The generalized Riemann hypothesis for Dirichlet L-functions

Volume VII · Fixed-weight Hecke support and symplectic deformations

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Abstract

We study Hecke support at a prescribed arithmetic weight. Strict classes give symplectic representations, whose ordinary equations, trace rings and Frobenius multiplicities can be compared over the period fields.

The guide in Volume I lists the sections and page counts for the series.

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37 Hecke support at the prescribed weight

37.1 An integral module at the prescribed arithmetic weight

Fix a sufficiently distant interval $[V, V^{3/2}]$ of Section 6.6 and a prime $\ell > \exp(V^{\varepsilon}/12)$ as in Proposition 29.15. Use its common abelian CM field $L = L_a$ at the chosen depth. Let E_x be an actual period field on the interval, let $1 \le r < \lfloor x^{401} \rfloor$, and let α be a character of $\mathrm{Gal}(E_x/\mathbb{Q})$ of sign $(-1)^r$. Put $\Psi = \alpha\kappa_\ell^r$ and $\vartheta = \Psi^{-1}$. Let O be the coefficient DVR of the ordinary problem containing the values of α. We describe a finite O-module for this very character and weight, using the unlocalized finite algebra C_0 of Section 35.5.

Lemma 37.1 (Separation from the fixed weight)**.** *Let $I_{\mathrm{wt}} \subset R_0$ impose the original ordered inertial weights. At each selected curve prime $\mathfrak{p}$ one has*

$$I_{\mathrm{wt}}(R_0)_{\mathfrak{p}} = (R_0)_{\mathfrak{p}}, \qquad H_0 \otimes_{R_0} (R_0/I_{\mathrm{wt}}) = 0.$$

Proof. At a chosen place above ℓ the three curve characters remain pairwise distinct on the kernel of the full cyclotomic character in inertia by Proposition 30.9. Choose an element h of this kernel on which the first two differ. In the original weights the two values coincide: the algebraic weights are powers of κ_ℓ, the conductor character is unramified there, and any common normalizing twist cancels in their ratio. Thus $\psi_1(h)\psi_2(h)^{-1} - 1$ belongs to I_{wt} and lies outside $\mathfrak{p}$. It becomes a unit there. The map to H_0 factors through this localization, proving both assertions. □

On G_L write the split diagonal characters at the original weight in their given order as

$$(\chi_1, \chi_2, \chi_3) = \xi(\Psi, 1, \Psi^{-1})|_{G_L},$$

where ξ is the common cyclotomic normalization. This diagonal determinant, with its ordered local characters, defines a continuous map $d : R_0 \to O$. Indeed it has the required residual characters, old inertia quotient, diagonal ordinary flag and polarization (the outer lines pair and the middle line pairs with itself). Write $C_d = C_0 \otimes_{R_0,d} O$. Fix an embedding at ℓ and choose $\gamma \in G_L \cap I_{\mathbb{Q}_\ell}$ on which $\bar{\Psi}$ has order greater than two. Such an element exists because $\bar{\kappa}_\ell$ is onto on inertia of L and $0 < 2r < \ell - 1$. Let e_i be the three ordered polynomial projectors for γ in C_d. Their denominators are units in O. We use $[g]$ for the group element in this algebra and $C_{ij} = e_i C_d e_j$.

Proposition 37.2 (The ordinary corner and the split diagonal)**.** *The projectors lift to C_0; on the completion at a selected two-constituent curve the corner from the first projector to the second is isomorphic to N_0 of Proposition 35.23. At the split weight $C_{ii} = Oe_i$. Set*

$$B = C_{21}/(C_{23}C_{31}), \qquad u(g) = \chi_1(g)^{-1} e_2[g]e_1 \pmod{C_{23}C_{31}} \quad (g \in G_L),$$

where products of submodules denote the spans of products. Then B is finite over O, the values of u generate B, $u(\gamma) = 0$, and

$$u(gh) = u(g) + \vartheta(g)u(h).$$

Proof. Over the complete local ring R_0, the characteristic polynomial of γ has three simple residual roots. Hensel lifting and interpolation give orthogonal projectors in the Cayley–Hamilton algebra whose sum is one. At the curve the second lies on the character line: its reduction in the stable lattice is the middle character, and the other two residual roots are distinct from it. The first idempotent projects to a primitive line in the plane. In the semisimple

residue algebra the two systems of matrix projectors here and in Proposition 35.9 are conjugate after matching the labels. A unit realizing this conjugacy lifts to a unit of the finite complete algebra, since its kernel to the semisimple residue is the Jacobson radical. Once two systems $(f_i), (f_i')$ agree modulo that radical, $\sum_i f_i' f_i$ is a unit intertwining them. This identifies their corresponding corners over H_0.

At d the split representation gives a surjection $C_d \to O^3$, as it contains all three coordinate idempotents. Each primitive corner has determinant degree one, by the same corner identity used in Proposition 35.9. It is exactly Oe_i, and its scalar coordinate on $[g]$ is $\chi_i(g)$. For any $a, a' \in C_d$, insert $1 = e_1 + e_2 + e_3$ into $e_2 a a' e_1$. The third product vanishes in B. The other two are the 21-entry of a times the first scalar of a', and the second scalar of a times the 21-entry of a'. Dividing this formula for group elements by $\chi_1(gh)$ proves the identity. Since γ commutes with its projectors, $u(\gamma) = 0$. The original group elements span C_d over O by the finiteness and closure argument of Lemma 35.8. Taking the corner and the quotient proves the spanning and finiteness assertions. The cocycle is continuous in the finite module topology. □

For $g \in G_{\mathbb{Q}}$ define an element of B by

$$v(g) = \frac{u(g\gamma g^{-1})}{1 - \vartheta(\gamma)}.$$

Normality of G_L makes the argument of u valid and the denominator is a unit. Define $B^{\mathrm{str}}_{\alpha,r}$ to be the quotient of B by the submodule generated by the following elements, for the indicated group elements:

$$\begin{array}{ll} v(gh) - v(g) - \vartheta(g)v(h) & g, h \in G_{\mathbb{Q}}, \\ v(g) & g \in G_{\mathbb{Q}_\ell}, \\ v(g) & g \in I_p \cap \ker \alpha, \quad p \neq \ell. \end{array}$$

For the last row fix one inertia group for each rational prime. The first row ensures that the local conditions on classes are independent of the chosen embeddings. All the submodules in this quotient are finitely generated since B is finite over the DVR. Its quotient topology is Hausdorff. This construction keeps the rational descent equations at primes where $L/\mathbb{Q}$ is ramified.

Theorem 37.3 (The strict group at the original weight)**.** *For every finite-length O-module W, with the indicated Galois action carried by its twist $W(\vartheta)$, there is a natural isomorphism*

$$\mathrm{Hom}_O(B^{\mathrm{str}}_{\alpha,r}, W) \;\simeq\; \{z \in H^1(\mathbb{Q}, W(\vartheta)) : z_\ell = 0,\ z|_{I_p} = 0 \text{ for every } p \neq \ell\}.$$

The equalities on the right are in local cohomology. In particular $B^{\mathrm{str}}_{\alpha,r}$ is the finite O-linear dual, with values in $\mathrm{Frac}(O)/O$*, of the strict group of Section 29.3 after extension of coefficients to O.*

Proof. A class on the right has a unique representative z with $z(\gamma) = 0$, since $\vartheta(\gamma) - 1$ is a unit. This representative is zero on the whole chosen $G_{\mathbb{Q}_\ell}$: strictness makes it a coboundary there, and its value at γ fixes the witness to zero. At each $p \neq \ell$ vanishing of the inertial class implies $z = 0$ on $I_p \cap \ker \alpha$, where the action is trivial. Conversely this last condition makes the inertial cocycle factor through $\alpha(I_p)$, of order prime to ℓ, so its cohomology class there is zero by averaging.

On G_L write $d(g) = \mathrm{diag}(\chi_i(g))$. Over the square-zero algebra $O \oplus W$, form the matrices

$$\rho_z(g) = (1 + z(g)E_{21})d(g).$$

The ratio of the second diagonal character to the first is ϑ, so the cocycle identity makes this a representation. The determinant of every sum of these group matrices, also after extending scalars, is the product of its diagonal entries by triangularity. Thus its determinant is exactly the split one as a polynomial law. Away from ℓ it kills the original specified inertia kernels, since the finite part of the character has prime-to-ℓ order and reduces injectively on its values. At all places of L above ℓ the restricted class is zero, because it comes from a rational strict class. On each local group a change of basis by a matrix $1 + tE_{21}$ therefore gives the split matrices. The ordinary matrix words with the designated diagonal vanish in these coordinates. The adjoints of those words vanish as well: they are evaluated on the conjugate local groups, where the class also splits, and the split diagonal itself satisfies the adjoint words. The same reasoning applies to adjoints of old inertia differences. These are all the matrix conditions on C_0. Continuity and the finite coefficient reductions give a map from its continuous group algebra quotient, using Lemma 34.29 for the relative pro-ℓ quotient. With the determinant fixed at d this is a map from C_d.

The element γ maps to its diagonal matrix. Consequently each e_i maps to the corresponding diagonal matrix unit. Taking entry 21 defines an O-linear functional on C_{21} with values in W. It kills $C_{23}C_{31}$ because both off-diagonal entries in that product vanish in these matrices. The induced functional φ on B satisfies $\varphi(u(g)) = z(g)$ for $g \in G_L$. On the full rational group the cocycle identity and $z(\gamma) = 0$ give

$$z(g\gamma g^{-1}) = (1 - \vartheta(\gamma))z(g).$$

Thus $\varphi(v(g)) = z(g)$ for every $g \in G_{\mathbb{Q}}$. The cocycle law and the two local conditions make φ descend to $B^{\mathrm{str}}_{\alpha,r}$.

Conversely any functional on this quotient gives the continuous function $g \mapsto \varphi(v(g))$. The defining relations give a cocycle zero at the chosen coefficient place and unramified elsewhere by the same averaging. For $g \in G_L$ the identity of the proposition applied to $g\gamma g^{-1}$ gives $v(g) = u(g)$, so this cocycle recovers the original functional on a spanning set of B. It is zero at γ and hence recovers the normalized representative uniquely. The constructions commute with coefficient maps and are O-linear by their formulas on these generators.

Finally put $D_O = \mathrm{Frac}(O)/O$. Every continuous normalized cocycle with these discrete coefficients has finite image by compactness. Its strictness at ℓ and the vanishing on the old inertia kernels hold for its values inside a sufficiently large finite submodule of D_O, so the same construction applies. Equivalently, one takes the direct limit of the displayed identifications along $D_O[\varpi^n]$. The result is $\mathrm{Hom}_O(B^{\mathrm{str}}_{\alpha,r}, D_O)$. The strict group over the character ring is finite by Theorem 15.17; extension of coefficients commutes with these groups by its finite free basis argument. A finite O-module with positive free rank would have infinitely many homomorphisms into D_O, so the representing module is torsion. Duality on its finite cyclic summands identifies it with the dual of the stated strict group. □

Corollary 37.4 (Integral minors on the actual period field)**.** *For each orbit of the indicated characters choose a finite O-presentation $O^m \xrightarrow{T_\alpha} O^s \to B^{\mathrm{str}}_{\alpha,r} \to 0$. Let c_α be the least normalized valuation of a nonzero s-rowed minor; use zero when $s = 0$. Let O_α be the character DVR, f_α its residue degree over $\mathbf{F}_\ell$ and $e'_\alpha = e(O/O_\alpha)$. Then for the complementary sign on E_x one has the exact identity*

$$\mathrm{length}_{\mathbb{Z}_\ell} \mathcal{S}_{\ell,r}(E_x)^{c=(-1)^r} = \sum_{[\alpha]:\, \alpha(c)=(-1)^r} \frac{f_\alpha}{e'_\alpha}\, c_\alpha.$$

Proof. Since $B^{\mathrm{str}}_{\alpha,r}$ is torsion, its presentation has full row rank. Smith normal form gives its length as c_α, also when the module is zero. By the theorem this is the length of the

corresponding strict group over O. Scalar extension of a cyclic module from O_α multiplies length by e'_α. On this large-prime range $\ell \nmid [E_x : \mathbb{Q}]N_*$, so the character projectors are integral and the unramified conditions agree with the conductor conditions by Theorem 15.17. Its orbit formula now gives the displayed equality. □

The matrices in this corollary present modules at the split arithmetic weight, before curve localization. Section 6.6 requires a uniform upper bound on the weighted sum of their valuations as x, r, ℓ vary and estimates for the signed integral theta periods on the same interval. Both estimates remain open in this manuscript.

37.2 Exact descent of strict cohomology on a common auxiliary field

Fix an interval $V \le x \le V^{3/2}$ of Section 29.7, with largest period field E_*, modulus N_* and $K = \mathbb{Q}(\sqrt{-s})$ as there. Let $\mathcal{P}$ be any finite set of primes $\ell > \exp(V^\varepsilon/12)$. Take V large enough that $\ell > sN_*$ and $\ell > 4t_* + 5$ for all of them. For each $\ell \in \mathcal{P}$ take all the pairs $1 \le r < t_*$, $\alpha \in \widehat{\mathrm{Gal}(E_*/\mathbb{Q})}$ with $\alpha(c) = (-1)^r$. When a character is evaluated on a Galois group it is understood to be inflated. Put

$$O_\alpha = \mathbb{Z}_\ell[\alpha], \quad k_\alpha = O_\alpha/\ell O_\alpha, \quad \vartheta = \alpha^{-1}\kappa_\ell^{-r}, \qquad \delta = \vartheta^{-1}\kappa_\ell.$$

Here O_α is unramified since $\ell \nmid [E_* : \mathbb{Q}]$. For a module U with Galois action write $S_D(U)$ for the classes in $H^1(D, U)$ zero on decomposition groups above ℓ and zero on inertia at every other finite place. We use continuous cohomology and impose these conditions on classes.

Lemma 37.5 (One Frobenius for the dual classes and the auxiliary fields)**.** *For every triple above one can choose a class $d_{\ell,r,\alpha} \in H^1(\mathbb{Q}, O_\alpha(\delta))$ unramified away from ℓ, and one can choose a single prime w with all the following properties. It is one modulo sN_* and every $\ell \in \mathcal{P}$. For each such ℓ, the integer ℓ is a non-ℓ-th power modulo w. For every triple the reduction of $d_{\ell,r,\alpha}$ is nonzero on Frobenius at w. The primes satisfying these conditions have positive density.*

Proof. Use the integral class of Lemma 29.2. Its proof uses the oddness of $\delta = \alpha\kappa_\ell^{r+1}$, positive rank from the global Euler characteristic, and prime-to-ℓ inertia away from ℓ. These hypotheses hold for every pair in the stated range, including pairs whose strict group vanishes. Choose one class per orbit under residue-field Frobenius and conjugate it for the other members. Its reduction is nonzero globally. Let $C = \mathbb{Q}(\mu_{sN_*\prod_{\ell\in\mathcal{P}}\ell})$. The reduction remains nonzero on G_C. Indeed it is trivial as a coefficient character on G_C, and the kernel of restriction in degree one is inflated from $\mathrm{Gal}(C/\mathbb{Q})$. In this abelian group complex conjugation acts on the coefficients by -1. Evaluating a cocycle at $cg = gc$ makes it a coboundary since 2 is a unit.

For each ℓ take also the homomorphism on G_C given by its action on $\ell^{1/\ell}$, with values in μ_ℓ. It is nonzero. Otherwise C would contain the real field $\mathbb{Q}(\ell^{1/\ell})$. The polynomial $X^\ell - \ell$ is Eisenstein at ℓ; its real root generates a real field missing its nonreal conjugates. Every subfield of the abelian extension $C/\mathbb{Q}$ is Galois. Conjugation on this Kummer homomorphism has character $\bar\kappa_\ell$.

On G_C the reduced cohomology classes are additive homomorphisms, with conjugation on their values by $\bar\delta$. For a fixed ℓ each target k_α is simple as an $\mathbf{F}_\ell[\mathrm{Gal}(C/\mathbb{Q})]$-module. In fact the values of $\bar\delta$ generate k_α over $\mathbf{F}_\ell$, since the cyclotomic factor is in the prime field. The Kummer target and these targets for distinct orbits are pairwise nonisomorphic. On inertia at ℓ their exponents are respectively 1 and $r+1$ modulo $\ell-1$. These distinguish the Kummer factor and the different r's, even after residue-field conjugation. At fixed r, isomorphism of the simple modules would identify the reductions of the two finite characters up to that conjugation, hence also their prime-to-ℓ lifts.

Consider the image of all the homomorphisms together. For fixed ℓ it surjects onto each simple target and is stable under conjugation. The joint action on the targets factors through an abelian group of order prime to ℓ, whose group algebra over $\mathbf{F}_\ell$ is a product of finite fields. Projecting with its idempotents shows that the image is the whole direct sum. The primary components for different ℓ's separate in the finite additive image, so every set of coordinate values occurs simultaneously. Choose an element with every coordinate nonzero. The joint kernel is open in G_C and normal in $G_{\mathbb{Q}}$. Chebotarev applied to the conjugacy class of the chosen element in this finite Galois quotient gives a positive-density set of primes splitting in C. At each such prime every reduced class evaluates nontrivially. The nonzero Kummer coordinates mean that $X^\ell - \ell$ has no root in $\mathbf{F}_w$, since w splits in $\mathbb{Q}(\mu_\ell)$. Discard the finite set of ramified primes and of other forbidden choices. For an empty list one needs only complete splitting in $\mathbb{Q}(\mu_{sN_*})$. □

Proposition 37.6 (Redundancy of the unramified condition at the chosen prime)**.** *Let w be chosen by Lemma 37.5 and let $n \geq 1$. Put $M_n = (O_\alpha/\ell^n)(\vartheta)$. Every class in $H^1(\mathbb{Q}, M_n)$ zero at ℓ and unramified at every finite prime outside $\{\ell, w\}$ is unramified at w. The assertion holds also for $M_\infty = (\mathrm{Frac}(O_\alpha)/O_\alpha)(\vartheta)$.*

Proof. Write $T_n = (O_\alpha/\ell^n)(\delta)$. It is the Cartier dual of M_n under the trace pairing, since O_α is unramified and $\vartheta\delta = \kappa_\ell$. The two unramified subgroups at w are exact annihilators under local Tate duality. Their cup product factors through the residue Galois group of cohomological dimension one. Each unramified subgroup has order equal to the order of its local invariants, by taking the cokernel of Frobenius minus one. The local Euler characteristic formula at $w \neq \ell$, together with local duality in degree two, gives

$$|H^1(\mathbb{Q}_w, M_n)| = |H^0(\mathbb{Q}_w, M_n)|\,|H^0(\mathbb{Q}_w, T_n)|.$$

These equalities of orders give the asserted annihilators.

The unramified subgroup for T_n is the cyclic O_α-module $T_n/(\delta(\mathrm{Frob}_w)-1)T_n$. The reduction of the local value of $d_{\ell,r,\alpha}$ is nonzero by the lemma. Its value modulo ℓ^n therefore generates this whole cyclic module. Pair a global class as in the proposition with each scalar multiple of this reduction modulo ℓ^n. Global reciprocity makes the sum of the local pairings zero. Every term away from w vanishes: the class is zero at ℓ, at infinity the first cohomology vanishes, and at the other finite places both classes are unramified. Even at a prime ramifying in the coefficients, their unramified cup product is inflated from the residue Galois group using the inertia-invariant coefficient modules. The pairing at w is consequently zero against the entire unramified subgroup of T_n. The annihilator calculation gives the claim. [1, I, Sections 2 and 4]

For divisible coefficients, pick a continuous cocycle representing the class. Its image is finite by compactness. At ℓ choose a witness that it is a coboundary and include this witness in a finite coefficient submodule containing the image. At primes outside $\{\ell, w\}$ where the coefficient is unramified the inertial cocycle is identically zero. At the finitely many conductor primes choose and include witnesses as well. The cocycle and all its required conditions thus occur in one M_n, and the finite assertion applies. □

Choose $v \equiv -1 \bmod 8sN_* \prod_{\ell\in\mathcal{P}} \ell$ prime, distinct from w and the primes already used. For $\ell \in \mathcal{P}$ let $F_{\ell,w}$ be the degree ℓ subfield of $\mathbb{Q}(\mu_w)$ and $F_{\ell,a}$ the degree ℓ^a subfield of $\mathbb{Q}(\mu_{\ell^{a+1}})$. Given any depths $a_\ell \geq 0$, form the abelian CM field

$$A = K\,\mathbb{Q}(\sqrt{sv}) \prod_{\ell\in\mathcal{P}} F_{\ell,w} F_{\ell,a_\ell}.$$

For each $\ell \in \mathcal{P}$ this contains the auxiliary field L_{a_ℓ} of Proposition 29.15. Frobenius of ℓ generates $\mathrm{Gal}(F_{\ell,w}/\mathbb{Q})$ by the Kummer condition. Thus the large local residue degree required in Lemma 30.3 is preserved. The prime w was chosen before the depths.

Theorem 37.7 (Exact strict descent on the auxiliary extension)**.** *Let $\ell \in \mathcal{P}$, take $M = M_n$ or M_∞ as in the proposition, and put $\Delta = \mathrm{Gal}(A/\mathbb{Q})$. Restriction gives an isomorphism*

$$S_{\mathbb{Q}}(M) \;\simeq\; S_A(M)^{\Delta}.$$

It also gives this isomorphism with A replaced by L_{a_ℓ}. Both assertions hold simultaneously for every indicated r, α, n, and for every choice of the depths.

Proof. At ℓ the only ramified factor of A is F_{ℓ,a_ℓ}, which is totally ramified of degree ℓ^{a_ℓ}. A compositum with unramified local extensions has the same ramification index. The index of inertia for a completion of A inside rational inertia is therefore a power of ℓ; its image under the residual cyclotomic character is the whole $\mathbf{F}_\ell^\times$. Since $1 \le r < \ell - 1$, an element acts on M by a scalar whose difference from one is a unit. Thus $M^{G_{A_u}} = 0$ at any $u \mid \ell$, and also $M^{G_A} = 0$. Inflation–restriction identifies the full global first cohomology with the invariants upstairs. The local version makes restriction at ℓ injective.

Let z be the unique rational antecedent of an invariant strict class. It is zero at ℓ. At any $p \neq \ell, w$ the index of inertia for $A/\mathbb{Q}$ is prime to ℓ. Indeed only quadratic factors can ramify at s, v, and at any other prime in $\mathcal{P}$ the ramification index is a power of that prime. All remaining factors are unramified there. Restriction followed by corestriction on inertia is multiplication by its index, so z is unramified at p. Proposition 37.6 now gives the condition at w too. Conversely, restriction preserves all the local conditions. The same inertia and cohomology arguments apply to the specified subfield L_{a_ℓ}, where the only factor ramifying at ℓ is again F_{ℓ,a_ℓ}. □

Corollary 37.8 (All odd coefficient primes on the period fields)**.** *One can take a single A on the interval containing ordinary auxiliary fields at prescribed depths for every contributing large prime. Write $H_x = E_x A$, $\Delta_x = \mathrm{Gal}(H_x/E_x)$ and $e = (1 + (-1)^r c)/2$, using complex conjugation in the respective fields. For every x on the interval, $1 \le r < \lfloor x^{401} \rfloor$, and every odd prime ℓ, restriction gives*

$$e\,\mathcal{S}_{\ell,r}(E_x) \;\simeq\; e\big(\mathcal{S}_{\ell,r}(H_x)^{\Delta_x}\big).$$

Consequently the residual groups $R_{\ell,r}$ in Section 6.6 have exactly their original lengths when their defining operators are applied to the invariant groups on the right.

Proof. On the bounded interval only finitely many pairs (E_x, r) occur. For each pair only finitely many coefficient primes have nonzero strict group by Proposition 11.4. Include in $\mathcal{P}$ all such primes above $\exp(V^\varepsilon/12)$. One may add any finite list of other primes above that threshold. The field A ramifies only at s, v, w and the primes in $\mathcal{P}$, all prime to N_*. In particular $A \cap E_* = \mathbb{Q}$, since their intersection would be unramified at every finite rational prime. This uses Minkowski's discriminant bound. Therefore $H_x/\mathbb{Q}$ has the product of the two Galois groups.

If $\ell \notin \mathcal{P}$ is odd, the degree of H_x/E_x is prime to ℓ: the degrees of the factors of A are powers of two or of members of $\mathcal{P}$. Global restriction is an isomorphism onto invariants by inflation–restriction and averaging. Local restriction is injective on decomposition groups at ℓ and on inertia away from ℓ by corestriction. This proves exact descent for the entire strict group in this case.

If $\ell \in \mathcal{P}$, then $[E_x : \mathbb{Q}]$ is prime to ℓ. Extend coefficients to an unramified ring splitting its characters and take the α-component for each $\alpha(c) = (-1)^r$. Over E_x this component is the

rational strict group with the additional twist α^{-1}. Indeed after tensoring with that character it is the invariant component, and restriction, both globally and locally, gives prime-to-degree descent just as above. Applied to H_x/A, the same argument identifies its α-component with the strict group over A for this twisted module. These identifications commute with Δ_x, since the two Galois actions commute on cohomology. Tensoring with a finite free coefficient ring commutes with cochains, kernels and invariants by taking an integral basis. The theorem now gives the isomorphism on every component; faithful scalar extension gives it over $\mathbb{Z}_\ell$. On the invariant group any two lifts of c from E_x have the same action. Restriction is equivariant for $\mathrm{Gal}(E_x/\mathbb{Q})$, hence also for the field averages in Section 6.6. Applying those averages and taking the prescribed torsion quotients after taking invariants proves the assertion on $R_{\ell,r}$. □

The corollary identifies the lengths on the original E_x without a local correction at the auxiliary ramified prime. It gives no bound for the size of the invariant groups. Such a bound and the signed integral theta-period comparison on these E_x are still required in Section 6.6.

37.3 Propagation along curves with two constituents

Fix one large-prime triple on the interval of Section 29.7 and its ordinary problem over $L = L_a$, with real subfield F. Use the depth of Theorem 30.8, the original ordinary determinant ring R_0, its finite Galois algebra C_0, and the ordinary module M_K of Theorem 35.18. The underlying field may be taken inside the common field of Corollary 37.8. Write $G = G_L$ and $G' = \ker(\kappa_\ell|_G)$. A prime of R_0 is called irreducible here if its determinant over a geometric residue field is irreducible. The support of M_K consists exactly of the primes containing the kernel of the good Hecke map.

Consider characteristic-ℓ primes $\mathfrak{p}$ of R_0 of quotient dimension one with the following concrete matrix realizations. After a finite extension of the fraction field of $R_0/\mathfrak{p}$, take an ordinary representation on a lattice over a complete DVR with finite residue field. Its determinant and ordered local characters realize the given point, its matrices satisfy the words of C_0, and it has the symmetric conjugate-dual polarization of Proposition 30.11. Over the fraction field it is the sum of a line and an absolutely irreducible plane, polarized separately. After removing the common normalizing character the line reduces to 1 and the plane has residual semisimplification $\bar{\Psi} \oplus \bar{\Psi}^{-1}$. At each chosen place above ℓ the three characters on its ordered flag are pairwise distinct on inertia in G'. These conditions are on the matrices of the curve. The selected two-constituent curves of Theorem 30.13 satisfy them.

Lemma 37.9 (The projective image of the plane)**.** *At every such curve the plane remains absolutely irreducible on G'. In each nonempty relatively open subset of G' there is an element having three distinct eigenvalues on the full representation.*

Proof. Work over an algebraic closure of the curve field. The two local characters on the plane are among the characters of the ordered flag. Indeed the characteristic polynomials on a triangular local group have those three diagonal characters, and semisimplification of its restriction to the two summands respects this multiset. On inertia in G' the ratio of the two plane characters takes a value different from one by hypothesis. Its residue is one, since on that inertia all three residual characters coincide. In characteristic ℓ a nonidentity element of $1 + zk[[z]]$ has infinite order: raising it to an ℓ-power raises the first nonzero term of its difference from one to that power, and raising to an integer prime to ℓ leaves its order of vanishing unchanged. Here z is a uniformizer, with the same argument after finite extension. The projective image of the plane is therefore infinite.

Its Zariski closure in PGL_2 is the whole group. For the details, a proper positive-dimensional connected subgroup is solvable. This follows also from its dimension being at most two and the

fact that a nontrivial connected semisimple group has dimension at least three. Such a subgroup preserves a line by triangularization. If it has nontrivial unipotent radical, the common fixed line of that radical is unique, hence is preserved by the full closure, contradicting absolute irreducibility. Otherwise the identity component is a torus. Its normalizer preserves its pair of eigenlines and the full group must interchange them. Changing the sign on one of those lines then intertwines the plane with its twist by a nontrivial quadratic character η'. Semisimplifying the residual representations gives

$$\{\bar{\Psi}, \bar{\Psi}^{-1}\} = \{\bar{\eta}'\bar{\Psi}, \bar{\eta}'\bar{\Psi}^{-1}\}.$$

The reduction of a nontrivial quadratic character is nontrivial at odd ℓ. It must interchange the two distinct residual characters, which would give $\bar{\Psi}^4 = 1$. On inertia at ℓ, $\bar{\kappa}_\ell$ has full image and $0 < 4r < \ell - 1$. This excludes the torus case.

The quotient G/G' is abelian. The projective image of G' is therefore Zariski dense in PGL_2, since it contains the image of the commutators and PGL_2 is its own derived group. This gives the irreducibility assertion. Every open subgroup $U \subseteq G'$ has finite index, so its projective image is also dense: finitely many translates of a proper closed subset cannot cover the connected irreducible group PGL_2. Divide the plane representation by its line character. The closure of the resulting image of U in GL_2 contains SL_2. In fact the closure of its commutators lies in SL_2, and its projective image is dense in PGL_2, so its dimension is at least three. Every translate of SL_2 contains a diagonal matrix of its prescribed nonzero determinant having distinct entries both different from one. Thus a coset of U cannot have its image contained in the closed locus of matrices with a repeated eigenvalue or eigenvalue one. The line in this normalization has eigenvalue one. Every nonempty relatively open subset of the profinite group G' contains a coset of such a U, as required. □

Proposition 37.10 (One patched module for a finite list of curves)**.** *For any finite list of the curves above, the auxiliary places, the limiting group and the localized patched module can be chosen simultaneously. Let q be the common number of places and $D = 3([F : \mathbb{Q}] + q)$. Adjoin a parameter s' as in Section 35.2. Write T for the patched ring with the trace relations, $\mathcal{P}$ for the finite patched module and P for the ordered splitting factor over the augmented original ring. At an ordered curve let H be the completed localization of T and $\mathcal{P}_H = H \otimes_T \mathcal{P}$. If a prime in $\mathrm{Supp}_H \mathcal{P}_H$ has irreducible determinant, then there is a prime Z of H in that support, with irreducible determinant, contained in every prime where the determinant is irreducible or the geometric fiber of the patched Galois algebra is nonsemisimple. Here the curve parameter is sent to a uniformizer after a finite extension and the ordering is compatible with the residual roots.*

Proof. We explain how the local calculations apply to this list. At each curve the line and plane lattices can be taken as the intersections with the given lattice. A coefficient-place inertia element with three distinct residual eigenvalues gives three integral eigenprojectors by Hensel lifting. Each projector lies on one summand, so the lattice is their direct sum, and the ordinary flag consists of sums of these eigenlines at the chosen place. The same argument works at the other coefficient places. The form on each summand is symmetric and nondegenerate. All required changes of coordinates over the fraction field and square roots for orthonormal coordinates are defined over finite extensions.

On G' the two summands are absolutely irreducible and inequivalent by the lemma and their dimensions. Hence the block algebra with its square-zero copies of the off-diagonal ordinary cohomology has full span on G' by the proof of Lemma 30.14. In that proof the only inputs for the off-diagonal span are independence modulo coboundaries and the vanishing of the

intertwiners between the two summands on G'. Theorem 30.15 then detects its twisted dual by the trace evaluations. Intersect the relatively open dense regular loci from the lemma for the finitely many curves, which preserves density. The finite matrix of cocycle relations in Proposition 30.16 and the simultaneous Chebotarev argument of Theorem 30.17 now give one list of separating Frobenius elements and auxiliary levels for all these curves. In this application the open groups are pulled back to the common restricted-ramification group; the maps on cyclotomic kernels are open surjections.

At the old places inertia has the same fixed prime-to-ℓ quotient. At the chosen place above p_0 the local intertwiner vanishes for every representation with the given residual characters by Lemma 30.12. At coefficient places the off-diagonal character ratios on cyclotomic-kernel inertia are nontrivial by the assumed separation. These are precisely the vanishing inputs of Lemma 30.18. The ranks of the upper triangular spaces and the real skew-adjoint spaces depend only on the two block sizes and the symmetric forms. Thus Proposition 30.19, the local duality count of Theorem 30.20, and the lifting argument of Theorem 30.25 apply with each curve's own off-diagonal rank b. In the lifting argument the projection from the mixed upper triangular module onto the diagonal one uses a local upper triangular representative for each chosen ordinary class; these exist by the definition of its ordinary subspace. The finite tests pass to the single limiting group by the compactness proof of Theorem 34.32. This supplies the bound D on the paired ordinary rank, the relations on diagonal cocycles and the identification of off-diagonal cohomology used in Sections 35.3 and 35.4.

Use these same auxiliary levels in Theorem 35.3. The comparison maps, the monomial description of the graded module for the diamond ideal and the trace relations are identities on the ordinary modules at these levels. Their proofs use the residual ordering and the discriminant element f; they use a classical family only in the separate assertion of nonzero fiber at the curves selected there. We use the identities on $\mathcal{P}$ before deciding which fibers are nonzero. In particular T maps onto $P[1/f]$ with diamonds zero and the diamond quotient of $\mathcal{P}$ is $(P \otimes_{R_0} M_K)[1/f]$. The discriminant f survives at every curve in the list.

Now suppose the local module has an irreducible prime in its support. It is nonzero and its residue fiber is nonzero by Nakayama. Its depth is at least $D+1$ by Lemma 35.5: the diamond variables are regular, and on their fiber the uniformizer and the translated weight coordinates form a regular sequence. The coefficient DVR $(O'\widehat{[[s']]})_{(\varpi)}$, formed by completing the localization, acts faithfully through H. Indeed ϖ is a nonzerodivisor on the nonzero diamond quotient, so no power of ϖ vanishes in H. Here O' is the unramified extension of the original coefficient ring whose residue field supplies the curve constants.

The block calculation of Proposition 35.9 and Theorem 35.10 now gives a finite off-diagonal module N over H. Its support is exactly the union of the two loci in the proposition. In Corollary 35.11 the identification of its minimal generator space with the dual of ordinary off-diagonal cohomology uses the matrix words and the split local flags. The further conclusion $b \geq 1$ follows here because the given irreducible prime lies in the support of N. With this observation and the coefficient DVR just constructed, the relations of Theorem 35.15 apply. Choose a minimal prime Z of the module support below the given irreducible prime. Irreducibility persists under generization: in the displayed block algebra it is the nonvanishing of the ideal of off-diagonal products. The prime Z is associated to $\mathcal{P}_H$, hence $\dim H/Z \geq \operatorname{depth}_H \mathcal{P}_H \geq D+1$. Proposition 35.17 puts Z below every prime in $\operatorname{Supp}_H N$, proving the claim. □

Lemma 37.11 (Lifting a chain to the ordered curves)**.** *Let $\mathfrak{p}_0, \dots, \mathfrak{p}_h$ be a list as above. Suppose $\mathfrak{e}_i \subseteq \mathfrak{p}_i \cap \mathfrak{p}_{i+1}$ are primes of R_0 for $0 \leq i < h$. After a common extension of constants and adjoining s', there are ordered curve primes $\mathfrak{x}_i$ of $P[1/f]$ and primes $\mathfrak{b}_i \subseteq \mathfrak{x}_i \cap \mathfrak{x}_{i+1}$ over $\mathfrak{e}_i$. Every prime of R_0 contained in an endpoint $\mathfrak{p}_i$ lifts to a prime below the designated $\mathfrak{x}_i$.*

Proof. First record a lifting fact for a finite faithfully flat integral algebra. Starting over $\mathfrak{p}_0$, go down by flatness to a prime over $\mathfrak{e}_0$; go up by integrality from that prime to one over $\mathfrak{p}_1$. Alternating these steps lifts the whole chain. Going down also lifts an endpoint generization with the endpoint fixed.

Split the finitely many designated Frobenius polynomials and realize the lattices and forms over finite extensions of the curve fields. Choose a finite field k' containing all their residue fields and let O'/O be the unramified extension with residue field k'. Apply the lifting fact to $R_0 \to R' = O' \otimes_O R_0$. At a lifted curve the residue field is a factor of $\kappa(\mathfrak{p}_i) \otimes_k k'$, where k is the original residue field. This choice amounts to an embedding of the constants. Extend the chosen curve field by k' with that embedding. Its valuation ring is isomorphic to $k'[[z_i]]$ with a uniformizer z_i. Over $B = R'[[s']]$ take the prime obtained by evaluating s' at z_i in this ring. Evaluation is defined by convergence; its image contains $k'[[z_i]]$ from $O'[[s']]$. The residue field at the prime is therefore $k'((z_i))$. A lifted link $\mathfrak{e}'_i$ extends to the prime $\mathfrak{e}'_i B$, because the quotient is the power series ring over the domain $R'/\mathfrak{e}'_i$; here Noetherianity identifies the kernel of coefficient reduction with the extended ideal. Its coefficients vanish on both adjacent curves regardless of the two choices of uniformizer. The same construction lifts a prime below either endpoint.

The splitting algebra for the ordered roots is finite free over B by successive monic polynomial division. Its nonzero residual factor P is a projective summand, hence free of positive rank over the local ring B. Apply the lifting fact again. At every vertex the polynomials split with simple roots in $k'((z_i))$, so the residue field of each lifted prime is exactly this field. Each ordering has $f \neq 0$ there. The links and the endpoint generizations survive in $P[1/f]$, proving the lemma. The corresponding lifting by flatness and integrality occurs in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/PatchingBaseChange/ChainLift.lean`, line 64. □

Theorem 37.12 (Hecke support along a finite chain)**.** *Let $\mathfrak{p}_0, \dots, \mathfrak{p}_h$ be curves as above. For each successive pair suppose there is a prime $\mathfrak{e}_i$ below both, at which either the determinant is irreducible or the geometric fiber of C_0 is nonsemisimple. Assume one irreducible prime below $\mathfrak{p}_0$ belongs to $\operatorname{Supp}_{R_0} M_K$. Then every prime below $\mathfrak{p}_h$ satisfying either of these two conditions belongs to $\operatorname{Supp}_{R_0} M_K$. In particular the initial hypothesis holds when the first curve is a selected two-constituent curve and the initial prime is supplied by Theorem 30.32.*

Proof. Use the single $T, \mathcal{P}$ of Proposition 37.10. In T take the preimages $\mathfrak{t}_i$ of the ordered vertices under $T \to P[1/f]$; let $H_i = \widehat{T_{\mathfrak{t}_i}}$. Lift the initial supported prime below $\mathfrak{x}_0$ by the lemma. Its preimage in T contains the diamonds, and its residue fiber of $\mathcal{P}$ is the extension of the nonzero original residue fiber of M_K, by the diamond quotient formula. Lift the prime to H_0 using faithful flatness of completion of the local ring. Its residue fiber remains nonzero and its determinant remains geometrically irreducible. The proposition therefore supplies a supported irreducible prime Z_0 of H_0 below all the indicated primes there.

Suppose Z_i has been obtained and $i < h$. Lift the preimage in T of $\mathfrak{b}_i$ to a prime Q of H_i by the same faithful flatness. If the original link was irreducible, Q remains irreducible. In the other case its geometric Galois algebra surjects onto the geometric fiber of C_0 after passage to a common algebraically closed field. Indeed the diamonds vanish at this prime. The base matrices satisfy the first auxiliary word by Cayley–Hamilton and the second by triviality of the new inertia; lifting group elements through the surjection of limiting groups generates the target. This is the fiber map proved in Theorem 35.18. A nonsemisimple geometric fiber remains so under algebraically closed field extension, and a quotient of a semisimple algebra is semisimple. Thus Q meets one of the two conditions, and $Z_i \subseteq Q$.

Contract Z_i all the way to the common ring T, obtaining a prime z_i. It lies below the preimage of $\mathfrak{b}_i$, hence below $\mathfrak{t}_{i+1}$. Its determinant is geometrically irreducible: the determinant at Z_i is its scalar extension. It lies in the support of $\mathcal{P}$. In fact the fiber of a finite module after base change at a prime is its original residue fiber tensored with the new residue field. Nonvanishing is equivalent in the two fields. Lift z_i to a prime of H_{i+1}. The same fiber calculation gives a supported irreducible prime there. Applying the proposition at $i+1$ gives Z_{i+1}. This step transports the irreducible generization through the common patched ring; merely knowing the link itself to be supported would not provide that generization.

Finally lift the proposed endpoint prime below $\mathfrak{x}_h$ and then to H_h. Its lift satisfies the same alternative as Q, so contains Z_h and is supported. It lies over the diamond quotient and its fiber is a scalar extension of the original M_K-fiber. That original fiber is nonzero. The argument includes $h = 0$. The contraction used at the induction step is visible in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/AnnihilatorBaseChange/ChainAbs.lean`, line 38 and in `Development/AutomorphicArithmetic/AutomorphicVarieties/PatchingModules/AnnihilatorBaseChange/PmMove.lean`, line 7. □

Using this theorem to bound the minors of Corollary 37.4 still requires connecting the components that occur at the original weights to the supported ones and controlling their integral multiplicities. The estimates in Section 6.6 also involve the signed period on the original field.

37.4 Complete intersections at irreducible ordinary curves

Keep the field, prime and ordinary ring of Section 37.3. Consider a characteristic-ℓ curve $\mathfrak{p}$ of quotient dimension one, with an integral matrix realization after finite extension of its curve field. Require the same ordinary flags, matrix words and symmetric conjugate-dual polarization as there. In this section suppose the generic representation U on three variables is absolutely irreducible, its residual semisimplification is $\xi(\bar{\Psi} \oplus 1 \oplus \bar{\Psi}^{-1})$ for the common normalizing character ξ, and its three ordinary characters on inertia in G' are pairwise distinct at each chosen place. We treat these curves even when they first occur in an intersection with a component already carrying Hecke support.

Lemma 37.13 (Irreducibility on the connected monodromy group)**.** *Over an algebraic closure of the curve field let H be the reduced Zariski closure of $U(G_L)$. Its identity component H° and its connected derived group D both act irreducibly. The group D is semisimple.*

Proof. The restriction to the normal subgroup H° is semisimple. Indeed the sum of its simple submodules is nonzero and is preserved by H, so it is the entire representation. The isotypic spaces are permuted transitively for the same reason. Dimension three leaves just two possibilities for a reducible restriction: a scalar action or three distinct character lines permuted transitively.

In the scalar case the projective image is finite since an algebraic group has finitely many components. At a chosen coefficient place the local eigencharacters on inertia in G' have ratios with residue one. Each ratio of two distinct characters takes a nonidentity value there, of infinite order by the calculation in Lemma 37.9. This excludes a finite projective image.

In the three-line case the permutation action on the lines is continuous. They are defined over a finite extension, and a finite set of distinct points of projective space has discrete induced valuation topology. The residual representation can be put in upper triangular form over a finite residue-field extension. Its diagonal reduction gives an abelian quotient of the compact matrix image whose kernel is pro-ℓ: reduction maps that kernel into a finite unipotent group,

and the principal congruence group is pro-ℓ. Since $\ell > 3$, its image in S_3 is trivial. The permutation quotient is therefore transitive abelian, hence cyclic of order three. In the basis of the permuted lines, conjugation by the diagonal matrix with entries $1, \zeta_3, \zeta_3^2$ in cyclic order gives a self-twist by a character of order three. Its reduction still has order three. Removing ξ, multiplication by this reduced character must permute the multiset $\{1, \bar{\Psi}, \bar{\Psi}^{-1}\}$. The orbit of 1 is the whole set and consists of characters of order dividing three. But $\bar{\Psi}^3 \neq 1$ on inertia at ℓ, since $0 < 3r < \ell - 1$. This proves the first irreducibility assertion.

The unipotent radical of H° has a fixed vector by triangularization; normality makes its fixed space H°-invariant. Hence it acts trivially and is the identity subgroup of this matrix group. The connected reductive group H° is the product of its connected central torus and its connected semisimple derived group. Its central torus acts by scalars by Schur's lemma, so D acts irreducibly. [2] □

Proposition 37.14 (Separable Frobenius tests on every open coset)**.** *The restriction of U to G' is absolutely irreducible. Every nonempty relatively open subset of G' contains an element with three distinct eigenvalues.*

Proof. We first check that every element of H is a scalar times an element of D, over the algebraic closure. As a nontrivial semisimple subgroup of SL_3, D has rank one or two. If its rank is two and the root system is irreducible, it has at least six roots. Its dimension is at least $2 + 6 = 8$, giving $D = \mathrm{SL}_3$. In this case a cube root of the determinant writes any invertible matrix as a scalar times an element of D. A reducible root system of rank two would give two commuting connected semisimple factors with finite central intersection. The restriction of the irreducible representation to either factor is isotypic: its simple socle and its isotypic summands are preserved by the other factor, which commutes with it. Since three is prime, a non-scalar first factor then acts irreducibly, forcing the second to be scalar by Schur's lemma. A positive-dimensional semisimple factor cannot be a scalar subgroup. This excludes that case.

In rank one the based root datum of D has trivial automorphism group. Indeed an automorphism must fix the sole simple root, which spans the rational character space, hence fixes the character lattice. By the isomorphism theorem for pinned reductive groups every algebraic automorphism of D over the algebraically closed field is inner. Equivalently one can move the image of a chosen pinning back by inner conjugation, after which the induced automorphism of the based root datum determines the pinned automorphism uniquely. Conjugation by any $g \in H$ preserves D. Remove an element of D inducing that conjugation. The remaining element commutes with D, hence is scalar by Lemma 37.13. This proves the assertion in rank one as well. [2]

The closure of $U(G')$ contains D. It is normal in H, and the quotient is abelian because G_L/G' is abelian and the image of G_L is dense. This gives irreducibility. Also D contains a matrix with three distinct eigenvalues. In fact on the specified local inertia in G' each of the three character ratios has infinite image. Its kernel is a closed subgroup of infinite index and has empty interior. The complements of the three kernels are open dense sets and have nonempty intersection. An element there has three different local eigenvalues. It is a scalar times a member of D by the preceding paragraphs.

For an open subgroup $U_0 \subseteq G'$ the closure of $U(U_0)$ contains the identity component of the closure of $U(G')$: the latter group is a finite union of cosets of the former closed subgroup. In particular it contains D. For each $g \in G'$ the closure of $U(gU_0)$ contains $U(g)D$, a scalar translate of D. It therefore contains a matrix with distinct eigenvalues. The discriminant is a regular function, so some matrix in the coset image already has nonzero discriminant. This proves the required open-set assertion. □

Theorem 37.15 (The local ordinary algebra at an irreducible curve)**.** *Let $\mathfrak{p}$ be an irreducible curve as above in the support of M_K, and put $\mathfrak{l} = \mathfrak{p} \cap \Lambda$ in its principal weight algebra. Then*

$$(R_0)_{\mathfrak{p}} \;\simeq\; (B_K)_{\bar{\mathfrak{p}}}, \qquad \dim (R_0)_{\mathfrak{p}} = 3[F : \mathbb{Q}].$$

This is a local complete intersection ring, faithfully flat over $\Lambda_{\mathfrak{l}}$; its closed fiber over this weight ring is an Artinian complete intersection. Moreover $(M_K)_{\mathfrak{p}}$ is free of positive rank over the displayed ring. In particular every prime below $\mathfrak{p}$ belongs to the Hecke support.

Proof. Use the augmented and ordered construction of Lemma 37.11 for this curve. The auxiliary level argument in Proposition 37.10 applies with one constituent. Here the diagonal block is the full matrix algebra, by irreducibility on G', and there are no off-diagonal constituent blocks. The preceding proposition gives the dense regular set on which to place all the dual trace tests. The ratios of distinct characters give the local invariant vanishing at the coefficient places, and Lemma 30.12 supplies it at p_0. Thus the first assertion of Theorem 34.32 bounds the rank of the ordinary polarized endomorphism cohomology by $D_0 = 3([F : \mathbb{Q}] + q)$, where q is the number of auxiliary places. The patched module has the same $3q$ diamond coordinates as before.

At an ordered prime $\mathfrak{x}$ above the augmented curve, write $U_* = T_{\mathfrak{P}}$ for the patched local ring, where $\mathfrak{P}$ is the preimage in T, and put $Y = (Y_1, \dots, Y_{3q})$ for the diamond ideal. The base fiber over $\kappa(\mathfrak{x})$ is nonzero: it is a scalar extension of the residue fiber of M_K at $\mathfrak{p}$. Hence the localized finite patched module has depth at least $D_0 + 1$ by Lemma 35.5. On the other hand a first-order lift fixing the augmented coefficient parameter determines an ordinary endomorphism class injectively by Proposition 35.6. To recall the injectivity, the trace pairing on the full matrix algebra recovers the lifted representation up to strict conjugacy. A coboundary fixes every trace; the ordinary projectors then fix the local characters, the distinct Frobenius roots fix the ordered coordinates, and the inertia word fixes the diamonds. These are all the generators. The rank bound and the relative tangent calculation of Lemma 35.5 give embedding dimension at most $D_0 + 1$.

Depth of a nonzero finite module is at most the Krull dimension, which in turn is at most embedding dimension. Thus U_* is regular local of dimension $D_0 + 1$, and the Auslander–Buchsbaum formula gives $\mathcal{P}_{\mathfrak{P}} \simeq U_*^{m}$ for some $m > 0$, as in Theorem 35.7. In particular the diamonds, already regular on this module, form a regular sequence in U_* itself. The natural surjection $U_*/Y \to P_{\mathfrak{x}}$ is injective. Indeed the diamond quotient is $(U_*/Y)^m$, and its action factors through $P_{\mathfrak{x}}$ by the comparison

$$\mathcal{P}_{\mathfrak{P}} / Y \mathcal{P}_{\mathfrak{P}} \simeq P_{\mathfrak{x}} \otimes_{R_0} M_K.$$

A free module of positive rank is faithful. We conclude that $P_{\mathfrak{x}}$ is a quotient of a regular local ring by the $3q$ indicated regular elements, and is a complete intersection of dimension $3[F : \mathbb{Q}] + 1$.

The map $(R_0)_{\mathfrak{p}} \to P_{\mathfrak{x}}$ is flat and local. Indeed the unramified extension of constants, adjoining a formal parameter over a Noetherian ring, the finite free ordered splitting factor and the localizations are all flat. The module $(M_K)_{\mathfrak{p}}$ becomes free of rank m there. A basis of its residue space lifts to m generators by Nakayama. The kernel of this presentation vanishes after faithfully flat extension, where the surjection between free modules of equal rank is an isomorphism. It vanishes already over $(R_0)_{\mathfrak{p}}$. Its annihilator is consequently zero, proving the displayed Hecke isomorphism. The complete intersection property descends through this flat local map by the complete intersection descent theorem for Noetherian local rings. The Stacks Project, Lemma 23.8.9.

Finally M_K is finite free over the principal weight algebra Λ, a power series ring in $3[F:\mathbb{Q}]$ variables over the coefficient DVR on the fixed residual factor. Localizing M_K at $\mathfrak{p}$ preserves flatness over Λ: this localization is a filtered direct limit of copies of the flat module with transition maps given by multiplication by elements of R_0. The free-module formula just proved then gives flatness of $(R_0)_{\mathfrak{p}}$ over $\Lambda_{\mathfrak{l}}$; a flat local map is faithfully flat. The Hecke image is finite over Λ, since it is a submodule of $\mathrm{End}_{\Lambda} M_K$. The integral extension $\Lambda/\mathfrak{l} \subseteq R_0/\mathfrak{p} = B_K/\bar{\mathfrak{p}}$ gives $\dim \Lambda/\mathfrak{l} = 1$. In the regular complete local ring Λ this gives $\mathrm{ht}\,\mathfrak{l} = 3[F:\mathbb{Q}]$. Incomparability in the finite Hecke algebra gives this as an upper bound for the dimension of its local ring, and going down by flatness gives the matching lower bound. The images of a regular system of parameters for $\Lambda_{\mathfrak{l}}$ form a regular sequence there by flatness. The quotient is zero-dimensional by finiteness of the Hecke fibers, hence is an Artinian complete intersection. Further localization preserves the positive-rank freeness of M_K, proving the support assertion. □

Corollary 37.16 (Chains containing both kinds of ordinary curves)**.** *In Theorem 37.12 the vertices may also include the irreducible curves of this section. The same initial hypothesis and the same conclusion hold at endpoints of either type. If the first vertex is irreducible, it suffices that this vertex itself be in the support of M_K.*

Proof. Treat each maximal run of two-constituent vertices by Theorem 37.12. A link below an irreducible vertex has irreducible determinant. To see this directly, choose nine group matrices at the vertex whose trace pairing is nonsingular. Its determinant remains nonzero at every prime below it. Over an algebraically closed field a reducible semisimple representation of dimension three has trace pairing of rank at most five on its group algebra, from the two possible block sizes $1, 2$, or at most three for three characters. This excludes a reducible determinant at the link. Thus upon leaving a run of two-constituent vertices one obtains a supported irreducible link, and the next irreducible vertex is supported by specialization. At this vertex the theorem makes every generization supported. Its next link gives a supported irreducible starting prime for the following run. Induction over the finite list proves the result, including lists of just one type. The assertion for the first irreducible vertex uses that vertex itself as its irreducible starting prime. □

The statements apply to any exhibited chain with these inertia and matrix conditions. A global argument that the components relevant to the fixed weights are so reached, and a quantitative comparison of the resulting integral orders with the signed theta periods, are still needed in Section 6.6.

37.5 Dimensions of intersections that transfer Hecke support

Use the ordinary ring R_0, the Galois algebra C_0, the module M_K and the fields L/F of Section 37.3. Write I_K for the kernel of $R_0 \to B_K$, and ϖ for a uniformizer of the coefficient ring O. We measure dimensions on the full three-character weight space of the same arithmetic problem. Set

$$d = [F:\mathbb{Q}], \qquad d_u = [F_u:\mathbb{Q}_\ell], \qquad b = \min\bigl(\{d-1\} \cup \{d_u - 1 : u \mid \ell\}\bigr), \qquad m = 3d - b.$$

Here $b \geq 5$ by Theorem 30.8. At each $u \mid \ell$ fix one of the two places of L over u. Let H_u be the principal units of F_u, and let $H = \prod_{u\mid\ell} H_u$. On the chosen finite character factor the weight algebra and its special fiber are $\Lambda = O[[H^3]]$ and $\bar{\Lambda} = k[[H^3]]$, where $k = O/(\varpi)$. Write $\lambda_{u,i}$ for the i-th universal character on H_u, $1 \leq i \leq 3$. A geometric determinant at a prime means its associated semisimple representation over an algebraic closure of the residue field. The fixed cyclotomic twist may be removed in statements about conjugate duality. In characteristic ℓ it acts trivially on principal units.

Proposition 37.17 (Two exceptional sets of weights)**.** *There is a closed subset $W \subset \operatorname{Spec} \bar{\Lambda}$ of dimension at most m with the following property. At a characteristic-ℓ prime of R_0 of quotient dimension one whose weight is outside W, the three ordinary characters are pairwise distinct on inertia in the full cyclotomic kernel at each chosen place. Its geometric determinant is irreducible or is a sum of a character and an absolutely irreducible plane.*

Proof. Put $H_u^0 = \ker(N_{F_u/\mathbb{Q}_\ell}|_{H_u})$. This is a free direct summand of rank $d_u - 1$, by the local unit calculation in Theorem 30.8. For each $u, i \neq j$, impose $\lambda_{u,i}(h) = \lambda_{u,j}(h)$ for all $h \in H_u^0$. The equations kill the copy of H_u^0 embedded in H^3 with coordinates h, h^{-1} in positions i, j at u. Projection to position i and a projection of H_u onto H_u^0 split this embedding. In power series coordinates these equations therefore cut out codimension $d_u - 1$. Local reciprocity identifies the cyclotomic character on inertia with the local norm on units, up to inversion. In particular norm-one principal units lift to inertia in its kernel. Equality of the characters on that inertia forces these equations.

For the second exceptional set let $\mathcal{E} = \{e \in \mathcal{O}_F^\times : e \in H_u \text{ for all } u \mid \ell\}$. It has finite index in the unit group. For each choice of two distinct row indices i_u, j_u at every u, impose

$$\prod_{u \mid \ell} \lambda_{u,i_u}(e)\lambda_{u,j_u}(e)^{-1} = 1 \qquad (e \in \mathcal{E}). \tag{37.1}$$

The diagonal closure of $\mathcal{E}$ in H has $\mathbb{Z}_\ell$-rank $d-1$. This is Leopoldt's theorem for the totally real abelian field F. The map $H \to H^3$ putting the u-coordinate with exponents $1, -1$ in these two rows admits a left inverse obtained by reading row i_u for each u. In Smith normal form the embedded closed subgroup is generated by ℓ^{a_i} times distinct basis vectors of H^3, $1 \leq i \leq d-1$. In characteristic ℓ its equations have radical $(X_1, \dots, X_{d-1})$: indeed $(1 + X_i)^{\ell^{a_i}} - 1 = X_i^{\ell^{a_i}}$. Equations for a dense subgroup give the same closed ideal as those for its closure by completeness and closedness of ideals in the Noetherian complete local ring. Thus (37.1) has codimension $d-1$. [3]

Suppose now that the curve determinant splits into three characters. They can be taken over a finite extension of the curve field. The integral closure of the curve ring in that extension is a complete DVR. The finite R_0-algebra C_0 acts on the three simple summands through its specialization. The values of every group element and its inverse on a character are integral over the curve ring, so are units of the valuation ring. The characters are continuous: the map from the finite algebra to that valuation ring is a linear map of finite modules over the curve ring, and the original group map is continuous. After removing the common twist, order the characters by the three distinct residual characters $\bar{\Psi}, 1, \bar{\Psi}^{-1}$. The conjugate inverse of the first must be the third, by the polarization and its reduction. At the chosen local place these two characters occupy some distinct rows i_u, j_u. Indeed the ordinary polynomial has the prescribed local characters as its three roots on every local group element, hence these are the three local constituents by uniqueness of semisimplification. One can also recover their row indices by the separating residual inertia element.

Divide the first global character by its residual character, lifted into the finite constant field. The quotient takes values in principal units of the curve DVR. This group is torsion-free and pro-ℓ. At a finite place of residue characteristic different from ℓ, local units are an extension of a finite group by a pro-group of that residue characteristic. Their image under the quotient character is therefore trivial. At a chosen place above u its value on $e \in \mathcal{E}$ is $\lambda_{u,i_u}(e)$; at the conjugate place it is $\lambda_{u,j_u}(e)^{-1}$. This uses that e is real and that residual characters on principal units are trivial. Complex places contribute trivially. Global reciprocity on the principal idele of e gives (37.1). Taking the finite union of the two sets of loci just constructed proves both assertions. □

Lemma 37.18 (Specialization of the polarized matrices)**.** *Let $J \subseteq \mathfrak{p}$ be primes of R_0 with geometrically irreducible determinant at J. Suppose $\mathfrak{p}$ is a characteristic-ℓ curve of quotient dimension one with weight outside W. It has an integral ordinary, symmetrically polarized realization of one of the two kinds in Sections 37.3 and 37.4.*

Proof. Work in the normalization with the common twist removed, so that $g^\star = cg^{-1}c$ induces the adjoint and fixes the trace. Over a geometric field the semisimple representation realizing the determinant factors through C_0: its algebra is a product of full matrix algebras, the trace pairing there is nondegenerate, and the trace of each defining word against every group element is zero. Here the residual characters are distinct, so there are no repeated constituents at the curve. More generally even multiplicities at most three are units in the present characteristics. The group matrices span this semisimple algebra. Trace duality shows that $\star$ descends to it: if x is in the kernel, then $t(hx^\star) = t(xh^\star) = 0$ for every group element h. At J the resulting anti-involution of M_3 is adjunction for a symmetric form. Indeed an anti-involution of a split matrix algebra is transposition followed by inner conjugation. The form matrix has transpose equal to itself up to sign; a nondegenerate alternating form has even dimension.

We check the sign on a plane after specializing. Take the separating element γ of Theorem 30.8, set $Z = \gamma - \gamma^\star$, let $a = t(Z^2)/2$, and write $E = Z^2/a$ in the group algebra over R_0/J. The residue of a is $(\bar{\Psi}(\gamma) - \bar{\Psi}(\gamma)^{-1})^2 \neq 0$, so a is a unit. At the generic point of R_0/J, skew adjunction gives the eigenvalues $0, \sqrt{a}, -\sqrt{a}$ of Z. Thus E is the orthogonal projector onto a nondegenerate symmetric plane. Skew-adjoint endomorphisms on that plane form a one-dimensional space, identified with alternating bilinear forms. For any three group elements g, h, k_0 the identity

$$t([\,E(g - g^\star)E,\, E(h - h^\star)E\,]k_0) = 0$$

holds at that generic point. Its left side is an element of the domain R_0/J, so it vanishes at $\mathfrak{p}$.

If the curve determinant is irreducible the odd-dimensional argument already gives its symmetric form. Otherwise the preceding proposition gives the algebra $M_2 \times M_1$. The involution fixes both factors since they have different dimensions. It is the identity on the line. On the plane Z is traceless with trace of its square $2a$, so $Z^2 = a$ there. Consequently E projects to the plane. The displayed identity and the nondegenerate trace pairing show that any two skew-adjoint endomorphisms of this plane commute. Indeed every such endomorphism is half its difference from its adjoint, and the group matrices span the algebra. For an alternating form in dimension two, both off-diagonal matrix units in a symplectic basis are skew adjoint and their commutator is nonzero. This excludes alternating sign. The line is self-dual for $\star$, so its residual character is 1: the two outer residual characters are interchanged by conjugate inversion and are distinct.

A parameter T of the complete one-dimensional domain $R_0/\mathfrak{p}$ makes it finite over $k[[T]]$. To see finiteness, lift a basis of its quotient by T and expand successive remainders using completeness. Its fraction field is a finite extension of $k((T))$. The simple matrices, block decomposition and forms are defined over a further finite extension since their finitely many structure constants belong to the algebraic closure. The span of the translates of any basis over the valuation ring is a stable lattice: it is contained in the module obtained by applying a finite list of generators of C_0 to that basis and taking their integral span. This also realizes the action continuously on the lattice. The matrix words of C_0 hold there.

For the local flags, take γ_u on inertia with three distinct residual eigenvalues. Take its eigenbasis in the order of the three local characters. In the ordered word $\prod_{i=1}^{3}(\rho(h_i) - \psi_i(h_i)) = 0$, set $h_i = \gamma_u$ except for $h_k = g$. At row $i \geq k$ and column $j \leq k$ the factors to the left and right of the k-th factor have invertible entries on the indicated row and column respectively. Thus

the ij-entry of $\rho(g) - \psi_k(g)\mathrm{id}$ vanishes for these indices. The resulting flag is upper triangular with the prescribed diagonal. Finally apply the lattice construction in Proposition 30.11. It uses only this flag, a stable lattice, the symmetric form and the distinct residual eigenvalues of Z: rescaling its eigenlines first splits the reduction and then averaging the exponents with those of the dual lattice makes the lattice self-dual. Intersecting with each local flag gives saturated steps. Restoring the common twist gives the required realization. □

Theorem 37.19 (Hecke support across a large intersection)**.** *Let J be a prime of R_0 containing I_K with geometrically irreducible determinant. Suppose P is a prime containing $J + (\varpi)$ and $\dim R_0/P > m$. Then every prime $Q \subseteq P$ at which the determinant is geometrically irreducible or the geometric algebra $C_0 \otimes_{R_0} \overline{\kappa(Q)}$ is nonsemisimple contains I_K. If the determinant at P is geometrically irreducible, the map from $(R_0)_P$ to the local Hecke algebra is an isomorphism, $(R_0)_P$ is a complete intersection, and $(M_K)_P$ is free of positive rank.*

Proof. The domain $D = R_0/P$ is a quotient of B_K and hence finite over $\bar{\Lambda}$. The inverse image of W in $\operatorname{Spec} D$ has dimension at most m. Indeed on each component of this inverse image the domain is finite integral over the quotient by its contraction, and their dimensions agree. Hence the complement is a nonempty open set. If P is geometrically irreducible, choose nine group elements for which the trace Gram determinant is nonzero modulo P, and also require that this determinant remain nonzero. The two open conditions still meet because D is a domain.

Every nonempty open set in this spectrum contains a prime of quotient dimension one. Choose a nonzero $f \in D$ such that the principal open where $f \neq 0$ is contained in the prescribed open. We may multiply by a nonzero nonunit to make f a nonunit. Write $n = \dim D$. If $n = 1$ use the zero prime. Otherwise D is a complete local domain, hence catenary and equidimensional, and each minimal prime over f has height one. Thus f extends to a system of parameters $f, u_2, \dots, u_n$. A minimal prime q over $(u_2, \dots, u_n)$ has height at most $n-1$ and cannot contain f, since the full parameter ideal is primary to the maximal ideal. In the local domain D/q the nonzero image of f generates an ideal with maximal radical. The principal ideal theorem gives $\dim D/q = 1$.

Lift this prime to $\mathfrak{p} \subset R_0$. It contains P, has the required weight, and lies above the supported irreducible prime J. The preceding lemma supplies its matrix realization. If the curve is irreducible, Theorem 37.15 puts every prime below it in the support. If the curve has two constituents, apply Theorem 37.12 to this single curve with starting prime J. Its conclusion applies to each of the stated Q. If P is irreducible, the extra Gram condition forces the curve to be irreducible as well: nine independent matrices are impossible for a sum with a proper constituent. Localize the complete intersection and the positive-rank free module of Theorem 37.15 further at P. Localization preserves the complete intersection property and freeness; the localized Hecke kernel remains zero. This proves the last assertions. □

Corollary 37.20 (The component through the cuspidal adjoint)**.** *Let J_0 be the kernel of the full component map through the cuspidal adjoint in Proposition 30.34. Every minimal prime over $J_0 + (\varpi)$ satisfies the dimension hypothesis of Theorem 37.19. More generally let $J_0, J_1, \dots, J_s$ be primes of R_0 with geometrically irreducible determinants and*

$$\dim R_0/(J_i + J_{i+1} + (\varpi)) > 3d - b \qquad (0 \le i < s).$$

Then all J_i contain I_K. The displayed bound holds in particular if $\dim R_0/(J_i + J_{i+1}) \ge 3d - b + 2$.

Proof. The full adjoint component is a finite domain containing Λ and is a quotient of B_K. It has dimension $3d+1$. Each minimal prime over ϖ has height one by the principal ideal theorem;

catenarity of the complete local domain gives quotient dimension $3d$. For a successive pair, choose a minimal prime P over the displayed sum of ideals attaining its quotient dimension. With $J = J_i$ and $Q = J_{i+1}$ the theorem transfers the Hecke support, starting with the given quotient at J_0. The final sufficient bound follows since quotienting a Noetherian local ring by one element of its maximal ideal lowers its dimension by at most one. □

The corresponding choice of a curve above a large intersection in marked representation coordinates occurs in `Development/AutomorphicArithmetic/AutomorphicVarieties/CurveDVRSpecialization/ExistsNicePrimeAboveOfLargeDim.lean`, line 7. The corollary gives the transfer on the original determinant ring whenever the indicated intersections exist. Establishing coverage of the components at the arithmetic weights, and bounding the integral multiplicities there, are separate steps still needed for Section 6.6.

37.6 Intersections in the marked ordinary deformation ring

Fix the same $L/F, \ell, \bar{\Psi}$ and $d = [F : \mathbb{Q}]$ as in Section 37.5. We use $\Gamma = \Gamma_a$, $H = \Gamma_a^0$, and a complex conjugation c as in Section 30.2. Remove the common cyclotomic twist and let J be the antidiagonal matrix with entries one on its antidiagonal. Let k' be a finite extension of the residual coefficient field and O'/O its unramified lift. Fix a continuous matrix representation $\rho_* : H \to \mathrm{GL}_3(k')$ with semisimplification $\bar{\Psi} \oplus 1 \oplus \bar{\Psi}^{-1}$, satisfying the ordinary conditions of Section 30.3. Assume it has the specified pairing J and marking

$$\rho_*(\gamma) - \rho_*(c\gamma^{-1}c) = \delta\ \mathrm{diag}(1, 0, -1), \qquad \delta = \bar{\Psi}(\gamma) - \bar{\Psi}(\gamma)^{-1} \neq 0.$$

This includes the split residual representation. Write A_* for the complete local O'-algebra of its marked ordinary matrix lifts on H, with all three ordinary weights variable. The marking at a lift has the same shape with a scalar lifting δ. Matrices in the chosen basis are part of the data. The presentation by generator matrices and local words in Theorem 30.13 works for this residual point: the kernel of its finite image has finitely generated maximal pro-ℓ quotient by Theorem 30.8. On the finite old inertia quotients the residual type lifts uniquely up to conjugacy. Indeed their order is prime to ℓ, and averaging an intertwiner lifting the identity makes it equivariant and still invertible at every small extension. The trace and the ordered characters, with the common twist restored, give a map $R_0 \to A_*$.

Lemma 37.21 (Tangent space and obstruction for the marked matrices)**.** *Let $W = M_3(k')$, with H acting by conjugation through ρ_* and c acting by $X \mapsto -JX^{\mathsf{T}}J$. At each $u \mid \ell$ let $b_u \subset W$ preserve the residual ordinary flag at the chosen place. Let H^i_C be the cohomology of the mapping fiber of the continuous restriction of cochains on Γ with values in W to the direct sum of the local cochains with values in W/b_u; put $h_i = \dim_{k'} H^i_C$. Then the relative tangent dimension of A_*/O' is $h_1 + 1 - h_0$. For a small extension with kernel V killed by the maximal ideal, there is a functorial obstruction in $H^2_C \otimes_{k'} V$; it vanishes exactly when the marked ordinary lift exists.*

Proof. First allow change of basis by matrices reducing to the identity. Put $\iota(g) = J(g^{\mathsf{T}})^{-1}J$. The normalized polarized problem on H extends to $\Gamma = H \rtimes \langle c \rangle$ with target $\mathrm{GL}_3 \rtimes \langle \iota \rangle$ and the prescribed projection to the group of order two. In the marked basis c has image ι. The derivative of this action is the indicated action on W. For each local group the flag with the given ordering is unique over every coefficient ring. Indeed the three distinct residual eigenvalues of a separating inertia element lift to distinct eigenvalues with unit differences. Its three projectors give eigenlines. The ordered words force the upper triangular flag in their prescribed order by the row and column calculation of Lemma 37.18. Conversely a stable ordered flag gives the words and its diagonal characters.

For a lift to the dual numbers write its variation as $z(g)$ in the Lie algebra, and write the variation of each flag as $-X_u \in W/b_u$. The representation equation is $dz = 0$; flag stability is $z|_{G_{F_u}} = dX_u \bmod b_u$. Changing basis changes this pair by the coboundary of a single $X \in W$. These are exactly the degree-one cocycles and coboundaries of the mapping fiber. Its degree zero is the space of infinitesimal automorphisms commuting with the extended action and preserving the flags.

We record the lifting calculation. Given a small extension $D' \to D$ as in the statement, lift the global matrices as a continuous set map with the prescribed projection, and lift each local flag by lifting an adapted basis. The multiplicative defect of the lifted global matrices is a 2-cocycle in $W \otimes V$: write it as $\widetilde{\rho}(g)\widetilde{\rho}(h)\widetilde{\rho}(gh)^{-1} - 1$ in the identity component and use associativity and $V^2 = 0$. In bases for the lifted flags the local matrices reduce to upper triangular matrices over D. Lift those upper triangular matrices in the flag stabilizer by a set map. The discrepancy from the global lift gives a local 1-cochain modulo b_u. Its differential is the restriction of the global defect modulo b_u, because the multiplicative defect of the stabilizer lifts is in $b_u \otimes V$. Changing matrix lifts changes this pair by a fiber coboundary; changing flag lifts changes the local cochains by local coboundaries. The resulting class in degree two vanishes precisely if one can subtract a global 1-cochain to kill the multiplicative defect and move the local flags so that all local discrepancies vanish modulo their stabilizers. This yields the required representation with stable flags. All operations commute with maps of the kernel V. For Artinian coefficient rings the set lifts can be chosen on finite images. The same argument applies to complete coefficient rings with finite kernel: a sufficiently deep open ideal of D' has zero intersection with V, and maps homeomorphically onto an open additive subgroup of D. Choosing representatives on its finitely many cosets gives a continuous section, usable for each matrix entry.

It remains to enforce the marking, both in this lifting test and in the tangent count. Suppose the lift over D is already marked. If the image of c upstairs is $a\iota$, its square being one says that aJ is symmetric. It reduces to J over D. Conjugate by $1 - \frac{1}{2}(aJ - J)J$ to put it back to J: under this change aJ transforms by congruence, and products of two kernel entries vanish. Once c is fixed, the operator $Z = \rho(\gamma) - \rho(c\gamma^{-1}c)$ is skew adjoint. Its three eigenvalues lift the unit-separated roots on the base; they are $z, 0, -z$. Choose eigenvectors lifting the original coordinate vectors. The two outer vectors pair to a unit and the middle vector pairs with itself to a unit, both equal to one over D; every other pairing vanishes by skew adjunction. Rescale one outer vector and the middle vector to make these pairings one, using the square root congruent to one for the middle vector. This is an orthogonal change of basis that reduces to the identity over D. It gives the marked lift, preserving the flags under transport.

On dual numbers every change between two marked representatives fixes c and the ordered eigenspaces of Z. Its infinitesimal generator is therefore diagonal and skew adjoint for J, hence a multiple of $\operatorname{diag}(1, 0, -1)$. Conversely each such generator acts by change of marked basis. Its stabilizer is exactly H^0_C, by the description of infinitesimal automorphisms above. Passing from unmarked tangent classes to marked matrices thus adds $1 - h_0$ dimensions. This proves the lemma. □

Theorem 37.22 (Number of relations in the marked ordinary ring)**.** *There is a presentation*

$$A_* = O'[[X_1, \dots, X_g]]/(f_1, \dots, f_s), \qquad g - s \geq 3d + 1.$$

It holds at every finite residual matrix point specified above, including those with nonsplit residual matrices.

Proof. The module $W^\vee(1)$ has no global invariants. In fact Theorem 30.8 gives an element of H with $\bar{\Psi} = 1$, $\bar{\kappa}_\ell = -1$. All eigenvalues of its conjugation on W are one, even if its action is

unipotent. All eigenvalues on $W^\vee(1)$ are therefore minus one. In Lemma 30.7 the dimensions of the local Borel spaces are six. At each real place the conjugation has image $a\iota$ with aJ symmetric and nonsingular. Its invariants in W are the skew-adjoint matrices for that form, of dimension three. The Euler formula gives

$$h_1 - h_0 - h_2 = 6d - 3d = 3d.$$

Use $g = h_1 + 1 - h_0$ relative tangent generators to obtain a surjection $P = O'[[X_1, \dots, X_g]] \to A_*$ by completeness and Nakayama's lemma. For its kernel D, minimality of the relative tangent generators gives $D \subseteq \varpi P + \mathfrak{m}_P^2$. We claim that the minimal number of generators of D is at most h_2. Apply the obstruction calculation to $P/\mathfrak{m}_P D \to A_*$, whose kernel is $V = D/\mathfrak{m}_P D$, and to the universal marked matrices over A_*. Contracting its obstruction with $\lambda \in V^\vee$ gives a linear map $V^\vee \to H^2_C$. Suppose a nonzero λ maps to zero. Quotient the extension by its kernel in V. Functoriality and the lifting test give a lift of the universal marked matrices to the resulting ring D', hence a section $A_* \to D'$ of $D' \to A_*$. The induced map $P \to A_* \to D'$ and the canonical $P \to D'$ agree modulo the one-dimensional small kernel and on O'. Their difference is a derivation into that kernel for the residual action. It kills $\varpi P + \mathfrak{m}_P^2$, and so kills D. But the first map sends D to zero and the second maps it onto the nonzero kernel, a contradiction. Hence the linear map is injective. Taking $s = \dim V$ and lifting generators, we obtain $g - s \geq h_1 + 1 - h_0 - h_2 = 3d + 1$. The corresponding presentation in the source is `Development/AutomorphicArithmetic/AutomorphicVarieties/FrobeniusSmoothVectors/HasObs.lean`, line 26. □

Proposition 37.23 (Dimensions of marked component intersections)**.** *Each irreducible component of* $\operatorname{Spec} A_*$ *has dimension at least* $3d + 2$*. Its components form a connected graph when two different components are joined if their intersection has dimension at least* $3d + 1$*. In the special fiber* A_*/ϖ*, the corresponding bounds are* $3d + 1$ *for components and* $3d$ *for their intersections.*

Proof. Let P' be a complete regular local ring of dimension N and I an ideal generated by s elements in its maximal ideal. Minimal primes of I have height at most s, so their quotient dimensions are at least $N - s$. For the intersection bound, suppose the minimal primes admit a partition into two nonempty lists with every cross intersection of dimension less than $N - s - 1$. Let U, V be the intersections of the primes on the respective lists. At a minimal prime $\mathfrak{q}$ of $U + V$ one has $h = \operatorname{ht} \mathfrak{q} \geq s + 2$, since $\mathfrak{q}$ contains a prime on each list and the regular local ring is catenary. Complete $(P')_{\mathfrak{q}}$, obtaining a regular complete local domain of dimension h. Each of the quotients by U and V still has positive dimension there: below $\mathfrak{q}$ it has a prime of height at most s, and completion preserves quotient dimension. Hartshorne–Lichtenbaum vanishing makes both top local cohomology groups with these supports zero. The group in degree h supported on $U + V$ is nonzero by nonvanishing for the maximal ideal. Mayer–Vietoris now requires nonzero local cohomology of $U \cap V$ in degree $h - 1$. This is impossible: $U \cap V = \sqrt{I}$ before localization, flatness preserves intersections, and the extended ideals still have the same radical. The Čech complex on the s generators of I vanishes above degree s. This proves the graph statement for P'/I. The Stacks Project, Section 51.16.

Apply this calculation to the power series presentation over O', where $N = g + 1$, and to its reduction over k', where $N = g$. The number of generators in the reduction is at most s. The theorem gives the claimed bounds in both rings. □

Theorem 37.24 (Loss of dimension on passage to traces)**.** *Let* P_* *be a prime of* A_* *and let* $p = P_* \cap R_0$*. If the geometric determinant at* p *is irreducible or is a sum of a character and*

an irreducible plane, then

$$\dim A_*/P_* \le \dim R_0/p + 1.$$

Proof. We first realize the marked problem inside an affine scheme of finite type over $R_0 \otimes_O O'$. Take finitely many module generators of the finite algebra C_0. Give each generator nine matrix coordinates and impose its module relations, multiplication table and identity. For the normalized group matrices impose J-adjunction, the traces of the universal determinant, and the prescribed diagonal form of Z. Its positive labeled eigenvalue is specified on the base: it is the root of $z^2 = t(Z^2)/2$ with residue δ, which exists uniquely by Hensel's lemma. Here the trace and Z are computed after twisting by the fixed character. All these are equations in the finite list of coordinates; arbitrary families of equations define a quotient of the same polynomial algebra. Denote the resulting algebra by S. The point of S for ρ_* is closed with residue field k'. The completion of its local ring is A_*. Indeed any lift of the matrices gives its determinant and ordered characters, hence its base map and a representation of C_0. Conversely such a matrix realization satisfies the defining words of C_0, hence the ordered flag by the separating local projectors. Over a local Artinian coefficient ring the group action is continuous by finiteness and continuity of C_0. These two constructions are inverse on all Artinian test rings.

Consider the fiber over an algebraic closure of $\kappa(p)$. Each marked realization there is semisimple under the stated hypothesis. This is immediate for an irreducible determinant. In the second case both distinct constituents are preserved by conjugate duality, since their dimensions differ. If a simple submodule U had degenerate restriction of the pairing, then $U \cap U^{\perp}$, being invariant, would be all of U. The quotient by $U^{\perp}$ would be its conjugate dual, isomorphic to U, giving a second copy among the composition factors. This contradicts multiplicity one. Hence U has an invariant orthogonal complement, proving semisimplicity.

Two matrices of representations in this geometric fiber are intertwined because their semisimple determinants agree. Their pairings are also intertwined after adjusting the isomorphism: on each self-dual simple summand the two invariant forms differ by a nonzero scalar by Schur's lemma, and one may rescale by a square root. Distinct summands are orthogonal because they are inequivalent and separately self-dual. The resulting intertwiner is an isometry of J commuting with the fixed diagonal matrix Z, whose three eigenvalues are distinct. Such isometries are $\mathrm{diag}(s, e, s^{-1})$ with $s \neq 0$ and $e = \pm 1$. Conjugation by this one-dimensional group therefore accounts for all geometric matrix points for each choice of the coefficient embedding. There are finitely many such embeddings. The finite-type geometric fiber has dimension at most one: its closed points are contained in the union of these orbit images, whose closures have dimension at most one.

Contract P_* to S, obtaining P_S. Write $D = R_0/p$. The finite-type domain S/P_S over D has generic transcendence degree at most one by the fiber calculation. To bound its dimension, present it as a quotient of $D[X_1, \dots, X_n]$ by a prime. After localization over $\mathrm{Frac}\, D$ this prime has height at least $n - 1$, so it already has height at least $n - 1$ over D. The polynomial ring has dimension $\dim D + n$, giving quotient dimension at most $\dim D + 1$. Localization at the residual matrix point and completion of that local quotient preserve the upper bound. The ring A_*/P_* is a quotient of this completion. This also treats primes in formal fibers, for which a direct count of parameters over the residue field would be insufficient. □

Corollary 37.25 (Transferring support along marked components)**.** *Let $P_0, \dots, P_t$ be components of* $\mathrm{Spec}\, A_*$*, written as minimal primes, whose contractions J_i to R_0 are geometrically irreducible. Suppose $I_K \subseteq J_0$. For each successive pair suppose that some prime Q above $P_i + P_{i+1}$ has $\dim A_*/Q \ge 3d + 1$ and its geometric determinant has at most two constituents. Then $I_K \subseteq J_t$.*

Proof. For $q = Q \cap R_0$ the preceding theorem gives $\dim R_0/q \geq 3d$. The special fiber of $R_0/(J_i + J_{i+1})$ therefore has dimension at least $3d - 1$, by the dimension inequality for a single element. Since $b \geq 5$, this exceeds $3d - b$. Apply the induction step of Corollary 37.20 with initial supported prime J_0. The connectedness assertion above supplies the dimensions of intersections in the marked ring. To use an entire connected component graph one must also treat intersections whose generic determinants split into three characters and components with reducible generic determinant. □

37.7 The triangular locus and support at adjacent lower entries

Use $A_*, R_0, L/F$ and the normalized matrices of Section 37.6. In this section the finite point ρ_* is lower triangular, with diagonal $(\bar{\Psi}, 1, \bar{\Psi}^{-1})$, and some strictly lower entry of some group element is nonzero. Denote by A_{low} the quotient on which all the matrices are lower triangular in the marked basis. Let h be the common upper bound for the ℓ-rank of the class groups of $T_a = L_a E_*(\mu_\ell)$ in Proposition 30.6. Here $d = [F : \mathbb{Q}] > 3h + 7$ by the choice of depth in Theorem 30.8.

Lemma 37.26 (The basis at a reducible determinant)**.** *At any prime of A_* whose geometric determinant is a sum of three characters all matrices are lower triangular. At a geometric determinant with one line and one irreducible plane all matrices preserve the middle coordinate line and the plane of the two outer coordinates.*

Proof. In the first case there is an invariant full flag over the algebraic closure. Every step is stable under $Z = \rho(\gamma) - \rho(c\gamma^{-1}c)$. Since Z is diagonal with three distinct eigenvalues, each step is a coordinate subspace. Thus there is an ordering of the coordinates in which all group matrices are triangular. Draw an arrow $j \to i$ for $i \neq j$ if some group element has nonzero ij-entry. The triangular flag precludes directed cycles. Adjunction by J pairs the entries $21, 32$ and the entries $12, 23$. Whenever one of a pair occurs the other occurs, possibly on another group element. A nonzero lower entry at the residual point is a unit in A_* and persists. If a lower adjacent entry occurs, the path $1 \to 2 \to 3$ prohibits every upper entry: an upper adjacent entry would give a two-cycle, and an upper 13 would close the path. If the lower 31 occurs, an upper 13 would reverse its arrow; the pair of upper adjacent entries would give the return path $3 \to 2 \to 1$. Thus no upper entry occurs in either case. In the second case Theorem 37.24 shows that the representation is an orthogonal direct sum. On the self-dual character line Z is zero by adjunction, so the line is the zero eigenspace, namely the middle coordinate. Its orthogonal complement is the outer plane. □

Proposition 37.27 (Dimension of the lower triangular locus)**.** *On the given field,*

$$\dim A_{\mathrm{low}} \leq 2d + 2 + 3h, \qquad \dim A_{\mathrm{low}}/\varpi \leq 2d + 1 + 3h.$$

Proof. For lower triangular lifts the outer diagonal characters are conjugate inverses and the middle diagonal character is its own conjugate inverse. Remove the finite residual characters by their Teichmüller lifts. The remaining variations take values in principal units and are unramified away from ℓ: at the old places the inertia quotient in H has order prime to ℓ. Let X_L be the maximal abelian pro-ℓ Galois group of L unramified outside ℓ. Use multiplicative notation for characters and additive notation for X_L. The first diagonal variation factors through X_L, and the middle one through its minus direct summand X_L^- for complex conjugation, since 2 is invertible.

The ray class exact sequence, followed by the inverse limit over powers of the primes above ℓ, identifies the image of the product of local principal units in X_L with its quotient by the

closure of the projected global units. Its cokernel is the ℓ-part of the ideal class group. The local unit product has ranks d, d in its plus and minus summands: its factors occur in conjugate pairs of equal local degree. The diagonal image of the real units has rank $d-1$ by Leopoldt for the abelian field F. The units of L have the same rank as those of F, by the unit theorem, so adjoining them changes this closure by at most finite index. Consequently X_L has rank $d+1$ and X_L^- has rank d. Both are finitely generated $\mathbb{Z}_\ell$-modules. The complete diagonal character algebra

$$D = O'[[X_L \times X_L^-]]$$

is finite over a power series ring on $2d+1$ variables over O'; its extra generators come from finite torsion groups. It has dimension $2d+2$, and its special fiber has dimension $2d+1$. The universal triangular diagonals give a map $D \to A_{\mathrm{low}}$. [3]

We bound the relative tangent dimension at this map. Let $\mathfrak{u}$ be the three-dimensional strictly lower triangular Lie algebra over k' with action of H by conjugation through ρ_*. A triangular infinitesimal lift with fixed diagonal differs from ρ_* by a cocycle in $Z^1(H, \mathfrak{u})$. This cocycle is a local coboundary at every place over ℓ. To see this at a chosen place, use the separating inertia matrix $\rho(\gamma_u)$. A lower triangular matrix with distinct diagonal entries having unit differences can be diagonalized by a unique lower unitriangular matrix of eigenvectors: solve for each column starting with its first nonzero coordinate equal to one and then solve successively below it. The ordinary flag is the increasing flag of these columns, since its ordered residual characters agree with the three labeled diagonal entries. The lower triangular action also preserves the opposite coordinate flag, which is unchanged by this lower unitriangular change of basis. The full local action therefore preserves both opposite flags and is diagonal in this basis. For an infinitesimal variation with fixed diagonal the change between the two diagonalizing bases is lower unitriangular and reduces to the identity. This gives the local coboundary in $\mathfrak{u}$. Conjugate places satisfy the same assertion by polarization, since inverse transposition followed by reversal preserves the lower unitriangular group.

The map from this relative tangent space to $H^1_{\mathrm{str}}(H, \mathfrak{u})$ is injective. If the cocycle is the global coboundary of $X \in \mathfrak{u}$, the two representations are strictly conjugate by $1 + \epsilon X$ up to changing the sign of X. Their Z's are diagonal. Hence $[X, Z_*]$ is diagonal, forcing $X = 0$ by distinctness of the three eigenvalues and strict lower triangularity. The local character variations are then zero as well, since the stable ordered flags are determined by their projectors.

The center $k'E_{31}$ of $\mathfrak{u}$ has character $\bar{\Psi}^{-2}$, and its quotient has two copies of $\bar{\Psi}^{-1}$. For either character the strict first cohomology on H has dimension at most h. Indeed restriction to T_a is injective, since $[T_a : L]$ is prime to ℓ. There the coefficient character is trivial and the strict class is an everywhere unramified additive homomorphism. It factors through the ideal class group, whose ℓ-rank is at most h. This proves the assertion for both powers. The quotient of $\mathfrak{u}$ by its center has zero local invariants above ℓ by the residual inertia characters. Its local exact sequence therefore injects the first cohomology of the center into that of $\mathfrak{u}$. A strict class on $\mathfrak{u}$ whose image in the quotient vanishes globally lifts by exactness to a class on the center, and local injectivity makes that lift strict. The total strict dimension is thus at most $3h$.

Finally lift generators of the relative cotangent space to the maximal ideal of A_{low}. They give a surjection from a power series ring on at most $3h$ variables over D: together with the image of the maximal ideal of D they generate the maximal ideal modulo its square, and successive approximation gives surjectivity by completeness. Taking the two quotient dimensions proves the proposition. □

Lemma 37.28 (Choosing a curve at a marked intersection)**.** *Let $Q \subset A_*$ be prime of quotient dimension at least $3d$, and suppose $I_K \subseteq Q \cap R_0$. There is a characteristic-ℓ curve $\mathfrak{p}$ in R_0,*

containing $Q \cap R_0$, with one of the two matrix realizations in Sections 37.3 and 37.4. It is the contraction of a one-dimensional marked specialization above Q.

Proof. Cut A_*/Q by ϖ if necessary and choose a prime Q' of the resulting special fiber attaining its dimension. Then $\dim A_*/Q' \geq 3d-1$. This is greater than $2d+1+3h$. The lemma on the marked basis and the proposition show that the generic determinant at Q' is not a sum of three characters. By Theorem 37.24, for $q' = Q' \cap R_0$ we have $\dim R_0/q' \geq 3d-2$. This exceeds $3d-b$ of Section 37.5. Since $I_K \subseteq q'$, R_0/q' is finite over the weight algebra. Its generic point avoids the exceptional weights of Proposition 37.17, by the finite-map dimension argument of Theorem 37.19.

Pull this open condition back to $\operatorname{Spec}(A_*/Q')$. It is a nonempty open in a complete local domain. The system of parameters argument of Theorem 37.19 gives a one-dimensional quotient there still satisfying the condition. Embed it in its normalization $k''[[z]]$. The contraction to R_0 has quotient dimension exactly one. In fact the weight is outside the exceptional set, so some principal unit weight ratio differs from one. Its difference from one gives an element $0 \neq f \in (z)$ of the image of R_0. This image contains $k[[f]]$ by completeness and continuity, with k the residue field of O. The ring $k''[[z]]$ is finite over $k[[f]]$ by expansion from a basis modulo f. The intermediate image is a finite module over this Noetherian DVR and has dimension one.

The marked matrices over the normalization are already polarized and ordinary. The determinant outside the exceptional set has at most two constituents. In the two-constituent case their representations split orthogonally as in Theorem 37.24, the line being the middle one. These statements remain true over a finite extension of the contracted curve field: the extension to the marked normalization is finite by the same argument with $k[[f]]$. Use Proposition 30.11 to obtain a marked self-dual lattice with split reduction after finite extension. The norm-one local characters are separated by construction, as required. □

Theorem 37.29 (All components at a nonzero adjacent lower entry)**.** *Suppose some matrix of ρ_* has a nonzero* 21- *or* 32*-entry. If a prime $P_s \subset A_*$ with $\dim A_*/P_s \geq 3d$ has Hecke-supported contraction, then $I_K A_*$ is nilpotent. In particular every marked specialization at this residual point has Hecke-supported determinant.*

Proof. The nonzero entry is a unit throughout the chart. A geometric determinant with one line and a plane is therefore impossible by Lemma 37.26. At every prime of quotient dimension at least $3d$ the determinant must be irreducible: the other possible case would lie in the lower triangular quotient, of dimension at most $2d+2+3h < 3d$. In particular this applies to P_s, to every minimal prime of A_*, and to primes of maximal dimension on the successive intersections of Proposition 37.23.

Take a minimal prime $P_0 \subseteq P_s$. Apply Lemma 37.28 with $Q = P_s$. Its curve is irreducible since a line and plane cannot occur anywhere in the marked chart. It is supported by specialization from $P_s \cap R_0$. Thus Theorem 37.15 makes every prime below the contracted curve supported, including $P_0 \cap R_0$. For an adjacent pair of components in Proposition 37.23, take a prime of maximal quotient dimension in their intersection. Once the first component is supported, this prime has supported contraction and quotient dimension at least $3d+1$. The same choice of curve and the same local theorem give support at the second component. Connectedness propagates this conclusion to all minimal primes. Their intersection is the nilradical, which is nilpotent in the Noetherian ring A_*; this gives the stated ideal containment and its consequence. □

Corollary 37.30 (Adjacent lower entries on the arithmetic fields)**.** *For every large-prime pair on the interval, finite residual matrix points with a nonzero adjacent lower entry as in the theorem exist on the layer L/F.*

Proof. Use the nonzero strict class of Proposition 30.5 with character $\bar{\Psi}^{-1}\eta$ on Γ, where η cuts out L/F. Choose a representative z with $z(c) = 0$; this is possible by a coboundary since the character takes value -1 at c. On H it has coefficient $\bar{\Psi}^{-1}$ and satisfies $z(cgc) = -z(g)$. Its class on H remains nonzero by restriction and corestriction. Put $N = E_{21} + E_{32}$ and $n(x) = 1 + xN + x^2N^2/2$. Then

$$g \longmapsto n(z(g))\,\mathrm{diag}(\bar{\Psi}(g), 1, \bar{\Psi}(g)^{-1})$$

is a representation of H, since $n(x)n(y) = n(x+y)$ and the diagonal conjugates N by $\bar{\Psi}(g)^{-1}$. It has the required polarization, since $JN^{\mathsf{T}}J = N$ and inverse transposition followed by reversal sends $n(x)$ to $n(-x)$. It is locally conjugate to the diagonal at the coefficient places by strictness, giving the prescribed ordered flags. The finite coefficient field gives a finite image. To mark it, take the eigenvectors of its lower triangular skew matrix Z, labeled by the three distinct diagonal entries. They can be chosen as a lower triangular basis, then rescaled to have form matrix J over a finite field extension. This only rescales the 21 extension class and changes its cocycle by a coboundary, as one sees on the quotient by the last coordinate line. The class is nonzero, so some adjacent entry persists. □

The last corollary constructs residual points. Hecke support for a prime of the size required in Theorem 37.29 is an additional assertion. A residual point whose only nonzero strictly lower entries are in position 31 can also admit one-dimensional plus two-dimensional constituents; its component intersections require the corresponding two-constituent argument.

37.8 Support through the chart with one long lower entry

Continue on the fields and at the depth of Section 37.7. Suppose all strictly lower entries at the finite residual point ρ_* vanish except for the 31-entry, which is nonzero on some group element. Then the residual matrices preserve the middle line and the outer plane. Let $D_0 = \mathrm{diag}(-1, 1, -1)$. Conjugation by D_0 fixes the residual matrices, preserves the form J and the marking Z, and induces an involution σ of A_* over R_0. A plus or minus sign below denotes the eigenvalue of this involution. Both eigenspace projections are available since 2 is invertible. A prime of R_0 will meet the hypothesis on a link in Theorem 37.12 if its determinant is geometrically irreducible or its geometric C_0-fiber is nonsemisimple.

Proposition 37.31 (Even and odd relations in the marked ring)**.** *There is a presentation of A_* as a quotient of*

$$P = O'[[x_1, \dots, x_a, y_1, \dots, y_g]], \qquad \sigma x_i = x_i, \quad \sigma y_j = -y_j,$$

by b even relations F_i and g odd relations $(My)_j$, where y is the column of the y_j and M is a square matrix of even elements of P. Here $a - b \geq 3d + 1$. The closed set $y = 0$ is exactly the locus on which the universal matrices preserve the middle line and the outer plane.

Proof. The residual adjoint of Lemma 37.21, its local flag stabilizers and their mapping fiber decompose under D_0. In fact D_0 commutes with the residual local action. The three separating eigenlines at the local places therefore lie individually in the line or the plane, and every ordered flag is preserved. Write $h_{i,\pm}$ for the two cohomology dimensions. The odd adjoint consists of the two off-diagonal blocks between a plane and a line. The odd dimension of a local flag

stabilizer is two: among the three eigenlines there are two unordered pairs with opposite signs, and each contributes one upper triangular entry. At a real place $c_\tau = hc$ the residual matrix of h preserves the blocks. The symmetric form defining adjunction there makes these blocks orthogonal; on the two off-diagonal blocks a skew-adjoint endomorphism is determined by one map from the line to the plane, of dimension two. The odd contribution at infinity is therefore two as well. The dual invariant vanishing used in Theorem 37.22 persists for this summand. Lemma 30.7 gives

$$h_{1,-} - h_{0,-} - h_{2,-} = 2d - 2d = 0.$$

In degree zero the marked automorphisms are contained in the diagonal torus Lie algebra from Lemma 37.21, which is even. Thus $h_{0,-} = 0$. The extra change-of-marking tangent directions there are also even. The relative odd tangent dimension is $g = h_{1,-} = h_{2,-}$, and the even tangent dimension a satisfies $a - h_{2,+} = 3d + 1$ by the full Euler formula.

Choose the generators of a minimal power series presentation with their parities, using the projections $(1 \pm \sigma)/2$. The odd generators can be chosen from off-diagonal block entries of the universal group matrices. Indeed the complete local algebra is generated by matrix entries and local characters as in Theorem 30.13. The characters and diagonal block entries are even and the off-diagonal block entries are odd. Therefore the indicated entries span the odd relative cotangent space. Every odd series belongs to (y), since its monomials have odd total degree in these variables. Consequently $y = 0$ is exactly the stated block locus.

Let J_P be the kernel of the presentation. The injection $(J_P/\mathfrak{m}_P J_P)^\vee \to H^2_C$ in Theorem 37.22 is equivariant: conjugating the universal matrices and lifted flags by D_0 conjugates both the multiplicative defect and its local projections, and these compute the obstruction class. Thus generators for J_P can be chosen with at most $h_{2,+}$ even and at most g odd ones. Lift eigenbases of its quotient by $\mathfrak{m}_P J_P$ with the prescribed parities and use Nakayama's lemma. Append zero odd relations to give exactly g of them. In each odd series factor the first y_j occurring in each monomial, in a fixed order; the remaining factor has even parity. Grouping these formal series gives My as asserted. □

Lemma 37.32 (A nonsemisimple fiber on the determinantal locus)**.** *Assume $g > 0$ and put $A^\dagger = A_*/(\det M)$, with $\det M$ evaluated in A_*. Every prime of A_* with geometrically irreducible determinant contains $\det M$. At every prime P_* of $A^\dagger$ of quotient dimension at least $3d$, the contraction to R_0 meets the hypothesis on a link.*

Proof. Multiplication by the adjugate in $My = 0$ gives $(\det M)y_j = 0$ for all j. At an irreducible determinant some y_j is nonzero because a block-preserving representation is reducible. This proves the first assertion. At a prime of the indicated dimension the determinant cannot split into three characters, by Lemma 37.26 and Proposition 37.27. If it has a line and plane, the same lemma gives $y = 0$ at the prime.

Take an algebraic closure K of its residue field, and evaluate the presentation at the resulting point. As $\det M = 0$, there is a nonzero $v \in K^g$ with $Mv = 0$ there. There is a lift of the evaluation to $K[\epsilon]$, $\epsilon^2 = 0$, given by keeping the x_i and sending y_j to ϵv_j. This substitution on formal series is defined by first reducing modulo $(y)^2$, then applying the given evaluation on $O'[[x]]$ to each coefficient. The even relations have no linear y-term, and the odd ones have first variation $Mv = 0$. The resulting map is defined on A_*. Its restriction to R_0 is the constant evaluation, because σ fixes R_0 and every invariant series has zero first variation at $y = 0$. An invariant element of the quotient has an invariant lift by averaging.

Let $q = P_* \cap R_0$. The universal group matrices now give a representation of $C_0 \otimes_{R_0} K$ over $K[\epsilon]$ with a nonzero variation in its off-diagonal blocks; one of the chosen y_j is such an entry. Suppose the geometric fiber of C_0 at q were semisimple. Its extension to K would be a

product of split matrix algebras. Every first-order variation of a representation of this product, with the algebra held fixed, is inner. Explicitly, in each matrix factor take $\sum_i E_{i1} \otimes E_{1i}$, and sum these tensors over the factors. Writing the result as $\sum a_i \otimes b_i$, we have $\sum a_i b_i = 1$ and $\sum x a_i \otimes b_i = \sum a_i \otimes b_i x$. For a derivation ∂ relative to the given representation ρ, applying ∂ to the first component of this identity and multiplying shows that $U = \sum \partial(a_i)\rho(b_i)$ satisfies $\partial(x) = U\rho(x) - \rho(x)U$. This is the asserted infinitesimal conjugation.

The marking Z has zero variation: its diagonal scalar is determined by the base trace and has its fixed choice of square root. Thus U commutes with Z and is diagonal, since the three eigenvalues are distinct. Commuting a diagonal matrix with a matrix that preserves the two blocks gives zero in the off-diagonal blocks. This contradicts the variation just constructed. The geometric fiber at q is therefore nonsemisimple, as required. □

Proposition 37.33 (Intersections after imposing the determinant)**.** *If an irreducible prime occurs in A_*, in the sense of its geometric determinant, then $g > 0$, $A^\dagger$ is nonzero and every component of* $\operatorname{Spec} A^\dagger$ *has dimension at least $3d+1$. These components form a connected graph under intersections of dimension at least $3d$.*

Proof. For $g = 0$ the block locus is the entire ring, so an irreducible determinant forces $g > 0$. The lemma then puts $\det M$ in the corresponding prime. Use the $a + g$ variables in the presentation over O' and the $b + g + 1$ relations $F, My, \det M$. The proof of Proposition 37.23, with $N = 1 + a + g$ and $s = b + g + 1$, gives component dimension at least $a - b$ and the connected graph with intersection dimension at least $a - b - 1$. Now $a - b \geq 3d + 1$. □

Theorem 37.34 (Hecke support of irreducible determinants at the long entry)**.** *Suppose $P_s \subset A_*$ is prime of quotient dimension at least $3d$ and its contraction to R_0 is both Hecke-supported and geometrically irreducible. Then $I_K A^\dagger$ is nilpotent. Every prime of A_* with geometrically irreducible determinant has Hecke-supported contraction.*

Proof. By Lemma 37.32 the prime P_s lies in the closed subscheme $\operatorname{Spec} A^\dagger$. Choose a minimal prime P_0 of this subscheme below P_s. To reach any other minimal prime, take a chain $P_0, P_1, \ldots, P_t$ supplied by the proposition, viewed as primes of A_*. Write $J_i = P_i \cap R_0$. Every J_i meets the link hypothesis by Lemma 37.32. Choose a marked curve above P_s using Lemma 37.28; write $\mathfrak{p}_0$ for its contraction to R_0. It lies above the given supported irreducible prime $P_s \cap R_0$. By Corollary 37.16 for this one curve, its indicated generizations belong to Hecke support, in particular J_0.

Suppose curves $\mathfrak{p}_0, \ldots, \mathfrak{p}_i$ have been obtained with $J_i \subseteq \mathfrak{p}_i$ supported, and with the two successive curves at each earlier step both above the corresponding J_j. In the intersection defined by $P_i + P_{i+1}$, choose a prime Q of quotient dimension at least $3d$. Its contraction contains J_i and is therefore supported. Lemma 37.28 now provides the next curve $\mathfrak{p}_{i+1}$ above $Q \cap R_0$. Both $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ contain J_i, which is an allowed link even when its geometric determinant has two constituents. The prime J_{i+1} is an allowed endpoint below the new curve. Apply Corollary 37.16 to the whole finite chain, using the original supported irreducible starting prime below $\mathfrak{p}_0$. Its conclusion gives $I_K \subseteq J_{i+1}$. Using the whole chain keeps the irreducible starting hypothesis available at stages where J_i has reducible determinant.

Induction proves the containment for every minimal prime of $A^\dagger$. Its nilradical is nilpotent by Noetherianity. The first assertion of Lemma 37.32 then proves the conclusion for every irreducible determinant of A_*. The related determinantal construction in the source appears in `Development/AutomorphicArithmetic/AutomorphicVarieties/FrobeniusSmoothVectors/ChainedOfComap.lean`, line 93. □

Corollary 37.35 (Local propagation at the nonzero lower points)**.** *At any marked lower triangular residual point with a nonzero strictly lower entry, a supported geometrically irreducible prime of marked quotient dimension at least* 3*d gives Hecke support at every geometrically irreducible prime in that chart. Both adjacent-entry and pure* 31*-entry residual points exist on the fields of Section 30.2.*

Proof. If an adjacent entry is nonzero, apply Theorem 37.29; existence in this case is Corollary 37.30. Otherwise apply the theorem just proved. To construct a pure 31-entry point, use the strict class with character $\bar{\Psi}^{-2}\eta$ in Proposition 30.5, choose $z(c) = 0$, and restrict to H. Its restriction remains nonzero. Use $N = E_{31}$ and $n(x) = 1 + xN$ in the proof of Corollary 37.30. Now $N^2 = 0$, conjugation by the residual diagonal has character $\bar{\Psi}^{-2}$ on N, and $JN^{\mathsf{T}}J = N$, giving the same local splitting and polarization. The skew marking is diagonalized within the outer plane and the middle vector is retained. On the plane the 31 extension class is nonzero and is only rescaled or changed by a coboundary in this basis. Thus the long entry occurs and both adjacent entries are zero. □

This proves local propagation at all the nonzero lower residual points, with the same group and local conditions at each step. Passing from a supported arithmetic family to all of these charts, and estimating the lengths at the prescribed integral weights, remain necessary to obtain the bound of Section 6.6.

37.9 Global Hecke support for irreducible ordinary determinants

Fix one of the large-prime triples and its layer L/F from Section 30.2. Write $X = \operatorname{Spec} S$ for the finite-type scheme over R_0 constructed in Theorem 37.24, using the original coefficients O. Thus a point specifies matrices of C_0 in dimension three with the base determinant, pairing J and diagonal marking Z, after removing the common cyclotomic twist. Let k be the finite residue field of R_0 and X_k the fiber at its closed point. The torus $\mathbf{G}_m$ acts on X by conjugation by $\operatorname{diag}(t, 1, t^{-1})$. Let U be the open in X_k where some strictly lower entry of a group matrix is nonzero. Within it let U_1 require some adjacent lower entry to be nonzero.

Lemma 37.36 (Marked points and the determinant map)**.** *Every geometrically irreducible prime of R_0 has a nonempty geometric fiber in X. Its geometric points form a single one-dimensional torus orbit. Every irreducible component of X is stable under the torus. At a closed point $x \in U$ with finite residue field k', the completed local ring is the marked ring A_* for the matrices of that point over the unramified coefficient extension with residue field k'.*

Proof. Over a geometric field the irreducible representation realizing the determinant factors through C_0, by the word identities and the trace pairing as in Lemma 37.18. Adjunction for its normalized group matrices is given by a nondegenerate symmetric form, by irreducibility and odd dimension. The skew operator Z has eigenvalues $z, 0, -z$, with z the specified square root of $t(Z^2)/2$ in the base. An eigenbasis has a nonzero pairing of the outer vectors and a nonzero norm of its middle vector; rescale them to obtain J. This gives a point. As in Theorem 37.24 any two such points are related by $\operatorname{diag}(s, e, s^{-1})$, $e = \pm 1$. If $e = -1$ multiply by the scalar -1 without changing conjugation. This is the given torus action. Its stabilizer on an irreducible representation is trivial: a commuting matrix is scalar and the middle entry of the torus element is one. Thus the orbit has dimension one. Every irreducible component C of X is stable because $\mathbf{G}_m \times C$ is irreducible and its image contains C at $t = 1$.

At a geometric point of X_k the determinant has three character constituents. Its invariant flag is a coordinate flag by the three eigenvalues of Z. The directed-cycle argument of Lemma 37.26 then shows that every point of U is lower triangular. The diagonal entries at

any such point are the prescribed $\bar{\Psi}, 1, \bar{\Psi}^{-1}$ on the group. Indeed for the i-th polynomial projector e_i of Z the entry is $t(e_i[g])$, whose value is fixed by the split trace on the closed fiber. The local conditions and marking are part of X. To pass to the completion at x, the unramified coefficient extension O'/O with residue field k' maps uniquely to $\widehat{\mathcal{O}}_{X,x}$ lifting the specified residue-field map, by Hensel's lemma. Equivalently, after this finite étale coefficient extension the completed factor at the chosen lift of x is the same ring. The Artinian test-ring identification in Theorem 37.24 gives exactly A_*. □

Proposition 37.37 (The lower triangular points in the closed fiber)**.** *The space U is nonempty and connected, and U_1 is dense in U.*

Proof. Work first over $\bar{k}$. The constructions of Corollary 37.35 give a fixed pure 31-entry point and a fixed adjacent-entry point. Write $d(g) = \mathrm{diag}(\bar{\Psi}(g), 1, \bar{\Psi}(g)^{-1})$. At a pure point the matrices are $d(g) + b(g)E_{31}$; denote the function for the fixed pure point by b_0. At any lower point write $a_{ij}(g)$ for its strictly lower entries. For an indeterminate T define

$$\rho_T(g) = d(g) + Ta_{21}(g)E_{21} + Ta_{32}(g)E_{32} + (T^2a_{31}(g) + (1-T)b_0(g))E_{31}.$$

This is a representation: scaling a lower entry by T^{i-j} respects matrix multiplication, and the additional 31 term satisfies the homogeneous equation $b_0(gh) = b_0(g)d_{11}(h) + d_{33}(g)b_0(h)$. There are no other products with that entry in the strictly lower part. Adjunction pairs the adjacent entries and preserves the long position, giving the original adjunction identity. The off-diagonal entries of Z are zero for both input matrices and remain zero under this formula.

The ordered local flags persist over $\bar{k}[T]$. At each chosen place the original local action is conjugate to d by a lower unitriangular matrix V, by the proof of Proposition 37.27. Scaling the entries below the diagonal of V by T^{i-j} gives a lower unitriangular polynomial matrix effecting the same scaling on the conjugated diagonal. The pure 31 function at this place has the form $b_0(g) = c_u(d_{11}(g) - d_{33}(g))$; one may diagonalize its outer plane directly. Conjugating the scaled matrices by $1 + (1-T)c_uE_{31}$ adds exactly the prescribed final term. Thus the words hold with the original local diagonal characters. The global determinant is constant since the matrices are triangular. This gives a morphism $\mathbb{A}^1_{\bar{k}} \to X_k$. To check factorization through the finite Galois algebra, the original group functions factor through finite quotients: they are specializations of the group map into the finite algebra $C_0/\mathfrak{m}_{R_0}C_0$ over a finite field. The new group action factors through their common finite quotient. Every group algebra element has the prescribed characteristic polynomial, by triangularity, so Cayley–Hamilton, the old inertia relations and the verified ordinary words give the required factorization.

At $T = 0$ this is the fixed pure point; at $T = 1$ it is the chosen lower point. If the latter belongs to U, the inverse image of U is a nonempty open subset of an affine line containing both endpoints. It is connected. Thus every closed point of $U_{\bar{k}}$ is connected in this space to the fixed one. Any nonempty open and closed summand contains a closed point, by finite type over $\bar{k}$; this proves connectedness. For density take any pure point and use its b in place of b_0, with the fixed adjacent point as the other input. The value at zero is the chosen pure point and all nonzero values have an adjacent entry. Thus every closed point of $U_{\bar{k}}$ is in the closure of $(U_1)_{\bar{k}}$. Closed points are dense in every closed subset, so this is the whole space. Surjectivity of the base-change map gives connectedness and density over k. The polynomial deformations are also used in `Development/AutomorphicArithmetic/AutomorphicVarieties/FrobeniusSmooth Vectors/ExistsNonzeroLowCocycle.lean`, line 144. □

Lemma 37.38 (Specialization to a nonzero lower entry)**.** *Every torus-stable closed irreducible subset $C \subseteq X$ with geometrically irreducible generic determinant contains a closed point of U.*

Proof. Let p be its generic contraction to R_0 and put $D = R_0/p$. It is a positive-dimensional complete local domain: the closed determinant has three distinct character constituents. The map $C \to \operatorname{Spec} D$ is dominant and of finite type, so its image contains a nonempty open by Chevalley's theorem. Intersect this open with the locus where a nine-element trace Gram determinant that is nonzero at p remains nonzero. By the parameter argument of Theorem 37.19, which applies in either characteristic, this open contains a prime giving a one-dimensional quotient of D. At its curve field choose a point of C over a finite extension of the field, using a closed point of the nonempty finite-type fiber. The representation there is absolutely irreducible. The normalized curve ring is a complete DVR with finite residue field: complete Noetherian local domains here are excellent, so normalization in dimension one is finite, and the Henselian property makes the normalization in a finite field extension local.

Over this DVR the group matrices have a stable lattice since the group algebra C_0 is finite over the base. By the construction in Proposition 30.11, after a finite extension we can take a marked self-dual basis with split reduction. In detail the unit-separated eigenvalues of Z split the lattice, a small spread of exponents splits the residual flag, and averaging with the dual exponents makes the lattice self-dual. The final unit rescalings restore J. That argument uses only the DVR, the pairing, and the distinct residual eigenvalues, and hence works in mixed characteristic as well. The normalized point in this basis stays in C: it differs from the initial point by the torus action by Lemma 37.36.

Write v for the discrete valuation. All nonzero strictly lower entries now have positive valuation, and at least one exists, since matrices all upper triangular would contradict irreducibility. Set

$$e = \min_{\substack{i>j,\ g\in H \\ \rho(g)_{ij}\neq 0}} \frac{v(\rho(g)_{ij})}{i-j} > 0.$$

The minimum is attained since the value group is discrete and there are only three positions. After a finite ramified extension take t of valuation e, with the valuation extended in the same normalization. Torus conjugation scales the ij-entry by t^{j-i}, so all entries stay integral, every upper entry still has positive valuation, and some lower entry is a unit. The resulting matrices give a point over the valuation ring: the algebra is spanned by group elements and the scalar relations and local characters keep their values. Its generic point is in C by stability; its closed point is in C by closedness and in U by the unit just found. Its residue field is finite, so the image is a closed point of X_k, and of X. □

Theorem 37.39 (Hecke support at every irreducible ordinary determinant)**.** *For the ordinary determinant ring R_0 over the field L and at the level of Section 37.3, every prime with geometrically irreducible determinant contains the kernel I_K of the good Hecke map.*

Proof. Let a component of X be called irreducible at its generic determinant if its generic point has that property. The locus of such determinants in X is open: a nonzero determinant of the trace pairing on nine group elements detects it, and one takes the union of these opens. Among components with this generic property, consider those on which the Hecke kernel vanishes in the reduced coordinate ring. We first exhibit one.

Take the full Hecke component through the cuspidal adjoint in Proposition 30.34. Its domain B is a quotient of R_0/I_K, has geometrically irreducible generic determinant, and is finite over and contains Λ. Thus $\dim B = 3d + 1$. In X take the reduced closure Y of the orbit over this generic point. The geometric fiber has one orbit of dimension one by Lemma 37.36; its reduced generic fiber is irreducible and its function field has transcendence degree one over $\operatorname{Frac} B$. The closure is irreducible, torus-stable, dominates B, and is supported by I_K. By the preceding lemma choose a finite closed $x \in Y \cap U$.

At this x the dimension of $\mathcal{O}_{Y,x}$ is $3d+2$. Here is the dimension calculation. Present the coordinate ring of Y as a domain quotient of $B[T_1,\dots,T_n]$ by a prime contracting to zero. Its height is $n-1$ by the transcendence degree over the fraction field of B: every prime below it contracts to zero as well. The inverse image of x has height $\dim B + n$, by a chain to $\mathfrak{m}_B B[T]$ followed by a height-n closed point in the residue polynomial ring, and the dimension upper bound for a polynomial ring. The polynomial ring over the complete local domain B is catenary, so the difference gives the required local dimension. Completion preserves this dimension. Choose a minimal prime in $\widehat{\mathcal{O}}_{Y,x}$ attaining it. This prime contracts to zero in $\mathcal{O}_{Y,x}$: completion is flat, and a nonzero element of the domain remains a nonzero divisor and is in no minimal prime. Its inverse image in $A_* = \widehat{\mathcal{O}}_{X,x}$ is therefore supported, has geometrically irreducible determinant, and has quotient dimension $3d+2$.

Apply Corollary 37.35. It puts every geometrically irreducible determinant of this chart in Hecke support. Some global component containing Y has geometrically irreducible generic determinant, by the Gram open. Every such component through x has a branch in the completion with the same generic contraction: by faithful flatness lift its local minimal prime and then take a minimal prime below the lift. It still contracts to that local minimal prime. Geometric irreducibility persists under residue-field extension. Thus the Hecke kernel vanishes at the generic point of this global component. This supplies the promised component.

The same argument transfers support at every finite closed point $z \in U$ shared by two components having geometrically irreducible generic determinants. If the first is supported, a minimal prime of $\widehat{\mathcal{O}}_{X,z}$ above its generic prime is a supported irreducible prime of quotient dimension at least $3d+2$, by Proposition 37.23. Applying Corollary 37.35 and contracting a branch of the second component proves that it is supported too. Moreover at each finite closed point of U_1 every component through the point has geometrically irreducible generic determinant. For any minimal branch its dimension is at least $3d+2$; the arguments of Lemma 37.26 and Proposition 37.27 exclude both a sum of three characters and a line with a plane at its generic point.

There are only finitely many components. Their subunion with geometrically irreducible generic determinant thus meets U in a closed subset containing all closed points of U_1; it contains U by the proposition and density of closed points. Intersect U with the union of the supported components in that list, and with the union of the others. These are two closed sets covering U. Their intersection is empty: if nonempty it has a finite closed point, since U is of finite type over a finite field, contradicting the transfer just proved. The first set is nonempty by the construction from the adjoint. Connectedness of U makes the second empty. Every component of the list meets U by Lemma 37.38, so every one is supported.

Finally take any geometrically irreducible prime of R_0. A geometric point above it exists by Lemma 37.36. Any component of X through this point has generically irreducible determinant by openness, hence is supported. The base prime contains I_K. The corresponding component theorem in the source is `Development/AutomorphicArithmetic/AutomorphicVarieties/CurveDVRSpecialization/PropagationTheoremGL3.lean`, line 213. □

Corollary 37.40 (The irreducible closure at the prescribed weights)**.** *Let R_{irr} be the quotient of R_0 by the intersection of all its geometrically irreducible primes. It is a finite Λ-algebra. At any original integral arithmetic weight $\Lambda \to O_w$, its base change is finite over O_w. Every geometric characteristic-zero point there occurs as an eigencharacter on the ordinary module at that weight, and their number is at most* $\operatorname{rank}_\Lambda M_K$.

Proof. The theorem makes R_{irr} a quotient of B_K, which is finite over Λ. Finiteness survives base change. A field point of this quotient has nonzero M_K-fiber, since M_K is finite and faithful

over B_K. Base change to the chosen geometric weight identifies this fiber with the residue fiber of the specialized ordinary module. Over an algebraically closed characteristic-zero field the specialized commutative finite algebra is a product of Artinian local algebras. Each such point gives a distinct nonzero summand of the module and an eigenvector there, by the nilpotence of its maximal ideal. The full dimension of the specialized module equals its stated rank by weight freeness. □

This finiteness holds at the original arithmetic weight even when the determinant there is reducible but lies in the indicated closure. The integral lengths in Corollary 37.4 involve a matrix corner at the split determinant; finiteness of the reduced trace closure alone gives no numerical bound for those lengths or for the signed projected periods.

37.10 The entire trace ring at the original arithmetic weight

Fix $V \le x \le V^{3/2}$ on an interval of Proposition 29.3. Put $K = \mathbb{Q}(\sqrt{-s})$. Take $\ell > P(x)$, $1 \le r < t(x)$, and a character α of E_x with $\alpha(c) = (-1)^r$. Write O for the unramified coefficient ring of this character, $J = \operatorname{Frac} O$, and T for the complete trace ring of the original marked fixed-weight problem on G_K. Its augmentation sends the universal trace t to the trace of $\Psi \oplus 1 \oplus \Psi^{-1}$, where $\Psi = \alpha\kappa_\ell^r$. Here K is independent of the interval, x, r, α and ℓ.

Theorem 37.41 (Finiteness at the original weight, including torsion)**.** *The ring T is finite over O.*

Proof. Pass first to the coefficient extension of Proposition 31.7 and denote the extended rings by the same letters. At the depth of Section 37.3 that proposition gives a surjection $R_0 \to T$. It sends the weight algebra Λ to the scalars by the original three ordered characters, with the fixed common normalization. Thus $T/I_K T$ is a quotient of the specialization of $B_K = R_0/I_K$ at this weight. It is finite over the coefficient DVR, since B_K is finite over Λ.

Take a prime p of T over the uniformizer. Over an algebraic closure of its residue field, the trace pairing on the image algebra of the semisimple representation is nondegenerate. On a simple matrix factor it is the matrix trace multiplied by the multiplicity, which is at most three and is invertible here. Consequently $I_9(G_K) \not\subseteq p$ precisely when the representation is absolutely irreducible. Lemma 31.5 gives $I_9(G_K) = I_9(G_L)T$. If $I_9(G_K) \not\subseteq p$, the preimage of p in R_0 therefore has geometrically irreducible determinant. Theorem 37.39 puts $I_K T$ inside p. In the other case p contains $I_9(G_K)$, and $T/I_9(G_K)$ is finite over the DVR by Theorem 33.7. In either case p is maximal since it lies in the special fiber of a finite algebra. The Noetherian special fiber of T is therefore Artinian.

Choose lifts of a residue-field basis of that special fiber. Their O-span surjects modulo the uniformizer ϖ. Lift successive remainders after division by ϖ, obtaining coefficients in O converging in the ϖ-adic topology. The errors lie in $\varpi^n T \subseteq \mathfrak{m}_T^n$, so separatedness gives equality in T. Thus the lifts generate the whole ring as a module. Finally the coefficient extension is finite faithfully flat and formation of the trace ring commutes with it by Proposition 31.7. Finiteness descends. For instance the original T embeds in the extended ring, which is a finite module over the original Noetherian DVR. □

Proposition 37.42 (Classical labels for every nonsplit trace)**.** *Each geometrically irreducible characteristic-zero point of T is represented, with the normalization of Corollary 31.8, by a conjugate-self-dual cuspidal representation of GL_3/K at level N_*. Each geometrically reducible point other than the augmentation is the sum of the trivial character and an irreducible plane. It has the primitive modular form label of Proposition 33.11 at a level dividing sN_*. For fixed ℓ, r, α, a label with its embedding and Frobenius convention determines at most one trace point.*

Proof. By finiteness, a geometric point is defined over a finite extension of J. It maps T into the integers there. This is a continuous local map: the inverse image of the maximal ideal is the unique maximal ideal over the uniformizer and its powers map into the corresponding powers. At an irreducible point the nine-by-nine minor argument in the preceding proof works also in characteristic zero. Hence restriction to G_L gives an irreducible eigencharacter of B_K at the original weight. Corollary 37.40 supplies the eigencharacter on the ordinary module. Ordinary control at the dominant algebraic weight identifies its characteristic-zero specialization with classical definite automorphic forms, exactly as in the proof of Corollary 30.37. At $r = 1$ the coefficient weight in this normalization is zero and the same control theorem applies. [4]

In the transfer of this system to GL_3/L, the attached semisimple Galois representation is the direct sum from its isobaric parameters (with their prescribed twists). Compatibility of good Frobenius traces and Chebotarev identify this representation with the normalized restriction of the point. Irreducibility forces a single rank-three cuspidal parameter. Descend it along a tower of cyclic prime-degree steps from L to K, as in Theorem 29.18. At each step the original Galois representation makes the upper system invariant, by strong multiplicity one. The possible descents differ by characters of the cyclic group. The normalized central character singles out one, since cubing is bijective on this group of order prime to three. Uniqueness makes this descent conjugate-self-dual. On the Galois side the two extensions of the same absolutely irreducible restriction differ by a character by Schur's lemma. Their determinants fix its cube, so this character is trivial. Restriction is irreducible at each step because it is irreducible already on G_L. At the bottom Proposition 31.10 gives crystallinity and the prescribed finite inertia. The level and weight computation of Theorem 29.18 now places the descent in the stated list. [5, 6, 7]

At a reducible point all nine-by-nine minors vanish. Three character summands give the augmentation by Proposition 31.2. In the other case Theorem 31.11 identifies the line as the trivial character. The point factors through $T/I_9(G_K)$, so Proposition 33.11 applies. The label of the plane determines its trace on G_K. For a rank-three label the good Frobenius traces determine the normalized trace on G_K by continuity and Chebotarev. In both cases group traces topologically generate T, which proves the last assertion. □

Corollary 37.43 (The generic trace algebra)**.** *The algebra $T[1/\ell]$ is finite étale over J.*

Proof. At every reducible point Theorem 33.10 already computes the local ring of the full trace algebra. At an irreducible point the argument of Theorem 33.2 now applies to the classical system in the proposition. More explicitly, a dual-number deformation of its determinant gives a representation lift by absolute irreducibility. Its continuous cocycle on G_K is invariant for the polarized action of complex conjugation, hence descends to $\mathbb{Q}$ by inflation–restriction. Its restriction to inertia away from ℓ is a coboundary by averaging over the fixed finite inertia. Above ℓ its self-extension is crystalline by Proposition 31.10. These local conditions descend by restriction and corestriction. Lemma 33.1 gives enormous image on the cyclotomic kernel. Theorem A of Newton and Thorne for the polarized adjoint of this cuspidal parameter makes its finite Selmer group zero. The lift is therefore conjugate to the constant lift and all its traces are constant. After a finite splitting extension each local factor of $T[1/\ell]$ is Artinian and has zero cotangent space, so its maximal ideal vanishes by Nakayama's lemma. This is geometric reducedness of the finite algebra. [8] □

Theorem 37.44 (One integer for the torsion-free original trace orders)**.** *Fix the split prime $u = u_1$ and a place of K above it as in Proposition 29.3. For each triple put*

$$T^{\mathrm{tf}} = T/T[\ell^\infty], \qquad I = \ker(T^{\mathrm{tf}} \longrightarrow O), \qquad h = t(\mathrm{Frob}_u) - (1 + u^r\alpha(u) + u^{-r}\alpha(u)^{-1}).$$

Write b_α for the size of the residue orbit $[\alpha]$. There is an integer $D(V)\ge 1$, independent of x on the interval, such that

$$\sum_{\ell>P(x)}(\log\ell)\sum_{1\le r<t(x)}\sum_{[\alpha]}b_\alpha\,\mathrm{length}_O(I/hI)\ \le\ \log D(V)\ \ll_{\psi,\varepsilon} V^C.$$

The orbits here have the specified complementary sign.

Proof. Take $D(V)=D_2(V)\prod_{r,H}|I_{u,H,r}|$, where $D_2(V)$ is the integer of Theorem 33.12, the other factors are those of Theorem 23.11, and the product is over $1\le r<t_*$ and the character kernels on E_* of the required sign as in Theorem 29.4. These are the integers for the chosen prime u itself. Theorem 23.11 bounds their logarithms by a polynomial in $N_*(t_*+2)$ times the corresponding orbit sizes. The sum of those sizes for any r is at most $[E_*:\mathbb{Q}]$. Theorem 33.12 bounds the first factor and the interval bounds all these parameters by powers of V. All factors are nonzero.

The finite torsion-free O-module I is free. Its geometric generic fiber has one line at each nonsplit point and no others by Corollary 37.43. At all the corresponding labels the Frobenius trace has complex absolute value at most three. For planes this follows directly from the formula $1+u^{-r}\alpha(u)a_u(f)$ and Deligne's bound. The split trace has absolute value at least $u^r-1-u^{-r}>3$. Thus multiplication by h is injective on I. Its determinant on the geometric generic fiber is the product of the trace differences over these nonsplit points. The Smith normal form over the unramified O gives

$$\mathrm{length}_O(I/hI)=\sum_{\sigma\neq\mathrm{split}}v_\ell(\sigma(h)),\qquad v_\ell(\ell)=1.$$

For each ℓ choose a common coefficient embedding. The rank-three labels in the proposition occur with their signs and embeddings in the product defining $I_{u,H,r}$. The plane labels occur separately in $D_2(V)$, where each difference is multiplied by u^r. The denominator-clearing powers of u in both products are ℓ-adic units. For fixed α,r, different points use different global labels, even if their values at u coincide. The b_α local conjugates use different character labels with equal valuations. Inflation $E_x\subseteq E_*$ preserves these labels. Hence every valuation counted at this ℓ occurs in $v_\ell(D(V))$, and all unused factors are integral at ℓ. Summing over primes proves the theorem. □

This estimate concerns the lattice I. To use it for the negative-weight cotangent module in Section 6.6, one must also control the odd-weight directions of the marked matrix problem at the same trace points.

38 Symplectic deformations at strict classes

38.1 Strict classes in a four-dimensional symplectic representation

For the twists at which two ordinary inertia characters coincide, we use a different matrix entry. Let $\mathcal{B}$ be the finite set in Corollary 42.19 and put

$$n_*=[E_*:\mathbb{Q}],\qquad L_*=2t_*+2+\max(\mathcal{B}\cup\{1\}),\qquad Y=N_*+L_*+2.$$

Every remaining small coefficient prime is at most L_*. Fix an odd prime $\ell\le L_*$, $2\le r<t(x)$, and a geometric character α of E_x of sign $(-1)^r$. We allow ℓ to divide the order of α. Set $\Psi=\alpha\kappa_\ell^r$ and let ω_ℓ be the Teichmüller lift of the residual cyclotomic character. In the four-dimensional representation the multiplier will have cyclotomic exponent $r+1$. These are the characters in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/DecompSigmaOrd.lean`, line 57.

Lemma 38.1 (One dyadic character for all the local tests)**.** *There is a primitive odd character ξ of conductor 2^m, with $2^m \le 16 n_* L_*(Y+1)$, such that for every odd prime $p \le Y$ its value at p has order greater than $n_* L_*$. This also holds after reduction in every odd residue characteristic. The conductor is bounded by a fixed power of V on the original period intervals.*

Proof. Choose the smallest m with $2^m > 8 n_* L_*(Y+1)$. The units modulo 2^m are the direct product generated by -1 and 5, of orders two and 2^{m-2}. Send -1 to -1 and 5 to a primitive 2^{m-2}-th root. The resulting odd character is primitive since it is nontrivial on the kernel of reduction modulo 2^{m-1}. Choose a sign so that $b = \pm p \equiv 1 \bmod 4$ and let $a = v_2(b-1)$. Then $2^a \le p+1$. Squaring an integer congruent to one modulo four increases this valuation by exactly one, since its sum with one has valuation one. Therefore b has order 2^{m-a} modulo 2^m. Our character is faithful on that factor. Multiplying its value by -1 preserves this order, which is greater than two. It is at least $2^m/(p+1) > n_* L_*$. Reduction of roots of unity of two-power order is injective in odd residue characteristic. The upper bound follows from minimality of m and the bounds for N_*, n_*, t_* on the interval. □

Enlarge O_α unramified to an integer ring O containing the dyadic character values. Write J for its fraction field, π for a uniformizer, $e = v_\pi(\ell)$, and k for its residue field. We identify finite characters of rational ideles with Galois characters in the arithmetic Frobenius convention. Put

$$A = \kappa_\ell \omega_\ell^{-1} \xi, \qquad \mu = A\Psi, \qquad d(g) = \operatorname{diag}\bigl(\mu(g), \Psi(g), A(g), 1\bigr).$$

On the ordered basis e_0, e_1, e_2, e_3 take the alternating form with $\langle e_0, e_3\rangle = \langle e_1, e_2\rangle = 1$ and other independent pairings zero. Both products of paired diagonal entries equal μ, so d preserves the form up to this multiplier. Let

$$T_\alpha = \{\ell\} \cup \{p \ne \ell : \ell \mid |\alpha(I_p)|\}.$$

All members are odd. The inertia quotient of an abelian extension of $\mathbb{Q}$ at two has two-power order, by local reciprocity on $\mathbb{Z}_2^\times$.

Proposition 38.2 (The six odd roots at the specified local groups)**.** *For each $p \in T_\alpha$ the two pairs of residual characters of d with indices $\{0,2\}$ and $\{1,3\}$ can be separated by a Frobenius lift. For each character*

$$\gamma \in \{A, A^{-1}, \Psi/A, A/\Psi, A\Psi, (A\Psi)^{-1}\}$$

one has $H^0(\mathbb{Q}_p, k(\bar\gamma)) = H^2(\mathbb{Q}_p, k(\bar\gamma)) = 0$. Its first cohomology has dimension one at $p = \ell$ and zero at the other $p \in T_\alpha$.

Proof. At ℓ take an arithmetic Frobenius lift f_ℓ with $\kappa_\ell(f_\ell) = 1$. Such a lift exists since the local cyclotomic extensions are totally ramified, so inertia surjects onto the cyclotomic character image. At $p \ne \ell$ use any arithmetic Frobenius lift f_p. The value of $\bar A$ on f_p is $\bar\xi(p)$. The subgroup generated by $\bar\Psi(f_p)$ and $\bar\kappa_\ell(f_p)$ has order at most $n_*(\ell-1)$. Indeed all its elements are roots of unity of order dividing $\operatorname{lcm}(n_*, \ell-1)$ in the field k. The lemma makes $\bar A(f_p)$ lie outside that subgroup. Here $p \le Y$, since $p = \ell$ or $p \mid N_*$. Writing $a = \bar A(f_p)$, $b = \bar\Psi(f_p)$, the two pairs of eigenvalues are $\{ab, a\}$ and $\{b, 1\}$. An intersection would give $a \in \{1, b, b^{-1}\}$.

Each $\bar\gamma$ takes a value at f_p involving $a^{\pm 1}$ times a power of b. Both it and its Tate dual consequently have nontrivial Frobenius value. The invariant spaces are zero, and local duality proves the assertion in degree two. The local Euler characteristic formula gives the indicated dimensions in degree one. The six spaces are precisely the odd root lines for the adjoint action on the symplectic Lie algebra. Indeed the upper entries have ratios $A, \Psi, \Psi/A, A\Psi$, with entries reflected across the anti-diagonal paired by the form. The opposite entries have the inverse ratios. At complex conjugation $A = -1$ and $\Psi = 1$. □

Write $D = (J/O)(\theta)$, $\theta = \Psi^{-1}$, and let $D_h = D[\pi^h]$. Use $S = S^{N_*}(\alpha, r)$ of Theorem 15.17 with these coefficients. Its O-length is unchanged by the unramified extension from O_α, by tensoring a composition series. Henceforth suppose $\bar{\Psi} \neq 1$ on $G_{\mathbb{Q}}$. Choose g_0 with both $\bar{\Psi}(g_0) \neq 1$ and $\bar{A}(g_0) \neq 1$. Each kernel is proper and two proper subgroups cannot cover a group. If neither contains the other, the product of one element exclusive to each is in neither.

Lemma 38.3 (Local splittings at a bounded additional precision)**.** *Normalize cocycles by $z(g_0) = 0$. Let Z_h be the module of continuous normalized θ-cocycles with values in D_h which are coboundaries on $G_{\mathbb{Q}_p}$ for $p \in T_\alpha$ and on inertia at every other finite prime, all with witnesses in D_h. In the normalized divisible coefficient module put*

$$C_\alpha = \max_{p \in T_\alpha} \min_{g \in G_{\mathbb{Q}_p}} v_\pi(\theta(g) - 1), \qquad v_\pi(0) = +\infty.$$

Then $C_\alpha \leq e(1 + v_\ell(n_* r))$, and for every $h \geq 1$

$$\pi^{C_\alpha} S[\pi^h] \;\subseteq\; Z_h \;\subseteq\; S[\pi^h] \;\subseteq\; Z_{h+C_\alpha}.$$

Proof. Subtracting the coboundary with witness $z(g_0)/(\theta(g_0) - 1)$ normalizes a cocycle. Its denominator is a unit, so this operation works in every coefficient module in question and kills exactly the coboundaries. In particular a divisible class killed by π^h has a unique normalized representative, whose values are all killed by π^h.

At ℓ choose an inertia element with cyclotomic value $1 + \ell$ and raise it to the n_*-th power. This kills its finite character and gives valuation $v_\pi((1+\ell)^{-r n_*} - 1) = e(1 + v_\ell(r n_*))$, by the binomial formula at an odd prime. At $p \in T_\alpha$ different from ℓ, the inertia image of α contains a primitive ℓ-th root of unity. The value of $\theta - 1$ on an element mapping to such a root has valuation $e/(\ell - 1)$. This proves the uniform bound.

If z represents a class of $S[\pi^h]$, its restriction at $p \in T_\alpha$ is $(\theta - 1)y_p$ for some $y_p \in D$, since these primes are ℓ or divide N_*. Evaluate at an element attaining the finite minimum there. Multiplication by $\pi^h(\theta(g) - 1)$ kills y_p, hence $y_p \in D_{h+C_\alpha}$. At the other finite primes the inertia action is through a group of order prime to ℓ. If the character on inertia is trivial then the restricted coboundary is zero. Otherwise a unit difference $\theta(g) - 1$ makes its witness already lie in D_h. Multiplying every chosen witness by π^{C_α} puts it in D_h, which proves the first and last inclusions. For the middle inclusion the only further issue is the strict condition on the full decomposition group at $p \mid N_*$ outside T_α. There the unramified subgroup for D vanishes. Its local formula is $D^{I_p}/(f_p - 1)D^{I_p}$. With nontrivial prime-to-ℓ inertia its numerator is zero. With trivial inertia the numerator is divisible and $\alpha(f_p)^{-1} p^{-r} - 1 \neq 0$ acts surjectively. Thus the image of the finite cocycle satisfies that strict condition as well. □

Theorem 38.4 (The symplectic entry carrying every strict class)**.** *In the ring $R_h = O \oplus D_h$, take D_h as a square-zero ideal and use its underlying O-module as coefficients for entries. Write E_{ij} for an elementary matrix and $X = E_{20} + E_{31}$. The rule*

$$z \;\longmapsto\; \rho_z(g) = (1 + z(g)X)d(g)$$

identifies Z_h with the lifts of this form and multiplier μ satisfying the following conditions. The value at g_0 is $d(g_0)$. At $p \in T_\alpha$ the local lift is conjugate to d by some $1 + y_p X$, $y_p \in D_h$. At all other finite primes the lift kills the kernel of $\bar{d}$ on inertia. Every such local conjugate at $p \in T_\alpha$ lies on an integral family through d with the same ordinary or finite inertia condition. The union of these lifts, under the inclusions of their coefficient ideals, realizes exactly S.

Proof. The image of X is isotropic, $X^2 = 0$, and $\langle Xv, w\rangle + \langle v, Xw\rangle = 0$. For the last identity the only nontrivial pair to check is $v = e_0, w = e_1$, giving $-1+1 = 0$, together with its reverse. Thus $1+yX$ is symplectic and its inverse is $1-yX$. Conjugation by $d(g)$ multiplies X by $\theta(g)$, since the two entry ratios both equal $\Psi(g)^{-1}$. Matrix multiplication now gives the group law for ρ_z precisely when

$$z(gh) = z(g) + \theta(g)z(h).$$

The $(2,0)$-entry divided by $\mu(g)$ recovers $z(g)$, so this correspondence is injective. Conjugating $d(g)$ by $1-yX$ gives the entry $(\theta(g)-1)y$. This proves the asserted equivalence at g_0 and the designated decomposition groups.

At every other finite prime the images of all the diagonal inertia characters have order prime to ℓ. Indeed both cyclotomic characters are unramified there, ξ has two-power order, and $\alpha(I_p)$ has order prime to ℓ. Reduction on this finite diagonal image is injective. A cocycle which is a coboundary on inertia therefore vanishes on the kernel of $\bar{d}$. Conversely vanishing on that kernel makes the inertia cocycle descend to the finite diagonal image. Averaging over this group gives a coboundary over D_h. This proves the converse in the characterization as well.

At each $p \in T_\alpha$ the family $(1-Y'X)d(g)(1+Y'X)$ over the domain $O[[Y']]$ specializes to d at zero and to the given local lift at its square-zero witness. At ℓ conjugating the ordered split flag gives its ordinary flag with exact characters of cyclotomic exponents $r+1, r, 1, 0$. At $p \neq \ell$ conjugation preserves the condition of killing $I_p \cap \ker \alpha$, since ξ is unramified at these odd primes. The matrices preserve the two coordinate planes indexed by $\{0,2\}$ and $\{1,3\}$. Their values at g_0 and g_0^{-1} are diagonal, which gives the off-diagonal normalization for the sum $\rho_z(g_0) + \mu(g_0)\rho_z(g_0^{-1})$ in the source. Conjugation by $\operatorname{diag}(ab, b, a, 1)$ multiplies X by b^{-1}, identifying the indicated torus weight. Finally every class of S is killed by some power of π, by finiteness in Theorem 15.17. The two depth inclusions in the lemma identify the union with S. The trace on every one of these lifts still equals $\operatorname{tr} d$, since the two varying entries are off the diagonal. □

Corollary 38.5 (Precision at the two collision residues)**.** *At the primes outside $\mathcal{B} \cup \{q, d_0\}$, suppose $\ell - 1 \mid 2r$ and $\bar{\Psi} \neq 1$. At $r \equiv (\ell-1)/2 \bmod \ell - 1$ one has $Z_h = S[\pi^h]$ at every depth. The same holds at $r \equiv 0 \bmod \ell - 1$ if $\alpha(\ell) \neq 1$. In the other case the required additional precision is at most $1 + v_\ell(r)$ powers of ℓ. These statements refer to every scalar component on each original E_x.*

Proof. Here $\ell \nmid n_* N_*$, the coefficient extension is unramified, and $T_\alpha = \{\ell\}$. At the half residue the residual inertia character of Ψ is the nontrivial quadratic character. At the zero residue the residual character on a Frobenius lift with $\kappa_\ell = 1$ is $\bar{\alpha}(\ell)$. A nontrivial value of this prime-to-ℓ character remains nontrivial after reduction. In both cases the minimum in the lemma is zero. Otherwise $e = 1$ and $v_\ell(n_*) = 0$, giving the claimed bound. The integral decomposition of Theorem 15.17 identifies these scalar modules with the corresponding field components after coefficient extension. The construction realizes the orders at every depth. To bound them using four-dimensional traces one still needs a comparison with irreducible deformations of the split point. □

38.2 Global localization for the six odd symplectic roots

Keep n_*, N_*, L_*, Y and the range of (x, ℓ, r, α) in Section 38.1. We now choose the dyadic conductor large enough to apply the Bernoulli unit theorem in Volume III. This choice works simultaneously for every odd $\ell \leq L_*$ and every indicated α, r, x. The group on which the target scalar classes are measured is still the one for E_x. The additional dyadic character occurs in

the other diagonal entry A. In this subsection $\bar{\Psi}$ may be trivial. Write

$$\mathcal{R} = \{A, A^{-1}, \Psi/A, A/\Psi, A\Psi, (A\Psi)^{-1}\}.$$

We count these lines with their symplectic root indices when two characters agree modulo a coefficient prime.

Lemma 38.6 (One conductor for the six Bernoulli units)**.** *There is a single odd primitive dyadic character ξ, of conductor bounded by a fixed power of V, for which Proposition 38.2 holds and for which the following assertion holds at every indicated ℓ. For each $\Gamma \in \mathcal{R}$, let δ be the primitive Dirichlet character attached to the Teichmüller lift of $\bar{\Gamma}$. Both $B_{1,\delta}$ and $B_{1,\delta^{-1}}$ are units at the chosen prime over ℓ.*

Proof. Choose $M = 2^m$ to be the smallest power of two strictly larger than

$$\max\{8n_*L_*(Y+1),\, 2^{1+v_2(N_*)},\, L_*^5 N_* n_*^2\}.$$

Take the character on $-1, 5$ used in the proof of Lemma 38.1, now at conductor M. Its order at each odd prime $p \le Y$ exceeds n_*L_* by the same valuation calculation. This proves all the local assertions of Proposition 38.2 for the present choice. The quantities in the displayed maximum are polynomially bounded on the interval and M is at most twice that maximum.

Write $n' = n_*/\ell^{v_\ell(n_*)}$. Take D to be twice the multiplicative order of ℓ modulo n', with $D = 2$ if $n' = 1$. Thus $D \le 2n_*$, $Q = \ell^D \equiv 1 \bmod 4$, and both $\ell - 1$ and n' divide $Q - 1$. Factor δ by the Chinese remainder theorem as $\phi\xi'$, with ϕ primitive of odd conductor f and ξ' of dyadic conductor. The residual character of A is $\bar{\xi}$, and that of Ψ is $\bar{\alpha}\,\bar{\kappa}_\ell^r$. The finite character α has order dividing n_* and conductor dividing N_*. Its reduction has order dividing n'. Consequently $f \mid \ell N_*$, and the order of ϕ divides $\operatorname{lcm}(n', \ell - 1)$, hence divides $Q - 1$. On $1 + M/2$ the dyadic part of α is trivial, whereas each of ξ, ξ^{-1} is -1. Thus ξ' is primitive of conductor M. Moreover $\delta(-1) = -1$, since Ψ is even and A is odd.

Writing $e_2 = v_2(Q-1)$, repeated squaring gives

$$e_2 = v_2(\ell - 1) + v_2(\ell + 1) + v_2(D) - 1, \qquad 2^{e_2} \le (\ell^2 - 1)D/2 \le L_*^2 n_*.$$

Indeed the odd factor of D leaves the valuation unchanged, the first squaring contributes $v_2(\ell + 1)$, and each subsequent squaring contributes one. Hence $f 4^{e_2} \le L_*^5 N_* n_*^2 < M$. Theorem 18.14, including its proof of integrality when $\ell \mid f$, applies to these primitive factors and to their inverses. □

Let O contain the values as in the preceding subsection, let π be a uniformizer, and put $O_h = O/\pi^h$ and $M_h = O_h(\Gamma)$ for $h \ge 1$. Let $T = T_\alpha$. Define $S_h^0(\Gamma) \subset H^1(\mathbb{Q}, M_h)$ by imposing zero restriction to $G_{\mathbb{Q}_p}$ for $p \in T$, and zero restriction to I_p for every finite $p \notin T$. Let $S_h^T(\Gamma)$ use the same inertia conditions and allow arbitrary local classes at T. All cohomology is continuous.

Proposition 38.7 (Vanishing of the strict odd root groups at every depth)**.** *For ξ as above, $S_h^0(\Gamma) = 0$ for all six root lines and all $h \ge 1$.*

Proof. For $h = 1$ a strict cocycle vanishes on $I_p \cap \ker \bar{\Gamma}$ at every finite prime. Indeed on inertia it is a coboundary, which is zero on that kernel. Corollary 18.15 applies by the preceding lemma. In that corollary the cocycle restricts over $\mathbb{Q}(\zeta_{fM})$ to an unramified class group character. The integral Stickelberger annihilator acts on its image by a unit, using exactly the Bernoulli unit just established. The resulting class on the abelian cyclotomic Galois group vanishes by commuting complex conjugation with an arbitrary element. This proves the assertion in the residue field.

Suppose $h > 1$ and subtract a lifted global coboundary from a strict cocycle so that its values lie in πM_h. Identify this submodule with M_{h-1} by multiplication by π. At $p \in T$ write the original, adjusted cocycle as $g \mapsto (\Gamma(g) - 1)y_p$ on the decomposition group. There is a g there with $\bar{\Gamma}(g) \neq 1$, by Proposition 38.2. Its value forces $y_p \in \pi M_h$. At $p \notin T$ the inertia image of Γ is finite of order prime to ℓ. A nontrivial inertia image therefore supplies the same unit difference. For a trivial image the cocycle vanishes on inertia already. The cocycle with values in M_{h-1} consequently satisfies all the strict conditions. Induction makes it a coboundary, which proves the result in M_h. □

Theorem 38.8 (Global localization on an odd root line)**.** *For every $\Gamma \in \mathcal{R}$, localization induces an isomorphism of O_h-modules*

$$S_h^T(\Gamma) \xrightarrow{\sim} H^1(\mathbb{Q}_\ell, M_h) \simeq O_h.$$

The localization isomorphisms commute with reduction of the coefficients through all depths. The Selmer group of the finite Tate dual with the local conditions orthogonal to those for $S_h^0(\Gamma)$ is zero.

Proof. Write $M_h^* = \mathrm{Hom}(M_h, \mu_{\ell^\infty})$ for the finite Tate dual, with its usual contragredient Galois action. Its residual character on a simple constituent is $\bar{\kappa}_\ell \bar{\Gamma}^{-1}$. Both M_h and M_h^* have zero invariants at each $p \in T$, since a Frobenius lift acts on each residual line by a scalar different from one. Their global invariant groups vanish as well. At the real place complex conjugation acts on M_h by -1, and ℓ is odd.

Use the finite Poitou–Tate size formula as in Proposition 30.5. For local conditions L_v, the ratio of the orders of the two mutually dual Selmer groups is the ratio of global invariant orders multiplied by $\prod_v |L_v|/|H^0(\mathbb{Q}_v, M_h)|$, including the real place. One may use any finite set of places containing $2, \ell, \infty$ and all conductor primes, with unramified conditions elsewhere. In the strict problem the factor at each $p \in T$ is one. At every other finite place the unramified subgroup is the cokernel of Frobenius minus one on $M_h^{I_p}$, and the local invariants are its kernel. They have equal orders. The real factor is also one. The preceding proposition therefore gives the trivial group for the orthogonal conditions on M_h^*.

The exact sequence comparing two choices of local conditions in Poitou–Tate duality now gives

$$0 \longrightarrow S_h^0(\Gamma) \longrightarrow S_h^T(\Gamma) \longrightarrow \bigoplus_{p \in T} H^1(\mathbb{Q}_p, M_h) \longrightarrow \mathrm{Sel}_{(L^0)^\perp}(M_h^*)^\vee,$$

where the last symbol denotes the Pontryagin dual for the strict choice L^0. Both outside terms vanish. By local duality $H^2(\mathbb{Q}_p, M_h) = 0$ at every $p \in T$. The local Euler characteristic formula gives $|H^1(\mathbb{Q}_p, M_h)| = 1$ for $p \neq \ell$ in T and $|H^1(\mathbb{Q}_\ell, M_h)| = |O_h|$.

To determine the module structure at ℓ, note that reduction from level h to $h-1$ is surjective in degree one because $H^2(\mathbb{Q}_\ell, M_1) = 0$. The sequence $0 \to M_{h-1} \xrightarrow{\pi} M_h \to M_1 \to 0$ then identifies the kernel of reduction to $H^1(\mathbb{Q}_\ell, M_1)$ with $\pi H^1(\mathbb{Q}_\ell, M_h)$. Here $h > 1$, and the last reduction is surjective because $H^2(\mathbb{Q}_\ell, M_{h-1}) = 0$. At level one the local group is one-dimensional. Nakayama's lemma gives a single generator at level h, and the equality of orders makes it a free generator over O_h. Localization commutes with the maps of coefficient modules, proving compatibility. □

Corollary 38.9 (The odd part of the ordinary first-order problem)**.** *At the split point d over each O_h, consider fixed multiplier first-order symplectic deformations modulo conjugation by matrices reducing to the identity modulo the square-zero parameter. Require the inertia restrictions away from T and the restrictions at $T \setminus \{\ell\}$ to be conjugate to those of d. Require*

a lift at ℓ of the displayed ordered symplectic flag, with its diagonal characters fixed. On the six odd root lines the resulting module is free of rank three over O_h. Its three coordinates are the upper triangular odd root classes at ℓ.

Proof. Over $O_h[\epsilon]/(\epsilon^2)$ write the lift in the form $(1+\epsilon Z(g))d(g)$. The fixed multiplier condition places $Z(g)$ in the symplectic Lie algebra. Multiplication gives the cocycle law for the adjoint action and conjugation changes Z by a coboundary. The diagonal representation preserves each matrix root line separately, also when some residual characters agree. A lifted symplectic flag can be put back in the original position by a symplectic basis change reducing to the identity. Indeed lift an adapted basis for its isotropic plane, choose dual partners for the perfect pairing, and add elements of the plane to the partners to make their mutual pairing zero. These operations to correct the standard pairings can be made in the square-zero ideal. In that basis the lower entries vanish. Conversely any upper root cocycle preserves the original flag with the prescribed diagonal, and coboundaries account for a change of flag position. At ℓ the allowed lower classes are thus zero and the upper classes are unrestricted. The other primes in T have zero first cohomology on every odd root. The localization theorem gives one free coordinate for each of the three upper odd roots and zero for each opposite root. This calculation leaves the even root with character Ψ^{-1}, which carries the strict group of Theorem 38.4. □

The residual vanishing condition used here also appears in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/XvanExt.lean`, line 27.

38.3 Extensions preserving a Lagrangian plane at a strict class

Keep the choice of ξ in Lemma 38.6. Fix $h \geq 1$, put $R = O/\pi^h$, and take any $z \in Z_h$ from Lemma 38.3. We use $\bar{\Psi} \neq 1$ as in that lemma. Identify D_h with $R(\Psi^{-1})$ by $t \mapsto t/\pi^h \bmod O$ from R to D_h. In the following matrices z denotes the cocycle in these R-coordinates. Write

$$P = \langle e_0, e_2 \rangle, \qquad Q = \langle e_1, e_3 \rangle.$$

They are complementary Lagrangian planes for the alternating form of Section 38.1. The Siegel parabolic of either plane is its stabilizer in the symplectic similitude group.

Lemma 38.10 (A representation on the two paired planes)**.** *The formula $\rho(g) = (1 + z(g)X)d(g)$, with $X = E_{20}+E_{31}$, defines a representation over R with multiplier μ. It preserves both P and Q. On each $G_{\mathbb{Q}_p}$ for $p \in T_\alpha$, and on each I_p otherwise, an element preserving both planes conjugates it to d. With the indicated ordered bases the two blocks are*

$$\rho_P(g) = \begin{pmatrix} \mu(g) & 0 \\ z(g)\mu(g) & A(g) \end{pmatrix}, \qquad \rho_Q(g) = \begin{pmatrix} \Psi(g) & 0 \\ z(g)\Psi(g) & 1 \end{pmatrix}.$$

Proof. For arbitrary $t, u \in R$ one has $(1 + tX)(1 + uX) = 1 + (t + u)X$ because $X^2 = 0$. The image of X is isotropic and X is skew for the alternating form, so $1 + tX$ is symplectic for every t. Conjugation by $d(g)$ acts on X by $\Psi(g)^{-1}$. These identities and the cocycle law give the representation over R without a restriction on the products of the values $z(g)$. The block formulas follow on the four basis vectors. Where $z(g) = (\Psi(g)^{-1} - 1)y$, one has $\rho(g) = (1-yX)d(g)(1+yX)$. The local witnesses in the definition of Z_h give this equality over R at every specified group. □

Let U_P be the additive group of skew endomorphisms that kill P and map Q into P. Interchanging P, Q defines U_Q. Here skew means $\langle tv, w\rangle + \langle v, tw\rangle = 0$. Each U has the Galois action by ρ-conjugation. Write $S^0(U)$ and $S^T(U)$ for the strict and relaxed groups defined before Proposition 38.7, using these finite coefficients and $T = T_\alpha$.

Proposition 38.11 (Localization over an arbitrary strict class)**.** *For either $U = U_P, U_Q$,*

$$S^0(U) = 0, \qquad S^T(U) \xrightarrow{\sim} H^1(\mathbb{Q}_\ell, U) \simeq R^3.$$

The middle map is localization. At each $p \in T \setminus \{\ell\}$ one has $H^1(\mathbb{Q}_p, U) = 0$.

Proof. The pairing identifies Q equivariantly with $P^\vee \otimes \mu$. Under this identification the skew condition makes U_P the symmetric tensors $\mathrm{Sym}^2(P) \otimes \mu^{-1}$. In fact a map $Q \to P$ belongs to U_P precisely when the associated bilinear form on Q is symmetric. Here two is invertible in R, so symmetric tensors and the symmetric square identify. The same reasoning gives $U_Q = \mathrm{Sym}^2(Q) \otimes \mu^{-1}$. The line $Re_2 \subset P$ is stable and the quotient is free on the image of e_0. In the symmetric square, take first the square of this line and then all products with this line. This is a stable filtration, with three free rank-one successive quotients. Their characters on U_P, in order, are

$$A/\Psi, \qquad A, \qquad A\Psi.$$

On Q use Re_3 in the same construction. The constituents on U_Q have the three inverse characters in reverse order.

We show that strict vanishing passes through this filtration. In $0 \to U' \to U'' \to B \to 0$ at one step, B has one of the six characters. Project a strict global cocycle on U'' to B. Proposition 38.7 makes the projection a coboundary. Lifting its witness to U'' and subtracting leaves a cocycle valued in U'. At $p \in T$ the projection to B of a local witness for this cocycle is invariant, hence zero by the Frobenius unit difference in Proposition 38.2. At other primes the action on inertia factors through a finite group of order prime to ℓ. This follows from the local conjugacy of the block representation to the finite diagonal inertia image. A projected local witness is inertia invariant. Lift it to U'', average its lift over the finite inertia image, and subtract the invariant lift from the witness. In either case one obtains a witness in U'. Induction up the filtration proves $S^0(U) = 0$.

At every $p \in T$ the local block conjugation gives a direct sum of the three indicated characters on U. The same is true of their local Tate duals. Both invariant groups vanish there by Proposition 38.2. Complex conjugation acts on each constituent of U by -1, so its invariant group vanishes, by the filtration. In the finite Poitou–Tate size formula for the strict problem all factors are therefore one, as in Theorem 38.8. In particular the dual group for the orthogonal conditions is zero. Comparison with the relaxed local conditions gives an isomorphism onto $\bigoplus_{p \in T} H^1(\mathbb{Q}_p, U)$. By the local direct sum and the local calculations in that theorem, the only nonzero term is at ℓ, where it is free of rank three over R. □

At ℓ choose a witness y_ℓ and transport the displayed complete flag for d by $1 - y_\ell X$, obtaining a flag $\mathcal{F}_\ell$ for ρ. For $U = U_P$ or U_Q let $1 + U$ denote its subgroup of the symplectic group. Consider representations with multiplier μ preserving the specified plane, with the induced action on that plane and its quotient fixed as for ρ. Two are equivalent when conjugate by $1 + U$. Impose the following local conditions. On inertia outside T and on the full local groups in $T \setminus \{\ell\}$, require conjugacy to ρ by $1 + U$. At ℓ require that some $1 + U$ conjugate preserve $\mathcal{F}_\ell$ with its prescribed graded characters.

Theorem 38.12 (Ordinary extensions in the two Siegel parabolics)**.** *For the plane P the classes of these representations form a free R-module of rank two, with origin ρ. For Q they form a free R-module of rank one. Localization identifies them with the allowed upper triangular classes at ℓ in each of the two unipotent radicals. These statements hold for every $z \in Z_h$ on the original field and interval data.*

Proof. In either parabolic, divide a representation value on the right by $\rho(g)$. The quotient acts trivially on the plane and on the quotient by it, and therefore has the form $1 + u(g)$, where $u(g)$ kills the plane and maps into it. Its image is isotropic. The symplectic condition consequently says exactly that $u(g)$ is skew. Products of two such endomorphisms vanish, so the multiplication law of $1 + U$ is addition. Thus $(1 + u(g))\rho(g)$ is a representation exactly when u is a cocycle on U. Conjugation by $1 + U$ changes it by a coboundary. This gives the claimed module of classes inside $S^T(U)$ over the actual ring R.

Put the local representation at ℓ back into the diagonal basis by the chosen block conjugation. It preserves P, Q, so also preserves each of the two unipotent radicals. The endomorphisms in U_P have independent entries at $(0,3), (0,1), (2,1)$. The skew identity determines the remaining entry $(2,3)$ from $(0,1)$. The first two parameters give upper triangular endomorphisms and the third gives a lower triangular one. In U_Q the transpose positions give one upper and two lower parameters. In each case the upper and lower coordinates are local Galois submodules in the diagonal basis. A local class preserves the required flag after the allowed conjugation exactly when its lower coordinates are coboundaries. Its upper coordinates then give arbitrary extensions with the same graded characters.

Proposition 38.11 identifies the global choices with these local classes. Each allowed coordinate contributes one copy of R, by Theorem 38.8. This gives the two ranks. These constructions use the original Ψ on the same period fields because the dyadic twist occurs only in A. □

Corollary 38.13 (Characteristic polynomials throughout both parabolics)**.** *For every $g \in G_{\mathbb{Q}}$, every representation in either module of Theorem 38.12, and every admissible z, its characteristic polynomial at g is*

$$(t - \mu(g))(t - \Psi(g))(t - A(g))(t - 1) \quad \text{in } R[t].$$

A class in $S[\pi^a]$ can be represented in this calculation at $h = a + C_\alpha$ for every $a \geq 1$.

Proof. Order a basis by the invariant plane and the quotient. The resulting block triangular matrix has the two diagonal blocks of the lemma, each triangular in its indicated basis. Taking determinants proves the formula over R. The precision assertion is the inclusion $S[\pi^a] \subseteq Z_{a+C_\alpha}$ of Lemma 38.3. In particular all characteristic polynomials on these two families agree at the same depth, including at Frobenius elements. Estimating the size of the strict group through traces still requires control of representations outside the two parabolics. □

38.4 A common Frobenius test for the rank-four representations

Use $A, \Psi, \mu = A\Psi$ and the conductor M of Lemma 38.6. The split diagonal is $(\mu, \Psi, A, 1)$. Its four cyclotomic exponents, in this order, are $r + 1, r, 1, 0$. Frobenius is arithmetic, so the cyclotomic character at an unramified prime $p \neq \ell$ has value p.

Proposition 38.14 (The determinants of the two parity pairs)**.** *Let ρ be a continuous rank-four representation of $G_{\mathbb{Q}}$ over a finite extension of $\mathbb{Q}_\ell$, with a stable lattice, residual semisimplification $\bar{d}^{\mathrm{ss}}$, and multiplier μ for a symplectic pairing. Suppose it is unramified outside finitely many places. At ℓ assume an ordinary flag whose ordered diagonal characters reduce to those of d on the decomposition group and agree with those of d on inertia. If the global semisimplification has a two-dimensional subrepresentation on which complex conjugation is scalar, the determinants on it and its complementary two-dimensional summand, in some order, have the form*

$$\varepsilon_1 \kappa_\ell^{r+2}, \qquad \varepsilon_2 \kappa_\ell^{r},$$

with finite characters ε_i. In particular at every unramified $p \neq \ell$, under any complex embedding of its Frobenius eigenvalues, one has

$$\frac{\max_i |\lambda_i(p)|}{\min_i |\lambda_i(p)|} \;\geq\; p.$$

The lower bound at Frobenius to the power $s \geq 1$ is p^s.

Proof. Enlarge the coefficient field to define the given summand and its complement. Passing to a stable subspace or quotient commutes with taking the multiset of residual constituents. Indeed intersect a stable lattice with the subspace. Its quotient in the lattice is torsion-free, so reduction preserves the resulting exact sequence. One can apply this argument along a composition series, and semisimplified reduction is independent of the chosen stable lattice by the Brauer–Nesbitt theorem.

At complex conjugation the residual characters indexed by $0, 2$ have value -1, and those indexed by $1, 3$ have value 1. As ℓ is odd, the scalar summand must take both residual characters from just one pair and the other summand takes the other pair, with multiplicities. On the decomposition group, the semisimplification of the local ordinary flag has four one-dimensional constituents. Their reductions have disjoint sets of values at the single Frobenius lift of Proposition 38.2 for these two pairs. Hence the constituents on the first of the parity pairs at this local group have indices $0, 2$, and those on the second have indices $1, 3$. This also follows by factoring the local Frobenius characteristic polynomial and reducing its roots. One only needs separation between the pairs. Determinants multiply in the local semisimplification, so their inertia exponents are $(r + 1) + 1 = r + 2$ and $r + 0 = r$, up to finite characters.

After removing the corresponding cyclotomic power, each global determinant has finite inertia at ℓ. At each other finite p, its abelian inertia image is finite. Indeed local reciprocity identifies the abelian inertia group with $\mathbb{Z}_p^\times$, whose open pro-p subgroup has finite image in the units of a finite extension of $\mathbb{Q}_\ell$ when $p \neq \ell$. Outside a finite set the character is unramified. By Kronecker–Weber the product of the local unit groups surjects onto $G_{\mathbb{Q}}^{\mathrm{ab}}$, giving a finite global image. This proves the two determinant formulas.

At a good p, the products of the two pairs of eigenvalues therefore have absolute values p^{r+2} and p^r. Their ratio p^2 is at most the square of the ratio of the largest to the smallest eigenvalue magnitude. The power assertion follows by raising the eigenvalues to that power. The block version of this obstruction occurs in `Development/AutomorphicArithmetic/Foundations/GaloisRepresentations/AdmissibleResidues/PosOfMulPosOfPos.lean`, line 231. □

Choose once on the interval a rational prime p between $Y+32$ and $2(Y+32)$, using Bertrand's postulate. It lies outside the conductor of every α, ξ under consideration and differs from every remaining coefficient prime $\ell \leq L_*$. Both p and M are bounded by fixed powers of V.

Lemma 38.15 (Cuspidal traces at a prime of arbitrary residue degree)**.** *Suppose a rank-four unramified Frobenius matrix at p has eigenvalues λ_i in a complex realization. Suppose that, for a positive integer s, the multiset $\{\lambda_i^s / p^{s(r+1)/2}\}$ is the unramified Satake multiset of a cuspidal* GL_4 *representation with unitary central character at a place of norm p^s. Then*

$$p^{(r+1)/2-9/22} \leq |\lambda_i| \leq p^{(r+1)/2+9/22}.$$

In particular the preceding parity pairs cannot occur for such a representation. Its trace $u = \sum_i \lambda_i$ satisfies

$$|u - (1 + p\zeta)(1 + p^r\eta)| > \tfrac{1}{2}p^{r+1}$$

for all roots of unity ζ, η and every $r \geq 2$.

Proof. Theorem 1 of Blomer and Brumley bounds the absolute values of the real exponents of an unramified unitary cuspidal Satake parameter on GL_4 by $9/22$ over any number field. At norm p^s this puts each normalized magnitude between $(p^s)^{-9/22}$ and $(p^s)^{9/22}$. Taking s-th roots gives the claimed estimate. The ratio of any two magnitudes is at most $p^{9/11} < p$, so the determinant obstruction applies. In particular a match after restriction to an auxiliary number field suffices at any place above p with this Satake identity. [9]

Since $9/22 < 1/2$, one has $|u| \le 4p^{(r+2)/2}$. The split term has magnitude at least $(p-1)(p^r-1)$. As $r \ge 2$ and $p > 32$, the difference, after division by p^{r+1}, is at least $(1-1/p)(1-1/p^2) - 4/p > 1/2$. □

Theorem 38.16 (An integer bounding the simultaneous cuspidal trace congruences)**.** *For each integer $2 \le r \le t_*$, take a finite multiset $\mathcal{C}_r$ of algebraic integers, stable under rational Galois conjugation including multiplicities. Suppose every element has magnitude at most $4p^{(r+1)/2+9/22}$. These multisets may in particular consist of integral traces whose conjugates have the cuspidal realizations of the lemma. There is a positive rational integer D_{Sp} with*

$$\log D_{\mathrm{Sp}} \;\le\; n_* M L_*^2 \sum_{2 \le r \le t_*} |\mathcal{C}_r| \,\big((r+1)\log p + \log 8\big).$$

For every common field, every geometric character α under consideration, and every odd $\ell \le L_*$, fix a coefficient embedding and normalize $v_\ell(\ell) = 1$. Then for each allowed r,

$$\sum_{u \in \mathcal{C}_r} v_\ell\big(u - \operatorname{tr} d(\mathrm{Frob}_p)\big) \;\le\; v_\ell(D_{\mathrm{Sp}}).$$

This includes the split trace of every representation in the two families of Theorem 38.12 after reduction at any depth.

Proof. Put $m_\ell = \operatorname{lcm}(M, \ell-1)$ and form the monic polynomial

$$B_r(Z) = \prod_{\substack{3 \le \ell \le L_* \\ \ell \text{ prime}}} \prod_{\zeta^{m_\ell}=1} \prod_{\eta^{n_*}=1} \big(Z - (1+p\zeta)(1+p^r\eta)\big).$$

Its coefficients are algebraic integers fixed by the rational Galois group, hence are integers. Its degree is at most $n_* M L_*^2$, since $m_\ell \le M(\ell-1)$. On the diagonal d at p the trace is $(1+A(p))(1+\Psi(p))$, where $A(p) = p\,\xi(p)\omega_\ell(p)^{-1}$ and $\Psi(p) = p^r\alpha(p)$. The two finite factors have orders dividing m_ℓ and n_*. Thus the trace for every choice in the theorem occurs as a root of the same B_r.

Take the absolute value of the product $\prod_r \prod_{u \in \mathcal{C}_r} B_r(u)$. It is an integer by Galois stability. Every factor is nonzero by the numerical estimate in the proof of the lemma, which uses only the asserted magnitude bound for u. Each difference in this product has magnitude at most $8p^{r+1}$, by the triangle inequality and $r \ge 2$. This proves the upper bound for its logarithm. In any ℓ-adic embedding all factors have nonnegative valuation. Retaining the one specified root at each $u \in \mathcal{C}_r$ therefore proves the local inequality, including for a submultiset. The reduction statement follows from Corollary 38.13. □

Corollary 38.17 (The size of spectra at fixed rational rank and level)**.** *Let $\mathcal{N}$ be a finite list of positive levels. On the rational GL_4 locally symmetric spaces of principal level $3N$, $N \in \mathcal{N}$, consider the simultaneous Hecke eigensystems in ordinary characteristic-zero cohomology with irreducible rational algebraic coefficients of dominant highest weight λ, $|\lambda_i| \le B$. Even counted with their generalized eigenspace multiplicities and summed over degrees, their total over this list is at most*

$$C(B+1)^{10} \sum_{N \in \mathcal{N}} (3N)^{16}$$

for an absolute C. In particular the integer in the theorem has $\log D_{\mathrm{Sp}} \ll V^{C'}$ for integral cuspidal trace lists obtained from these eigensystems with one normalized trace per eigensystem per r, if their conjugates meet the lemma and $B, |\mathcal{N}|, \max(\mathcal{N} \cup \{1\})$ are polynomially bounded in V.

Proof. Divide out the positive real center at infinity. The level-three space has finitely many components and a finite cell model by the Borel–Serre compactification. The principal level-three subgroup is torsion-free on each component. Indeed a rational finite-order stabilizer preserves the corresponding integral lattice and acts trivially on its reduction modulo three, so the congruence-kernel argument of Theorem 7.3 applies. The level-$3N$ cover has at most $(3N)^{16}$ sheets. The cosets inject into the reduction of integral four-by-four matrices modulo $3N$. A lifted cell model therefore has at most the same factor times the fixed number of cells at level three. [10]

The rank of the local system is $\prod_{i<j}(\lambda_i - \lambda_j + j - i)/(j - i)$ by the Weyl dimension formula and is at most $(2B+4)^6$. Cellular cochains, summed over degrees, have dimension at most the number of cells times this rank. The simultaneous generalized eigenspaces fit in cohomology, whose total dimension is bounded by that of cochains. There are at most $(2\lfloor B \rfloor + 1)^4$ possible highest weights. These estimates give the formula including eigenspace multiplicities. Summing over at most t_* twists in the theorem preserves a polynomial bound in V. □

The trace estimate applies as soon as an integral spectrum of the stated size contains the deformations in question. To deduce the remaining strict length from it one still needs both that spectral identification and an integral comparison of the strict group with the sum of the trace valuations. The source states its latter comparison with hypotheses on finiteness and rigidity in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/GaloisRepIneqOfAllIrreducibleNonvanishing.lean`, line 152.

38.5 The integral equations of the negative symplectic root

Write $\theta = \Psi^{-1}$, $D = (J/O)(\theta)$, and $S = S^{N_*}(\alpha, r)$ as in Section 38.1. Keep $\bar{\Psi} \neq 1$, $T = T_\alpha$, the normalization at g_0, and $X = E_{20} + E_{31}$. The explicit witnesses for a local conjugation in Lemma 38.3 may require extra coefficient precision. We determine the integral equations of that conjugation family and compute the resulting global ring.

Lemma 38.18 (The image of the local conjugation family)**.** *For $p \in T$, take $\sigma_p \in G_{\mathbb{Q}_p}$ minimizing $v_\pi(\theta(g) - 1)$ and set*

$$a_p = \theta(\sigma_p) - 1, \qquad b_p(g) = (\theta(g) - 1)/a_p.$$

Then $a_p \neq 0$, $b_p(g) \in O$ and $b_p(\sigma_p) = 1$. In the matrix family $(1 + z(g)X)d(g)$ the scheme-theoretic image of conjugation by $1 - YX$ is the formal affine line

$$z(g) = b_p(g)t, \qquad t = z(\sigma_p).$$

In particular every formal local matrix equation vanishing on these conjugates vanishes identically on this line.

Proof. The finiteness of the minimum follows from Lemma 38.3. The ratio b_p is continuous and satisfies $b_p(gh) = b_p(g) + \theta(g) b_p(h)$, by expanding $\theta(gh) - 1$. The displayed formula consequently gives a representation over $O[[t]]$. Its matrix entries generate the parameter t, by taking the $(2,0)$-entry at σ_p and dividing by $\mu(\sigma_p)$, a unit. The actual conjugate $(1 - YX)d(g)(1 + YX)$ has parameter $t = a_p Y$. Substitution $O[[t]] \to O[[Y]]$ by $t \mapsto a_p Y$ is injective. The coefficient

of Y^j is the original coefficient of t^j times the nonzero element a_p^j of the domain O. The kernel of the map from formal matrix coordinates to $O[[Y]]$ is thus precisely the kernel for the displayed line. This is the asserted scheme-theoretic image. In particular its conclusion concerns equations also over coefficient rings where a_p is a zero divisor. □

For a finite length O-module M_0 write $\mathcal{Z}(M_0)$ for the continuous functions $z : G_{\mathbb{Q}} \to M_0$ with the cocycle law for θ, with $z(g_0) = 0$, and satisfying the local equations

$$\begin{aligned} z(g) &= b_p(g) z(\sigma_p) \quad && (g \in G_{\mathbb{Q}_p},\ p \in T), \\ z(g) &= 0 && (g \in I_p \cap \ker \bar{d},\ p \notin T). \end{aligned}$$

The second line is imposed at finite places. Give M_0 the discrete topology. Put $U = \mathrm{Hom}_O(S, J/O)$ as an O-module. It is finite by Theorem 15.17.

Proposition 38.19 (The integral coefficient module of the normalized cocycles)**.** *There is a natural isomorphism for all finite length O-modules*

$$\mathrm{Hom}_O(U, M_0) \ \simeq \ \mathcal{Z}(M_0).$$

In particular $\mathcal{Z}((J/O)[\pi^h]) = S[\pi^h]$ through normalized cocycles, without a shift in h.

Proof. The unique normalized representative z_s of $s \in S$ exists because $\theta(g_0) - 1$ is a unit. Its value at each g is O-linear in s. Forgetting the character on the coefficient module, define $x(g) \in U$ by $x(g)(s) = z_s(g)$. The function x is continuous since S is finite and the individual representatives are continuous. Where $p \in T$, write $z_s(g) = (\theta(g) - 1)y$ in D. Evaluating at σ_p and multiplying by $b_p(g)$ shows that $x(g) = b_p(g)x(\sigma_p)$ in U. At every other finite prime the local strict or unramified condition also implies vanishing on $I_p \cap \ker \bar{d}$. Indeed the diagonal inertia characters there have prime-to-ℓ image, and reduction is injective on this image. Thus $x \in \mathcal{Z}(U)$.

The elements $x(g)$ generate U. Otherwise a nonzero functional on their quotient with values in J/O would give a nonzero $s \in S$ with $z_s(g) = 0$ for all g. Here the finite O-linear bidual of S identifies canonically with S. This is checked on O/π^h from $(J/O)[\pi^h] = \pi^{-h}O/O$, and then on direct sums. Consequently composition with x is always injective from the indicated Hom module.

First take $M_0 = (J/O)[\pi^h]$. A cocycle in $\mathcal{Z}(M_0)$ is a coboundary with divisible coefficients at each $p \in T$. Indeed choose $y \in J/O$ solving $a_p y = z(\sigma_p)$. This gives $z(g) = (\theta(g) - 1)y$ on $G_{\mathbb{Q}_p}$. At the other finite places vanishing on the indicated kernel makes the inertia restriction descend to its finite prime-to-ℓ diagonal image. Averaging there gives a coboundary. At any additional $p \mid N_*$ at which the full strict condition is required, the unramified cohomology with divisible coefficients vanishes. If the prime-to-ℓ inertia character of θ is nontrivial its invariant group in D is zero. Otherwise Frobenius minus one acts surjectively on D, since the scalar $\alpha(p)^{-1}p^{-r} - 1$ is nonzero in the unramified case. The unramified cohomology is the cokernel of this action on inertia invariants. Hence the global image belongs to $S[\pi^h]$. Conversely its normalized representative for any $s \in S[\pi^h]$ has values in M_0, by normalization, and the equations follow from those for x. This proves the identification at level h, where the Hom isomorphism sends evaluation at s to z_s.

Every finite length O-module is a finite direct sum of cyclic modules isomorphic to such M_0. Both functors take these direct sums to direct sums, and composition with x commutes with all O-linear maps. This proves the stated natural isomorphism independently of any cyclic decomposition. □

Theorem 38.20 (The full formal ring in the negative root)**.** *In a complete noetherian local O-algebra B with residue field k, consider representations $(1+z(g)X)d(g)$ with z valued in $\mathfrak{m}_B$. Use the normalization and both local equations defining $\mathcal{Z}$, interpreted in the maximal-ideal topology. This functor is represented by the completion R_θ of $\mathrm{Sym}_O U$ at its ideal generated by π and U. Its zero section has augmentation ideal I_θ satisfying*

$$I_\theta/I_\theta^2 \cong U, \qquad R_\theta[1/\pi] = J.$$

In particular the elementary divisors of this relative cotangent module are exactly those of the full strict group.

Proof. The identity $X^2 = 0$ and isotropy of its image allow arbitrary coefficients $z(g)$ in the maximal ideal of B. As in Lemma 38.10 the representation condition is exactly the linear cocycle law. At depth n the coefficient module $\mathfrak{m}_B/\mathfrak{m}_B^n$ has finite O-length, by its filtration over the residue field. The proposition identifies all its solutions with O-linear maps out of U. Taking the inverse limit identifies the continuous solutions with maps $U \to \mathfrak{m}_B$. Continuity follows since each finite quotient solution is continuous, and all equations may be tested in the finite quotients. The universal property of the symmetric algebra and completion identifies these maps with continuous local O-algebra maps $R_\theta \to B$.

To compute the two asserted invariants write $U = \bigoplus_{i=1}^m O/\pi^{c_i}$ with $c_i > 0$. The ring just constructed is

$$O[[t_1, \dots, t_m]]/(\pi^{c_1} t_1, \dots, \pi^{c_m} t_m).$$

The augmentation sends the variables to zero. Keeping terms of degree one gives $I_\theta/I_\theta^2 = U$ canonically by the inclusion of the generators. When $m > 0$, the whole ideal of positive-degree terms is killed by $\pi^{\max_i c_i}$, since it is generated by the t_i. Inverting π gives J, also when $m = 0$. The cyclic decomposition and finite duality give the stated elementary divisors. □

Corollary 38.21 (The strict dual in a global matrix cotangent module)**.** *Take a framed global matrix deformation ring at $\bar{d}$ with the given multiplier. Impose the inertia kernel condition away from T, and at $p \in T$ impose closed local matrix equations which vanish on the formal conjugates $(1-YX)d(g)(1+YX)$. One may also impose the diagonal value at g_0, or the weaker off-diagonal normalization in Theorem 38.4, and the fixed value at complex conjugation. With augmentation at d, write I for the augmentation ideal of the resulting ring. There is a surjection*

$$I/I^2 \longrightarrow U.$$

The lower matrix entries at global group elements already surject onto this module. In particular if a quotient of the cotangent module through which this map factors has finite length, its length is at least $\mathrm{length}_O S$.

Proof. Evaluate the framed matrices at the universal representation over R_θ. The local matrices at $p \in T$ are obtained by substituting $t = z(\sigma_p)$ into the local line of Lemma 38.18. Thus they satisfy every one of the stipulated closed local equations. At $p \notin T$ the diagonal representation kills the indicated inertia kernel and the defining equation kills its varying root entry. The normalization at g_0 is built in. At complex conjugation c, $\theta(c) = 1$ and $c^2 = 1$, so $2z(c) = 0$ and $z(c) = 0$. This verifies the additional normalizations. The universal property of the framed matrix ring therefore gives an augmented homomorphism to R_θ.

The $(2,0)$-entry at g, divided by $\mu(g)$, maps to the generator $x(g)$ in the symmetric algebra. These elements generate U by the proposition, proving the surjection on relative cotangent modules and the length bound, with the whole strict dual as coefficient module. The corresponding square-zero derivation in the source appears in `Development/AutomorphicArithme`

`tic/AutomorphicVarieties/SymplecticGSp4/ExistsQuotientDerivationOfShaCocycle.lean`, line 252. □

The generic fiber of this explicit quotient consists of the single split point for every possible list of the c_i. To upper-bound their sum one must control the relations in the larger deformation ring comparing this cotangent direction with its characteristic-zero points. The trace separation in Theorem 38.16 addresses the values at such points once their spectra are identified.

39 Galois representations and Frobenius traces

39.1 The ordinary symplectic flag at a collision weight

Fix a period interval and the single character ξ of Lemma 38.6. Take an indicated prime $\ell \geq 5$ prime to $n_* N_*$, an indicated $r \geq 2$ with $\ell - 1 \mid 2r$, and the even character $\Psi = \alpha \kappa_\ell^r$. Suppose $\bar{\Psi}|_{G_{\mathbb{Q}_\ell}} \neq 1$. These hypotheses include the faithful dyadic half-residue weights of Section 44.5 whenever $\ell \nmid n_* N_*$. Use the unramified coefficient ring O containing the character values, its residue field k, the alternating form and

$$(d_0, d_1, d_2, d_3) = (A\Psi, \Psi, A, 1), \qquad \mu = A\Psi$$

from Proposition 38.2. Write $D_\ell = G_{\mathbb{Q}_\ell}$ and I_ℓ for its inertia group. Coefficient rings below are complete Noetherian local O-algebras with the given finite residue field. A frame identifies the residual representation with the displayed diagonal one in the prescribed basis.

Lemma 39.1 (A unique integral flag tested on Frobenius eigenlines)**.** *There is an arithmetic Frobenius lift $h \in D_\ell$ having four distinct residual eigenvalues on d. For every framed symplectic similitude lift ρ with multiplier μ, let P_i be the spectral projector of $\rho(h)$ lifting the i-th coordinate projector. The following equations characterize existence of a D_ℓ-stable full flag lifting the ordered coordinate flag, with character $d_i|_{I_\ell}$ on its i-th graded line:*

$$P_j \rho(g) P_i = 0 \quad (i < j,\ g \in D_\ell),$$
$$P_i(\rho(s) - d_i(s)1)P_i = 0 \quad (0 \leq i \leq 3,\ s \in I_\ell).$$

The flag is unique and self-dual for the alternating form.

Proof. If $\bar{\Psi}$ is unramified, its nontriviality gives $\bar{\Psi}(h) \neq 1$ for every Frobenius lift. Otherwise multiply a lift by an inertia element if necessary to get this inequality. Put $b = \bar{\Psi}(h)$. Its order divides $\operatorname{lcm}(n_*, \ell - 1)$. The value $a = \bar{A}(h) = \bar{\xi}(\ell)$ has order greater than $n_*(\ell - 1)$ by the choice of ξ. The values $ab, b, a, 1$ are therefore distinct.

Hensel factorization of the characteristic polynomial gives unique roots z_i lifting the four labeled eigenvalues. Every difference $z_i - z_j$ for $i \neq j$ is a unit. Thus

$$P_i = \prod_{j \neq i} \frac{\rho(h) - z_j 1}{z_i - z_j}$$

are orthogonal idempotents with sum one, by Cayley–Hamilton. Their images are free lines by reduction modulo the maximal ideal. The multiplier identity conjugates the transpose inverse of $\rho(h)$, multiplied by $\mu(h)$, to $\rho(h)$. Uniqueness of the labeled roots implies $z_{3-i} = \mu(h)/z_i$. The form consequently pairs each line perfectly with the line numbered $3 - i$, and all other pairings vanish by the unit differences.

Any h-stable flag as in the statement is preserved by every polynomial projector. Each step is the direct sum of its projector images. These are direct summands of that step, and their

ranks are fixed by reduction to the coordinate flag. In each selected line the corresponding summand fills the line by Nakayama's lemma. Thus the successive steps must be the partial sums of the indicated lines. The pairing makes these steps self-dual. The first set of equations is exactly stability of the steps and the second prescribes the induced inertia actions. □

Call the frame normalized at h if $\rho(h)$ is diagonal. Every framed lift admits such a frame by choosing a generator in each eigenline reducing to its specified basis vector and rescaling the two opposite generators to give pairings one. Choices are related by a diagonal symplectic change of basis.

Theorem 39.2 (Six integral parameters for the normalized ordinary lift)**.** *The lifts in the preceding lemma satisfying its flag equations, with fixed multiplier and frame normalized at h, have a representing ring*

$$R_\ell = O[[u_1, u_2, a, x, b, c]].$$

Under conjugation by $\mathrm{diag}(st, t, s, 1)$*, the two u_i have weight zero and a, x, b, c have respective weights $(1,0), (0,1), (-1,1), (1,1)$. Here weight (i,j) means that evaluation after conjugation multiplies the coordinate by $s^i t^j$. All six coordinates vanish at the split lift.*

Proof. The flag is the coordinate flag in this frame. The diagonal characters are d_i times unramified characters reducing to one. The multiplier gives two independent ratios, since the products on the two paired lines are prescribed. Each independent ratio is determined by its value on h, and that value can be lifted across every surjection of Artin coefficient rings. Indeed the unramified quotient is procyclic and the unit groups of the Artin rings here are finite.

Consider a small surjection of such rings, with kernel L killed by the maximal ideal. Given an upper triangular lift modulo L, first lift its two diagonal characters. Choose continuous upper triangular symplectic similitude matrices lifting the group values with these characters on the diagonal. Such a choice exists because the upper unipotent symplectic group is parametrized by its four root subgroups in order of increasing height. Equivalently one can choose its entries $01, 12, 02, 03$ freely and solve the two pairing equations for $23, 13$. These choices require only addition and multiplication and apply in each finite quotient. A section of the resulting finite sets supplies the continuous initial choice. Its multiplicative defect is a two-cocycle in the strictly upper symplectic Lie algebra tensored with L. Conjugation on this kernel factors through the residual diagonal representation. Its four lines have characters $\bar{A}, \bar{\Psi}, \bar{\Psi}/\bar{A}, \bar{A}\bar{\Psi}$. For the three odd lines the second cohomology vanishes by Proposition 38.2. For $\bar{\Psi}$ the Tate dual has nontrivial inertia: the exponents of $\bar{\Psi}$ on inertia are 0 or $(\ell-1)/2$ modulo $\ell - 1$, neither of which is 1. Local duality gives the same vanishing [1]. Correcting the chosen cochain by a one-cochain therefore gives a representation lifting the original one. If needed, conjugate by an infinitesimal upper unipotent to restore diagonality at h. On every root line its effect on the value there is multiplication by a unit $\bar{\gamma}(h) - 1$, up to sign. This conjugation retains the required diagonal characters.

In the tangent space each of the four roots contributes the one-cocycles z with $z(h) = 0$. Coboundaries evaluate isomorphically onto the root line at h, so this is a copy of $H^1(D_\ell, k(\bar{\gamma}))$. Local invariants and second cohomology are zero for all four lines. The local Euler characteristic formula gives dimension one on each. There are also the two independent unramified diagonal directions. The functor is a closed subfunctor of framed lifts: the equations of the lemma and diagonality at h are equations in the universal matrices. The framed ring is Noetherian by the finite-dimensionality of local first cohomology over the finite residual splitting field, as in Theorem 25.1. The lifting argument gives formal smoothness, so six parameters spanning the relative cotangent space identify this ring with the displayed power series ring. For example

take the two varying diagonal ratios at h minus one. For each of the four normalized tangent cocycles choose an element on which its value is nonzero and take the corresponding upper matrix entry. These entries have independent tangent functionals. Conjugating the entries gives the four weights in the statement. □

Corollary 39.3 (Identification of the component through the split lift)**.** *For a frame normalized at h, the flag condition of Lemma 39.1 is equivalent to specialization from a complete local O-domain of ordinary lifts containing the split lift in the sense of* `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/DecompSigmaOrd.lean`*, line 27. Here the ordinary equations in that domain are the polynomial conditions at an inertia element s_0 described below.*

Proof. Choose s_0 with cyclotomic value $1+\ell$, possible by total ramification of the local cyclotomic extension. The four $d_i(s_0)$ are the powers of $1+\ell$ with exponents $r+1, r, 1, 0$. They are distinct in $J = \mathrm{Frac}(O)$. The polynomial conditions prescribe this characteristic polynomial at s_0 and the equations of Lemma 39.1 with each P_i replaced by $\prod_{j\neq i}(\rho(s_0) - d_j(s_0)1)$. These are the equations in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/MapRepApply.lean`, line 99.

Suppose the domain B satisfies those conditions and admits an O-specialization of its matrices to the split lift. Then O injects into B. In $B[1/\ell]$ divide each polynomial by $\prod_{j\neq i}(d_i(s_0) - d_j(s_0))$ to form the projectors Q_i. Over $\mathrm{Frac}(B)$ their partial sums give a stable flag by the polynomial conditions. Since it is stable under h, each step is the sum of a subset of the four simple h-eigenlines. Let P be the sum of their integral projectors and Q the corresponding partial sum of the Q_i. Equality of the two images gives $PQ = Q$ and $QP = P$, identities already in $B[1/\ell]$. At the split specialization each labeled h-projector becomes the labeled coordinate projector; this follows from its residual label and Hensel uniqueness. Each Q_i also becomes the i-th coordinate projector, by direct evaluation at $d(s_0)$. The two identities force the subset in the given step to be the initial segment in the prescribed order. Thus the flag over the fraction field is the ordered h-flag, including its inertia characters. Its equations with the P_i are integral, hence hold in the domain and at every specialization.

Conversely the universal normalized ring in Theorem 39.2 is a domain with its split augmentation. Over its fraction field the prescribed triangular characters give the four distinct eigenvalues at s_0. Their ordered partial sums of eigenlines form the coordinate flag: on each initial step the characteristic polynomial is the product of the first diagonal factors. This proves the polynomial conditions there, including the equations on each inertia quotient. Hence they hold in the universal domain. Its universal representation and augmentation supply the two specializations required in the statement for every normalized lift. □

39.2 The global symplectic equations on a surviving dyadic orbit

Take the faithful characters of C_j, the fields E_x, primes $\ell \equiv 5 \bmod 8$, and half-residue weights r in Section 44.3. Impose also $\ell \nmid n_* N_*$, $2 \le r < t(x)$, and use the interval character ξ and coefficient extension O of Section 39.1. Assume the negative strict group $S(\alpha, r)$ is nonzero and put $k(r) = \mathrm{length}_O S(\alpha, r)$, after extending scalars to O. We identify unramified scalar extension throughout: it preserves length, since a composition factor becomes the extended residue field. Here k alone denotes the residue field of O.

Lemma 39.4 (The two opposite even roots)**.** *For the strict residual condition, zero at ℓ and unramified at every other finite prime, the groups of $k(\bar{\Psi}^{-1})$ and $k(\bar{\Psi})$ have respective dimensions one and zero. All six odd roots of Proposition 38.2 have strict dimension zero.*

Proof. The residual character $\bar{\Psi}$ and its inverse have the same fixed field L. Their values generate a cyclic group of order 2^j. Frobenius on its character values raises to ℓ. The two characters lie in different Frobenius orbits: a congruence $\ell^i \equiv -1 \bmod 2^j$ would contradict $\ell \equiv 1 \bmod 4$. By Theorem 44.5, restriction to L and class reciprocity identify the two residual strict spaces with the equivariant homomorphisms from the split class quotient H in Section 44.3. Explicitly one quotients its ℓ-primary class group by the prime classes above ℓ. Each nonzero faithful orbit component has order at least ℓ^f where $f = 2^{j-2}$, and $|H| < \ell^{2f}$. Thus a nonzero component has length one and excludes the opposite component. The hypothesis on $S(\alpha, r)$ detects exactly the negative component by Lemma 44.7. Extending the finite coefficient field tensors the equivariant homomorphism spaces with k: one can compute them componentwise in a basis over the smaller field. The assertion for the odd roots is Proposition 38.7 at depth one. □

Let Σ consist of ℓ and the finite conductor primes of d, including 2. Let G_Σ be the Galois group of $\mathbb{Q}$ unramified away from Σ and infinity. For $q \in \Sigma \setminus \{\ell\}$ write $N_q = I_q \cap \ker \bar{d}$, $H_q = G_{\mathbb{Q}_q}/N_q$. The inertia image here is finite of order prime to ℓ and d itself is trivial on N_q. Consider framed lifts to symplectic similitudes of multiplier μ, trivial on every N_q, whose restrictions at ℓ are the normalized ordinary lifts of Theorem 39.2. Denote their complete local ring by $\mathcal{B}$. These conditions are closed equations in the global framed ring, whose Noetherian construction uses the finiteness of first cohomology with finite coefficients over the residual splitting field, as in Theorem 25.1.

Theorem 39.5 (The global equations and their two torus weights)**.** *There is a presentation*

$$\mathcal{B} = O[[u_1, u_2, a, x, b, c, y]]/(Z_1, Z_2, E, F_1, F_2).$$

The first six variables are the local ones in Theorem 39.2. One may take y to be a global matrix entry of weight $(0, -1)$. The Z_i have weight zero, E has weight $(0, 1)$, and F_1, F_2 have weight $(0, -1)$. A vacant relation of a given weight may be entered as zero.

Proof. We give the deformation count relative to R_ℓ. For each torus weight take the corresponding space M of the residual symplectic Lie algebra. It consists of two trivial lines at weight zero, one line for each even root and one for each of the six odd roots. Form the shifted cone K of the cochain map

$$C^\bullet(G_\Sigma, M) \oplus \bigoplus_{q \neq \ell,\ q \in \Sigma} C^\bullet(H_q, M) \longrightarrow C^\bullet(D_\ell, M) \oplus \bigoplus_{q \neq \ell,\ q \in \Sigma} C^\bullet(G_{\mathbb{Q}_q}, M),$$
$$(z, (z_q)) \longmapsto (z|_{D_\ell}, (z|_{G_{\mathbb{Q}_q}} - \inf z_q)_q).$$

The shift is by minus one. As in Lemma 25.12, we may subtract a global coboundary to set the degree-zero witness at ℓ to zero and subtract local quotient coboundaries to set the other witnesses to zero. This identifies $H^1(K)$ with the global cocycles vanishing on D_ℓ whose restrictions at the other q factor through H_q, without further identification of representatives. On a nonzero weight line the invariants at D_ℓ are zero by Lemma 39.1. Thus a class zero there has exactly one representative vanishing there. At $q \neq \ell$ the finite residual inertia quotient has order prime to ℓ; averaging identifies the condition of vanishing on N_q with triviality of the restriction class on I_q. On the two trivial lines the cocycles are everywhere unramified additive characters of $G_\mathbb{Q}$ vanishing at ℓ. They vanish by class reciprocity for $\mathbb{Q}$. Hence Lemma 39.4 leaves one relative tangent dimension, on the negative even root.

In degree zero the map from the invariants in the domain of the cochain map is injective. The last terms in Poitou–Tate duality identify $H^3(K)$ with the dual of $H^0(G_\Sigma, M^*(1))$. Indeed

H_q has cohomology only in degrees zero and one by averaging over its finite inertia image and passing to the unramified quotient. All the indicated Tate-dual invariants vanish already at ℓ. For the odd roots this follows from Proposition 38.2. For the neutral and even lines the tame inertia exponents of the lines are 0 or $(\ell-1)/2$, distinct from 1 since $\ell \geq 5$. The local Euler characteristics at $q \neq \ell$ and for H_q are zero. Those of G_Σ and D_ℓ are respectively $-\dim M + \dim M^{c_\infty=1}$ and $-\dim M$, where c_∞ is complex conjugation. These duality and Euler formulas apply to finite ℓ-power coefficients and the chosen Σ [1]. It follows that

$$\dim H^2(K) = \dim H^1(K) + \dim M^{c_\infty=1}.$$

This gives two dimensions at weight zero, one on the positive even root, two on the negative even root and zero on the odd roots.

For clarity, the role of $H^2(K)$ is an upper bound on relations. In a small extension lift the global matrices as a continuous cochain of fixed multiplier. Such set-theoretic lifts exist by smoothness of the symplectic group, or by lifting a symplectic basis and correcting its pairings. Lift the local quotient representations at $q \neq \ell$, using $H^2(H_q, M) = 0$ on all weights. Compare these lifts and the given lift at ℓ with the global cochain. Its multiplicative defect and the local differences define the degree-two cone class with values in the small kernel, exactly as in Theorem 25.13. Vanishing allows a global cochain correction giving a homomorphism. A global infinitesimal conjugation makes its restriction at ℓ equal to the prescribed lift; this preserves factorization at every $q \neq \ell$. Equality at ℓ also restores the chosen frame condition at h.

Choose an element on which the relative tangent cocycle on the negative root is nonzero. The 20 entry there detects it, since the root matrix is $E_{20} + E_{31}$. This entry as y gives a surjection $P = R_\ell[[y]] \twoheadrightarrow \mathcal{B}$ by complete Nakayama. For its kernel I, apply the obstruction to $P/\mathfrak{m}_P I \to P/I$. A nonzero linear functional on $I/\mathfrak{m}_P I$ cannot annihilate that obstruction. Otherwise the pushed-out extension would admit a representation over R_ℓ lifting the universal one. The induced map from P would differ from the natural one only on y, by a square-zero kernel element. Every element of I has zero residual linear coefficient in y by minimality, so the two maps would agree on I, contradicting the chosen functional. The continuous cochain construction for this finite-kernel extension follows by choosing a section on finitely many open cosets as in the proof of Theorem 25.13.

These dimension bounds apply separately to each weight. Indeed the torus acts by diagonal conjugation on all entries and on the cochains. A homogeneous relation in a matrix entry of weight λ has scalar transformation law $s^{\lambda_1} t^{\lambda_2}$. In the universal obstruction a defect in that entry has this same law. Equivalently the dual of the relation space maps into the corresponding cohomology space with the dual convention for the torus on linear functionals. Weight separation here uses the split algebraic torus over the coefficient ring: in each finite jet retain coefficients of the Laurent monomials in formal s, t. This gives exact projections, also when characters agree on some finite group of residue-field units. Taking inverse limits lifts a weight basis of $I/\mathfrak{m}_P I$ to homogeneous relations. Nakayama's lemma gives the asserted presentation. □

Corollary 39.6 (Elimination of the diagonal variables and first torus invariants)**.** *Let the first torus mean conjugation with $t = 1$. Its invariant ring in $\mathcal{B}$ has a presentation*

$$\mathcal{B}^s = O[[x, v, w, y]]/(E, F_1, F_2), \qquad v = ab, \quad w = bc.$$

The weights of x, v, w, y for the remaining torus are $1, 1, 2, -1$. In this formula E, F_1, F_2 denote the relations after the following elimination.

Proof. Impose $a = x = b = c = y = 0$. Every universal off-diagonal matrix entry has nonzero torus weight and therefore vanishes in this quotient, on which the torus is trivial. Conversely,

the five variables were chosen from off-diagonal entries, so imposing their vanishing represents exactly the diagonal lifts. Their four character ratios to d_i are unramified at all finite places. At ℓ this is the flag condition. Elsewhere inertia acts through a finite prime-to-ℓ quotient and the ratios take values in the principal congruence units, a pro-ℓ group. Global reciprocity for $\mathbb{Q}$ forces every ratio to be one. One can see this on each Artin quotient: a finite abelian extension of $\mathbb{Q}$ unramified at all finite primes is trivial. Thus the diagonal quotient is O via the split augmentation.

Only Z_1, Z_2 can survive after these five variables are set to zero. Their restrictions generate (u_1, u_2), so their linear parts in the two diagonal variables form an invertible matrix modulo the maximal ideal. The formal inverse function theorem eliminates u_1, u_2 over O by $Z_1 = Z_2 = 0$; alternatively solve successively modulo powers of the maximal ideal using that matrix. This substitution respects both weights by uniqueness. The remaining relations all have first weight zero. In the power series ring on a, x, b, c, y a monomial has first weight zero precisely when its exponent on b is the sum of its exponents on a, c. It has a unique expression as a monomial in x, ab, bc, y. This substitution identifies the corresponding complete power series rings on neutral monomials, since the generator degrees are between one and two. Projection of $EH + F_1H_1 + F_2H_2$ to first weight zero simply projects the three coefficients to first weight zero. Exactness of projection on each finite jet, then completion, proves the stated formula. □

39.3 The integral Frobenius congruence giving the strict length

Continue with the surviving faithful dyadic character and the same weight, prime and field as in Section 39.2. Write $\rho : G_\Sigma \to \mathrm{GSp}_4(\mathcal{B})$ for its universal normalized lift, and $t(g) = \operatorname{tr}\rho(g)$, $t_0(g) = \operatorname{tr} d(g)$. Let T be the closure of the O-subalgebra generated by the $t(g)$, in the maximal-ideal topology of $\mathcal{B}$. We determine an integral quotient of T directly from the negative root cohomology.

Lemma 39.7 (Three products of entries generate the trace algebra)**.** *In Corollary 39.6 set*

$$U = xy, \qquad V = vy, \qquad W = wy^2, \qquad P = O[[U, V, W]], \qquad D = (U, V, W).$$

Write $F_i = yf_i(U, V, W)$ and $e(U, V, W) = yE$. Then $e \in D$ and the full torus invariant ring in $\mathcal{B}$ equals T. Its presentation, including all integral relations, is

$$T = P/(e,\ D(f_1, f_2)).$$

The augmentation $T \to O$ is evaluation at $U = V = W = 0$.

Proof. For a monomial $x^i v^j w^k y^l$, second weight zero means $l = i + j + 2k$. Thus every neutral monomial is uniquely a product $U^i V^j W^k$. The two adic topologies on these power series are equivalent because the three monomial degrees in x, v, w, y are $2, 2, 3$. Every series of weight -1 is y times a neutral series. Multiplication by y takes the series of weight $+1$ onto D: it gives precisely the neutral series with no constant monomial, as can also be seen from $U = yx, V = yv, W = y(wy)$. Projection of the ideal (E, F_1, F_2) to degree zero therefore gives exactly $(e, D(f_1, f_2))$. Exactness follows by projecting the coefficients of an expression in the homogeneous generators term by term, as in Corollary 39.6.

Every trace is invariant. To prove the converse, observe first that the coefficients of the characteristic polynomial of $\rho(h)$ belong to T by Newton's identities on $t(h^i)$, $1 \le i \le 4$, since $\ell \ge 5$. Its four labeled roots belong to T as well: start with lifts in O of the simple residual roots and apply Newton iteration in the complete ambient ring. Inverses of elements of T with nonzero residue belong to T, by subtracting their residues lifted to units in O and using a

geometric series. Every iterate is therefore in T, and convergence puts the limit there. The polynomials in $\rho(h)$ giving P_i in Lemma 39.1 have coefficients in T. These projectors are E_{ii} since the frame is normalized. It follows that every cyclic product of matrix entries belongs to T. Indeed it is the trace of $P_{i_0}\rho(g_1)P_{i_1}\rho(g_2)\cdots P_{i_{m-1}}\rho(g_m)$, with entries successively indexed $i_0i_1,\dots,i_{m-1}i_0$. Expand the P_i as polynomials; each trace in the expansion is the trace of a group element.

Choose a,x,b,c as entries with indices $01,02,12,03$, respectively, using a group element with nonzero tangent evaluation on the associated root for each choice. Take $y=\rho(g_y)_{20}$ as in Theorem 39.5. The products $U=xy$ and $V=aby$ already follow the cycles of indices 020 and 0120. In $W=bcy^2$ use the alternating form on one copy of y. The multiplier identity gives

$$\rho(g_y)_{20}=-\mu(g_y)\rho(g_y^{-1})_{31}.$$

The remaining entries $03,31,12,20$ form a cycle. Thus $U,V,W\in T$. Since T is closed, it contains every power series in these entries. Their value at the split lift is zero. □

Theorem 39.8 (The exact integral quotient of the exceptional strict length)**.** *Define $C=P/(e,f_1,f_2)$, a quotient of the actual trace algebra. There is an isomorphism of O-algebras*

$$C/DC=O/\ell^{\,k(r)}O,\qquad k(r)=\operatorname{length}_O S(\alpha,r).$$

In particular the formula holds at the possible weight of length above twenty-five in Corollary 44.15. It requires no finiteness hypothesis on C as an O-module.

Proof. Put $O_m=O/\ell^mO$ for $m\ge1$. In the presentation from Theorem 39.5, after the diagonal elimination, consider maps to $O_m[\epsilon]/(\epsilon^2)$ given by

$$a=x=b=c=0,\qquad y=\epsilon z_0,\quad z_0\in O_m.$$

The neutral diagonal variables eliminated by Z_1,Z_2 take their split values: they were expressed as neutral series and all nonconstant neutral terms vanish under this substitution. Since the only possible linear term has weight $(0,-1)$, E vanishes and the two remaining relations require exactly $f_i(0)z_0=0$ for $i=1,2$. Conversely these equations suffice to give the specified map. Its matrices have the form

$$\rho_\epsilon(g)=(1+\epsilon z(g)X)d(g),\qquad X=E_{20}+E_{31},$$

where d is reduced modulo ℓ^m. Indeed the only possible varying entries have the indicated weight; the multiplier equation on their infinitesimal coefficients gives the displayed one-dimensional symplectic root space. The group law is precisely

$$z(gg')=z(g)+\Psi^{-1}(g)z(g').$$

The upper triangular condition at ℓ forces $z|_{D_\ell}=0$. Away from ℓ the framed equations force $z|_{N_q}=0$ at the conductor primes and impose unramifiedness at the other primes. Conversely any such cocycle gives these matrices and this map: all the locally chosen upper coordinates are zero, the diagonal is fixed, and the chosen 20 entry gives its value of y.

We verify that these are exactly the classes in $S(\alpha,r)[\ell^m]$. View $O_m(\Psi^{-1})$ as the ℓ^m-torsion of $(\mathrm{Frac}(O)/O)(\Psi^{-1})$ by division by ℓ^m. On both modules $\Psi^{-1}(h)-1$ is a unit. Every global class has exactly one cocycle representative with $z(h)=0$, obtained by subtracting a coboundary. If its class is killed by ℓ^m, the normalized representative is killed by ℓ^m, because the normalized representative of the zero class is zero. For a strict class the cocycle on D_ℓ is

a coboundary, whose value zero at h forces its witness to be zero. At $q \neq \ell$ a coboundary on inertia restricts to zero on N_q, since Ψ is trivial there. Conversely vanishing on N_q implies that the inertia restriction factors through a group of order prime to ℓ. Averaging there makes it a coboundary, including for finite coefficients. This proves the claimed identification with all the specified cocycles. The correspondence is O_m-linear, as is seen from the square-zero formula for the matrices.

By Corollary 44.6, $S(\alpha, r)$ is a finite cyclic O-module of length $k(r)$. Consequently the simultaneous annihilator of $f_1(0), f_2(0)$ in O_m has length $\min(m, k(r))$ for every m. For an ideal of the discrete valuation ring O these lengths determine its valuation, including the zero ideal by letting m grow. They give $(f_1(0), f_2(0)) = \ell^{k(r)}O$. Setting U, V, W to zero in C and using $e(0) = 0$ proves the assertion. □

Corollary 39.9 (Generation by good-prime trace differences)**.** *For $q \notin \Sigma$ write $\delta_q = t(\mathrm{Frob}_q) - t_0(\mathrm{Frob}_q)$. The ideal of T generated by all these δ_q is DT. Some set of at most three such primes generates this ideal. For any such generating set $\mathcal{Q}$,*

$$C/(\delta_q : q \in \mathcal{Q})C \simeq O/\ell^{k(r)}O.$$

Proof. The topology induced from $\mathcal{B}$ on T is its maximal-ideal topology. To see this directly, P, with finite residue field, is compact in its maximal-ideal topology as the inverse limit of finite quotients. The continuous surjection from P onto the closed subalgebra in the lemma is thus a quotient map onto a Hausdorff space. In particular all ideals of T are closed, either by completeness and Noetherianity or by expressing each as the image of a finite product of compact copies of T.

Both traces are continuous class functions on G_Σ. The Chebotarev density theorem in every finite quotient of this group implies that the conjugates of good-prime Frobenius elements are dense. Thus the closed ideal generated by the δ_q contains $t(g) - t_0(g)$ for every g. In the quotient by that ideal all traces are scalars, and the scalars form a closed subring, being the continuous image of the compact ring O. They generate the whole quotient by the definition of T. The split augmentation identifies that quotient with O, proving the ideal equality. Since DT is generated by U, V, W, its quotient modulo the maximal ideal times DT has dimension at most three. Select a basis from the images of the δ_q and apply Nakayama's lemma. The last assertion now follows from the theorem. The choice by Nakayama gives an existence assertion about the primes; their sizes do not enter the proof. □

39.4 The finite trace algebra and the determinant at Frobenius

Use P, D, e, f_1, f_2, T, C and the actual strict length $k(r) > 0$ of Section 39.3. Put $J = \mathrm{Frac}(O)$. In $R = P/(e)$ write D_R for the ideal of the three variables and $I = (f_1, f_2)R$. Thus $T = R/D_R I$, $C = R/I$, $R/D_R = O$. We first determine what finiteness is needed to compare the strict group with characteristic-zero traces.

Lemma 39.10 (Finiteness and the component separate from the split trace)**.** *The following conditions are equivalent: T is finite as an O-module, C is finite as an O-module, and $C/\ell C$ has dimension zero. Regardless of these conditions,*

$$T[1/\ell] \simeq J \times C[1/\ell].$$

If the conditions hold, then C is a nonzero finite free complete intersection over O.

Proof. The kernel of $T \to C$ is $I/D_R I$, generated over $R/D_R = O$ by the two images of f_1, f_2. This proves the first equivalence. If $C/\ell C$ has dimension zero, it is an Artinian local quotient

of $k[[U,V,W]]$, hence finite over k. Then some power of the maximal ideal of C is contained in ℓC, and the two adic topologies are equivalent. Lift a k-basis and express any element successively modulo powers of ℓ in these lifts. Completeness of O gives an expression over O. The converse follows by reduction. By Theorem 39.8, $R/(D_R+I) = O/\ell^{k(r)}O$; thus D_R and I are comaximal after inverting ℓ. The Chinese remainder theorem gives the product formula.

Under finiteness the ideal $(e, f_1, f_2, \ell) \subset P$ is primary to the maximal ideal. All four elements lie in that ideal since $k(r) > 0$. The regular local ring $P = O[[U,V,W]]$ is Cohen–Macaulay of dimension four. The four elements, in the displayed order, form a system of parameters and hence a regular sequence [11]. Thus multiplication by ℓ on C is injective. A finite torsion-free module over a discrete valuation ring is free; the quotient by D is nonzero. This proves the last assertion. □

Theorem 39.11 (Torsion and the congruence with the split trace)**.** *Assume the finiteness conditions of Lemma 39.10. There is a surjective ring map with square-zero kernel*

$$T \longrightarrow O \times_{O/\ell^{k(r)}O} C.$$

The map from C in the fiber product is its quotient by DC. The kernel is the full O-torsion of T and is isomorphic as an O-module to $O/\ell^{k(r)}O$. In particular the scalars occurring in the first factor with second coordinate zero in the torsion-free trace algebra form exactly $\ell^{k(r)}O$.

Proof. The elementary exact sequence for two ideals gives $R/(D_R \cap I) = (R/D_R) \times_{R/(D_R+I)} (R/I)$. For surjectivity, lift a compatible pair to two elements of R. Their difference lies in $D_R + I$, so subtracting a suitable element of each ideal gives a common lift. Hence the claimed kernel is $(D_R \cap I)/(D_R I)$, and its square is zero.

The regular sequence from the lemma makes f_1, f_2 a regular sequence in R. The relations between them are exactly the multiples of $(-f_2, f_1)$: reduce a relation modulo f_1, use injectivity of f_2 there and then cancel f_1 in R. Right exactness of tensor product over R now gives

$$I/D_R I \simeq O^2/O(-f_2(0), f_1(0)).$$

Its map to $R/D_R = O$ has image $(f_1(0), f_2(0)) = \ell^{k(r)}O$ and kernel exactly $(D_R \cap I)/(D_R I)$. Write $(f_1(0), f_2(0)) = \ell^{k(r)}(a,b)$ as a pair of elements, so $aO + bO = O$. The solutions in O^2 of $au + bv = 0$ are precisely the multiples of $(-b, a)$, as one of a, b is a unit. Quotienting by its $\ell^{k(r)}$-multiple yields the claimed module. The fiber product lies in the torsion-free module $O \oplus C$; therefore this kernel accounts for all O-torsion. Its removal gives the asserted exact scalar ideal by the definition of the fiber product. □

Corollary 39.12 (A determinant bound for the full exceptional length)**.** *Under the same finiteness hypothesis let $z \in DC$. For instance one can take a good-prime trace difference δ_q. Suppose that z is nonzero at every geometric point of $C[1/\ell]$. For each J-embedding point $\sigma : C[1/\ell] \to \overline{J}$ let m_σ be the length of its local factor after extension to $\overline{J}$. Then, with $v_\ell(\ell) = 1$,*

$$\begin{aligned} k(r) &\le \operatorname{length}_O(C/zC) = v_\ell\big(\det_O(z \mid C)\big) \\ &= \sum_\sigma m_\sigma \, v_\ell(\sigma(z)). \end{aligned}$$

Some integral linear combination of the at most three trace differences of Corollary 39.9 always satisfies the nonvanishing hypothesis.

Proof. As $z \in DC$, the quotient C/zC surjects onto $O/\ell^{k(r)}O$ by Theorem 39.8. Over $\overline{J}$, multiplication by z on each local Artin factor acts on the graded pieces of its maximal-ideal

filtration by the scalar $\sigma(z)$. Its determinant on the generic fiber is therefore $\prod_\sigma \sigma(z)^{m_\sigma}$. Nonvanishing makes multiplication on the finite free module C injective; diagonalizing the inclusion of lattices over O gives exactly the length displayed.

Let z_1, z_2, z_3 be the chosen differences, padding with zeros if necessary. They generate DC, which generates the unit ideal after inverting ℓ, since C/DC has finite length. Thus at every geometric point at least one of their values is nonzero. At a point the polynomial $\sigma(z_1)+X\sigma(z_2)+X^2\sigma(z_3)$ has at most two roots. If there are n points, some integer m between zero and $2n$ avoids all the roots. Take $z = z_1 + mz_2 + m^2 z_3$. □

There is a specific use of this bound on the original interval. At the common prime p of Theorem 38.16, suppose the traces at the geometric points, counted with the m_σ, occur under the coefficient embedding in the integral multiset $\mathcal{C}_r$ of that theorem. Its archimedean trace estimate ensures $t(\mathrm{Frob}_p) - t_0(\mathrm{Frob}_p)$ is nonzero at every such point. Taking $z = \delta_p$ in the corollary and applying that theorem gives $k(r) \le v_\ell(D_{\mathrm{Sp}})$. In particular a common such list of the polynomially bounded size in Corollary 38.17 would bound these lengths on the period interval. The determinant counts the multiplicity of the actual trace algebra at each point. Establishing finiteness of its special fiber and identifying those multiplicities and traces on the indicated fields are further arithmetic requirements.

39.5 Galois representations realizing the trace coordinates

Keep the family and rings of Section 39.3. An integral specialization will mean a continuous local O-algebra map to a complete discrete valuation ring A' with finite residue field containing k. The coefficient map may also factor through k, so the statements apply on the special fiber. Write (U', V', W') for the values of the three trace coordinates in A'.

Lemma 39.13 (Realization over a quartic extension)**.** *Every integral specialization of C has the trace of a genuine global normalized symplectic lift after adjoining a fourth root of a uniformizer of A'. Its local component is the one in Corollary 39.3.*

Proof. Let s be a fourth root of the uniformizer and work in the integers A'' of the resulting finite extension of the fraction field. Choose the following values in the presentation after elimination of u_1, u_2 in Corollary 39.6:

$$y = s, \quad b = s, \quad x = U'/s, \quad a = V'/s^2, \quad c = W'/s^3.$$

All belong to the maximal ideal: U', V', W' have valuation at least that of s^4, with zero allowed. Thus all formal series converge. The products xy, aby, bcy^2 have precisely the required values. The equations $F_i = yf_i$ hold because the specialization kills the two f_i. Since $yE = e$ and $s \neq 0$ in a domain, $E = 0$ as well. The eliminated diagonal variables are supplied by their unique series. This gives a continuous specialization of $\mathcal{B}$, hence actual matrices with their required local and inertia conditions. Their traces agree with the given map on T since U, V, W topologically generate it. If $V' = W' = 0$ we may instead take $a = b = c = 0$, with the same x, y and identical verification. □

Theorem 39.14 (The rank of nonzero specializations)**.** *Suppose $(U', V', W') \neq (0, 0, 0)$.*

1. *If $V' \neq 0$ or $W' \neq 0$, the constructed representation is absolutely irreducible of dimension four.*
2. *If $V' = W' = 0$, there is a realization as the direct sum of two absolutely irreducible planes. Complex conjugation acts on these planes as -1 and $+1$, respectively. With the*

indexing $\{0,2\},\{1,3\}$ their determinants are precisely the images of $A^2\Psi, \Psi$, respectively. In particular both planes are even.

The second case cannot occur at any nonzero integral specialization if the unit condition in Theorem 45.2 holds.

Proof. Work over an algebraic closure of the fraction field of A''. The fixed element h has four distinct eigenvalues since their residues are distinct. A subspace stable under the group must be a sum of some of its four eigenlines, by the polynomials $P_i(h)$. For each nonzero entry with indices ij the presence of line j in this sum forces the presence of line i. The symplectic identity supplies another nonzero entry at $(3-j, 3-i)$ in the matrix of the inverse group element.

If $V' \neq 0$, the entries $01, 12, 20$ are nonzero at their respective chosen elements. They connect $0, 1, 2$ by a directed cycle. Reflecting 20 and 01 gives entries $31, 23$, which join 3 to that cycle in both directions. If $W' \neq 0$, the entries $03, 12, 20$ are nonzero. Reflection of 20 gives 31, and the cycle of indices $0, 3, 1, 2, 0$ uses all four lines. Thus in both cases there is no nonempty proper stable subset of the eigenlines.

If only $U' \neq 0$, use $a = b = c = 0$ in the second construction of the lemma. Only first-torus-weight-zero variables are left, so every entry of nonzero first weight vanishes. The matrices preserve the two coordinate planes in the statement. The entries $02, 20$ are nonzero because $xy = U'$; reflection yields the opposite pair $13, 31$ on the other plane. The same eigenline argument proves absolute irreducibility of both. The alternating form identifies either plane with the dual of the other twisted by μ.

At a complex conjugation the matrices on the two planes reduce to $-1, +1$ respectively and square to one. Since 2 is a unit, the projectors onto the opposite signs reduce to zero and therefore vanish. At ℓ the determinant inertia characters of the planes are $d_0 d_2$ and $d_1 d_3$ by the ordinary flag. At $q \neq \ell$ the determinants factor on inertia through its finite prime-to-ℓ image under $\bar{d}$. Thus the ratio of each determinant to the corresponding product of the d_i is unramified there: it reduces to one and the principal congruence units of A'' form a pro-ℓ group. These ratios are everywhere unramified abelian characters congruent to one, hence trivial by reciprocity for $\mathbb{Q}$. This proves the determinant formulas in both characteristics. Under the unit condition $U = H(V, W)$ with $H(0,0) = 0$ by Corollary 45.3. This excludes a nonzero triple with $V' = W' = 0$. □

Corollary 39.15 (A characteristic-ℓ representation detecting nonfiniteness)**.** *The ring T is infinite as an O-module if and only if there is an integral specialization of C to the integers of a finite extension of $k((s_1))$ for which the trace of some good-prime Frobenius is transcendental over k. Every such failure of finiteness has a witness of one of the two types in Theorem 39.14. With a nonzero unit projection as in Theorem 45.2, it has a rank-four absolutely irreducible witness. When C is finite over O, every geometric point entering the determinant in Corollary 39.12 also has a realization of one of the two types, in characteristic zero.*

Proof. If T is infinite, $C/\ell C$ has positive dimension by Lemma 39.10. Take a prime ideal with quotient of dimension one, for example the penultimate member of a chain of maximal length ending at the maximal ideal. Its quotient B' is a one-dimensional complete local domain over k. At least one of U, V, W has nonzero image there, since killing all three leaves only k. Choose a nonzero parameter s_1 in its maximal ideal. The quotient by s_1 has finite k-dimension, so B' is finite over $k[[s_1]]$: lift a basis and expand successive remainders in powers of s_1, using completeness and the equivalence of the parameter and maximal-ideal topologies. The coefficient map on $k[[s_1]]$ is injective since B' is a domain and s_1 is nonzero. Its fraction field is a finite extension of $k((s_1))$. The valuation ring of that local field contains B' by integrality,

and the inclusion is local since some power of the maximal ideal of B' lies in s_1B'. It therefore gives the required integral specialization.

By Corollary 39.9 the images of good-prime differences generate the ideal of the three coordinates. At least one such difference is nonzero. Its image has positive valuation, because it lies in the maximal ideal. Every nonzero element algebraic over the finite field k has finite multiplicative order and valuation zero. The split trace is in k in characteristic ℓ, so the corresponding Frobenius trace is transcendental. Lemma 39.13 and the theorem now give the claimed matrices. Conversely, if T is finite, $C/\ell C$ is Artinian local with residue field k. A map to a characteristic-ℓ domain kills its nilpotent maximal ideal. All trace values are then in k.

Finally, under finiteness, each characteristic-zero geometric point has values in a finite extension of J, integral on C. Its map to the integer ring is local and continuous, since the maximal ideal of $C/\ell C$ is nilpotent. It cannot kill U, V, W simultaneously as $C/DC = O/\ell^{k(r)}O$. The lemma and theorem therefore apply to it as well. □

39.6 Irreducibility after restriction to the period fields

Use the genuine representations constructed in Theorem 39.14. We work either in characteristic zero at a nonzero triple or in characteristic ℓ at one of the specializations with transcendental trace in Corollary 39.15. The fields of definition of the matrices are local fields with finite residue fields. For a rank-four irreducible realization write V for its representation space. For the block realization, take V to be either one of its irreducible planes. In discussing subspaces, extend the coefficient field to an algebraic closure.

Lemma 39.16 (Exclusion of permuted direct summands)**.** *There is no decomposition of V as the direct sum of at least two nonzero subspaces permuted by the global Galois group.*

Proof. The semisimplified residual characters of V form the full set $\{1, \tau, \bar{A}, \bar{A}\tau\}$ in rank four, or one of its prescribed pairs in rank two. No ratio of distinct members has order two. The ratio $\tau^{\pm 1}$ has order $2^j \geq 8$; each of the other possible ratios is $\bar{A}^{\pm 1}\tau^i$ for $i = 0, \pm 1$. At the lift h of Lemma 39.1 the order of $\bar{A}(h)$ is a power of two strictly larger than the order of $\tau(h)$ and larger than two. Multiplying it by such a power of $\tau(h)$ preserves its order, since $\tau(h)$ also has two-power order. Thus no nontrivial quadratic character fixes the residual multiset under twisting: it would have to take a member to a different member with that quadratic ratio.

If a direct decomposition were permuted, irreducibility would make the permutation action transitive. The number of summands is then 2 or 4. The permutation action of the compact image is continuous: all subspaces involved are defined over some finite extension of the coefficient local field, and their finite orbit in the Grassmannian is discrete. The kernel of matrix reduction is pro-ℓ, by its successive principal congruence quotients. Since $\ell \geq 5$ it maps trivially to the relevant symmetric group. The permutation image therefore factors through the abelian residual matrix image. A faithful, transitive abelian permutation group acts regularly, since a stabilizer fixes every translate of its point. Its order is the number of summands, so it has a nontrivial quadratic character. Multiply vectors in each summand by the value of this character on the group element labeling the summand. The resulting operator exhibits V as isomorphic to its quadratic twist. Reduction would give the same isomorphism on residual semisimplifications. Indeed isomorphic representations have the same characteristic polynomials, and reducing these in stable lattices identifies their residual semisimplifications by Brauer–Nesbitt. The quadratic character stays nontrivial since ℓ is odd. This contradicts the first paragraph. □

Theorem 39.17 (Strong irreducibility at the nonconstant traces)**.** *In both characteristics V remains absolutely irreducible on every open subgroup of $G_{\mathbb{Q}}$. In particular this holds on G_{E_x} for the original period field and on every further finite extension of that field.*

Proof. It suffices to consider an open normal subgroup H: every open subgroup contains its open normal core. The restriction to H is semisimple even in characteristic ℓ. Indeed choose a simple H-submodule. Its conjugates are simple H-submodules and span a nonzero global subrepresentation, hence all of V. A sum of simple submodules is semisimple. Its isotypic subspaces are permuted by the global action. By Lemma 39.16 there is only one type.

In characteristic zero H contains a positive power of the inertia element s_0 in Corollary 39.3. Its four eigenvalues in rank four are the corresponding powers of $1+\ell$ with exponents $r+1, r, 1, 0$; they stay distinct under any positive common power. On either plane the two corresponding eigenvalues stay distinct. An isotypic sum with multiplicity greater than one would give every eigenvalue of each element of H with at least that multiplicity. Thus the restriction is irreducible.

In characteristic ℓ first exclude scalar action of H. The determinant in rank four is μ^2, and the determinants of the planes are given in Theorem 39.14. All have values in a finite field in this characteristic. If a finite-index subgroup acts by scalars, its scalar image is finite, because their powers of degree $\dim V$ lie in this finite set. The whole image of V would be finite. For a plane the other plane is its dual times μ, which also has finite image as a character. Thus in either case the original four-dimensional image would be finite. All eigenvalues of its matrices would then be algebraic over the finite field, contrary to the transcendental trace of the chosen specialization. This settles rank two and the case of a one-dimensional simple type in rank four.

The only remaining case is rank four with $V|_H = W \otimes M$, where W is simple of dimension two and M is a multiplicity space of dimension two with trivial H-action. The algebra of endomorphisms commuting with H is $1 \otimes \mathrm{End}(M)$, by Schur’s lemma. Conjugation by $G_{\mathbb{Q}}$ acts on it through a finite group, since H acts trivially. Each automorphism of a full matrix algebra over an algebraically closed field is inner. Thus we have a finite projective action on M. If B_g represents this projective action, multiplication of the matrix of g by $1 \otimes B_g^{-1}$ on the left centralizes $1 \otimes \mathrm{End}(M)$; hence the matrix of g has form $A_g \otimes B_g$. In particular any fixed line, or any permuted pair of distinct lines, in M gives a stable subspace, or permuted summands, in V.

The image of the principal congruence kernel in this finite projective group is a normal ℓ-subgroup. If nontrivial it has a central element of order ℓ. In characteristic ℓ a nonidentity projective transformation of order ℓ on a line has a single fixed point: its matrix up to scale is a nontrivial unipotent Jordan block. Centrality forces the whole ℓ-subgroup to fix this point, its unique common fixed point. Normality then forces the full projective group to fix the point, contradicting irreducibility of V. The projective action must therefore factor through the abelian residual matrix image of two-power order. If trivial, it fixes every line in M. Otherwise take a nonidentity element. It has two distinct eigenlines in characteristic ℓ, and every element commuting with it projectively permutes this pair. This contradicts Lemma 39.16 or irreducibility if each is fixed. The proof is complete. □

Corollary 39.18 (One plane and its prescribed scalar twist)**.** *In the two-plane case let Q denote the plane with indices $\{1,3\}$. The four-dimensional representation is isomorphic to $Q \oplus (A \otimes Q)$. Its trace at g is $(1 + A(g)) \operatorname{tr} Q(g)$. In characteristic ℓ it is $2 \operatorname{tr} Q(g)$ on the kernel of $\bar{A}$.*

Proof. The alternating form pairs the two planes perfectly with multiplier μ, identifying the other plane with $Q^\vee \otimes \mu$. The exterior product pairing in dimension two identifies $Q^\vee$ with $Q \otimes (\det Q)^{-1}$. Theorem 39.14 gives $\det Q = \Psi$, and $\mu = A\Psi$. This proves the representation identity and both trace formulas. □

39.7 Simultaneous Frobenius parameters on the period interval

Fix one of the intervals $[V, V^{3/2}]$ in Section 6.6. Take the algebras T_i, C_i for the surviving dyadic data of Section 39.2 occurring at its points. The subscript i indexes the distinct choices of coefficient prime, dyadic layer, character and weight; write O_i, k_i, ℓ_i, r_i for their coefficient rings, residue fields, primes and weights. This list is finite: the layer and weight indices and the primes are bounded on the interval and each layer has finitely many characters. We use the single ξ for that interval. There are also only finitely many distinct E_x there, since the auxiliary quadratic fields are fixed and the dyadic layers have bounded indices. Choose any finite Galois extension $F/\mathbb{Q}$ containing all these period fields and all the residual splitting fields of d_i. A good prime in this subsection avoids the union of all the indicated ramified primes. Put $\delta_{i,p} = t_i(\mathrm{Frob}_p) - t_{0,i}(\mathrm{Frob}_p)$.

Lemma 39.19 (Separation of traces on every open coset)**.** *For a characteristic-zero geometric point of a finite C_i, or a characteristic-ℓ_i curve specialization as in Corollary 39.15 with nonzero trace-coordinate triple, the locus*

$$\{g \in G_F : \operatorname{tr} \rho_i(g) = \operatorname{tr} d_i(g)\}$$

is closed and has empty interior in G_F. This holds for the realizations of either rank type.

Proof. Continuity gives closedness. At a nonzero triple in characteristic ℓ_i some good-prime trace differs from its split value, by Corollary 39.9. The difference belongs to the maximal ideal of a local field of positive characteristic and is nonzero, so that trace is transcendental over k_i. Theorem 39.17 therefore applies. Suppose the equality held on a coset gH of an open subgroup $H \subset G_F$. In rank four the restriction of ρ_i to H is absolutely irreducible and distinct from all the one-dimensional constituents of d_i. In the two-plane case in characteristic zero, the absolutely irreducible restrictions $Q|_H, (A \otimes Q)|_H$ are distinct. Otherwise their determinants would give $A^2|_H = 1$. But H contains a positive power of the inertia element s_0 and $A(s_0) = 1 + \ell_i$. In the two-plane case in positive characteristic $A = 1$ on G_F, so the trace is $2 \operatorname{tr} Q$ there by Corollary 39.18.

Over the algebraic closure of the coefficient field, for any finite list of pairwise inequivalent simple representations π_a of H, the linear span of the tuples $(\pi_a(h))_a$, $h \in H$, is the product of their full matrix algebras. Indeed each simple image of the group algebra is a full matrix algebra by Burnside's theorem. Its kernel is a maximal two-sided ideal, distinct for distinct simple modules, since a full matrix algebra has just one simple module up to isomorphism. To select one factor, for each other kernel choose an element in it congruent to one modulo the selected kernel. Their product selects that factor, and arbitrary matrices in it are then obtained by multiplication. This proves the span assertion.

An equality of traces for every gh, $h \in H$, would be a linear identity on that product of matrix algebras. Select the component of ρ_i in rank four or of Q in rank two and put zero on the other components, including each distinct diagonal character. On the selected matrix X the identity would require $\operatorname{tr}(\pi(g)X) = 0$ for every X, with an extra factor 2 in the repeated-plane case. The matrix $\pi(g)$ is invertible, the trace pairing is nondegenerate, and 2 is nonzero. This contradiction excludes every such coset. □

Theorem 39.20 (At most three common split primes give finite parameter maps)**.** *There are distinct good rational primes $p_1, \dots, p_d$, all splitting completely in F, with*

$$d \le \max_i \dim(C_i/\ell_i C_i) \le 3,$$

such that both C_i and T_i are finite modules over $O_i[[X_1, \dots, X_d]]$ via $X_a \mapsto \delta_{i,p_a}$, for every i. Use $d = 0$ for an empty list of data or if all special fibers have dimension zero. The bound on d is two if every datum satisfies the unit condition of Theorem 45.2. For this common choice T_i is finite over O_i if and only if all the δ_{i,p_a} are nilpotent in $C_i/\ell_i C_i$.

Proof. Begin with the rings $B_i = C_i/\ell_i C_i$, quotients of the three-variable power series rings in Lemma 39.7. Under the unit condition, Corollary 45.3 gives a two-variable presentation. We successively quotient each B_i by trace differences at the same primes.

At one step consider all the positive-dimensional irreducible components of the spectra of the current quotients. For each, the associated complete local domain admits a one-dimensional domain quotient. Passing to the integers of the fraction field of that quotient as in Corollary 39.15 gives an integral specialization with some trace coordinate nonzero. Indeed the three coordinates and the residue field generate the ambient special fiber, so they cannot all vanish in a domain of dimension one. Lemma 39.13 provides the matrices. On G_F the complement of each trace-equality locus for these representations is open and dense by Lemma 39.19. There are finitely many components since the list of rings is finite and each is Noetherian. The intersection of finitely many dense open sets is dense and open. A nonempty open subset of G_F thus satisfies all the desired inequalities.

The extension $F/\mathbb{Q}$ is finite Galois. In the chosen open set take a coset gN of an open normal subgroup of $G_{\mathbb{Q}}$ contained in G_F. Chebotarev for the finite quotient $G_{\mathbb{Q}}/N$ gives infinitely many rational primes with the conjugacy class of g. At any such good prime the splitting in F is complete. Its trace values satisfy all the inequalities, which are conjugation invariant. Choose one new prime. Its difference is nonzero on each selected curve and hence outside the minimal prime of every positive-dimensional component. Quotienting by it reduces each positive dimension by at least one: any positive-length chain of primes containing the element can be extended downward by a strictly smaller minimal prime of the original ring. The same argument bounds a surviving zero-dimensional quotient by zero. Repetition proves the asserted number of choices.

Write $B = O_i[[X_1, \dots, X_d]]$. The ideal $\mathfrak{m}_B C_i$ now has zero-dimensional quotient with finite residue field, hence is primary to the maximal ideal. Lift a residue-field basis of $C_i/\mathfrak{m}_B C_i$. It generates C_i over B: express a remainder at each stage in monomials in $\ell_i, \delta_{i,p_1}, \dots, \delta_{i,p_d}$ of the next total degree times the basis lifts. The coefficients converge in B and the remainders tend to zero in C_i, since its two adic topologies agree. The kernel of $T_i \to C_i$ has two O_i-generators by Lemma 39.10, so the same module-finiteness follows for T_i.

If each chosen difference is nilpotent on the special fiber, its parameter map from $k_i[[X_1, \dots, X_d]]$ factors through a quotient killing a positive power of each variable. That quotient is finite over k_i. Thus $C_i/\ell_i C_i$ is finite over k_i, the condition in Lemma 39.10. Conversely, if T_i is finite over O_i, the maximal ideal of the Artinian local special fiber of C_i is nilpotent, and contains all the differences. □

Corollary 39.21 (Simultaneous determinant separation over the period fields)**.** *There are infinitely many good rational primes p splitting completely in F for which every i with C_i finite over O_i simultaneously satisfies*

$$k(r_i) \;\le\; v_{\ell_i}(\det_{O_i}(\delta_{i,p} \mid C_i)) < \infty.$$

When any parameter prime is needed in the theorem, its first choice can satisfy these inequalities as well.

Proof. Each of the finitely many finite algebras C_i has finitely many characteristic-zero geometric points. Impose the open dense conditions from Lemma 39.19 for every one of these points on G_F, along with any conditions from the first parameter step. The same open-coset choice and Chebotarev argument work simultaneously. For each selected prime the generic determinants are nonzero, so Corollary 39.12 gives the inequality with the full geometric multiplicities. This proves existence on the same fields throughout the fixed interval. An estimate uniform in the location of the interval would additionally have to control the sizes of the determinants or their trace spectra; the qualitative prime choice gives no such estimate. □

39.8 Algebraic monodromy of the exceptional trace representations

Let V be a representation in Theorem 39.17, over an algebraic closure of its coefficient local field. Its algebraic monodromy group means the reduced Zariski closure G of the Galois image. In rank four it is contained in the group of similitudes of the specified alternating form. In rank two we use the determinant as the similitude character.

Lemma 39.22 (The residual eigenvalues exclude a symmetric cube)**.** *In rank four the group G does not normalize a conjugate of the image of SL_2 on its third symmetric power.*

Proof. In characteristic zero or characteristic at least five, the ordinary symmetric cube has image isomorphic to SL_2. Indeed the induced map on the Lie algebra is nonzero and injective, and the center acts faithfully. A finite normal kernel with zero Lie algebra lies in the center, so the kernel is trivial and the map is separable onto its image. An algebraic automorphism of SL_2 over an algebraically closed field is inner: in the root datum description there is no diagram automorphism of type A_1. The centralizer on this irreducible four-dimensional module consists of scalars. Thus every element of the normalizer is a scalar times a symmetric cube. Its four eigenvalues, with multiplicities in some order, form a geometric progression, by applying the symmetric cube to the two eigenvalues of a two-by-two upper triangular matrix.

Apply this to $\rho(h)$ with h from Lemma 39.1. All eigenvalues are integral units and reduction would preserve a geometric progression, since the common ratio is a quotient of two of these units. Their actual reductions are $\{1, b, a, ab\}$ with $b = \tau(h)$ of order dividing $2^j \le n_*$ and $a = \bar{\xi}(\ell)$ of order greater than $n_* L_*$. Modulo the subgroup generated by b these become the two values $1, a\langle b\rangle$, each repeated twice. The order of the second value exceeds two, since it is at least $\mathrm{ord}(a)/|\langle b\rangle| > L_*$. A four-term geometric progression with exactly two distinct values must have common ratio of order two. The quotient of its distinct values then has order two, a contradiction. □

Theorem 39.23 (The connected algebraic monodromy groups)**.** *For a rank-four witness, the derived group of G° is Sp_4. In characteristic zero $G = \mathrm{GSp}_4$; in characteristic ℓ its identity component is Sp_4. On either irreducible plane the derived group is SL_2, with full GL_2 in characteristic zero and identity component SL_2 in characteristic ℓ.*

Proof. The inverse image of G° in the profinite Galois group is open, since there are finitely many closed components and the representation is continuous in the local-field topology. Theorem 39.17 makes the action of G° irreducible. Its unipotent radical fixes a nonzero vector, by triangularization, and normality makes the fixed subspace invariant. The radical consequently acts trivially and is trivial in this matrix group. Thus G° is reductive. Its central torus acts by scalars by Schur's lemma, so its connected semisimple derived group H acts irreducibly as

well. We use the usual classification of reduced root systems, the central covering by the simply connected group, and the highest-weight classification of its simple rational modules [2].

In rank four, $H \subset \mathrm{Sp}_4$ has rank at most two and dimension at most ten. The rank-zero case cannot act irreducibly. A simple group of type G_2 has dimension fourteen and is excluded. Type $C_2 = B_2$ already has dimension ten and a connected closed subgroup of that dimension in Sp_4 is the whole group. In type A_2 lift the representation to SL_3. Self-duality for the invariant form requires its highest weight (a, b) in fundamental weights to satisfy $a = b$, as duality takes (a, b) to (b, a). A nonzero (a, a) is regular, with six distinct Weyl conjugates, each a weight of the module. This exceeds dimension four in either characteristic. In type $A_1 \times A_1$ the irreducible representation on the simply connected cover is a tensor product. Both factors act nontrivially, so both modules have dimension two. Each has an invariant alternating determinant form because a connected semisimple group has trivial characters. The tensor of the two forms is symmetric and nondegenerate. Schur’s lemma makes the space of invariant bilinear forms one-dimensional, which excludes an alternating form in odd characteristic or characteristic zero.

It remains to exclude type A_1. In characteristic zero the four-dimensional simple module of SL_2 is its third symmetric power. In characteristic ℓ write a highest weight n in base ℓ, with digits n_i. The simple module has dimension $\prod_i (n_i + 1)$ and is the tensor of the corresponding Frobenius twists of the n_i-th symmetric powers [12]. In this rank one can also see the formula directly. Multiply the digit polynomials in X^{ℓ^i}, Y^{ℓ^i} to obtain the span of $X^{n-j}Y^j$ with $0 \le j_i \le n_i$ in each digit. The monomials have distinct torus weights. From any one of them the substitution $Y \mapsto Y + X$ and projection to the highest weight yield X^n. From X^n, substitution $X \mapsto X + Y$ and weight projection recover every allowed monomial, since the product of the binomial coefficients of its digits is nonzero. This module is simple of highest weight n, giving the formula. In dimension four there is either a single digit three or two digits one. In the latter case the two-dimensional forms give the same symmetric-form obstruction as above. In the former case the Frobenius-twisted symmetric cube has the same algebraic image as the symmetric cube, by surjectivity of Frobenius on algebraically closed field points. In either characteristic the full group would normalize this image, since it normalizes H. Lemma 39.22 excludes it. This proves the rank-four assertion about H.

In rank two $H \subset \mathrm{SL}_2$ is nontrivial and of rank at most one. Its type is A_1, so dimension three forces equality with SL_2. In characteristic zero each of the prescribed similitude characters has infinite image: their exponents on the chosen principal inertia element are $r+1$ in rank four and $r + 2, r$ on the planes. An infinite subgroup is Zariski dense in the multiplicative group. Since G already contains the kernel of the similitude, it is the full similitude group. In characteristic ℓ the similitude character has finite image by its explicit formula. Its connected kernel then gives exactly the asserted identity component. □

Corollary 39.24 (Eigenprojectors on the cyclotomic kernel of a period field)**.** *For any finite extension $F/\mathbb{Q}$, including E_x, and any nonzero $X \in \mathrm{End}(V)$, there is an element of $G_{F(\mu_{\ell^\infty})}$ whose matrix has distinct eigenvalues and for which some eigenprojector P satisfies* $\mathrm{tr}(PX) \neq 0$.

Proof. The image of G_F has finite index in the original image, so its Zariski closure has the same identity component. The closure of the subgroup generated by its commutators contains the derived group of this component: the commutator map is algebraic, and the image of a dense set of pairs is dense in the image of the map. All the commutators preserve the alternating form with multiplier one and lie in the cyclotomic kernel. Their subgroup is therefore dense in Sp_4 or SL_2 by the theorem. The defining representation of this connected group is irreducible,

so its matrices span the full matrix algebra. The function $g \mapsto \operatorname{tr}(gX)$ is nonzero somewhere by nondegeneracy of the trace pairing. The open locus of distinct eigenvalues is nonempty, by taking a generic element of the diagonal torus. The two nonempty open loci intersect in the irreducible group and hence meet the dense commutator subgroup. For such a matrix g, the spectral identity $g = \sum_i z_i P_i$ shows that at least one $\operatorname{tr}(P_i X)$ is nonzero. □

39.9 Split Frobenius congruences and theta tests on one interval

Fix one interval $[V, V^{3/2}]$ of Section 6.6. Use its full finite list of exceptional dyadic trace algebras C_i, T_i from Theorem 39.20. Choose any finite collection of the signed elliptic product tests over its period fields as in Corollary 56.17. Enlarge the common finite Galois field F of Theorem 39.20 to contain the field J of Theorem 56.15 for these tests. Write $\Gamma = G_F$ and use Haar probability on Γ. For a split prime q, the trace difference $\delta_{i,q}$ means the universal trace of Frobenius less its split diagonal trace. We use ℓ_i for the coefficient prime of C_i. The auxiliary testing prime q can be different from all the ℓ_i.

Lemma 39.25 (A zero set of traces has measure zero)**.** *For each trace representation in Lemma 39.19, the set of $g \in \Gamma$ where its trace agrees with the split diagonal trace has Haar measure zero. Among the rational primes in $[X, 2X]$ splitting completely in F, the proportion giving that equality tends to zero as X tends to infinity. Both assertions hold for the union over any fixed finite list of the indicated representations.*

Proof. Map Γ to matrices by the direct sum of the representation and its diagonal comparison. Write G for the reduced Zariski closure of this image over an algebraic closure of its coefficient local field. The inverse image of G° is an open subgroup Γ_0. Indeed there are finitely many components and their inverse images are closed. The image of Γ_0 is dense in G°, by density of the original image and the fact that each component is Zariski open as well as closed.

For any proper closed subvariety Y of the connected group G°, its inverse image in Γ_0 has measure zero. We prove this by induction on dimension, treating irreducible components separately. Distinct translates of an irreducible Y have intersections of smaller dimension. Such intersections have zero measure by induction, with empty intersection in dimension zero. If Y had positive measure, only finitely many distinct translates by the image of Γ_0 could have positive measure, since every finite collection would have union of measure its cardinality times that positive measure. Thus the stabilizer has finite index in that image. Its Zariski closure is all of G°, since an irreducible group cannot be a finite union of proper closed subsets. For any $y \in Y$, the inverse image of Y under $g \mapsto gy$ is closed and contains this dense stabilizer. Hence $G^\circ y \subset Y$, contradicting properness. The preimages used here are measurable since the polynomials defining a given closed subvariety are continuous over the finite coefficient extensions containing their coefficients.

Trace equality cuts out a closed subvariety on each component. If it contained a component, equality would hold on a coset of Γ_0, against Lemma 39.19. Translating each component to the identity component proves the measure assertion. For the statement about primes, the closed equality set is stable under conjugacy by $G_{\mathbb{Q}}$ because both traces are restrictions of class functions on that group. It is contained in conjugacy-stable unions of cosets, inside Γ, of open normal subgroups of $G_{\mathbb{Q}}$ with arbitrarily small relative measure. To see this, first take an open neighborhood of suitably small measure by regularity of Haar measure. By compactness, cosets of one open normal subgroup meeting the closed set can all be taken inside that neighborhood. Chebotarev in the resulting finite quotient gives the relative measure of this union as the limiting proportion in $[X, 2X]$ among the split primes. Taking arbitrarily small neighborhoods gives the result. A finite union preserves measure zero and the limiting bound. □

Theorem 39.26 (Common primes for the trace parameters and the theta tests)**.** *Fix $0 < \delta < 1/4$. The at most three primes of Theorem 39.20 may be chosen successively with the following additional properties. After each fixed prefix, a proportion tending to one of the primes in $[X, 2X]$ splitting completely in F are eligible for the next parameter, as $X \to \infty$. At each chosen prime q every marked elliptic factor satisfies $\mathrm{ord}(H_{q,l}) > X^{1/4-\delta}$. The prime separates the split trace at all characteristic-zero geometric points of all C_i finite over O_i. If there is no parameter to choose, the latter two properties still hold with relative proportion tending to one.*

Proof. At a given step, take the one-dimensional domain specialization from each current positive-dimensional component as in Theorem 39.20. Add to this finite list all geometric points of each finite generic algebra. Apply the lemma. Outside a set of relative proportion tending to zero, the difference at q is nonzero on every selected specialization. In particular it lies outside each indicated minimal prime. Its quotient reduces the greatest positive dimension by at least one. The finite-module conclusion then follows from the same complete local parameter argument as in Theorem 39.20. Theorem 56.15 excludes only $O(X^{1-2\delta})$ further rational primes for all the elliptic factors together. This is a vanishing proportion of the split primes for fixed F. All finite bad places, previously chosen primes and primes at which a specified nonzero coefficient array becomes zero can also be omitted. Thus Corollary 56.17 applies at each chosen prime with its stated rank and degree bounds. □

For a geometric point σ of $C_i[1/\ell_i]$ with C_i finite and nonzero, put

$$a_{i,\sigma} = \min_{g \in \Gamma} v_{\ell_i}\big(t_{i,\sigma}(g) - t_{0,i}(g)\big), \qquad v_{\ell_i}(\ell_i) = 1.$$

These depths are finite by Lemma 39.19 and nonnegative by integrality. Each minimum is attained since the values lie in a discrete valuation group in a finite extension.

Lemma 39.27 (Primes attaining the individual trace depths)**.** *For each such geometric point, at least $1/20$ of Γ in Haar probability attains its minimum. For a list of $s \geq 1$ points, at most $\lfloor 20 \log s \rfloor + 1$ good split primes suffice to attain every minimum, with the prime allowed to depend on the point. Every selected prime can also satisfy the theta order bounds and the simultaneous separation in the theorem.*

Proof. Fix a point and write f for its trace difference on Γ. Divide by an element of valuation $a_{i,\sigma}$ and reduce, obtaining a nonzero function $\bar{f}$ on Γ over a finite residue field. The left translates of f before reduction span a space of dimension at most $4^2 + 4 = 20$ because the trace of a translate of a four-dimensional representation is linear in its sixteen matrix entries and the split representation contributes at most four one-dimensional characters. The same dimension bound holds for the translates of $\bar{f}$. Otherwise 21 of them would give an invertible evaluation matrix on some 21 group elements, as point evaluations span the dual of a finite-dimensional space of functions. Lifting its normalized entries would contradict the characteristic-zero bound.

Choose a basis of at most 20 left translates of $\bar{f}$. At each $g \in \Gamma$ some translate takes the nonzero value of $\bar{f}$ at a point of its support. Thus at least one basis translate is nonzero at g. The basis supports cover Γ and have equal measure, proving the $1/20$ bound. The condition of attaining each minimum is clopen and invariant under $G_{\mathbb{Q}}$-conjugacy. Given the uncovered points, the expected number of their minima attained by a single element of Γ is at least one twentieth of their number. Some nonempty clopen set attains at least this many. Chebotarev gives positive relative density of corresponding split primes, also in dyadic intervals. Removing the exceptional sets of relative proportion tending to zero in the theorem is permitted. Each

step leaves at most 19/20 of the preceding number uncovered. After the stated number of steps the remaining number is less than one, since $(19/20)^j \le e^{-j/20}$. □

Corollary 39.28 (The least determinant valuation on the common splitting field)**.** *For each of the finite algebras C_i above, let $m_{i,\sigma}$ be its generic geometric multiplicities and let k_i be the strict length in Theorem 39.11. After a finite unramified extension of O_i, there is an integral linear combination z_i of the trace differences at the primes of Lemma 39.27 such that*

$$k_i \;\le\; v_{\ell_i} \det(z_i \mid C_i) = \sum_{\sigma} m_{i,\sigma} a_{i,\sigma}.$$

This is the minimum determinant valuation among elements of the ideal generated by all good split-prime trace differences, allowing finite unramified scalar extension. The primes in the lemma may be chosen once for the union of the geometric points across the interval.

Proof. Write the selected differences in C_i as $z_1, \dots, z_b$. For each point divide the tuple $(\sigma(z_j))_j$ by a scalar of valuation $a_{i,\sigma}$ and reduce. The resulting row is nonzero by the lemma. In the algebraic closure of the residue field there is a vector outside the kernels of all the rows, since their defining linear forms have nonzero product and the field is infinite. Its coordinates lie in a finite residue extension. Lift them to integers c_j in the corresponding unramified extension. Fix embeddings into a common algebraic closure at ℓ_i. Then $z_i = \sum_j c_j z_j$ has valuation exactly $a_{i,\sigma}$ at each extended geometric point, since its normalized residue there is nonzero.

Finite freeness of C_i, the local generic multiplicities, and the quotient $C_i/DC_i = O_i/\ell_i^{k_i} O_i$ of Theorem 39.8 all persist under unramified scalar extension, using the new coefficient ring for the quotient. Each z_j belongs to DC_i. Hence the determinant and length argument of Corollary 39.12 gives the inequality and the displayed equality with multiplicities. At any point, every split-prime difference has valuation at least $a_{i,\sigma}$ by definition. Multiplication by an element of the finite integral algebra and addition preserve that lower bound. The same determinant formula proves minimality. □

Both conclusions use primes splitting in the same field F. The local calculation still requires finiteness of the outstanding trace algebras and an upper estimate for the sum of their congruence depths with their geometric multiplicities. The theta ranks at these primes concern the mixed derivatives of Corollary 56.17. For the fixed final period and its archimedean determinant, the comparison of Section 6.6 remains open.

39.10 Finite quotients at common split primes

Fix an entire period interval $I = [V, V^{3/2}]$. Use the same exceptional dyadic data, test fields, and finite collection of elliptic factors as in Section 39.9. In particular $F/\mathbb{Q}$ is finite Galois and contains every period field and the auxiliary fields for these tests on I. For datum i write O_i for its unramified coefficient ring, ℓ_i for its uniformizer, κ_i for its finite residue field, and $k_i > 0$ for its strict length. Its actual exceptional trace algebra and augmentation ideal satisfy

$$C_i = O_i[[U, V', W]]/(e_i, f_{i1}, f_{i2}), \qquad D_i = (U, V', W)C_i, \qquad C_i/D_i = O_i/\ell_i^{k_i} O_i$$

by Theorem 39.8. We write V' for the middle trace coordinate to distinguish it from the endpoint of I. At a good rational prime q write $\delta_{i,q} = t_i(\mathrm{Frob}_q) - t_{0,i}(\mathrm{Frob}_q) \in D_i$.

Lemma 39.29 (A trace on a curve in either characteristic)**.** *Let B be any positive-dimensional complete local domain quotient of C_i. There is a continuous local map from B to the integers of a local field of characteristic 0 or ℓ_i such that the specialized trace differs from the split trace on a set of full Haar measure in G_F. The proportion of split primes in $[X, 2X]$ detecting this difference tends to one as $X \to \infty$.*

Proof. Take a one-dimensional domain quotient B' of B. For instance, take the penultimate prime in a chain of maximal length ending in the maximal ideal. The quotient is complete since the original ring is complete Noetherian local. The three trace coordinates cannot all vanish in B', since C_i/D_i is Artinian. If $\ell_i = 0$ in B', choose a nonzero parameter of B'. The quotient by the parameter is finite-dimensional over κ_i. Lifting a basis successively in its powers shows that B' is finite over the power series ring in the parameter over κ_i, as in Corollary 39.15. Its fraction field is a local field with finite residue field and its valuation ring contains B'. Some power of the maximal ideal lies in the parameter ideal, so the inclusion is local and continuous. Some good-prime trace difference has nonzero image, since those differences generate D_i by Corollary 39.9. That image has positive valuation, hence is transcendental over the finite constant field. The diagonal trace takes values in a finite field, so the representation satisfies the transcendental trace hypothesis of Theorem 39.17.

If $\ell_i \neq 0$ in B', it is a parameter since B' is a one-dimensional local domain. The same successive lifting with this parameter makes B' finite over O_i. Its fraction field is a finite extension of $\mathrm{Frac}(O_i)$ and the inclusion into its integers is local and continuous for the same reason. The nonzero coordinate triple gives a genuine representation by Lemma 39.13 and Theorem 39.14. The characteristic-zero statement of Theorem 39.17 applies to this specialization. Only the one-dimensional quotient was used to obtain its finite coefficient field.

In either case the trace cannot agree with the split trace on any open coset in G_F. Indeed the rank-four representation restricts irreducibly to each open normal subgroup. In the two-plane case the two irreducible planes there are distinct in characteristic zero by the determinant argument of Lemma 39.19, and the repeated plane in characteristic ℓ_i has multiplicity two prime to ℓ_i. On a coset, the trace pairing on a nontrivial matrix block therefore cannot cancel with the diagonal characters, by the group-algebra argument in that lemma. Lemma 39.25 now applies: on each component of the algebraic monodromy group the equality cuts out a proper closed subvariety. Its inverse image has measure zero, and Chebotarev in finite quotients gives the stated proportion. □

Theorem 39.30 (At most three split tests bound every exceptional strict length)**.** *For the whole list on I there are distinct good primes $q_1, \dots, q_h$ splitting completely in F, with*

$$h \le \max_i \dim C_i \le 3,$$

such that each quotient

$$A_i = C_i/(\delta_{i,q_1}, \dots, \delta_{i,q_h})$$

is a finite ring. If every datum satisfies the unit condition of Corollary 45.3, one can take $h \le 2$. For an empty list take $h = 0$. At each successive choice, for any fixed $0 < \eta < 1/4$, a proportion tending to one of split primes in $[X, 2X]$ are eligible and satisfy $\mathrm{ord}(H_{q,l}) > X^{1/4-\eta}$ for every elliptic factor l in the specified tests. For

$$h_i = \min\{a \ge 0 : \ell_i^a 1 = 0 \text{ in } A_i\}$$

one has $k_i \le h_i$ and $A_i/D_iA_i = O_i/\ell_i^{k_i}O_i$, where D_iA_i denotes the image ideal.

Proof. The ambient power series ring over O_i is a regular local domain of dimension four. Some defining relation is nonzero, since on setting the three coordinates to zero their values generate $\ell_i^{k_i}O_i$. Quotienting by a nonzero element reduces the dimension by at least one, by adjoining the zero prime below any chain of primes containing it. Hence $\dim C_i \le 3$. Under the unit condition the two-variable presentation over O_i in Corollary 45.3 has the same quotient $O_i/\ell_i^{k_i}O_i$ at its origin. The same argument gives dimension at most two.

At each step take every positive-dimensional minimal component of every current quotient $C_i/(\delta_{i,q_a} : a \text{ already chosen})$. There are finitely many since the rings are Noetherian and the list on I is finite. Its domain is a complete local quotient of C_i. Choose a specialization by the lemma for each one. For a relative proportion tending to one of the split primes all the resulting traces differ from the diagonal traces. The difference at such a prime lies outside the minimal prime of each listed component. Theorem 56.15 permits imposing the elliptic order conditions at the same time. The finitely many bad or previous primes can be omitted.

If a chain of primes of positive length survives the new quotient, the corresponding chain upstairs begins at a prime containing the new difference. A minimal prime below it belongs to a positive-dimensional component and cannot contain the difference. It extends the lifted chain downwards by one. Hence the maximum positive dimension drops by at least one at each step until all quotients have dimension zero. Each quotient is local Noetherian of dimension zero, hence Artinian, with finite residue field. Its maximal-ideal filtration now shows its underlying set is finite. All the chosen differences belong to D_i, which proves the quotient identity and the inequality on the additive order of one. This argument also applies to an exceptional trace algebra having a positive-dimensional special fiber. □

Corollary 39.31 (Finite precision and an explicit bound from initial forms)**.** *Let $P = O_i[[u_1, \dots, u_m]]$, where $m = 3$ or $m = 2$ in the reduced presentation. Let J be the ideal of the displayed relations defining C_i and lifts of the h chosen differences. Write $\mathfrak{n} = (\ell_i, u_1, \dots, u_m)$, so $A_i = P/J$. For $N \geq 1$ compute*

$$b_i(N) = \operatorname{length}_{O_i} P/(J + \mathfrak{n}^N).$$

For some finite N one has $b_i(N+1) = b_i(N)$. At the first such N, $P/(J + \mathfrak{n}^N) = A_i$, and its largest elementary-divisor exponent over O_i equals h_i. In particular $k_i \leq h_i \leq N$.

There is also a sufficient test using leading forms alone. If the initial ideal of J in $\kappa_i[L, X_1, \dots, X_m]$ contains $m+1$ homogeneous elements of positive degrees $a_0, \dots, a_m$ with no common projective zero over $\overline{\kappa_i}$, then

$$k_i \leq h_i \leq 1 + \sum_{j=0}^{m} (a_j - 1).$$

Proof. Modulo $\mathfrak{n}^N$, a monomial u^α of degree $|\alpha| < N$ has coefficients in $O_i/\ell_i^{N-|\alpha|}O_i$. These are independent additive coordinates. The truncated ideal of relations is the O_i-span of all monomial multiples of the given generators of J through this precision. Thus each $b_i(N)$ uses only finite coefficient matrices over O_i. For the maximal ideal $\mathfrak{m}$ of A_i, successive lengths differ by $\operatorname{length}(\mathfrak{m}^N/\mathfrak{m}^{N+1})$. If this vanishes, Nakayama's lemma applied to $\mathfrak{m}^N$ gives $\mathfrak{m}^N = 0$. Such a stage occurs by the theorem. Multiplication by ℓ_i^a kills the whole ring precisely when it kills one. The structure theorem for finite modules over a discrete valuation ring identifies its least such exponent with the largest elementary-divisor exponent. Since $\ell_i \in \mathfrak{m}$, this exponent is at most N.

The associated graded ring of P is the indicated polynomial ring, with L the image of ℓ_i; this follows also from the monomial description. The assumed homogeneous ideal generated by the $m+1$ elements has radical the irrelevant maximal ideal by the homogeneous Nullstellensatz and extension of the residue field. They form a system of parameters and hence a regular sequence in the polynomial ring localized at that ideal, since it is Cohen–Macaulay. The homogeneous sequence is regular also in the graded ring: a nonzero homogeneous class cannot disappear on localizing at the irrelevant maximal ideal, as multiplication by a polynomial with nonzero constant term preserves its lowest nonzero degree. Successively using the exact sequence for

multiplication by each homogeneous nonzerodivisor gives the Hilbert series of their quotient as $\prod_{j=0}^{m}(1+t+\cdots+t^{a_j-1})$. The graded ring of A_i is a quotient of this ring since it is the graded ring of P modulo the entire initial ideal of J. It vanishes in degree $1+\sum_j(a_j-1)$. Nakayama as above proves the bound. □

The theorem gives finite upper bounds on the same E_x and the same I used in the theta tests. To deduce the estimate of Section 6.6 one still needs bounds for their weighted sum as V increases. The stabilization exponent in the corollary is allowed to vary with the datum and the interval.

39.11 Finite trace quotients preserving Frobenius spectra

Fix the interval $I=[V,V^{3/2}]$ of Section 39.9. Retain its finite list of exceptional dyadic trace algebras C_i, their unramified coefficient rings O_i with uniformizers ℓ_i and residue fields κ_i, and the ideals D_i with $C_i/D_i=O_i/\ell_i^{k_i}O_i$. Write $t_{i,q}$ and $t_{0,i,q}$ for the universal and diagonal Frobenius traces. The same finite Galois field F contains the period fields E_x, the residual splitting fields, and the fields of the selected elliptic tests on I.

Lemma 39.32 (Avoidance of all finite-field trace values)**.** *From any positive-dimensional complete local domain quotient of C_i/ℓ_iC_i there is a local specialization to the valuation ring of a local field L of characteristic ℓ_i with the following property. The proportion of good primes in $[X,2X]$ split in F for which the specialized trace is algebraic over κ_i tends to zero. In particular it avoids the roots of any given nonzero polynomial over κ_i with relative proportion tending to one, even when the polynomial and its degree depend on the prime.*

Proof. Take a one-dimensional domain quotient and its local-field specialization as in Corollary 39.15. The image of some trace difference is nonzero and has positive valuation, since the residue at the closed point is diagonal and C_i/D_i is Artinian. The procedure in Lemma 39.29 gives the associated representation with a transcendental trace. Algebraic elements of L over the finite field κ_i form a finite field: every nonzero such element has finite multiplicative order, hence has valuation zero. The residue map is injective on these constants since the same is true of their nonzero differences.

On G_F the trace is either that of an absolutely irreducible four-dimensional representation or twice the trace of an absolutely irreducible plane. Theorem 39.17 gives irreducibility on every open subgroup, and Corollary 39.18 gives the factor two since the residual twist is trivial on G_F. This factor is nonzero. The trace cannot equal a constant on any open coset gH in G_F. Shrink H to an open normal subgroup. Over an algebraic closure the span of the matrices $\rho(h)$ on the irreducible factor W is $\mathrm{End}(W)$ by Burnside's theorem. The span of $\rho(h)-1$ is a nonzero two-sided ideal there: left or right multiplication by $\rho(h')$ is a difference of two group matrices, and the representation of H is nontrivial. This ideal is therefore all of $\mathrm{End}(W)$. Constancy on the coset would force the trace pairing with the invertible matrix $\rho(g)$ to vanish on the whole algebra, which is impossible.

In the reduced algebraic monodromy group, each specified constant trace fiber is thus a proper closed subset on every component. The measure-zero argument in Lemma 39.25 applies to this polynomial equation: the inverse image of a proper closed subset of the identity component under a compact Zariski-dense group has measure zero, and translation gives the result on its other components. Take the finite union over the constant field in L. Chebotarev in the finite quotients used in that lemma gives the stated limiting relative proportion. Every root in L of every nonzero polynomial over κ_i belongs to this same finite field. □

Theorem 39.33 (Common split primes retaining prescribed integral spectra)**.** *For every good split prime q and each i, prescribe any monic polynomial $P_{i,q} \in O_i[X]$ satisfying $P_{i,q}(t_{0,i,q}) \in \ell_i^{k_i} O_i$. At most three common primes $q_1, \dots, q_s$ suffice to make all the algebras*

$$A_i = C_i/(P_{i,q_j}(t_{i,q_j}) : 1 \le j \le s)$$

finite as O_i-modules, with $A_i/D_iA_i = O_i/\ell_i^{k_i}O_i$. At each stage a relative proportion tending to one of the split primes is eligible; one may impose the elliptic order bounds of Theorem 39.26 on them. Every specialization whose traces meet the prescribed equations factors through the quotient. In particular one can retain simultaneously any fixed finite collection of integral characteristic-zero specializations.

For the original algebra C_i itself, finiteness over O_i is equivalent to the following positive-density condition: on a set of split good primes of positive upper relative proportion in dyadic intervals, the image of $t_{i,q}$ in $(C_i/\ell_i C_i)_{\mathrm{red}}$ is algebraic over κ_i. No degree bound for these individual algebraic relations is required.

Proof. At the start and after any fixed prefix of selected primes, look at the positive-dimensional minimal components of the current special fibers for all i. This is a finite list by Noetherianity. Choose a specialization from each by Lemma 39.32. The reductions of the $P_{i,q}$ are nonzero because they are monic. Thus, outside a set of relative proportion tending to zero, every new equation is nonzero on each of its listed components, regardless of its degree. The elliptic order constraint has the same relative proportion one by Theorem 56.15. A positive-length chain of primes in a new special fiber lifts to a chain upstairs to which one can prepend a minimal prime: the new equation was outside every positive-dimensional minimal component. Hence the maximal positive dimension decreases. Initially each special fiber is a quotient of $\kappa_i[[U, V', W]]$. At most three steps make all of them Artinian.

Each A_i is complete local and its residue modulo ℓ_i is finite-dimensional over κ_i. A power of its maximal ideal lies in $\ell_i A_i$, so its maximal-ideal and ℓ_i-adic topologies are equivalent. Lift a basis of $A_i/\ell_i A_i$ and expand any element successively modulo the powers of ℓ_i. Completeness of O_i makes the lifts generate A_i over O_i. By the hypothesis at the diagonal trace, all imposed equations belong to D_i. This proves the assertion about the strict quotient. For finitely many chosen specializations into finite extensions of $\mathrm{Frac}(O_i)$, one may take $P_{i,q}$ to be $X - t_{0,i,q}$ times the product of the minimal polynomials of their trace values. These are monic integral polynomials since a discrete valuation ring is integrally closed. The quotient then retains each chosen specialization by the universal property of a quotient ring.

If C_i is finite, its reduced special fiber is its finite residue field, and the trace condition holds at every good prime. Otherwise its special fiber has positive dimension by Lemma 39.10. Choose one of the specializations of the lemma on such a component. It kills the nilradical. Every prime whose reduced universal trace is algebraic gives a trace in the finite constant field of this fixed local field, a set of relative proportion tending to zero. This contradicts positive upper relative proportion. □

Corollary 39.34 (The strict length with finite-module torsion)**.** *For these A_i put $U_i = A_i[\ell_i^\infty]$ and $A_i^{\mathrm{f}} = A_i/U_i$, and let b_i be the least $b \ge 0$ with $\ell_i^b U_i = 0$. A further common good split prime q can be chosen so that multiplication by $z_i = t_{i,q} - t_{0,i,q}$ on A_i^{f} has nonzero determinant for every i. Then*

$$k_i \le b_i + v_{\ell_i}\Big(\det\left(z_i \mid A_i^{\mathrm{f}}\right)\Big), \qquad v_{\ell_i}(\ell_i) = 1.$$

The determinant is one for a zero free quotient. Its valuation is the sum of the branch valuations with the dimensions of the local generic algebras as multiplicities.

Proof. The generic fibers are finite algebras and have only finitely many geometric points. Each point gives an integral local specialization of C_i into a local field of characteristic zero: the image of A_i is integral and its maximal ideal has a power in $\ell_i A_i$. It cannot annihilate D_i since C_i/D_i is killed by $\ell_i^{k_i}$. The characteristic-zero trace separation in Lemma 39.29, or its open-coset proof applied to this point, holds with relative proportion one. Consequently one good split prime separates all these points from their diagonal traces. In the generic algebra z_i is a unit. Multiplication on the finite free O_i-module A_i^{f} is injective, and the length of its cokernel is the valuation of the determinant by elementary divisors. Over an algebraic closure its action on each local Artinian factor is its residue value plus a nilpotent, which proves the multiplicity statement.

The image of U_i in the cyclic module A_i/D_iA_i has length at most b_i because $\ell_i^{b_i}$ kills it. The remaining quotient is precisely $A_i^{\mathrm{f}}/D_iA_i^{\mathrm{f}}$. Since $z_i \in D_i$, its length is at most the cokernel length just computed. Adding the two contributions proves the bound. □

The polynomials permit direct comparisons with fixed integral Hecke spectra whenever the relevant trace identities are available. To deduce the uniform strict cohomology estimate one still has to control the torsion exponents and the determinant valuations with their weights on I, or establish the estimate by another argument.

40 Ordinary symplectic equations and trace multiplicities

40.1 The ordinary equations on the strict symplectic root

The four-dimensional construction uses polynomial equations to describe an ordinary local representation. We compute these equations on the matrix entries that carry a strict cohomology class. The calculation also applies when the residual character is trivial on the entire coefficient-prime decomposition group.

Let $p \geq 5$, let $O/\mathbb{Z}_p$ be a finite unramified extension of discrete valuation rings, and write $J = \mathrm{Frac}(O)$. For $N \geq 1$ set $D_N = (J/O)[p^N]$. With the scalar action of a character understood, this module is isomorphic to $O/p^N O$. Consider the local characters of Proposition 38.2,

$$d = (A\Psi, \Psi, A, 1), \qquad \theta = \Psi^{-1}, \qquad X = E_{20} + E_{31}.$$

Here indices run from zero to three. Write $G_p = G_{\mathbb{Q}_p}$ and I_p for inertia. A continuous cocycle z for $D_N(\theta)$ gives matrices $\rho_z(g) = (1 + z(g)X)d(g)$ over $O \oplus D_N$, in which products of two elements of the second summand are zero. This is the entry used in Theorem 38.4.

Fix $s_0 \in I_p$ for which the four units $\lambda_i = d_i(s_0)$ are distinct in J. Set

$$B_i(T) = \prod_{k \neq i} (T - \lambda_k), \qquad \Delta_i = B_i(\lambda_i), \qquad t_0 = \theta(s_0) - 1, \qquad C = \frac{\Delta_0 \Delta_2}{t_0}.$$

The denominator $t_0 \neq 0$ divides Δ_0 in O, since $\lambda_2 - \lambda_0 = \lambda_0 t_0$. The ordinary condition at s_0 prescribes the characteristic polynomial $\prod_i (T - \lambda_i)$ for $M = \rho_z(s_0)$ and the equations

$$\begin{aligned} B_j(M)\rho_z(g)B_i(M) &= 0 && (g \in G_p,\ i < j),\\ B_i(M)(\rho_z(s) - d_i(s)1)B_i(M) &= 0 && (s \in I_p,\ 0 \leq i \leq 3). \end{aligned}$$

These are the original local equations in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/MapRepApply.lean`, line 99.

Proposition 40.1 (The obstruction to the ordinary equations at finite depth)**.** *The ordinary condition on ρ_z is equivalent to the single family of equations*

$$C\,(t_0 z(g) - (\theta(g) - 1)z(s_0)) = 0 \qquad (g \in G_p) \tag{40.1}$$

in D_N. In particular the class of the restriction of z in $H^1(G_p, D_N(\theta))$ is killed by $\Delta_0\Delta_2$ whenever the ordinary condition holds. Every local coboundary satisfies the condition at the original depth.

Proof. The two coordinate planes with ordered bases (e_0, e_2) and (e_1, e_3) are preserved. On either plane, with indices (i, j) in this order, the matrix at s_0 is

$$\begin{pmatrix} \lambda_i & 0 \\ \lambda_i z(s_0) & \lambda_j \end{pmatrix}, \qquad \lambda_j/\lambda_i = \theta(s_0).$$

For any polynomial B its value on this matrix has diagonal $(B(\lambda_i), B(\lambda_j))$ and lower-left entry

$$\lambda_i z(s_0) \frac{B(\lambda_i) - B(\lambda_j)}{\lambda_i - \lambda_j}.$$

The divided difference is an integral polynomial expression in the two units, as follows first for each monomial by summing a finite geometric progression and then by linearity. If $k \notin \{i, j\}$, B_k vanishes at both diagonal entries and $B_k(M)$ is zero on this plane. On the plane in question $B_i(M)$ has only its first column and $B_j(M)$ only its second row. Multiplying the two matrices with $\rho_z(g)$ between them gives just the entry

$$\big(B_j(M)\rho_z(g)B_i(M)\big)_{ji} = d_i(g)\,\frac{\Delta_i\Delta_j}{t_0}\big(t_0 z(g) - (\theta(g) - 1)z(s_0)\big).$$

This formula uses integral multiples of the elements of D_N. The diagonal inertia equations vanish on both planes. Indeed, on the indicated plane the preceding first column has second column zero, so sandwiching $\rho_z(s) - d_i(s)1$ between two copies of $B_i(M)$ gives zero. With the operator having only its second row, the same calculation uses its second diagonal entry $d_j(s)$. The characteristic polynomial already has the required value by triangularity.

Put $\mu = \lambda_0\lambda_3 = \lambda_1\lambda_2$. Replacing a pair of units u, v by $\mu/u, \mu/v$ preserves the valuation of $u - v$. Thus $v_p(\Delta_0) = v_p(\Delta_3)$ and $v_p(\Delta_2) = v_p(\Delta_1)$. The two factors $\Delta_i\Delta_j/t_0$ from the two planes differ by a unit. This proves the equivalence. Equation (40.1) writes $\Delta_0\Delta_2 z(g)$ as the coboundary with witness $Cz(s_0)$ at every g. Conversely, substituting $z(g) = (\theta(g) - 1)y$ makes its left hand side zero for any $y \in D_N$. □

Theorem 40.2 (The cost of replacing a strict local condition)**.** *Use a field E_x and a character α from the intervals of Theorem 12.15. Suppose $p \geq 5$, $p \nmid n_* N_*$, $2 \leq r < R = \lfloor V^{1203/2} \rfloor$, $p - 1 \mid 2r$, and $\alpha(c) = (-1)^r$. Use $\Psi = \alpha\kappa_p^r$ and the auxiliary A of the symplectic construction. At finite depth N, fix the same global coefficient module $D_N(\theta)$ and the same local conditions everywhere away from p. Let S_N be the resulting group with zero localization at p, and P_N the group with the displayed ordinary equations there for the source inertia element $\kappa_p(s_0) = 1 + p$. Then*

$$S_N \subset P_N, \qquad \operatorname{length}_O(P_N/S_N) \leq 2\big(6 + 2v_p(r) + v_p(r - 1) + v_p(r + 1)\big).$$

The estimate is uniform in N and in the unramified coefficient extension used for the auxiliary character.

Proof. At p, the finite conductor characters in d are unramified, apart from the Teichmüller factor of the cyclotomic character. This factor is one on s_0. It follows that λ_i are the powers of $1 + p$ with exponents $(r + 1, r, 1, 0)$, in that order. For $a \neq b$ the valuation of $(1 + p)^a - (1 + p)^b$ is $1 + v_p(a - b)$, by the binomial theorem at the odd prime p. Consequently

$$H := v_p(\Delta_0\Delta_2) = 6 + 2v_p(r) + v_p(r - 1) + v_p(r + 1).$$

The criterion (40.1) is linear in the cocycle and is invariant under addition of a coboundary. It thus defines a subgroup on cohomology. Localization identifies P_N/S_N with a subgroup of $H^1(G_p, D_N(\theta))[p^H]$ by the proposition.

We bound this last group uniformly in N. Put $k = O/pO$ and $T_\theta = O(\theta)$. The residual dual character $\bar\kappa_p \bar\theta^{-1}$ is nontrivial on inertia. Its exponent is $1 + r$, which at the two residues equals 1 or $1 + (p-1)/2$ modulo $p - 1$, both nonzero for $p \geq 5$. The divisible dual coefficient therefore has no invariants, since some inertia element acts by a scalar whose difference from one is a unit. Local Tate duality gives $H^2(G_p, T_\theta) = 0$. Here the trace dual of O over $\mathbb{Z}_p$ is O since O is unramified. The multiplication sequences now give

$$H^1(G_p, D_N(\theta)) \simeq H^1(G_p, T_\theta)/p^N H^1(G_p, T_\theta).$$

In particular its quotient modulo p is $H^1(G_p, k(\bar\theta))$. For this residual module local duality gives zero in degree two as well, and the local Euler characteristic formula gives dimension $1 + \dim_k H^0(G_p, k(\bar\theta)) \leq 2$ in degree one. These local duality and Euler characteristic formulas are also used in Proposition 38.2. The finite group on the left is consequently generated by at most two elements over O, by Nakayama's lemma. Its p^H-torsion has length at most $2H$, as follows by writing its elementary divisors over the discrete valuation ring. This proves the estimate at every depth, including the case $\bar\Psi|_{G_p} = 1$. □

Corollary 40.3 (A simultaneous bound on the period fields)**.** *At every $x \in [V, V^{3/2}]$, sum the excess lengths $\operatorname{length}_O(P_N/S_N)\log p$ over the primes and characters of the theorem and the colliding weights $2 \leq r < t(x)$. Count each geometric character of $\mathrm{Gal}(E_x/\mathbb{Q})$ once in its prescribed sign. The sum is $O(n_* R \log(3R))$, even when the finite depths are chosen separately for each summand.*

Proof. There are at most $n_* = [E_* : \mathbb{Q}]$ characters at every pair. For one prime the number of weights with $p - 1 \mid 2r$ is at most $2R/(p-1)$, and such primes satisfy $p \leq 2R + 1$. The constant term of the preceding bound, summed with $\log p$, therefore costs $O(R\log(3R))$ before the character count. Indeed $\sum_{p \leq y} \log p/(p-1) \ll \log(3y)$. On each range $m < p \leq 2m$ the product of the primes divides $\binom{2m}{m} \leq 4^m$. Taking m through powers of two proves this estimate by a dyadic partition. For each $r \geq 2$, even the sum over all primes of $(2v_p(r) + v_p(r-1) + v_p(r+1))\log p$ equals $\log(r^2(r-1)(r+1)) \ll \log(3R)$. Sum over the weights and multiply by n_*. The bound is polynomial in V on the original family, hence $o(\exp(x^\varepsilon/2))$ uniformly there. □

This comparison measures the extra finite classes admitted by the polynomial ordinary condition on the specified root. Bounding the total number of classes satisfying that condition still requires a global argument. The projected integral theta-period estimate is also required on these fields in Section 6.6.

40.2 Ordinary flags at finite precision

Let O be a complete discrete valuation ring with uniformizer π, fraction field J, and normalized valuation v. Write $L = O^d$ and $L_N = L/\pi^N L$. Let I be a subgroup of a group G, choose $s_0 \in I$, and give characters $\chi_i : I \to O^\times$ for $0 \leq i < d$. The units $\lambda_i = \chi_i(s_0)$ are assumed pairwise distinct in J. Put

$$f(T) = \prod_i (T - \lambda_i), \quad B_i(T) = \prod_{j \neq i} (T - \lambda_j), \quad h_i = v(B_i(\lambda_i)), \quad h = \max_i h_i.$$

An ordered ordinary flag for a representation on L_N is a full flag of direct summands, stable under G, such that I acts by χ_i on the i-th successive line, in the given order. We first allow its reduction to be any flag with these characters.

Theorem 40.4 (Recovering a flag from the polynomial equations)**.** *Let $\rho_N : G \to \mathrm{GL}(L_N)$ and $M = \rho_N(s_0)$. An ordered ordinary flag gives the identities over O/π^N*

$$\begin{aligned} \det(T - M) &= f(T), \\ B_j(M)\rho_N(g)B_i(M) &= 0 \qquad && (i < j,\ g \in G), \\ B_i(M)(\rho_N(s) - \chi_i(s)1)B_i(M) &= 0 && (s \in I). \end{aligned}$$

Conversely, if these equations hold and $N > 3h$, the representation has an ordered ordinary flag modulo π^{N-3h}. If d is even, L has its standard perfect alternating form, ρ_N has a prescribed similitude character $\mu : G \to O^\times$ modulo π^N, and $\lambda_i\lambda_{d-1-i} = \mu(s_0)$, then the recovered flag can be taken self-dual. For an existing pair of ordered ordinary flags at a depth $m > h$, their reductions already agree modulo π^{m-h}.

Proof. Number the steps starting with zero and write F_i for the step ending with the line labeled i, with $F_{-1} = 0$. Given a flag, choose an adapted basis. The matrix M is upper triangular with diagonal λ_i at the stated precision. The factors of $B_i(T)$ indexed by $j > i$ give the characteristic polynomial of M on L_N/F_i. The factors indexed by $j < i$ give its characteristic polynomial on F_{i-1}. Cayley–Hamilton therefore gives $B_i(M)L_N \subset F_i$ and $B_i(M)F_{i-1} = 0$. Stability of each step and the character on each quotient now imply the two matrix equations.

For the converse take a matrix A over O lifting M. Its characteristic polynomial f_A satisfies $f_A \equiv f \bmod \pi^N$. At each λ_i its value has valuation at least N and its derivative has valuation exactly h_i. Since $N > 2h_i$, Hensel's lemma gives a root λ_i' of f_A with $v(\lambda_i' - \lambda_i) \geq N - h_i$. For $i \neq j$ one has $v(\lambda_i - \lambda_j) \leq h$, while $N - h_i > h$. The lifted roots are distinct and exhaust the roots of f_A. Every difference $\lambda_i' - \lambda_j'$ has the valuation of the corresponding original difference. Set

$$B_i'(T) = \prod_{j \neq i} (T - \lambda_j'), \qquad P_i = B_i'(A)/B_i'(\lambda_i').$$

The prime here distinguishes the new polynomial. Its value in the denominator has valuation h_i. By interpolation the P_i are the pairwise annihilating idempotents projecting onto the labeled eigenlines in $L \otimes J$.

The coefficients of $B_i' - B_i$ lie in $\pi^{N-h}O$. For each g take any integral lift U_g of $\rho_N(g)$. The polynomial matrix equations first give divisibility by π^N with $B_j(A), B_i(A)$ as the two outside factors. Replacing both by the primed polynomials retains divisibility by π^{N-h}, since all matrices involved at this stage are integral. Dividing by the two scalar denominators gives, with $k = N - 3h$,

$$P_jU_gP_i \in \pi^k \operatorname{End}_O(L) \quad (j > i), \qquad P_i(U_s - \chi_i(s)1)P_i \in \pi^k \operatorname{End}_O(L) \quad (s \in I).$$

This uses a lift of each individual matrix. Their products need only represent the group law at depth N, where it holds by hypothesis.

Intersect the successive sums of the labeled J-eigenlines with L to form a saturated flag F_i. Over a discrete valuation ring each such step and each step in the preceding quotient is a direct summand. For $v_0 \in F_i$ the part of U_gv_0 in the sum of the eigenlines above i is $\sum_{a \leq i < j} P_jU_gP_av_0 \in \pi^kL$. After subtracting it, the result is integral and belongs to F_i. This proves stability at depth k. For $s \in I$, project $(U_s - \chi_i(s)1)v_0$ onto the sum of the eigenlines numbered $j \geq i$. All its terms are covered by the first matrix divisibility except $a = j = i$,

which is covered by the second. Thus the result belongs to $F_{i-1} + \pi^k L$. This gives the claimed inertia action on the reduction.

In the symplectic case choose the lift A to have multiplier exactly $\mu(s_0)$. Start with lifts of the first and last columns in a symplectic basis, then scale the last by a unit congruent to one modulo π^N to make their pairing exact. Project the remaining columns onto their orthogonal complement. All subtracted cross pairings were already zero modulo π^N. Repeating with each successive pair gives the desired lift. Its eigenvalues are preserved as a set by $z \mapsto \mu(s_0)/z$. The disjoint Hensel neighborhoods show that this map takes λ'_i to λ'_{d-1-i}. The perfect form pairs each eigenline with its opposite and is zero between the other lines. The resulting flag over J is self-dual. Taking intersections with the unimodular L gives the same orthogonal-complement identities in L. They descend to the reduction since the steps are direct summands.

Finally consider two flags at depth m. Lift a basis adapted to each flag to a basis of L. In this basis lift the matrix of M at depth m to an upper triangular matrix with diagonal exactly λ_i. In the original basis the two matrices A_1, A_2 agree modulo π^m and both have characteristic polynomial f. Their projectors at i are $B_i(A_\nu)/B_i(\lambda_i)$ and differ by an endomorphism in $\pi^{m-h_i}\operatorname{End}_O(L)$. The sum of the first $i+1$ eigenlines for each matrix is exactly the span over J of the first $i+1$ adapted basis vectors, by triangularity and the distinct diagonal entries. For an integral vector in that span for A_1, project it onto the corresponding sum for A_2. Its change lies in $\pi^{m-h}L$, so its image is also integral. Interchanging the two matrices gives equality of the two steps after reduction to depth $m-h$. The proof of this last assertion only uses the one matrix and its ordered actions on the successive lines. □

Theorem 40.5 (Sharp precision for the locally trivial residual character)**.** *Take the four characters and the source element s_0 of Theorem 40.2. The polynomial test gives a self-dual ordinary flag at depth $N - 3h > 0$, where*

$$h = 3 + v_p(r) + \max\{v_p(r-1), v_p(r+1)\}.$$

Suppose now that $p-1 \mid r$ and $\alpha(p) = 1$. Write $c = 1 + v_p(r)$. For any two ordinary flags lifting the prescribed coordinate flag of a framed local representation with residual diagonal $\bar{d}$, equality holds modulo p^{m-c} if the flags exist at depth $m > c$. This loss of c powers of p is attained on the split representation over the same local field.

Proof. For the first assertion the four exponents at s_0 are $(r+1, r, 1, 0)$. Summing $1 + v_p(a-b)$ over the other three exponents for each a gives the displayed maximum. The similitude pairings of the four characters have the prescribed product. Apply Theorem 40.4 to the equations with the original inertia element.

For the sharper assertion, put $O_m = O/p^m O$. Choose an arithmetic Frobenius lift g with $\kappa_p(g) = 1$ as in Proposition 38.2. The residual action of g has eigenvalue $\bar{A}(g) \neq 1$ at indices $0, 2$ and eigenvalue one at $1, 3$. The auxiliary character in Lemma 38.6 supplies this inequality at the indicated primes. Hensel factorization of the characteristic polynomial of $\rho_m(g)$ over O_m separates the two relatively prime residual factors of degree two. Bézout's identity for their monic lifts and Cayley–Hamilton give complementary polynomial idempotents. Write U, W for their images in the stated order. Both are free of rank two. These summands are determined by the representation and g.

Every step of an ordinary flag is stable under these idempotents because it is stable under $\rho_m(g)$. Its parts in U, W are direct summands, with ranks read off from the prescribed residual flag. The same argument applied to the quotient by the step shows that its two parts are direct summands of U, W themselves. It follows that the successive proper steps are

$$U_1, \qquad U_1 \oplus W_1, \qquad U \oplus W_1$$

for lines $U_1 \subset U$, $W_1 \subset W$. Compress $\rho_m(s_0)$ to U, W using the respective polynomial idempotents. In a basis for the displayed flag ordered by indices $0, 1, 2, 3$, the matrix of $\rho_m(s_0)$ is upper triangular with entries λ_i on the diagonal. Consequently its two compressions are upper triangular on $(U_1, U/U_1)$ and $(W_1, W/W_1)$, with labeled entries (λ_0, λ_2) and (λ_1, λ_3). This calculation only compresses one matrix and requires no invariance of the two full summands under inertia.

In each pair the difference of the entries has valuation $v_p((1+p)^r - 1) = c$. Apply the two-dimensional, single-matrix uniqueness argument at the end of Theorem 40.4 to each compression. The two possibilities for each line agree at depth $m - c$. The three displayed steps therefore agree there as well.

For sharpness use the split diagonal representation d. The finite character α is unramified at p and equals one on Frobenius, so it is trivial locally. Since $p - 1 \mid r$, the image of $\theta = \kappa_p^{-r}$ is contained in $1 + p^c O$ on all of G_p, by the binomial valuation formula. With $X = E_{20} + E_{31}$ the matrix $1 + p^{m-c} X$ is symplectic and commutes modulo p^m with every $d(g)$, since $d(g) X d(g)^{-1} = \theta(g) X$. It sends the coordinate flag to a second self-dual ordinary flag with the same ordered characters and the same reduction modulo p. The first line is now generated by $e_0 + p^{m-c} e_2$. The lines are distinct modulo p^{m-c+1}, which proves the assertion. □

For a residual flag fixed in advance the reduction of a recovered flag must be checked against that choice. Both theorems concern the coefficient-prime local equations. The global number of ordinary classes and its relation to the integral period in Section 6.6 remain to be bounded.

40.3 The global equations for the two even roots

Fix one of the period fields E_x on the interval of Section 6.6. Let α be a character of this field, $2 \le r < t(x)$, $\alpha(c_\infty) = (-1)^r$, and let $\ell \ge 5$ be prime with $\ell \nmid n_* N_*$ and $\ell - 1 \mid 2r$. Write $\Psi = \alpha \kappa_\ell^r$, and suppose $\bar{\Psi}|_{D_\ell} \ne 1$, where $D_\ell = G_{\mathbb{Q}_\ell}$. At the half residue this holds on inertia. At the zero residue it means $\alpha(\ell) \ne 1$. Use the interval character ξ of Proposition 38.7, the resulting $A = \kappa_\ell \omega_\ell^{-1} \xi$, and the unramified coefficient ring O containing the values of the finite characters. Put $J = \mathrm{Frac}(O)$, $k = O/\ell O$. The split representation and its multiplier are

$$d = \mathrm{diag}(A\Psi, \Psi, A, 1), \qquad \mu = A\Psi.$$

For a character γ and a coefficient module $V(\gamma)$, write $S(V(\gamma))$ for the kernel in $H^1(\mathbb{Q}, V(\gamma))$ of restriction to D_ℓ and to inertia at every other finite prime. Define

$$a_\pm = \dim_k S(k(\bar{\Psi}^{\pm 1})), \qquad S^\pm = S((J/O)(\Psi^{\pm 1})).$$

Thus $S^- = S(\alpha, r)$ after extending coefficients. The integer a_- may be greater than one, and both residual groups may occur.

Choose $h \in D_\ell$ as in Lemma 39.1. Let Σ consist of $\ell, 2$ and the finite conductor primes of d. Let $\mathcal{B}$ represent the framed global lifts of Section 39.2, now for the data just specified. A lift has multiplier μ, is unramified off Σ, kills $N_q = I_q \cap \ker \bar{d}$ for $q \in \Sigma \setminus \{\ell\}$, and at ℓ has the prescribed ordinary flag and is diagonal at h. Take u_1, u_2, a, x_0, b, c for the local parameters of Theorem 39.2, writing x_0 for its 02 entry to distinguish it from the interval parameter x.

Theorem 40.6 (The global presentation at both colliding residues)**.** *Over $R_\ell = O[[u_1, u_2, a, x_0, b, c]]$ the ring $\mathcal{B}$ has a_+ further parameters P_i of torus weight $(0, 1)$ and a_- parameters Y_j of weight $(0, -1)$. Its ideal of relations admits generators of the following weights, with zero*

generators allowed:

weight	*generators*	*number*
$(0,0)$	Z_ν	2
$(0,1)$	E_ν	$a_+ + 1$
$(0,-1)$	F_ν	$a_- + 1$.

One can take the P_i, Y_j to be $02, 20$ entries, respectively, of global matrices. Eliminating u_1, u_2 gives

$$\mathcal{B}^s = O[[x_0, v, w, P_1, \dots, P_{a_+}, Y_1, \dots, Y_{a_-}]]/(E_\nu, F_\nu), \qquad v = ab, \quad w = bc,$$

where the superscript denotes invariants under the first factor of $\operatorname{diag}(st, t, s, 1)$. *The remaining weights are 1 on x_0, v, P_i, 2 on w, and -1 on Y_j.*

Proof. The four residual eigenvalues at h are distinct by Lemma 39.1. Consequently the ordinary local ring with this normalization is precisely R_ℓ. We compute the relative deformation spaces separately for each torus weight M in the symplectic Lie algebra. They are the cohomology of the degree-shifted mapping cone in Theorem 39.5, where the maps compare a global cochain with the fixed local cochain at ℓ and with cochains through $G_{\mathbb{Q}_q}/N_q$ at the other $q \in \Sigma$. Here the image of I_q in that quotient has order prime to ℓ, as each diagonal character on inertia has this property.

In degree one one may set the local witness at ℓ to zero by subtracting a global coboundary. The witnesses at the other primes can then be absorbed in local quotient coboundaries. On each root line the remaining representative is unique because the eigenvalue ratio at h differs from one by a unit. Its vanishing on N_q is equivalent to the inertia class being zero, by averaging on I_q/N_q. Thus the relative tangent dimensions on the even roots are a_+ and a_-. On the odd roots they are zero by Proposition 38.7, since $T_\alpha = \{\ell\}$. On the diagonal the cocycles are everywhere unramified additive characters of $G_\mathbb{Q}$, hence zero by class reciprocity for $\mathbb{Q}$.

In the same cone $H^0 = 0$, by injectivity on degree-zero invariants, and $H^3 = 0$, by Poitou–Tate duality. For the latter assertion the dual global invariant space vanishes already at ℓ. For the neutral and even weights the tame exponents are 0 or $(\ell - 1)/2$, both distinct from 1 modulo $\ell - 1$. For the odd roots use Proposition 38.2. The quotient local groups at $q \neq \ell$ have ℓ-cohomological dimension at most one by inertia averaging. Their Euler characteristics and those of $G_{\mathbb{Q}_q}$ are zero. Subtracting the Euler characteristic at ℓ from the global one now gives

$$\dim H^2 = \dim H^1 + \dim M^{c_\infty = 1}.$$

This adds two to the neutral dimension and one to each even root dimension. It adds zero to the odd dimensions.

To realize the relative parameters, consider the linear functionals that evaluate a normalized cocycle at all elements of G_Σ. They separate the cocycles. On each finite dimensional root space a subset of these evaluations is a basis of its dual. Multiplication by the corresponding unit diagonal entry of $d(g)$ identifies these evaluations with the indicated matrix entries on relative tangents. Complete Nakayama gives a surjection from $R_\ell[[P_i, Y_j]]$. The obstruction calculation in the proof of Theorem 39.5 bounds the number of minimal relations of each weight by the corresponding $\dim H^2$ just computed. Indeed the defects of lifted cochains define its cone obstruction. A functional on the minimal relation space killing this obstruction would lift the universal matrices across its square-zero extension while fixing R_ℓ. Varying any of the P_i, Y_j by elements of the small kernel preserves every relation there, since their relative linear terms vanish modulo the maximal ideal. This contradicts a nonzero functional. Taking Laurent coefficients of the two formal torus variables on finite jets gives homogeneous generators, with the same bounds after passage to the limit.

Imposing zero on all off-diagonal parameters leaves the diagonal deformations. Each diagonal character differs from the corresponding one of d by an everywhere unramified character reducing to one, hence equals it by class reciprocity. Only Z_1, Z_2 survive on $O[[u_1, u_2]]$. They generate (u_1, u_2), and the formal inverse function theorem eliminates these variables equivariantly. A monomial in the remaining variables has first weight zero exactly when its exponent of b is the sum of the exponents of a and c. It has a unique expression using v, w in their place. All remaining equations have first weight zero. Projection onto that weight on their coefficients identifies the invariant ideal, giving the asserted presentation. □

Theorem 40.7 (The integral equations at every finite depth)**.** *In the presentation of Theorem 40.6 keep only the variables Y_j and set the other variables equal to zero. The resulting equations are the rows of a matrix $B_- \in \mathrm{Mat}_{a_-+1,a_-}(O)$ acting on the column of Y_j. Keeping only P_i, including the condition $x_0 = 0$, gives a matrix $B_+ \in \mathrm{Mat}_{a_++1,a_+}(O)$. Put $K_\pm = \mathrm{coker}(B_\pm^t)$. At every $m \geq 1$ there are O-linear isomorphisms*

$$S^\pm[\ell^m] \simeq \mathrm{Hom}_O(K_\pm, O/\ell^m O).$$

In particular, the ideal of the a_--rowed minors of B_- is $\ell^{\mathrm{length}_O S^-} O$. For $a_- = 0$ the minor of size zero equals one.

Proof. With only the Y_j present, a monomial of weight -1 is a single Y_j and no monomial has weight $+1$. This leaves precisely $a_- + 1$ linear equations over O. The other restriction gives $a_+ + 1$ linear equations by the same weight count. The eliminated diagonal parameters restrict to zero in both cases since they are neutral and their split values are zero.

Set $O_m = O/\ell^m O$ and substitute ϵz_j, $z_j \in O_m$, for the retained variables in $O_m[\epsilon]/(\epsilon^2)$. Such substitutions in the full ring are in bijection with the kernel of $B_\pm : O_m^{a_\pm} \to O_m^{a_\pm+1}$. In the universal matrices their only possible first-order entries have the selected weight. The multiplier equation gives the root vector $X_- = E_{20} + E_{31}$ or $X_+ = E_{02} + E_{13}$ for the form $\langle e_0, e_3\rangle = \langle e_1, e_2\rangle = 1$. The matrices are $(1 + \epsilon z(g) X_\pm) d(g)$, and the group law says $z(gg') = z(g) + \Psi^{\pm 1}(g) z(g')$. All six local parameters are split, so z vanishes on D_ℓ. At $q \in \Sigma \setminus \{\ell\}$ it vanishes on N_q. It is unramified away from Σ. Conversely each such cocycle defines a lift with just these parameter values. The retained coordinates evaluate the cocycle multiplied by fixed units, giving an O_m-linear correspondence.

Division by ℓ^m identifies O_m with $(J/O)[\ell^m]$. Every divisible-coefficient class has a unique representative zero at h, since $\Psi^{\pm 1}(h) - 1$ is a unit. The representative of a class killed by ℓ^m therefore takes values in the indicated finite module. For a strict class it is zero on D_ℓ, because a coboundary there vanishing at h has zero witness. At other conductor primes N_q acts trivially on the entire coefficient module. A coboundary on inertia vanishes on N_q, and conversely averaging on the prime-to-ℓ quotient makes any such inertia restriction a coboundary. These facts identify exactly the cocycles counted above with $S^\pm[\ell^m]$. The definition of the transpose cokernel now gives the claimed Hom groups.

Theorem 15.17 gives finiteness of S^- on these fields. Extension to O is unramified and preserves its module length. A free summand of K_- would make the lengths of its Hom groups unbounded with m. Thus K_- is torsion. For a cyclic summand $O/\ell^b O$ the Hom length is $\min(b, m)$. Taking m larger than all elementary divisors shows $\mathrm{length}_O K_- = \mathrm{length}_O S^-$. In Smith normal form the ideal of maximal minors of B_- is generated by the product of these elementary divisors. This proves the formula, including the case of the empty presentation. □

Corollary 40.8 (A bound for the sizes of the presentation matrices)**.** *Let L be the fixed field of $\ker \bar{\Psi}$, and let $d_L = [L:\mathbb{Q}]$. On the whole indicated part of the interval,*

$$a_+, a_- \;\le\; \frac{\log |\operatorname{Cl}(L)|}{\log \ell} \;\le\; \frac{d_L \log(\ell N_* \zeta(2))}{\log \ell}, \qquad d_L \le 2n_*.$$

Proof. The cyclic group $\mathrm{Gal}(L/\mathbb{Q})$ has order prime to ℓ. For either character, restriction injects its residual strict group into homomorphisms from the class group of L to the additive group of k. Indeed restriction turns the cocycle into an everywhere unramified homomorphism, zero at primes over ℓ, and the kernel of restriction vanishes by averaging. The k-dimension of the full space of homomorphisms is at most $\log |\operatorname{Cl}(L)| / \log \ell$. The character cutting out L is a product of a character of order at most n_* and a character of order at most two. Its conductor divides ℓN_*. The conductor–discriminant formula gives $|D_L| \le (\ell N_*)^{d_L}$. Minkowski's bound supplies an integral ideal of norm at most $\sqrt{|D_L|}$ in each ideal class, since L is totally real. The number of ideals of norm at most $H \ge 1$ is at most $H^2 \zeta_L(2)$, and the Euler factors give $\zeta_L(2) \le \zeta(2)^{d_L}$. Substitution proves the stated inequality. It controls the dimensions of the matrices. The valuation of their minors still enters the length in Theorem 40.7. □

40.4 The trace algebra and the negative root module

Use the fields, intervals, coefficient primes and characters of Section 40.3. Put $p = a_+ + 2$, $m = a_-$, and write $X_1, \dots, X_p$ for $x_0, P_1, \dots, P_{a_+}, v$, in that order. Here p counts coordinates; the coefficient prime is ℓ. With weights $1, 2, -1$ on X_i, w, Y_j, respectively, the first torus invariant algebra has the presentation

$$B = \mathcal{B}^s = B_0/(E_\nu, F_\nu), \qquad B_0 = O[[X_1, \dots, X_p, w, Y_1, \dots, Y_m]].$$

The weights of E_ν, F_ν are $+1, -1$. A weight space of a completed ring means the series transforming by the specified character on every finite jet. Let T be the closed algebra in $\mathcal{B}$ generated over O by the traces of the universal representation ρ. Write D for its split augmentation ideal and $k_- = \operatorname{length}_O S(\alpha, r)$.

Theorem 40.9 (Trace generators and integral relations)**.** *The remaining torus invariants in B form T. They are topologically generated over O by*

$$U_{ij} = X_i Y_j \quad (1 \le i \le p,\ 1 \le j \le m), \qquad W_{jk} = w Y_j Y_k \quad (1 \le j \le k \le m).$$

Write $W_{jk} = W_{kj}$ when $j > k$. The relations among these monomials in B_0 are generated by

$$U_{ij} U_{ht} = U_{it} U_{hj}, \quad U_{ij} W_{kt} = U_{ik} W_{jt}, \quad W_{ij} W_{kt} = W_{ik} W_{jt},$$

with indices in the ranges of their respective symbols. To obtain T, impose also the neutral equations

$$Y_j E_\nu = 0, \qquad X_i F_\nu = 0, \qquad w Y_j F_\nu = 0.$$

Each left-hand side is expressed in the monomial generators. The images of all U_{ij}, W_{jk} generate D.

Proof. For a neutral monomial assign one of its negative factors to each X and two to each w. Equality of weights uses up every Y. Two monomials in the resulting symbols have the same image precisely when the multiplicities of every X_i, of w, and of each Y_j agree. Match the X-factors by type and match the w-factors in any order. The allocations of the Y's now differ by a permutation of the occupied slots. A transposition exchanges two single slots, a single

and a double slot, or slots of two double factors. The displayed identities give these exchanges, with symmetry handling an exchange in one double factor. Grouping polynomial terms with the same image shows that these binomials generate the whole polynomial kernel over O. The identities persist on completion. Indeed for a fixed image the number of symbols is the total number of its X's and w's; their degrees as original monomials are two or three. Thus the two filtrations, including powers of ℓ, are cofinal. The finitely generated binomial ideal is closed in the complete Noetherian ring.

Every monomial of weight -1 has a Y_j which leaves a neutral factor when removed. Each monomial of weight $+1$ has either an X_i or a factor wY_j with the same property. Indeed, in the absence of an X_i, the number of Y's is twice the number of w's minus one. Grouping series by a chosen such factor proves the two generation statements also for power series. Project an expression in the homogeneous ideal (E_ν, F_ν) onto weight zero. Its coefficients on the E_ν have weight -1 and those on the F_ν have weight $+1$. This gives exactly the listed additional equations.

It remains to recover these invariant functions from traces. Newton's identities applied to $\operatorname{tr}\rho(h^j)$ for $1 \le j \le 4$ give the coefficients of the characteristic polynomial of $\rho(h)$, since $\ell \ge 5$. Newton iteration from its four distinct residual roots gives their labeled lifts in T: each denominator has unit residue, its inverse belongs to the closed algebra by a geometric series, and the iterations converge. The four coordinate projectors e_{ii} are therefore polynomials in $\rho(h)$ over T. Inserting such projectors between matrices and taking the trace shows that any cyclic product of matrix entries belongs to T. Expanding the projectors expresses it in traces of group elements.

The entries for x_0, P_i have indices 02, and those for Y_j have indices 20. Thus their pairwise products are cycles. Since $v = ab$ uses $01, 12$, its product with Y_j is a cycle as well. For $W_{jk} = bcY_jY_k$ use the multiplier identity

$$\rho(g_j)_{20} = -\mu(g_j)\rho(g_j^{-1})_{31}, \qquad Y_j = \rho(g_j)_{20}.$$

The indices $03, 31, 12, 20$ now form a cycle. All generators belong to T, and every trace is invariant under conjugation. The invariant series converge in the original topology. This proves equality with the closed trace algebra. Evaluating at the split representation kills exactly its ideal of monomials of positive degree in the displayed generators. □

Theorem 40.10 (The Fitting ideal at the split trace)**.** *Let M be the weight -1 space of $B = \mathcal{B}^s$, viewed as a module over T. It is generated by the Y_j. Its zeroth Fitting ideal $I = \operatorname{Fitt}_0^T(M)$ satisfies*

$$DI = 0, \qquad M/DM \simeq K_-, \qquad (I + D)/D = \ell^{k-}O \ \subset\ T/D = O.$$

In particular $C = T/I$ is defined by the module itself and $C/DC \simeq O/\ell^{k-}O$.

Proof. Before imposing any equations in B_0, extract the coefficients of the monomials $Y_1, \dots, Y_m$ from a series of weight -1. This maps the weight space onto O^m. Its kernel consists of the products of this space with the neutral augmentation ideal: subtracting the linear terms, each remaining monomial is Y_j times a nonconstant neutral monomial. Such expressions for series follow by distributing the monomials among the finitely many generators. In the quotient B, the additional relations in this O^m come exactly from the coefficients of Y_j in the F_ν. A multiplier of an F_ν has weight zero and only its scalar constant can contribute. A multiplier of an E_ν has weight -2; every such monomial has at least two Y's and gives no linear term. The resulting rows are those of B_- in Theorem 40.7. Exact projection onto weight -1 commutes with taking the homogeneous quotient, so $M/DM = \operatorname{coker}(B_-^t)$.

The module is finite over the Noetherian ring T, hence finitely presented. Its zeroth Fitting ideal is generated by the maximal minors of any relation matrix on a finite generating set. This operation commutes with base change by tensoring the presentation. Such minors annihilate the module, by applying the adjugate to each square subsystem. The convention for zero generators gives the unit ideal for the zero module. Applying base change to $T \to O$ and using Theorem 40.7 gives the stated image of I. Since I kills every Y_j in B, it kills their products U_{ij} and W_{jk}, which generate D. Taking the quotient by $I + D$ gives the last assertion. □

Corollary 40.11 (Separation over the fraction field and good-prime traces)**.** *There is a surjection*

$$T \twoheadrightarrow O \times_{O/\ell^{k-}O} C.$$

Its kernel has square zero and is killed by ℓ^{k-}. In particular $T[1/\ell] = J \times C[1/\ell]$, and T is finite over O if and only if C is finite over O. At most

$$pm + \frac{m(m+1)}{2}$$

good-prime trace differences $\delta_q = \operatorname{tr}\rho(\mathrm{Frob}_q) - \operatorname{tr} d(\mathrm{Frob}_q)$, with $q \notin \Sigma$, generate D. For such a choice the quotient of C by these differences is $O/\ell^{k-}O$.

Proof. The difference of two representatives of a compatible pair lies in $D + I$; subtracting appropriate elements of these two ideals gives a common lift. The kernel is $D \cap I$, whose square vanishes by $DI = 0$. Lift the scalar ℓ^{k-} to an element of I using Theorem 40.10. Its difference from that scalar is in D. Both terms annihilate $D \cap I$, so ℓ^{k-} does too. Inverting ℓ gives the product. As T is Noetherian and $DI = 0$, the ideal I is finitely generated over $T/D = O$. The sequence with quotient C gives the finiteness equivalence.

The trace algebra is a continuous image of a formal power series ring in the stated number of variables over O. The latter is compact, so its quotient topology agrees with the subspace topology in $\mathcal{B}$, which is Hausdorff. Ideals in T are closed. By Chebotarev in the finite quotients of G_Σ, the ideal of the good-prime differences contains the differences of traces at every group element. In its quotient all traces are scalars. The scalar subring is compact and closed, and the traces topologically generate, so this quotient is O by the split augmentation. Thus these differences generate D. Their images span $D/\mathfrak{m}_T D$, of dimension at most the number of monomials generating D. Take a basis from these images and apply Nakayama's lemma. The final quotient follows from Theorem 40.10. □

40.5 Integral realization of Frobenius traces

Keep the arithmetic data and the indexing of Theorem 40.9; in particular $X_p = v = ab$. Let H be a complete discrete valuation ring with finite residue field and a local coefficient map from O. We allow equal or mixed characteristic. Fix a continuous local O-algebra map $\sigma : T \to H$, and write ν for the valuation on H normalized by $\nu(H^\times) = 0$ and $\nu(\varpi_H) = 1$. Define three minima of valuations of its trace coordinates:

$$\nu_2 = \min_{i<p,\, j} \nu\big(\sigma(U_{ij})\big),$$
$$\nu_3 = \min_{j} \nu\big(\sigma(U_{pj})\big), \qquad \nu_4 = \min_{j,k} \nu\big(\sigma(W_{jk})\big).$$

The value of zero and the minimum of an empty set are $+\infty$.

Theorem 40.12 (The exact ramification index for an integral realization)**.** *Let H'/H be the valuation rings in a finite extension of fraction fields, with ramification index e. There is a*

continuous local $\mathcal{B} \to H'$ inducing σ on traces if and only if

$$e\nu_i \geq i \qquad (i = 2, 3, 4).$$

In particular the least possible index is $\max(1, \lceil 2/\nu_2 \rceil, \lceil 3/\nu_3 \rceil, \lceil 4/\nu_4 \rceil)$*, taking* $1/\infty = 0$*. It is achieved by a totally tamely ramified extension of index at most four.*

Proof. In an integral local realization every chosen off-diagonal parameter lies in the maximal ideal. The coordinates in the three minima are products of respectively two, three and four such entries: they are X_iY_j for $i < p$, abY_j, and bcY_jY_k. Their valuations in H' are therefore at least two, three and four. This proves necessity.

If all coordinates vanish, take the split representation. Otherwise form the matrix over H whose rows are the vectors $(\sigma(U_{ij}))_j$ and $(\sigma(W_{kj}))_j$. All 2-rowed minors vanish by Theorem 40.9, including for one row of each kind by the symmetry of W. Choose a nonzero row and an index j_0 where its entries have minimal valuation. Divide it by its j_0-th entry to obtain $\lambda = (\lambda_j) \in H^m$ with $\lambda_{j_0} = 1$. All rows are multiples of this one over the fraction field, hence each row is its own j_0-th entry times λ. In particular symmetry gives

$$\sigma(U_{ij}) = u_i\lambda_j, \qquad \sigma(W_{jk}) = w_0\lambda_j\lambda_k, \qquad u_i = \sigma(U_{ij_0}), \quad w_0 = \sigma(W_{j_0j_0}).$$

Thus $\nu_2 = \min_{i<p} \nu(u_i)$, $\nu_3 = \nu(u_p)$, and $\nu_4 = \nu(w_0)$.

Take any uniformizer s of H'. In the full global presentation after elimination of the neutral variables set

$$Y_j = s\lambda_j, \quad X_i = u_i/s \ (i < p), \quad b = s, \quad a = u_p/s^2, \quad c = w_0/s^3.$$

The inequalities put each assigned value in the maximal ideal. They give $v = u_p/s$, $w = w_0/s^2$ and all the prescribed trace coordinates. Every relation E_ν evaluates to zero since $Y_{j_0}E_\nu$ is an invariant relation and $Y_{j_0} = s \neq 0$. Some X_i or wY_{j_0} has nonzero value because the trace coordinates are not all zero. Its product with each F_ν is an invariant relation, so F_ν evaluates to zero too. The neutral substitutions converge, giving a continuous local map on all of $\mathcal{B}$. If $u_p = w_0 = 0$, the same verification permits $a = b = c = 0$.

Every finite ν_i is a positive integer. The least integer in the assertion is consequently at most four and is prime to the residual characteristic $\ell \geq 5$. Adjoining a root of $Z^e - \varpi_H$ gives a totally ramified extension of that index by Eisenstein's criterion. This also proves attainability in equal characteristic. □

Theorem 40.13 (The two-plane locus and its Frobenius determinants)**.** *Suppose the trace coordinates of σ are not all zero. If some $\sigma(U_{pj})$ or $\sigma(W_{jk})$ is nonzero, its realization above is absolutely irreducible. If all these coordinates are zero, it has a realization as the direct sum $V_- \oplus V_+$ of two absolutely irreducible planes with*

$$c_\infty|_{V_\pm} = \pm 1, \qquad \det V_- = A^2\Psi, \qquad \det V_+ = \Psi.$$

In characteristic zero take any finite extension $F/\mathbb{Q}$, a place above a rational prime $q \notin \Sigma$ of residue degree f, and an embedding of the eigenvalue field into $\mathbb{C}$. For the four eigenvalues of arithmetic Frobenius on these planes let M_q, m_q be the largest and smallest of their complex absolute values. Then $M_q/m_q \geq q^f$.

Proof. The eigenvalues of h are distinct in the fraction field because they are distinct in the residue field. Any invariant subspace over an algebraic closure is a sum of coordinate eigenlines. A nonzero entry ij forces any such sum containing line j to contain line i. The multiplier

identity supplies also a nonzero entry $3-j, 3-i$ at the inverse group element. If some $\sigma(U_{pj}) \neq 0$, entries $01, 12, 20$ all occur. They connect the first three lines in both directions, and their reflections connect the fourth in both directions. If some $\sigma(W_{jk}) \neq 0$, entries $03, 12, 20$ occur, and reflection provides 31. Their cycle visits all four lines. Neither set of edges permits a proper nonempty invariant subset.

In the remaining case set $a = b = c = 0$ as in the proof of Theorem 40.12. Every entry of nonzero first torus weight vanishes. The two planes with indices $\{0, 2\}$ and $\{1, 3\}$ are therefore invariant. A nonzero remaining $\sigma(U_{ij})$ gives entries $02, 20$ and their reflections $13, 31$, proving absolute irreducibility on each plane. Complex conjugation reduces to the stated scalars on these planes. Its idempotent for the opposite sign reduces to zero and hence is zero, since 2 is a unit. At ℓ the ordinary flag gives the determinant inertia characters $d_0 d_2, d_1 d_3$. At $q \neq \ell$ the inertia quotients have order prime to ℓ; a determinant ratio reducing to one is trivial on these groups since the principal congruence units form a pro-ℓ group. The two ratios to the displayed characters are therefore everywhere unramified global characters, hence trivial by class reciprocity for $\mathbb{Q}$.

The determinants at the indicated Frobenius have absolute values $q^{f(r+2)}$ and q^{fr}. Indeed each finite character value has absolute value one, and $\kappa_\ell(\mathrm{Frob}) = q^f$. On V_- the larger absolute value is at least $q^{f(r+2)/2}$. On V_+ the smaller is at most $q^{fr/2}$. Dividing proves the bound. In particular, any Frobenius estimate giving relative spread strictly less than q^f at even one such place excludes this two-plane realization. □

40.6 Strict length and Frobenius multiplicities

Fix the period field, character, weight and coefficient prime of Section 40.3. Use $T, D, I, C = T/I$ of Theorem 40.10. Let $\overline{J}$ be an algebraic closure of J, with $v_\ell(\ell) = 1$. When T is finite over O, let σ range over the geometric points of $C[1/\ell]$, and set

$$n_\sigma = \dim_{\overline{J}} (C \otimes_O \overline{J})_\sigma .$$

Here the subscript denotes the local Artin factor. This convention counts the several embeddings of each residue field separately.

Theorem 40.14 (A strict-length bound with the generic multiplicities)**.** *If T is finite over O and $d \in D$ is nonzero at all the σ, then*

$$\operatorname{length}_O S(\alpha, r) \;\leq\; 2 \sum_\sigma n_\sigma v_\ell(\sigma(d)).$$

Such a d exists in the integral span of the good-prime trace differences. The assertion includes finite trace algebras with ℓ-torsion and with nonreduced generic fiber.

Proof. Write $m = a_-$, $p = a_+ + 2$. When $m = 0$, the strict group vanishes by Theorem 40.7. Assume $m \geq 1$. Take the direct sum A of the integer weight spaces of the ambient power series ring $B_0 = O[[X_i, w, Y_j]]$ from Section 40.4. Its degree-zero ring A_0 is the complete monomial algebra in U_{ij}, W_{jk}. Each weight space is finite over A_0. Indeed, remove factors $X_i Y_j$ and $w Y_j Y_k$ from any monomial. What remains is purely positive, purely negative, a scalar, or $w^b Y_j$ with $b > 0$. There are only finitely many such remainders in a fixed weight. Grouping the corresponding terms of a series proves the claim. In particular A is generated over its Noetherian neutral ring by the X_i, w, Y_j. Taking the quotient by E_ν, F_ν in A gives exactly the weight spaces of the actual completed solution ring. To check this for one weight, project the coefficients of an expression in the finitely many homogeneous relations to the required weights. Its neutral quotient is T.

In the notation of Sections 24.3 and 24.5, the numbers of negative variables of weights $-1, -2$ are $m, 0$. The numbers of positive variables of weights $1, 2$ are $p, 1$; the positive equation counts are $p-1, 0$, and there are $m+1$ negative equations of weight -1. There are no neutral variables or equations in the formal chart. The chart map is coefficient extension into the same power series ring over J. Its cotangent is freely generated by the listed variables: all monomials of ordinary degree at least two in one weight can be grouped by two of their factors to put the corresponding series in the square of the augmentation ideal. Thus the negative cotangent over O, modulo the linear parts of the negative equations, is K_- of Theorem 40.7. It has finite length k_-. The rational negative spanning condition follows. The split point is the unique geometric fixed point, since killing all nonzero weights kills all the U_{ij}, W_{jk} and leaves the scalars.

We give the change to the multiplicity calculation in Theorem 24.8. A mixed point has both a nonzero Y_j and a nonzero X_i or w, so some neutral monomial $g = U_{ij}$ or W_{jj} is nonzero. Conversely, on a principal open defined by any U_{ij} or W_{jk} we can write $g = fY_j$ with f of weight 1. Inverting g in the graded solution algebra makes Y_j invertible. Every homogeneous element there is a neutral element times a power of Y_j, and the grading makes a sum of distinct powers direct. Consequently this localized algebra is

$$T[1/g][Y_j, Y_j^{-1}].$$

Over the generic fiber the opens cover precisely the nonsplit factor $C[1/\ell]$, by Corollary 40.11. There is one scaling orbit over each σ; the localized weight pieces at that geometric invariant character all have dimension n_σ. These statements can first be checked over a finite extension splitting the finite generic algebra.

Lift d to the ambient neutral ring with split value zero. In the construction of Lemma 24.4, the modules in large multidegrees are finite over O and have geometric generic support only at mixed characters. Its proof uses finiteness of the neutral quotient, uniqueness of the fixed point and rational spanning on the negative side, all verified above. For completeness, at the split character a relevant prime in the construction would give a purely negative solution with some $Y_j \neq 0$. Setting all positive variables to zero leaves $B_-Y = 0$, and full column rank over J excludes it. At the other characters a product of opposite weights is invertible, exactly the localization just described.

At such a character the upper estimate of Theorem 24.5 uses the neutral piece of the equations with the $m+1$ negative and $p-1$ positive weight-one equations squared. Before squaring, every localized weight piece has dimension n_σ. Squaring one equation h of weight b at most doubles a common bound for those dimensions. In fact for the ring L with the other current equations imposed the sequence

$$(L/(h))_{j-b} \xrightarrow{h} (L/(h^2))_j \longrightarrow (L/(h))_j \longrightarrow 0$$

is exact and remains so on localization. The resulting upper bound for the local dimension in each sufficiently large multidegree is $2^{m+p} n_\sigma$. On a finite O-module the difference of cokernel and kernel lengths of multiplication by d equals its determinant valuation on the torsion-free quotient, since the two lengths cancel on its torsion. On the geometric local factors of this quotient the valuation is the sum of the dimensions times $v_\ell(\sigma(d))$. These valuations are nonnegative by integrality. This gives the upper bound $2^{m+p} \sum_\sigma n_\sigma v_\ell(\sigma(d))$ for the same difference used in Theorem 24.8.

For the lower bound the integral chart and the equations just checked are exactly those in its proof. More explicitly, take Smith normal form of B_- with diagonal entries $\ell^{c_1}, \dots, \ell^{c_m}$, so $\sum c_i = k_-$. In the three graded modules of that proof the factors stripped from the corresponding squared equations contribute $2 \sum c_i$. The remaining finite difference of the monomial

count is $2^{m-1+(p-1)}$, by the calculation at the end of Section 24.5 with no negative variable of weight -2 and one positive variable of weight 2. Thus the lower bound is $2^{m+p-1}k_-$. The lower calculation requires only the chart, the linear terms, and finite length after imposing d on the large pieces. The preceding upper calculation gave this last property even at a multiple point. Comparing and dividing gives the claimed inequality.

Finally the finitely many good-prime differences in Corollary 40.11 generate D, and $DC[1/\ell]$ is the unit ideal. At every σ at least one difference is nonzero. Use these differences as the coefficients in order of a polynomial in a new variable. Each of the finitely many specialized polynomials has only finitely many roots, so some integer gives a nonzero value at all the points. The empty set of points allows $d = 0$; in the lower calculation one can use the ambient lift adjusted by zF_i as in the proof of Theorem 24.8. □

Corollary 40.15 (A global representation forced by a strict class)**.** *If $S(\alpha, r) \neq 0$, at least one of the following global ordinary representations exists, with the conductor and multiplier conditions of Section 40.3:*

1. *a representation over the integers of a finite extension of J with nonsplit trace;*
2. *a representation over the integers of a finite extension of $k((t))$ with a good-prime Frobenius trace transcendental over k.*

Each has one of the two representation types in Theorem 40.13.

Proof. If T is finite, absence of any geometric point of $C[1/\ell]$ would give zero on the right in Theorem 40.14. Thus some such point exists. It takes values in the integers of a finite extension of J. Its specialization map is local and continuous, since the maximal ideal of the finite local algebra $T/\ell T$ is nilpotent. It cannot kill D, since C/DC is torsion. Apply Theorem 40.12.

If T is infinite over O, its special fiber has positive dimension. Indeed, a dimension-zero special fiber is Artinian, so a power of the maximal ideal lies in ℓT. The two adic topologies then agree, and lifting a finite basis modulo ℓ by successive approximation makes T finite over O. Take a prime quotient of the special fiber of dimension one. It is a complete local domain over k. For any nonzero parameter t in its maximal ideal the same approximation applied modulo t makes it finite over $k[[t]]$; the map from this discrete valuation ring is injective. Inclusion into the valuation ring of its fraction field gives a local map from T to the integers of a finite extension of $k((t))$. The image of D is nonzero because the quotient by D of the special fiber is k. Some good-prime difference therefore has nonzero image of positive valuation. This element is transcendental over the finite field, since each nonzero element algebraic over k is a root of unity. The split trace is constant in k, so the corresponding Frobenius trace is also transcendental. Apply Theorem 40.12 in equal characteristic and then Theorem 40.13. □

Corollary 40.16 (The contribution at a common Frobenius prime)**.** *Use the common prime and multisets of Theorem 38.16. At any triple (ℓ, r, α) under consideration suppose T is finite and the multiset of nonsplit Frobenius traces, with the n_σ multiplicities and the chosen coefficient embedding, occurs in $\mathcal{C}_r$. Then $\operatorname{length}_O S(\alpha, r) \leq 2v_\ell(D_{\mathrm{Sp}})$. For all such triples on one E_x their total scalar length, summed over geometric characters and weighted by $\log \ell$, is at most $2n_* t_* \log D_{\mathrm{Sp}}$.*

Proof. The size bound on every member of $\mathcal{C}_r$ in Theorem 38.16 separates it from the split trace. Apply Theorem 40.14 to the trace difference at that same prime. The integer product in Theorem 38.16 bounds the sum of its valuations with the indicated multiplicities, giving the first inequality. At each ℓ there are at most $n_* t_*$ character and weight pairs. Summing and

using $\sum_\ell v_\ell(D_{\mathrm{Sp}})\log\ell = \log D_{\mathrm{Sp}}$ proves the second. Establishing finiteness and the asserted occurrence of traces throughout the interval still requires arithmetic information about the global representations. □

40.7 The two marks in the symplectic trace algebra

On the fields E_x and intervals of Section 6.6, take the four characters $d = (A\Psi, \Psi, A, 1)$, multiplier $\mu = A\Psi$, and the auxiliary character ξ of Lemma 38.6. Let $r \geq 2$, $\ell \geq 5$, $\ell \nmid n_* N_*$, and $\ell - 1 \mid 2r$. In the first theorem it suffices that $\bar{\Psi} \neq 1$ globally. The coefficient ring O is the unramified integer ring containing the finite character values. Use any complete Noetherian local O-algebra R with the same finite residue field and a continuous matrix lift $\rho : G_{\mathbb{Q}} \to \mathrm{GSp}_4(R)$ of $\bar{d}$ with multiplier μ. Write u^* for the adjoint of a matrix under the fixed alternating pairing; it sends $\rho(g)$ to $\mu(g)\rho(g^{-1})$.

Choose an arithmetic Frobenius lift $s \in D_\ell$ with $\kappa_\ell(s) = 1$, and take $g_0 \in G_{\mathbb{Q}}$ with $(\bar{A}(g_0) - 1)(\bar{\Psi}(g_0) - 1) \neq 0$. Such a g_0 exists because two proper subgroups cannot cover a group. By the choice of ξ, $\bar{A}(s)$ is outside the subgroup generated by $\bar{\Psi}(s)$. The residual eigenvalues of s at positions $0, 2$ therefore form a set disjoint from those at $1, 3$; repetitions within each set are allowed. Define

$$\phi = \lim_{j\to\infty} s^{\ell^{j!}}, \qquad H = \rho(g_0) + \mu(g_0)\rho(g_0^{-1}).$$

The limit exists in the profinite cyclic closure of s. On its finite quotients the exponents are eventually zero on the ℓ-part and one modulo the prime-to-ℓ part. The marking in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/DecompSigmaOrd.lean`, line 140 requires that $\rho(\phi)$ preserve the two coordinate planes $02, 13$ and that $H_{02} = H_{20} = 0$.

Theorem 40.17 (Neutral entries and traces with the two marks)**.** *There is a symplectic change of frame congruent to the identity that gives the indicated marking. Any two resulting marked bases differ by diagonal factors with product one on each opposite pair. For a marked lift let $T_0 \subset R$ be the O-algebra generated by the traces of $\rho(g)$. Let $M_0 \subset R$ be the O-algebra generated by the products of matrix entries of total weight zero under conjugation by* $\mathrm{diag}(st, t, s, 1)$*, where s, t here are independent torus variables. There are $z \in M_0$ and $\Delta = z^2 \in T_0$, both units of R, such that*

$$M_0[1/\Delta] = T_0[1/\Delta, z], \qquad \overline{M_0} = \overline{T_0}.$$

Closures use the maximal-ideal topology of R. In particular, M_0 is finite as an O-module exactly when $\overline{T_0}$ is. In this case $M_0 = \overline{T_0}$.

Proof. Let m be the order of the residual diagonal matrix of s. It is prime to ℓ. The congruence kernel in every Artinian quotient of R is an ℓ-group, by its filtration by powers of the maximal ideal. Hence $\rho(\phi)^m = 1$, and $\rho(\phi)$ reduces to $\bar{d}(s)$. The m-th roots occurring in the residue field have Teichmüller lifts in O; here m divides the order of the multiplicative group of that field. Interpolating at these roots, with unit differences, gives a projector P over O as a polynomial in $\rho(\phi)$: it takes value one on the two eigenvalues indexed by $0, 2$, and zero on the others. Unoccupied eigenspaces are zero by Nakayama's lemma. The two summands $U = PR^4$, $U' = (1 - P)R^4$ are free of rank two. They are isotropic and dual to each other. Indeed two eigenspaces can pair only when their eigenvalues multiply to $\mu(\phi)$; the paired eigenvalues lie in opposite sets, as one checks after reduction. Thus $P^* = 1 - P$.

Now $H = H^*$. The endomorphism $H_U = PHP$ on U has distinct residual eigenvalues $\bar{A}(g_0)\bar{\Psi}(g_0)+1$ and $\bar{A}(g_0)+\bar{\Psi}(g_0)$. Hensel factorization of its quadratic characteristic polynomial gives two unique rank-one eigensummands labeled by $0, 2$. The adjoint projectors give the dual eigensummands on U' labeled by $3, 1$. Choose generators lifting the residual basis in U and

the paired dual generators in U'. This is the desired symplectic basis. In any marked basis the two planes must be U, U', since the projector is a polynomial and the two residual spectra are disjoint. The two off-diagonal equations then select exactly the eigensummands of H_U. This proves the uniqueness assertion about bases.

Write s'_0, s'_2 for the H_U-eigenvalues and put $z = s'_0 - s'_2$. The symbols s'_i in this calculation are scalars. In the marked frame their difference is a difference of diagonal entries of H, so belongs to M_0. Since the polynomial P has coefficients in O, $\operatorname{tr} H_U, \operatorname{tr}(H_U^2)$ belong to T_0: expand the matrix products in elements of the group. Here the full trace agrees with the trace on U because PHP vanishes on U'. Therefore

$$\Delta = 2\operatorname{tr}(H_U^2) - (\operatorname{tr} H_U)^2 = z^2 \in T_0.$$

Its residue is nonzero. The two coordinate projectors inside P are $(H_U - s'_2 P)/z$ and its complement in P, with $s'_2 = (\operatorname{tr} H_U - z)/2$. Taking adjoints gives the other two. Since $\rho(g)^* = \mu(g)\rho(g^{-1})$, the four projectors belong to the span of the matrices $\rho(g)$ over $T_0[1/\Delta, z]$. Tracing products with these projectors inserted shows that the product of the entries on any directed cycle belongs to this scalar ring.

For a neutral product regard an entry ij as an arrow from j to i. If δ_i is the number of incoming arrows minus the number of outgoing arrows at i, neutrality and $\sum_i \delta_i = 0$ give $(\delta_0, \delta_1, \delta_2, \delta_3) = (d, -d, -d, d)$. The multiplier identity permits replacing an arrow $j \to i$ by $3 - i \to 3 - j$ for the inverse group element, at the cost of a signed unit in O. For $d > 0$ there are at least $2d$ arrows from $\{1, 2\}$ to $\{0, 3\}$. Replacing d of them reduces every imbalance to zero; one replacement subtracts one at both $0, 3$ and adds one at both $1, 2$. For $d < 0$ use $-d$ arrows in the opposite direction. The resulting finite directed graph is a union of directed cycles, by following arrows until returning to a vertex and then removing the cycle. This gives the inclusion of M_0 in $T_0[1/\Delta, z]$. Diagonal entries have weight zero, so $T_0 \subset M_0$, proving the localized equality.

A unit of R in $\overline{T_0}$ has its inverse there by a geometric series after lifting its residue to O. Starting with a lift of the residue of z, Newton iteration for the square root of Δ in the closed ring gives z: the two candidate squares agree and the sum of the two roots with this residue is a unit. This proves the equality of closures. If M_0 is finite over O it is compact, hence closed in R. Conversely M_0 is a submodule of $\overline{T_0}$ and O is Noetherian. This proves the finiteness assertion. □

Proposition 40.18 (The local component in the two ordinary markings)**.** *Impose in addition $\bar{\Psi}|_{D_\ell} \neq 1$. Choose the separating Frobenius h of Lemma 39.1. On a framed local lift with fixed multiplier, the following conditions are equivalent over every coefficient ring R as above.*

1. *The lift has the ordered ordinary flag of Lemma 39.1, with graded inertia characters d_i.*
2. *The lift is a specialization of a lift over a complete local O-domain that also specializes to the split lift $d|_{D_\ell}$ over O. The domain lift satisfies the ordinary polynomial equations at the marked inertia element of Theorem 40.4.*

These are the local component maps in `Development/Automorphic Arithmetic/Automorphic Varieties/Symplectic GSp 4/Decomp Sigma Ord.lean`*, line 25 and its coefficient-prime condition on line 126.*

Proof. For the first implication choose a symplectic frame reducing to the given residual frame in which the separating h is diagonal. The flag consists of the successive labeled eigenlines. Theorem 39.2 realizes the normalized matrices by specializing $O[[u_1, u_2, a, x, b, c]]$, with split

specialization at the origin. To allow the inverse frame change, adjoin ten formal variables for X in the symplectic Lie algebra. The matrix $(1+X)(1-X)^{-1}$ is symplectic and parametrizes every symplectic change C congruent to one: $X=(C-1)(C+1)^{-1}$ is skew adjoint and has entries in the maximal ideal. This uses invertibility of two. Conjugate the universal local lift by this matrix. The resulting ring is a power series domain through the split point. The ordinary flag gives the polynomial equations over this domain by the forward implication in Theorem 40.4, equivalently by Cayley–Hamilton on its successive submodules and quotients.

Conversely let B be such a domain and K its fraction field. The split specialization $B\to O$ is the identity on O, so K has characteristic zero. At the specified inertia element the four values d_i are distinct in K: its principal cyclotomic value has infinite order and the exponents $r+1, r, 1, 0$ are distinct. The polynomial equations therefore give the ordered stable flag over K by ordinary spectral interpolation, with the stated inertia actions. Independently, $\rho_B(h)$ has four distinct residual roots, so its four labeled spectral projectors are integral over B and belong to $\mathrm{End}_B(B^4)$ by Hensel factorization. Every step of the flag over K is a sum of some of these eigenlines. For the resulting permutation, its stability equations, and its inertia equations on each graded line, are matrix identities in these integral projectors. They hold over B because $B\hookrightarrow K$. Specialize to the split lift over O, then pass to $\mathrm{Frac}\,O$. Its four distinct labeled inertia eigenvalues force the permutation to be the original one. The equations over B thus give the required ordered direct summands after specialization to R, including when R has nilpotents. □

Corollary 40.19 (The finiteness condition for the strict-length inequality)**.** *For the triples of Proposition 40.18 let T be the trace ring of Theorem 40.14. Its finiteness over O is equivalent to the following arithmetic assertion for the same characters. For every framed global lift with multiplier μ, the two marks above, the split local component at ℓ, and the given inertia kernels away from ℓ, M_0 is finite over O.*

Proof. At a finite $q\neq\ell$ the residual diagonal inertia image has prime-to-ℓ order since $\ell\nmid n_*N_*$ and ξ has two-power order. Its kernel is the imposed inertia kernel in both descriptions; no tame ℓ-part of α is present. Given a lift in the two marks, diagonalize $\rho(h)$ by a symplectic frame reducing to the identity. Proposition 40.18 identifies the local condition. The resulting lift factors through the universal ring of Theorem 40.6. If T is finite over O, its image is compact and contains every trace. Theorem 40.17 proves the stated finiteness for M_0. Conversely take the universal lift in the h-diagonal marking and apply the basis construction of Theorem 40.17 over its own coefficient ring. It satisfies the two marks and the split local component. Conjugation preserves all of its traces, so the asserted finiteness for this one lift makes T finite. □

The arithmetic assertion in the corollary is the finiteness clause spelled out in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/DecompSigmaOrd.lean`, line 172 for this datum. The conductor choices in the source's individual cases are specified separately in `Development/AutomorphicArithmetic/AutomorphicVarieties/AutomorphicCases/SymplecticGSp4.lean`, line 15. The equivalence proves the change of marking for a fixed datum. An arithmetic bound on its trace algebra and its Frobenius multiplicities is still required in the interval argument.

40.8 Infinitesimal traces at the ordinary symplectic points

Fix $E_x,\alpha,r,\ell,O,J,\mathcal{B},T$ as in Section 40.3. In particular $\ell\geq 5$, $\ell\nmid n_*N_*$, $\ell-1\mid 2r$, and the four characters are separated at the element $h\in D_\ell$. Assume T is finite over O. Let σ be a geometric point of the generic fiber with an absolutely irreducible rank-four realization in

Theorem 40.13. Take a finite extension L/J that splits the generic trace algebra and contains the coefficients of an integral realization of σ. Such an extension exists by Theorem 40.12. Write V for its representation space over L and $\mathfrak{s} = \{X \in \mathrm{End}_L(V) : X^* = -X\}$, with Galois acting by conjugation. Write $\mathfrak{m}_\sigma$ for the maximal ideal of the local factor of $T \otimes_O L$ at σ.

Lemma 40.20 (Lifting a tangent of the trace algebra)**.** *Every L-algebra map from the local factor to $L[\epsilon]/(\epsilon^2)$ with reduction σ is the trace of a continuous symplectic-similitude lift of V with the original multiplier. The lift preserves the ordered local flag, its graded inertia characters, and the inertia kernels at the other primes. After fixing the integral realization, one can choose the lift so that its first derivative depends linearly on the tangent vector.*

Proof. Inside $\mathcal{B}$ take the T-algebra A_0' generated by the entries of all universal matrices. Each generator is homogeneous for the two torus variables in Theorem 40.6. Finite sums of its weight spaces are direct. Indeed after separating the Laurent characters in the torus action modulo each power of the maximal ideal, a relation between distinct weights is zero in each weight at every depth. The piece of weight zero is T, by Theorem 40.9.

Draw an arrow $j \to i$ if a matrix entry in row i, column j has nonzero value at the chosen realization of V. If a vertex cannot reach another, its reachable vertices span a nonzero proper invariant subspace. Absolute irreducibility therefore makes the graph strongly connected. The product of entries along a path has weight equal to the difference of the endpoint weights and is nonzero at the realization. The four vertex weights are $(1,1),(0,1),(1,0),(0,0)$. Choose path products x_+, x_-, y_+, y_- of respective weights $(1,0),(-1,0),(0,1),(0,-1)$. Their product $b \in T$ satisfies $\sigma(b) \neq 0$. All four become units on inverting b. Multiplication by suitable powers of x_+ and y_+ then makes every homogeneous element have weight zero. A fraction of weight zero still has numerator of weight zero as b is neutral. Directness of the grading gives an isomorphism

$$A_0'[1/b] = T[1/b][x_+^{\pm 1}, y_+^{\pm 1}].$$

Extend the given map on T through this Laurent algebra by sending x_+, y_+ to their constant values in the chosen realization. Reduction agrees with that realization on the whole algebra. Applying the extension to each universal entry gives the lift. Multiplication, the pairing, zeros below the local flag, its diagonal inertia actions, and triviality on the specified inertia kernels are identities between these entries. They hold after applying the map. To check continuity, choose for each entry weight a fixed product of the four path elements of opposite weight. It has invertible value in the dual numbers. Multiplying the entry by this element gives a continuous T-valued function of the group element. Indeed T is compact, so its topology as a finite O-module agrees with its subspace topology in the Hausdorff ring $\mathcal{B}$. The map on this finite O-module is continuous. Dividing by the fixed invertible value proves continuity of each lifted entry. Holding the two Laurent variables constant extends every derivation linearly in the original derivation, which proves the last assertion. □

Proposition 40.21 (The finite local condition at the coefficient prime)**.** *At an integral point of the specified global chart, the ordinary representation and every continuous first-order lift with the same ordered flag and graded inertia characters are crystalline over $K = \mathbb{Q}_\ell(\mu_\ell)$. For a first-order lift, crystallinity concerns its underlying eight-dimensional L-representation. Its self-extension class at ℓ belongs to $H^1_f(\mathbb{Q}_\ell, \mathrm{End}_L V)$.*

Proof. On D_ℓ the diagonal characters of the point are $d_i u_i$, where $d = (A\Psi, \Psi, A, 1)$ and the u_i are unramified characters congruent to one. Those of a first-order lift differ further by unramified characters in $1 + \epsilon L$. Write $w = (r+1, r, 1, 0)$ for the successive cyclotomic exponents. Over K the tame character ω_ℓ is trivial. Both α and ξ are unramified at ℓ. Thus on

each graded part the representation over K is $\kappa_\ell^{w_i}$ times an unramified representation. In the first-order case the unramified part has two identical character factors. Continuous unramified representations are crystalline of weight zero, so all graded parts are crystalline.

Let $s \in D_\ell$ be the arithmetic Frobenius with $\kappa_\ell(s) = 1$ used in Section 40.7. It lies in G_K and induces Frobenius on the residue field, since $K/\mathbb{Q}_\ell$ is totally ramified. Whenever $i < j$ and $w_i - w_j = 1$, the ratio of the unramified character factors on the two graded parts is nontrivial. For the pairs 01 and 23 its value at s reduces to $\bar{\xi}(s)$. For the remaining possible pair 12, which occurs when $r = 2$, it reduces to $\bar{\Psi}(s)/\bar{\xi}(s)$. Neither residue equals one because $\bar{\xi}(s) \notin \langle \bar{\Psi}(s) \rangle$.

Suppose an initial part P of the flag is crystalline over K, and let Q be the next graded part. Set $W = \mathrm{Hom}_L(Q, P)$. This is crystalline, and all its cyclotomic exponents are positive. In the convention where κ_ℓ has Hodge–Tate weight -1, this gives $\mathrm{Fil}^0 D_{\mathrm{dR}}(W) = 0$. Also $H^0(K, W^\vee(1)) = 0$. Indeed a character factor of the dual twist coming from a difference greater than one is nontrivial on an open subgroup of inertia. At a difference equal to one, nontriviality follows from the Frobenius calculation. The finite filtrations on P, Q, refined by the two scalar factors on a dual-number block, give a filtration on $W^\vee(1)$. Vanishing of invariants on every factor implies vanishing on the whole module by left exactness.

The Bloch–Kato dimension formula and the local Euler characteristic formula with Tate duality now give

$$\dim_L H^1_f(K, W) = [K : \mathbb{Q}_\ell] \dim_L W + \dim_L H^0(K, W) = \dim_L H^1(K, W).$$

Thus every extension in the flag has finite local class. For crystalline end terms this says precisely that the middle term is crystalline. Indeed splitting the extension after tensoring with B_{cris} gives the required crystalline periods by the comparison isomorphisms for the two end terms. Induction proves the assertion for both the point and its first-order lift. [13, 1]

The exact self-extension over K therefore splits over B_{cris}. One may take a linear splitting on the crystalline period spaces and extend scalars by their comparison isomorphisms. The class over $\mathbb{Q}_\ell$ with coefficients in $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_\ell} \mathrm{End}_L V$ restricts to zero over K. Corestriction multiplies it by $[K : \mathbb{Q}_\ell]$, which is invertible over L. This gives the asserted finite local condition over $\mathbb{Q}_\ell$. □

Theorem 40.22 (A Selmer bound for the tangent at an irreducible trace)**.** *There is an L-linear injection*

$$\mathrm{Hom}_L(\mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2, L) \hookrightarrow H^1_f(\mathbb{Q}, \mathfrak{s}).$$

Here the finite global group uses the unramified condition away from ℓ and the kernel of scalar extension to B_{cris} at ℓ. If this group vanishes, the multiplicity n_σ in Theorem 40.14 is one.

Proof. Write a lifted matrix as $(1 + \epsilon z(g))\rho(g)$. The group law makes z a continuous one-cocycle, and the fixed multiplier gives $z(g)^* = -z(g)$. At each finite prime $q \neq \ell$ the cocycle is zero on the prescribed kernel of inertia. Its restriction to inertia factors through a finite group and is a coboundary by averaging. At ℓ the class is finite in $\mathrm{End}_L V$ by Proposition 40.21. The equivariant projection $X \mapsto (X - X^*)/2$ is a left inverse to the inclusion of $\mathfrak{s}$. Applying it after scalar extension to B_{cris} gives the same local condition in $\mathfrak{s}$.

These operations and the choice in Lemma 40.20 are linear on derivatives. If the cohomology class is zero, conjugation by a constant first-order change of basis makes all the matrices equal to the original ones. Thus every trace has zero derivative. The O-algebra generated by the traces is an O-submodule of the finite module T, hence finite and compact. Its closure is T, so the traces generate T algebraically. The derivative therefore vanishes on all of T, proving injectivity. If the right side is zero, Nakayama's lemma applied to the maximal ideal of the

local Artin algebra gives $\mathfrak{m}_\sigma = 0$. This factor is L. At a nonsplit point an element of the augmentation ideal D is invertible. The identity $D\,\mathrm{Fitt}_0^T(M) = 0$ in Theorem 40.10 identifies the local factor of C with this same field. Its multiplicity is one. □

40.9 Irreducibility and multiplicity on the period fields

Take the triples of Section 40.3. In mixed characteristic let V be the absolutely irreducible four-dimensional representation of Theorem 40.13, or either of the irreducible planes in its second case. We work over a finite extension of J over which the matrices admit the integral model of that theorem. All statements about invariant subspaces allow extension to the algebraic closure.

Proposition 40.23 (Irreducibility over every finite extension)**.** *For every finite extension $F/\mathbb{Q}$, the restriction of V to G_F is absolutely irreducible.*

Proof. Write $n = \dim V$. There are two local elements with different roles. At h the n residual diagonal characters have pairwise distinct values $b_1, \dots, b_n$. Choose an inertia element s_0 in the kernels of the finite characters with $\kappa_\ell(s_0)$ of infinite order. Such elements exist because inertia maps onto the principal cyclotomic units after passage to a subgroup of finite index. The eigenvalues of s_0 are the powers with the prescribed exponents from $r+1, r, 1, 0$. Every positive power of s_0 therefore has simple spectrum. The ordinary flag supplies a D_ℓ-stable line, also for either plane by intersection with the coordinate flag.

Suppose the global group permutes a direct-sum decomposition into $m > 1$ nonzero subspaces. It acts transitively on them by irreducibility. These finitely many subspaces are defined over a finite extension of the coefficient field. Their orbit in its Grassmannian is finite and Hausdorff, so the permutation action is continuous. The kernel of reduction on the compact matrix image is pro-ℓ. It acts trivially on the summands because $\ell \geq 5$ and $m \leq 4$. Thus the permutation image is a quotient of the residual diagonal group and is abelian. A faithful transitive abelian permutation group is regular, since a point stabilizer fixes every translate of that point.

Take a power of s_0 fixing the summands. Each summand is a sum of its eigenlines by simple spectrum. The D_ℓ-stable line is one such eigenline, so lies in a unique summand. Consequently D_ℓ fixes that summand and acts trivially on all summands by regularity. Choose g inducing a nonidentity permutation. Every gh^j permutes the summands without a fixed one, so its trace is zero. On reduction, writing a_i for the value at g of the i-th residual character, we obtain

$$\sum_{i=1}^{n} a_i b_i^j = 0 \qquad (0 \leq j < n).$$

The Vandermonde determinant is a unit since the b_i are distinct. This forces $a_i = 0$ for all i, whereas character values are units. This rules out any permuted decomposition.

For an open normal subgroup U of $G_\mathbb{Q}$, choose a simple U-submodule of V. Its global conjugates are simple U-submodules and span V. Thus $V|_U$ is semisimple. Its isotypic components would give a permuted decomposition if there were several. Within its single isotypic component a multiplicity greater than one would repeat every eigenvalue of every element of U. A positive power of s_0 lies there and has simple spectrum. This proves absolute irreducibility on U, and the normal core gives it on any open subgroup. □

Theorem 40.24 (The cyclotomic image at the two colliding residues)**.** *For every finite $F/\mathbb{Q}$, the image of $G_{F(\mu_{\ell^\infty})}$ on V is enormous in the sense of Newton and Thorne. More explicitly, after a finite extension of coefficients there are n^2 elements of this image with distinct eigenvalues on each element, whose matrices span* $\mathrm{End}(V)$.

Proof. Let G be the Zariski closure of the global image over the algebraic closure of the coefficient field, and let G° be its identity component. The intersection of the image of G_F with G° is Zariski dense in G°. Indeed it has finite index in the global image, and finitely many translates of a proper closed subset cannot cover a connected algebraic group. The preimage of G° is open in the profinite group, since algebraic components are closed in the local field topology and there are finitely many. Thus G° acts absolutely irreducibly by Proposition 40.23. Its unipotent radical has a nonzero fixed vector. Normality and irreducibility make all vectors fixed, so this radical is trivial. The connected reductive group G° is the product of its central torus and its connected derived group H. Schur's lemma makes the central torus scalar, hence H acts absolutely irreducibly. A power of s_0 as in the proposition has image in G° with simple spectrum. Removing its scalar central factor gives an element of H with simple spectrum.

Write D for the Zariski closure of the image of the indicated cyclotomic kernel. It contains H. In fact the commutators of elements of G_F lie in the kernel of the cyclotomic character. Apply the commutator morphism to the dense subset just used inside $G^\circ \times G^\circ$. Closedness of D puts all commutators of G° in D, which contains the closed group they generate. Thus D° contains the connected group H. It follows that D° acts irreducibly and that the nonvanishing locus of the characteristic-polynomial discriminant in D° is nonempty. The actual kernel elements in this open locus are Zariski dense there and hence in D°. Burnside's theorem shows that the linear span of D° is the whole matrix algebra, so the actual simple-spectrum elements also span it. Select a basis and extend coefficients once to split its characteristic polynomials.

For any nonzero matrix X, nondegeneracy of the trace pairing gives a selected matrix M with $\operatorname{tr}(MX) \neq 0$. Write $M = \sum_a a P_a$, where P_a is its spectral projector at the eigenvalue a. Some $\operatorname{tr}(P_a X)$ is nonzero. Applying this to a nonzero vector of each simple submodule of the endomorphism space gives precisely the condition in Definition 2.23 of Newton and Thorne. [8] □

Theorem 40.25 (Multiplicity one at potentially automorphic rank-four points)**.** *Suppose T is finite over O. At a rank-four point σ, suppose that on some totally real or CM number field F its representation V is attached to a regular algebraic cuspidal polarized automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$, in the Galois normalization of Newton and Thorne. Then $n_\sigma = 1$ in the strict-length inequality of Theorem 40.14.*

Proof. Write $F^+ = F$ in the totally real case, and take the maximal totally real subfield otherwise. The multiplier of the automorphic pairing on G_F agrees with $\mu|_{G_F}$. To see this, in the CM case the conjugate of $V|_{G_F}$ is isomorphic to $V|_{G_F}$, since V is defined on $G_{\mathbb{Q}}$. In either case the two pairings make $V|_{G_F}$ isomorphic to its twist by the ratio ν of the multipliers. Determinants give $\nu^4 = 1$. On the open kernel of ν, the intertwiner is scalar by Proposition 40.23. It intertwines on all of G_F, so $\nu = 1$.

Over a totally real F the automorphic pairing is consequently a scalar multiple of the original alternating pairing, by Schur's lemma. The fixed-multiplier Lie algebra $\mathfrak{s}$ is a direct summand of the Lie algebra of its similitude group. Indeed $X \mapsto (X - X^*)/2$ is an equivariant projection onto it. By Theorem 40.24 the cyclotomic image is enormous. Theorem 5.3 of Newton and Thorne makes the finite Selmer group of this similitude Lie algebra over F zero, hence also that of $\mathfrak{s}$.

Over a CM field choose a complex conjugation $c \in G_{F^+}$ and put $C = \rho(c)$. Let J_0 be the matrix of the original alternating pairing. Since $C^2 = 1$ and $\mu(c) = -1$, the matrix $B = J_0 C$ is symmetric. It pairs V with its conjugate over G_F, since

$$\rho(g)^{\mathsf{T}} B\, \rho(cgc^{-1}) = \mu(g) B \qquad (g \in G_F).$$

The equality of multipliers and irreducibility give uniqueness up to scalar. In the polarized adjoint on $\mathrm{End}(V)$ the action of c is $X \mapsto -B^{-1}X^{\mathsf{T}}B$. On $\mathfrak{s}$ this is $X \mapsto CXC^{-1}$, precisely the original action of c. The self-adjoint matrices are stable under this operation and under G_F, giving an invariant complement. Theorem 5.2 of Newton and Thorne and Theorem 40.24 give vanishing of the finite Selmer group of the polarized adjoint over F^+. The direct summand $\mathfrak{s}$ has the same vanishing. Both theorems use the unramified local condition away from ℓ and the finite crystalline-period condition at ℓ. [8]

Restriction takes $H^1_f(\mathbb{Q}, \mathfrak{s})$ into $H^1_f(F^+, \mathfrak{s})$ by functoriality of these conditions. Corestriction composed with restriction is multiplication by $[F^+ : \mathbb{Q}]$, a nonzero scalar in the coefficient field. The global group over $\mathbb{Q}$ is therefore zero. The multiplicity assertion follows from Theorem 40.22. $\square$

Thus a rank-four point equipped with such a cuspidal companion contributes its Frobenius valuation once. Finiteness of the trace algebra, existence of companions throughout the interval, and the treatment of the two-plane points are further inputs to the strict-length bound.

41 References

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