = \int_R b(rg)J_B(rg;\beta)\,dr$. Integration over the quotient gives $J_{\pi(b)B}(1;\beta)$. The g-integration is now over a compact set and the two orbital formulas provide an integrable majorant on each remaining fiber. Their finite orbital integrations give the stated factor. Lemma 66.10 takes the last test to the raw character by a fixed metaplectic action and a permutation of marked terms with scalars of absolute value one. It preserves the bound after changing $A_{i,h}$. □

66.6 A finite character on a proper block of the definite core

Let $H = U(C_m)$ have rank $N = k + m(r+6)$, and choose T as in Lemma 72.26. Each $H(F_v)$, $v \mid T$, is $\mathrm{GL}_N(\mathbb{Q}_T)$. Choose the split factor and write its core lattice as $L = O^k \oplus O \oplus O^{N-k-1}$, $O = \mathbb{Z}_T$. Write the coordinates w, y, u in this order. Fix a nonzero residue covector with integral lift ℓ_v. Define

$$a_v(w,y,u) = (w, \ell_v(w) + Ty, u), \qquad z(w,y,u) = (zw, y, u) \quad (z^{T-1} = 1).$$

These scalars form a copy Z_v of $\mathbb{F}_T^\times$. Set $a = (a_v)_{v|T}$, extended by one elsewhere, $K = \prod_{v|T} \mathrm{GL}(L)$, and $Z = \prod_{v|T} Z_v$. For $\lambda = \prod_v \lambda_v$ with every $\lambda_v \neq 1$, write $S_\lambda = \sum_{z\in Z} \overline{\lambda(z)} R(z)$ and $D_\lambda = S_\lambda R(a)$.

Proposition 66.6 (Compression on the definite automorphic quotient)**.** *Let e_K be Haar probability averaging over K. For K-fixed v, w on $[H]$,*

$$\langle R(a)v, D_\lambda w\rangle = \langle v, L_T w\rangle, \qquad \langle D_\lambda v, D_\lambda w\rangle = |Z|\langle v, L_T w\rangle.$$

Here $L_T = \prod_{v|T}(1 - M_v)$ and M_v is convolution on fixed vectors with the probability on the double coset of $\mathrm{diag}(T, T^{-1}, 1, \dots, 1)$*. On every real isotypic subspace nontrivial on at least one real special unitary factor, the spectrum on the fixed vectors satisfies*

$$(1 - c_T)^f I \ \le L_T \le (1 + c_T)^f I, \qquad c_T = \frac{3T-1}{T(T+1)}.$$

Proof. The image $a_v L$ is given by $y = \ell_v(w) \bmod T$. For $z \neq 1$, its translate by z has equation $y = z^{-1}\ell_v(w) \bmod T$. These are distinct hyperplanes modulo T with sum L upon taking inverse images. Their intersection has index T in each lattice. Consequently the elementary divisors of $a_v^{-1} z a_v$ in L have one exponent 1, one exponent -1, and all others zero. Compact compression therefore gives M_v for every $z \neq 1$ and the identity for $z = 1$. The coefficients on $z \neq 1$ sum to -1. Multiplication at distinct places proves $e_K R(a)^* S_\lambda R(a) e_K = L_T e_K$. The identity $S_\lambda^* S_\lambda = |Z| S_\lambda$ gives the second pairing.

We verify the absence of local derived-group invariants needed for the spectral estimate. The special unitary group H^1 of the core is simply connected and absolutely almost simple over F, and $H^1(F_v) = \mathrm{SL}_N(\mathbb{Q}_T)$ is noncompact. Strong approximation gives density of $H^1(F)$ in the adeles of H^1 away from v. V. Platonov, The problem of strong approximation and A. Rapinchuk, Strong approximation for algebraic groups, Theorem 2.3. If a smooth automorphic function is right invariant under $H^1(F_v)$, left rational invariance and this density give invariance under $H^1(F_\infty)$. Indeed for $g \in H(\mathbb{A}_F)$ and $s \in H^1(F_\infty)$, approximate the components

of gsg^{-1} away from v by rational elements of H^1. At v correct each rational element by a local element of $H^1(F_v)$. Conjugating this correction through g_v preserves the derived group. The corrected translates have the value at g and converge to gs. Smoothing proves the same assertion for invariant vectors in L^2, since the local invariant subspace is stable under every translation. An irreducible real type nontrivial on the derived group has zero invariant subspace there. The local matrix coefficient argument of Lemma 74.1 now applies. Its three-coordinate root subgroup proof is valid for $N \geq 3$ in every unitary representation without SL_N-invariants. It gives $\|M_v\| \leq c_T$. Each M_v is self-adjoint since its double coset equals its inverse. The factors at different places commute, which proves the spectral bounds. □

66.7 A projected Fourier coefficient on the original row group

In Theorem 72.27 write $G_0 = U(S_0)$, $H_0 = U(W_0)$, of ranks $r = k + m - 1$ and k. Write $F_0A(h) = \int_{[N_0]} A(nh)\overline{\chi_\varpi(n)}\,dn$. On G_0 use the original terminal graph $\Lambda_{\ell_{0,v}}$ and the trivial compact row character at $v \mid T$. For λ nontrivial at every such v, put $A_\lambda = \sum_{z\in Z} \overline{\lambda(z)} R_0(z)A$. Let $w \geq 0$ be integral and denote the right-hand side of Theorem 67.8 with $w_m = w$ by $\mathcal{D}_w$. Use $N, d_\rho, d_0, a_0, B, L_0$ of Theorem 66.12 and set

$$\mathcal{C}_w = \frac{\sqrt{|Z|d_\rho}\,(1-c_T)^f\,\mathcal{D}_w}{2(1+c_T)^{f/2}N^{d_0}L_0^2B^{6a_0}}.$$

Theorem 66.7 (Descent to the original column rank and terminal graph)**.** *For each unit bottom row of Theorem 67.8 and each such λ, there is a smooth left $G_0(F)$-invariant row A on $G_0(\mathbb{A}_F)$, fixed by all the stated terminal graph stabilizers, such that*

$$|F_0A_\lambda(1)| \geq (1+D_m)^w\mathcal{C}_w.$$

Here D_m is the final Hermite degree of a horizontal lift from that bottom row. In particular the projection to the specified central character of H_0 is nonzero on the original field, frequency and terminal graph. The row A is a restriction of a translated horizontal lift.

Proof. Choose the raw lift B_m and its integral coefficient from Theorem 67.8. The raw period has modulus at least $(1+D_m)^w\mathcal{D}_w$. Choose the coordinates on the core as in Theorem 72.27. Put $B_\lambda = \sum_z \overline{\lambda(z)}R(za)B_m$ on the large row and $P = F_{\chi_+}B_\lambda$ on its definite core. The compression calculation of Theorem 66.12 works directly with χ_+, before its rational change of frequency. In detail, projection of $F_{\chi_+}B_m$ to the original finite level and conjugate real type has norm at least the raw period modulus divided by its column norm. Applying L_T costs at most the lower spectral factor $(1-c_T)^f$, and extraction against the integral list gives a column Q for which

$$|\langle D_\lambda Q, P\rangle| \geq \frac{|Z|(1-c_T)^f(1+D_m)^w\mathcal{D}_w}{\sqrt{h_m}\,N^{d_0}L_0B^{4a_0}}.$$

This is the same two-sided compression identity, since F_{χ_+} commutes with translations on H. The upper frame bounds give $\|Q\| \leq L_0B^{2a_0}/\sqrt{h_md_\rho}$, and Proposition 66.6 gives $\|D_\lambda Q\| \leq \sqrt{|Z|}(1+c_T)^{f/2}\|Q\|$. Cauchy–Schwarz in probability measure shows $\sup_{[H]}|P| \geq 2(1+D_m)^w\mathcal{C}_w$.

The continuous P attains this supremum on the compact quotient. Choose a maximizing representative $g \in H(\mathbb{A}_F)$ whose components at $v \mid T$ are all one. Indeed P is left $H(F)$-invariant by rational conjugation in its integral, and right invariant under an open subgroup at those places by smoothness of B_λ. The weak approximation argument of Theorem 72.27 supplies a single rational left translate putting the components in that open subgroup. Right translation inside it then sets them all to one. The same argument sets $g_v = 1$ at any further prescribed finite set of split places, including the split primes marking the theta tests.

Proposition 68.3 supplies an archimedean $n(x)$ for which the small integral of $B_\lambda(n_0 n(x) g)$ has modulus at least $|P(g)|/2$.

Define $A(y) = B_m(yn(x)ga)$ for $y \in G_0(\mathbb{A}_F)$ using the embedded subgroup. Restriction preserves smoothness and left rational invariance. The terms $n(x)$ and g are trivial at $v \mid T$. At those places the last test of B_m is spherical for the large lattice, so the graph stabilizers of Theorem 72.27 fix A. The scalars z commute with both $n(x)$ and g, being supported at the terminal places. Hence $A_\lambda(y) = B_\lambda(yn(x)g)$. Applying F_0 at one gives exactly the small integral just bounded. Finally $F_0 A_\lambda = |Z| p_{\lambda,H_0} F_0 A$ by right equivariance for H_0, as in Corollary 74.7. □

66.8 A prescribed real projection after restriction

Keep $B_m, \mathcal{C}_w, D_m$ and the common fields of Theorem 66.7. Write $K = H(F_\infty)$ for the large definite group. Let $K_J \subset H_0(F_\infty)$ be the product of the small definite groups at any subset J of the real places, with identity elsewhere. Let ρ be the real type of $D_\lambda Q$ in the proof of that theorem. For a type ν of K_J, let m_ν be its multiplicity in $\rho|_{K_J}$ and $d_\nu = \dim \nu$. Put $q_\nu = m_\nu d_\nu / d_\rho$. All compact group measures in this subsection are probabilities.

Theorem 66.8 (Extraction in an occurring real type)**.** *If $q_\nu > 0$, there is a smooth left rationally invariant row A on the original G_0, of real K_J-type ν and fixed by the original terminal graph stabilizers, with*

$$|F_0 A_\lambda(1)| \;\ge\; \tfrac{1}{2}\sqrt{q_\nu}\,(1 + D_m)^w \mathcal{C}_w.$$

For $w = \lceil d_{\mathrm{osc}}/2 \rceil$, its coefficient on $H_0(F_\infty)$ at the identity finite component has L^2 norm at least $\sqrt{q_\nu}\mathcal{C}_w/2$. The construction permits compact real projections of the lift and of its restriction.

Proof. Apply the orthogonal real projector e_ρ for K to B_m, obtaining $\widetilde{B}$. This is convolution with $d_\rho \overline{\chi_\rho}$ and preserves the final Hermite degree. It commutes with the terminal finite operators and with the large Fourier map, which is H-equivariant. Write $P = F_{\chi_+} D_{\lambda,G} \widetilde{B}$. In the proof of Theorem 66.7, the test $D_\lambda Q$ already has type ρ. The pairing with it is unchanged, and dividing its lower bound by $\|D_\lambda Q\|$ gives $\|P\|_{L^2([H])} \ge 2(1 + D_m)^w \mathcal{C}_w$.

At a finite level fixing P, the compact quotient is a finite union of components $\Gamma_c \backslash K$ for finite Γ_c. On normalized measures its squared norm is a weighted mean of the squared Haar norms of the functions $p_c(s) = P(s, c)$ on K. This follows by invariance of the conditional measure on each compact orbit. Choose a finite representative c with $\|p_c\|_2 \ge \|P\|_2$. Its function $p = p_c$ is a sum of matrix coefficients of ρ by Peter–Weyl. For $b, s \in K$ set

$$I(b, s) = d_\nu \int_{K_J} \overline{\chi_\nu(h)}\, p(bhs)\, dh.$$

Then $\int_{K^2} |I(b,s)|^2 \, db\, ds = q_\nu \|p\|_2^2$. Indeed write $p(s) = \mathrm{tr}(M\rho(s))$. The integral inserts the orthogonal projection P_ν onto the ν-isotypic space between $\rho(b)$ and $\rho(s)$. Schur orthogonality in s gives $d_\rho^{-1} \|M\rho(b) P_\nu\|_{\mathrm{HS}}^2$. The average of $\rho(b) P_\nu \rho(b)^*$ is scalar, and its trace makes that scalar q_ν. Another application of orthogonality gives the identity. Thus some b, s give modulus at least $\sqrt{q_\nu}\|p\|_2$.

We realize the left rotation b while restoring the terminal components to one. Choose $\gamma \in H(F)$ with γ_∞^{-1} arbitrarily close to b and $\gamma_v c_v$ inside the right level of P at each $v \mid T$. For weak approximation use the Cayley chart in Theorem 72.27. At a real compact factor the matrices with no eigenvalue -1 are dense, as is seen by a small scalar rotation. At the split finite places the same chart is dense. Approximation in its rational vector-space coordinates

gives both requirements together. Take $g_\infty = s$, $g_f = \gamma_f c$ and set $g_v = 1$ at $v \mid T$ by right level invariance. For $h \in K_J$, rational invariance gives $P(hg) = p(\gamma_\infty^{-1} hs)$. By continuity the integral of this function against $d_\nu \overline{\chi_\nu(h)}$ has modulus at least $\sqrt{q_\nu}\|P\|_2/2$.

Apply Proposition 68.3 at each hg and integrate in h. The integrands are uniformly bounded since $[N]$ and K_J are compact and g is fixed. Indeed all factors $n_0 n(x)$ inside the row belong to the large unipotent group. Dominated convergence applies. Every h commutes with the real $n(x)$ in that proposition. The triangle inequality supplies an x for which the character integral of the small Fourier coefficients has modulus at least $\sqrt{q_\nu}\|P\|_2/4$. Start with $y \mapsto \widetilde{B}(yn(x)ga)$ on G_0. Project its right translates by K_J to ν. The resulting A is smooth, retains the terminal graph invariance and gives exactly the selected expression after the finite character sum and F_0, as these operations commute with the compact real projector. This proves the pointwise estimate. The orbit under $H_0(F_\infty)$ still spans at most $(1 + D_m)^{d_{\mathrm{osc}}}$ dimensions. The compact projections preserve the real degree space, and the last projection is inside the existing finite orbit span on the small group. The evaluation bound of Corollary 66.13 gives the norm estimate. □

66.9 Unfolding the first nontrivial Fourier–Jacobi period

Keep $S = C \perp Ez$, $T = D \perp (Ee + Ee')$ and the oscillator of Proposition 62.5. Put $G = U(S)$, $H = U(C)$, $U = U(D)$. Let N be the unipotent radical of the stabilizer of Ee in $U(T)$, and denote rational adelic quotients by brackets. In the mixed position $x = X(e)$ write a Schwartz test as $a(x)\beta$, where β belongs to the residual oscillator on $\mathrm{Hom}(-D, S)$. On this test the action of $g \in G(\mathbb{A}_F)$ has the form $a(g^{-1}x)\Omega(g)\beta$. This formula defines the residual action Ω with the splitting factor of the actual kernel. It follows by conjugating position translations and dual modulations, since G stabilizes both hyperbolic summands and preserves the measure of the position space.

Write $\Theta_q(n, k)$ for the theta series on the line part $\mathrm{Hom}(-D, Ez)$ with Schwartz test q, evaluated at $n \in N(\mathbb{A}_F)$, $k \in U(\mathbb{A}_F)$. Here the Heisenberg action is the one on the fiber $x = z$. On U the line uses the splitting attached to the odd line η^{-1}. In the Hermitian direct sum the residual theta uses the core kernel on C and this line kernel on Ez. Give the compact unipotent quotient $[N]$ Haar probability and use dual adelic measures on Heisenberg position and frequency spaces, with rational additive quotients of volume one.

Proposition 66.9 (The orbit formula at the prescribed character)**.** *Let B on $[G]$ and A on $[U]$ be smooth compact-finite cusp forms. Write $\beta = b \otimes q$ in the direct sum, with $q \neq 0$. The Fourier–Jacobi period of the plain horizontal lift is*

$$\begin{aligned} P(a, \beta) &= \int_{[U]} A(k) \int_{[N]} \overline{\Theta_q(n, k)} \int_{[G]} K_{T,S}(nk, g; a \otimes \beta)\, B(g)\, dg\, dn\, dk \\ &= \int_{H(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} a(g^{-1}z)\, \mathcal{R}(g)\, d\dot{g}. \end{aligned}$$

The quotient measure disintegrates the two group measures. With b_g the contraction of $\Omega(g)\beta$ against q in the line variable, linear in β, one has

$$\mathcal{R}(g) = \int_{[H]} \int_{[U]} B(hg)\, A(k)\, K_{D,C}(k, h; b_g)\, dk\, dh.$$

The function $\mathcal{R}$ is continuous. At the base point it satisfies

$$\mathcal{R}(1) = \|q\|_2^2 \int_{[H]} B(h) A_2(h)\, dh, \qquad A_2(h) = \int_{[U]} A(k) K_{D,C}(k, h; b)\, dk.$$

Proof. The center of N shears e' by a trace-zero multiple of e. Parameterize it by $c \in \mathbb{A}_F$ so that it acts on the position x by $\psi(ch_S(x,x))$. Indeed a shear by αe, $\bar{\alpha} = -\alpha$, adds αx to the dual column. In the Schrödinger formula its phase is half the alternating cross pairing with x. Thus c is $\mathrm{Tr}_{E/F}(\xi_0\alpha)/2$ up to the common sign of the dual coordinate. This is an invertible F-linear change of the center parameter. The line test has character $\psi(-c)$ in these coordinates. Integrating the product with its conjugate over $F\backslash\mathbb{A}_F$ therefore selects precisely the rational positions x of norm -1. Witt extension over E/F makes these one $G(F)$-orbit, with stabilizer $H(F)$. Unfolding the sum gives the domain $H(F)\backslash G(\mathbb{A}_F)$ with representative position z. The rational frame scalings in Proposition 62.5 apply to both kernels, so the character selected by this calculation is the grid character.

On this fiber the remaining action of N is the full Heisenberg action in the line variable. To check this, modulo the center an element of N is determined by its functional on D taking values in Ee. Under composition with X it adds this functional times z to the residual columns. These columns run through $\mathrm{Hom}(-D, Ez)$; the C-columns are fixed. The commutator on the line is the alternating trace pairing, with the central action just computed. The map is an isomorphism on their vector quotients and on their one-dimensional centers, hence it identifies the two Heisenberg groups over F. The Jacobi action is given by the ordinary Weyl operators. In coordinates with central term one half of the commutator, the noncentral generators are obtained by adding a functional to the D-columns and inserting its Hermitian adjoint in the e'-column; the quadratic correction in that column is the usual half self-pairing. This gives the translation and modulation on the line with the same phase in the two kernels.

On the subgroup $H \times U$ the residual oscillator factors as the core oscillator and the line oscillator just specified. The symplectic direct sum gives this factorization up to a character of the two groups. At a split place the scalar on the chosen components for $h \in H,\ k \in U$ is

$$\frac{\gamma_2(\det h)}{(\gamma_2\eta^{-1})(\det k)} = \frac{\gamma_2(\det h)}{\gamma_2(\det k)}\,\eta(\det k).$$

The first factor on the right is the core splitting and the second the line splitting. Hyperbolic substitution in the x-variable has Jacobian one. Rational changes of polarization preserve the adelic theta functional by Poisson summation, so the character ratio is trivial on rational points. At almost all nonsplit finite places the integral models match on the two hyperspecial subgroups. A character trivial there is trivial on the whole local group by the Cartan and hyperbolic-plane argument of Proposition 61.9. The remaining local factors are trivial by weak approximation for these unitary groups, as in that proposition. This proves the factorization on the adelic groups, including its phases.

For line tests v, q, integration of the two Heisenberg theta series on $[N]$ gives their inner product $\langle v, q\rangle$. In rationally dual position and frequency coordinates, integration over the frequency quotient makes the two rational summation indices equal. The sum of the resulting translated position integrals over a fundamental domain is the full position integral of $v\bar{q}$. The common central phase cancels and the center quotient has volume one. This computation applies for every k, since its line action is unitary on both tests. It also applies to a general residual Schwartz vector by integration in its line variable, or by Schwartz-continuous extension from finite sums of tensors. The resulting core test is b_g. Disintegrating $H(F)\backslash G(\mathbb{A}_F)$ over $H(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ now gives the formula. Equivariance makes the inner expression invariant under left translation of g by $H(\mathbb{A}_F)$.

Here are convergence details for the two indefinite groups. Uniformly on a compact fundamental set of N, each absolute theta sum grows at most polynomially in the heights on Siegel sets for U and G. This follows by bounding the rational lattice sum by finitely many real Schwartz seminorms after fixing the finite supports, as in Theorem 59.14. It also majorizes

the sum restricted to the norm shell by using the absolute sum before the central integration. Rapid decrease of A, B therefore justifies unfolding and Fubini. For continuity of the inner expression, compact variation of g keeps b_g in a bounded set of Schwartz tests. The restrictions $B(hg)$ decrease rapidly on Siegel sets of H, uniformly in such g. Indeed extend an isotropic flag defining one of these sets in C to a full rational isotropic flag in S. The split torus coordinates embed with additional entries one after the original positive entries. The original lower bounds place them in a Siegel chamber of G with a fixed lower bound. The bounded unipotent coordinates embed in the larger parabolic. Bounded anisotropic Levi coordinates commuting with the split torus can be moved into the right compact set. Thus each of the finitely many reduction sets lands in a Siegel set of G with a fixed compact factor, also after the indicated translations. Rapid decrease there supplies an integrable majorant for any polynomial growth of the core kernel. Dominated convergence proves continuity. Finally $b_1 = \|q\|_2^2 b$, which proves the value at the identity. S. Miller and W. Schmid, Theorem A and Theorem 3.1. For the corresponding norm-orbit unfolding in the symplectic theta correspondence see H. Xue, Lemma 8.3.1. □

66.10 The Bessel character in the terminal coordinates

Let $m = 2s + 2 \geq 4$, so $S_m = V_{m,0}$ is the direct sum of C_m and a staircase of length $2s + 1$. Denote the first s unscaled staircase columns by b_j and the middle column by z. Let N be the flag unipotent radical and $H = U(C_m)$, embedded by fixing all staircase columns. For $n \in N$ write $c_j(n)$ for the coefficient of b_j in $(n-1)b_{j+1}$, $j < s$. Write $c_s(n)$ for the coefficient of b_s in $(n-1)z$. The character in the supplied terminal Bessel datum and its bilinear period are

$$\chi_+(n) = \psi_E\left(\xi_0 \sum_{j=1}^{s} c_j(n)\right),$$
$$\mathcal{B}^+(e, B) = \int_{[H]} e(h) \int_{[N]} B(nh)\overline{\chi_+(n)}\, dn\, dh.$$

These formulas read off the isotropic columns, the middle column and the coefficients of the given datum in the raw Hermitian extension. In a convention conjugating the column form one takes that form to be $\bar{e}$. `Development/AbelianVarietyTheory/Definitions/Divisors/CmHostImaginaryGeneratorNeZero.lean`, line 189–218. `Development/AbelianVarietyTheory/Definitions/Divisors/BplusOfEquivariant.lean`, line 269–291.

Lemma 66.10 (Passage to the prescribed terminal character)**.** *A nonzero terminal Bessel period in the scaled frames of Theorem 66.16 yields a nonzero $\mathcal{B}^+(e, B')$ with the same e and the same earlier horizontal tests. The last horizontal test of B' can have one original marked coset at each marked place and one product of real Hermite functions.*

Proof. Scaling b_j by d^j preserves its flag. In the scaled coordinates the coefficient in the j-th simple position is $d^{j+1}d^{-j}c_j = dc_j$. The last vector is $d^{s+1}z$, so the same ratio applies to the last coefficient in $d^s b_s$. Thus the scaled character is $\psi_E(d\sum_j c_j) = \overline{\chi_+(n)}$. The embedded core fixes both the raw and the scaled columns, and is exactly H.

Define a rational unitary involution δ by multiplying b_j and its opposite column by $(-1)^{j-1}$, multiplying z by $(-1)^s$, and fixing C_m. Opposite columns still pair to one. Conjugating n by δ negates each $c_j(n)$, including $c_s(n)$. The involution preserves the rational unipotent quotient and its Haar probability and commutes with H. For the right translate $B'(g) = B(g\delta)$, rational left invariance gives $B'(nh) = B(\delta^{-1}n\delta\, h)$. Changing variables in the displayed integral therefore identifies it with the scaled-character period of e, B, with the same measure on H.

If the probability on the compact terminal core is used instead, both periods acquire the same positive normalizing factor.

Right translating a theta lift by δ acts on its last Schwartz function by the Weil action on S_m, by the defining theta kernel. At a marked split place this is integral matrix substitution times a unit-modulus determinant scalar. The substitution preserves the original matrix lattice and its depth because each staircase summand is preserved. It therefore sends a single coset to a single coset. At the unmarked places δ fixes the original vacuum by Proposition 66.15. The scalars can be put in the real test. At infinity the Weil action gives a Schwartz function. Its terminal period with the chosen finite data is continuous in this function, by Proposition 66.4 and the same rational conjugation. Expansion in the original Hermite basis selects a single product with nonzero period. This affects only the last edge. □

66.11 Weighted selection of an individual Hermite test

Fix a field and an edge $i \geq 2$ as in Lemma 66.5. Let d_i be the total real position dimension of its horizontal oscillator, and $g_i = \dim_{\mathbb{R}} G_i(\mathbb{R} \otimes_{\mathbb{Q}} F)$, where $G_i = U(V_{i,0})$ is regarded here as a group over F. Write T_i for the number of terms in the finite partition on the edge. Its coefficients have absolute value one. Put $t_i = 1$ on an odd-index edge and $t_i = \|q\|_2^2$ on an even-index edge, with the full line norm and measures of Proposition 62.6. Then $J_B(1;\beta) = t_i P_i$, where P_i is the specified incoming period.

At each intermediate edge fix the polynomial Fock realization for compatible maximal compact subgroups of the row pair. Such coordinates can be fixed before the input forms are chosen. Indeed the two compact groups have compact product image in the real symplectic group. Average a positive inner product over this product. The skew operator for the symplectic form in that metric, divided by the positive square root of minus its square, is an invariant compatible complex structure, with the sign chosen for positivity. An orthonormal complex basis gives normalized monomials, or equivalently product Hermites in its Schrödinger coordinates. Use the same maximal compact of a row on its two incident edges. The underlying Schwartz space and its seminorms transform by a fixed metaplectic map when changing coordinates. At the very last edge the original terminal Hermite coordinates may be used for the expansion.

Proposition 66.11 (A weighted one-edge period bound)**.** *Suppose $i \geq 3$ and the preceding test is a normalized product Hermite of total degree D in the compatible coordinates. For each integer $w \geq 0$ there is a constant $C_i(w) > 0$ depending only on the local real data and the specified row type. If $P_i \neq 0$, one original finite choice μ and one normalized product Hermite h_ν on this edge satisfy*

$$\frac{|P_{i+1}(\mu, h_\nu)|}{(1+|\nu|)^w} \;\geq\; \frac{c_{i,f} v_{i,f} t_i}{2 T_i C_i(w)} \frac{|P_i|}{(1+D)^{E_i(w)}}, \qquad E_i(w) = d_{i-1} + 4(w + d_i + 1) + 4g_i + 4.$$

For $i = 2$ take any finite list of compact types containing the second row. With M_2, H_2 of Proposition 84.7, the same inequality holds with the degree factor $(1+D)^{E_i(w)}$ replaced by $M_2 H_2^{4(w+d_2+1)+4g_2+4}$.

Proof. For $i \geq 3$ the compact orbit of the preceding test is contained in the Fock polynomials of degrees at most D. The action of a unitary symplectic change of basis on this space is substitution on the variables times its action on the vacuum line. On each degree its differential is the sum of the infinitesimal actions on at most D tensor factors and the fixed scalar action on the vacuum. Thus each fixed compact generator has operator norm at most $C_i(1+D)$. The dimension of this space is $\binom{D+d_{i-1}}{d_{i-1}}$. The theta map with the preceding row fixed intertwines the outgoing compact action on its test with right translation on its output. Therefore the compact

types of the incoming form on edge i occur in this finite-dimensional space. Their distinct dimensions sum to at most its dimension, and $C_K(\tau) \le C_i'(1+D)^2$ on them, by summing the squared generator norms. Theorem 61.1 fixes the discrete series at every real place. Thus the scalar for Ω_G on all these forms is fixed even when there are several automorphic copies. The parameters of Proposition 84.7 satisfy

$$M \le C_i''(1+D)^{d_{i-1}}, \qquad H \le C_i'''(1+D).$$

At the second row its finite type list instead gives M_2, H_2 directly.

Take b from that proposition with $\pi(b)B = B$, and let $a_\nu = \langle \Phi_b, h_\nu \rangle$ be the coefficients of the test in Lemma 66.5. Use the raw transform before expansion on the last edge. Set $j = w + d_i + 1$. Cauchy–Schwarz and Parseval give

$$\sum_{\nu} |a_\nu|(1+|\nu|)^w \le \left(\sum_{\nu} (1+|\nu|)^{-2(d_i+1)} \right)^{1/2} \|(1+\mathcal{N}_i)^j \Phi_b\|_2.$$

At degree l there are $\binom{l+d_i-1}{d_i-1}$ indices, so the numerical series converges. By Lemma 66.5 the last norm is at most $A_{i,j}\|b\|_{C^{2j}}$. The compact convolution estimate now bounds the whole weighted sum by $C_i(w)(1+D)^{E_i(w)}$. At $i=2$ it gives $C_2(w)M_2H_2^{4j+4g_2+4}$. All the constants just used involve the charts, fixed residual tests, coordinates, measures, group and representation parameters.

The period evaluated on the full finite partition and the real test has modulus $c_{i,f}v_{i,f}t_i|P_i|$. Its expansion in the individual $P_{i+1}(\mu, h_\nu)$ is absolutely convergent. In fact the period has finite Schwartz order by Proposition 84.6, the term on h_ν grows at most polynomially in $|\nu|$, and the test coefficients have summable moments of every order. The same continuity holds after a fixed coordinate change. If every weighted period were smaller than the stated right-hand side, the triangle inequality and the weighted coefficient sum over the T_i finite choices would give at most $c_{i,f}v_{i,f}t_i|P_i|/2$. This contradiction proves the assertion. A selected nonzero form on an intermediate edge is compact-finite and cuspidal by Theorem 61.1. □

66.12 A character-projected period at the larger terminal rank

Use the bottom vector and data of Theorem 67.8 and take $w_m = 0$. Denote its positive lower bound by $\mathcal{D}$. Use T, a, t, and Z from Lemma 72.26 and Proposition 66.6. On $G = U(S_m)$ use the same translations $D_{\lambda,G}$ as D_λ, via the inclusion of $H = U(C_m)$. Define $\mathcal{P}_\varpi(B,Q) = \langle Q, F_{\chi_\varpi} B \rangle_{[H]}$ with inner product conjugate-linear in Q. Put

$$\mathcal{D}' = \frac{(1-c_T)^f \, \mathcal{D}}{\sqrt{h_m}\, N^{d_0} L_0 B^{4a_0}},$$

where B in this denominator is the lattice constant of Lemma 89.2.

Theorem 66.12 (Projected terminal Bessel periods on a proper scalar block)**.** *There is a single horizontal lift B_m from the prescribed bottom row and a single column Q from the conjugate list in Proposition 68.2 such that simultaneously for every character λ nontrivial at all $v \mid T$,*

$$\left|\mathcal{P}_\varpi(D_{\lambda,G}R(t)B_m,\, D_\lambda Q)\right| \;\ge\; |Z|\,\mathcal{D}'.$$

With the integral column $T^{a_0}D_\lambda Q$ the lower bound is multiplied by T^{a_0}. With orthogonal character projections on both translated arguments the lower bound is $\mathcal{D}'/|Z|$ before this clearing of denominators.

Proof. Theorem 67.8 gives B_m and a column e_0 with $|\int_{[H]} e_0 F_{\chi_+} B_m| \geq \mathcal{D}$. The two upper frame bounds in Proposition 62.7 bound each original vector and covector norm by the square root of its upper frame bound. The class norm formula gives $\|e_0\| \leq L_0 B^{2a_0}/\sqrt{h_m d_\rho}$. Let Y be the orthogonal projection of $F_{\chi_+} B_m$ to the real type of $\bar{e}_0$ and the full original finite core level. This function has finite norm by smoothness and compactness of the definite and unipotent quotients. Since the projected space contains $\bar{e}_0$, $\|Y\| \geq \mathcal{D}/\|e_0\|$.

The real type is nontrivial on the special unitary group. At the terminal real place the parameter of Theorem 59.16 contains the cloud entries $v_j - 1/2$. Consecutive cloud entries have difference $4\Delta > 1$, with $k \geq 3$, and every added string lies above the cloud by the core gap. Subtracting the ordered half sum of roots leaves an unequal pair of highest weights. The conjugate type has the same nontriviality. Hence $\|L_T Y\| \geq (1 - c_T)^f \|Y\|$ by Proposition 66.6. Apply Proposition 68.2 to $L_T Y$. Substituting the bound for $\|e_0\|$ gives one Q with $|\langle Q, L_T Y\rangle| \geq \mathcal{D}'$.

The rational change of flag gives $F_{\chi_\varpi} R(t) B_m = F_{\chi_+} B_m$, as functions before real projection. Right translation of either row by an element of H passes inside its unipotent integral as right translation of the resulting function. Thus the new period pairs $D_\lambda Q$ with $D_\lambda F_{\chi_+} B_m$. By Lemma 72.26 the latter input before translation is K-fixed. Compression between K-fixed inputs gives $|Z|\langle Q, L_T F_{\chi_+} B_m\rangle$ by Proposition 66.6. The real projector and the finite-level projectors away from T commute with L_T. They fix Q, so this last pairing equals $|Z|\langle Q, L_T Y\rangle$. In particular Q works for all the characters. Dividing each sum S_λ by $|Z|$ gives its orthogonal projection. The column integrality follows from Proposition 68.2. On the row the finite sums act on the last theta test by equivariance. The rational element t also translates its real factor. □

Here the ambient and column ranks are $N + m - 1, N$, with $N = k + m(r + 6)$. The graph mixes a k-dimensional block with one further direction *in the definite core*. In Corollary 74.7 the ranks are $k + m - 1, k$, and the graph uses the negative middle line of the ambient space. There the scalars are central in the entire local column group. Translations within that group preserve every central character, since they commute with each scalar. In particular they preserve a zero projection to a prescribed central character. A comparison with the period on that graph and its integral row normalization is still required.

66.13 A lower bound on the original definite real group

Keep the original ranks, frequency and terminal graph of Theorem 66.7. Use on the last oscillator the Hermite coordinates for a positive metric compatible with the compact core $H(F_\infty)$. Let d_{osc} be the total number of real position variables across the real places, and put $w = \lceil d_{\mathrm{osc}}/2 \rceil$. Denote by $H_{0,\infty}$ the product of the definite real column groups and give it Haar probability.

Corollary 66.13 (A definite real norm with the prescribed finite character)**.** *For the row provided by Theorem 66.7,*

$$\left(\int_{H_{0,\infty}} |F_0 A_\lambda(h_\infty, 1_f)|^2 \, dh_\infty \right)^{1/2} \geq \mathcal{C}_w.$$

The right side is independent of the chosen final Hermite index and of the real translation in the unipotent averaging. On the compact adelic column quotient $F_0 A_\lambda$ has a nonzero pairing with an integral column in some real type and the prescribed central character, at a sufficiently fine finite level.

Proof. In signed orthonormal coordinates the compact group of a definite summand preserves the positive metric on the oscillator and its compatible complex structure. Its Weil action preserves total Hermite energy. Equivalently, the compact group acts linearly on creation operators, and acts by a scalar on the vacuum. The span of the real compact translates of the final test is consequently of dimension at most $(1 + D_m)^{d_{\mathrm{osc}}}$, by counting multi-indices of degree at most D_m. Theta equivariance gives the same bound for the $H(F_\infty)$-orbit span of B_m. Finite translations and their sum in B_λ commute with this real group. The element $n(x)$ acts only between the extra space D and the flag spaces and fixes W_0. It commutes with $H_{0,\infty}$. Also $g_\infty \in H(F_\infty)$. Thus a right real column translation k of $B_\lambda(yn(x)g)$ is obtained from the translation of B_λ by $g_\infty^{-1} k g_\infty$ on the large row. Taking the small Fourier coefficient and evaluating on the real column group are linear. The same dimension bound holds for the right translates of the function inside the displayed norm.

In a right-translation-invariant finite-dimensional space of continuous functions on a compact group, the sum of the squares of an orthonormal basis at a point is constant. A translation changes the basis unitarily, and integration shows the constant is the dimension for Haar probability. Cauchy–Schwarz therefore bounds the absolute value at the identity by the square root of that dimension times the norm. Apply this to the preceding span and use Theorem 66.7. The choice of w absorbs the final Hermite degree and gives the assertion.

The nonzero real restriction lies on a finite adelic neighborhood of positive measure, by the finite smooth level of the row. The full function on $[H_0]$ is compact-finite at infinity by the same orbit-span argument. Take a finite level fixing it and refine inside a positive principal congruence group at an odd split prime. As in Proposition 73.4, the class stabilizers are trivial since they are finite and this congruence group is torsion-free. Each occurring compact unitary type has integral highest weights. Its Schur and determinant model and the dual admit integral lattices spanning their spaces, so their matrix coefficients span the type on each class. Some class coefficient Q thus pairs nontrivially with the function. The sum $\sum_{z \in Z} \overline{\lambda(z)} R(z) Q$ multiplies this pairing by $|Z|$, because the function has character λ. One can use a common finite refinement under the translates. The scalars preserve the standard column lattices at T, so pulling back integral matrix coefficients uses no denominator in the fixed-right-argument convention of Proposition 68.2. This proves existence at that level. □

The norm just estimated uses the real group on one finite component. The lower estimate in Corollary 74.7 requires a prescribed global cuspidal row, a prescribed real column type, the original levels away from T, and a comparison in the specified integral normalization. The restriction constructed here proves the original terminal graph and frequency conditions. Its projection to those global spaces and the required quantitative comparison remain open.

66.14 Passage from a Bessel period to a Jacobi period

Use $i = 2r + 2$, $\ell = r + 1$, and the groups and measures of Proposition 62.6. Write $s = \dim_E S$ and $t = \dim_E D$. Write $c_{\ell,f}$ and v_f for the base scalar and compact-orbit measure of Theorem 68.6 at this edge.

Theorem 66.14 (Transfer to the next Fourier–Jacobi period)**.** *Fix a nonzero Bessel period P_i of a compact-finite cusp form B_i at the prescribed level against the specified descent $A_i = \mathrm{desc}_b A_{i+1}$. There is a fixed distinguished line test q_ℓ, chosen before the two forms, and one horizontal test at the original depth with one original coset at each marked place, such that*

$$|P_{i+1}(q_\ell)| \;\geq\; \frac{c_{\ell,f} v_f \, \|q_\ell\|_2^2}{2(N\mathfrak{m})^{\ell s + rt}} \, |P_i|.$$

The real cutoff inducing this estimate has quotient integral one. For nonvanishing the horizontal real test may be a single product Hermite function, with the same q_ℓ.

Proof. The middle ideal in S is $I_\ell^{(2r+1)} = \mathfrak{l}^\ell$ in the notation of Proposition 99.4. The undeepened line summand of the residual map lattice from D to $S_0 = C \perp L$ is therefore $\mathfrak{Al}^\ell \operatorname{Hom}(\Lambda_D, \mathcal{O}_E z)$. The Jacobi line rule multiplies this by $\mathfrak{m}/\bar{\mathfrak{m}}$, as in Proposition 61.4. At an unmarked place take its ordinary vacuum. At a marked place its vacuum is the indicator of the zero coset of depth a_v in the original line position lattice. Indeed the position lattice acquires the factor π^{a_v} and the opposite lattice the dual factor π^{-a_v}. Translation and modulation then give exactly this invariant line. Take a fixed nonzero Gaussian at every real place. These choices define q_ℓ and have positive finite L^2-norm.

Use $b \otimes q_\ell$ as residual test. Its finite part has one original marked coset, by the direct sum $C \perp L$. For the remaining finite variables use the two position tests of Proposition 99.4. Their integrated value at the identity is $c_{\ell,f}(b \otimes q_\ell)_f$, and their finite support projects to the orbit of measure v_f. Represent this orbit by the level group. Right invariance of the test and of B_i makes the finite part constant there after disintegration, as in Theorem 66.16.

At infinity choose a residual Schwartz test, in both the opposite position and the residual oscillator, whose value at the zero opposite position is $(b \otimes q_\ell)_\infty$. On real local sections use the continuous function obtained by acting on this test and inserting its evaluated value in J_{q_ℓ} of Proposition 62.6. At the base point the function has value $\|q_\ell\|_2^2 P_i$. Choose a product neighborhood throughout which the difference from this value is less than half its modulus. Lemma 83.1 realizes a nonnegative quotient bump of integral one supported in this neighborhood. Its upper-group integral has the phase required by the flag character. The orbital formula now gives a new period of modulus at least $c_{\ell,f} v_f \|q_\ell\|_2^2 |P_i|/2$.

The marked expansion has exactly $(N\mathfrak{m})^{\ell s + rt}$ terms with coefficients of absolute value one. The line is already a single original coset at each of those places. The triangle inequality gives the assertion for an individual term. Its period is a continuous functional of the horizontal real test by Proposition 62.6. The Hermite expansion and cuspidality argument of Theorem 66.16 now give a compact-finite lift with nonzero Jacobi period. As there, the numerical lower bound uses the localized smooth test. □

66.15 Local integrals for horizontal pairs of opposite parity

Use the original pair $V_{i+1,0}, V_{i,0}$ on a field of Theorem 59.10. Its ranks are $N = u+s$, $u = r+is$, where $s = r+6$ is odd. Attach $\gamma_i = \chi_{m-i}$ to the even space and $\gamma_i \eta^{-1}$ to the odd space. Use the character of Proposition 59.18, or any parity character with the same infinite type, unramified off the marked set S_0 and trivial on its prescribed principal unit groups. Write Λ_N, Λ_u for the actual Hermitian lattices. These include the fractional ideals in the orthogonal extensions. `Development/AbelianVarietyTheory/Definitions/Divisors/BasePlaceNotMemChiGridBadPlaces.lean`, line 168–188,401–439.

Proposition 66.15 (The horizontal finite tests at the original level)**.** *On every horizontal edge of the bottom row there is a fixed product of finite tests satisfying the original depth equations. At each marked place its factor is one integral coset indicator, and its local doubling form on vectors fixed by $K_u(a_v)$ is $D_{i,v}$ times the source inner product, where*

$$D_{i,v} = (Nv)^{-a_v(uN+u^2)} \prod_{h=1}^{u} \bigl(1 - (Nv)^{-h}\bigr)^{-1}.$$

At the other finite places its normalized doubling section gives the spherical integral. The tests are invariant under the given principal congruence groups on both sides at marked places and the two lattice stabilizers elsewhere.

Proof. The full map lattice $\mathfrak{A}\operatorname{Hom}(\Lambda_N, \Lambda_u)$ is self-dual for the trace pairing by Proposition 100.7. At a marked place its two halves are dual rectangular matrix lattices. Take the indicator of $(I_u \quad 0) + \varpi_v^{a_v} \operatorname{Mat}_{u,N}(\mathcal{O}_v)$ in lattice bases on the chosen half. The marked characters have exponents at most a_v, by the level of γ_i and the choice of η. Their determinant factors are therefore one on both congruence groups. Left translation by h gives overlap with this rectangle precisely when $h \in K_u(a_v)$. Indeed the first u columns of both an intersecting matrix and its translate are congruent to I_u, so their quotient is in $K_u(a_v)$. In that subgroup the entire coset is preserved. The integral is the additive volume $(Nv)^{-a_v u N}$ times the group volume of $K_u(a_v)$. Counting invertible matrices modulo $\varpi_v^{a_v}$ gives the stated product, with the measures of Theorem 80.7. The same theorem's translation and modulation calculation proves the depth equations. Projectivity and the Chinese remainder theorem lift all the chosen marked matrix residues at once, including for the fractional lattices.

At an unmarked split place the position indicator is that of the integral rectangle in the two lattice bases, dilated by the local generator of $\mathfrak{A}$ in the chosen component. Dilation commutes with both substitutions and their determinant characters. Its normalized coefficient is thus the usual split unramified coefficient, also at the dyadic places.

At a nonsplit v the extension is unramified, the residue characteristic is odd, and the change of variables in Proposition 100.7 gives a trace-zero unit and a character of conductor $\mathcal{O}_v$ for the original tensor lattice. Both Hermitian lattices split integrally into hyperbolic planes and, in odd rank, one unit line. The proof is successive lifting of isotropic residue vectors and their partners as in Proposition 100.8. Polarize the even space by its two integral isotropic halves. In the resulting position space take the lattice indicator and give its lattice volume one. Translations and dual modulations fix it. On the odd space the group acts by substitution with trivial determinant twist, since the character attached to the even space is unramified with trivial restriction to $F_v^{\times}$, and a uniformizer is common to the two fields.

On the even space the integral Siegel Levi preserves the indicator by change of basis. Its character is trivial on unit determinants. Integral unipotents multiply it by a phase of an integral quadratic form on its support. A Weyl element interchanging isotropic coordinates acts by partial Fourier transform with unramified unit factors. The Fourier transform preserves the corresponding self-dual lattice indicator. In the Weil formula the index of a unimodular quadratic form here is one. Indeed the integral of its quadratic phase over any dual-lattice coset outside the lattice vanishes by orthogonality of its nontrivial linear term, whereas the integral over the lattice is its volume one. This evaluates the stabilizing Gaussian integral defining the index. The other splitting factor on an integral Weyl representative is the character of a unit. These elements generate the lattice stabilizer. Reduction modulo the maximal ideal gives the Siegel Bruhat decomposition over the residue field, whose parabolic factors and partial exchanges lift integrally. An element left congruent to the identity has an invertible diagonal block, so block elimination factors it into integral upper and lower unipotents and a Levi element. This proves invariance for both compact groups with the specified splitting. For the character and Weil-index factors in the Schrödinger splitting see S. Kudla, Theorem 3.1 and S.-Y. Pan, Sections 2.1 and 3.2.

The sign on the integration group can now be checked explicitly. Put $Q = Nw = (Nv)^2$. In a Cartan representative of $U(u)$ write the hyperbolic entries as $\varpi^{t_j}, \varpi^{-t_j}$, $t_j \geq 0$, and set $T = \sum_j t_j$. If N is even, the position space consists of $N/2$ copies of the small space. Substitution on each gives normalized overlap Q^{-T}. If u is even, the small torus acts through

its Siegel Levi on $u/2$ copies of the large space. On the j-th copy dilation by ϖ^{t_j} has normalized overlap $Q^{-Nt_j/2}$, by the square root of the lattice index. Its splitting character on the chosen uniformizer to power t_j is $(-1)^{Nt_j}$ by restriction to $F_v^\times$. Thus in both cases the coefficient is

$$(-1)^{NT} Q^{-NT/2}.$$

In the doubled group the primitive exterior generator of the diagonal u-plane has lattice norm Q^T after applying $(h, 1)^{-1}$. This follows by choosing one column from each coordinate pair in the exterior product on a Cartan representative. The Siegel Levi determinant consequently has valuation T. At the theta point the normalized unramified section evaluates its unramified character on this determinant and raises its absolute value to $s/2 + u/2 = N/2$. This gives the displayed coefficient. Bi-invariance on the small compact group, with the unramified determinant character trivial there, extends the equality to all h. Flat extension multiplies both restrictions by the same power of the Iwasawa height. The unramified doubling calculation therefore applies with exactly the omitted places S_0. □

66.16 Passage from a Jacobi period to a Bessel period

Let $i = 2r + 1$, $r \geq 1$, $i + 1 \leq m$. Use the finite and quotient measures of Theorem 68.6 and Proposition 66.4. Denote by $c_{r,f}$ the positive scalar in the former theorem's base value, and by v_f its support probability on the product of compact quotient orbits. Put $s = \dim_E S$, $t = \dim_E T_0$, and $s_0 = \dim_E C$.

Theorem 66.16 (Transfer to the next Bessel period)**.** *Take a compact-finite row cusp form B_i at the prescribed level. Let A_i be the descent of A_{i+1} by the specified core test of Theorem 59.13. Suppose its Jacobi period $P_i(q)$ against B_i is nonzero, with q in the distinguished finite line vacuum and a fixed real Schwartz test. There is a horizontal test with one original integral coset at every marked place, and the original vacuum elsewhere at finite places, whose lift satisfies*

$$|P_{i+1}| \;\geq\; \frac{c_{r,f}\, v_f}{2(N\mathfrak{m})^{\,r(s+t)+s_0}}\, |P_i(q)|.$$

Here P_{i+1} is the Bessel period against A_{i+1} at the grid's distinguished character. The real first-position cutoff induces a bump of quotient integral one by Lemma 83.1. The other real tests use the specified core and line vectors. For nonvanishing a single product of real Hermite functions can be used on the entire horizontal edge.

In particular, for $m \geq 4$ the datum from Theorem 61.6 gives a nonzero fourth Bessel period on the same field, interval, and finite levels, retaining its original terminal integral coefficient and its preceding three horizontal tests when those give a compact-finite third row.

Proof. Form the residual test as the tensor product of the specified descending core test and the conjugate transfer of q from Proposition 61.4. At the finite places take the full new line indicator and put its normalization scalar into the real line test. Denote the resulting residual product by $\beta_f \otimes \beta_\infty$. At infinity choose a Schwartz test ρ_∞ in the second-position and core variables with value β_∞ at p_*. For instance multiply β_∞ by a product of real bumps of value one at the required rows.

At the finite places take the two positions of Proposition 99.4. Apply Theorem 68.6 termwise to the marked partition of the full new line. The coefficient is supported on $R_f K_f$ and its value at 1 is $c_{r,f}\beta_f$. On the image of K_f in the quotient use representatives in K_f. Both the full horizontal test and B_i are right invariant there, by Proposition 66.15 and the level hypothesis. Consequently integration of Proposition 66.4 in the finite variable gives the factor $v_f c_{r,f}$.

Use real sections $s(y)$ through one. Before multiplication by the first-position cutoff, the remaining scalar function of y on these sections is

$$J(y) = \mathcal{J}\big((s(y),1);[\Omega(s(y))\rho_\infty](p_*)\otimes\beta_f\big).$$

It is continuous by Proposition 66.4, since the real oscillator action and evaluation are continuous on Schwartz space. The last assertion of that proposition gives $J(1) = P_i(q)$. Take a product neighborhood where $|J(y) - P_i(q)| < |P_i(q)|/2$. Lemma 83.1 supplies a first-position test that multiplies $J(y)$ by a nonnegative cutoff a supported there, with quotient integral one. Outside its support the integrand vanishes. The upper-group bump in that lemma also has integral one, in the actual disintegrated measures. Thus the period for the finite sum is $c_{r,f}v_f\int a(y)J(y)\,dy$, of modulus at least $c_{r,f}v_f|P_i(q)|/2$.

The two finite positions contribute $(N\mathfrak{m})^{r(s+t)}$ terms with unit-modulus coefficients. The full new line contributes $(N\mathfrak{m})^{s_0}$ more choices, each with coefficient one. They occupy disjoint matrix blocks, and the core test already has one marked coset. Linearity and the triangle inequality give the displayed bound for an individual product of original marked cosets. At that finite choice the period is a continuous linear functional of the real test by the Schwartz seminorm estimate in Proposition 66.4. The real Hermite expansion converges in Schwartz space. Nonvanishing on the selected test therefore implies nonvanishing on an individual product Hermite function in the original polarization. The corresponding output is a compact-finite cusp form by Theorem 61.1. The quantitative inequality uses the smooth cutoff, whose derivatives can depend on the two forms.

For $i = 3$, choose the compact-finite third row and nonzero Jacobi period of Theorem 61.6. Its q has exactly the distinguished finite vacuum, its descent is the one just used, and its level is the original one. The construction changes only the test on the fourth horizontal lift. This gives the stated fourth period. □

67 The prescribed initial and terminal periods

67.1 Nonvanishing at the finite level of the initial restriction

Use $G, H, \iota, z, \mathfrak{m}$ from Theorem 89.1 and write $J = \iota^{-1}K(\mathfrak{m})$ on the finite adeles. Take Haar product measures, with counting measure on rational points. Let ϕ be a bounded continuous automorphic function on H invariant under J. For a real-place function F_∞ on G_∞ put $\mathcal{K} = F_\infty\mathbf{1}_{K(\mathfrak{m})}$. Define the Poincaré series and its restricted period at $g \in H(\mathbb{A}_F)$ by

$$P_g(t) = \sum_{\gamma\in G(F)} \mathcal{K}(\iota(g)^{-1}\gamma t), \qquad \mathcal{P}_g = \int_{H(F)\backslash H(\mathbb{A}_F)} P_g(\iota h)\,\phi(h)\,dh.$$

The initial restriction pairing in the source uses such a series of a prescribed discrete-series coefficient. Its contribution from the identity coset, divided by $\operatorname{vol} J$, is

$$A_g = \int_{H_\infty} F_\infty(\iota k)\,\phi(g(k,1))\,dk.$$

Theorem 67.1 (The initial restriction period and its nonidentity terms)**.** *Suppose F_∞ is measurable and for some $B \ge 0$ and every $t \in G_\infty$ satisfies*

$$\int_{H_\infty} |F_\infty(t\iota k)|\,dk \le B\prod_\tau |t_\tau z|_\tau^{-(2n+5)}.$$

Here the norms are in the adapted frames at the identity, $n = \dim_E X$, and $f = [E^+ : \mathbb{Q}]$. Then the restricted series is absolutely integrable and

$$\left|\frac{\mathcal{P}_g}{\operatorname{vol} J} - A_g\right| \le \|\phi\|_\infty B\,C_n^f(b_g\sqrt{N\mathfrak{b}})^{-(2n+4)}$$

with the constants in Theorem 89.17.

For every g with $\phi(g) \neq 0$ there also exists $F_\infty \in C_c^\infty(G_\infty)$ for which all nonidentity terms vanish and $\mathcal{P}_g \neq 0$ at this same finite level. If ϕ belongs on the right to a specified isotypic component σ of a maximal compact $K_H \subset H_\infty$, the function F_∞ can be taken of right type $\bar{\sigma}$ under ιK_H.

Proof. Group the sum first by the right cosets of $H(F)$ in $G(F)$. Left invariance of ϕ unfolds the integral for each such coset to one integral over $H(\mathbb{A}_F)$. For its representative γ, the finite variables satisfying the support condition form either the empty set or one right coset $h_0 J$: the product of the inverse of one allowed group element with another is $\iota(h_0^{-1} h_f) \in K(\mathfrak{m})$. Thus each nonempty finite integral of the absolute indicator has volume $\operatorname{vol} J$. The archimedean argument for the coefficient is $t = \iota(g_\infty)^{-1} \gamma_\infty$ followed by ιk, so the integral of its absolute value is at most $B D_g(\gamma z - z)$. By Theorem 89.17 these bounds are summable, including the identity term, whose weight is one. Applying Tonelli's theorem first with absolute values justifies unfolding the full series on the quotient and then multiplying by the bounded ϕ. For the identity term change variables from h to gh'; the finite component of h' runs over J, on which ϕ is right invariant. This gives $\operatorname{vol} J\, A_g$. Summing the other terms gives the displayed bound.

For the existence assertion, the diagonal image of $\mathfrak{b}\Lambda_g$ in the real vector space $\prod_\tau X_\tau$ is a lattice. After applying $\iota(g_\infty)^{-1}$ it is still discrete. Choose a neighborhood U of zero meeting this transformed lattice only in zero. Choose a nonnegative smooth bump on G_∞, positive at the identity, supported in a sufficiently small compact neighborhood there such that $tz - z \in U$ throughout its support. If an unfolded term meets the finite test, $w = \gamma z - z \in \mathfrak{b}\Lambda_g$ by Theorem 89.1. Every argument $t = \iota(g_\infty)^{-1} \gamma_\infty \iota k$ of its real test satisfies $tz - z = \iota(g_\infty)^{-1} w$. Hence the only class meeting both supports is the identity. Compact support makes the original series locally finite. Its unfolding is absolutely integrable on the restriction by the same support argument and compactness of the intersection with the closed subgroup ιH_∞.

By continuity shrink the bump so that $|\phi(g(k,1)) - \phi(g)| < |\phi(g)|/2$ wherever $F_\infty(\iota k) \neq 0$. The integral of the bump on H_∞ is positive. The triangle inequality now gives $|A_g| \geq \frac{1}{2} |\phi(g)| \int_{H_\infty} F_\infty(\iota k)\, dk > 0$. Finally apply the right $\bar{\sigma}$-isotypic projector on the bump if ϕ has the stated type. In the bilinear pairing the projector on the bump moves to the σ-isotypic projector on ϕ, by right translation and $\chi_\sigma(k^{-1}) = \overline{\chi_\sigma(k)}$. It therefore preserves the period. Right multiplication by ιK_H fixes z, so this average still excludes every nonidentity term. □

The compactly supported construction gives a nonzero restricted period for a given nonzero row and its finite level. For its use in the theta chain, the core form must also lie in a specified G_∞ discrete series and in the image of the descending column lifts. The first bound quantifies the error for a proposed coefficient in that series. Its identity action and the uniform constants for that coefficient still enter the comparison.

67.2 The first restriction on the original intervals

Take the fields and intervals of Theorem 13.13 and the first pair $H = U(Y) \subset G = U(X)$ of Theorem 89.1. Here $n = \dim_E X = k + m$, $f = [E^+ : \mathbb{Q}]$, and $b = \exp(x^\varepsilon)$. Use the original lattice, finite modulus $\mathfrak{m}$ and its restriction $J = \iota^{-1} K(\mathfrak{m})$ from Theorem 67.1. Write $m_0 = g_1 g_2$, the product of the two marked level primes, and fix $C > 0$. Put $Q = C f n^4 \log b$. For a real-place function F_∞ on G_∞ set $V(h) = F_\infty(\iota h)$ on H_∞.

Theorem 67.2 (The restriction bound for a prescribed real coefficient)**.** *Let u be a nonzero bounded smooth cuspidal row fixed on the right by $K_H(m_0)$ and J. Suppose a measurable F_∞*

and a constant $c \neq 0$ satisfy the following three conditions, with the measures of Theorem 67.1.

$$\int_{H_\infty} V(h)u(sh)\,dh = c\,u(s) \qquad (s \in H(\mathbb{A}_F)),$$

$$\int_{H_\infty} |F_\infty(t\iota h)|\,dh \le |c|e^{Q}\prod_{\tau}|t_\tau z|_\tau^{-(2n+5)} \qquad (t \in G_\infty),$$

$$\int_{H_\infty} |V(\exp(-A)h) - V(h)|\,dh \le |c|e^{Q}\|A\|_1 \qquad (A \in \mathrm{Lie}(H_\infty)).$$

For all sufficiently distant intervals there is a point g at which the original restricted period satisfies

$$|\mathcal{P}_g| \ge \tfrac{1}{4}\,\mathrm{vol}(J)|c|\,\|u\|_\infty.$$

For fixed C the start of this range is uniform in the row, its supremum, the coefficient and its scalar c.

Proof. Put $U = \|u\|_\infty$ and choose g with $|u(g)| > U/2$. Apply the first hypothesis at $s\exp A$ and at s. The change of variable $h \mapsto \exp(-A)h$ in the first integral and the third hypothesis give

$$|u(s\exp A) - u(s)| \le Ue^{Q}\|A\|_1.$$

The tail bound at $t = 1$ supplies the absolute convergence needed here. Theorem 84.4 therefore applies with $L = Ue^{Q}$, $q_* = n$, and $d = 2f(n-1)$.

Choose the cyclotomic modulus M in Lemma 72.22. On these intervals $\log M = o(\log b)$ and $\log m_0 = O(\log b)$. The ranks in Theorem 6.29 give $f \le n$ for large x, $n \to \infty$, and $\log n = O(\log x)$. The different bound in Lemma 84.1 gives $\mathrm{rd}(E) \le M$. In particular

$$\log\kappa = \log m_0 + 2\log M + (d+5)\log(2d), \qquad -\log b_g \ll_C f^2n^5\log b.$$

For the second inequality take logarithms in Theorem 84.4. The terms are at most $f\log(4\sqrt{fM})$ plus $fn(\log(4\kappa) + Q + \log 2)$, with the displayed bound since $d \le 2fn$, $f \le n$ and $\log b = x^{\varepsilon}$.

Let p_0 be the auxiliary prime at which the original grid specifies its depth, and write p_1, p_2 for its two type primes. For the fixed positive depth constant C_{dep} the exponent at the selected prime above p_0 is $\lfloor S/\log p_0\rfloor + 1$, where

$$S = C_{\mathrm{dep}}f^2n^6\log\left(bnm_0p_1p_2|\operatorname{disc}E|\right).$$

These are the original parameters in `Development/AbelianVarietyTheory/Definitions/Divisors/LeastSplitAuxiliaryPrimeThetaChainSpec.lean`, line 250–256. The ideal norm of that prime is at least p_0. Hence $\log N\mathfrak{m} > S \ge C_{\mathrm{dep}}f^2n^6\log b$. The height loss above is at most half this lower bound once n is large enough. We obtain $b_gN\mathfrak{m} \ge (N\mathfrak{m})^{1/2}$.

The identity term divided by $\mathrm{vol}\,J$ is $c\,u(g)$. By Theorems 89.17 and 67.1, with $\sqrt{N\mathfrak{b}} = N\mathfrak{m}$, the other terms together have absolute value, after the same division, at most

$$U|c|\exp\{Q + f(6n+9) - (n+2)\log N\mathfrak{m}\}.$$

By the depth lower bound the exponent tends uniformly to $-\infty$. For large x the error is at most $U|c|/4$. Subtracting this from $|c\,u(g)| > U|c|/2$ gives the claimed estimate. □

The hypotheses specify the remaining real-place calculation at this step. The source lists the corresponding local branching coefficient, its integrated tail and its derivative in `Development/AutomorphicArithmetic/ThetaCorrespondence/RealBands/BranchDefs.lean`, line 11–19. For the designated row one must obtain such a coefficient in its prescribed discrete series. The restricted period must then be related to the integral pairing through all later lifts and the prescribed projection. The unrestricted cohomology sum in (13.12) is a separate input on these same intervals.

67.3 A single real estimate for the prescribed initial row

On the fields and intervals of Theorem 67.2 keep its $G, H, n, f, \mathfrak{m}, m_0, J$. Let the discrete series σ of H_∞ have the spaced row parameters in (9.1) and the lists following it. Take an isometric H_∞-equivariant realization T of σ in cuspidal L^2 and a unit vector w in its lowest compact type S. Write $u_w = Tw$, using its smooth representative, and assume the boundedness and the two finite invariances of Theorem 67.2. Use the larger real representation π in the convention of Theorem 59.15, with a multiplicity-one isometric embedding $j : \sigma \to \pi|_{H_\infty}$. Since F_v is a conjugate coefficient, the representation generated on the right by this kernel is contained in the contragredient of π. Thus for a prescribed core series one takes π to be its contragredient in this convention.

Theorem 67.3 (A normalized overlap at the original grid level)**.** *At every x in every sufficiently distant interval, each point g with $|u_w(g)| > \|u_w\|_\infty/2$ satisfies $b_g N\mathfrak{m} \ge (N\mathfrak{m})^{1/2}$, uniformly in T, w. At any such point, for every v with $\mathcal{T}(v) < \infty$ and $\alpha = \langle v, jw\rangle \neq 0$, choose $F_{v'}$ as in Theorem 59.15. Then the restricted period satisfies*

$$\left| \frac{\mathcal{P}_g}{\operatorname{vol} J} - \frac{|\alpha|^2}{d_\sigma} u_w(g) \right| \le \|u_w\|_\infty d_S \mathcal{T}(v) C_n^f (N\mathfrak{m})^{-(n+2)}.$$

In particular $|\mathcal{P}_g|$ is at least $\operatorname{vol}(J)\|u_w\|_\infty |\alpha|^2/(4d_\sigma)$ whenever

$$\frac{d_\sigma d_S \mathcal{T}(v)}{|\langle v, jw\rangle|^2} \le \frac{(N\mathfrak{m})^{n+2}}{4C_n^f}.$$

All quantities in this comparison belong to the same initial pair and the same interval. Rescaling the Haar measure of H_∞ changes the integral in $\mathcal{T}$ and the formal degree by reciprocal factors.

Proof. Let $B \ge 1$ bound the absolute Harish-Chandra coordinates of the row across the real places. The construction following (9.1) gives $\log(2+B) = O(\log b)$ uniformly. Proposition 84.5 gives the Lipschitz hypothesis of Theorem 84.4 with $L/\|u_w\|_\infty \le 40\sqrt{n-1}(B+2n)$, independently of v. Take $q_* = n$, $d = 2f(n-1)$. Taking logarithms of the cusp bound yields

$$-\log b_g \ll fn\{\log(2m_0 M^2 f(2+B)) + fn\log(2fn)\}$$

for every g above half the supremum. Indeed $\operatorname{rd}(E) \le M$, $\log\kappa = \log m_0 + 2\log M + (d+5)\log(2d)$, and $\log(1+L/\|u_w\|_\infty) \ll \log(2+B)+\log(2n)$. Such points exist by the definition of the positive supremum. Since $f \le n$, $\log n = O(\log x)$, $\log m_0 = O(\log b)$ and $\log M = o(\log b)$, the displayed bound is $o(f^2 n^6 \log b)$. The depth computation in the proof of Theorem 67.2 now gives the stated height at g.

Write $V = F_{v'}|_{H_\infty}$. Since $V \in L^1$ and T is bounded and equivariant, the identity $\int V(h) u_w(\cdot\, h)\, dh = (|\alpha|^2/d_\sigma)u_w$ holds in L^2 by Theorem 59.15. The integral is also pointwise continuous by dominated convergence and boundedness of u_w. Pairing against a compactly supported continuous function and applying Fubini identifies it with the Hilbert integral. The equality therefore holds everywhere, since quotient Haar measure has full support. Use this identity for the main term of Theorem 67.1 and the bound $\mathcal{T}(v') \le d_S \mathcal{T}(v)$ for the remaining terms. The height estimate proves the first inequality of the theorem. Under the last displayed hypothesis in its statement the error is at most one quarter of $\|u_w\|_\infty |\alpha|^2/d_\sigma$. The lower bound follows from the choice of g. □

There is a fixed-parameter existence argument for vectors in the last comparison. Suppose the adjacent gaps of the larger series exceed $s + 2D$, where $s = 2n+5$ and $D > 4n^2$ is an

integer. Since compact-finite vectors are dense, some such v has $\langle v, jw\rangle \neq 0$. Lemma 64.19 bounds its coefficient by a constant times $\prod_\tau \|g_\tau\|_{\mathrm{op}}^{-(s+2D)}$. Here a compact-finite vector in the product is a finite sum of tensors, by compact admissibility at each of the finitely many places. To integrate a single real factor write $t = b_1 h_0$, where h_0 fixes z and $\|b_1\|_{\mathrm{op}} = \|b_1^{-1}\|_{\mathrm{op}} \le 2|tz|$. Indeed for a positive line the positive and negative coordinate lengths of tz are $\cosh l, \sinh l$. The boost on a signed plane sends z to these coordinates and has norm e^l. A block unitary aligns them with tz. For a negative line exchange the signs, and in a definite space a compact isometry suffices. The norm of $b_1 h$ is at least $|tz|$ and at least $\|h\|_{\mathrm{op}}/(2|tz|)$. Raising these bounds to powers $s + D$ and D, respectively, gives the integrable majorant $2^D |tz|^{-s} \|h\|_{\mathrm{op}}^{-D}$ for the integrated power. This proves $\mathcal{T}(v) < \infty$ for the fixed parameters.

The compact dimension itself satisfies $\log d_S = O(f n^2 \log(2 + B + n))$ by the Weyl dimension formula. Each numerator is at most $2(B + n) + n$, and each denominator is a positive integer. Theorems 84.9, 84.2 and 59.6 now control the decay constants and the prescribed component together as the series vary. The source obtains a translate by estimating a second moment of a branching coefficient in `Development/WeilBound/RepresentationTheoryGL/HolomorphicDiscreteSeries/SchurEigenProjections/HolomorphicMoments/MomentHigherEstimateHolds.lean`, line 16–26. Its use to select a nonzero component appears in `Development/AutomorphicArithmetic/ThetaCorrespondence/FockModels/RealBranch.lean`, line 140–169. The inequality above requires the component along the specified w. It concerns the first restriction. The later projected integral and the cohomology sum in (13.12) remain to be bounded on the same intervals.

67.4 A quantitative period after two horizontal lifts

On the original fields use the period and measures of Theorem 61.5. Take $m \ge 2$, as holds on the intervals in question. Write $A_2(e)$ for descent from a terminal class coefficient $e = e_{c,a,b}$ of Theorem 61.2 to C_2, with the given tests on all edges above it. Let b_1 be the prescribed test on the second core edge. For a Hermite product h_α on the first edge, denote its two factors on Y and Ez by p_α and q_α, respectively. Use the finite components of Theorem 59.3. First lift a prescribed smooth row u on Y to C_1 using p_α. Lift the resulting form to $Z = C_2 \perp Ez'$ with the test $b_1 \otimes U q_\alpha$ of Proposition 61.9, replacing its marked line component by the individual residue indicator ν. Write $B_{2,\alpha,\nu}$ for the resulting form, and set

$$J_{e,\alpha,\nu} = \int_{[U(C_2)]} A_2(e)(k)\, B_{2,\alpha,\nu}(k \oplus 1)\, dk.$$

The count of allowed ν is $D_L = (\mathrm{Nm}\,\mathfrak{m})^{\dim_E C_1}$.

Theorem 67.4 (Second horizontal periods with individual integral tests)**.** *Let $\ell(A) = \mathcal{B}(A, u)$ on $H(v)$. With the terminal constants of Theorem 61.2 and the first real test ϕ of Proposition 59.4, one has*

$$\sum_e \sum_\alpha \frac{1}{D_L} \sum_\nu |J_{e,\alpha,\nu}|^2 \;\ge\; \frac{e^{-\delta} C_{\mathrm{fix}} \|\ell\|_*^2}{h_m d_\rho B^{4a_0} \|\phi\|_2^2}.$$

The sum over Hermite indices denotes the supremum of finite partial sums. For every $0 < \eta < 1$ and $\|\ell\|_* > 0$, one terminal coefficient, one Hermite product and one marked residue satisfy

$$|J_{e,\alpha,\nu}| \;\ge\; \frac{(1-\eta) e^{-\delta/2} \sqrt{C_{\mathrm{fix}}}\, \|\ell\|_*}{H_* \sqrt{d_\rho}\, N^{d_0} B^{2a_0} \|\phi\|_2 C(v, \phi)}.$$

At the two horizontal steps the forms occurring in this nonzero restriction are cuspidal. For any nonzero row u of the prescribed realization there is a pure compact-finite v to which the individual lower bound applies.

Proof. For the single product h_α, factor the two kernels by Proposition 61.9. The first restriction of the core descent equals the period on C_2 with the full Fourier-transformed line test. In detail, the descent to C_1 is the integral of its second core kernel against $A_2(e)$. Upon inserting it into the restriction, first integrate the Y-kernel against u to form the first horizontal lift. This lift is cuspidal or zero by the real first occurrence in Theorem 61.5. It and $A_2(e)$ give rapid decay at the two variables of the remaining integral, with the intervening theta kernels of polynomial growth. The two orders of integration and the replacement by the second horizontal kernel are therefore justified as in that theorem. This argument gives the identity for every product test, including those with a vanishing lift. By Proposition 80.8 are scalars ϵ_ν of modulus one such that

$$\mathcal{B}(F_e(h_\alpha), u) = D_L^{-1} \sum_\nu \epsilon_\nu J_{e,\alpha,\nu}.$$

The cosets are those of the actual negative line lattice and the other marked factor is the specified rectangle. This is the finite Fourier decomposition at the original levels. The first finite averaging of Theorem 59.3 also preserves the first period since the two input forms have the required level invariances.

Put $z_e = \mathcal{B}(P_v T^\dagger e, u)$. Apply Proposition 59.4 simultaneously to all terminal coefficients. For every e, z_e is the limit of the finite sums of the last display with coefficients $c_{j,\alpha}$. Cauchy–Schwarz in the Hermite index and the equal-weight average over ν bound the squared modulus of the j-th sum by

$$\|\Phi_j\|_2^2 \sum_{\alpha \in \operatorname{supp} c_j} D_L^{-1} \sum_\nu |J_{e,\alpha,\nu}|^2.$$

Using the upper limit for $\|\Phi_j\|_2$ gives a lower bound $|z_e|^2/\|\phi\|_2^2$ for the full sum at this e. Summing and applying Theorem 61.2 proves the first inequality.

For the same finite sums the triangle inequality and the absolute Hermite coefficient bound give

$$|z_e| \le C(v,\phi)\|\phi\|_2 \sup_{\alpha,\nu} |J_{e,\alpha,\nu}|.$$

Choose the single e of Theorem 61.2 and take an entry within the indicated factor $1-\eta$ of the resulting lower bound for the supremum. This choice works also when that supremum is infinite. Since the period is nonzero, both horizontal lifts are nonzero. Their cuspidality then follows from real first occurrence at each corresponding target and the tower property in Theorem 61.1. The existence of v for a prescribed smooth row follows from Theorem 74.14. □

The estimate reaches the restriction on C_2 with one integral terminal input and one integral residue at the marked places. The factor $\sqrt{C_{\mathrm{fix}}}/\|\phi\|_2$ records the normalization of the actual even core transforms. For a general compact-finite v, the weighted integral in $C(v,\phi)$ and the norm of its restriction functional still enter the bound. The final arithmetic projection and the full residual cohomology estimate on these intervals remain open in the argument.

67.5 Nonvanishing of the terminal Bessel period

Fix a primitive Dirichlet character, one of the sufficiently distant intervals of Theorem 62.4, and any one of its admissible fields. Keep the original horizontal spaces $V_{i,0}$, core spaces C_i, congruence levels and the characters of Proposition 59.18. Let $m \ge 4$ be the even chain length on this field.

Theorem 67.5 (Nonvanishing at the prescribed terminal character)**.** *Given a unit vector in the lowest compact type of a prescribed old cuspidal row, one can choose a terminal integral*

class coefficient e, a descending core chain starting with e, and a horizontal chain starting with that row vector such that

$$\mathcal{B}^{+}(e, B_m) \neq 0.$$

Here $\mathcal{B}^{+}$ is the global bilinear Bessel period with the original raw terminal flag, middle vector, character and embedded core of Lemma 66.10. After the two initial lifts of Theorem 62.4, each horizontal edge can have a single original coset indicator at every marked place and a single product of real Hermite functions. Away from the marked places it uses the original lattice vacua. The coefficient e and the entire descending chain are the ones selected for the second restriction.

Proof. Theorems 59.13 and 62.4 give the terminal coefficient, descents A_i, and initial tests for a compact-finite B_2 with $\int_{[U(C_2)]} A_2(h)B_2(h \oplus 1)\, dh \neq 0$. For the staircase of length one the Bessel radical is trivial, so this is the required even-index period at $i = 2$. Theorem 61.6 yields a compact-finite cusp form B_3 with nonzero Jacobi period for the distinguished integral line vacuum. Equivalently one may use Theorem 66.14 at $r = 0$.

Suppose first that the current index $i = 2r + 1 \geq 3$ is odd. The induction hypothesis gives the nonzero Jacobi period for the distinguished test. Since A_i is the specified descent of A_{i+1}, Theorem 66.16 gives the next nonzero Bessel period, with the same terminal coefficient. Take its individual Hermite test. Its output is cuspidal and compact-finite at the prescribed level, as required at the next step. If $i = 2r + 2 < m$ is even, the input is that nonzero Bessel period, and Theorem 66.14 gives the next nonzero Jacobi period, now for q_{r+1}, with a compact-finite cuspidal output. At each step the previous tests are fixed. This finite induction reaches $i = m$, with the Bessel period paired against $A_m = e$. Lemma 66.10 puts it in the required raw character convention, using only the last horizontal test. All the constructions took place at the field and finite levels fixed at the start.

The finite costs in the two transfer estimates can be tracked along this chain. For clarity let M be the largest E-dimension of any row or core space. At edge i let r_i, ℓ_i be the two flag lengths of Proposition 99.4. The number of marked terms is at most $(N\mathfrak{m})^{(\ell_i + r_i)M + M}$, including the full line partition on the odd-index edges. The reciprocal finite orbit measure is at most $(N\mathfrak{m})^{2\ell_i M}$ by Theorem 68.6. The reciprocal squared norm of the finite zero-coset line on an even-index edge is at most $(N\mathfrak{m})^{M}$ when each undeepened marked position lattice has volume one and the unmarked vacua have norm one. This follows by integrating its indicator. There are fewer than m edges under consideration and $r_i, \ell_i \leq m/2$. Hence the product of these three finite costs over the chain is at most $(N\mathfrak{m})^{3m^2 M}$. The separate base scalars $c_{\ell_i, f}$ and their measure conventions are given exactly in Theorem 68.6. □

In the assertion just proved e is integral and the period is a nonzero complex number. A uniform lower estimate for the designated integral projection in Corollary 74.7 requires a comparison in its fixed arithmetic normalization. For example, the Hermite selection in each transfer proves nonvanishing without bounding the index or the normalized coefficient of the selected Hermite function. The uniform strict cohomology sum in (13.12) is also open. These estimates are needed to reach the prime-sum bound (W) for Main Theorem 1.1.

67.6 Normalized Bessel functionals under a weight shift

Keep the Euclidean Witt frame, dual vector ω, evaluation map ev, and spectral projections of Proposition 83.12. Give the dual defining space the invariant Hermitian form dual to h, so pairing against ω evaluates at U_1. Give its tensor product with the row the resulting invariant Hermitian form $[\,,\,]$. All positive Hilbert norms below belong to the discrete series or the

compact column. Use the same Haar measures for every row, and write d_μ for its formal degree.

Proposition 67.6 (The coefficients for a single lowering step)**.** *Start with a row of signs* (65.4) *and parameters strictly interlacing the compact column in the order*

$$u_1 > \mu_1 > u_2 > \cdots > \mu_{k-1} > u_k > \mu_k > \cdots > \mu_r.$$

Suppose the row gaps exceed two. Write $P_i = \iota_i p_i$, where ι_i includes the shifted unitary row and $p_i\iota_i = 1$. The maps can be normalized so that $[\iota_i y, \iota_i y] = \epsilon_i \|y\|^2$ with $\epsilon_i \in \{1, -1\}$. Take Λ_μ in the normalization of Theorem 84.20. If $\Lambda_\mu \operatorname{ev} \iota_i$ is nonzero, it equals $\beta_i \Lambda_{\mu - e_i}$ in the same normalization, and

$$|\beta_i|^2 \frac{d_\mu}{d_{\mu - e_i}} = \frac{(2\pi)^{2s}}{2} \frac{\prod_{a=1}^{k} |\mu_i - 1/2 - u_a|}{\prod_{j \neq i} |\mu_i - 1 - \mu_j|}.$$

If $\mu_i - 1/2$ is a column parameter, the map is zero. All nonzero terms retain the strict interlacing above. On the original fields and intervals, after at most $4r^2$ successive lowerings, each displayed coefficient is bounded by $(2\pi)^{2s}(1 + 2L)^2$, where L bounds the absolute row and column entries and may be taken to satisfy $\log(2 + L) = O(x^\varepsilon)$.

Proof. Different summands are orthogonal for the invariant tensor form. For example, the split Casimir of Proposition 83.12 is symmetric for this form and has distinct real eigenvalues. Its symmetry follows by taking the difference of the three quadratic Casimirs for invariant forms, each symmetric as the real infinitesimal generators are skew adjoint. The tensor form is nondegenerate on the compact-finite module, so its restriction to each summand is nondegenerate. An invariant Hermitian form on an irreducible unitarizable Harish–Chandra module is a real scalar multiple of its Hilbert form, by Schur's lemma applied to the maps into the conjugate compact-finite dual. Rescale the inclusions to obtain the indicated signs.

Write $L_i v = p_i(\omega \otimes v)$ and $R_i y = \operatorname{ev} \iota_i y$. Both commute with B. In the Hilbert forms they satisfy $\langle R_i y, v\rangle_\mu = \epsilon_i \langle y, L_i v\rangle_i$ on compact-finite vectors. This follows by pairing $\iota_i y$ with $\omega \otimes v$ in the invariant tensor form. Continuity gives the equality on smooth vectors, since the inclusions and projections act on smooth globalizations. We use the convention that the Hilbert pairing is linear in the first variable in this proof.

To track the modulus in passing from Hilbert vectors to Bessel functionals, include a dual compact column vector and write $\ell_\nu(v \otimes z) = z(\Lambda_\nu v)$. In the proof of Theorem 84.20, the integral of a matrix coefficient over the period group against its character is precisely $\ell_\nu(a)\overline{\ell_\nu(b)}$. The compact column and the measures are the same on the two sides. Extend R_i, L_i by the identity on its dual. Since they commute with the period group, the adjoint equality holds for the integrands and gives

$$\ell_\mu(R_i y)\overline{\ell_\mu(v)} = \epsilon_i \, \ell_i(y)\overline{\ell_i(L_i v)}$$

whenever the shifted Bessel model exists. Here ℓ_i denotes the normalized functional on $\mu - e_i$. On compact-finite test vectors the integrals apply directly. The maps preserve compact finiteness, since contraction or insertion of a vector in a finite-dimensional compact representation enlarges its compact span by a finite factor. The matrix-coefficient integral is the one justified in Theorem 84.20. Smooth density then gives the display on all smooth inputs.

Multiplicity at most one for these real Bessel models (the local theorem used in Theorem 65.7) gives $\Lambda_i L_i = \kappa_i \Lambda_\mu$, since L_i is B-equivariant. The last display with a nonzero $\ell_\mu(v)$ gives $\Lambda_\mu R_i = \epsilon_i \overline{\kappa_i} \Lambda_i$. If the shifted model is absent, B-equivariance gives $\Lambda_\mu R_i = 0$ directly. In the former case the signed scalar of Proposition 83.12 equals $\epsilon_i |\kappa_i|^2$, by composing the two maps. In the latter case that scalar is zero. In particular a zero of the numerator kills the

entire outgoing map. When the map is nonzero, $|\beta_i| = |\kappa_i|$. By Theorem 72.9, the ratio of formal degrees is the ratio of absolute Vandermondes of the two rows, since the measure factors cancel. Only the factors at position i change. Multiplying by their ratio gives the asserted formula.

A row coordinate is integral and every column coordinate is half-integral. A unit downward step can cross a column coordinate only when $\mu_i - 1/2$ equals that coordinate. Thus every contributing shift keeps the column order fixed. The original adjacent row gaps exceed $8(f + 1)(r + 1)^3$ by the spacing in Section 64.7. After $4r^2$ unit steps these gaps still exceed two for sufficiently large x. The original size bound follows from (9.1) and its endpoint formulas. Increasing it by $4r^2$, which is polynomial in x, preserves $\log(2 + L) = O(x^{\varepsilon})$.

For the numerical bound set $t = \mu_i - 1/2$ and $a_j = \mu_j + 1/2$. The numerator distances are $|t - u_a|$, and the denominator distances are $|t - a_j|$ for $j \neq i$. After omitting a zero numerator assume first that $i < k$. We have $u_{i+1} < t < u_i$. For $2 \leq a \leq i$ pair u_a with $a_{a-1} \geq u_a$. For $i + 1 \leq a < k$ pair u_a with $a_a \leq u_a$, using half-integrality and $\mu_a < u_a$. Each paired denominator distance is at least the numerator distance and the paired indices are distinct from i. Only the two endpoints remain. For $i \geq k$ we have $t < u_k$ and can pair u_a with a_{a-1} for $2 \leq a \leq k$, leaving only u_1. All unused denominator distances are at least one by the row gaps. Each unpaired numerator distance is at most $1 + 2L$. This proves the bound, including equality cases in the distance comparisons. □

67.7 A terminal period bound from the second restriction

Fix the original field, levels, unit lowest-type row, terminal integral coefficient e and descending core tests of Theorem 62.4. Keep the selected second row B_2 and its nonzero restriction P_2. Let $m \geq 4$ be the even row length. Fix the line tests on the subsequent Jacobi edges as in Theorem 67.5. Fix their real coordinate and measure conventions as in Proposition 66.11. Let M_2, H_2 be the two compact spectral parameters for the initial B_2. For instance these can be taken from the compact orbit of its second horizontal oscillator test, since the theta lift intertwines that action. These parameters require only the finite compact types and the specified infinitesimal character.

Theorem 67.7 (A lower bound along the whole horizontal row)**.** *Choose an integer $w_m \geq 0$. In descending order from $m - 1$ to 3 put $w_i = E_i(w_{i+1})$, and put $s_2 = 4(w_3 + d_2 + 1) + 4g_2 + 4$. On these same data one can select one normalized product Hermite and one original marked choice at each edge after the second row. The resulting raw terminal Bessel period obeys*

$$\frac{|\mathcal{B}^+(e, B_m)|}{(1 + D_m)^{w_m}} \;\geq\; \frac{|P_2|}{M_2 H_2^{s_2}} \prod_{i=2}^{m-1} \frac{c_{i,f} v_{i,f} t_i}{2T_i C_i(w_{i+1})}.$$

Here D_m is the total degree of the final test in the original terminal coordinates. All constants in the product are local data of the specified edges. Taking $w_m = 0$ gives a lower bound for the complex period in terms of the initial restriction with no intermediate Hermite degrees or periods on its right-hand side.

Proof. At edge two take the test given by Proposition 66.11 with $w = w_3$. Denote by D_3 the degree selected there. The result is a lower bound for $|P_3|/(1 + D_3)^{w_3}$ by

$$\frac{|P_2|}{M_2 H_2^{s_2}} \frac{c_{2,f} v_{2,f} t_2}{2T_2 C_2(w_3)}.$$

In particular $P_3 \neq 0$. Suppose one has reached row i with its tests fixed and with this kind of bound for $|P_i|/(1 + D_i)^{w_i}$. Apply the same proposition with input degree D_i and weight w_{i+1}.

The degree exponent required on the input is exactly w_i. The lower bound just obtained is multiplied by $c_{i,f}v_{i,f}t_i/(2T_iC_i(w_{i+1}))$, and now bounds $|P_{i+1}|/(1+D_{i+1})^{w_{i+1}}$. This selects one actual test and preserves all earlier choices. Finite induction proves the displayed result. The final raw character is included before the last expansion by Lemma 66.5. Every step uses the finite partitions and unmarked vacua of the original field. The descent ending in e was fixed throughout. □

The right-hand side can be combined directly with the quantitative estimate for P_2 in Theorem 62.4. Its factors $C_i(w)$ involve derivatives of real frame extensions and of fixed oscillator vectors at the indicated orders. Their growth as the field, ranks and length vary on the common interval has yet to be bounded on the required scale. A separate identity or estimate must relate this complex period to the specified integral character projection in Corollary 74.7. The resulting comparison and the upper bound for the cohomology mean in Theorem 8.3 are both needed by the proposed proof of Main Theorem 1.1.

67.8 A terminal lower bound from the original bottom vector

Take the original fields and levels of Theorem 62.4, a unit lowest-type bottom row u, and a fixed pure lowest-type core vector v_0. Choose its polynomial even core tests as in that theorem. Use $B, H_*, N, d_0, a_0, d_\rho, \delta, C_{\mathrm{fix}}$ from Theorem 61.2, in the terminal probability normalization. The letters J, K_f, d_σ, d_π have the measure conventions of Proposition 84.3. That proposition provides a polynomial $P_{\psi,\varepsilon}$ for this family. Define the positive quantity

$$Z_0 = \frac{e^{-\delta/2}\sqrt{C_{\mathrm{fix}}}}{H_*\sqrt{d_\rho}\,N^{d_0}B^{2a_0}}\frac{\mathrm{vol}(J)}{d_\sigma}\left(\frac{d_\pi}{\mathrm{vol}(K_f)}\right)^{1/2}\|u\|_\infty e^{-P_{\psi,\varepsilon}(x)}.$$

In Lemma 84.8 bound $W(a_\tau)$ by $T_0^{Cn^3}$ as in Proposition 84.3, giving η, S from prescribed data. Fix the subsequent real core and line tests. Choose $w_m \geq 0$ integral, use w_i, s_2 of Theorem 67.7, and set $q = d_1 + s_2$.

Theorem 67.8 (From a unit bottom row to the raw Bessel period)**.** *There is a terminal integral coefficient e and a chain of individual horizontal tests at the original finite levels for which*

$$\frac{|\mathcal{B}^+(e,B_m)|}{(1+D_m)^{w_m}} \;\geq\; \frac{Z_0}{2SH_0^{s_2}(1+L_1+q/\eta)^q}\prod_{i=2}^{m-1}\frac{c_{i,f}v_{i,f}t_i}{2T_iC_i(w_{i+1})}.$$

Here H_0, L_1 are from Proposition 68.1. In particular at $w_m = 0$ the right side involves only the original bottom vector, field, group parameters, prescribed local tests and measures. It is independent of every selected Hermite index and of every individual intermediate period.

Proof. Choose the real translate a from Proposition 84.3. With $\ell_a(F) = \mathcal{B}(R(a)F, u)$, that proposition and Theorem 61.2 give one e such that $\left|\ell_a(P_{v_0}T^\dagger e)\right| \geq Z_0$. Keep this same e. Lemma 59.5 identifies this scalar with the first restriction on $\omega(a)\phi^0$. Expand that test in the original bottom and line Hermites. For each product in its expansion the two-kernel identity of Theorem 67.4 expresses the first restriction as $D_L^{-1}\sum_\nu \epsilon_\nu J_{e,\alpha,\nu}$, where there are D_L terms and $|\epsilon_\nu| = 1$. Schwartz convergence permits passage from finite Hermite sums to the first restriction by Proposition 59.4. If all individual terms satisfied

$$|J_{e,\alpha,\nu}| < (Z_0/(2S))\,\exp(\eta|\alpha|),$$

the triangle inequality and Lemma 84.8 would bound the modulus of every finite sum by $Z_0/2$. The limit contradicts its lower bound. Thus one pair α, ν gives at least the displayed size. Set $P_2 = J_{e,\alpha,\nu}$ for this choice. Its two earlier horizontal tests and e are now fixed.

Put $j = |\alpha|$. Proposition 68.1 gives

$$M_2 H_2^{s_2} \le H_0^{s_2}(1 + L_1 + j)^q.$$

For every real $j \ge 0$, $(1 + L_1 + j)^q e^{-\eta j} \le (1 + L_1 + q/\eta)^q$. Indeed the left side decreases for $j \ge q/\eta$, and below that point the power alone is at most the right side. Hence $|P_2|/(M_2 H_2^{s_2})$ is at least the scalar before the product in the theorem. Apply the proof of Theorem 67.7 starting at this nonzero restriction. That proof uses only its compact types and nonvanishing, so it applies to the selected α, ν. It yields the asserted individual tests, measures and raw terminal character. Finally $\|u\|_\infty$ may itself be replaced below by $\mathrm{vol}([H])^{-1/2}$, since u has unit L^2 norm. □

The weights in this estimate satisfy $w_i \ge 4w_{i+1} + 1$ for $3 \le i < m$. Thus the derivative orders requested of the fixed local data already grow at least geometrically with the number of transfers. Establishing the needed uniform scale, the upper bound for the full theta period and the lower bound in the designated integral projection remains open on the intervals of Section 6.6.

68 Restriction, levels and Fourier coefficients

68.1 The initial degrees and the second row spectrum

Use the first two horizontal edges of Theorem 67.4. A Hermite product on the first core edge factors as $h_\alpha = p_\alpha \otimes q_\alpha$ in the bottom hyperplane and line. The second horizontal test at infinity is $b_1 \otimes U q_\alpha$, where b_1 is the fixed second core polynomial and U is the opposite-line intertwiner. Let L_1 be the total energy of b_1 across the real places. Use signed orthonormal Hermitian bases respecting the two direct sums in Proposition 61.9. Write d_1 for the total real position dimension on this second horizontal edge, whose incoming index is 1.

Proposition 68.1 (Compact spectral bounds before the remaining transfers)**.** *For every nonzero $B_{2,\alpha,\nu}$ in that construction the M_2, H_2 of Proposition 84.7 can be chosen to satisfy*

$$M_2 \le (1 + L_1 + |\alpha|)^{d_1}, \qquad H_2 \le H_0(1 + L_1 + |\alpha|).$$

The constant H_0 can be fixed from the real group, its specified infinitesimal character, and its compact action on the creation space and vacuum before choosing e, α, ν or the bottom cusp vector.

Proof. At a real place the adjoint on the line summand sends a row of complex coordinates to a conjugate column times signature signs. The line generators have respective norms $1, -1$. Thus it is an isometry for the positive Hilbert–Schmidt metrics, formed by making every signed basis vector of length one. The sign switch identifies the alternating forms as in Proposition 61.9. Both normalized Heisenberg metrics use the same absolute trace scalar of the additive character. An isometry carrying one symplectic form to the other carries the positive complex structure to the other, by the polar decomposition of its skew operator. Hence the unitary change of real oscillator model intertwines the two number operators. This also follows by transporting the sum of squares of positions and frequencies, then subtracting the same vacuum energy. The splitting scalar has no effect. Thus the energy on the second test is at most $L_1 + |\alpha|$. The compatible compact of the second row acts on this space preserving the total number operator, since the signed bases on the core and new line together are a signed orthonormal basis for that row.

Set $h = L_1 + |\alpha|$. There are at most $(1+h)^{d_1}$ multi-indices of degree at most h, since each individual coordinate is between zero and h. Equivariance in the second horizontal test puts every compact type of its output among those of this polynomial space. This bounds the sum of the distinct type dimensions. To see the dependence of the eigenvalues explicitly, let A bound the operator norms of the compact orthonormal Lie generators on the creation space and let B bound the absolute values of their vacuum scalars. Their norms through degree h are at most $B + Ah$ by the symmetric tensor action. Thus $C_K(\tau) \le (\dim K)(B+Ah)^2$ for each type in the list. One may take

$$H_0 = 1 + \sqrt{|\Omega_G(\pi_2)|} + \sqrt{2 \dim K}\,(A+B),$$

with the metric and Casimir of Proposition 84.7. Theorem 61.1 fixes its scalar $\Omega_G(\pi_2)$ on the entire second row. This gives the claimed H_2 bound for every allowed finite residue. □

68.2 Integral coefficient lattices on the graph correspondence

Use the coefficient representation W of Proposition 73.4. For $b \in H(\mathbb{A}_{F,f})$ put $\Lambda_b = C_m(E) \cap b\widehat{\Lambda}$, where $C_m(E)$ here denotes the underlying E-vector space of the core. Write $W(\Lambda_b)$ for the tensor over the chosen embeddings of its integral Young images and determinant powers. Enlarge the coefficient field to include all finite character values. A common choice for the right argument of a matrix coefficient is the fixed lattice $W^\circ = W(\Lambda_1)$. At a class represented by b take the left covector in $W(\Lambda_b)^\vee$. This convention respects left rational translation, which acts only on the left argument. It also defines a lattice at any finer finite level contained in the stabilizer of $\widehat{\Lambda}$.

Proposition 68.2 (A common integral lattice for the translated columns)**.** *There is a list $\mathcal{Q}$ of $h_m N^{2d_0}$ integral columns at the original level. For every vector Y of its full real type and finite level, some $Q \in \mathcal{Q}$ satisfies*

$$|\langle Q, Y\rangle| \ge \frac{\|Y\|}{h_m \sqrt{d_\rho}\, N^{d_0} B^{2a_0}}.$$

For the graph elements of Proposition 66.6, $T^{a_0} D_\lambda Q$ is an integral column in the same fixed-right-argument convention at the intersection of the original level with all its conjugates by za, $z \in Z$. Both assertions hold also for the conjugate coefficient representation.

Proof. Keep the N^{d_0} covectors of Proposition 62.7 for each class and use its N^{d_0} vectors on the identity class for the right list at every class. Their lower frame bounds are both B^{-2a_0}. The upper bounds are B^{2a_0} for the covectors and $L_0^2 B^{2a_0}$ for the vectors. Form all products and extend each class coefficient by zero on the other classes. The class stabilizers are trivial by Proposition 73.4. With the probability on $[H]$, compact Schur orthogonality divides the tensor-product frame operator by $h_m d_\rho$. Thus the sum of all squared pairings with Y is at least $B^{-4a_0}\|Y\|^2/(h_m d_\rho)$. Divide by the list length and take square roots.

For $u = za$, the elementary divisors of u at the chosen split half are units and one T, and those at the conjugate half are their inverse duals. At all other places $u = 1$. Consequently for every b

$$T\Lambda_b \subset \Lambda_{bu} \subset T^{-1}\Lambda_b.$$

At an embedding of degree-d_τ polynomial tensors, the two inclusions with exponent d_τ pass to the image of the same integral Young operator. In the determinant line the corresponding exponent is at most N, by taking top exterior powers. Dualization reverses inclusions and gives the same two-sided estimate for negative determinant powers. Tensoring over the embeddings gives $T^{a_0} W(\Lambda_b) \subset W(\Lambda_{bu}) \subset T^{-a_0} W(\Lambda_b)$, since $a_0 = \sum_\tau (d_\tau + N|j_\tau|)$.

Write $bu = \gamma t_c k_f$ with $\gamma \in H(F)$, t_c the old representative, and k_f in the old level. If Q has support on c, its translate at b has real value $l_c(\rho(\gamma_\infty^{-1}k)v)$, where k is the real argument and v is its fixed right vector. The left covector is integral on $W(\Lambda_{bu})$, since $\Lambda_{bu} = \gamma\Lambda_{t_c}$. The displayed inclusion therefore makes T^{a_0} times this covector integral on $W(\Lambda_b)$. If the support condition fails, the value is zero. The common intersection fixes every translated form, and character values are units in the enlarged coefficient ring. Summing the translated columns proves integrality. Complex conjugation preserves inclusions, norms and the integer T, proving the last assertion. □

68.3 Restriction of a Fourier coefficient to a smaller row

In the embedding of Theorem 72.27, let I be the isotropic flag space. The large middle space has the orthogonal decomposition $W = W_1 \perp D$, with $W_1 = W_0 \perp Ez'$ belonging to the small row. Set $X_D = \mathrm{Hom}_E(D, I)$, regarded as an additive F-group. For $x \in X_D(F_\infty)$ let $n(x)$ be the real unipotent having this matrix as its middle-to-flag block and zero skew-adjoint parameter. Its finite components are one. Let $\eta(x) = \chi_+(n(x))$. Use probability measures on the compact rational unipotent quotients. A box B_R is $[-R, R]^{\dim_{\mathbb{R}} X_D(F_\infty)}$ in any fixed real linear coordinates, with Lebesgue volume $|B_R|$.

Proposition 68.3 (Averaging the small Fourier integrals)**.** *For any smooth automorphic function B on the large group and g in its adelic points, one has*

$$\int_{[N]} B(ng)\overline{\chi_+(n)}\,dn = \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} \overline{\eta(x)} \int_{[N_0]} B(n_0 n(x) g)\overline{\chi_\varpi(n_0)}\,dn_0\,dx.$$

In particular some real x gives a small integral of modulus at least half the modulus of any nonzero left side.

Proof. In the ordering $I, W, I^\vee$ the radical fixing I pointwise has block matrices

$$\begin{pmatrix} 1 & X & Z - XX^*/2 \\ 0 & 1 & -X^* \\ 0 & 0 & 1 \end{pmatrix}, \qquad Z^* = -Z.$$

The adjoint uses the Hermitian form on the middle space and the pairing of the two outer spaces. The isometry condition on the upper right entry is precisely that its sum with its adjoint equal $-XX^*$. The full flag radical also has the upper unitriangular group on I, acting dually on $I^\vee$. The subgroup L given by $X = 0$ and arbitrary Z is normal and belongs to both radicals. Both characters are trivial on L. Modulo L, the middle blocks add, and conjugation by a flag matrix A sends X to AX. Hence $\bar{N} = N/L = X_D \rtimes \bar{N}_0$, where $\bar{N}_0 = N_0/L$. The same description holds on rational points, by the displayed matrix representatives.

Average $B(ng)$ on the left over $[L]$ and denote the resulting function on $[\bar{N}]$ by $b(\bar{n})$. Translations in $L(\mathbb{A}_F)$ preserve its probability, and rational conjugation preserves it by normality. This proves that b is well defined. It is continuous and bounded because all the unipotent quotients here are compact. The quotient measure in $[\bar{N}]$ integrates first over $[X_D]$ on the left of a representative of $[\bar{N}_0]$. Indeed the semidirect product coordinates give a fundamental set from fundamental sets for the two rational subgroups. The flag matrices have determinant one in their action on X_D, preserving additive Haar measure. Normalization gives the two probabilities. Averaging over $[L]$ in N_0 uses the same probability as in N.

For fixed $\bar{n}_0$, let A be its flag matrix. Then $\bar{n}_0\,\bar{n}(x) = (A_\infty x)\bar{n}_0$ in the quotient. Since χ_+ is a character, its restriction η to X_D is invariant under this conjugation, is trivial on rational points and extends the displayed real character. The continuous function $y \mapsto b(y\bar{n}_0)\overline{\eta(y)}$ on

$[X_D]$ therefore occurs on the right through $y = A_\infty x$. The box mean along this real subspace tends to its Haar mean. In fact the characters of the compact additive quotient are indexed by its rational dual under trace. Each nontrivial one has nonzero real frequency on the full archimedean space, also after the invertible transformation A_∞. Integration in a coordinate of nonzero frequency makes its box mean tend to zero. Finite sums of characters are uniformly dense in the continuous functions by Stone–Weierstrass. This proves the convergence for each $\bar{n}_0$. The bound $\|b\|_\infty$ permits integration over $[\bar{N}_0]$ by dominated convergence. The small character is the restriction given in Theorem 72.27. Restoring the common L-integration gives the identity. Absolute values and the triangle inequality prove the final assertion. □

68.4 The degree required at the high real places

At a real place τ write D_τ for the degree of the final Hermite test in the compatible coordinates of Corollary 66.13. For a product test the total degree in Theorem 67.7 is $D_m = \sum_\tau D_\tau$. We consider the unrestricted horizontal lift before projecting to the special large compact type ρ.

Proposition 68.4 (A necessary Hermite degree for the full column)**.** *In the restrictions $y \mapsto B_m(yn(x)ga)$ of Theorem 66.7, every occurring real column highest weight μ_τ satisfies*

$$\mu_{\tau,1} - \mu_{\tau,k} \;\le\; D_\tau.$$

The bound survives the terminal character sum, the small Fourier integral, compact projections on the large core before restriction or on the column afterwards, and any determinant twist. For the prescribed full word, every high place must therefore have $D_\tau \ge a + b_*$. Hence $D_m \ge h(a+b_*)$. A test in the span of all total degrees less than this last threshold gives zero after the full word projection for every such restriction.

Proof. Consider in the real column $U(k)$ the circle $\mathrm{diag}(e^{it}, 1, \dots, 1, e^{-it})$. Its weights on the defining representation and its dual are $1, 0, -1$, with the zero weight omitted when $k = 2$. On the complexified real symplectic space of the last oscillator its weights are drawn from these three integers. Indeed this space comes from the tensor of the last two Hermitian spaces. The circle acts on the endpoint factor through a copy of the defining representation, or its dual, and trivially on the complementary coordinates. It is trivial on the other tensor factor. Complexification of the underlying real tensor adds the conjugate representation and gives the same three possibilities.

For a positive compatible metric the creation space is an invariant subspace of the complexified symplectic space. The vacuum spans a line under the compact group. Its character, including the chosen splitting, is trivial on the displayed circle because it lies in $SU(k)$, a connected group with semisimple Lie algebra. The homogeneous Hermites of degree D_τ are spanned by length-D_τ products of creation operators applied to this vacuum. Each adds a weight of absolute value at most one. Thus the largest circle weight is at most D_τ. The conclusion also holds on every conjugate of the circle in the large definite $U(N)$, by equivariance of the compatible metric.

The map from the final oscillator test to the horizontal lift intertwines the real endpoint action. In the small restriction the real column commutes with $n(x)$ and acts through its conjugate by g_∞ inside the large definite group. The terms at the finite places commute with all these actions. Applying the small Fourier integral intertwines the column translations. Each successive compact projection mentioned in the statement stays in the corresponding degree space or its image. A determinant character is trivial on each of the circles. If an irreducible column type occurs, its highest vector has circle weight $\mu_{\tau,1} - \mu_{\tau,k}$, which proves the inequality. At the high places the word has width $a + b_*$ by Theorem 74.10. A product test of smaller total

degree misses this width at some high place. Linearity on homogeneous components proves the last assertion. □

The lower bound in Theorem 74.10 uses the compact type paired with the integral large-core column. Its high-place projection vanishes identically. A different component of the full real theta lift can enter there only at the degrees just estimated. A quantitative pairing with that component and with the prescribed global cuspidal row at the original levels is still required for the integral comparison of Corollary 74.7.

68.5 The finite lattices in the restricted group

Fix the rational embedding of Theorem 72.27 on a field of Theorem 12.15. Write $S = S_m$, $C = C_m$, $H = U(C)$, and let Λ_0, Λ_W be the original signed row lattice and its definite column lattice. At a finite place v of $F = E^+$ let π_v be a uniformizer of F_v. All ideal inclusions below use the integer ring of $E \otimes_F F_v$, with both factors at a split place. The raw lattice in the staircase is the direct sum of the lines $I_j b_j, I_j^{\mathrm{op}} b_j^\vee, I_z z$ and the lattice in C. Here $b_j^\vee$ is the opposite vector paired to b_j with value one. These are the actual local fractional ideals of Proposition 99.4.

Proposition 68.5 (A finite translate compatible with the column at all places)**.** *There is a finite adele $u \in H(\mathbb{A}_{F,f})$, equal to one above T, such that $L_v = u_v \Lambda_{S,v}$ contains $\Lambda_{W,v}$ as an orthogonal summand at every finite place, and v_0 of Theorem 72.27 belongs to $u_v \Lambda_{C,v}$ everywhere. For $v \nmid T$ write $\pi = \pi_v$ and let e_v be the smallest nonnegative integer satisfying the inclusions*

$$\pi^{e_v} \sigma^{j-s-1}/2 \in I_j, \qquad \pi^{e_v} \bar{\sigma}^{\,s+1-j} \in I_j^{\mathrm{op}} \quad (1 \le j \le s), \qquad \pi^{e_v} \bar{\varpi} \in I_z.$$

Then $\pi^{e_v} \Lambda_{0,v} \subset L_v$ and $\pi^{e_v} \operatorname{pr}_{S_0} L_v \subset \Lambda_{0,v}$. In particular $e_v = 0$ at every odd unmarked place where σ and the staircase ideals are units. The choices of u and e_v depend only on the embedded spaces and lattices.

Proof. The core and small column lattices are Hermitian self-dual. The extension E/F is unramified at every finite place by the inertia argument of Lemma 71.7, and its dyadic places split by Proposition 100.7. First make a local model of the prescribed W_0, v_0 inside $\Lambda_{C,v}$. At a split place a self-dual lattice is a free module and its dual under evaluation. Embed the lattice of W_0 in the first k coordinates with their duals. In the remaining coordinates take $(b, (T-1)b^\vee)$, with $b^\vee(b) = 1$. This integral vector has norm $T-1$, also when v divides $T-1$.

At an unramified field place of odd residue characteristic, a self-dual lattice has an orthonormal basis. Its nonsingular residue form has a vector of nonzero norm, since polarization and nonsingularity rule out identically zero norm. Lift that vector and scale the lift to norm one. The norm on units is surjective by the residue norm and lifting successive digits by the surjective residue trace. The unit line splits off by orthogonal projection. Induction gives the basis. Embed the column by the first k vectors. The complement has rank at least two. In two orthonormal coordinates choose $c\bar{c} = -1$. The vectors $l = w_1 + cw_2$ and $l' = (w_1 - cw_2)/2$ in these coordinates are isotropic and pair to one. The vector $l + (T-1)l'/2$ is integral of norm $T-1$.

In both cases the resulting copy of $W_0 \perp Ev_0$ is nondegenerate over the local field. Witt extension carries it by a local unitary map to the given rational copy with the specified coordinates. Apply this map to the core lattice. At the terminal places the original core already has the two properties by Theorem 72.27. At almost every other place the initial rational embedding maps the column lattice and v_0 integrally into the original core. Self-duality of the

column then splits it off by orthogonal projection. Thus the maps can be chosen to be the identity outside a finite set and at all $v \mid T$. They give u.

In each original signed plane write $u_j = e_+ + e_-$ and $u_j^\vee = (e_+ - e_-)/2$. The integral generators are $u_j/2 + u_j^\vee$ and $u_j/2 - u_j^\vee$. Under the rational embedding the two flag vectors go to $\sigma^{j-s-1} b_j$ and $\bar{\sigma}^{s+1-j} b_j^\vee$. The middle generator goes to $\bar{\varpi} z + v_0$. The line ideals in the statement and the integrality of v_0 now put $\pi^{e_v} \Lambda_{0,v}$ inside L_v. For $x \in L_v$ and $y \in \Lambda_{0,v}$, pairing x with $\pi^{e_v} y$ gives an integer. This is the pairing of $\pi^{e_v} \operatorname{pr}_{S_0} x$ with y. Self-duality of the signed lattice gives the other inclusion. The formulas make e_v finite, with finite support. □

68.6 The finite Fourier coefficient at every horizontal step

Fix $2 \le i < m$ on any of the fields and intervals of Theorem 61.6. Use the two position tests of Proposition 99.4. Take a nonzero residual product β_f on $\operatorname{Hom}(-T_0, S_0)$ with the original vacua away from the marked set and one original integral coset per marked place. Write K_f for the original level of $U(S)$, including all its marked depths. In the notation of Proposition 80.1 put $R_f = R(\mathbb{A}_{F,f})$, where R preserves the scaled ordered frame up to an upper unitriangular transformation. Use the Haar measure giving O_{E_v} volume one in each upper entry of $U_\ell(E_v)$. Let d_E be the absolute field discriminant and $s = \dim_E S$.

Theorem 68.6 (Support and value of the finite Fourier coefficient)**.** *The restricted product of the local integrals $\mathcal{E}_v(g_v)$ for these tests has nonzero set exactly $R_f K_f$. Its value at 1 in the residual oscillator is*

$$\mathcal{E}_f(1) = |d_E|^{\binom{\ell+1}{3}} (N\mathfrak{j})^{\binom{\ell}{3}} \beta_f.$$

At a marked v, let p_v be the ordered-frame probability in Lemma 80.6. Give the compact orbit of the full lattice stabilizer in $R_v \backslash U(S)(F_v)$ probability one. The measure of the projected level in that orbit is $(Nv)^{a_v \ell(\ell-1)} p_v$. With these compact-orbit measures at all finite places, the measure of the support in the quotient is at least

$$(N\mathfrak{m})^{-(2s\ell - 2\ell^2 + \ell)}.$$

Proof. The unmarked support is Proposition 80.1. At a marked place the full product test is fixed by $K(a_v)$. Indeed its split expansion in Proposition 99.4 consists of individual original integral cosets, each preserved by matrix substitution at that depth. The determinant character is trivial on this congruence group by Proposition 66.15. If the integral is nonzero there is a u such that the columns of $g^{-1} X_* u$, after division by d^j, lie in $b_j + \pi^{a_v} L_{j,v}$. Here the substitution in the first position is valid for the whole group, independently of the residual action. The columns have the original Gram matrix. Indeed the Gram matrix of X_* is zero or is supported only at the last diagonal entry, and an upper unitriangular matrix has last row equal to the last coordinate functional. Lemma 80.6 supplies a $k \in K(a_v)$ with $kX_* = g^{-1} X_* u$. Hence $gk \in R_v$. Conversely right invariance and the left covariance proved in Proposition 80.1 apply at the marked places as well. It suffices to calculate the base value.

At 1 the row evaluation leaves β_v unchanged. In column j, the coefficient of b_k for $k < j$ is $d^{k-j} u_{kj}$. Thus the first indicators give, independently in every entry,

$$u_{kj} \in \mathfrak{j} (dAB)^{j-k} O_{E_v} = \mathfrak{D}_{E/\mathbb{Q}}^{-(j-k)} \mathfrak{j}^{1-(j-k)} O_{E_v}.$$

For $j = k+1$ this is the inverse different, so the entire character is trivial there. The integral is the product of the positive entry volumes times the residual vector. It is the given vacuum at almost all places. Multiplying finite volumes of a fractional ideal I gives $(NI)^{-1}$ in the specified measures. There are $\ell - m'$ entries with $j - k = m'$. Summing $m'(\ell - m')$ and $(m' - 1)(\ell - m')$

for $1 \le m' < \ell$ gives $\binom{\ell+1}{3}$ and $\binom{\ell}{3}$, respectively, by the formulas for the first and second powers of successive integers. This proves the base value and the product support. In fact at almost all places the input group element lies in the full stabilizer, so the factorizations into $R_v K_v$ assemble adelically.

It remains to pass from the ordered frame to its orbit modulo unitriangular transformations. Write K for the full marked lattice stabilizer and H for the pointwise stabilizer of X_*. The homomorphism $R_v \to U_\ell(E_v)$, $gX_* = X_* u$, has kernel H. Its image on $R_v \cap K$ is the subgroup of upper unitriangular matrices with entries in $(dAB)^{j-k} O_{E_v}$, $k < j$. Necessity follows by applying the lattice condition to each of the two split components, as in the column calculation. Conversely these conditions make the normalized columns and the normalized rows in Lemma 80.6 integral. Within each of the two ordered spanning sets their change is triangular with unit diagonal, so both still extend to bases. They have the same prescribed pairing. The residue-field extension argument and the integral right-inverse correction in that lemma now give an element of K inducing u. The same lemma identifies the image of $R_v \cap K(a_v)$ by multiplying each entry ideal by π^{a_v}. Both necessity and sufficiency here use congruence of the entire ordered frame.

The index of the latter image in the former is $(Nv)^{2a_v\binom{\ell}{2}}$ by product Haar measure, since $E_v = F_v \times F_v$. The preimages have kernels $H \cap K$ and $H \cap K(a_v)$. Thus replacing H by R_v multiplies the marked probability of the congruence orbit by this index. Indeed that probability for a subgroup D is $[D \cap K \,:\, D \cap K(a_v)]/[K \,:\, K(a_v)]$, by counting cosets in $K/K(a_v)$. This proves the asserted local factor. The ordered-frame count in Lemma 80.6 is at most $(Nv)^{a_v(2s\ell-\ell^2)}$, by replacing each difference of powers by its leading power. Unmarked quotient probabilities are one. Multiplication over the marked places gives the stated lower bound. □

The base value here is an identity for the finite part of the mixed Fourier coefficient. The global period also integrates the real coefficient against the two automorphic forms. In particular the lower bound in this theorem measures the finite orbit. A bound for the integral projected global period still requires control of that pairing in the prescribed integral normalization.

68.7 The prescribed column type at every real place

Give the compact quotients and definite real groups probability measure. In Theorem 72.28 put $M_B = |P_{\mathrm{raw}}|/\|e\|_{L^2([H])} > 0$. Write ρ for the real type of $\bar{e}$. Let μ be the prescribed product of real word types, $d_\mu = \dim \mu$, and let ν be its product with $\prod_\tau \det^{-j_\tau}$ as in Proposition 72.30, using ρ in that proposition. Put $m_\nu = [\rho|_{H_0(F_\infty)} : \nu] \ge 1$. As before $Z = \prod_{v|T} \mathbb{F}_T^\times$ and $c_T = (3T-1)/(T(T+1))$.

Theorem 68.7 (The projected coefficient in the full prescribed real type)**.** *For each word and each $\lambda = \prod_{v|T} \lambda_v$ with all $\lambda_v \ne 1$, there is a smooth left rationally invariant row A on G_0, fixed by its prescribed terminal graph stabilizers, whose central character sum at the original frequency satisfies*

$$\left(\int_{H_0(F_\infty)} |F_0 A_\lambda(h, 1_f)|^2 \, dh \right)^{1/2} \;\ge\; \frac{\sqrt{|Z|}(1-c_T)^{f/2}}{4} \sqrt{\frac{m_\nu}{d_\rho d_\mu}}\, M_B.$$

The function under the norm has precisely real type μ. The row is obtained from the modified horizontal lift by real compact projection, translation, restriction and the determinant twist of Proposition 72.30.

Proof. The characters of the modified grid are unramified at T. Thus $Y = e_\rho F_{\chi_+} B$, where e_ρ is the real type projector on the large compact core, is fixed under each original maximal

compact subgroup there. Its pairing against $\bar{e}$ has modulus $|P_{\mathrm{raw}}|$, so $\|Y\| \ge M_B$. The compact type ρ is nontrivial on a real special unitary factor. In its parameter the adjacent cloud entries have difference $4\Delta > 1$. They differ by $4\Delta - 1 > 0$ in its highest weight. The same holds after duality. Apply the finite translation $D_\lambda = S_\lambda R(a)$ defined in Proposition 66.6, with the coordinate choice of Theorem 72.27. That proposition gives

$$\|D_\lambda Y\|^2 = |Z|\langle Y, L_T Y\rangle \;\ge\; |Z|(1 - c_T)^f M_B^2.$$

It applies to the whole real isotypic subspace, independently of the size of the weights. The geometric embedding of Theorem 72.27 uses only the Hermitian spaces, their lattices, and the additive character. They are the same for the modified grid. The large character on the small radical is therefore exactly χ_{ϖ}, and the resulting row has the required terminal stabilizers.

Apply the compact averaging calculation in the proof of Theorem 66.8 to $P = D_\lambda Y$ and the full $H_0(F_\infty)$. To recall the constants, choose a finite component of real norm at least $\|P\|$. Schur orthogonality in the two real translates gives a projected value of modulus at least $\sqrt{m_\nu d_\mu / d_\rho}\,\|P\|$. Rational weak approximation realizes its left rotation with at least half this modulus and sets $g_v = 1$ for every $v \mid T$. The averaging of the small Fourier integrals in Proposition 68.3 then retains at least half again at one real translate. The small real projector commutes with the terminal translations. Finally the determinant twist is trivial on the unipotent and on the finite scalars. It converts ν into μ and preserves the modulus at the identity. This gives a value of $|F_0 A_\lambda(1)|$ at least $\sqrt{m_\nu d_\mu / d_\rho}\,\|P\|/4$.

On the real group at this finite component the function is in the matrix coefficient space of μ, of dimension d_μ^2. An orthonormal basis of that space has sum of squared absolute values constant by translation and equal to d_μ^2 by integration. Evaluation at one thus costs at most d_μ times the L^2 norm. This proves the inequality and uses no bound on the Hermite degree of the selected test. □

The dimension estimate in Proposition 72.30 bounds the logarithmic loss from the real projections by $O_{\psi,\varepsilon}(fN^2 \log b)$. The positive quantity M_B is the normalized raw period for the enlarged auxiliary weights. Its size at the arithmetic scale of Section 6.6 still requires an estimate.

68.8 A bounded translation at one marked place

Let $H = U(C)$ be the definite group of rank $N \ge 3$. Choose one marked split place v_* of residue size Q, and on its first split factor let $K_* = \mathrm{GL}(L_*)$ for the core lattice. Fix a product finite level J with $J_{v_*} \subset K_*$. Let $\omega > 0$ be the smallest mass of a finite class at level J on $[H]$ in probability measure. Each class carries Haar probability on its real orbit conditional on that class. Put $K_\infty = H(F_\infty)$. Let P be a continuous right J-invariant function of a single irreducible real type ρ, of dimension d. In the restriction to $K_0 = U(W_0)(F_\infty)$ let ν have multiplicity $m_\nu > 0$ and dimension d_ν. Set $\theta = m_\nu d_\nu / d$.

Choose a rational line in the perpendicular of $W_0 \perp Ev_0$ in C. The homomorphism D of the norm-one torus into H acting by its argument on that line and by the identity on its orthogonal complement is a section of the determinant. It centralizes the entire embedded $U(S_0)$. Write $c_Q = (3Q - 1)/(Q(Q + 1))$, and put

$$j = \left\lceil \frac{\log(2d^3/\omega)}{-\log c_Q} \right\rceil.$$

Theorem 68.8 (Detection with bounded denominators at one split place)**.** *There are a finite norm-one idele δ, a real rotation $s \in K_\infty$, and a translation t supported at v_* with*

$t_*^{\pm 1} L_* \subset \pi_*^{-j} L_*$, *such that*

$$\left| d_\nu \int_{K_0} \overline{\chi_\nu(h)} P(hs, D(\delta)t)\, dh \right| \;\geq\; \sqrt{\theta/2}\, \|P\|_{L^2([H])}.$$

The integer j is independent of the choice of P in its type and level.

Proof. For $x \in [H]$ average the squared real projection over its right argument. With all compact groups given probability measure set

$$U(x) = \int_{K_\infty} \left| d_\nu \int_{K_0} \overline{\chi_\nu(h)} P(xhs)\, dh \right|^2 ds.$$

On a finite class c, write its real function as $p_c(b) = \operatorname{tr}(M_c \rho(b))$. Then $\|M_c\|_{\mathrm{HS}} = \sqrt{d}\, \|p_c\|_2$, with the norm on the real group. The character integral replaces $\rho(h)$ by the orthogonal projection of rank $m_\nu d_\nu$. Orthogonality in s, b, as in Theorem 66.8, gives mean $\theta \|p_c\|_2^2$ for U on this class. The trace inequality bounds the integrand before squaring by $\sqrt{d\, m_\nu d_\nu} \|p_c\|_2$. Since each class has mass at least ω, these statements imply

$$\int_{[H]} U = \theta \|P\|_2^2, \qquad \|U\|_\infty \leq d^2 \theta \|P\|_2^2/\omega, \qquad \|U\|_2 \leq d\theta \|P\|_2^2/\sqrt{\omega}.$$

The last inequality uses $U \geq 0$. In the real variable U is a sum of matrix coefficients of $\rho \otimes \bar{\rho}$ and is invariant under the real center.

Let M be right convolution with probability on $K_* \operatorname{diag}(\pi_*, \pi_*^{-1}, 1, \ldots, 1) K_*$. Let Π be projection onto the functions fixed by both $\mathrm{SL}_N(F_{v_*})$ and K_*. These projections commute by normality. On local special-linear invariants M equals averaging over K_*, since the middle diagonal has determinant one. On their orthogonal complement the three-coordinate estimate in Lemma 74.1 gives norm at most c_Q. Its proof applies to any unitary representation on this complement and uses the residue size Q. It follows that $\|(M^j - \Pi)U\|_2 \leq c_Q^j \|U\|_2$.

Both images are fixed by J and have their real coefficients in the same matrix coefficient space as U. That space has dimension at most d^4. On a compact group an orthonormal basis of a translation-invariant coefficient space has constant sum of squared moduli, equal to its dimension by integration. Thus point evaluation on any one class costs at most $d^2/\sqrt{\omega}$ times the global L^2 norm. In particular

$$|(M^j - \Pi)U(x)| \leq c_Q^j d^3 \theta \|P\|_2^2/\omega \leq \tfrac{1}{2}\theta \|P\|_2^2 \qquad (x \in [H]).$$

The continuous function ΠU depends at finite places only on the determinant and is constant in its real argument. Indeed strong approximation for the simply connected group $\mathrm{SU}(C)$ holds away from v_*, where the group is split and noncompact. For $g \in H(\mathbb{A}_F)$ and $a \in \mathrm{SU}(C)(\mathbb{A}_F)$, approximate gag^{-1} away from v_* by rational special unitary elements. At v_* the difference can be corrected exactly by a right local special unitary element, since that group is normal. Rational left invariance, local right invariance and continuity give $(\Pi U)(ga) = (\Pi U)(g)$. This is the strong approximation theorem applied in Proposition 66.6. At infinity every unitary matrix is a scalar times a special unitary matrix, by taking an N-th root of its determinant. The real central invariance of U passes through Π.

Projection preserves the mean since it fixes the constants. Some $x = (1, D(\delta))$ therefore has $\Pi U(x) \geq \theta \|P\|_2^2$. At that point $M^j U$ is at least half this number. It is the mean of $U(xt)$ over a j-fold product of the indicated compact double coset. Each factor and its inverse takes L_* into $\pi_*^{-1} L_*$. Every element in the product satisfies the asserted inclusions. Taking one value at least the mean in the combined compact averages over t, s proves the theorem. $\square$

68.9 Matching rectangular tests at a split finite place

In a square of the theta comparison the two spaces of matrices have different row ranks. Fix a split finite place of the period field and write k for the local field, O for its integers, π for a uniformizer, and $q = |O/\pi O|$. Normalize $|\pi| = q^{-1}$. Additive measure gives O mass one. Multiplicative Haar measure gives $K_m = \mathrm{GL}_m(O)$ mass one in $\mathrm{GL}_m(k)$. Write $M_{m,n}$ for the space of m-by-n matrices. An integral matrix with m rows is *primitive* if it has rank m modulo π. Equivalently it has an integral right inverse. A test of *level* e is supported on $M_{m,n}(O)$ and is constant on cosets of $\pi^e M_{m,n}(O)$.

Take $a > b \geq 1$ with $n = a + b$ even, $z = n/2$, and $\delta = z - b$. Let α, β be the smooth unitary splitting characters on $k^\times$, and put $\nu = \alpha/\beta$. Set $X_0 = (I_a, 0)$, $Y_0 = (0, I_b)$. For $h \in \mathrm{GL}_n(k)$ write $X_h = X_0 h^{-t}$ and $Y_h = Y_0 h$, where h^{-t} is the transpose of the inverse. The two sections in the linear coordinates are

$$\begin{aligned} A_\Psi(Y) &= \int_{\mathrm{GL}_b(k)} \Psi(gY)\nu(\det g)|\det g|^z \, d^\times g, \\ S_\Psi(h) &= \alpha(\det h)|\det h|^{b/2} A_\Psi(Y_h), \\ B_\Phi(h) &= \beta(\det h)|\det h|^{-a/2} \int_{\mathrm{GL}_a(k)} \Phi(gX_h)\nu(\det g)|\det g|^z \, d^\times g. \end{aligned}$$

We first construct Φ for which the last integral has compact support in g. We treat spherical normalization separately below.

Lemma 68.9 (A finite determinant average)**.** *Suppose that $\nu|_{O^\times}$ has positive conductor exponent c. Thus c is the least $j \geq 0$ for which ν is trivial on U_j, where $U_0 = O^\times$ and $U_j = 1 + \pi^j O$ for $j > 0$. Let Ψ have level $e \geq c$. Write $R_e = O/\pi^e O$, $\gamma_m = \prod_{j=1}^m (1 - q^{-j})$, and $H_m(e) = q^{em^2}\gamma_m$. For $T \in M_b(R_e)$ let $0 \leq t_1 \leq \cdots \leq t_b \leq e$ be its Smith exponents, counting each zero pivot as e, and $s(T) = \sum_i t_i$. At every primitive Y,*

$$A_\Psi(Y) = \frac{1}{H_b(e)} \sum_{\substack{T \in M_b(R_e) \\ t_b(T) \leq e - c}} \Psi(\widetilde{T}Y)\nu(\det \widetilde{T})\, q^{-\delta s(T)}.$$

Each summand is independent of the lift $\widetilde{T}$.

Proof. The right inverse of Y shows that gY is integral precisely when g is integral. On invertible matrices $d^\times g = \gamma_b^{-1}|\det g|^{-b}\, dg$. Indeed multiplication on either side has additive Jacobian $|\det g|^b$, and the proportion of invertible matrices over the residue field is γ_b, by choosing independent columns in order. Singular matrices have additive measure zero, by induction and Fubini applied to the nonzero determinant polynomial. Since $\delta > 0$, the resulting integral over integral matrices converges absolutely. The same conclusion for any full-rank Y follows by changing its row basis to a basis of the saturated integral row lattice.

Partition integral g by reduction modulo π^e. Each class has mass q^{-eb^2}, and $\Psi(gY)$ is fixed on it. Integral invertible row and column operations diagonalize a representative modulo π^e. If $t_b \leq e - c$, every lift is invertible over k, has determinant valuation $s(T)$, and its inverse has entries in $\pi^{-t_b}O$. Changing the lift changes its determinant by a factor in U_{e-t_b}. If $t_b > e - c$, choose $u \in U_{e-t_b}$ with $\nu(u) \neq 1$. In diagonal coordinates multiplying the last row by u preserves the whole class, its additive measure and $|\det g|$. The integral over that class equals $\nu(u)$ times itself and vanishes. Substitution of the class masses gives the formula, since $H_b(e) = |\mathrm{GL}_b(R_e)|$. □

Theorem 68.10 (A matched section at every group element)**.** *For each Ψ of the preceding lemma there is a linear choice $\Phi = \mathcal{T}\Psi$ of level e, supported on primitive a-frames, such that $B_\Phi(h) = S_\Psi(h)$ for all h. On primitive-supported tests transforming under left K_b by $\nu^{-1} \circ \det$, this construction is an isomorphism onto the corresponding K_a-covariant tests. Over the ring generated by the values of ν on units, it permutes basis vectors indexed by direct summands and multiplies each by a root of unity.*

Proof. Right invariance of Haar measure gives $A_\Psi(CY) = \nu(\det C)^{-1}|\det C|^{-z} A_\Psi(Y)$. Every full-rank a-frame is X_h for some h. Set

$$F(X_h) = \nu(\det h)|\det h|^{z} A_\Psi(Y_h).$$

If h' gives the same X_h, then $h'h^{-1}$ has block form $\left(\begin{smallmatrix} I_a & * \\ 0 & D \end{smallmatrix}\right)$, as is seen by fixing its first a columns. Its determinant is $\det D$ and $Y_{h'} = DY_h$. The preceding covariance makes F well defined. Replacing h by $\operatorname{diag}(C^{-t}, I_b)h$ gives $F(CX) = \nu(\det C)^{-1}|\det C|^{-z} F(X)$.

Set $\Phi(X) = F(X)$ on primitive frames and zero elsewhere. For such X, completing its rows to an integral basis gives $h \in K_n$. If the first rows change by multiples of π^e, keep the remaining rows of the completion unchanged and take the inverse transpose. This gives $h' \equiv h \bmod \pi^e$. The unit character and the finite formula in Lemma 68.9 give equal values at the two frames. Primitivity is decided modulo π, so Φ has level e. For arbitrary X_h, the primitive bases of its row space are exactly the integral bases of the intersection of that space with O^n. Consequently the elements g making gX_h primitive form one coset $K_a g_0$, of mass one. On it the twisted integrand equals $F(X_h)$. The coset also lies in a single double coset of K_a, so summation by elementary-divisor shells gives the same value. Multiplying by the prefactor gives $S_\Psi(h)$, since $\beta\nu = \alpha$ and $z - a/2 = b/2$.

For a primitive-supported covariant Ψ and primitive Y, only $g \in K_b$ contributes, and $A_\Psi(Y) = \Psi(Y)$. On K_n the formula is therefore $\Phi(X_h) = \nu(\det h)\Psi(Y_h)$. Modulo π^e the row spans of X_h and Y_h are orthogonal complements for the standard perfect pairing. Every free direct summand admits an extended basis, and passage to its orthogonal complement is a bijection. A covariant function on either set of frames is determined by one value on a basis of each summand. Changing this basis multiplies the value by a root of unity, since ν on units factors through $R_e^\times$. This proves the assertion over the specified ring, including invertibility. □

Proposition 68.11 (The two normalized spherical tests)**.** *At a split place with unramified α, β, take the integral lattice indicator in each rank $m = a, b$. Replace z inside the determinant integral by s, multiply it by $c_m(s) = \prod_{j=1}^{m}(1 - \nu(\pi)q^{m-j-s})$, and continue to $s = z$. The resulting B and S sections agree at every h and equal one at the identity.*

Proof. For additive probability on $M_m(O)$ the generating function of $v(\det G)$ equals

$$\int u^{v(\det G)}\, dG = \prod_{j=1}^{m} \frac{1 - q^{-j}}{1 - uq^{-j}} \qquad (|u| < q).$$

The least valuation of the entries in the first column is r with probability $(1 - q^{-m})q^{-mr}$. Move the resulting primitive column to the first basis vector by an integral invertible change of basis. Conditionally on this column the other columns still have independent additive Haar distribution. Expanding the determinant removes the first row and column, adds r to its valuation, and proves the product by induction. Substituting $u = \nu(\pi)q^{m-s}$ and dividing by γ_m gives $c_m(s)^{-1}$ on a primitive frame for $\Re s > m - 1$. This is the determinant integral of the matrix zeta construction of Godement and Jacquet [6], with the present Haar measures.

For arbitrary h, choose $k_0 \in K_n$ whose bottom rows span the saturated bottom row lattice of h. Then $h = \left(\begin{smallmatrix} A & * \\ 0 & D \end{smallmatrix}\right) k_0$, so $X_h = A^{-t} X_{k_0}$ and $Y_h = D Y_{k_0}$. The changes of basis in the two integrals give the normalized factors $\nu(\det A)|\det A|^s$ and $\nu(\det D)^{-1}|\det D|^{-s}$. Both are entire in s. At $s = z$, multiplying by the defining prefactors gives equal sections, using $|\det k_0| = 1$, $\nu(\det k_0) = 1$, and $a + b = 2z$. □

68.10 A fixed functional on the matched frames

A global class sum can prescribe different, even opposite, coefficients on different local frames. We compute the local effect of matching on each such functional. Retain the linear charts, $c \geq 1$, $e \geq c$, and the level ring R_e of Section 68.9. Write $\mathcal{G}$ for the rank-b direct summands of R_e^n. For each $W \in \mathcal{G}$ choose $k_W \in K_n$ such that the rows of $Y_W = Y_{k_W}$ give a basis of W after reduction. Let $X_W = X_{k_W}$. These large frames represent the complementary summands. Evaluation at them gives coordinates on the primitive covariant output of the transfer $\mathcal{T}$.

Theorem 68.12 (Coefficients of a prescribed functional)**.** *Fix scalars λ_W and put $\Lambda(\Phi) = \sum_{W \in \mathcal{G}} \lambda_W \Phi(X_W)$. Write $H = |\mathrm{GL}_b(R_e)|$ and $u = \nu(\pi) q^{-\delta}$. Define D_Z by $H\Lambda(\mathcal{T}\Psi) = \sum_{Z \in M_{b,n}(R_e)} D_Z \Psi(Z)$. Give a rectangular matrix Z its b Smith exponents $0 \leq t_1 \leq \cdots \leq t_b \leq e$, with e for a zero pivot. Then $D_Z = 0$ if $t_b > e - c$. Otherwise, put $s = \sum t_i$. For each W containing the rows of Z, write $Z = T_{Z,W} Y_W \bmod \pi^e$. There is a unique $T_{Z,W} \in M_b(R_e)$, and*

$$D_Z = u^s \sum_{W \supseteq \mathrm{row}(Z)} \epsilon_{W,Z} \lambda_W,$$
$$\epsilon_{W,Z} = \nu(\det k_W)\, \nu(\pi^{-s} \det \widetilde{T}_{Z,W}).$$

Each $\epsilon_{W,Z}$ is a well-defined root of unity. For primitive Z the sum has one term. Its coefficient is a unit multiple of $\lambda_{\mathrm{row}(Z)}$.

Proof. Insert the finite sum of Lemma 68.9 in the evaluation of Theorem 68.10 at X_W. The map $T \mapsto T Y_W \bmod \pi^e$ is injective, since Y_W has an integral right inverse. Its image consists of all matrices whose rows lie in W, by expansion in the chosen basis. An invertible column operation on R_e^n takes Y_W to $(I_b, 0)$, hence the Smith pivots of Z are those of T. The determinant of any contributing lift is π^s times a unit. The independence of the unit character from this lift was proved in Lemma 68.9. Collecting the coefficient of $\Psi(Z)$ gives the displayed formula. A primitive Z spans a direct summand with cardinality $|R_e|^b$. A rank-b direct summand containing it has the same cardinality, so equals it. Every W occurs this way, for instance at Y_W. □

Corollary 68.13 (A lower bound for every functional on the image)**.** *Give functionals the norm dual to the additive L^2 norm, restricting the large-side functional to the covariant output space. Then*

$$\|\Lambda\mathcal{T}\|_*^2 = \frac{p_a}{p_b} \|\Lambda\|_*^2 + \frac{q^{ebn}}{H^2} \sum_{\substack{Z \text{ imprimitive} \\ t_b(Z) \leq e-c}} |D_Z|^2.$$

Thus every nonzero functional on the matched frames survives pullback. At $e = c$ the squared norm ratio is p_a/p_b for every such functional. In the original partial Fourier models the same lower bound and the same minimal-level ratio hold, with the functional restricted to the actual image.

Proof. Put $G = |\mathcal{G}|$. The images of bases of an individual W account for H matrices. Thus $p_b = GHq^{-ebn}$. On the large side all G complementary summands have the same measure and

changes of basis have character of absolute value one. Consequently $\|\Lambda\|_*^2 = (G/p_a)\sum_W |\lambda_W|^2$. Each small residue class has measure q^{-ebn}, which makes its evaluation functional have squared norm q^{ebn}, and the evaluations at different classes are orthogonal. The primitive terms in $q^{ebn}H^{-2}\sum_Z |D_Z|^2$ contribute $q^{ebn}H^{-1}\sum_W |\lambda_W|^2 = (p_a/p_b)\|\Lambda\|_*^2$. All other terms are as displayed. In particular the rows can have any prescribed signs. The image is the full covariant space because the transfer already maps the small primitive covariant subspace onto it. The two changes of Fourier chart in Theorem 74.19 are scalar multiples of isometries with the product of their moduli, including the scalar σ_P, equal to one. Conjugating the transfer by the isometries gives the stated bound on the transformed image. □

Theorem 68.14 (Compact translates at the conductor level)**.** *Let $\mathcal{H}_e$ be the covariant space on primitive b-frames at level e, with its invariant Hilbert norm. Let $G_e = \mathrm{GL}_n(R_e)$ act by right changes of coordinates, and let P_c be the average of the kernel of $G_e \to G_c$. Its image is the level-c frame space, of dimension*

$$R = q^{(c-1)ab}\prod_{i=0}^{b-1}\frac{q^n - q^i}{q^b - q^i}.$$

This representation of G_c is irreducible. For every individual $f \in \mathcal{H}_e$ and every functional Λ,

$$\frac{1}{|G_e|}\sum_{g\in G_e}|\Lambda(gf)|^2 \;\geq\; R^{-1}\|\Lambda P_c\|_*^2\,\|P_c f\|^2.$$

At $e = c$ this is an equality. At that level the character projection for the full stabilizer of any specified W, with the character by which it acts on the basis line of W, is the coordinate projection onto that one line. The same assertions hold on the matched large frame space with the induced action.

Proof. At level c there is one basis line for every summand W. The stabilizer acts on that line by the determinant character of its action on W, or its inverse according to the action on functions. For distinct summands W, W' take $x \in W'$ whose image modulo W is nonzero. Since W is a direct summand, there is a linear form l on R_c^n vanishing on W for which $l(x)$ has valuation $t < c$. The endomorphism $g = 1 + d\,xl$ fixes W pointwise and preserves W'. On both W' and the ambient module its determinant is $1 + d\,l(x)$, by the rank-one determinant formula. Choose $d \in R_c$ so that this determinant is a unit with nontrivial ν-value. Such a unit exists in $1 + \pi^t R_c$ by the definition of the conductor, using any unit of nontrivial character when $t = 0$. Then g is invertible and acts by different phases on the two basis lines. The W, W' entry of every commuting endomorphism must vanish. Transitivity on the summands makes its diagonal entries equal. Since an invariant subspace of a unitary representation has a commuting orthogonal projection, the representation is irreducible. For the stabilizer of a fixed W the same element g forces the W' coordinate of any vector of its specified character to vanish for each $W' \neq W$. The W basis vector has that character. Averaging the action against the inverse character is the orthogonal projection onto this eigenspace and hence keeps exactly the W coordinate. In particular this projector preserves the integral frame lattice even at primes dividing the stabilizer order.

The congruence kernel acts transitively on the lifts of each primitive frame modulo π^c. Indeed extend a frame to an invertible matrix and change its first rows to the prescribed lift, keeping the remaining rows fixed. The change of coordinates is the identity modulo π^c. This proves the description of P_c. At residue level one the number of summands is the number of independent b-tuples divided by the number of bases of a b-space. Every such summand has $q^{(c-1)ab}$ lifts, by writing a lift as a graph into a complementary free a-space. This gives R.

At level c the average of the operators $v \mapsto \langle v, gf\rangle\, gf$ (inner product linear in the first variable) commutes with G_c. Its trace is $\|f\|^2$, so its scalar is $\|f\|^2/R$. Evaluating on the Riesz vector of Λ proves the equality. For $e > c$ fix a coset of the congruence kernel and apply the squared-mean inequality to the values $\Lambda(gf)$ in this coset. Their mean is $\Lambda(gP_c f)$. The remaining average is over G_c, to which the equality applies with row ΛP_c. The correspondence of complementary frames transports the representation to the large side, up to the common unitary determinant character in Theorem 68.10. Tensoring with that character preserves irreducibility and the squared absolute values. □

68.11 The cost of the prescribed terminal Hecke modification

Fix one ordinary word with the notation of Theorem 60.34. Define S'_T by taking $H_w^m S_w$ at every $w \mid T$ and taking S_w elsewhere. This is the terminal-prime prescription before the additional scalar rescaling of the source. The local lattices define a full global sublattice of S. Indeed they have finite index at finitely many places. Reduction modulo a common integer annihilating the local quotients and the Chinese remainder theorem realize their intersection with exactly the specified localizations. Write h for the number of high embeddings. At any finite place over a rational prime p, let $v_w(p) = 1$.

Theorem 68.15 (Absolute index of the required terminal input lattice)**.** *For the coded weights on the common period intervals,*

$$\frac{\log[S : S'_T]}{[J : \mathbb{Q}]d} = \frac{m \log T}{2}\left(hb_* + \frac{h(k-1)}{2} + \frac{1}{k}\sum_{\tau\ \mathrm{high}} |\lambda_\tau|\right) \ll_{\psi,\varepsilon} b\, x^{-1/2-9\varepsilon/8}.$$

Here the index is the cardinality of the underlying finite abelian quotient. Let $\mathcal{P}$ be the joint signed arithmetic period on any selection of words and translates at this x taking values in one test space. In the following inequality use the direct sums of their respective ordinary lattices S and terminal modifications S', and their total source dimension D. Then

$$0 \ \le\ \frac{\log[\mathcal{P}(S) : \mathcal{P}(S')]}{[J : \mathbb{Q}]D} \ \ll_{\psi,\varepsilon}\ b\, x^{-1/2-9\varepsilon/8}.$$

The field, the terminal primes and all the signed maps here are the ones used for these columns. No lower bound for their image dimensions is required.

Proof. Apply the determinant formula of Theorem 60.34 to the m-th power, which multiplies its valuation by m even when the elementary divisors of the power change their ordering. The logarithm of the abelian index is the sum of these valuations times $[J_w : \mathbb{Q}_T]\log T$, by Smith normal form and the residue degrees. A given high τ occurs exactly when $v_w(\tau(\pi)) = 1$. This value is either zero or one since $\pi\bar{\pi} = T$ is split. Its average with weights $[J_w : \mathbb{Q}_T]/[J : \mathbb{Q}]$ is $1/2$. Indeed $\tau(\pi)$ is a unit away from T and each archimedean conjugate has magnitude $\sqrt{T}$. Taking its absolute norm to $\mathbb{Q}$ proves the assertion. This gives the exact index.

At a high embedding $\lambda_1 = a$, $\lambda_k = -b_*$, and every interior entry is polynomially bounded in x by (9.1). Thus $|\lambda_\tau|/k \le a/k + C(x)$ for some fixed polynomial $C(x)$. The index contributions of b_* and of any fixed polynomial, after multiplication by $mf\log T$, are $o(b/x^A)$ for every fixed $A > 0$. The remaining contribution is at most a constant times $mfa\log T/k$, since $h \le f$. By the rank and prime estimates in the proof of Theorem 105.3, this is

$$O_{\psi,\varepsilon}\big(mb\, x^{-1-17\varepsilon/16}\big) = O_{\psi,\varepsilon}\big(b\, x^{-1/2-9\varepsilon/8}\big).$$

The bounds are uniform in the selected columns. On their direct sum the source indices multiply, so the normalized logarithmic index is their dimension-weighted average. Finally

the map on the finite quotients $S/S' \longrightarrow \mathcal{P}(S)/\mathcal{P}(S')$ is surjective. This proves the second inequality for the actual signed map, including zero image. □

Proposition 68.16 (The cost per image dimension at the terminal prime)**.** *For one word take any map B on its ordinary initial space, of image dimension $1 \le j \le d$. At $w \mid T$ let $e_w(B)$ be the valuation of the determinant of $B(H_w^m S_w)$ in $B(S_w)$. With $s = |A_w|$,*

$$0 \le \frac{e_w(B)}{j} \le m \left[s\{b_* + k - 1 + C(x)\} + \frac{2(a+k)}{k-1}\{s \log 2 + \log(d/j)\} \right],$$

where $C(x)$ may be chosen as in Theorem 105.3. In particular for $\log(d/j) \le x^{2+8\varepsilon}$ the logarithmic cost per image dimension is $O_{\psi,\varepsilon}(b\, x^{-1/2-9\varepsilon/8})$. This applies to the signed column on the prescribed tests.

Proof. In the reindexing of Theorem 60.34, first use the reference coefficient lattice on the vector markings. The product incidence and its transpose are integral on the augmentation lattices, and their product in both orders is $T^{s(k-1)}I$. Hence passing from this reference to the target by incidence shifts every increasing source exponent upward by a number in $[0, s(k-1)]$. This is exactly the min–max argument in Theorem 98.8. Relative to the reference, the exponents are sums of s independently sampled coordinate weights, counted with their Weyl multiplicities. The scalar in H_w adds sb_*. With $t = (k-1)/(2(a+k))$, Lemma 98.10 therefore bounds the average of $e^{t\gamma}$, for these single-step exponents γ, by

$$\exp\{ts(b_* + k - 1 + C(x))\}\, 2^s.$$

Their lower endpoint is nonnegative by integrality. Apply the unit-minor argument in the proof of Theorem 105.3 to any rank j map on this space. It bounds the change of its image determinant per dimension by the mean of the largest j of these exponents. Jensen's inequality bounds that mean by the logarithm of the last display plus $\log(d/j)$, all divided by t. Now use this single-step bound for $B, BH_w, \dots, BH_w^{m-1}$. They all have the same rank because H_w is invertible over the local field. Their successive image indices telescope. This proves the displayed inequality. Multiplying by $\log T$ and using $s \le f$ and the same field parameters as in Theorem 105.3 gives its last assertion. □

In particular the prescribed terminal Hecke modification of the input has a controlled index after projection. The unmodified image can itself have a large index in the designated test lattice, or have small rank. Its comparison with the analytic Bessel pairing and the bounds at the other coefficient primes required by Section 6.6 must also be established.

68.12 The determinant of the normalized input on a common terminal prime

For a single word write S for its ordinary initial lattice of rank d. Let $S_T^{\#}$ be the lattice obtained from S by the Hecke operation and the scalar normalization at its terminal prime, with S at the other rational primes. Use A and the common embedding of K_d of Theorem 74.31. For two nested full lattices $L' \subset L$ in one J-space, let $\mathfrak{d}(L', L)$ be the integral ideal satisfying $\det L' = \mathfrak{d}(L', L) \det L$. Locally it is generated by the determinant of any basis of L' in a basis of L.

Proposition 68.17 (The determinant ideal of the normalized source)**.** *With the initial weights λ_τ, put*

$$\theta = m \left(h b_* + \frac{h(k-1)}{2} + \frac{1}{k} \sum_{\tau \text{ high}} |\lambda_\tau| \right).$$

Then $d\theta$ is a nonnegative integer, and

$$\mathfrak{d}(S'_T, S) = (\pi)^{d\theta}, \qquad \mathfrak{d}(S^{\#}_T, S) = (\pi)^{d(A+\theta)}.$$

In particular these equalities hold in the coefficient integers even after a ramified extension above T.

Proof. All coefficient embeddings in the type restrict to one embedding of K_d. When $w \mid \pi$, every high embedding therefore induces a place dividing the original π; when $w \mid \bar{\pi}$, none does. Above π the induced places of the high embeddings are pairwise distinct by complete splitting of T in E_x. Theorem 60.34 gives $v_w \det H^m_w = d\theta$ above π and zero above $\bar{\pi}$. Integrality of H_w gives nonnegativity.

The scalar part $dmhb_*$ is integral, as is $dmh(k-1)/2$ since m is even. For the remaining part, on each tensor of coefficient modules the mean of a single diagonal coordinate at τ is $|\lambda_\tau|/k$. Its product with the dimension of the tensor is the sum of integer weights with multiplicity. In the initial space each such tensor is repeated with the integer multiplicities of the class and scalar-augmentation bases from Theorem 60.34. Hence $d|\lambda_\tau|/k$ is integral. Returning to the ordinary discrete valuation gives $e_w d\theta$ above π, exactly the order of $\pi^{d\theta}$. Localization away from T gives the unit ideal by definition. This proves the first identity. Multiplication of a rank-d lattice by π^A multiplies its determinant by π^{Ad}. Theorem 74.31 now gives the second. □

Theorem 68.18 (Removing the common terminal scalar after signed projection)**.** *At fixed x take any selected words and translates sharing one generator π, with signed arithmetic period maps to a common J-space. Denote their joint map by P, their total source dimension by D, and its image dimension by j. Use direct sums for the three input lattices. Then*

$$\frac{\log[P(S) : P(S^{\#}_T)]}{[J : \mathbb{Q}]} = \frac{Aj}{2} \log T + \frac{\log[P(S) : P(S'_T)]}{[J : \mathbb{Q}]}.$$

The last summand lies between zero and $O_{\psi,\varepsilon}(Db\, x^{-1/2-9\varepsilon/8})$. For the selected fixed-weight words of one translate, write $\delta = \sum_\tau \delta_\tau$ for the normal degree of Corollary 105.2, and P_0 for the same projected class sum with the normal-coordinate scalar removed. On these words the joint lattice identity is

$$P(S^{\#}_T) = \pi^A \bar{\pi}^{-\delta}\, P_0(S'_T).$$

Consequently the absolute scalar on its j-th exterior power at each complex embedding is $T^{j(A-\delta)/2}$.

Proof. Theorem 74.31 gives the same integral scalar π^A on every summand of the joint domain. Linearity gives $P(S^{\#}_T) = \pi^A P(S'_T)$ after taking the sum of the images. The relative index of the latter scalar multiplication on a rank-j lattice is $|N_{J/\mathbb{Q}}\pi|^{Aj}$. This follows locally from the determinant, also for a projective lattice without a global basis. All archimedean conjugates of π have absolute value $\sqrt{T}$. Taking logarithms and using multiplicativity of the index gives the first identity. The bounds for the residual index are those of Theorem 68.15.

At one translate the prescribed words all have the same Hamming weight. Their last restriction includes $\prod_\tau \tau(\bar{\pi})^{-\delta_\tau} = \bar{\pi}^{-\delta}$ with the same δ on every summand by Corollary 105.2. All remaining factors of that restriction, the earlier theta maps and the tests belong to P_0. Pulling the common scalar through the class sum proves the lattice identity. Its absolute value at infinity follows from $|\pi| = |\bar{\pi}| = \sqrt{T}$ there. All identities include the zero image by taking the degree-zero determinant to be one. □

The determinant just computed compares two inputs to the same signed arithmetic period. For Section 6.6 one still needs the size and rank of its image in the designated analytic Bessel test lattice, and the bound on the residual strict groups in Proposition 6.11.

69 References

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