The generalized Riemann hypothesis for Dirichlet L-functions

Volume IX · Cuspidal companions and ordinary Hecke fibers

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Abstract

We construct cuspidal points on auxiliary towers from quadratic induction and symmetric cubes. At the prescribed weights we study their traces, their Hecke fibers and the dual Selmer conditions on these families.

The guide in Volume I lists the sections and page counts for the series.

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47 Cuspidal companions and the two even planes

47.1 Cuspidal companions over one imaginary quadratic field

Fix $K = \mathbb{Q}(i)$ throughout the interval of Section 6.6. Let V be one of the characteristic-zero representations of Proposition 40.23, of dimension $n = 4$ or 2. Fix $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$. We say that a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ matches V if π is regular algebraic and the following equality of multisets holds at all but finitely many places w. Write q_w for the residue cardinality, S_w for the unramified Satake parameters of π_w, and g_w for geometric Frobenius. The equality is

$$\iota(\,\mathrm{Eig}(V(g_w))) = q_w^{(n-1)/2} S_w.$$

This is the Frobenius convention of the source in `Development/AutomorphicArithmetic/AutomorphicVarieties/AutomorphicForms/BlochKatoSelmerVanishes.lean`, line 66. Below we specify when we take the inverse, arithmetic Frobenius.

Proposition 47.1 (Polarization from a single cuspidal match)**.** *A matching π admits a polarization whose Galois multiplier on G_K equals λ. Here $\lambda = \mu|_{G_K}$ in dimension four and $\lambda = (\det V)|_{G_K}$ in dimension two. The Galois representation attached to the polarized pair is isomorphic to $V|_{G_K}$. In dimension four, when T is finite, it follows that $n_\sigma = 1$.*

Proof. Denote the same two multipliers on $G_{\mathbb{Q}}$ by λ_0. They are products of a finite Dirichlet character and a power of κ_ℓ, by the definition of μ and the two determinant identities in Theorem 40.13. In either dimension there is an alternating pairing showing $V^\vee \otimes \lambda_0 \simeq V$. Conjugation by the nontrivial automorphism of $K/\mathbb{Q}$ leaves the isomorphism class of the restriction unchanged, since it extends to $G_{\mathbb{Q}}$.

Take the algebraic Hecke character χ of $\mathbb{Q}$ corresponding to $\lambda_0 \kappa_\ell^{n-1}$ by class field theory in geometric Frobenius normalization. It is a finite character times a norm power. Its base change $\chi_K = \chi \circ N_{K/\mathbb{Q}}$ has uniformizer value $\iota((\lambda_0 \kappa_\ell^{n-1})(g_w))$ at every good place. Since $\kappa_\ell(g_w) = q_w^{-1}$, dualizing the display defining a match gives

$$S_{cw} = \iota(\lambda_0(g_w)) q_w^{1-n} S_w^{-1} = \chi_K(\varpi_w) S_w^{-1}.$$

Here c acts on places by complex conjugation. Strong multiplicity one for cuspidal GL_n gives $\pi^c \simeq \pi^\vee \otimes (\chi_K \circ \det)$. If necessary, multiply χ by the quadratic character of $K/\mathbb{Q}$ to give the required real sign in the definition of a polarized pair. This quadratic character is trivial after base change, so the character on G_K in the pairing is still $\kappa_\ell^{1-n} r_{\ell,\iota}(\chi)|_{G_K} = \lambda$. The existence and unramified compatibility theorem for regular algebraic cuspidal polarized pairs now gives the associated semisimple Galois representation. Its good Frobenius polynomials give the same traces as $V|_{G_K}$. Chebotarev, continuity and the characteristic-zero character criterion give the isomorphism, since the latter is irreducible by Proposition 40.23. Apply Theorem 40.25 in dimension four. [1, 2] □

Theorem 47.2 (The sign obstruction for either even plane)**.** *Neither of the two characteristic-zero planes in Theorem 40.13 has a matching regular algebraic cuspidal representation over K.*

Proof. If such a match existed, Proposition 47.1 would identify the plane over G_K with the irreducible representation attached to a polarized cuspidal pair. Its conjugate-dual pairing with multiplier $(\det V)|_{G_K}$ is symmetric by the sign theorem of Bellaïche and Chenevier, in the polarized formulation of Barnet-Lamb, Gee, Geraghty and Taylor, Theorem 2.1.1(1). Here the lift c of complex conjugation has order two. Choose an alternating determinant matrix J_2 on the plane. The matrix $J_2 V(c)$ is also a conjugate-dual pairing with this very multiplier, since

$$V(g)^{\mathsf{T}} J_2 V(c)\, V(cgc^{-1}) = (\det V(g)) J_2 V(c) \qquad (g \in G_K).$$

But $V(c)$ is either 1 or -1 on each plane by Theorem 40.13. This second pairing is alternating. Irreducibility over K and Schur's lemma make the space of such pairings one-dimensional. A nonzero symmetric form and a nonzero alternating form cannot be proportional in characteristic zero. This proves the assertion. [3, 2] □

Theorem 47.3 (The integral Frobenius eigenvalues at a matching point)**.** *Suppose a rank-four point has a cuspidal match over K. For every rational prime $p \neq \ell$ at which V is unramified, each eigenvalue of its arithmetic Frobenius is an algebraic integer. Under every complex embedding every such eigenvalue has absolute value $p^{(r+1)/2}$. In particular the Frobenius trace u appearing in Theorem 38.16 is an algebraic integer, and every conjugate of u has absolute value at most $4p^{(r+1)/2}$.*

Proof. Choose a place w of K above p of residue degree f. The eigenvalues of arithmetic Frobenius there are the f-th powers of the rational ones, since V is unramified at p. By Proposition 47.1 the restricted representation belongs to the compatible system of a polarized cuspidal pair. At $w \nmid \ell$ the Weil–Deligne parameter of this representation is unramified with zero monodromy. Local-global compatibility identifies its Frobenius semisimplification with the normalized local Langlands parameter of π_w. In particular π_w is unramified. Strict purity for this compatible system implies that the Frobenius eigenvalues are algebraic numbers and all eigenvalues at w have the same complex absolute value under every embedding. This holds also for their inverses in the arithmetic convention. The determinant on V is μ^2, whose value at this arithmetic Frobenius has absolute value $p^{2f(r+1)}$. Taking first fourth roots and then f-th roots gives the stated absolute value at p. [2]

At a coefficient place above any prime $p' \neq p$, the system is unramified at w and the same polynomial is realized on a stable lattice by local-global compatibility. Its arithmetic eigenvalues are therefore integral there. It remains to check coefficient places above p. The p-adic member is crystalline at w since π_w is unramified. Its crystalline Frobenius, with the residue degree included, realizes the geometric Frobenius polynomial. These assertions are Theorem 2.1.1(4) of the same reference. Its Hodge–Tate numbers at all embeddings, with the cyclotomic character of weight -1, are

$$-(r+1), \quad -r, \quad -1, \quad 0.$$

Indeed the ordinary flag at ℓ gives these numbers for the original representation by Proposition 40.21. They are the same at both embeddings of K since the representation comes from $\mathbb{Q}$. The algebraic weight formula of Theorem 2.1.1(3) gives the same multiset for the p-adic member, including after permutation of the embeddings.

Use the dual p-adic member, whose labeled Hodge–Tate numbers are all nonnegative. Its filtered isocrystal is weakly admissible. The induced Hodge filtration on its smallest-slope subisocrystal has only nonnegative jumps, so the Hodge number of this subobject is nonnegative. Weak admissibility puts its Newton number at least as high. Hence its smallest slope is nonnegative, which gives nonnegative valuation for every eigenvalue of its residue-degree Frobenius. These are the arithmetic eigenvalues at w for the original member. One may forget the coefficient action in the slope argument, since this retains all conjugate eigenvalues. Integrality now holds at every finite place. The f-th powers of the rational Frobenius eigenvalues are thus algebraic integers. Each eigenvalue itself is integral by transitivity of integrality. Taking the trace proves the last assertion. □

The source's condition in `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/BaseChangeUnits/IsAutQ.lean`, line 13 asks for a cuspidal match over every

imaginary quadratic field on which the restriction is irreducible. For the conclusions above the single field K suffices throughout the interval. The sign obstruction concerns the existence of a cuspidal match for an individual plane. Eliminating the plane from the deformation space itself requires a further argument.

47.2 A common level for the cuspidal contribution to strict length

Keep $K = \mathbb{Q}(i)$ and the common family E_x of Section 6.6. Use V for the lower endpoint of the interval $[V, V^{3/2}]$, M for the dyadic conductor of Lemma 38.6, and N_*, n_*, t_*, L_*, Y as in Section 38.1. Write ρ_σ for a rank-four point admitting a match in the sense of Section 47.1. For a coefficient prime put $d_\ell = \ell M N_*$. Principal level b on $\mathrm{GL}_4(\mathbb{A}_{K,f})$ means the kernel of reduction modulo b in $\mathrm{GL}_4(\widehat{\mathcal{O}}_K)$.

Proposition 47.4 (The level and weight of any matching system)**.** *Every match π of ρ_σ over K has a nonzero vector of principal level d_ℓ^4. It occurs in cuspidal cohomology with coefficients of highest weights bounded in absolute value by $r + 4$ at both embeddings of K.*

Proof. Fix a finite place w of K over q. First take $q \neq \ell$. The inertia of the global lift at q factors through the finite abelian group given by its residual inertia image. This group has order prime to ℓ, and the conjugation action of Frobenius on it is trivial since the residual global image is diagonal. The multiplicities of the inertia characters of the lift equal those of d. Indeed the character idempotents of this finite group act on the integral lattice, and their ranks can be read after reduction. Thus the semisimplified Weil representation restricts on inertia to characters of conductor dividing MN_*. The monodromy operator vanishes since the inertia image is finite. Under restriction to K_w each local character is composed with the norm. This preserves the bound by the extended ideal. In fact the norm of a unit congruent to one modulo $q^k \mathcal{O}_{K_w}$ is congruent to one modulo $q^k \mathbb{Z}_q$, by taking the determinant of multiplication on the integral lattice.

For $q = \ell$, the ordered local representation is potentially crystalline by Proposition 40.21. The Weil–Deligne functor on potentially semistable representations is exact. On each of the four graded characters the unramified factor and the cyclotomic powers have unramified Weil parameter. The only possible ramified factor is ω_ℓ^{-1}, of conductor dividing ℓ. After restricting to K_w the same norm argument applies. The semisimplified Weil parameter of π_w, up to the unramified norm twist, agrees with this parameter by local-global compatibility up to semisimplification at the coefficient prime. Away from ℓ use Theorem 2.1.1(2) of Barnet-Lamb, Gee, Geraghty and Taylor. Thus at every w the semisimplified inertia parameter has four character factors, each of conductor exponent at most $b_w = v_w(d_\ell)$. When $b_w = 0$, the argument at $q \neq \ell$ also gives zero monodromy on the automorphic parameter and π_w is unramified. [4, 2]

When $b_w \geq 1$, write a_j for the four character conductor exponents. The Artin conductor formula for a Weil–Deligne parameter (R, N) adds $\dim R^{I_w} - \dim(\ker N)^{I_w}$ to the Weil conductor. The first dimension is the number of unramified character factors. Consequently the local conductor exponent of π_w is at most

$$\sum_{a_j > 0} a_j + \#\{j : a_j = 0\} \;\leq\; 4b_w.$$

Here we use compatibility of local Langlands with conductors. The local components of a cuspidal GL_4 representation are generic. Their conductor newvectors are fixed by the subgroup whose last row is that of the identity modulo the local conductor. It contains the principal congruence subgroup at exponent $4b_w$. The tensor product of these local vectors proves the level assertion. [5]

The labeled Hodge–Tate multiset at both embeddings is the one computed in Theorem 47.3. In the algebraic weight formula the four highest-weight entries are obtained by subtracting $3, 2, 1, 0$ from the ordered Hodge–Tate numbers. Taking the dual coefficient system reverses the entries and changes their signs. In either convention the absolute bound is $r+4$. The cohomological characterization of regular algebraic cuspidal representations gives nonzero relative Lie algebra cohomology with this algebraic coefficient system. Cuspidal cohomology injects into the cohomology of the locally symmetric space, with the action of the finite Hecke operators. [6, 7] □

In Section 38.4 choose the common prime p in $(Y + 32, 2(Y + 32))$ to split in K. For all sufficiently large V such a choice is possible by the prime number theorem in the fixed class $1 \bmod 4$. It is prime to $3d_\ell$ for every ℓ in the indicated range. Fix one of its two places w in K.

Theorem 47.5 (One finite list for every matching trace on the interval)**.** *For each $2 \le r \le t_*$ there is a rationally Galois-stable multiset $\mathcal{C}_r$ of algebraic integers, all of whose complex absolute values are at most $4p^{(r+1)/2}$, with*

$$|\mathcal{C}_r| \;\le\; C(r+5)^{20} \sum_{\substack{5 \le \ell \le L_* \\ \ell \text{ prime}}} (3d_\ell^4)^{32}$$

for an absolute constant C. Fix any triple in Section 40.3 with T finite and one coefficient embedding ι. The arithmetic Frobenius traces at p of all the distinct rank-four points having a match under ι, counted once per point, form a submultiset of $\mathcal{C}_r$ under that embedding.

Proof. Use the locally symmetric spaces for GL_4/K modulo the positive real split center at infinity, at principal levels $3d_\ell^4$. Take all dominant pairs of coefficient weights, one for each embedding of K, with entries bounded by $r+4$. The complex algebraic coefficient representation with such a pair of weights descends to K. Indeed the restriction of scalars of GL_4 splits into two copies over K, and each highest-weight representation is defined over the split field. Restrict scalars of this coefficient space to $\mathbb{Q}$ to obtain rational local systems. Their complexifications include every indicated coefficient space and its conjugate. The finite Hecke correspondences act over $\mathbb{Q}$ on the resulting cohomology.

The Borel–Serre compactification gives a finite cell model at the fixed level three. A stabilizer in the symmetric space modulo the positive real center preserves an integral lattice of rank eight. Its determinant is a Gaussian unit, of complex absolute value one, so it belongs to a compact stabilizer before taking that quotient. Its discreteness makes it finite. A finite-order integral matrix congruent to the identity modulo three is the identity. Thus the level-three arithmetic groups act freely. The level-$3d_\ell^4$ spaces are covers of degree at most $(3d_\ell^4)^{32}$, by reduction of sixteen entries in $\mathcal{O}_K$. There are $O((r+5)^8)$ highest-weight pairs. The Weyl dimension formula on the two factors gives rank $O((r+5)^{12})$ for each coefficient system, including restriction of scalars. Multiplying by the number of cells bounds the sum of all cohomology dimensions by the displayed estimate up to an absolute constant. [8]

Give the maximal compact subgroup at w volume one. Let U_w be the Hecke operator of the double coset of $\mathrm{diag}(1, 1, 1, \varpi_w^{-1})$. Use $p^{-3}U_w$ on rational cohomology. In the convention in which the double coset of $\mathrm{diag}(\varpi_w, 1, 1, 1)$ has eigenvalue $p^{3/2}\sum_j \beta_j$ on Satake parameters $\{\beta_j\}$, the inverse coset has eigenvalue $p^{3/2}\sum_j \beta_j^{-1}$. These are the two minuscule formulas of the normalized Satake isomorphism. After division its eigenvalue is $\sum_j (p^{3/2}\beta_j)^{-1}$, the arithmetic Frobenius trace for a match. The choice of a split prime identifies the Frobenius at w with the rational one on this representation.

Take the product of the characteristic polynomials on all the cohomology groups, levels and weights above. It is a rational polynomial. Keep each root which is an algebraic integer and whose every conjugate has absolute value at most $4p^{(r+1)/2}$, with twice its multiplicity. This defines $\mathcal{C}_r$ and proves its size and stability assertions. Proposition 47.4 and Theorem 47.3 put each matching trace among the retained roots.

It remains to account for repeated values. Distinct cuspidal systems supply linearly independent simultaneous Hecke eigenvectors in the direct sum, because strong multiplicity one distinguishes them away from the union of the levels. Thus a value shared by j such systems has algebraic multiplicity at least j before doubling. Two rank-four points with the same matching system have isomorphic restrictions to G_K by Proposition 47.1. Fix an isomorphism on this subgroup. For each $g \in G_{\mathbb{Q}}$ its conjugate by the two actions is another such intertwiner, since G_K is normal. Their ratio is scalar by Proposition 40.23 and Schur's lemma. These scalars form a character of $G_{\mathbb{Q}}/G_K$, which has only two characters. There are therefore at most two global isomorphism classes for the given system. The traces of a global class determine its point of T. The factor two supplies the required upper bound for repeated values. □

Corollary 47.6 (The interval bound under finiteness and cuspidal matching)**.** *On each E_x sum over any set of triples from Section 40.3 with the following property. For each triple T is finite, and for some common embedding for that triple, each nonsplit geometric point is realized by a rank-four representation with a cuspidal match over K. There is one positive integer D_{Sp} for the whole interval, with $\log D_{\mathrm{Sp}} \ll V^{C'}$ for a fixed $C' > 0$, such that*

$$\sum_{(\alpha,r,\ell)} \operatorname{length}_O S(\alpha, r) \log \ell \;\le\; 2n_* t_* \log D_{\mathrm{Sp}}.$$

The same conclusion holds if the matching assumption is imposed instead on every irreducible constituent of the semisimple realizations in Theorem 40.13.

Proof. Theorem 38.16 applies to the lists just constructed. All its size factors, including p, M, N_*, L_*, n_* and t_*, are bounded by fixed powers of V, so its integer has the asserted logarithmic bound. At every nonsplit point the multiplicity is one by Proposition 47.1. The trace difference at the same p is nonzero by Lemma 38.15 or by its triangle inequality using the stronger magnitude bound just proved. The strict-length inequality of Theorem 40.14 now gives a bound of $2v_\ell(D_{\mathrm{Sp}})$ for the individual length. At a fixed ℓ there are at most n_* geometric characters and at most t_* weights. Summing and using the prime factorization of D_{Sp} gives the formula. Under the alternative hypothesis, Theorem 47.2 excludes the two-plane realization at each point, giving the first hypothesis there. □

The corollary constructs the level, spectral list and common integer from field and interval data. Finiteness and existence of the indicated cuspidal matches are still needed to apply it to all cases in Section 6.6.

47.3 The prescribed auxiliary conductors on the period interval

Use the polynomial-conductor intervals of Theorem 12.15. Write F for the conductor of the original character ψ, and o for its order. At $x \in [V, V^{3/2}]$ let

$$E_x = K_\psi C_J(\sqrt{-q}, \sqrt{-d_0}), \qquad N = \operatorname{lcm}(2^{J+2}, F) q d_0.$$

Let $N^\natural$ use the largest J on the interval in this formula, and let $n_* = [E_* : \mathbb{Q}]$ be the degree at that layer. Fix a nontrivial character α of E_x and a weight $2 \le r < t(x)$ with $\alpha(-1) = (-1)^r$. Throughout this subsection impose

$$\ell \ge 5, \qquad \ell \notin \mathcal{B}_\psi, \qquad \ell \nmid n_* N^\natural, \qquad \ell - 1 \mid 2r, \qquad (\bar{\alpha}\bar{\kappa}_\ell^{\,r})|_{D_\ell} \ne 1.$$

Here $\mathcal{B}_\psi$ consists of 3 and the odd primes dividing $oFp_Kh_*p_1$. The fixed integers p_K, h_* are the auxiliary prime and quadratic class number of the source. The integer p_1 is its least prime congruent to one modulo the odd part of o and prime to $2Fp_K$. Dirichlet's theorem gives such a prime. Keep $\Psi = \alpha\kappa_\ell^r$, and let ω_ℓ be the Teichmüller lift of $\bar\kappa_\ell$.

The chosen modulus gives exactly the required field. Indeed, the subgroup of $(\mathbb{Z}/N)^\times$ acting trivially on the four factors is cut out by $\ker\psi$ modulo F, by $\{\pm1\}$ modulo 2^{J+2}, and by the squares modulo each of q, d_0. The last two descriptions use $q, d_0 \equiv 3 \bmod 4$. The fixed field of the intersection is the compositum. The auxiliary primes also satisfy the size condition of the source on this very interval. In the proof of Theorem 12.15 both exceed H^{401W}, while $V^2 \le H^{K_0}$ and $W > K_0 + 2$. Since $t_* = \lfloor V^{1203/2} \rfloor$, they exceed $2t_* + 1$ and every fixed integer F, o, p_K, h_*, p_1 at large V.

Write $\alpha = \alpha_2\alpha'$ for its factorization into the characters at two and at the odd primes. Put $2^{c_2} = \operatorname{cond}(\alpha_2)$, allowing $c_2 = 0$. Then $\operatorname{ord}(\alpha') \mid 2o$. In fact $\mathrm{Gal}(E_x/\mathbb{Q})$ embeds in the product of the four Galois groups. A character of a subgroup of a finite abelian group extends to the group with values in the roots of unity. To see this, adjoin one generator at a time, choosing a root of the prescribed value of its first power in the subgroup. Hence α is a product of a power of ψ, a dyadic character and the two quadratic characters. Taking the odd-prime components proves the divisibility claim. An α of odd order factors through K_ψ, since $[E_x : K_\psi]$ is a power of two. It has conductor dividing F, trivial dyadic part and order dividing the odd part of o. In general $\operatorname{ord}(\alpha) \mid 2^{c_2+1}o$.

Define

$$f_0 = \operatorname{ord}_{\operatorname{lcm}(4,\operatorname{ord}\alpha')}(\ell), \qquad n_0 = v_2(\ell^{f_0} - 1).$$

Here the order is defined since $\operatorname{ord}\alpha'$ divides $\operatorname{ord}\alpha$, which divides n_*. On the indicated triples the source calls the even-order case II and the odd-order case III. Its component away from two is α', since α has order prime to ℓ. Its component away from ℓ is all of α, since $\ell \nmid N$. Thus its exact dyadic conductor M_α (which also depends on ℓ, x) is the least power of two at least the respective quantity

$$\begin{aligned} &\max\{\ell N 4^{n_0},\ 2^{c_2+6}o\ell^3F^2\}, && \operatorname{ord}\alpha \text{ even},\\ &\max\{\ell F p_1 4^{n_0},\ 8\ell^2\}, && \operatorname{ord}\alpha \text{ odd}. \end{aligned}$$

These are the definitions in `Development/AutomorphicArithmetic/AutomorphicVarieties/HeightAndVanishing/DirichletPrimeToLPartMulLPowerPart.lean`, line 211 and the character prescription in `Development/AutomorphicArithmetic/AutomorphicVarieties/AutomorphicCases/SymplecticGSp4.lean`, line 13.

Lemma 47.7 (Local separation at the prescribed conductor)**.** *There is an odd character ξ unramified at ℓ whose dyadic component has conductor exactly M_α. In case II it is purely dyadic. In case III its only other component has conductor p_1 and order $\operatorname{ord}\alpha$. For $A = \kappa_\ell\omega_\ell^{-1}\xi$ the following holds. At an arithmetic Frobenius $s \in D_\ell$ with $\kappa_\ell(s) = 1$, $\bar A(s)$ is outside the subgroup $\bar\Psi(D_\ell)$. There is an arithmetic Frobenius h separating all four residual entries of $d = (A\Psi, \Psi, A, 1)$.*

Proof. Write $M_\alpha = 2^d$ in this proof. Send -1 to -1 and 5 to a primitive 2^{d-2}-th root to define ξ_2 on the units modulo 2^d. Its restriction to the kernel of reduction modulo 2^{d-1} is nontrivial, so it is primitive. In II use $\xi = \xi_2$. In III the cyclic group of units modulo p_1 admits a character ξ_0 of order $\operatorname{ord}\alpha$, since this nontrivial odd order divides $p_1 - 1$. It is even and primitive. Set $\xi = \xi_2\xi_0$. All finite character values used here have order prime to ℓ and are available after a finite unramified coefficient extension. Under such an extension a finite length over the integer ring is preserved. For the strict cohomology groups this base change follows by writing the new

ring as a finite free module over the old one, which commutes with continuous cochains and their restriction maps.

Inertia at ℓ surjects onto the cyclotomic image, so we may choose s with the asserted normalization. Put $a = \pm\ell \equiv 1 \bmod 4$ and $b = v_2(a-1)$, so that $2^b \le \ell + 1$. The inequality $2^d > \ell + 1$ gives $b < d$. Each squaring of a adds one to the valuation of its difference from one. Since ξ_2 is faithful on the cyclic subgroup congruent to one modulo four, its value at a has order $2^{d-b} \ge M_\alpha/(\ell+1) > 2$. Multiplication by a sign preserves this order. The odd-order factor in III also preserves its two-part, as does reduction in odd characteristic. Thus the two-part of the order of $\bar{A}(s) = \bar{\xi}(\ell)$ is at least $M_\alpha/(\ell+1)$.

On s, $\bar{\Psi}$ has the value of $\bar{\alpha}$; its inertia image has order at most two since $\ell - 1 \mid 2r$. Consequently the two-part of the order of $\bar{\Psi}(D_\ell)$ is at most $2^{c_2+1}o$ in II and at most two in III. The second term in the defining maximum for M_α makes $M_\alpha/(\ell+1)$ larger than these bounds. If $\bar{\Psi}(s) \neq 1$ take $h = s$. Otherwise the local nontriviality assumption gives an inertia element by which one can multiply s to obtain $\bar{\Psi}(h) \neq 1$. The residual $\bar{A}$ is unramified there. Therefore $\bar{A}(h)\bar{\Psi}(h), \bar{\Psi}(h), \bar{A}(h), 1$ are pairwise distinct. □

Proposition 47.8 (The six odd strict groups with the source characters)**.** *Set $\mu = A\Psi$. For each of*

$$\Gamma = A^{\pm 1}, \quad (\Psi/A)^{\pm 1}, \quad (A\Psi)^{\pm 1}$$

the strict group for $(O/\ell^j)(\Gamma)$ vanishes at every $j \ge 1$. The strict conditions are zero on D_ℓ and zero on inertia at every other finite prime. The prescribed characters and marks satisfy the local and residual hypotheses of `Development/Automorphic Arithmetic/Automorphic Varieties/Symplectic GSp 4/DecompSigmaOrd.lean`*, line 90.*

Proof. The Teichmüller character lifting $\bar{\Gamma}$ is odd. Factor its primitive Dirichlet character as $\phi\eta$ with odd and dyadic conductors f and M_α. For the latter assertion observe that its dyadic factor is $\xi_2^{\pm 1}$ times a character of conductor at most 2^{c_2}, and $d = \log_2 M_\alpha > c_2$ in both cases. It is nontrivial on $1 + 2^{d-1}$. At the odd primes the factors come from α', ω_ℓ^r and, in III, ξ_0, or their inverses. Each has order dividing $\ell^{f_0} - 1$, since $\operatorname{ord}\xi_0 = \operatorname{ord}\alpha'$ in III and $\ell - 1 \mid \ell^{f_0} - 1$. The conductor f divides ℓN in II and $\ell F p_1$ in III. Hence $M_\alpha \ge f 4^{n_0}$, and $\ell^{f_0} \equiv 1 \bmod 4$. Theorem 18.14 makes both $B_{1,\phi\eta}$ and its inverse-character counterpart units above ℓ.

The residual symplectic diagonal has prime-to-ℓ inertia image away from ℓ. Consequently a cocycle unramified on the kernel of that diagonal on inertia gives zero as a class on inertia, by averaging on the finite quotient. A strict residual cocycle thus vanishes on $I_v \cap \ker\bar{\Gamma}$ at every finite v, since a coboundary vanishes on that subgroup. Corollary 18.15 applies. Its class-group argument uses the unit just obtained in the integral Stickelberger annihilator. To lift the vanishing, subtract a lift of a global coboundary from a cocycle at depth $j > 1$ to make its values divisible by ℓ. At D_ℓ each $\Gamma(s) - 1$ is a unit by Lemma 47.7. The element exhibiting the local coboundary is therefore also divisible by ℓ. Elsewhere inertia has finite prime-to-ℓ image. A nontrivial inertia action supplies the same unit difference, and at a trivial one the cocycle already vanishes on inertia. Division by ℓ gives a strict cocycle at depth $j - 1$, so induction finishes the proof of vanishing.

For the remaining data take all prime divisors of $2\ell N$ in II and of $2\ell N p_1$ in III. Outside these primes the finite characters are unramified. The auxiliary character is odd, Ψ is even, and the only prime in the distinguished local set is ℓ, since $\ell \nmid \operatorname{ord}\alpha$. At s one has $\bar{\xi}(s)\bar{\Psi}(s)^j \neq 1$ for all integers j. Both each odd root and its Tate dual have nontrivial reduction there, using $\bar{\kappa}_\ell(s) = 1$. Choose g_0 outside the union of the two proper kernels of $\bar{A}, \bar{\Psi}$. In the procyclic closure of s the sequence $s^{\ell^{j!}}$ converges to its prime-to-ℓ part, since this holds in each finite quotient. Finally the intersection of inertia with the kernels of the finite characters has finite

index in inertia. Its cyclotomic image contains an open subgroup of $1+\ell\mathbb{Z}_\ell$, giving the required marked element of infinite cyclotomic order. This constructs the two marks of Section 40.7 together with the separating h of Lemma 47.7. □

Theorem 47.9 (A common strict bound from the prescribed arithmetic data)**.** *For each x select any of the above triples for which the following three assertions hold for the characters just constructed. Use the global multiplier μ, the two marks, the ordinary local component through d at ℓ, and the kernel of the residual diagonal on inertia away from ℓ.*

1. *In every such integral lift over a complete Noetherian local coefficient algebra, the subalgebra generated by products of matrix entries of total torus weight zero is finite over O.*
2. *For each characteristic-zero specialization preserving both coordinate planes $02, 13$, the matrices on plane 02 span a proper subspace of its endomorphisms.*
3. *There is a single coefficient embedding for the triple under which every absolutely irreducible characteristic-zero specialization has a regular algebraic cuspidal match over $\mathbb{Q}(i)$, with the good Frobenius polynomials of Section 47.1.*

Then a single positive integer $D_{\mathrm{Sp}}^{\mathrm{src}}$ for the interval satisfies, for every x on it,

$$\sum_{\text{selected } (\alpha,r,\ell) \text{ on } E_x} \operatorname{length}_O S(\alpha,r)\log\ell \;\le\; 2n_*t_*\log D_{\mathrm{Sp}}^{\mathrm{src}} \;\ll\; V^C$$

for a constant C depending only on the fixed interval construction. The automorphic requirement of the source over imaginary quadratic fields implies assertion 3 here.

Proof. First apply Corollary 40.19 with these data. Its argument uses the two disjoint residual Frobenius pairs, the two distinct residual eigenvalues of the compressed self-adjoint mark, and the separated h, all verified above. It identifies the finiteness in assertion 1 with finiteness of the trace ring T. Proposition 40.18 identifies its local component with the ordered ordinary flag, also over nonreduced quotients. In the proof of Theorem 40.6, the odd-root inputs are strict vanishing and the local dimensions. The former is Proposition 47.8. At ℓ the vanishing of the invariants of a residual odd root and its Tate dual gives these dimensions by local duality and the Euler characteristic formula. The localization calculation at each finite depth now follows from the same finite duality formula as in Theorem 38.8. The even root is still Ψ, so the integral negative-root module measures exactly $S(\alpha,r)$. The trace presentation and the strict-length inequality of Theorem 40.14 therefore apply with the prescribed ξ.

At a nonsplit point the two-plane alternative in Theorem 40.13 has an integral block realization. Putting it in the two-mark frame preserves those blocks. Indeed the projector from the prime-to-ℓ part of s is the identity on the first block and zero on the second, as follows by reduction and its prime-to-ℓ eigenvalues. The second change of frame diagonalizes a compression within the first block and uses the dual basis on the second. The first plane is absolutely irreducible. Its image spans an algebra since group elements are closed under multiplication, and this algebra is the full matrix algebra by Burnside's theorem. Assertion 2 excludes this case. For a rank-four point Proposition 40.23 gives irreducibility on $G_{\mathbb{Q}(i)}$, using the ordinary line, the separated h, and the simple spectrum of every positive power of the marked inertia element. This proves the last claim about the source's matching requirement. Under assertion 3, Proposition 47.1 gives multiplicity one at each such point.

It remains to bound the changing conductors in a common integer. Put L_* as in Section 38.1. Since $f_0 \le 4n_*$, repeated squaring gives $2^{n_0} \le (\ell^2-1)f_0$. Indeed for f_0 odd the valuation is

$v_2(\ell-1)$, and for f_0 even it is $v_2(\ell-1)+v_2(\ell+1)+v_2(f_0)-1$. Also $2^{c_2}\le N^\natural$, $N \mid N^\natural$, and $\ell \le L_*$. Taking the least power of two above a positive real number costs a factor of at most two. The maximum $M_{\max}$ of the prescribed dyadic conductors on the interval is therefore polynomial in V and a multiple of each; take it to be one for an empty family. The conductor of ξ divides $p_1 M_{\max}$ and its order divides $n_* M_{\max}$.

Use $d'_\ell = \ell N^\natural p_1 M_{\max}$ in Proposition 47.4. It bounds all four local character conductors, including the tame character ω_ℓ^{-1} at ℓ. Thus its fourth power is a common principal level at that ℓ. Choose one rational prime $p \equiv 1 \bmod 4$ between Y' and $2Y'$ with $Y' = 32+L_*+N^\natural+p_1$, using the prime number theorem in this fixed progression. It is prime to every $3d'_\ell$. Theorem 47.5 now supplies one multiset of integral cuspidal Hecke traces for each weight, with polynomial cardinality and conjugate magnitudes at most $4p^{(r+1)/2}$. Use the polynomial in the proof of Theorem 38.16 with the first root of unity of order dividing $\operatorname{lcm}(\ell-1, n_* M_{\max})$ and the second of order dividing n_*. It includes the split trace for every one of the present choices of ξ. Its values on the Hecke multiset have nonzero integral product, by the same archimedean separation and Galois stability. Multiplying their absolute products over the weights gives the common integer. The polynomial degrees, the multiset sizes, the weights and $\log p$ are all polynomial in V. The strict-length inequality at multiplicity one bounds each length by $2v_\ell(D_{\mathrm{Sp}}^{\mathrm{src}})$. There are at most $n_* t_*$ pairs (α, r) on a given field at each ℓ. Prime factorization proves the stated bound. □

The conductor choices and the odd strict vanishings in this subsection are unconditional. The three global arithmetic assertions in Theorem 47.9 are hypotheses. They concern the actual deformation spaces for the fields used by the integral projected theta period.

47.4 Two character sums for the even planes

Use the fields E_x, the character α, the colliding weight r and the prime ℓ of Section 47.3. In the first two results we also allow $\bar\Psi|_{D_\ell} = 1$. Put $k = O/\ell O$, $\tau = \bar\Psi = \bar\alpha \bar\kappa_\ell^r$ and let $\omega = \bar\kappa_\ell$. The global τ is even and nontrivial. Indeed at the half residue it is nontrivial on inertia and at the zero residue it is the nontrivial $\bar\alpha$. For $\epsilon \in \{+1,-1\}$ set

$$M_\epsilon = k(\tau^\epsilon), \qquad \delta_\epsilon = \omega \tau^{-\epsilon}.$$

Let $H_{\mathrm{rel}}(M)$ denote the classes in $H^1(\mathbb{Q}, M)$ whose restriction to inertia is a coboundary away from ℓ. Let $S(M)$ impose in addition zero restriction as a class at D_ℓ.

Lemma 47.10 (The two ordinary class groups for relaxed extensions)**.** *Let L_ϵ be the cyclic field cut out by δ_ϵ and $\Delta_\epsilon = \mathrm{Gal}(L_\epsilon/\mathbb{Q})$. Then*

$$\dim_k H_{\mathrm{rel}}(M_\epsilon) = \dim_k \mathrm{Hom}_{\Delta_\epsilon}(\mathrm{Cl}(L_\epsilon), k(\delta_\epsilon)).$$

The homomorphisms on the right are additive. Write f_ϵ for the conductor of the primitive Dirichlet character lifting δ_ϵ by Teichmüller representatives. It divides ℓN and is divisible by ℓ. Take $1 \le b_\epsilon \le f_\epsilon$ prime to f_ϵ with $b_\epsilon \bmod \ell \ne \delta_\epsilon(b_\epsilon)$. Such a choice exists. If the finite sum

$$C_\epsilon = \sum_{a \in (\mathbb{Z}/f_\epsilon)^\times} \left\lfloor \frac{b_\epsilon [a]}{f_\epsilon} \right\rfloor \delta_\epsilon(a)^{-1} \quad \in k$$

is nonzero, then $H_{\mathrm{rel}}(M_\epsilon) = 0$. Here $[a]$ is the representative in $[1, f_\epsilon]$. Whether the sum is nonzero is independent of the allowed b_ϵ.

Proof. The Tate dual character of M_ϵ is δ_ϵ. It is odd and its inertia character at ℓ is $\omega^{1-\epsilon r}$. The collision condition gives $r \equiv 0$ or $(\ell-1)/2 \bmod \ell-1$; for $\ell \geq 5$ the indicated inertia character is nontrivial. This proves the assertion about f_ϵ, since α has conductor dividing N. Global invariants of both modules vanish.

Use the finite Poitou–Tate order formula as in Theorem 38.8. At each finite $q \neq \ell$ the unramified local conditions are orthogonal complements. Indeed their cup product factors through the unramified quotient of cohomological dimension one. On either module the unramified dimension equals the local invariant dimension, by the kernel and cokernel formula for Frobenius minus one on inertia invariants. These dimensions add to the full local H^1 dimension by local duality and the local Euler characteristic. Thus the order-formula factor at $q \neq \ell$ is one. At ℓ the relaxed group takes all of H^1. Its dimension exceeds the invariant dimension by one, since the dual has no local invariants. This holds also when M_ϵ is locally trivial. At the real place the factor is $1/|k|$, since M_ϵ is even and ℓ is odd. These two factors cancel. The orthogonal condition at ℓ is zero, so $\dim_k H_{\mathrm{rel}}(M_\epsilon) = \dim_k S(k(\delta_\epsilon))$.

The group Δ_ϵ has order prime to ℓ. Inflation–restriction and averaging identify classes on the right with equivariant homomorphisms on G_{L_ϵ} unramified everywhere and zero on decomposition at primes over ℓ. The same argument on inertia and decomposition groups verifies the local converse. By class reciprocity these homomorphisms factor through the ordinary class group, with the prime classes above ℓ killed. In fact every equivariant homomorphism on the ordinary class group already kills those classes. An inertia subgroup at ℓ fixes each such prime in the abelian field, whereas it acts nontrivially on the target line. This gives the dimension formula.

For the finite test abbreviate $\delta = \delta_\epsilon$, $f = f_\epsilon$, and use the full cyclotomic field $\mathbb{Q}(\zeta_f)$. A strict cocycle restricts there to an unramified additive homomorphism on the class group, equivariant for δ. Set $\Theta_f = \sum_a ([a]/f)\sigma_a^{-1}$, where $\sigma_a(\zeta_f) = \zeta_f^a$. The integral Stickelberger element $(b-\sigma_b)\Theta_f$ annihilates that class group. Its coefficient at σ_a^{-1} is $(b[a]-[ba])/f$. Hence its action on the target is precisely the scalar C_ϵ. If this scalar is nonzero the restricted homomorphism is zero. The original class then inflates from the abelian cyclotomic Galois group. Its H^1 on this line is zero: commuting complex conjugation with each group element gives a coboundary since δ is odd and two is a unit. This proof uses no restriction on the index $[\mathbb{Q}(\zeta_f) : L_\epsilon]$.

Since $\ell \mid f$, the character ω also factors modulo f. Its ratio to δ is the nontrivial τ^ϵ. An allowable b therefore exists. Evaluating the same group-ring element at the Teichmüller lift $\widetilde{\delta}$ gives

$$(b - \widetilde{\delta}(b))\, B_{1,\widetilde{\delta}^{-1}}, \qquad B_{1,\widetilde{\delta}^{-1}} = f^{-1} \sum_a [a]\, \widetilde{\delta}(a)^{-1}.$$

The first factor is a unit in O, and the product is integral by the floor formula. Reduction proves the asserted independence of the choice of b. [9] □

Theorem 47.11 (Triangularity and a vanishing test at the colliding weights)**.** *If either $C_+ \neq 0$ or $C_- \neq 0$, a continuous two-dimensional lift with diagonal residual characters of ratio τ and prime-to-ℓ inertia image away from ℓ is triangular after a change of frame. This holds over a complete Noetherian local coefficient ring with finite residue field k, including in positive characteristic. In particular there is no absolutely irreducible plane of this kind. If $C_- \neq 0$, then also*

$$S(\alpha, r) = 0$$

for the full strict group with divisible coefficients, even when $\tau|_{D_\ell} = 1$.

Proof. Let R be the coefficient ring and $\mathfrak{m}$ its maximal ideal. Choose $h \in G_{\mathbb{Q}}$ with $\tau(h) \neq 1$. The matrix at h has two distinct residual roots. Hensel factorization and the two spectral

idempotents give a basis of its free eigenlines lifting the original coordinates. In that frame write

$$\rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}, \qquad \bar{a}/\bar{d} = \tau.$$

Let J_b be the ideal generated by the $b(g)$. If it is nonzero, Nakayama's lemma supplies a k-linear functional λ on $J_b/\mathfrak{m}J_b$ nonzero on some entry. Reduction of $b(gg') = a(g)b(g') + b(g)d(g')$ then makes $z(g) = \lambda(b(g))/\bar{d}(g)$ a nonzero cocycle on $k(\tau)$. It is continuous. Indeed Artin–Rees gives $J_b \cap \mathfrak{m}^N \subseteq \mathfrak{m}J_b$ for large N, so the finite linear quotient is continuous in the subspace topology. The cocycle is zero at h. A coboundary with this value is zero everywhere, because $\tau(h) - 1$ is a unit. On inertia away from ℓ the cocycle factors through its finite prime-to-ℓ matrix image and is a coboundary there by averaging. Thus $J_b \neq 0$ implies $H_{\mathrm{rel}}(M_+) \neq 0$. Interchanging the two lines proves that a nonzero lower-entry ideal implies $H_{\mathrm{rel}}(M_-) \neq 0$. Lemma 47.10 now kills the corresponding ideal under the asserted test. This proves triangularity over R.

For the strict group put $D = (\mathrm{Frac}(O)/O)(\Psi^{-1})$. Its global invariants vanish because $\tau^{-1} \neq 1$. The coefficient sequence for multiplication by ℓ gives a unique finite-coefficient class in $H^1(\mathbb{Q}, D[\ell])$ above every element of $S(\alpha, r)[\ell]$. At any finite $q \neq \ell$ the inertia action on D is finite of order prime to ℓ. Its invariant group is either zero or the entire divisible module. Thus $D^{I_q}/\ell D^{I_q} = 0$. The finite-coefficient class is unramified there by the same coefficient sequence on inertia. At ℓ impose the relaxed condition. We have obtained an injection of $S(\alpha, r)[\ell]$ into $H_{\mathrm{rel}}(M_-)$. When $C_- \neq 0$ the latter vanishes. Every continuous cocycle from a compact group to the discrete D has finite image, so its class is killed by a power of ℓ. A nonzero such group contains an element of order ℓ. This gives the full vanishing. □

Corollary 47.12 (Removing the plane hypothesis on the original interval)**.** *In Theorem 47.9, on triples with $C_+ \neq 0$ one may omit assertion 2. For triples with $C_- \neq 0$ the strict length is zero independently of all three assertions. The two tests use a polynomial number of finite-field summands on the whole interval.*

Proof. The two-plane alternative of Theorem 40.13 provides an integral lift of the plane 13 with residual characters $\tau, 1$ and prime-to-ℓ inertia image away from ℓ. Its absolute irreducibility contradicts Theorem 47.11 when $C_+ \neq 0$. The proof of Theorem 47.9 uses assertion 2 only to remove this alternative. The remaining length claim follows directly from Theorem 47.11. On the interval all characters come from E_*, so there are at most n_* geometric choices. At each of the at most L_* coefficient primes, ω^r has only the two indicated possibilities. Each conductor divides $\ell N^\natural$. The two sums require together at most $4n_*L_*^2N^\natural$ terms. All these factors are polynomial in V; the calculation uses ordinary residue arithmetic even at moduli divisible by the coefficient prime. □

For example take $\psi = 1$, $\ell = 13$, and the even quartic character modulo 16 specified by $\bar{\alpha}(5) = 5 \in \mathbb{F}_{13}$. Its field $\mathbb{Q}(\zeta_{16})^+$ occurs in each sufficiently large E_x. For $12 \mid r$ in the stated range one has $\tau = \bar{\alpha}$ and $\tau(13) = 8$. Both primitive conductors are $16 \cdot 13 = 208$, and $b = 3$ is allowed for both. The rows below give the data for grouping the sum by c mod 16. In the last row sum over $a = c + 16j$, $0 \leq j \leq 12$, omitting $13 \mid a$; each term is $\lfloor 3a/208 \rfloor / a$ in $\mathbb{F}_{13}$.

c	1	3	5	7	9	11	13	15
$\tau(c)$	1	8	5	12	12	5	8	1
sum in the class c	9	0	3	7	7	3	0	9

Multiplying the last row by $\tau(c)$ and summing gives $C_+ = 8$. Using $\tau(c)^{-1}$ gives $C_- = 0$. Thus one test alone excludes the plane on every such weight. A zero value of the other sum

leaves open the strict-length question for that test. More generally, simultaneous zero sums are necessary for either even plane to occur. Global finiteness and cuspidal realization of rank-four points still enter the length bound when the negative-root test gives zero.

47.5 Critical values and the trace congruence at an even plane

Work on the fields and triples of Section 47.3, with the separating $h \in D_\ell$ and the prescribed ξ. For a nonsplit characteristic-zero point with the two-plane realization of Theorem 40.13, write P for the plane on coordinates 13 in its integral h-eigenbasis. Its determinant is $\Psi = \alpha\kappa_\ell^r$, and its ordered residual lines have characters $\bar{\Psi}, 1$. Take a finite coefficient field J' of realization, with integer ring O', uniformizer π, and $e' = v_\pi(\ell)$. We use $v_\ell = v_\pi/e'$ on J'. Put

$$P(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}, \qquad \mathfrak{b} = (b(g) : g \in G_{\mathbb{Q}}), \qquad \mathfrak{c} = (c(g) : g \in G_{\mathbb{Q}})$$

as ideals in O'. Both ideals are nonzero by irreducibility of the plane and both lie in the maximal ideal by reduction. Denote the entire four-dimensional realization by ρ and the split representation by d_{sp}.

Theorem 47.13 (The exact trace ideal and its critical-value bound)**.** *The ideal generated by* $\operatorname{tr}\rho(\mathrm{Frob}_p) - \operatorname{tr} d_{\mathrm{sp}}(\mathrm{Frob}_p)$ *at the unramified rational primes away from* ℓ *and the prescribed level is exactly* $\mathfrak{bc}$*. Writing* $m = v_\pi(\mathfrak{bc})$*, one has*

$$0 < \frac{m}{e'} \le v_\ell(L(1-r,\alpha)).$$

Here L *is the primitive Dirichlet* L*-function in the specified coefficient embedding. The result does not require automorphic realization of the plane.*

Proof. Modulo $\mathfrak{bc}$, the two diagonal functions on P are multiplicative by the matrix product law. On inertia at ℓ they are $\Psi, 1$ because the local representation is upper triangular in the ordered h-frame. At every other finite prime the inertia matrix image has finite prime-to-ℓ order. A character on it congruent to one has trivial image in the principal units of O'/π^m. Indeed those units form an ℓ-group. Apply this to the ratios of the two diagonal characters to $\Psi, 1$ on inertia, using also that the finite character α has order prime to ℓ. Both ratios are everywhere unramified at finite primes. Abelian class reciprocity over $\mathbb{Q}$ gives triviality, since the narrow class number of $\mathbb{Q}$ is one. Consequently $a \equiv \Psi$, $d \equiv 1 \bmod \pi^m$ globally.

The alternating pairing identifies the other plane with $\mu P^\vee \simeq A \otimes P$, since $\det P = \Psi$ and $\mu = A\Psi$. Thus all four-dimensional trace differences are in $\mathfrak{bc}$. For the converse take $\mathfrak{t}$ generated by these differences on all of $G_{\mathbb{Q}}$. Modulo $\mathfrak{t}$, the four eigenvalues of $\rho(h)$ are the corresponding entries of $d_{\mathrm{sp}}(h)$. In fact their characteristic polynomials agree by the traces at h, h^2, h^3, h^4 and Newton's identities, with $1, 2, 3, 4$ invertible. Each actual eigenvalue is congruent to its labeled split value: evaluation in the split polynomial makes all the other factors units, by residual separation. For any g, the four trace identities at $h^j g$, $0 \le j \le 3$, now say that the four diagonal differences of the matrices at g are killed by the Vandermonde matrix of those split eigenvalues. Its determinant is a unit. In particular $a(g) \equiv \Psi(g)$ and $d(g) \equiv 1 \bmod \mathfrak{t}$. The matrix product then gives $b(g)c(g') = a(gg') - a(g)a(g') \in \mathfrak{t}$ for all g, g'. This proves equality with $\mathfrak{bc}$. Some element attains its least valuation m. Reduction of the two continuous representations modulo π^{m+1} has finite image. Chebotarev supplies a good rational prime in the same conjugacy class in that finite quotient, so the asserted good-prime ideal also has valuation m.

Choose $j = v_\pi(\mathfrak{b})$. The map $z(g) = \pi^{-j} b(g) \bmod \pi^m$ is a cocycle for $(O'/\pi^m)(\Psi)$. This follows from $b(gg') = a(g)b(g') + b(g)d(g')$ and the diagonal congruences just proved. Some

$z(g)$ is a unit. Since $z(h) = 0$ and $\Psi(h) - 1$ is a unit, a scalar multiple of z that is a coboundary must vanish identically. Thus its class generates a copy of O'/π^m. It is unramified away from ℓ by inertia averaging. The map from these finite coefficients into $(J'/O')(\Psi)$ by division by π^m is injective on global H^1, by the multiplication sequence and the vanishing of the global invariants of $(J'/O')(\Psi)$. Its image belongs to the relaxed positive-twist group. Theorem 21.6 computes the length of that group over O before this coefficient extension. Its hypotheses hold because $r \geq 2$, the sign of α is $(-1)^r$, and $\bar{\Psi} \neq 1$. Tensoring with O' commutes with continuous cochains and the inertia conditions, since O' is finite free over O; each O-length is multiplied by e' as an O'-length. The length of the cyclic submodule is m. This proves the inequality. □

Lemma 47.14 (One integral trace test for finitely many points)**.** *Let O be unramified over $\mathbb{Z}_\ell$, with residue field of cardinality q_0. Let $t_1, \dots, t_j$ be integral trace differences evaluated at a finite set of geometric points σ, with nonnegative integer weights n_σ. Suppose that at each point some difference is nonzero and put $w_\sigma = \min_i v_\ell(\sigma(t_i))$. Some O-linear combination t is nonzero at every point and satisfies*

$$\sum_\sigma n_\sigma v_\ell(\sigma(t)) \;\leq\; \sum_\sigma n_\sigma \left(w_\sigma + \frac{1}{q_0 - 1} \right).$$

Proof. Choose the coefficients independently with normalized additive Haar measure on O. At a point fix an index attaining the minimum and condition on all other coefficients. After dividing by the value of that entry, the linear combination at the point is $c - z$ with c uniform in O and z a fixed element of the valuation ring of a finite extension. For $n \leq t < n + 1$, $n \geq 0$, the condition $v_\ell(c - z) > t$ confines c to at most one class modulo $\ell^{n+1}O$. In fact any two solutions differ by an element of integral valuation greater than t. The probability is at most q_0^{-n-1}. Integration over $t \geq 0$ bounds the expectation of the excess over w_σ by $1/(q_0 - 1)$. An exact zero has probability zero. Summing the expectations over the finitely many points proves the existence claim. □

Corollary 47.15 (Counting the plane contribution by its multiplicity)**.** *Suppose T is finite for each selected triple and that all its absolutely irreducible rank-four points admit the cuspidal matches of Theorem 47.9, with one embedding per triple. Let $P_{\mathrm{pl}} = \sum n_\sigma$ on the two-plane points of that triple, and let H_{irr} be its number of irreducible points in the nonsplit quotient. There exist two positive integers $D_{\mathrm{cusp}}, D_{\mathrm{crit}}$ for the entire interval, with logarithms polynomial in V, such that on every selected triple*

$$\begin{aligned} \operatorname{length}_O S(\alpha, r) \;\leq\; 2\Big(& v_\ell(D_{\mathrm{cusp}}) + P_{\mathrm{pl}}\, v_\ell(D_{\mathrm{crit}}) \\ & + \frac{H_{\mathrm{irr}} + P_{\mathrm{pl}}}{|k| - 1} \Big). \end{aligned}$$

In particular a polynomial bound on P_{pl} suffices for a polynomial bound on the weighted sum of strict lengths, even if some even planes occur.

Proof. For each $2 \leq r \leq t_*$ and each nontrivial geometric character γ of E_* of sign $(-1)^r$, let f_γ be its primitive conductor. Define D_{crit} to be the absolute value of the product, over these pairs, of

$$r f_\gamma (r + 1)!\, L(1 - r, \gamma).$$

The primitive functional equation and the Euler product at $r > 1$ give nonvanishing with this parity. Each factor is an algebraic integer. Indeed the generalized Bernoulli formula is

$$L(1 - r, \gamma) = -\frac{f_\gamma^{r-1}}{r} \sum_{a=1}^{f_\gamma} \gamma(a) B_r(a/f_\gamma).$$

In the polynomial expansion the Bernoulli denominators through order r divide $(r+1)!$. This follows by induction from the Bernoulli recurrence after multiplication by $(j+1)!$ at order j. Multiplication also by f_γ clears the one possible remaining power of the conductor. The product is Galois invariant, hence rational integral. Every conjugate factor has magnitude at most $\exp(Cr\log(2rf_\gamma))$. For example the expansion of $z/(e^z-1)$ on $|z|=1$ gives $|B_j|\le C_0 j!$ by Cauchy's estimate; insertion in the displayed finite sum gives the claimed bound. Therefore

$$\log D_{\mathrm{crit}} \ll n_* t_*^2 \log(2t_* N^\natural).$$

At the chosen embedding all the other factors are integral. Since the clearing multiplier is an ordinary integer, Theorem 47.13 bounds the minimum valuation at each plane by $v_\ell(D_{\mathrm{crit}})$.

Use the common prime and the cuspidal multiset in the proof of Theorem 47.9. Their construction uses only the matching points and gives D_{cusp} even when additional points have two-plane realizations. The matching points have multiplicity one by Proposition 47.1. The sum of their valuations at this prime is at most $v_\ell(D_{\mathrm{cusp}})$ by Theorem 38.16. Enlarge this single test by a good prime attaining the minimum of Theorem 47.13 for each plane point. Finiteness of T makes this a finite set. At the matching points the minimum can only decrease; at each of the other points it has the ideal valuation already computed. Apply Lemma 47.14 with the multiplicities of Theorem 40.14. The resulting integral linear combination belongs to its trace-difference ideal and is nonzero at all nonsplit points. That theorem gives the displayed bound. With an empty nonsplit fiber the same theorem gives zero directly.

Finally H_{irr} is bounded by the size of the common Hecke multiset, which is polynomial on the interval. If also $P_{\mathrm{pl}} \le V^C$, sum the displayed estimate with weight $\log \ell$. At each ℓ there are at most $n_* t_*$ pairs (α, r). For each integer D the sum of $v_\ell(D)\log\ell$ is $\log D$, and $\sum_{5\le \ell\le L_*} \log\ell/(|k|-1) \ll (1+\log L_*)^2$ by comparison with the sum over all integers. These bounds give the claimed polynomial estimate. □

The critical value bounds the depth at each even plane without an assumption on its origin. The total multiplicity of the plane points, finiteness of the trace algebra, and the cuspidal matches of the rank-four points remain to be controlled in general.

47.6 The total congruence depth on the one-variable plane locus

For this subsection put

$$a_\pm = \dim_k S(k(\bar\Psi^{\pm 1})), \qquad a_+ = 0, \quad a_- = 1,$$

on the fields and triples of Section 47.3. These are conditions on the two strict residual groups. The positive relaxed group can still be nonzero. We bound all the two-plane trace congruences together when each point is counted with the multiplicity coming from the plane equations.

Theorem 47.16 (The sum over the plane equations)**.** *Let T_{pl} be the quotient of the trace ring obtained by setting the two rank-four trace coordinates V, W to zero, retaining the block coordinate U defined below. Its quotient by ℓ-power torsion is finite over O. For a geometric characteristic-zero point with $U_\sigma \neq 0$ put*

$$m_\sigma^{\mathrm{pl}} = \dim_{\overline{J}} (T_{\mathrm{pl}} \otimes_O \overline{J})_\sigma.$$

Then

$$\sum_{U_\sigma \neq 0} m_\sigma^{\mathrm{pl}} v_\ell(U_\sigma) \;\le\; v_\ell(L(1-r,\alpha)).$$

At each such point $v_\ell(U_\sigma)$ is the minimum valuation of the good-prime trace differences of Theorem 47.13.

Proof. In Theorem 40.6 the three positive coordinates after taking the first torus invariants are x, v, w of weights $1, 1, 2$; the single negative coordinate is y of weight -1. Here x is the upper even-root coordinate on the local ordinary flag. There is one relation E of weight 1 and two F_1, F_2 of weight -1. The trace coordinates of Theorem 40.9 are $U = xy,\ V = vy,\ W = wy^2$. On the block locus set $v = w = 0$. Equivalently one may first set the three off-block entries to zero in the full two-torus chart. Their invariant products are v, w; every neutral monomial involving either of these contains V or W. Projecting the equations onto their weights therefore gives the same quotient on trace coordinates. For some $g, f_1, f_2 \in O[[U]]$, one has

$$E(x, 0, 0, y) = xg(xy), \qquad F_i(x, 0, 0, y) = yf_i(xy).$$

Multiplication by the opposite-weight monomials gives precisely the neutral ideal

$$T_{\mathrm{pl}} = O[[U]]/(Ug(U),\, Uf_1(U),\, Uf_2(U)).$$

The scalar $g(0)$ has valuation exactly $v_\ell(L(1-r, \alpha))$. To prove this, over $(O/\ell^j)[\varepsilon]/(\varepsilon^2)$ use only the positive root and put $x = \varepsilon z$. The sole equation is $g(0)z = 0$ in O/ℓ^j. In matrix entries these lifts have the form $(1+\varepsilon b(g')X_+)d_{\mathrm{sp}}(g')$ with $X_+ = E_{02}+E_{13}$ as in Theorem 40.7. The group law gives cocycles for $(O/\ell^j)(\Psi)$. They are normalized by $b(h) = 0$. Conversely every relaxed class has a unique representative with this normalization because $\Psi(h) - 1$ is a unit. On D_ℓ an upper-root cocycle preserves the ordinary flag and fixes its graded characters. Away from ℓ the inertia averaging used in Theorem 40.7 gives precisely the prescribed kernel condition. Thus these lifts give all the relaxed classes, with linear correspondence since $\varepsilon^2 = 0$. The other coordinates vanish on these lifts: their first-order entries have other weights and there are no additional relative positive even-root coordinates since $a_+ = 0$.

Multiplication by ℓ^j on $(\mathrm{Frac}(O)/O)(\Psi)$ identifies this finite group with the ℓ^j-torsion of the divisible-coefficient relaxed group. The global invariants vanish and the invariants on inertia away from ℓ are divisible, giving the identification also with those local conditions. Theorem 21.6 gives a finite relaxed group of length $v_\ell(L(1-r, \alpha))$. For all sufficiently large j its full length is attained, whereas the length of the kernel of multiplication by $g(0)$ is $\min(j, v_\ell(g(0)))$. This proves the claim, in particular $g(0) \neq 0$.

Let μ_0 be the least valuation of a coefficient of g. Weierstrass preparation gives $g = \ell^{\mu_0} u_0 H$ with u_0 a power series unit and H a monic distinguished polynomial, possibly one. Its constant term is nonzero. In the torsion-free quotient of T_{pl} one has $UH(U) = 0$. This quotient is therefore a quotient of $O[[U]]/(UH)$, finite free as a module before taking that further quotient by Weierstrass division. At a nonzero geometric root of H, its local dimension after imposing the remaining equations is at most the root multiplicity of H. All roots of a nonconstant distinguished polynomial have positive valuation. Counting with the latter multiplicities, the sum of their valuations equals $v_\ell(H(0)) = v_\ell(g(0)) - \mu_0$. This proves the inequality with m_σ^{pl} and does not impose finiteness on the full trace ring.

At a point with $U_\sigma \neq 0$ the ideal of specialized trace differences is $U_\sigma O'$ in any integral field of realization. Indeed the augmentation ideal in the block trace ring is generated by U and by the trace differences, using Theorem 40.9. Equivalently, to pass from closed trace generation to this identity after specialization, approximate U by polynomials in the traces and subtract their split values. The resulting ideals in the integer ring O' are closed. Chebotarev as in Theorem 47.13 detects the valuation at a good prime. □

The local multiplicity of the full characteristic-zero trace algebra at such a point can also include the directions V, W. We first record an exact form of the simultaneous choice of a trace test that works over the original normalized valuation.

Lemma 47.17 (Simultaneous minimum valuations after extending coefficients)**.** *For any triple of Section 47.3 with finite trace ring, let Σ' be the geometric points of the nonsplit quotient in Theorem 40.14. Take a finite list of integral trace differences t_i with some nonzero evaluation at each point. Then*

$$\operatorname{length}_O S(\alpha, r) \;\le\; 2 \sum_{\sigma \in \Sigma'} n_\sigma \min_i v_\ell(\sigma(t_i)).$$

There is no restriction on a_+, a_- in this lemma.

Proof. Choose a finite unramified extension O_1/O whose residue field k_1 has cardinality greater than $|\Sigma'|$, inside the fixed algebraic closure of J. The presentations and the strict-length inequality of Theorems 40.6 and 40.14 may be tensored with O_1. In detail, finite free scalar extension commutes with the power series rings, their homogeneous weight spaces, the finite lists of relations and the minors in the negative-root module. It preserves lengths over the respective unramified integer rings. The continuous cochains and all the local restriction maps computing the strict group also commute with it, by a finite free basis. The resulting generic algebra tensored with $\overline{J}$ is the original geometric algebra, with the same multiplicities. These are exactly the algebraic data used in the strict-length inequality.

At each σ divide the list of evaluations by one attaining the displayed minimum. The residues of the resulting integral entries give a nonzero linear form on the list of coefficient variables over $\overline{k}$. On k_1^j, where j is the list length, its zero set has at most $|k_1|^{j-1}$ elements: fix all coordinates except one with nonzero coefficient. The union of the bad sets misses a tuple in k_1^j. Lift that tuple to O_1^j. Its linear combination belongs to the extended trace-difference ideal and attains the minimum valuation at every point. The strict-length inequality over O_1 proves the result. For an empty point set the same inequality gives zero directly. □

Corollary 47.18 (The contribution of the two transverse coordinates)**.** *Suppose $a_+ = 0, a_- = 1$, T is finite, and each rank-four point has the cuspidal match in Corollary 47.15. At a two-plane point let $\mathfrak{j}_\sigma = (V, W)$ in the local geometric Artin algebra of the nonsplit quotient. Take $d \ge 1$ with $\mathfrak{j}_\sigma^d = 0$ at every such point. Then*

$$\operatorname{length}_O S(\alpha, r) \;\le\; 2v_\ell(D_{\mathrm{cusp}}) + d(d+1)v_\ell(L(1-r, \alpha)).$$

The same fields admit a polynomial bound on the weighted sum over any such triples for which d is polynomial in the interval parameter V_0, where V_0 here denotes the left endpoint of the interval. If all the local geometric algebras at plane points are reduced, one can take $d = 1$.

Proof. At $U_\sigma \neq 0$ the trace ring and its nonsplit quotient have the same local geometric algebra. This follows directly from the neutral relations: the equations $Uf_i = 0$ give $f_i = 0$ when U is a unit. Equivalently the split summand has disappeared after localization. Quotienting the local algebra by V, W therefore gives precisely the corresponding local algebra of T_{pl}. In the filtration by $\mathfrak{j}_\sigma^i$ the i-th quotient is generated over this block algebra by the $i+1$ monomials of degree i in V, W. Summing through $i = d-1$ gives

$$n_\sigma \le \tfrac{1}{2} d(d+1) m_\sigma^{\mathrm{pl}}.$$

Nilpotence holds at each point since V, W are zero in its residue field. In a reduced local Artin algebra their ideal is zero.

Use the common prime at the rank-four points and one prime attaining each plane minimum in Lemma 47.17. The matching points have multiplicity one and their total valuation is bounded by $v_\ell(D_{\mathrm{cusp}})$ as in Corollary 47.15. For the others combine the multiplicity inequality with Theorem 47.16. This proves the individual bound without any factor counting distinct plane

points. To sum over each field and all the indicated weights, bound the critical valuations by those of the common integer D_{crit} of Corollary 47.15 and apply prime factorization to the two common integers. There are at most $n_* t_*$ terms at each prime. □

The one-variable block calculation gives unconditional control of the weighted sum internal to that block. The two directions in the full algebra still require a uniform nilpotence bound when nonreduced plane points occur.

47.7 Ordinary extensions between the two even planes

Fix a triple and its characters on the fields of Section 47.3. In the order $0, 1, 2, 3$ used for the local flag, put $P = \langle e_1, e_3 \rangle$ and $P' = \langle e_0, e_2 \rangle$. In this subsection the given representation is block diagonal on $P' \oplus P$. The two summands are Lagrangians, and the pairing identifies $P' = \mu P^\vee$. We allow a finite local Artin coefficient ring R over O with residue field k' extending k. Assume the block representation reduces to the specified diagonal, is ordinary in the specified frame at ℓ, is diagonal at h, and kills $N_q = I_q \cap \ker \bar{d}$ at the finite level primes $q \neq \ell$. It is unramified outside the level and ℓ. Reductions of the integral realizations of Theorem 47.13 satisfy these hypotheses.

An infinitesimal symplectic map with only its P-to-P' block nonzero is a symmetric form on P with values in the multiplier line. Reversing the two spaces gives the other off-diagonal block. Denote the resulting $G_{\mathbb{Q}}$-modules by

$$M_+ = \mu \operatorname{Sym}^2(P^\vee), \qquad M_- = \mu^{-1} \operatorname{Sym}^2(P).$$

Both have rank three. The signs in $M_\pm$ specify the direction of a map and are independent of the notation $S^\pm$ for the even character groups. Let $B_\pm \subset M_\pm$ be the strictly upper triangular part in the original order $0, 1, 2, 3$. Conjugation by an upper triangular block representation preserves it on D_ℓ. In B_+ one may freely specify entries 01 and 03, with 23 determined by 01. In B_- the free entry is 12. Thus these are direct summands as R-modules of ranks two and one. Write $H_{\mathrm{rel}}(N)$ for the global H^1 classes unramified at every finite place away from ℓ, and $H_{\mathrm{str}}(N)$ for the kernel of their restriction to D_ℓ.

Lemma 47.19 (Localization for the entire off-diagonal block)**.** *For either $M = M_\pm$ and every finite R-module C, restriction gives a natural isomorphism*

$$H_{\mathrm{rel}}(M \otimes_R C) \xrightarrow{\sim} H^1(D_\ell, M \otimes_R C).$$

The right side is $H^1(D_\ell, M) \otimes_R C$, where $H^1(D_\ell, M)$ is free of rank three. For $B = B_\pm$, the map from $H^1(D_\ell, B \otimes_R C)$ to this group is injective. Its source and cokernel are obtained by tensoring free R-modules of ranks $\operatorname{rank} B$ and $3 - \operatorname{rank} B$, respectively, with C.

Proof. The residual lines of M_+ have characters $A/\Psi, A, A\Psi$ after reduction. Those for M_- have their inverses. Each line is odd and has zero strict group by Proposition 47.8. Scalar extension to k' preserves this calculation by additivity of cochains. The residual eigenvalues of the element s of that proposition on these six lines and on their finite Tate duals all differ from one. Hence the local invariants at ℓ vanish on every module built by extensions of these lines and likewise on its Tate dual. This applies also to the successive constituents of B and M/B, since the residual representation is diagonal in the specified frame.

Take a composition series of C and argue by induction. For a short exact sequence $0 \to C_1 \to C \to C_2 \to 0$ in such a series, tensoring with M is exact. The image of a strict class in the last module vanishes by induction. Subtract a global coboundary lifting a witness in the quotient. The resulting cocycle takes values in $M \otimes C_1$. At ℓ it is a coboundary, and a witness

for that coboundary belongs to $M \otimes C_1$ because its image in $M \otimes C_2$ is invariant there. Away from ℓ, being unramified is equivalent to the cocycle vanishing on the kernel of the residual inertia action: the kernel acts trivially here, and its quotient in inertia has order prime to ℓ, on which averaging kills H^1. Vanishing on this kernel survives passage to the submodule. The class in the submodule is therefore strict, and induction kills it. The split residual lines give the starting case. This proves $H_{\mathrm{str}}(M \otimes C) = 0$.

Apply the finite Poitou–Tate order formula and the exact sequence for enlarging local conditions as in Theorem 38.8 [10]. For the strict conditions both global invariant groups vanish by the local invariant calculation. The factor at ℓ is one. Away from ℓ, Frobenius minus one on inertia invariants has kernel and cokernel of equal orders. At the real place all residual characters are odd, so the invariants vanish there as well, since two is invertible. The order formula gives zero dual Selmer group. The exact sequence from the strict to the relaxed conditions now gives the asserted localization isomorphism.

For each of $D = M, B, M/B$, local duality gives $H^2(D_\ell, D \otimes C) = 0$, and H^0 vanishes as above. Consequently $C \mapsto H^1(D_\ell, D \otimes C)$ is exact. Using a finite presentation of C and additivity on free modules identifies this functor with $H^1(D_\ell, D) \otimes_R C$. The local Euler characteristic at the residue field gives dimension $\operatorname{rank}_R D$. Exactness up a composition series then gives $|H^1(D_\ell, D)| = |R|^{\operatorname{rank}_R D}$. Nakayama's lemma gives a surjection onto this group from a free module of that rank, and equality of orders makes it an isomorphism. Applying the local cohomology sequence to $0 \to B \to M \to M/B \to 0$ proves the last assertion. □

Theorem 47.20 (One-sided ordinary extensions with fixed blocks)**.** *For every finite R-module C, the global off-diagonal classes with prescribed ordinary flag at ℓ in the two directions are respectively*

$$H^1(D_\ell, B_+) \otimes_R C, \qquad H^1(D_\ell, B_-) \otimes_R C.$$

They have unique global cocycle representatives whose value at h is zero. Each such representative defines an actual one-sided symplectic extension with the original diagonal blocks. More precisely, for a finite local R-algebra R' and a nilpotent ideal $I \subset R'$, use $C = I$ and pull the given block representation to R'. These representatives parametrize all the indicated extensions in that frame which reduce to the given representation modulo I. Every one of them has the original trace at every group element.

Proof. On M, conjugation by the diagonal matrix at h minus the identity is invertible: every eigenvalue is a ratio of distinct residual roots. The same holds on B and M/B. Subtracting a unique coboundary therefore normalizes any cocycle to have zero value at h. A normalized cocycle on D_ℓ belongs to $B \otimes C$ if and only if its class comes from $H^1(D_\ell, B \otimes C)$. Indeed a representative in $B \otimes C$ can be normalized there, and normalized representatives in $M \otimes C$ are unique. The original flag is the coordinate flag in this normalized frame by its four distinct residual roots. Thus this is precisely the ordinary condition on an off-diagonal extension. Lemma 47.19 supplies the unique global class with each prescribed local value.

Write ρ_0 for the block representation. For a cocycle z with values in $M_+ \otimes I$ put $\rho_z(g) = (1 + z(g))\rho_0(g)$. Two maps supported in this same off-diagonal block have zero product, even if $I^2 \neq 0$. The multiplication law for ρ_z is therefore exactly the cocycle law. Its multiplier is the prescribed one because the linear symplectic condition is the symmetry defining M_+, and the quadratic pairing of two images is zero on the Lagrangian P'. The restriction at ℓ preserves the flag, the normalization at h, and the graded characters. At every other finite prime the cocycle kills the required inertia kernel by the averaging argument in the lemma. Conversely, multiply any extension of this form by $\rho_0(g)^{-1}$ on the right and subtract one. The result is the required normalized cocycle, by the same matrix identities. Interchanging P, P' proves the

assertion for M_-. A matrix supported in just one off-diagonal block changes neither diagonal block in this formula, proving the trace assertion. □

Corollary 47.21 (The three ordinary directions at a plane point)**.** *At an integral two-plane point over a finite extension J' of the coefficient field, the normalized one-sided ordinary classes over O' are free of ranks two and one. Over J' their dimensions are two and one. Both spaces are invisible to all trace functions on one-sided extensions.*

Proof. Reduce a fixed integral block realization through $R = O'/\pi^j$. The local cohomology groups for $B_\pm$ commute with these reductions by Lemma 47.19. A basis at residue level lifts compatibly to bases at every level by freeness and Nakayama's lemma. The normalized global representatives also reduce compatibly by uniqueness. Taking their inverse limits gives continuous cocycles with values in the integral lattice, and every normalized integral cocycle gives such a system. Over J', a continuous cocycle on the compact global group has bounded image. Multiplying it by a power of π puts it in the lattice. The normalized local flag and inertia conditions are preserved by this operation and by division in J'. This identifies the two rational spaces with the integral ones tensored with J'. The last assertion follows from the matrix formula in Theorem 47.20. □

47.8 The mixed obstruction at an even plane

Let ρ_0 be an integral two-plane realization as in Section 47.7. Work over a finite coefficient field L containing its entries. Let X_+ and X_- be the spaces of normalized global one-sided ordinary cocycles over L. They have dimensions two and one by Corollary 47.21. An element g acts on matrix coefficients by conjugation by $\rho_0(g)$. We take cochains on $G_\Sigma = \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$, where $\mathbb{Q}_\Sigma$ is the maximal extension unramified outside the finite level primes, ℓ, and the real place.

On the neutral block of the fixed-multiplier Lie algebra we identify $N = \mathrm{End}_L(P)$ with the endomorphisms acting by w on P and by $-w^\vee$ on P'. Write $N_0 = \{w : \mathrm{tr}_P\, w = 0\}$. For $D = N, N_0$ let $\mathcal{C}^1(D)$ be the space of continuous D-valued one-cochains on G_Σ with the following constraints. They vanish on all N_q at level primes $q \neq \ell$. On D_ℓ they are upper triangular on P, at h they are diagonal, and on I_ℓ both diagonal entries are zero. For a cochain set

$$\partial w(g, g') = w(gg') - w(g) - gw(g').$$

Given $u \in X_+$ and $v \in X_-$, form the continuous two-cochain on the neutral block

$$H(u, v)(g, g') = \tfrac{1}{2}\big([u(g), gv(g')] + [v(g), gu(g')]\big).$$

The bracket is the matrix commutator. Project onto N_0 by subtracting half the trace on P times the identity on P. Write $K_v(u)$ for the class of this projection in

$$C^2_{\mathrm{cts}}(G_\Sigma, N_0)/\partial\mathcal{C}^1(N_0).$$

This quotient allows the obstruction to be stated directly in continuous cochains. The normalization at h makes $K_v(u)$ bilinear without any choices of representatives.

Proposition 47.22 (The traceless obstruction to mixing two extensions)**.** *Put $D_2 = L[\varepsilon, \eta]/(\varepsilon^2, \eta^2)$. The two one-sided deformations u, v on the two axes extend to one ordinary fixed-multiplier representation over D_2, with the prescribed inertia kernels and diagonal frame at h, if and only if $K_v(u) = 0$.*

Proof. Every matrix congruent to the identity modulo (ε, η) admits a logarithm by the degree-two power series. In fixed multiplier this logarithm lies in the symplectic Lie algebra: taking the symplectic adjoint intertwines logarithm, and the logarithm of the inverse has opposite sign. Write a candidate lift as

$$\exp(\varepsilon u(g) + \eta v(g) + \varepsilon\eta w(g))\rho_0(g).$$

Multiplying the two truncated exponentials gives the group law exactly when $\partial w = H(u, v)$. This also shows that $H(u, v)$ is a two-cocycle, either by associativity of three such products or by the cocycle identities for u, v. The right side is neutral. If any w works, its neutral component works, since the block decomposition is stable, and projection to its diagonal blocks preserves the stated local constraints. For neutral w, the constraints are precisely $w \in \mathcal{C}^1(N)$. Indeed u, v already vanish on the forbidden inertia kernels and at h. Locally at ℓ they are strictly upper triangular, so every quadratic product there has zero diagonal and respects the ordered flag.

It remains to solve for the scalar part of w. The scalar coefficient on P has trivial Galois action. We claim $H^2(G_\Sigma, L) = 0$. At every finite place in Σ the dual invariant group for $L(1)$ is zero, hence so is the local H^2. Global duality reduces the claim to the injectivity of localization on $H^1(G_\Sigma, L(1))$ [10]. By Kummer theory and the trivial ideal class group of $\mathbb{Z}[1/\prod_{q\in\Sigma_f} q]$, the latter space is spanned freely over L by the Kummer classes of the rational primes in the finite subset Σ_f. Their valuations at those same primes form the identity matrix. The claim follows.

There is consequently a continuous scalar primitive w_s for the scalar component of $H(u, v)$. At $q \neq \ell$ the two cocycles and the action there factor through D_q/N_q. This local quotient has finite inertia of order prime to ℓ and procyclic unramified quotient, so its H^2 for L is zero. Choose a scalar primitive on that quotient. Its difference from $w_s|_{D_q}$ is an additive character. By local reciprocity every such L-valued character of $\mathbb{Q}_q^\times$ kills the units, since their open pro-q subgroup and their finite quotient both have zero image. Thus w_s already vanishes on N_q. At ℓ the scalar two-cochain is zero because both products in the bracket are strictly upper triangular. The restriction of w_s is now an additive character. Its restriction to inertia is a scalar multiple of the restriction of $\log_\ell \kappa_\ell$, by local reciprocity and the rank one of the principal units of $\mathbb{Q}_\ell$. Subtract that multiple of the global cyclotomic logarithm. The resulting scalar primitive has all the required local conditions. Adding it to any traceless primitive proves sufficiency. Conversely a permitted neutral primitive projects to a permitted traceless one. This proves necessity. □

Theorem 47.23 (The transverse trace tangent space)**.** *Suppose $a_+ = 0$, $a_- = 1$ and the trace algebra T is finite. At a nonzero two-plane point σ let A_σ be its local characteristic-zero trace algebra after a finite extension L splitting the point. Put*

$$I = (V, W), \qquad m = \dim_L(A_\sigma/I), \qquad n = \dim_L A_\sigma.$$

For any $0 \neq v \in X_-$, the image of the tangent space of A_σ in the two trace coordinates V, W is isomorphic to $\ker K_v$. In particular

$$\dim_L \mathfrak{m}_\sigma/\mathfrak{m}_\sigma^2 = \dim_L \ker K_v + \mathbf{1}_{\{m>1\}},$$

where $\mathfrak{m}_\sigma$ is the maximal ideal of A_σ.

Proof. Use the two-torus chart of Theorem 40.6 after eliminating the two neutral parameters. Its variables are a, x, b, c, y. Before taking invariants the equations are E, F_1, F_2. They depend only on x, ab, bc, y by their zero first torus weight. As in Theorem 47.16, the trace variables are

$$U = xy, \qquad V = yab, \qquad W = y^2bc.$$

There are series e, f_1, f_2 in U, V, W such that $yE = e$ and $F_i = yf_i$. Indeed yE is neutral and every monomial of weight -1 is y times a neutral monomial. The integral block realization gives values x_0, y_0 in the maximal ideal of the coefficient integers with $x_0y_0 = U_\sigma \neq 0$; all of a, b, c are zero there. Enlarge L to contain these values if needed. At this point the trace equations are exactly e, f_1, f_2, because U is invertible in A_σ. A tangent vector (dU, dV, dW) must annihilate the differentials of these three series at $(U_\sigma, 0, 0)$, and that condition suffices. The first-order Taylor evaluations converge since U_σ has positive valuation. Conversely every map from $T \otimes_O L$ to dual numbers at this point is continuous on the finite O-module T; approximating its series by polynomials gives the same Taylor rule.

At the block realization the first torus acts trivially on the point. Its weights on a, c are $+1$, on b are -1, and on x, y are zero. Therefore on a one-sided positive tangent only a, c can vary, say by a_1, c_1, and on a negative tangent only b can vary, say by b_1. These coordinates detect the normalized cocycles: if their variations all vanish the local six-parameter chart is unchanged, since the eliminated neutral coordinates also have zero derivative on a nonzero weight. The normalized local cocycle is then zero, hence so is the global one by Lemma 47.19. By the dimensions in Corollary 47.21, the maps to $(a_1, c_1) \in L^2$ and $b_1 \in L$ are isomorphisms. In particular $b_1 \neq 0$ for the chosen v.

If a mixed lift exists, the trace is constant on each axis. On D_2 its trace coordinates consequently change only in degree $\varepsilon\eta$, with

$$dV = y_0a_1b_1, \qquad dW = y_0^2c_1b_1.$$

To apply the chart also to a continuous lift over D_2, choose a uniformizer π of L. The coefficients of its three nilpotent terms on the compact Galois group are bounded. For large k all entries lie in the finite free integer algebra on $\pi^{-k}\varepsilon, \pi^{-k}\eta$ inside D_2, with the prescribed residual representation. Reduction of this algebra through every power of π and completeness give the desired chart evaluation.

Conversely, take a tangent vector with the displayed dV, dW and some dU. In the original chart set

$$y = y_0, \quad x = x_0 + \varepsilon\eta\, dU/y_0, \quad a = \varepsilon a_1, \quad c = \varepsilon c_1, \quad b = \eta b_1.$$

All power series converge by Taylor expansion about the integral point. Their traces are precisely the assigned tangent vector multiplied by $\varepsilon\eta$. Thus $e = f_1 = f_2 = 0$ there, and invertibility of y_0 over L gives all three original equations. The specialization of the universal matrices is continuous, as one also sees by scaling the nilpotent generators as above and evaluating in their complete integer algebra. Its two axis restrictions have exactly the prescribed normalized cocycles by the coordinate isomorphisms. Proposition 47.22 gives the stated equivalence. Since $y_0b_1 \neq 0$, the displayed map from (a_1, c_1) to (dV, dW) is an isomorphism of the two ambient planes.

Finally, tangents with zero projection to V, W factor through A_σ/I. This local block algebra is a quotient of the one-variable completed local ring at U_σ, by Theorem 47.16. Its tangent dimension is zero if $m = 1$ and one otherwise. Taking dimensions proves the formula. □

Corollary 47.24 (Multiplicity when the mixed obstruction is injective)**.** *If $m = 1$, the full trace algebra at σ is reduced exactly when K_v is injective. For arbitrary m, if K_v is injective then $n \leq md$ whenever $I^d = 0$. Thus if all the two-plane points in Corollary 47.18 have injective K_v, its length estimate improves to*

$$\operatorname{length}_O S(\alpha, r) \leq 2v_\ell(D_{\mathrm{cusp}}) + 2d\, v_\ell(L(1 - r, \alpha)).$$

Proof. For $m = 1$, the tangent space vanishes exactly when the maximal ideal is zero, by Nakayama's lemma. For $m > 1$ and K_v injective, the cotangent space is a line generated by the class of $t = U - U_\sigma$: it maps onto the cotangent line of the block quotient, where this class generates. Nakayama's lemma and nilpotence then give $A_\sigma = L[t]/(t^n)$. Indeed t generates the maximal ideal, successive powers span its nonzero graded pieces, and the nilpotence order of t equals the length of the algebra. Its ideal with quotient of length m is (t^m). Hence $I^d = 0$ implies $n \le md$. Use this inequality in the sum over two-plane points in Theorem 47.16; the rank-four term is the one in Corollary 47.18. □

The scalar part of the mixed equation always has a solution. The action of the traceless obstruction on the two-dimensional space X_+ still has to be determined arithmetically.

48 Auxiliary towers and cuspidal points

48.1 An explicit tame tower for the odd-order characters

Return to the fields E_x and triples of Section 47.3. Consider its case III, where $m = \mathrm{ord}(\alpha) > 1$ is odd. The character α has conductor dividing the fixed conductor F, is unramified at two, and r is even. Write

$$\tau = \bar{\alpha}\,\bar{\omega}_\ell^{\,r}, \qquad \chi = \bar{A} = \bar{\xi}_2\bar{\xi}_0, \qquad \Delta = \chi^2/\tau.$$

The character ξ_2 is odd with conductor M_α a power of two; ξ_0 has order m and conductor p_1. Bars denote residual characters. All the finite character values occurring here have order prime to ℓ. We construct the auxiliary totally real and CM fields in `Development/AutomorphicArithmetic/AutomorphicVarieties/WeakShimuraTheory/NearSplit.lean`, line 141 for the ordered residual tuple $(\chi\tau, \tau, \chi, 1)$.

Lemma 48.1 (One auxiliary quadratic pair throughout the interval)**.** *There are distinct primes $p, p' \equiv 7 \bmod 8$, the same for all the triples on the interval, such that every prime in the prescribed levels and every coefficient prime of those triples splits in both $K = \mathbb{Q}(\sqrt{-p})$ and $K' = \mathbb{Q}(\sqrt{-p'})$. The prime p also splits in K'. They can be chosen with*

$$\log(pp') \ll_\psi (t_* + 1)\log(2t_* N^\natural).$$

Proof. Take the set $\mathcal{S}$ of prime divisors of $2N^\natural F p_1$ together with all primes at most $2t_* + 1$. A coefficient prime is included since $\ell - 1 \mid 2r$ and $r < t(x) \le t_*$. Let $B = 8\prod_{q \in \mathcal{S},\, q \text{ odd}} q$. Choose primes $p \equiv -1 \bmod B$ and $p' \equiv -1 \bmod Bp$. The discriminants are respectively $-p, -p'$, both one modulo eight. At every odd prime of $\mathcal{S}$ these discriminants are one modulo that prime, and $-p'$ is also one modulo p. The quadratic splitting criterion gives all the assertions about places. Linnik's least-prime theorem gives $p \ll B^c$ and $p' \ll (Bp)^c$ for an absolute c. The logarithm of B has the asserted bound by summing the logarithms of the integers up to $2t_* + 1$ and adding $\log(8N^\natural F p_1)$. □

Theorem 48.2 (An abelian totally real base with separated residual characters)**.** *Let H_0 be the cyclic field cut out by Δ, and set*

$$H = H_0\,\mathbb{Q}(\sqrt{pp'}), \qquad H' = HK.$$

Then H is totally real and abelian over $\mathbb{Q}$ of even degree, and H'/H is CM quadratic and unramified at every finite place. One has

$$[H' : \mathbb{Q}] = 4\,\mathrm{ord}(\Delta) \le 4M_\alpha m, \qquad \ell \nmid [H' : \mathbb{Q}].$$

The degree and logarithmic conductor of H' are bounded by a polynomial in the interval parameter, uniformly in the triple. The following properties hold for these fields.

1. *On G_H one has $\tau = \chi^2$. On $G_{H'}$ there is an element g_0 with $\bar{\omega}_\ell(g_0) = 1$ at which χ has order $2m$. The four ordered residual characters are separated there.*

2. *At every place of H above ℓ the residual diagonal is trivial on inertia, while $\bar{\omega}_\ell$ is nontrivial on that inertia. The ramification index over $\mathbb{Q}_\ell$ divides two, and the place splits in H'.*

3. *On $G_{H'}$ all eight symplectic roots are nontrivial, and each differs from $\bar{\omega}_\ell$. The only quadratic character factoring through $\bar{\omega}_\ell$ which permutes the four residual characters by multiplication is the trivial one.*

Proof. The residual character Δ is even since τ is even and χ is odd. Its field H_0 is totally real and unramified outside $2Fp_1\ell$. It is disjoint from $\mathbb{Q}(\sqrt{pp'})$ since the latter ramifies at p. Their compositum has degree $2\,\mathrm{ord}(\Delta)$. Adjoining K doubles this degree since H is totally real. The resulting quadratic extension is totally imaginary, with conjugation fixing H. A finite prime different from p is unramified in $K/\mathbb{Q}$, and remains unramified or splits after base change. At p one has $H' = HK'$ since H already contains $\sqrt{pp'}$. The splitting of p in K' gives splitting there after base change as well.

The order of Δ divides $M_\alpha m$: its dyadic factor has order dividing M_α, its odd-order factors have orders dividing m, and $\bar{\omega}_\ell^r$ has order at most two. Also Δ takes values in the multiplicative group of a finite field of characteristic ℓ. These facts give the degree claims. A cyclotomic modulus for H' divides $\ell M_\alpha F p_1 pp'$. In the formula for M_α of Section 47.3, f_0 is bounded in terms of ψ since $\mathrm{ord}\,\alpha \mid o$. Thus $4^{v_2(\ell^{f_0}-1)} \le \ell^{2f_0}$ and M_α is polynomial in $1+\ell$. Lemma 48.1 now gives the logarithmic conductor bound.

To construct g_0, choose a residue at $\mathrm{cond}(\alpha)$ where $\bar{\alpha}$ equals a primitive m-th root z. Squaring is an automorphism of the group of m-th roots, so choose its unique solution $y^2 = z$ in that group. Choose a residue at p_1 where $\bar{\xi}_0 = y$, the residue -1 at M_α, and the residue one at ℓ, p, p'. These moduli are pairwise coprime: p_1 is prime to $2F$, α has odd conductor, and the other exclusions follow from Section 47.3 and the two progressions. The Chinese remainder theorem and the Galois group of the cyclotomic field give an element of $G_{\mathbb{Q}}$ with all these values. It has $\chi = -y$, $\tau = z = \chi^2$, and fixes both quadratic fields. It therefore belongs to $G_{H'}$. The order of $-y$ is $2m \ge 6$. On G_H, by the definition of H_0, the four residual characters are $\chi^3, \chi^2, \chi, 1$; their values at this element are distinct.

At ℓ both χ and $\bar{\alpha}$ are unramified. The restriction of Δ to inertia is $\bar{\omega}_\ell^{-r}$, of order at most two, and the two auxiliary quadratic fields split there. This proves the ramification bound and, on inertia over H, forces $\tau = \chi = 1$. Since $\bar{\omega}_\ell$ has order $\ell - 1$ on rational inertia and this inertia subgroup has index at most two, its restriction has order at least $(\ell-1)/2 > 1$. Splitting at ℓ allows this same element of inertia to be used on $G_{H'}$.

The roots are $\chi^{\pm 1}, \tau^{\pm 1}, (\tau/\chi)^{\pm 1}, (\chi\tau)^{\pm 1}$, with their indicated multiplicities as root spaces. On G_H their exponents as powers of χ belong to $\{\pm 1, \pm 2, \pm 3\}$. They are all nontrivial at g_0, while they are all trivial at the inertia element just chosen where $\bar{\omega}_\ell \neq 1$. Finally a character factoring through $\bar{\omega}_\ell$ has value one at g_0. Multiplying by it can fix the set of the four characters only by fixing each character, because their values there are distinct. Cancellation gives the trivial character. □

These calculations also check the remaining elementary conditions on the tower in `Development/AutomorphicArithmetic/AutomorphicVarieties/WeakShimuraTheory/NearSplit.lean`, line 211. Use the original order of the four lines. Each rational level prime splits in K. For every place of H over ℓ choose an extension to the algebraic closure inducing the prescribed one of the two places of K; this is possible since K is split there. The root χ has order greater than four at g_0. At a prime away from ℓ, triviality of the residual diagonal on an inertia element

implies $\bar{\alpha} = 1$ there since $\bar{\omega}_\ell$ is unramified. It implies $\alpha = 1$ since its order is prime to ℓ. The same order m kills α even on the entire group. Thus there is no distinguished ramification prime besides ℓ. The reduction of the second root equals the square of the first root on G_H. This verifies the field and character requirements in case III; the construction of the automorphic family over this base is a further step.

Corollary 48.3 (Descent of the strict conditions to the period field)**.** *Put $B = E_x$ or $B = \mathbb{Q}$, and $B' = BH'$. For a discrete ℓ-primary G_B-module W, restriction identifies the global H^1 classes over B that are zero at the places above ℓ and unramified elsewhere with the* $\mathrm{Gal}(B'/B)$*-invariants of the group defined by the same conditions over B'. The assertion also holds with no condition at ℓ.*

Proof. The extension B'/B is Galois of degree dividing $[H' : \mathbb{Q}]$, hence prime to ℓ. Cohomology in positive degree of this finite group with ℓ-primary coefficients is zero by averaging. Inflation–restriction identifies the two global groups before imposing local conditions. At each coefficient place restriction on H^1 is injective by the same argument for the local extension. At every other finite place the inertia subgroups have quotient of order dividing the global degree, and the same injectivity holds on their H^1. Consequently a global class has each of the specified local restrictions zero exactly when its restriction upstairs does. This gives the assertion on the original E_x for each x in the interval. □

48.2 The unramified quadratic character in the even-order tower

For case II of Section 47.3 put $m = \operatorname{ord} \alpha$. Thus m is even, $\ell \nmid m$, $\tau = \bar{\alpha}\bar{\omega}_\ell^r$, and $\chi = \bar{A} = \bar{\xi}_2$ is the reduction of the odd primitive dyadic character, of conductor $M_\alpha = 2^d$. Write 2^{c_2} for the conductor of the dyadic part of α, and set $b = \max(2, c_2)$. Retain the primes p, p', modulus B, and imaginary quadratic fields K, K' of Lemma 48.1.

Lemma 48.4 (A real quadratic field common to all even-order triples)**.** *One may choose a single prime $p_3 \equiv 1 \bmod Bpp'$ throughout the interval. Write $K_0 = \mathbb{Q}(\sqrt{p_3})$ and ϕ for its quadratic character. This field splits at every prime of $\mathcal{S}$, and p_3 splits in both K, K'. Moreover $\log p_3 \ll_\psi (t_*+1)\log(2t_*N^\natural)$. Let H_0 be the field of the character τ/ϕ, and put $H = H_0\mathbb{Q}(\sqrt{pp'})$, $H' = HK$. Then*

$$[H : \mathbb{Q}] = 2m, \qquad [H' : \mathbb{Q}] = 4m, \qquad \ell \nmid [H' : \mathbb{Q}].$$

Here H is totally real abelian, H'/H is CM quadratic and unramified at every finite place, and every place of H above ℓ splits in H'.

Proof. Use Linnik's theorem once more for the residue class one modulo Bpp', using the bound on $\log(pp')$ from Lemma 48.1. The discriminant p_3 is one modulo eight and one at each odd prime in $\mathcal{S}$, so these primes split in K_0. The Dirichlet characters of K, K' have conductors p, p', and both take the value one on p_3.

The character τ/ϕ is even and its order divides m, since $\bar{\omega}_\ell^r, \phi$ have order at most two. Choose a residue giving $\bar{\alpha}$ order m and prescribe one at ℓ, p_3. These three moduli are pairwise coprime, so this gives a value of order m for τ/ϕ. The field H_0 is therefore totally real of degree m and unramified at p, p'. The real quadratic field $\mathbb{Q}(\sqrt{pp'})$ ramifies at p, so their intersection is trivial. Adjoining K doubles the degree again. At p write $H' = HK'$ and use the splitting of p in K'. At every other finite place $K/\mathbb{Q}$ is unramified. The conclusions about H'/H follow, including splitting at ℓ since ℓ splits in K. □

Theorem 48.5 (The tame even-order tower with its two local witnesses)**.** *On G_H the even root τ equals the character of HK_0/H. This character is nontrivial, unramified at every finite place, and trivial on each decomposition group above ℓ. The following further properties hold.*

1. *There is a single $g_0 \in G_{H'}$ with $\bar{\omega}_\ell(g_0) = 1$, $\tau(g_0) = -1$, and $\chi(g_0)^4 \neq 1$. The residual entries $(\chi\tau, \tau, \chi, 1)$ there are distinct.*

2. *At every place above ℓ the inertia index over $\mathbb{Q}_\ell$ divides two. The residual diagonal is trivial there on inertia, and $\bar{\omega}_\ell$ is nontrivial on that same inertia. Above ℓ the decomposition group contains an element on which $\chi^4 \neq 1$.*

3. *At every place above two the inertia group contains an element on which $\chi^4 \neq 1$.*

These assertions about local groups hold over H; the witnesses above two and ℓ may also be taken over H'.

Proof. The equality $\tau = \phi$ on G_H is its defining character equation. Away from p_3 the character ϕ is unramified. At p_3 the character τ is unramified, so their common restriction to inertia over H is trivial there too. Since ℓ splits in K_0, τ is trivial on the entire decomposition group over ℓ.

We first record the available variation at two. The group $1+2^b\mathbb{Z}_2$ in the units modulo 2^d has order 2^{d-b}. It is generated by $5^{2^{b-2}}$: one has $v_2(5^{2^j} - 1) = j + 2$ by induction on j. Primitivity of the dyadic character makes it faithful on the cyclic subgroup $1 + 4\mathbb{Z}_2$ modulo 2^d. Indeed any nontrivial kernel there contains its subgroup of order two, which is the kernel of reduction from 2^d to 2^{d-1}. Reduction in characteristic $\ell \neq 2$ preserves the orders. Thus on $1 + 2^b\mathbb{Z}_2$ the order of the image of χ is exactly $2^{d-b} > 4$, by the lower bound for M_α in case II. This whole subgroup acts trivially on α_2.

Choose a residue at $\operatorname{cond}\alpha$ giving $\bar{\alpha} = -1$, which is possible since its image is cyclic of even order with faithful reduction. Lift its dyadic component to the units modulo 2^d, and vary it by $1 + 2^b\mathbb{Z}_2$. The value of $\bar{\alpha}$ stays fixed. The possible values of χ form a coset of size greater than four, so at least one has fourth power different from one. Prescribe one modulo ℓ, p, p' and choose a nonsquare modulo p_3, so that $\phi = -1$. The Chinese remainder theorem applied to the separate prime powers and cyclotomic Galois theory realize all these values on one g_0. It fixes H_0, K, K', satisfies $\tau(g_0) = -1$, and belongs to $G_{H'}$. This proves nontriviality of $\phi|_{G_H}$. The indicated four entries at g_0 are $-\chi(g_0), -1, \chi(g_0), 1$, pairwise distinct since ℓ is odd and $\chi(g_0)^2 \neq 1$.

At ℓ, the restriction of τ/ϕ to rational inertia is $\bar{\omega}_\ell^r$, of order at most two. The quadratic fields K, K' split there. Hence the ramification index divides two, χ is unramified, and $\tau = \phi$ is trivial on inertia over H. The cyclotomic reduction still has order at least $(\ell - 1)/2 > 1$ there. For the decomposition group choose the arithmetic Frobenius lift s at ℓ with $\kappa_\ell(s) = 1$ from Lemma 47.7. It satisfies

$$\operatorname{ord}(\chi(s)) \geq M_\alpha/(\ell + 1), \qquad j = \operatorname{ord}(\tau(s)) \leq 2^{c_2+1}o.$$

The second inequality uses $\tau(s) = \bar{\alpha}(\ell)$. Then s^j fixes H_0, K, K', since ϕ, K, K' are split at ℓ. The order of $\chi(s^j)$ is at least $M_\alpha/((\ell + 1)j) > 4$ by the case II conductor bound. Conjugating the chosen decomposition group gives a witness at every place since the characters are abelian.

Finally, rational inertia at two acts on the 2^d-th roots of unity through all the dyadic units, by total ramification. Choose an element mapping to a generator of the subgroup $1 + 2^b\mathbb{Z}_2$ modulo 2^d. Its $\bar{\alpha}$-value is one by the preceding computation and unramifiedness of the odd-conductor part. The cyclotomic character at ℓ is unramified there. Also K_0, K, K' split at two. The element belongs to inertia over H' and has $\chi^4 \neq 1$. Conjugation again supplies the other places. □

Corollary 48.6 (Uniform descent in both source cases)**.** *For all the triples of Section 47.3 the two original auxiliary field cases admit the ordered tower, residual separation, and inertia conditions of* `Development/AutomorphicArithmetic/AutomorphicVarieties/WeakShimuraTheory/NearSplit.lean`*, line 211. The finite characters used for K, K', K_0 may be fixed on the entire interval. Each tower has polynomial degree and polynomial logarithmic conductor, and the strict and relaxed groups on E_x descend from E_xH' by the invariant formula of Corollary 48.3.*

Proof. The odd-order assertions are Theorem 48.2. In the even-order case a cyclotomic modulus for H' divides $\ell N p_3 p p'$. Its logarithm has the required bound by the two prime constructions, and its degree is $4m \le 4n_*$ by Lemma 48.4. The extension is prime to ℓ, which proves descent by the same inflation–restriction argument.

For the remaining character conditions use the original ordered tuple. All symplectic roots are ratios of distinct entries at g_0 and hence nontrivial there. The inertia witness above ℓ has all residual entries one and cyclotomic reduction different from one; in particular every monomial in the residual roots differs from the cyclotomic character. At g_0 the latter is one, so a quadratic character factoring through it can permute the four characters only by fixing each, as in the odd-order case. The root χ has two-power order greater than four on g_0. The field K_0 required for the even root is split at two and at ℓ. Away from ℓ triviality of the residual diagonal on inertia forces $\alpha = 1$ on that element, since α has order prime to ℓ; its order also kills it globally. The prescriptions for the marked places above ℓ are possible because K splits there, and all level primes split in K. This proves the stated conditions. □

48.3 One splitting field for the trace algebras on the interval

Keep the period fields E_x, the upper field E_*, and both auxiliary field constructions of Corollary 48.6. We compare integral trace algebras before taking any characteristic-zero points. For a finite Galois extension $B/\mathbb{Q}$, write T_B for the closed O-subalgebra of the global trace algebra T generated by the traces on G_B. We use the topology induced by the complete local representation ring.

Proposition 48.7 (A common field of polynomial Galois exponent)**.** *There is a single finite abelian field $\mathcal{F}/\mathbb{Q}$ for the interval with the following properties. It contains all E_x and all H' of both source cases, and every residual diagonal character of every indicated triple is trivial on $G_{\mathcal{F}}$. The degree is prime to each coefficient prime of the triples. Its Galois exponent D and its logarithmic conductor are polynomial in V. In particular the strict and relaxed groups at each E_x can be recovered as the* $\mathrm{Gal}(\mathcal{F}/E_x)$*-invariants of the corresponding groups over this single field.*

Proof. Let M_* be the largest of the dyadic conductors M_α on the interval, using 8 if there are none. Let o_{odd} be the odd part of o, and let C_1 be the unique subfield of degree o_{odd} in $\mathbb{Q}(\mu_{p_1})$, allowing $C_1 = \mathbb{Q}$. This exists since the Galois group is cyclic and $p_1 \equiv 1 \bmod o_{\mathrm{odd}}$. Let $\mathcal{L}$ consist of the coefficient primes that occur. Take the compositum of

$$E_*, \quad \mathbb{Q}(\mu_{M_*}), \quad C_1, \quad K, \quad K', \quad K_0, \quad \mathbb{Q}\big(\sqrt{(-1)^{(\ell-1)/2}\ell}\big) \quad (\ell \in \mathcal{L}).$$

The last fields are the quadratic subfields of the prime cyclotomic fields. They kill $\bar{\omega}_\ell^r$ since $\ell - 1 \mid 2r$. The field E_* kills $\bar{\alpha}$; the second field kills all dyadic $\bar{\xi}_2$; and C_1 kills every $\bar{\xi}_0$ since its order divides o_{odd}. This proves triviality of the residual diagonal. The displayed fields kill both characters χ^2/τ and τ/ϕ used in the respective constructions and contain $KK'K_0$. Thus they contain every required H'.

The Galois group of the compositum embeds in the product of those of the constituent fields. All their degrees are prime to each $\ell \in \mathcal{L}$ by the exclusions $\ell \nmid n_* o$ and $\ell \geq 5$. Their exponents give

$$D \mid \operatorname{lcm}(n_*, M_*, o_{\mathrm{odd}}, 2).$$

For both formulas of Section 47.3, f_0 is bounded in terms of the fixed ψ, $4^{n_0} \leq \ell^{2f_0}$, and $2^{c_2} \leq N^\natural$. Thus M_* is polynomial in V, since $N^\natural$ and t_* are polynomial and $\ell \leq 2t_* + 1$. A cyclotomic modulus containing the compositum is

$$8N^\natural M_* p_1 p p' p_3 \prod_{\ell \in \mathcal{L}} \ell.$$

Its logarithm is polynomial by Lemmas 48.1 and 48.4. Finally apply the inflation–restriction argument of Corollary 48.3 to $\mathcal{F}/E_x$. □

Theorem 48.8 (Integral traces from primes split on the entire interval)**.** *For every indicated triple the global trace algebra T is a finite module over $T_{\mathcal{F}}$. This subalgebra is generated topologically by good rational Frobenius traces at primes split completely in $\mathcal{F}$. Every trace $\operatorname{tr}\rho(g)$ satisfies a monic equation of degree D^4 with coefficients in $T_{\mathcal{F}}$, including on nonreduced coefficient rings. In particular T is finite over O exactly when $T_{\mathcal{F}}$ is finite over O.*

Proof. For a positive integer d and indeterminates X_i form the polynomial

$$\prod_{\zeta_1,\dots,\zeta_4 \in \mu_d} \left(Z - \sum_{i=1}^{4} \zeta_i X_i \right).$$

Its coefficients as a polynomial in Z, X_i are cyclotomic integers fixed by every automorphism of the cyclotomic field, hence rational integers. Multiplying any one X_i by a d-th root of unity preserves the product. Every exponent of X_i in a nonzero monomial is consequently divisible by d. Permutation of the four variables also preserves the product. The fundamental theorem of symmetric polynomials now writes the coefficients in Z as integral polynomials in the elementary symmetric functions of the X_i^d. The polynomial is monic of degree d^4 and vanishes at $Z = \sum_i X_i$.

For any four-by-four matrix A this gives an equation for $\operatorname{tr} A$ in the characteristic coefficients of A^d. To justify it over arbitrary coefficient rings, first take the generic matrix over the polynomial ring in sixteen independent entries over $\mathbb{Z}$. The equation holds over a splitting field of its characteristic polynomial by the eigenvalue computation. Its two sides belong to the integer polynomial ring, which injects into that field. Specialization therefore gives the identity even over rings with nilpotents.

Use $d = D$ and $A = \rho(g)$. Then $g^D \in G_{\mathcal{F}}$. Newton's identities express the characteristic coefficients of A^D using the traces of its first four powers with denominators invertible in O, since $\ell \geq 5$. These coefficients all lie in $T_{\mathcal{F}}$. By Corollary 40.11 the augmentation ideal of T is generated by a finite list of good-prime trace differences. Their traces generate T topologically over O: adjoining ℓ to those differences generates its maximal ideal, and successive approximation in that ideal gives the assertion. If the list has length b, adjoining its traces to $T_{\mathcal{F}}$ as an ordinary algebra requires at most D^{4b} module generators by the monic equations. This algebra is already closed: the coefficient algebra $T_{\mathcal{F}}$ is compact, and the finite list of monomials gives a continuous map from a finite power of it onto their module span. Thus the span is all of T. One direction of the finiteness equivalence follows by transitivity. In the other, $T_{\mathcal{F}}$ is a submodule of a finite module over the Noetherian ring O.

For any $g \in G_{\mathcal{F}}$ and any finite quotient of the integral representation through a power of the maximal ideal, take the compositum of its fixed field with $\mathcal{F}$. The conjugacy class of g in this

finite quotient restricts to the identity on $\mathcal{F}$. Chebotarev gives a good rational prime in the class, with the same trace in the quotient. Varying the precision proves the asserted generation by split primes. □

Proposition 48.9 (Equality of the integral algebras over a separated base)**.** *For the particular auxiliary field H' of each triple, one has $T = T_{H'}$. The augmentation ideal is generated as a closed ideal by the good-prime trace differences at primes split in H'. In case II with $\ell - 1 \nmid r$, both equalities hold with $E_x H'$ in place of H'.*

Proof. Here is an integral group algebra argument. Let $\mathcal{R}$ be the complete Noetherian local coefficient ring of the given lift with finite residue field, and G its compact matrix image. Suppose a normal subgroup has closed image G_1 of index prime to ℓ, containing an element whose residual diagonal has four distinct entries. The residual image D_0 of G is a finite group of diagonal matrices of order prime to ℓ. Its kernel P is pro-ℓ, since each successive principal congruence quotient embeds in an additive group in residual characteristic. Hence $P \subset G_1$.

There exists a finite subgroup of G mapping isomorphically to D_0. In detail, lift the identity section of D_0 successively through the finite images modulo powers of the maximal ideal. At each step the inverse image of the previous section is an extension of D_0 by an abelian ℓ-group. A set-theoretic section has a multiplicative defect in its second cohomology, which vanishes by averaging since $|D_0|$ is invertible on the kernel. Correcting the section gives a homomorphism lifting the preceding one. Compactness of G identifies the inverse limit of the compatible sections with a section into G. Denote its image by D_1, so $G = PD_1$.

Take $s \in D_1$ lifting the separated residual element. It is in G_1 because $P \subset G_1$. The eigenvalues of s are the Teichmüller lifts of its four residual entries in O: its finite order is prime to ℓ, and the corresponding simple roots of unity lift uniquely. More explicitly the roots of the prime-to-ℓ order polynomial split over O for this element since its order is the order of its diagonal reduction. Lagrange interpolation in s gives four projectors in $O[s]$ with rank-one free images over $\mathcal{R}$. Every element of the abelian group D_1 acts by a scalar on each such line. Its scalar is again the Teichmüller lift of its residual character, by uniqueness of the prime-to-ℓ root of unity in that residue class. The coefficient ring O contains these character values by the choice of coefficients in Section 47.3. It follows that $D_1 \subset O[s]$, whence

$$O[G] = O[G_1] \quad \text{inside } M_4(\mathcal{R}).$$

Taking traces shows that every global trace is already an O-linear combination of traces on the subgroup. Applying the augmentation of the universal lift to that same equality and subtracting gives the assertion for trace differences. Chebotarev at finite precision replaces the subgroup elements by split rational primes, as above.

For $G_{H'}$ the two requirements are the prime-to-ℓ index and the element g_0 of Theorem 48.2 or 48.5. In the stated subcase of case II prescribe one modulo the odd part of N and one modulo $2^{v_2(N)}$. This fixes E_x. At the dyadic modulus M_α, vary the latter residue by $1+2^{\max(2,v_2(N))}\mathbb{Z}_2$. Its χ-image has order $M_\alpha/2^{\max(2,v_2(N))} > 4$ by the conductor bound and the cyclic calculation in Theorem 48.5. Choose $\chi^2 \neq 1$. Independently choose $\bar{\omega}_\ell^r = -1$, possible because $\ell-1 \mid 2r$ and $\ell - 1 \nmid r$. Require $\phi = -1$ at p_3 and one at p, p'. These prescriptions give $\bar{\alpha} = 1,\ \tau = -1 = \phi$, so the element fixes $E_x H'$ and its four entries $-\chi, -1, \chi, 1$ are distinct. The field $E_x H'$ has degree prime to ℓ. The same proof applies. □

The finite-module result uses all the primes split in one explicit field on the interval. Estimating the resulting restricted trace algebra is still an arithmetic requirement.

48.4 An integral denominator for traces at common split primes

Keep the single field $\mathcal{F}$ of Proposition 48.7. Every residual diagonal character is trivial on its Galois group. We use the known ordinary inertia characters in the integral lift to recover the four diagonal projectors with a bounded denominator. Write G for the compact image of $G_{\mathbb{Q}}$ in the universal normalized representation over $\mathcal{R}$, and P for its kernel of reduction. Write $O[G]$ for the subalgebra of $M_4(\mathcal{R})$ algebraically generated by the group matrices over O, and similarly for subgroups.

Lemma 48.10 (The diagonal finite subgroup in the ordinary frame)**.** *One has $P = \rho(G_{\mathcal{F}})$. The group G contains the finite diagonal lifts of its residual image in the frame normalized at h: for each residual matrix it contains the diagonal matrix of the Teichmüller lifts of its four entries on the diagonal.*

Proof. Triviality of the residual diagonal gives $\rho(G_{\mathcal{F}}) \subset P$. The quotient $G/\rho(G_{\mathcal{F}})$ has order dividing $[\mathcal{F} : \mathbb{Q}]$, which is prime to ℓ. The kernel P is pro-ℓ, as in Proposition 48.9. Its image in this quotient is therefore trivial, proving equality.

Let D_0 be the residual image and $C_0 \subset D_0$ the cyclic subgroup generated by $\bar{\rho}(h)$. Its lift by the diagonal Teichmüller matrix s is already in G. Indeed $\rho(h)$ is diagonal and $\rho(h)^{\ell^{j!}}$ tends to s: in each finite quotient the principal-unit part is killed for large j, and $\ell^{j!}$ is one modulo the prime-to-ℓ order for all sufficiently large j.

We can choose a section $D_0 \to G$ agreeing with this specified lift on C_0. Use the successive finite congruence quotients from Proposition 48.9. Suppose the agreement holds at one level. After lifting the section to the next level using the vanishing of H^2, its restriction and the specified section of C_0 differ by a one-cocycle in the abelian congruence kernel. Averaging over C_0 makes that cocycle a coboundary. Conjugating the new section by the corresponding kernel element restores agreement and leaves its previous reduction fixed. Induction and compactness give the section into G. Its image commutes with s since D_0 is abelian. The diagonal entries of s differ pairwise by units; commuting with s thus forces all off-diagonal entries to be zero. The diagonal entries have order prime to ℓ and the prescribed residues, so they are their unique Teichmüller lifts. This proves the assertion in the specified frame. □

Theorem 48.11 (The common-split matrix denominator)**.** *For $c = 3 + v_\ell(r(r-1)(r+1))$ one has*

$$\ell^c O[G] \;\subseteq\; O[\rho(G_{\mathcal{F}})] \;\subseteq\; O[G].$$

The same inclusions hold after every specialization of the global representation.

Proof. Rational wild inertia at ℓ acts trivially on $\mathcal{F}$, whose degree is prime to ℓ. Its cyclotomic image is $1 + \ell\mathbb{Z}_\ell$: full inertia surjects onto $\mathbb{Z}_\ell^\times$ since the local cyclotomic extensions are totally ramified, and the quotient by wild inertia has order prime to ℓ in every finite quotient. Choose an element u of wild inertia with $\kappa_\ell(u) = 1 + \ell$. In the ordered ordinary flag the diagonal of $\rho(u)$ consists of

$$\lambda_i = (1+\ell)^{a_i}, \qquad (a_0, a_1, a_2, a_3) = (r+1, r, 1, 0).$$

The finite inertia factors are trivial on this element. If N is the order of the diagonal matrix s in Lemma 48.10, then

$$Z = \frac{1}{N} \sum_{j=0}^{N-1} s^j \rho(u) s^{-j} = \operatorname{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \quad \text{belongs to } O[\rho(G_{\mathcal{F}})].$$

Indeed N is a unit and every conjugate belongs to the normal subgroup P. Each off-diagonal entry is multiplied in the sum by the powers of a nontrivial N-th root of unity whose difference from one is a unit. Their sum is zero.

Set $d_i = \prod_{j \neq i}(\lambda_i - \lambda_j)$. For distinct nonnegative integers a, b,

$$v_\ell\big((1+\ell)^a - (1+\ell)^b\big) = 1 + v_\ell(a-b).$$

In fact raising a unit $1+z$, $v_\ell(z) \geq 1$, to a power prime to ℓ preserves the valuation of its difference from one, by the leading term of the binomial expansion. Raising it to ℓ increases that valuation by one, since the other terms have strictly larger valuation for odd ℓ. Applying these two facts to $|a-b|$ proves the identity. The absolute differences at each of our four exponents are either $1, r, r+1$ or $1, r, r-1$. At most one of $r-1, r, r+1$ is divisible by $\ell \geq 5$. Thus $\max_i v_\ell(d_i) = c$. For the coordinate projector e_i this gives the identity of integral matrices

$$\ell^c e_i = \frac{\ell^c}{d_i} \prod_{j \neq i} (Z - \lambda_j).$$

The scalar coefficient on the right lies in O. The identity is checked on the four diagonal entries, so also holds in quotients with ℓ-torsion.

Every $g \in G$ has the form tp, with $p \in P$ and t the finite diagonal lift of its reduction by Lemma 48.10. The product $\ell^c t$ is an O-linear combination of the $\ell^c e_i$. It follows that $\ell^c g$ belongs to $O[P]$. Taking O-linear combinations gives the first inclusion since the algebra of a matrix group is its linear span. All statements pass to images of the matrices under coefficient maps. □

Corollary 48.12 (Trace ideals and preservation of characteristic-zero multiplicities)**.** *Let $\mathfrak{a}$ be the ideal in T generated by all differences from the split trace and $\mathfrak{a}_{\mathcal{F}}$ the ideal in T using only good rational primes split in $\mathcal{F}$. Then*

$$\ell^c \mathfrak{a} \subseteq \mathfrak{a}_{\mathcal{F}} \subseteq \mathfrak{a}, \qquad T_{\mathcal{F}}[1/\ell] = T[1/\ell].$$

At any integral point whose least normalized valuation of a good-prime trace difference is finite and equals v, the least valuation using these common split primes lies between v and $v + c$. Uniformly on the interval,

$$c \log \ell \;\leq\; 3\log(2t_* + 1) + 3\log(t_* + 1).$$

The equality of generic algebras includes their nilpotents.

Proof. For each group element the matrix theorem writes $\ell^c \rho(g)$ as a finite O-linear combination of matrices of elements of $G_{\mathcal{F}}$. Take traces, then apply the split augmentation to the same expression and subtract the resulting equality. This puts the scaled trace difference in the ideal generated by the restricted differences. At each finite precision any restricted trace difference is approximated by one at a good split rational prime as in Theorem 48.8, taking both the universal and split matrices in the finite quotient if necessary. Ideals in the complete Noetherian local ring T are closed. Its intrinsic topology agrees with that in the representation ring by compactness and Hausdorffness. This gives the asserted inclusions.

For the equality after inversion, let b be the size of the finite trace generating list in Theorem 48.8. Each generator multiplied by ℓ^c belongs to $T_{\mathcal{F}}$ by the matrix identity. That theorem realizes T as the ordinary $T_{\mathcal{F}}$-module spanned by monomials in the generators of exponents less than D^4 in each variable. In particular

$$\ell^{cb(D^4-1)}\, T \;\subseteq\; T_{\mathcal{F}},$$

which proves equality after inversion. Thus all characteristic-zero local algebras, including their multiplicities when finite, coincide. Evaluating the two ideal inclusions in a coefficient DVR gives both valuation bounds with $v_\ell(\ell) = 1$. Finally $\ell \le 2t_* + 1$ and $v_\ell(r(r-1)(r+1)) \log \ell \le 3 \log(t_* + 1)$. □

48.5 Nonsplit block witnesses on the auxiliary towers

On a triple of Section 47.3, let H, H' be its fields in Corollary 48.6. Use P for the plane on coordinates 13, and P' for the plane on coordinates 02 in a block realization, as in Section 47.7. The following construction gives the orientation of the nonsplit extension appearing in `Development/AutomorphicArithmetic/AutomorphicVarieties/WeakShimuraTheory/RedIdeal.lean`, line 65.

Lemma 48.13 (Invariant subspaces on the auxiliary cyclotomic kernel)**.** *Let an integral realization over a complete DVR with finite residue field reduce to the specified diagonal. The coefficient field may have characteristic zero or ℓ. Every invariant subspace over the algebraic closure of its fraction field for $G_{H'(\mu_{\ell^\infty})}$ is already invariant for $G_{\mathbb{Q}}$.*

Proof. The elements g_0 in Theorems 48.2 and 48.5 were chosen using prescribed cyclotomic residues, with the residue one at ℓ. In their construction one can prescribe the entire ℓ-adic cyclotomic component to be one. Indeed all other moduli used in specifying this element are prime to ℓ, and $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) = \prod_p \mathbb{Z}_p^\times$. Lifting this element to $G_{\mathbb{Q}}$ gives $g_0 \in G_{H'(\mu_{\ell^\infty})}$ with the same four distinct residual diagonal entries.

Hensel's lemma lifts these four simple roots for $\rho(g_0)$ to the coefficient DVR. Its four integral eigenlines give a basis by their unit-separated spectral projectors, and their reductions are the original coordinate lines. Every subspace invariant under the indicated subgroup is a sum of a subset of these eigenlines, since the projectors are polynomials in $\rho(g_0)$. The subgroup is normal in $G_{\mathbb{Q}}$ because the underlying extension is abelian. Hence any $g \in G_{\mathbb{Q}}$ carries such a sum to another sum of the same eigenlines. The reductions of the two sums are identical: the original residual matrix at g is diagonal, and an invertible integral matrix transports the saturated lattice of a subspace to that of its image. Distinct subsets have distinct reductions. Thus g preserves the original subspace. This argument also covers all subspaces over the algebraic closure, since a sum of the specified eigenlines is already defined over the coefficient field. □

Theorem 48.14 (An ordinary nonsplit extension with exactly the plane traces)**.** *In either characteristic, every two-plane DVR realization $\rho_0 = P' \oplus P$ of Theorem 40.13 admits another integral realization ρ_1 with the same multiplier, residual representation, ordered local conditions and all Frobenius polynomials. Over the fraction field it is an extension*

$$0 \longrightarrow P \longrightarrow \rho_1 \longrightarrow P' \longrightarrow 0.$$

Its only nonzero proper invariant subspace over the algebraic closure is P. The same holds on $G_{H'(\mu_{\ell^\infty})}$.

Proof. Write R for the DVR and π for a uniformizer. At each R/π^j, Theorem 47.20 gives a free rank-one space of ordinary normalized cocycles in the direction $P' \to P$. Reduction maps these spaces onto one another by Lemma 47.19. Lift a nonzero residual generator compatibly through all levels. Uniqueness of representatives normalized at h gives an actual continuous inverse limit cocycle z with values in M_-. It has a unit entry at some group element since its reduction is nonzero. No part of this inverse-limit argument depends on the characteristic of the DVR.

Put $\rho_1(g) = (1 + \pi z(g))\rho_0(g)$. Products of two of these off-diagonal corrections vanish even in R: their images are in P, on which they are zero. Thus the cocycle law gives an actual representation. Symplecticity, the ordered local conditions and the inertia kernels follow from the same matrix verification as in Theorem 47.20. The factor π retains the original residual diagonal. The characteristic polynomial of a block triangular matrix is the product of those of its diagonal blocks, so all characteristic polynomials agree with those of ρ_0.

An invariant complement of P would be a graph from P' to P. Since $z(h) = 0$ and the two blocks at h have disjoint spectra, a graph invariant at h must have zero graph map. But P' itself is moved by an element where $z \neq 0$. Thus there is no complement, even after scalar extension. The two blocks are absolutely irreducible by Theorem 40.13. The intersection of a stable subspace with P is therefore zero or all of P, and its image in P' is zero or all of P'. The only additional nontrivial possibility would be the excluded complement. Lemma 48.13 preserves this conclusion on the cyclotomic kernel. □

Corollary 48.15 (Detecting failure at a common split prime)**.** *If T is infinite over O, there is an ordinary lift over a complete finite-residue-field DVR of characteristic ℓ with a transcendental trace at a good rational prime split completely in the single field $\mathcal{F}$. It is either absolutely irreducible or a nonsplit extension with exactly the proper subspace in Theorem 48.14. In characteristic zero every nonzero two-plane trace point has a witness of the latter kind for which, at every unramified place of every finite extension over a good rational prime q, the ratio of the largest to the smallest complex Frobenius absolute value is at least q^{f_q}, where f_q is the residue degree.*

Proof. In case of nonfiniteness take the one-dimensional special-fibre quotient and its integral realization in the proof of Corollary 40.15. At least one rational Frobenius trace is transcendental over the finite residual coefficient field. If every common split-prime trace were algebraic over that field, each would lie in the constant field of the target DVR. Indeed reduction is injective on its elements algebraic over a finite field: the nonzero ones are roots of unity of order prime to ℓ. They thus form a finite field in the target, since its residue field is finite. It is closed, so the whole image of $T_{\mathcal{F}}$ would lie in this finite field by Theorem 48.8. The monic equation in that theorem would make every trace algebraic, a contradiction. Thus one can take a good prime split in $\mathcal{F}$.

Use the rank-four realization when it is irreducible, and replace the block realization by the nonsplit one of Theorem 48.14 otherwise. This preserves the transcendental trace. In characteristic zero the same replacement preserves all Frobenius polynomials, so the determinant comparison on the two planes in Theorem 40.13 gives the asserted ratio for any coefficient embedding into $\mathbb{C}$. □

We record precisely how the finite-weight condition on a family would exclude the equal-characteristic witness. Let Y be a local algebra over a coefficient DVR O_1 of characteristic zero and residue characteristic ℓ, with finite residue field. Suppose a specialization y to a field of characteristic ℓ supplies the traces of the witness on G_B for a finite extension $B/\mathbb{Q}$ as elements of $y(Y)$. Suppose further that an ideal $J \subseteq \ker y$ and a collection of weight elements $\theta_{v,i}(s) \in Y$ satisfy

$$\mathfrak{m}_Y^N \subseteq J + \mathfrak{m}_{O_1} Y + \big(\theta_{v,i}(s) - 1\big)_{v,i,s}, \qquad y(\theta_{v,i}(s)) = 1$$

for some $N \geq 1$, with s ranging over inertia above ℓ. These are the parts of the finite-weight condition in `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/ThresholdArguments/DenseThresholdCases/PotentialModularityTower.lean`, line 38 used here. When the weights specialize to the prescribed diagonal inertia on H, their reductions there are one by Theorems 48.2 and 48.5.

Indeed y kills $\mathfrak{m}_{O_1}$, since a power of its uniformizer is a unit times ℓ, and a field has zero nilradical. The inclusion gives $y(\mathfrak{m}_Y) = 0$ and hence a finite image $y(Y)$. For any $g \in G_{\mathbb{Q}}$ some positive power g^j lies in G_B, by its action on the finite coset space. The traces on g^{ij} for $1 \le i \le 4$ are now algebraic over the finite field. Newton's identities and $\ell \ge 5$ show that all eigenvalues of $\rho(g^j)$ are algebraic. Those of $\rho(g)$ are j-th roots of these eigenvalues and are therefore algebraic as well. This contradicts the common split-prime witness. Thus such family realizations for all the indicated equal-characteristic witnesses would prove finiteness of T. In characteristic zero the corollary gives the separate contradiction for a family giving Frobenius purity at even one good place of a nonsplit block witness. Establishing the required family realizations and the rank-four cuspidal matches remains necessary for this argument.

48.6 A cuspidal symmetric cube on the odd-order towers

Use the odd-order case of Section 48.1, the real field H, its CM extension HK, and the element g_0 of Lemma 48.13. On G_H the ordered residual characters are $(\chi^3, \chi^2, \chi, 1)$, where $\chi = \bar{\xi}$, and $\chi(g_0)$ has order $2m \ge 6$. Use the cyclotomic character κ_ℓ with arithmetic Frobenius convention. The required multiplier is $\mu = A\Psi = \kappa_\ell^{r+1}\omega_\ell^{-1}\xi\alpha$. We construct an initial automorphic point for these towers with its integral representation and ordered flags. Its weight is allowed to vary in the definition of the initial point.

Lemma 48.16 (One level prime and an ordinary modular plane)**.** *There is one rational prime w split completely in $\mathcal{F}$ of Proposition 48.7, common to the triples on the interval, with $\log w$ polynomial in V. For each odd-order triple and each positive multiple j of $\ell - 1$, choose a character θ of order ℓ and conductor w. There is an ordinary cuspidal newform of weight $j+1$, exact level $\mathrm{cond}(\xi)w$ and character $\xi\theta$. Its Galois plane W has determinant $\kappa_\ell^j\xi\theta$, residual semisimplification $\chi \oplus 1$, and Zariski-dense projective image in PGL_2.*

Proof. Choose $w \equiv 1 \bmod B$, where B is a common multiple of the conductor of $\mathcal{F}$, $2N^\natural M_* p_1$, and all the primes up to $2t_*+1$; the least such multiple will do. Proposition 48.7 and Lemma 48.1 bound $\log B$ by a polynomial in V. Linnik's theorem supplies $w \ll B^C$ with an absolute C. In particular $w \equiv 1 \bmod D\ell$ for $D = \mathrm{cond}(\xi)$, so $(\mathbb{Z}/w)^\times$ admits the required character. Construct forms for any fixed identification of the algebraic closures used for their complex and ℓ-adic coefficients.

Here j is even and ξ is primitive and odd. At weight $k = j+1 > 2$ the normalized $E_k(1, \xi)$ has integral constant term at ℓ by Lemma 43.5 and integral nonconstant coefficients. On the prime-to-ℓ full cover the expansion principle gives integrality at the cusps, as in Proposition 43.6. At level Dw reduce $E(z) - E(wz)$. For a cusp represented by a primitive column $(b, c)^{\mathsf{T}}$, set $d = \gcd(w, c)$. The second constant term is the first-level constant at $(wb/d, c/d)^{\mathsf{T}}$ times $(d/w)^k$. The two cusp columns agree modulo D since $w \equiv 1 \bmod D$, and the factor reduces to one since $w \equiv 1 \bmod \ell$. The difference is consequently cuspidal in the residue field. Its first coefficient is one. It is an eigenvector for the Hecke operators at primes $p \nmid Dw$ with eigenvalues the reductions of $1 + \xi(p)p^j$, including the unit eigenvalue at ℓ.

The character lifting argument of Proposition 43.6 applies with the single prime w. More explicitly, the ℓ-Sylow diamond group acts freely on $X_1(Dw)$, since smooth-point stabilizers have order divisible only by two and three and cusp stabilizer orders divide $2Dw$. On the quotient, descend the weight-k cusp line with the prescribed ℓ-power character. Its residual twist is trivial; the other diamond characters are separated by integral projectors. Serre duality on the special fibre gives vanishing of the first cohomology of the descended cusp line, since its dual sections pull back to weight $2 - k < 0$. Thus the residual eigensystem occurs in the reduction of the torsion-free cusp module with character $\xi\theta$. A minimal prime of its finite Hecke

algebra below this eigencharacter has characteristic zero: a uniformizer is a nonzero divisor on that algebra by its faithful action on the torsion-free module. This yields a characteristic-zero cuspidal eigensystem. The character is primitive of conductor Dw, so its newform has this exact level. Deligne's representation and the ordinary local description give the determinant, the stated residual semisimplification by Chebotarev, and an unramified local quotient with unit Frobenius value reducing to one. [11, 12]

The plane of the cuspidal newform is absolutely irreducible. The two Hodge–Tate weights $j, 0$ exclude finite projective image. If the connected projective Zariski closure were a proper positive-dimensional subgroup, it would be solvable. A nontrivial unipotent radical has a unique fixed line preserved by the whole group, contradicting irreducibility. Otherwise the connected group is a torus, whose normalizer permutes two lines. A transitive permutation supplies a quadratic self-twist by conjugating with the diagonal matrix with entries $1, -1$ on these lines. Reduction would give a nontrivial quadratic character permuting $\{1, \chi\}$ by multiplication. This forces $\chi^2 = 1$, contrary to its value at g_0. This proves the density. □

Proposition 48.17 (The nonsplit residual point and its ordinary flags)**.** *There is a lattice in W in which g_0 is diagonal and the residual plane is a nonsplit extension*

$$0 \longrightarrow \mathbf{1} \longrightarrow \bar{W} \longrightarrow \chi \longrightarrow 0.$$

On G_H its symmetric cube is lower triangular with diagonal $(\chi^3, \chi^2, \chi, 1)$ and has both simple negative root entries nonzero as functions on the group. At every place above ℓ its ordinary flag is represented by a lower unitriangular symplectic matrix in this frame. These assertions persist on every finite totally real abelian extension $H_1/\mathbb{Q}$ containing H with $[H_1 : H]$ an ℓ-power, using a power $g_0^{\ell^a} \in G_{H_1}$ for the normalization.

Proof. Start with a stable lattice over a coefficient DVR O large enough for W and the characters, with uniformizer π. Since g_0 has distinct residual eigenvalues, Hensel's lemma and its spectral projectors give an integral eigenbasis ordered by $\chi(g_0), 1$. The traces at g and $g_0 g$ determine the two diagonal entries modulo π as $\chi(g), 1$ by a two-by-two system of unit determinant. Write b_g, c_g for the upper and lower off-diagonal entries. The first diagonal entry at gg' gives $b_g c_{g'} \equiv 0 \bmod \pi$ for all g, g'. By irreducibility both sets of entries have finite minimum valuation. If $c = \min_g v_\pi(c_g)$, replace the second basis vector by π^c times itself. Now all the upper entries vanish on reduction, some lower entry is a unit, and g_0 remains diagonal. A stable complement of the second residual line would be the first eigenline of g_0; the nonzero lower entry moves it.

The extension class belongs to $H^1(G_{\mathbb{Q}}, k_O(\chi^{-1}))$, where k_O is the residue field. Restriction to G_{H_1} is injective on this group by inflation–restriction, since $\chi(g_0^{\ell^a}) \neq 1$ for every a, and some such power belongs to G_{H_1}. This applies also to H. Thus some lower entry on the restricted group remains nonzero. In the monomial basis of the symmetric cube the first two lower subdiagonal entries of a residual matrix with plane first column $(\chi(g), c_g)^{\mathsf{T}}$ are $3\chi(g)^2 c_g$ and $2\chi(g) c_g$. Both are nonzero when $c_g \neq 0$, since $\ell \geq 5$.

For the symplectic frame use the monomials $X^3, X^2Y, -3XY^2, Y^3$. Indeed the alternating form induced on the symmetric tensors by the cube of the determinant pairing has values $1, -1/3$ on the two ordered complementary pairs in the unscaled monomial basis. This follows by averaging the three tensor factors: among the pairings of X^2Y with XY^2 one third are nonzero, with negative sign. The rescaling gives the prescribed antidiagonal form; it changes the two subdiagonal entries only by units. The multiplier of the symmetric cube is the cube of the determinant.

At any rational place above ℓ the ordinary subline of W has residual character χ, since the unramified quotient reduces to one. This character differs from one on the rational decomposition group by Lemma 47.7, which gives $\chi(\ell) \neq 1$. The reduction of the saturated subline therefore differs from the global invariant residual line. It has an integral generator $X + a_v Y$ in the ordered plane basis. The lower unitriangular matrix sending X to this vector and fixing Y gives the ordinary flag, and its symmetric cube gives the required symplectic flag. At a place upstairs use the restriction of this same rational ordinary line. Conjugating the place transports the line because the positive-weight ordinary subcharacter is distinguished from its weight-zero quotient. The flags therefore restrict compatibly through all the indicated towers, including at places where the two local residual characters become equal. □

Theorem 48.18 (A cuspidal initial point for every odd-order tower)**.** *Let $C_{w,\ell}$ be the degree-ℓ subfield of $\mathbb{Q}(\mu_w)$. For every H_1 of Proposition 48.17 containing $C_{w,\ell}$ there is an ordinary symplectic lift of that same residual point with multiplier μ. It is absolutely irreducible on $G_{H_1K(\mu_{\ell^\infty})}$, and over H_1K an explicit scalar twist of it is attached to a regular algebraic cuspidal representation of* GL_4*. Its ordered local characters, weights, and ramification satisfy the initial-point conditions of* `Development/AutomorphicArithmetic/AutomorphicVarieties/WeakShimuraTheory/NearSplit.lean`*, line 312 on the towers of Section 48.1. One can take $H_1 = HC_{w,\ell}$.*

Proof. The character $\mu/(\det W)^3$ on G_H is one on reduction because $\tau = \chi^2$ there. Let b be its unique square root with values in $1 + \mathfrak{m}_O$. Existence and uniqueness follow from Hensel's lemma since ℓ is odd. Uniqueness proves multiplicativity, and applying the same lifting modulo all powers proves continuity. Use the integral representation

$$R = b \operatorname{Sym}^3 W$$

on G_H and its restrictions, in the basis of the proposition. Its multiplier is μ and its residual point is exactly the constructed one. Every prescribed normalizing power of g_0 is diagonal. Before adjoining $C_{w,\ell}$ allow the single extra ramification prime w.

At every prime $p \mid Dw$ the conductor exponent of the plane equals that of its determinant by primitivity of the character and local-global compatibility for newforms. For a ramification group the codimension of fixed vectors is at least that for the determinant. Equality of the Artin conductors forces equality for inertia, including the nonnegative monodromy contribution. More explicitly the inertial fixed space has codimension at least one since the determinant is ramified, hence exactly one. Finite inertia then has characters $1, \xi\theta$, distinct on inertia, and a commuting nilpotent monodromy must be zero. At the other primes away from ℓ the plane is unramified. These are the conductor and inertia arguments of Theorem 43.7. [13] On G_{H_1} the character θ is trivial. At an inertia element away from ℓ in the kernel of the residual four characters, ξ and α are both one: their reductions are one and their orders are prime to ℓ. The plane, the multiplier and its scalar correction are therefore trivial at this element. On H the same calculation works outside w, ℓ.

Above ℓ use the integral flags of the proposition. The four residual local characters in their given order are $\chi^3, \chi^2, \chi, 1$. The two ordinary characters of the plane are an unramified character and $(\det W)$ times its inverse. On elements whose image in the local abelianization has ℓ-power order the unramified characters are one, since the unramified quotient is torsion-free. The cyclotomic character also takes value one there as $\mathbb{Z}_\ell^\times$ has no ℓ-power torsion, and the finite factors in $\mu, \det W$ are unramified there or of order prime to ℓ. Thus all four ordinary characters and b take value one on these elements. On an open subgroup of inertia the squared ordered characters of R are the cyclotomic powers with exponents

$$h_i = 2(3 - i)j + r + 1 - 3j, \quad 0 \leq i \leq 3.$$

Their consecutive differences are $2j \geq 4$. The inverse twist b^{-1} has square on the same subgroup equal to κ_ℓ^a, $a = 3j - r - 1$, and $a + h_0 = 6j$ is even.

The projective image of W on the absolute commutator subgroup is Zariski dense in PGL_2. Indeed the closure contains all commutators of the original Zariski closure by density and the commutator morphism, and PGL_2 is its own derived group. Since $H_1K(\mu_{\ell^\infty})/\mathbb{Q}$ is abelian, its absolute Galois group contains these commutators. In characteristic zero the third symmetric power of the standard SL_2-module is irreducible, so no proper subspace of R can be stable on this group.

The plane comes from a non-CM form by the density in Lemma 48.16. Symmetric power functoriality in degree three gives a regular algebraic cuspidal transfer over $\mathbb{Q}$. In the algebraic Frobenius normalization its four parameters at a good prime are the degree-three monomials in the two parameters of W. This is Theorem A of Newton and Thorne, *Symmetric power functoriality for holomorphic modular forms, II*. Use cyclic base change successively through the finite abelian extension $H_1K/\mathbb{Q}$. Cuspidality could be lost at a cyclic step of prime degree only by a nontrivial self-twist of the representation before that step. The equality of good polynomials and Chebotarev would then give a self-twist of $\mathrm{Sym}^3 W$ on that field. An intertwiner would commute with its restriction to the kernel of the twisting character. This kernel has finite index and the restriction is absolutely irreducible by the same density argument (or by connected projective monodromy). The intertwiner would be scalar, which forces the twisting character to be trivial. Thus the base changes stay cuspidal. Base change preserves the distinct algebraic weights and raises the good Frobenius eigenvalues to the residue degrees. The resulting representation over H_1K is attached to $b^{-1}R$ there. [14]

Finally, the characters of $G_\mathbb{Q}$ of order dividing ℓ unramified away from w all factor through $\mathrm{Gal}(C_{w,\ell}/\mathbb{Q})$. By rational class field theory their conductors divide w, since the wild units at w are pro-w, $\ell \neq w$, and ℓ is odd. The tame quotient $(\mathbb{Z}/w)^\times$ is cyclic. Hence killing these characters is exactly the condition $C_{w,\ell} \subset H_1$. This subfield is totally real and $HC_{w,\ell}/H$ is cyclic of degree ℓ, since $[H : \mathbb{Q}]$ is prime to ℓ. Together with the restriction of the flags, the two nonzero negative root entries and the inertia calculation above, this proves the claimed initial-point conditions. □

The prime w is auxiliary to this construction; the cohomology and signed periods of Section 6.6 are still taken over E_x. The theorem supplies one cuspidal initial point for each odd-order triple, together with its restrictions to the required towers. Proving that the components through the other ordinary deformations occur in an automorphic family remains a separate step.

48.7 Opposite ordinary quotients over the real quadratic field

Consider an even-order triple of Section 47.3. Write $D_0 = K_0$ for the real quadratic field of Lemma 48.4, ϕ for its quadratic character and HK for the auxiliary CM field. The rational prime ℓ splits in D_0 as v_0v_1, as does two. Let λ be the finite Hecke character of D_0 obtained by restricting the Galois character ξ, or equivalently by composing its idele character with the norm. It is totally odd, of conductor $\mathfrak{n}$ supported at the two places over two, and unramified at ℓ. Its values at v_0, v_1 are both $\xi(\ell)$. Their common reduction differs from one by Lemma 47.7. Use the arithmetic convention for the eigenvalues of Frobenius.

Lemma 48.19 (Raising the weight with two different unit roots)**.** *For arbitrarily large positive j divisible by $2(\ell - 1)$ there is a parallel-weight $j + 1$ Hilbert Hecke eigensystem over D_0 of character λ, tame level $\mathfrak{n}$, and Iwahori level at both places over ℓ. Its residual $T_\mathfrak{q}$-eigenvalues away from $\mathfrak{n}\ell$ are $1 + \bar{\lambda}(\mathfrak{q})$. Its two U-eigenvalues reduce to $\bar{\lambda}(v_0), 1$ at v_0, v_1, respectively.*

Proof. Start with the parallel weight-one Eisenstein series $E_1(1,\lambda)$, with nonconstant ideal coefficients $\sum_{\mathfrak{d}\mid\mathfrak{m}}\lambda(\mathfrak{d})$; extend λ by zero at ideals meeting its conductor. It exists for this totally odd primitive character of nontrivial finite conductor. At each $v \mid \ell$ its Hecke roots are $1, \lambda(v)$. The oldform V_vE has coefficient at $\mathfrak{m}$ equal to that of E at $\mathfrak{m}/v$, interpreted as zero when the latter ideal is nonintegral. On the span of E, V_vE, the formula for U_v gives $U_vV_vE = E$ and $U_vE = (1+\lambda(v))E - \lambda(v)V_vE$. Thus apply $1 - V_{v_0}$ and $1 - \lambda(v_1)V_{v_1}$. The resulting form E^* has the two required roots, integral nonconstant coefficients and first coefficient one. The operators at the two distinct places commute. Its constants at the standard cusps are algebraic. We use the ideal-coefficient conventions and Eisenstein expansions in Dasgupta, Darmon and Pollack, *Hilbert modular forms and the Gross–Stark conjecture*, Sections 2.1–2.2.

Take $j = 2(\ell-1)\ell^a$ for a tending to infinity. The level-one parallel form $E_j(1,1)$, for $j \geq 4$, has normalized constant term $2^{-2}\zeta_{D_0}(1-j)$ on each standard component. Its remaining coefficients are the integral divisor sums with exponent $j-1$. The valuation of this constant tends to minus infinity. Indeed $\zeta_{D_0}(1-j) = \zeta(1-j)L(1-j,\phi)$. Von Staudt–Clausen gives $v_\ell(\zeta(1-j)) = -1 - v_\ell(j)$. On this residue class of weights the Kubota–Leopoldt interpolation formula identifies $(1-\phi(\ell)\ell^{j-1})L(1-j,\phi)$ with $L_\ell(1-j,\phi)$. The latter tends to $L_\ell(1,\phi) \neq 0$ by the ℓ-adic class number formula. For this real quadratic field the regulator is nonzero: the fundamental unit in either completion $\mathbb{Q}_\ell$ has nonzero logarithm, since a unit of $\mathbb{Q}_\ell$ with zero logarithm is a root of unity. The Euler factor tends to one. This proves the valuation claim. The interpolation and the abelian ℓ-adic class number formula are the classical formulas of Chapter 5 of Washington, *Introduction to Cyclotomic Fields*.

Normalize $E_j(1,1)$ by its constant term. Its expansion on every standard component is congruent to one to arbitrarily large ℓ-adic precision, uniformly in the Fourier index. To see this also in the unnormalized power series, take the finitely many narrow class representatives prime to ℓ. The changes between ideal coefficients and these power series multiply by powers of their rational ideal norms, which are ℓ-adic units. Changing the weight by j changes these factors by powers $N(\mathfrak{t})^{\pm j/2}$. These are one on reduction by the choice of j. Multiply E^* by the normalized form. In the product all nonconstant coefficients reduce to those of E^*: the precision can exceed the denominators of the finitely many constants of E^*, and the other coefficients are integral. The coefficient convolutions at fixed totally positive index are finite. This gives a genuine weight-$j+1$ form with the indicated residual coefficients.

At a good prime the second term in the Hecke coefficient formula now acquires the factor $N(\mathfrak{q})^j$, one on reduction. At $v \mid \ell$ the U_v coefficient is the coefficient at $v\mathfrak{m}$ in either weight. Consequently the product has the required eigencharacter on reduction. For clarity, work in its Hecke span over a finite coefficient field completed at ℓ, and let $\mathcal{L}$ be the lattice of forms in that span with integral nonconstant ideal coefficients. Finitely many coefficient functionals embed the span in a finite-dimensional coordinate space: vanishing of all such coefficients would give a constant function on each classical component, impossible in positive weight unless it is zero by the modular transformation law. Thus $\mathcal{L}$ is a finite module over the coefficient DVR. It spans the Hecke span since the coefficient formulas preserve integrality and the span is generated by translates of the product. Divisibility of every nonconstant coefficient by a uniformizer is exactly divisibility in $\mathcal{L}$. The unit first coefficient therefore gives a nonzero eigenvector on reduction. The finite Hecke algebra acts faithfully on $\mathcal{L}$, so a uniformizer is a nonzero divisor on it. A minimal prime below the resulting maximal ideal lies in characteristic zero. In its fraction field the corresponding system occurs in the Hecke span by the faithful action and simultaneous triangularization. This proves the lemma at arbitrarily large weights. □

Theorem 48.20 (A cuspidal plane with opposite ordinary quotients)**.** *Every eigensystem just obtained is cuspidal. Its plane W has determinant $\kappa_\ell^j\lambda$, semisimple reduction $1 \oplus \bar{\lambda}$, projectively*

dense image, and an unramified ordinary quotient at each of v_0, v_1. These quotients reduce to $\bar{\lambda}, 1$, respectively. Its exact finite conductor away from ℓ is $\mathfrak{n}$; the underlying automorphic newform is unramified above ℓ.

Proof. In parallel weight $j+1 > 2$ the good eigenvalues of an Eisenstein constituent are $\eta(\mathfrak{q}) + \nu(\mathfrak{q})N(\mathfrak{q})^j$ for finite-order characters η, ν, with $\eta\nu = \lambda$. This is the classical Eisenstein decomposition for holomorphic Hilbert forms of parallel weight. The unweighted character is one global character: the absolute-value exponents of a Hecke character at the real places must be parallel by the unit theorem. Oldforms retain the same pair. At a place over ℓ the Iwahori invariants require the two characters to be unramified. Indeed ramification of either forces ramification of both by the unramified central character, giving principal-series conductor exponent at least two. Its two oldform roots at that place are therefore $\eta(v), \nu(v)N(v)^j$. Exactly the first is a unit.

By Chebotarev the good residual traces and determinants identify $\{\bar{\eta}, \bar{\nu}\} = \{1, \bar{\lambda}\}$ as characters, since $\bar{\kappa}_\ell^j = 1$. For example one may compare the characteristic polynomials in a common finite quotient and apply the semisimple character criterion. It follows that the reduction of the unit root at both places comes from 1 at both places or from $\bar{\lambda}$ at both places. This contradicts Lemma 48.19 since $\bar{\lambda}(v_0) = \bar{\lambda}(v_1) \neq 1$. An eigencharacter in the quotient by cusp forms would be Eisenstein, so the eigensystem is cuspidal.

At a tame prime the conductor is bounded by the given level and bounded below by the conductor of the central character. These exponents agree. Above ℓ the only ramified Iwahori-spherical cuspidal local possibility is an unramified twist of Steinberg. In the arithmetic normalization its U_v-eigenvalue has square $\lambda(v)N(v)^{j-1}$, of positive valuation. Both prescribed roots are units, so the local newforms are unramified. The ordinary Galois representation theorem for Hilbert newforms gives the determinant and the unramified quotient with Frobenius eigenvalue the chosen unit root. The good polynomials identify the residual semisimplification by Chebotarev. These are the ordinary local descriptions of Wiles and the local-global compatibility theorem for Hilbert newforms. [13]

The plane is absolutely irreducible by the Galois representation theorem for regular cuspidal Hilbert newforms. Its distinct weights $j, 0$ exclude finite projective image. A nontrivial quadratic self-twist would permute $\{1, \bar{\lambda}\}$ on reduction, forcing $\bar{\lambda}^2 = 1$. At either place over two its inertia has the full order of the original primitive dyadic character on inertia since two splits in D_0; that order exceeds four by Section 47.3. The algebraic subgroup argument in Lemma 48.16 now gives projective density. □

Corollary 48.21 (Cuspidal quadratic induction at parallel weight)**.** *Put $U = \mathrm{Ind}_{G_{D_0}}^{G_{\mathbb{Q}}} W$. Its restriction to $G_{HK(\mu_{\ell^\infty})}$ is absolutely irreducible. Over HK it has the good Frobenius polynomials of a cuspidal quadratic automorphic induction in the arithmetic normalization. On G_H an integral scalar twist of U has multiplier μ and residual semisimplification $(\chi\tau, \tau, \chi, 1)$.*

Proof. For the nontrivial conjugation σ of $D_0/\mathbb{Q}$, W^σ is inequivalent to W. Otherwise the unramified ordinary quotients at the two places over ℓ would coincide, which is excluded on reduction by Theorem 48.20. The quotient at each place is intrinsic, since the other character of the ordinary flag has strictly positive weight.

Both planes are absolutely irreducible on $G_{\mathbb{Q}^{\mathrm{ab}}}$. In fact this group contains the commutators of G_{D_0}, whose projective images are Zariski dense by the proof of Theorem 48.18. If they became isomorphic there, the line of intertwiners would be stable under G_{D_0}, since this subgroup is normal. Thus $W^\sigma \simeq W \otimes \delta$ for a character δ on G_{D_0}. Their determinants agree because λ is the restriction of ξ. Hence $\delta^2 = 1$; any nontrivial choice again contradicts the residual pair $\{1, \bar{\lambda}\}$. The trivial choice contradicts the preceding paragraph.

Let $G' = G_{HK(\mu_{\ell^\infty})}$. The group $G' \cap G_{D_0}$ contains $G_{\mathbb{Q}^{\mathrm{ab}}}$, so the restriction of U there splits into the inequivalent irreducibles W, W^σ. The element g_0 of Lemma 48.13 in the even-order construction belongs to G' and has $\tau(g_0) = \phi(g_0) = -1$. It interchanges these summands. Hence no nonzero proper subspace is G'-invariant. For the automorphic assertion, base change the Hilbert cusp form to HKD_0. All the cyclic steps remain cuspidal, since a self-twist on an intermediate field would give a scalar intertwiner on the kernel of its character by projective density on open subgroups. The cuspidal base change is inequivalent to its conjugate over HK, as is seen from good polynomials and the proved inequivalence on $G_{\mathbb{Q}^{\mathrm{ab}}}$. Cyclic automorphic induction along HKD_0/HK is therefore cuspidal. At split primes its parameters are the union of the two pairs, and at inert primes its Frobenius polynomial is obtained from the plane polynomial at the square by substituting X^2. These are the polynomials of U, in the normalization in which the plane has weight j. [14]

Write $d = \kappa_\ell^j \xi$ on $G_{\mathbb{Q}}$ and take any $t \notin G_{D_0}$. In the two plane summands one can write $t(x, y) = (W(t^2)y, x)$. Give the two planes the alternating determinant forms with coefficients $1, -d(t)$, respectively. On G_{D_0} both determinants are d. At t, evaluation of the two pairings after applying this matrix gives multiplier $-d(t)$, since $\det W(t^2) = d(t)^2$. Thus the multiplier of U is $d\phi$. Its residual semisimplification is $1 \oplus \phi \oplus \chi \oplus \chi\phi$ by induction of the two residual characters. On G_H one has $\tau = \phi$, so $d\phi$ and μ have the same reduction. The unique square root of $\mu/(d\phi)$ congruent to one on this group supplies the integral twist. □

The two labelled weights of the plane at each real embedding are still $j, 0$. Thus the four weights of the induction are $j, j, 0, 0$, and a common scalar twist preserves both repetitions. The regular initial point in the even-order case requires different weight gaps at the two embeddings. The result above supplies the cuspidal system with the opposite ordinary quotients on the particular real quadratic field where that change of weights is needed.

48.8 A regular cuspidal point on the even-order tower

Retain D_0, v_0, v_1, λ and one parallel exponent j from Theorem 48.20. Choose a positive multiple t of $\ell - 1$. We raise the weight gap at v_1. At v_0 the old ordinary quotient reduces to $\bar{\lambda}$; at v_1 it reduces to one. Fix the two embeddings of D_0 indexed by v_0, v_1 under the chosen identification of coefficient fields.

Proposition 48.22 (A change of one weight at the exact tame level)**.** *There is a cuspidal Hilbert plane W with determinant $d = \kappa_\ell^j \xi|_{G_{D_0}}$, residual semisimplification $1 \oplus \bar{\lambda}$, and tame conductor exactly $\mathfrak{n}$. At the two places it is crystalline and ordinary. Its ordered weights, with the subcharacter first, are $(j, 0)$ at v_0 and $(j + t, -t)$ at v_1. The two ordered residual pairs there are $(1, \bar{\lambda})$ and $(\bar{\lambda}, 1)$, respectively.*

Proof. Use the quaternion algebra over D_0 ramified at its two real places and split at every finite place. The Jacquet–Langlands correspondence realizes the cuspidal system of Theorem 48.20 on this algebra, with the same finite Hecke action and both chosen Iwahori refinements. We use the algebraic modular forms realization of this correspondence as in Hida, *On p-adic Hecke algebras for* GL(2) *over totally real fields*, Theorem 2.1. We give the finite-module argument for changing the coefficient at v_1.

Over a sufficiently large coefficient DVR, the original algebraic coefficient at each split place is Sym^{j-1}. At v_1 replace it by $\mathrm{Sym}^{j+2t-1} \otimes \det^{-t}$, leaving the v_0-factor as it is. For each factor $\mathrm{Sym}^n \otimes \det^m$, the holomorphic weight is $n+2$, the ordered arithmetic weights are $(m+n+1, m)$, and the algebraic central exponent is $n + 2m$. Thus both coefficient systems have central exponent $j - 1$ at both places. At finite places fix the same level, finite central character and tame nebentypus as for the initial form.

Here are details of the integral comparison. In the quaternionic model integral forms are functions of the finite adelic argument satisfying left rational invariance, right level equivariance through the inverse coefficient action, and the prescribed central action. These conventions result by sending a form equivariant on the rational left through the algebraic coefficient ρ to its product by $\rho(g_\ell)^{-1}$ at the argument g. The central action in this integral model is the same for the two coefficients, since their central exponents coincide and their finite characters are fixed. It has unit values: the valuation at ℓ of the algebraic central factor cancels the corresponding valuation of the finite adelic norm to exponent $j-1$. Away from ℓ impose the original character on level units.

Values at a finite set of double-coset representatives identify this module with a direct sum of invariant submodules of the coefficient lattice. The relevant stabilizers modulo the center are finite by definiteness and discreteness. Their orders are prime to ℓ. Indeed a rational projective quaternion element of order ℓ would have two eigenvalues of ratio a primitive ℓ-th root ζ. Its squared reduced trace divided by its reduced norm, minus two, equals $\zeta+\zeta^{-1}$ in D_0. This has degree more than two for $\ell \geq 7$; for $\ell = 5$ it generates $\mathbb{Q}(\sqrt{5})$, whereas $D_0 = \mathbb{Q}(\sqrt{p_3})$ with $p_3 \neq 5$. Cauchy's theorem excludes divisibility by ℓ. The compatible central character has already been imposed in taking this quotient. Averaging over each stabilizer now shows that the reduction of the integral function module is the full module of residual functions with the same equivariance, for either weight.

At a place above ℓ use the upper Iwahori I and the double coset $I\,\mathrm{diag}(1,\ell)I$. Its elements reduce to matrices of rank one with image the first coordinate line. With the coefficient action just described the operator sends a function f at g to $\sum_h \rho(h) f(gh)$ with h running through right-coset representatives. For $\det^m$ normalize this operator by ℓ^{-m}. This multiplies the determinant factor by the inverse of its uniformizer part, leaving $(\det(h)/\ell)^m$ a unit. For $m = 0$ this is the usual arithmetic U with eigenvalues the two roots of the good Hecke polynomial for a spherical representation. One may use either Iwahori direction for that calculation: the element interchanging the two ends of the Iwahori edge conjugates the two double cosets, and the roots are those of the same spherical Hecke polynomial. For general m the operator before normalization is the same unweighted adelic double coset under the displayed change of model. The scalar ℓ^{-m} consequently gives the normalized roots ℓ^{-m} times the arithmetic ones. The spherical Hecke normalization is common to these weights since the central exponent $j-1$ is fixed.

On residual coefficients the image of the normalized operator in its factor lies in the line spanned by X^n, where X is the first basis vector. Reduction of $\mathrm{Sym}^n h$ has this image or is zero on a particular input. The Iwahori preserves the line, and on it a level matrix acts by $h_{11}^n(\det h)^m$. For a double-coset summand the scalar is $h_{11}^n(\det h/\ell)^m$ on reduction. Both h_{11} and the indicated determinant unit have residues in $\mathbb{F}_\ell^\times$, since the places split. Thus the line with its level and normalized operator actions is identical for the two choices of (n,m) at v_1; their exponent differences $2t, -t$ are divisible by $\ell-1$. The tensor of the two residual lines is also the same for the actions at all other places and at the center.

Choose a primitive integral eigenvector of the old system. Its reduction takes values in both lines because its two normalized U-eigenvalues are units. Identify the lines in the new coefficient and obtain a nonzero eigenfunction there with the same residual eigenvalues. The reduction surjectivity proved above lifts it into the new integral module. The finite algebra generated by the integral good Hecke operators and the two normalized U's acts faithfully on that free module. The characteristic-zero minimal-prime argument in Lemma 48.19 lifts the eigencharacter. One-dimensional quaternionic automorphic representations have trivial type on the norm-one group at infinity. Neither coefficient here has trivial type. Thus the recovered system transfers to a cuspidal Hilbert form. Its tame conductor equals the lower bound from

the same primitive central character at every tame prime. The determinant in the arithmetic normalization is $\kappa_\ell^j\lambda$. The good Hecke polynomials give the same residual pair.

At a place of exponent $m = 0$ or $-t$, the new normalized root is a unit. Steinberg would give valuation $(j-1)/2 - m > 0$ by the fixed central character, so the underlying representation is spherical. The Hilbert local-global compatibility theorem makes its Galois plane crystalline. After the local twist by κ_ℓ^{-m} the two weights are $n+1, 0$ and there is a unit Frobenius root with the specified reduction. The unit-root criterion gives an unramified quotient with this root: in the dual crystalline convention the slope-zero eigenline has Hodge weight zero by weak admissibility and hence corresponds to a crystalline line. Dualizing gives the stated quotient in the arithmetic convention. Twisting back contributes $\bar\kappa_\ell^m = 1$ on reduction. Taking the determinant determines the two complementary subcharacters, giving exactly the orders stated in the proposition. These compatibility statements apply to both motivic weight systems, including when the totally real degree is even. C. Skinner, *A note on the p-adic Galois representations attached to Hilbert modular forms*. □

Theorem 48.23 (A regular cuspidal initial point for every even-order triple)**.** *In the fields of Theorem 48.5 there is an integral ordinary symplectic lift of multiplier μ whose residual matrices and local flags have the prescribed even-order initial-point conditions. It is absolutely irreducible on $G_{HK(\mu_{\ell^\infty})}$; a scalar twist over HK is attached to a regular algebraic cuspidal representation of* GL_4*. The quadratic field for the induction is the single D_0 already fixed on the interval.*

Proof. Induce the plane in Proposition 48.22 from G_{D_0} to $G_{\mathbb{Q}}$. Its projective image on G_{D_0} is dense by the argument in Theorem 48.20; residual orders at two and distinct Hodge–Tate weights again exclude a torus normalizer and finite image. The two conjugate planes are inequivalent even after finite-order twisting, since their weight gaps at v_0 are j and $j+2t$. Their determinants agree. The intertwiner argument of Corollary 48.21 gives inequivalence on $G_{\mathbb{Q}^{\mathrm{ab}}}$, irreducibility of the induction on the required cyclotomic kernel, and cuspidal automorphic induction over HK. Its four weights are now the strictly decreasing list

$$m_0 = j + t, \qquad m_1 = j, \qquad m_2 = 0, \qquad m_3 = -t.$$

This list at every complex embedding also gives regular algebraicity in the arithmetic Satake normalization of that corollary.

We check the lattice and flags. Write $s = g_0$ in the even-order tower; $\phi(s) = -1$ and $\chi(s)^4 \neq 1$. In the plane choose an integral eigenbasis e, f of $W(s^2)$ with residual eigenvalues $1, \chi(s)^2$. Take the stable-lattice argument of Proposition 48.17 with the opposite entry normalized: rescale until the minimum valuation of the upper entries is zero. The products of upper and lower entries lie in the maximal ideal by the two diagonal trace equations at s^2. Thus all lower entries reduce to zero and the residual plane is a nonsplit upper extension of χ by one. This remains nonsplit on G_{HD_0}: inflation–restriction is injective since $s^2 \in G_{HD_0}$ and $\chi(s)^2 \neq 1$.

On the two summands of the induction $s(x, y) = (W(s^2)y, x)$. Choose square roots α_0, β_0 of the two eigenvalues, with $\alpha_0 \equiv 1$ and $\alpha_0\beta_0 = -d(s)$, writing $d = \kappa_\ell^j\xi$ also on $G_{\mathbb{Q}}$. This is possible by Hensel's lemma and $\det W(s^2) = d(s)^2$. Use in order the four vectors

$$(\beta_0 f, f), \quad (-\alpha_0 e, e), \quad (\beta_0 f, -f), \quad (\alpha_0 e, e).$$

They form an integral basis since $2, \alpha_0, \beta_0$ are units. For the alternating determinant forms on the two summands with coefficients $1, -d(s)$ as in Corollary 48.21, both ordered antidiagonal pairings are $-2\alpha_0\beta_0$. Divide the form by this unit. The matrix at s is diagonal with entries

$(\beta_0, -\alpha_0, -\beta_0, \alpha_0)$. The unique square root $b \equiv 1$ of $\mu/(d\phi)$ on G_H gives the desired multiplier after twisting by b.

On G_{HD_0} let c, c' denote the two residual upper entries of the plane and its conjugate, evaluated at the same group element. Their diagonal pairs are both $(1, \chi)$. Reading the induced action in the four vectors above, its only possible off-diagonal positions are $10, 12, 30, 32$, with indices starting at zero. The entries at $10, 30, 12$, in that order, are

$$\tfrac{1}{2}(c' - \gamma c), \qquad \tfrac{1}{2}(c' + \gamma c), \qquad -\tfrac{1}{2}(c' + \gamma c), \qquad \gamma = \overline{\beta_0/\alpha_0}.$$

The support assertion extends to the other coset by multiplying by the diagonal at s. The full residual diagonal is $(\chi\tau, \tau, \chi, 1)$ since $\phi = \tau$ on G_H. Either the 10-entry occurs or $c' = \gamma c$ identically, in which case both $12, 30$ occur, since some $c \neq 0$. These are exactly the two alternatives for the residual entries in the even-order initial point.

Above ℓ distinguish the summand with weight gap $j + 2t$ and the other summand. The top line in the former reduces to a generator $f + ae$, since it has residual character χ different from one on the decomposition group at the split rational quadratic-field place. In the other summand the top line reduces to e: it has character one and this eigenline is unique at an element separating χ and one. The complete flag in descending weight order consists successively of the first of these lines, both top lines, and the first line together with the whole other summand. These are isotropic in dimension two and self-dual as a full flag for the displayed form.

Normalize its first column to have coordinate zero equal to one. Its coordinate two is then 1 or -1, according to the summand containing the first line: the two f-coordinates of a vector in either single summand have this ratio in the displayed basis. In the other summand the second top vector has, modulo the maximal ideal, zero in coordinates zero and two and a unit in coordinate one. Subtract a multiple of the first column to clear coordinate zero, and divide by the remaining unit pivot. Its coordinate two is still zero on reduction. The third subspace is the perpendicular of the first line. In it the vector with first two coordinates zero and coordinate two one has uniquely determined last coordinate, since the first pivot is one. Use it as third column and the last coordinate vector as fourth. The resulting lower unitriangular matrix is symplectic: the two prescribed pairings are one by the pivots and all other pairings vanish by isotropy, perpendicularity and the first two zero coordinates. Thus its 20-entry is a unit and its 21-entry reduces to zero. All other lower positions are allowed. In this flag the ordered residual local characters are $(\chi, 1, \chi, 1)$, as required since $\tau = 1$ on the decomposition groups of H at ℓ.

Away from ℓ the tame conductor of W equals that of λ; its inertia has the two characters $1, \lambda$ with zero monodromy by the conductor argument of Theorem 48.18. At every finite place HD_0/H is unramified by Theorem 48.5. Inertia over H therefore acts through the two summands; it acts trivially where the residual diagonal is trivial, since ξ has order prime to ℓ, and the same holds for the scalar correction there. At ℓ the crystalline graded characters on the split quadratic-field places are cyclotomic powers times unramified characters. They and the correction kill the ℓ-power torsion of local abelianization, as in Theorem 48.18. On open inertia the squared characters after correction have exponents $h_i = 2m_i + r + 1 - j$. The inverse twist b^{-1}, which recovers the automorphic induction, has squared exponent $a = j - r - 1$. We have $a + h_0 = 2m_0$ and the consecutive gaps in the m_i are at least two. This verifies the weight and twist conditions with the original multiplier. The field, matrix and flag conditions just proved are those in `Development/AutomorphicArithmetic/AutomorphicVarieties/WeakShimuraTheory/NearSplit.lean`, line 305. □

Corollary 48.24 (Initial points in both residual character cases)**.** *For every triple of Section 47.3, the required auxiliary character case admits a regular cuspidal initial point with its indicated integral residual shape on the towers of Corollary 48.6. The fields K, K', D_0 and the*

single auxiliary prime needed in the odd-order construction can be fixed throughout the period interval.

Proof. Apply Theorem 48.23 for even order and Theorem 48.18 for odd order. The common base choices are those of Lemmas 48.1, 48.4 and 48.16. □

The initial points supply cuspidal points with particular residual extensions and ordinary flags. To bound the strict lengths at every triple by this route one must still propagate automorphic support to the deformations that witness a failure of the bound. The signed integral period comparison remains a second requirement on the original fields.

48.9 Ordinary deformations at the two cuspidal points

Write $F = H$ in Theorem 48.23 and $F = H_1$ in Theorem 48.18, with $C_{w,\ell} \subset H_1$. Put $L = FK$, a CM field with maximal totally real subfield F. Let R be the indicated four-dimensional representation of G_F and put $U = b^{-1}R$, removing the scalar correction of the construction. Choose a finite coefficient extension $E/\mathbb{Q}_\ell$ over which the matrices, the alternating form J, and the ordinary flags are defined. Fix the multiplier of R. The corresponding Lie algebra of infinitesimal changes is

$$V = \{X \in \mathrm{End}_E(E^4) : X^{\mathsf{T}}J + JX = 0\},$$

with G_F acting by conjugation. R and U give the same action. All cohomology below is continuous.

Proposition 48.25 (The two finite Selmer groups on the auxiliary field)**.** *At each of these points both $H^1_f(F, V)$ and $H^1_f(F, V^\vee(1))$ vanish. Here the finite condition is the unramified condition away from ℓ and the Bloch–Kato crystalline-period condition above ℓ.*

Proof. On G_L the representation U comes from a regular algebraic cuspidal representation Π, and its restriction to $G_{L(\mu_{\ell^\infty})}$ is absolutely irreducible by the two theorems just cited. We specify the polarization to which we apply adjoint vanishing. The multiplier ν of U is $(\det W)^3$ for the symmetric cube and $d\phi$ for the quadratic induction. In both cases it is an integral cyclotomic power times a finite character of G_F, and $\nu(c) = -1$ at every real conjugation c. For one such c put $C = U(c)$. The form $B = JC$ is symmetric since $C^2 = 1$ and $C^{\mathsf{T}}JC = -J$. It satisfies

$$U(g)^{\mathsf{T}} B\, U(cgc^{-1}) = \nu(g) B \qquad (g \in G_L).$$

In particular the good parameters of Π^c are the dual parameters of Π times a character from F. Indeed the display gives this identity in the Galois normalization. Restoring the Satake normalization changes the twisting character by an integral norm power, the square of the normalizing factor in rank four. A finite character times a norm power is an algebraic Hecke character, by class field theory. Strong multiplicity one gives the corresponding global identity. One can normalize the real sign of the polarizing character by the quadratic character of L/F, which is trivial on norms from L. This verifies polarizability with the specified restriction of the multiplier. For that restriction B is unique up to scalar by absolute irreducibility. These are the conventions of Barnet-Lamb, Gee, Geraghty and Taylor, Section 2.1 of *Potential automorphy and change of weight*.

On the polarized adjoint over F, G_L acts by conjugation and c acts by $X \mapsto -B^{-1}X^{\mathsf{T}}B$. If $X \in V$ the latter map equals CXC^{-1}. The self-adjoint endomorphisms for J give a stable complement, as the formula also shows. Thus V with its original G_F-action is a direct summand. Thorne’s vanishing theorem over the CM field L applies using the irreducibility on the

cyclotomic kernel verified above. It annihilates the finite Selmer group of the polarized adjoint over F, hence that of V. This argument applies to the induced representation even though it splits over the inducing field. J. A. Thorne, *On the vanishing of adjoint Bloch–Kato Selmer groups of irreducible automorphic Galois representations*, Theorem 6.1.

For the dual assertion apply the global Selmer dimension formula with the finite local conditions, which are mutual annihilators under local duality. Both global invariant terms vanish. For V this follows from Schur's lemma, since a scalar in V is zero. The trace form identifies $V^\vee$ with V. A nonzero invariant in $V(1)$ would give an isomorphism $U \simeq U(1)$ by absolute irreducibility. Determinants would then give $\kappa_\ell^4 = 1$ on G_F. At a finite $v \nmid \ell$ unramified cohomology has the same dimension as the local invariants. At $v \mid \ell$, V is crystalline since U is crystalline. Its ten weights at each embedding have four positive root weights, their negatives and two zeros, using the convention that κ_ℓ has weight one. The local difference $h^1_f - h^0$ is consequently $4[F_v : \mathbb{Q}_\ell]$ by the Bloch–Kato dimension formula. At a real place the two eigenspaces of C are complementary Lagrangians since C reverses the form. An infinitesimal symplectic transformation commuting with C acts on one by an arbitrary two-by-two matrix and on the other by its negative transpose in dual bases. The real term is therefore -4. Summing gives zero, since F is totally real. Thus the dual finite Selmer group has the same dimension as the first. [15, 10] □

For $v \mid \ell$ let $\mathfrak{b}_v \subset V$ preserve the chosen flag and let $\mathfrak{n}_v$ be its strictly upper triangular part. The quotient $\mathfrak{t} = \mathfrak{b}_v/\mathfrak{n}_v$ is the trivial rank-two module of infinitesimal diagonal changes with fixed multiplier.

Lemma 48.26 (Ordinary and fixed-weight local directions)**.** *Write $G_v = G_{F_v}$. The map $H^1(G_v, \mathfrak{b}_v) \to H^1(G_v, V)$ is injective. Restriction of its diagonal characters to local units gives an exact sequence*

$$0 \longrightarrow H^1_f(F_v, V) \longrightarrow H^1(G_v, \mathfrak{b}_v) \longrightarrow \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}_{F_v}^\times, E)^2 \longrightarrow 0.$$

Over a small extension of local Artin E-algebras every ordinary local deformation lifts with any prescribed lift of its two diagonal inertia characters.

Proof. The ordinary characters of U are unramified characters times cyclotomic powers. Their consecutive weight gaps are at least two. Thus $\mathfrak{n}_v$ has a filtration by four positive-root characters, each of weight at least two, and $V/\mathfrak{b}_v$ has the four opposite weights. It follows that $H^0(G_v, V/\mathfrak{b}_v) = 0$. Local duality gives $H^2(G_v, \mathfrak{n}_v) = H^2(G_v, \mathfrak{b}_v) = 0$, since the duals twisted by one have no constituent of weight zero. The first vanishing of invariants gives the injection. The vanishing of $H^2(G_v, \mathfrak{n}_v)$ gives surjectivity onto $H^1(G_v, \mathfrak{t})$. By local reciprocity, restriction from $F_v^\times$ to its units is onto on additive characters, as one can choose their values at a uniformizer.

Put $d_v = [F_v : \mathbb{Q}_\ell]$. The kernel of the map to units has dimension $h^0(G_v, \mathfrak{b}_v) + 6d_v - 2d_v$, by the local Euler characteristic formula. It contains $H^1_f(F_v, \mathfrak{b}_v)$ since the finite local condition on the trivial module $\mathfrak{t}$ is unramified. The Bloch–Kato dimension formula for the crystalline submodule $\mathfrak{b}_v$ gives exactly the displayed kernel dimension. Under the injection to $H^1(G_v, V)$ this finite subspace maps into $H^1_f(F_v, V)$. The two spaces have equal dimension, since $h^0(G_v, V) = h^0(G_v, \mathfrak{b}_v)$ and both have four positive weights. This proves the asserted identification.

For the lifting assertion choose a symplectic basis for the lifted flag, possible by lifting a basis of each successive summand and adjusting the dual vectors. First lift the two diagonal characters by prescribing the given values on units and lifting their values at a uniformizer. The

fixed multiplier then determines the other two. For a small extension with kernel I annihilated by the maximal ideal, arbitrary upper triangular symplectic-similitude matrix lifts with these diagonals have multiplication defect a continuous two-cocycle in $\mathfrak{n}_v \otimes_E I$. Continuous choices exist using coordinates on the upper unipotent group and an E-linear section of the coefficient surjection. Its class vanishes by the H^2 computation above. Subtracting a one-cochain whose coboundary is the defect gives the required homomorphism. □

Theorem 48.27 (The formal ordinary family at either initial point)**.** *Consider deformations of R on G_F with fixed multiplier, the indicated ordered ordinary flags at ℓ, and inertia conjugate to the original finite inertia away from ℓ. Take continuous deformations over local Artin E-algebras with residue field E, up to symplectic conjugation reducing to the identity. For each such algebra the restriction to units of the first two diagonal characters at every $v \mid \ell$ identifies this functor with the functor of arbitrary lifts of these unit characters. In particular the unframed completed deformation ring at the point is a power series ring over E in $2[F : \mathbb{Q}]$ variables. Fixing all the inertial weights gives the local ring E.*

Proof. Enlarge a finite set S to include the archimedean places, those over ℓ and all ramification of R. Outside S require unramifiedness. At finite $v \in S$ away from ℓ the tangent condition for fixed inertia is unramified cohomology. Indeed a class restricts to zero on inertia exactly when an infinitesimal conjugation makes the two inertia actions agree. This local condition lifts over small extensions. Having identified the inertia action with the original one, a Frobenius matrix is the original Frobenius times an element centralizing finite inertia in the symplectic group. This centralizer is a smooth algebraic group in characteristic zero and its elements lift. The resulting matrix, together with finite inertia, defines a continuous lift: the closure of the powers of the Frobenius matrix is compact. In fact its eigenvalues are units and the powers of each unipotent Jordan block are bounded by the integrality of the binomial coefficients. This argument uses the faithful E-linear realization of matrices over the finite-dimensional Artin algebra. At a real place the group of order two has zero higher cohomology.

Now let $A' \to A$ be a small extension with kernel I. Start with a global deformation over A and a lift of its unit characters. At places over ℓ choose local lifts by Lemma 48.26. At the other places of S use the preceding paragraph. The obstruction to a global fixed-multiplier lift belongs to $H^2(G_{F,S}, V) \otimes_E I$; it is the multiplication defect of arbitrary continuous matrix lifts, computed as in the lemma with the full Lie algebra. The chosen local lifts make its restrictions zero. By Poitou–Tate duality the kernel of this localization in degree two is dual to the kernel of localization in degree one for $V^\vee(1)$. The latter kernel is contained in its finite Selmer group, which vanishes by Proposition 48.25. Hence a global lift exists before imposing the local choices.

Compare this lift with the chosen ones at S. Their differences are local classes modulo $H^1_f(F_v, V)$, tensored with I, by Lemma 48.26 and the fixed-inertia calculation. The same duality sequence gives a surjection

$$H^1(G_{F,S}, V) \longrightarrow \bigoplus_{v \in S,\ v \text{ finite}} H^1(F_v, V)/H^1_f(F_v, V),$$

since the dual finite Selmer group is zero. Correct the global lift by a preimage of the discrepancies. It now realizes the local conditions with the chosen weights. Two such corrected lifts of the same deformation differ up to strict conjugation by a class in $H^1_f(F, V) \otimes_E I$, which is zero. A lift of the ordered flag for a fixed matrix representation is unique at each small extension: the difference of two is invariant in $(V/\mathfrak{b}_v) \otimes_E I$, whose invariants vanish by the lemma. Induction on the length of the algebra proves the asserted bijection, naturally in the algebra.

Finally the formal ratios of lifted unit characters to the originals have values in $1 + \mathfrak{m}_A$. The finite logarithm identifies this group with the additive $\mathfrak{m}_A$. Local units modulo their finite torsion have a pro-ℓ free part of rank d_v; their open principal units have this rank by the local logarithm. Two characters at every place therefore give $2\sum_{v|\ell} d_v = 2[F : \mathbb{Q}]$ formal parameters. Sending all of them to zero gives the fixed-weight assertion. □

The coefficient field in this theorem has characteristic zero. The remaining support problem includes components in characteristic ℓ, where the ordinary weights can collide. The theorem gives no bound for the integral lengths of such components on the original period fields.

49 Quadratic induction and symmetric cubes

49.1 Quadratic induction throughout the even-order ordinary family

Take an even-order triple on the interval of Corollary 48.24. Write $F = H$, $M = FD_0$, and let ϕ be the nontrivial character of $\mathrm{Gal}(M/F)$. The extension is unramified at every finite place and split above ℓ, by Theorem 48.5. Use a coefficient field E as in Theorem 48.27. In its ordered ordinary flags the initial representation has the description

$$R_0 = b\,\mathrm{Ind}_{G_M}^{G_F} W_0, \qquad \det W_0 = d|_{G_M}, \qquad b^2 d\phi = \mu.$$

Here W_0 is the restriction to M of the Hilbert modular plane of Proposition 48.22. Fix the first summand of the induction over G_M to be the one labeled W_0. All Artin algebras below are local E-algebras with residue field E.

Theorem 49.1 (Induction on the entire formal ordinary family)**.** *Induction followed by multiplication by b gives a natural bijection from the ordinary deformations of W_0 over M of fixed determinant $d|_{G_M}$ and fixed inertia away from ℓ to the ordinary deformations of R_0 of Theorem 48.27. For the plane one retains the indicated upper line at each place over ℓ and uses strict changes of basis. For the four-dimensional representation one uses the original order of the four graded lines. On the coset $G_F \setminus G_M$ the trace vanishes in every coefficient algebra, including those with nilpotents.*

Proof. Write the induced module in the two summands specified above, and let S act by 1 on the first and by -1 on the second. The alternating form in Theorem 48.23 is an orthogonal sum of determinant forms on the two planes. Thus S is symplectic and satisfies

$$SR_0(g)S^{-1} = \phi(g)R_0(g), \qquad S^2 = 1.$$

It preserves the initial flags. In fact at each split place the first line is in the plane with larger weight gap, the second subspace is the sum of the two upper lines, and the third subspace is the sum of the first line and the entire other plane.

Let R_A be an ordinary deformation over A, with flags $\mathcal{F}_v$. The matrices $\phi(g)SR_A(g)S^{-1}$ with flags $S\mathcal{F}_v$ give another deformation of the same point and multiplier. At a finite place away from ℓ, ϕ is trivial on inertia and S centralizes the original inertia. Conjugating a strict inertia identification by S gives one for the new matrices. At $v \mid \ell$ the character ϕ is trivial on the whole local group. The ordered graded characters, in particular their restrictions to units, agree for the two deformations. The uniqueness in Theorem 48.27 supplies a strict symplectic change of basis P from the new deformation to the given one. It carries the indicated flags as well. Consequently $J_A = PS$ preserves those flags, reduces to S, and satisfies

$$J_A R_A(g) J_A^{-1} = \phi(g)R_A(g).$$

The commutant of $R_A(G_F)$ consists of scalars. Indeed absolute irreducibility of R_0 implies that the E-span of its group matrices is the entire matrix algebra. This follows from Burnside's theorem after extending to an algebraic closure and then by equality of dimensions over E. Choose sixteen group elements giving a basis. The determinant of their lifted coordinates is a unit in A, so the lifted group matrices span $\mathrm{Mat}_4(A)$. The square J_A^2 is therefore a symplectic scalar reducing to 1. Such a scalar u satisfies $(u-1)(u+1) = 0$ with $u+1$ a unit. Thus $J_A^2 = 1$. Two symplectic intertwiners reducing to S likewise differ by a strict symplectic scalar and coincide. This uniqueness commutes with change of coefficients and strict conjugation.

The two idempotents $(1 \pm J_A)/2$ give free summands of rank two, since their images are projective over a local ring and their reductions have dimension two. The summands are orthogonal and the form on each is perfect. The group G_M preserves both and the other coset interchanges them. On the positive summand let W_A be the restriction of $b^{-1}R_A$ to G_M. Its determinant is $b^{-2}\mu|_{G_M} = d|_{G_M}$, since on a plane the multiplier of a determinant form is the determinant. The map from the induction which sends $g \otimes x$ to $b(g)^{-1}R_A(g)x$ is well-defined and equivariant for $b^{-1}R_A$. On each of the two coset summands it is an isomorphism onto the corresponding plane. The interchange of the planes gives the asserted zero trace.

Each of the two idempotents preserves every local flag submodule above ℓ. On each free rank-one graded quotient an idempotent is either zero or one and has precisely its original value on reduction. Applying the idempotents to the flag therefore recovers the two plane filtrations with their prescribed upper lines. For the two places of M above one place of F, these are the filtrations on the first and on the conjugate plane, identified by a fixed element of $G_F \setminus G_M$. At a place away from ℓ, identify the four-dimensional inertia with its constant lift by strict conjugation. In these coordinates the positive projector p and its reduction $p_0 = (1+S)/2$ both commute with the constant inertia. The matrix $p_0 p + (1-p_0)(1-p)$ also commutes with it, is the identity on reduction, and intertwines p with p_0. It identifies the inertia on the lifted plane with the original one. Multiplication by the fixed scalar character b^{-1} preserves this identification.

Conversely fix $\sigma \in G_F \setminus G_M$ and start with such a plane W_A. In a basis lifting the original basis, use the determinant forms with coefficients $1, -d(\sigma)$ on the two induced summands and the original overall scalar normalization. On the subgroup both determinants are d. The matrix of σ sends (x, y) to $(W_A(\sigma^2)y, x)$; it multiplies this form by $-d(\sigma)$ because $\det W_A(\sigma^2) = d(\sigma)^2$. The full multiplier after twisting is μ. Interleave the two lifted upper lines in the order described at the beginning of the proof. This produces the required self-dual flag above ℓ. Away from ℓ, inertia lies in G_M. The strict inertia identifications in the two planes may each be given determinant one by multiplying by a scalar congruent to one, since the square map is invertible on $1 + \mathfrak{m}_A$ by the finite binomial series. Their orthogonal sum is symplectic.

The same scalar adjustment makes strict plane changes of basis compatible with symplectic equivalence. In the inverse construction choose a basis of the positive summand lifting its original basis with the prescribed determinant pairing, by rescaling one vector by a strict unit. Transport it to the other summand by $b(\sigma)^{-1}R_A(\sigma)$. The two resulting forms have the coefficient ratio $-d(\sigma)$ just computed, so the induction isomorphism is a strict symplectic one after identifying the original bases. It respects the recovered flags. Finally, on an induced lift the sign operator on its two summands already has the defining properties of J_A. Uniqueness recovers its first plane. This proves both inverse identities and naturality. □

Corollary 49.2 (All self-twisting characters at the two initial points)**.** *For every deformation in Theorem 48.27, a continuous $A^\times$-valued character giving an isomorphism with its twist is trivial in the odd-order case and belongs to $\{1, \phi\}$ in the even-order case. Both characters in*

the latter set give isomorphisms. In particular M is the unique quadratic inducing field of an even-order lift.

Proof. Taking determinants shows that every value of the character is a fourth root of unity. Reduction identifies these roots with those in E because a root lifts as a constant and the quotient of two lifts of the same root is a fourth root congruent to one, hence one. Reducing the isomorphism now gives a self-twist of R_0. In the odd-order case the projective image of the underlying plane stays dense in the connected group PGL_2 on every finite-index subgroup. Its third symmetric power remains absolutely irreducible there, as in Theorem 48.18. An intertwiner commutes with the restriction to the kernel of the character, so it is scalar and the character is trivial.

In the even-order case restrict to the intersection of that kernel with G_M. Both plane constituents remain absolutely irreducible there by connected projective density. They remain inequivalent because at any extension of the embedding in Proposition 48.22 their Hodge–Tate weight gaps are j and $j+2t$, which remain distinct on every finite field extension. A common scalar twist cancels in this comparison. Hence the intertwiner over E is scalar on each plane, and commutes with all of G_M. The twisting character is therefore trivial on G_M. Theorem 49.1 realizes the remaining possibility ϕ over A. Any induction from a quadratic field has the self-twist by its nontrivial quadratic character, as one sees by the sign operator in the two coset summands. This gives the uniqueness assertion. □

Corollary 49.3 (Trace identities on components through an even-order point)**.** *Let a family with generic coefficient characteristic zero have a point with residue field E and Noetherian local ring B. Suppose its reductions over all $B/\mathfrak{m}_B^n$ are deformations in Theorem 49.1 in compatible coordinates. For every $g \in G_F \backslash G_M$ its trace vanishes in B, hence on each irreducible component through the point whenever the trace is defined on the ambient family.*

Proof. The trace belongs to every power of $\mathfrak{m}_B$ by the theorem. Their intersection in a Noetherian local ring is zero. In an affine neighborhood where the trace is defined it is therefore killed by an element invertible at the point. That element is outside every minimal prime contained in the prime of the point, so the trace lies in each such minimal prime. □

This identifies the ordinary family at the specified characteristic-zero point for every even-order triple on the interval. Bounding the integral local cohomology still requires control of the components carrying the torsion classes.

49.2 The symmetric cube and the prescribed integral weights

In the odd-order construction write $F = H_1$, $R_0 = b\,\mathrm{Sym}^3 W_0$, and $\mu = b^2(\det W_0)^3$, as in Theorem 48.18. Here W_0 denotes the plane after restriction to G_F. At $v \mid \ell$ number the four ordered diagonal characters from zero to three, starting with the top of the ordinary flag. Given an ordinary lift as in Theorem 48.27, let $\epsilon_{v,i}$ be the ratio of its i-th character on local units to the corresponding character of R_0.

Theorem 49.4 (The equations for the formal symmetric cubes)**.** *Over every local Artin E-algebra with residue field E, the lifts of R_0 obtained by applying $b\,\mathrm{Sym}^3$ to ordinary lifts of the plane of fixed determinant and fixed inertia away from ℓ, with the induced ordered flags, are precisely the points satisfying*

$$\epsilon_{v,0} = \epsilon_{v,1}^3 \qquad (v \mid \ell).$$

They form a smooth closed formal subscheme of codimension $[F:\mathbb{Q}]$ *in the ordinary deformation space. The same equations hold if the plane determinant and scalar twist are both allowed to vary subject to their original reductions and the fixed four-dimensional multiplier.*

Proof. For the plane put $V_2 = \mathfrak{sl}(W_0)$. The derivative of its symmetric cube embeds V_2 into the fixed-multiplier adjoint V of Proposition 48.25. The trace pairing on this image equals ten times $(X,Y) \mapsto \operatorname{tr}_{W_0}(XY)$. Indeed both pairings are invariant, and the invariant symmetric forms on $\mathfrak{sl}_2$ are multiples of its trace form. This last fact follows from invariance under the diagonal torus and the brackets of the upper and lower root vectors. For $X = Y = \operatorname{diag}(1,-1)$ the eigenvalues in the cube are $3, 1, -1, -3$, giving 20 in place of 2. The orthogonal complement therefore splits off over E, even when $\ell = 5$. The finite local conditions commute with direct sums, as do their duals. The two vanishings of Proposition 48.25 imply

$$H^1_f(F, V_2) = H^1_f(F, V_2^\vee(1)) = 0.$$

We record why every choice of the single plane weight at each place is attained. Above ℓ the trace-zero flag-preserving algebra $\mathfrak{b}_2$ has its upper root $\mathfrak{n}_2$ and a trivial one-dimensional quotient. The upper root has weight $j \geq 2$, and the quotient $V_2/\mathfrak{b}_2$ has weight $-j$. Thus the latter has no invariants, while local duality gives $H^2(F_v, \mathfrak{n}_2) = H^2(F_v, \mathfrak{b}_2) = 0$. It follows that ordinary classes inject into $H^1(F_v, V_2)$ and map onto the local diagonal classes, hence onto additive characters of local units by local reciprocity. The kernel of the map to units has dimension $h^0(F_v, \mathfrak{b}_2) + [F_v : \mathbb{Q}_\ell]$, by the local Euler characteristic formula. It contains the finite subspace of $\mathfrak{b}_2$, since a finite class in the trivial diagonal quotient is unramified. By the crystalline dimension formula this finite subspace has the displayed dimension, as does $H^1_f(F_v, V_2)$, since the local invariants coincide and there is one positive weight. Thus the kernel is precisely the finite local condition. Over a small extension the lifting obstruction with a given diagonal lift lies in the second cohomology of the upper root and vanishes. Away from ℓ keeping finite inertia fixed gives the unramified tangent condition and lifts over a small extension by lifting Frobenius in its smooth centralizer, as in Theorem 48.27.

With these local lifts the global obstruction restricts to zero at every place. By Poitou–Tate duality the kernel in degree two is dual to the everywhere locally zero classes for $V_2^\vee(1)$, all of which are finite. It vanishes. The discrepancy between an unrestricted global lift and the prescribed local lifts, viewed modulo the local finite conditions, can be removed by a global class because the dual finite Selmer group is zero. With the weights fixed the ambiguity is the first finite Selmer group, also zero. The local flag itself is unique over a small extension since $V_2/\mathfrak{b}_2$ has no invariants. Induction on the length of the coefficient algebra gives a unique plane up to strict equivalence for each choice of its upper character on units.

Write z_v for the ratio of the lifted upper character to the original one. The lower ratio is z_v^{-1}. The first two ratios for the cube are z_v^3, z_v. Conversely take $z_v = \epsilon_{v,1}$ and use the plane just constructed. Its twisted cube has the required multiplier, inertia and flags. If the equations in the statement hold, its first two characters agree with the given ones on units, so the two lifts agree up to strict symplectic conjugation by Theorem 48.27. In logarithmic weight coordinates the equations subtract three times the second coordinate from the first at each principal-unit generator. Each has coefficient one on a distinct first coordinate. There are $\sum_{v \mid \ell} [F_v : \mathbb{Q}_\ell] = [F : \mathbb{Q}]$ such generators. This proves the assertion on formal schemes. Finally multiply a plane of variable determinant by the inverse square root of its determinant ratio. This square root is a character given by the finite binomial series on strict units. Absorb its third power in the outside twist. After this operation the determinant is fixed and the remaining twist divided by b is a strict character of square one, hence is identically one. □

We now use the prescribed integral characters, independently of the existence of a lift of the initial point. Take any triple of Section 47.3. Write

$$A = \kappa_\ell \omega_\ell^{-1} \xi, \quad \Psi = \alpha \kappa_\ell^r, \quad \mu = A\Psi, \qquad (\delta_0, \delta_1, \delta_2, \delta_3) = (A\Psi, \Psi, A, 1).$$

Let O be a coefficient DVR containing their values. For a local inertia group I, let $\mathfrak{c}(I) \subset O$ be generated by all $\delta_1(s)^3/(\mu(s)\delta_0(s)) - 1$ for $s \in I$. For an ordered scalar-twisted cube of a plane with diagonal characters u, v and scalar b', its first two characters are $b'u^3, b'u^2v$ and its natural multiplier is $(b')^2(uv)^3$. They give exactly zero in this equation over any coefficient ring.

Theorem 49.5 (Exact integral depth of the ordered weight equation)**.** *At each $v \mid \ell$ of the original period field E_x, throughout the common interval,*

$$\mathfrak{c}(I_{E_{x,v}}) = \begin{cases} O, & \ell - 1 \nmid r, \\ \ell^{1+v_\ell(r-2)} O, & \ell - 1 \mid r. \end{cases}$$

In the second case $r > 2$. In the odd-order case take the auxiliary $F = HC_{w,\ell}$ of Theorem 48.18. At all places over ℓ of either F or FE_x the ideal is $\ell^{1+v_\ell(r-2)}O$ for $r > 2$, and it is zero for $r = 2$. These are equalities of ideals regardless of the ramification index of O.

Proof. The character in the definition of $\mathfrak{c}$ before subtracting one equals Ψ/A^2. Write $\langle \kappa_\ell \rangle = \kappa_\ell/\omega_\ell$. Since E_x is unramified at ℓ and α, ξ are also unramified there, the character on this inertia is

$$\omega_\ell^r \langle \kappa_\ell \rangle^{r-2}.$$

The extension generated by the ℓ-power roots of unity over $\mathbb{Q}_\ell$ is totally ramified, so this inertia realizes every value in $\mathbb{Z}_\ell^\times$ under κ_ℓ. A nontrivial tame factor gives an element whose character value differs from one in the residue field, hence the unit ideal. If the tame factor is trivial, $r \geq \ell - 1 \geq 4$. The pro-ℓ part has topological generator $1 + \ell$. For a nonzero integer n the ideal generated by $u^n - 1$ as u runs over $1 + \ell\mathbb{Z}_\ell$ is $\ell^{1+v_\ell(n)}O$. All values have that divisibility by the binomial expansion; equality occurs at $u = 1 + \ell$, since a prime-to-ℓ power preserves the first nonzero digit and an ℓ-th power raises its valuation by one. This proves the first formula.

On G_H the finite characters ξ^2 and $\alpha\omega_\ell^r$ coincide. They have prime-to-ℓ order and have the same reduction by Theorem 48.2. Thus $\Psi/A^2 = \langle \kappa_\ell \rangle^{r-2}$ on G_H. The local ramification index of H divides two, and $C_{w,\ell}$ and E_x are unramified at ℓ. Adjoining unramified extensions locally preserves this ramification index. The inertia subgroups upstairs thus have index prime to ℓ in the original rational inertia. Their images onto the pro-ℓ cyclotomic factor are still surjective. The same binomial calculation gives the asserted ideal; at exponent zero the character is identically one. □

Corollary 49.6 (Comparison with the torsion cutoff on the period fields)**.** *On the triples of Theorem 49.5 the normalized lengths of the evaluated weight quotients $O/\mathfrak{c}(I)$ on E_x, summed with weight $\log \ell$ over the odd-order characters and $2 \leq r < t = t(x)$, are $O_\psi(t \log^2(2t))$. The same bound holds over the indicated odd-order auxiliary fields upon omitting $r = 2$. For all sufficiently large x, the image in $U_{\ell,r}$ of any $O/\mathfrak{c}(I)$-module under a $\mathbb{Z}_\ell$-linear map is killed by the quotient $U_{\ell,r} \to R_{\ell,r}$ of Section 6.6 in these finite-length cases. At $r = 2$ the odd-order character summands are already removed by the averaging operator of that section.*

Proof. Normalized length means O-length divided by $v_O(\ell)$, with v_O equal to one at a uniformizer. For $r > 2$ it is at most $1 + v_\ell(r - 2)$. All odd-order characters factor through the fixed cyclic field K_ψ, since E_x/K_ψ has two-power degree. Their number is at most $[K_\psi : \mathbb{Q}]$.

A contributing prime satisfies $\ell - 1 \mid 2r$. For a fixed prime the number of such $r < t$ is at most $2t/(\ell - 1)$. Consequently the contribution of the constant one in the bound is at most a fixed multiple of

$$2t \sum_{2 \le n \le 2t+1} \frac{\log n}{n-1} \ll t \log^2(2t).$$

For each $r > 2$ the sum of $v_\ell(r-2)\log \ell$ over its contributing primes is at most $\log(r-2)$. At $r = 2$ the only possible prime is five and the quotient on E_x has length zero.

Finally $\ell \le 2t + 1 \le P$ and $1 + \log_5 t < b_0$ for all sufficiently large x, by the definitions in Section 6.6. Every evaluated module in the finite cases is annihilated by ℓ^{b_0}, so its image is in the kernel $U_{\ell,r}[\ell^{b_0}]$. At $r = 2$ one has $r \le x^\eta$ eventually and $K_\psi \subset L_x$. On the character summand the entire group $\mathrm{Gal}(E_x/L_x)$ therefore acts trivially, so its average acts as the identity. The same holds for the inverse character if one uses the dual indexing convention. □

The length here measures an equation on the ordered characters. To deduce the full strict length bound one must still account for the cohomology carried by representations outside the symmetric cube.

49.3 Quadratic induction at the rational fixed weights

Take an even-order triple of Section 47.3. Use its rational fixed-weight trace ring T of Theorem 40.9, its coefficient DVR O with residue field k, and the universal trace t. Write $F = H$, $M = FD_0$, and let ϕ denote the rational character of D_0. On G_F it is the nontrivial character of M/F. Define the ideal

$$J_\phi = \big(t(g) : g \in G_F,\ \phi(g) = -1\big) \ \subset T.$$

Theorem 49.1 gives these trace identities at the varying-weight even-order initial point. Here we evaluate them in the rational trace space with its prescribed weights.

Proposition 49.7 (The exact intersection with the split trace)**.** *Under the split evaluation $T \to O$ the image of J_ϕ is $\ell^{1+v_\ell(r)}O$. In T the ideal can also be generated by good rational Frobenius traces at primes split completely in F and inert in D_0.*

Proof. On G_F one has $\alpha\omega_\ell^r = \phi$, by the equality on reduction of finite prime-to-ℓ characters in Theorem 48.5. Write $\langle \kappa_\ell \rangle = \kappa_\ell/\omega_\ell$. For $g \in G_F \setminus G_M$ the split trace is

$$(1 + A(g))(1 + \Psi(g)) = (1 + A(g))(1 - \langle \kappa_\ell(g) \rangle^r).$$

Every value has the stated divisibility. Choose g_0 on this coset with $\kappa_\ell(g_0) = 1$ and $\bar{A}(g_0)^4 \neq 1$, as in Lemma 48.13. The ramification index of F above ℓ divides two and D_0 splits there. Its inertia over F contains an element $s \in G_M$ with $\kappa_\ell(s) = 1 + \ell$, since the wild cyclotomic factor is pro-ℓ and surjective on rational wild inertia. At $g_0 s$ the first factor is a unit and the second has valuation $1 + v_\ell(r)$ in the normalization $v_\ell(\ell) = 1$, by binomial expansion. For the generating assertion apply Chebotarev in every finite Galois quotient which records membership in G_F, G_{D_0} and the trace modulo an arbitrarily high power of the maximal ideal. Such quotients exist by continuity on the compact group, using an open normal subgroup on whose cosets this reduction is constant. Conjugacy preserves the trace. Ideals of the complete Noetherian local ring T are closed by the Krull intersection theorem applied to the quotient. Hence the good traces generate the ideal. □

Theorem 49.8 (Which fixed-weight representations can have the quadratic trace identity)**.** *Let $\sigma : T \to O'$ be a continuous local specialization to a complete DVR with finite residue field, and suppose it kills J_ϕ. Its coefficient field has characteristic ℓ. If some trace coordinate of Theorem 40.9 is nonzero at σ, the realization is absolutely irreducible of rank four,* $\operatorname{ord}\alpha = 2$*, and after a finite coefficient extension it is induced over $\mathbb{Q}$ from a plane over the real quadratic field cut out by $\tau = \bar{\Psi}$. This quadratic field composited with F is M. In particular, every nonzero two-plane specialization in either characteristic has a nonzero trace at a good rational prime split completely in F and inert in D_0.*

Proof. If all trace coordinates are zero, the evaluation factors through the split trace. The proposition excludes characteristic zero. Otherwise Theorems 40.12 and 40.13 realize the trace, after a finite coefficient extension, by an absolutely irreducible four-dimensional representation or a sum of two absolutely irreducible planes $P' \oplus P$. We work over an algebraic closure of the fraction field. All these irreducibles stay irreducible on G_F. Indeed a subspace invariant there is invariant on $G_{FK(\mu_{\ell^\infty})}$, and hence is rationally Galois invariant by Lemma 48.13.

The restrictions of the semisimple realization and of its ϕ-twist have the same traces on G_F. Newton's identities applied to powers up to four give equality of characteristic polynomials, since 4! is invertible. Brauer–Nesbitt gives an isomorphism there. In the plane case real conjugation acts as opposite scalars on the two planes, and both F and M are totally real. Twisting therefore preserves each constituent separately. Let Z denote the full irreducible representation in the first case and the plane P, with residual characters $1, \tau$, in the second. In both cases $Z|_{G_F}$ is absolutely irreducible and isomorphic to its ϕ-twist.

An intertwining operator for this twist has scalar square by Schur's lemma. Over a finite extension rescale it to an involution. Its two eigenspaces are preserved on G_M and exchanged on the other coset. Both occur, since otherwise the operator would be scalar. They have equal dimension, and inducing the first eigenspace from G_M recovers $Z|_{G_F}$. The first eigenspace is irreducible on G_M, because a proper nonzero subspace stable there, together with its conjugate in the other eigenspace, would give a proper G_F-submodule. The two constituents on G_M are inequivalent. Indeed Frobenius reciprocity identifies the endomorphisms of the induction with the maps from the first constituent to the sum of the two on G_M; the endomorphism space has dimension one.

Since $M/\mathbb{Q}$ is abelian, G_M is normal in $G_\mathbb{Q}$. The whole rational group permutes the two isotypic summands of $Z|_{G_M}$. It gives a quadratic character η whose restriction to G_F is ϕ. This character is continuous because it factors through $\mathrm{Gal}(M/\mathbb{Q})$. Writing D for its quadratic field, we have $FD = M$ since $\eta|_{G_F}$ cuts out M/F. The two summands also express Z as an induction over $\mathbb{Q}$ from D and yield the rational self-twist by η. These constructions descend to a finite extension of the coefficient field, since the intertwiner is found by linear equations over that field and its eigenvalues require only a finite extension.

Reduction gives a rational residual self-twist. For the plane its set of residual characters is $\{1, \tau\}$; for rank four it is $\{1, \tau, \chi, \chi\tau\}$, with $\chi = \bar{A}$. The nontrivial quadratic character must belong to that set since 1 does. In the rank-four set neither χ nor $\chi\tau$ is quadratic, as an inertia element over two in Theorem 48.5 has $\tau = 1$ and $\chi^4 \neq 1$. Thus $\eta = \tau$ in both cases, identifying the signs in odd characteristic. On the rational group τ has order $\operatorname{ord}\alpha$ because its two factors have coprime conductors and the Teichmüller factor has order at most two, dividing the even order of α. It follows that $\operatorname{ord}\alpha = 2$.

Suppose the coefficient characteristic is zero. The rational decomposition group at ℓ contains an element with $\eta = -1$, by the standing condition $\bar{\Psi}|_{D_\ell} \neq 1$. This element exchanges the two summands. Choose a wild inertia element with cyclotomic value $1 + \ell$. It belongs to G_D since its image in a group of order two is trivial. Its four eigenvalues in the full ordinary realization

are

$$(1+\ell)^{r+1}, \quad (1+\ell)^{r}, \quad 1+\ell, \quad 1.$$

They are distinct because $r \geq 2$. The same simplicity holds on the invariant subrepresentation Z. In either case Z has a line stable under the rational decomposition group, by intersecting the ordinary flag with Z and taking the first nonzero step. This line is an eigenline for the chosen inertia element. An eigenline for a simple eigenvalue of an operator preserving both summands belongs to one of them. The exchanging element cannot preserve it, a contradiction.

Finally suppose the coefficient characteristic is ℓ and Z is the plane. It is induced from a character of G_D. That character has finite inertia at every place. Above ℓ its values on inertia are among those of the fixed residual graded characters of the ordinary flag. Away from ℓ the fixed inertia kernel has finite index, and the representation is unramified outside the prescribed finite set. Some power of the character is therefore everywhere unramified at finite places and factors through the finite narrow class group by class field theory. The original character has finite order. In characteristic ℓ its values are roots of unity algebraic over the prime field. All traces of its induction are likewise algebraic over the prime field. On such constants reduction is injective, since a nonzero element has finite multiplicative order and valuation zero. The residual trace is $1+\tau$, whose values are constants. Equality of traces on reduction thus gives equality with $1+\tau$ in the fraction field. The degree-two Newton identities and Brauer–Nesbitt now make Z reducible, a contradiction. This proves the classification and, by the good-prime generating assertion, the last conclusion. □

Corollary 49.9 (The integral intersection in the rational trace ring)**.** *The complete algebra $C = T/J_{\phi}$ is killed by a power of ℓ. It has finite O-length if $\operatorname{ord}\alpha > 2$. For every even-order triple its quotient obtained by imposing $U_{pj} = W_{jk} = 0$ for all j, k has finite O-length. These are the coordinates cutting out the two-plane locus in Theorem 40.13.*

Proof. Write π for a uniformizer of O. If π were not nilpotent in C, a quotient by a prime avoiding it would give a complete local domain in which π is nonzero and belongs to the maximal ideal. While its dimension is at least two, choose in its maximal ideal an element outside all minimal primes of (π). These primes have height one. Mod out by a minimal prime of the chosen element. It too has height one and cannot contain π, as otherwise it would be a minimal prime of (π). The dimension strictly decreases, and π survives. This yields a one-dimensional complete local domain B with $\pi \neq 0$. Its quotient modulo π is Artinian and finite over k. The π-adic and maximal-ideal topologies on B coincide, since some power of the maximal ideal lies in πB. Lifting a basis modulo π and expanding successively in powers of π shows that B is finite over O. The integral closure of O in its fraction field is a complete DVR, and the resulting specialization is local and continuous. This is a forbidden characteristic-zero point of Theorem 49.8. Thus a power of π, and hence a power of ℓ, is zero in C.

If the special fibre has positive dimension, quotient by a prime and cut down to a one-dimensional complete local domain of characteristic ℓ. One can choose a prime immediately below the maximal ideal in a chain of maximum length. In that domain choose a nonzero parameter z in its maximal ideal. The same successive expansion gives finiteness over $k[[z]]$, which embeds because a nonzero series is a power of z times a unit. Passing to the integral closure in the fraction field gives an equal-characteristic complete DVR with finite residue field. At least one trace coordinate has nonzero image since these coordinates topologically generate the special fibre. If $\operatorname{ord}\alpha > 2$ the theorem excludes such a point. After imposing the displayed two-plane equations it excludes such a point for every even order. In either situation the quotient is zero-dimensional and Noetherian, hence Artinian with finite residue field and finite O-length. □

The split evaluation determines an explicit exponent in the first proposition. A uniform bound for the nilpotence exponent in the full trace intersection, and control of the traces outside that intersection, are still needed for the strict-length argument.

49.4 Strict classes over the quadratic inducing field

In the last possible case of Theorem 49.8 let D be the rational quadratic field of $\tau = \bar{\Psi}$. It has a single place w over ℓ, since $\tau|_{D_\ell} \neq 1$. Write $\chi = \bar{A}$. The character χ is nontrivial on G_{D_w}. Indeed $FD = M$ and Theorem 48.5 gives a place over ℓ at which M/F is split and a local element of G_F with $\chi^4 \neq 1$. With compatible choices of places this element lies in G_{D_w}.

Lemma 49.10 (Vanishing of both strict groups over the inducing field)**.** *For $\lambda = \chi|_{G_D}$ and for its inverse, the subgroup of $H^1(D, k(\lambda))$ restricting to zero on G_{D_w} and on inertia at all places away from ℓ is zero. The assertion persists under finite residue field extension.*

Proof. Let λ_0 be the corresponding rational character χ or χ^{-1}. Induction from G_D gives $k(\lambda_0) \oplus k(\lambda_0\tau)$ as a rational module. Indeed the permutation module of the two cosets splits into its two sign eigenspaces because 2 is a unit. Shapiro's isomorphism carries the strict conditions into the rational strict conditions on both summands. For a rational decomposition group the local isomorphism is a sum over its orbits on the two cosets, with stabilizers the local groups upstairs. The same coset description for inertia gives the restrictions to the conjugate inertia subgroups upstairs. Thus a class satisfying the stated vanishings has zero local images in both rational summands. Their global strict groups vanish by Proposition 47.8 at residual precision because the characters are among the reductions of $A^{\pm 1}, (\Psi/A)^{\pm 1}, (A\Psi)^{\pm 1}$, since $\tau^2 = 1$. Extension of the finite coefficient field commutes with these kernels by a basis expansion on cochains. □

Proposition 49.11 (The local ordinary line is globally invariant)**.** *Let B be a complete Noetherian local coefficient ring with finite residue field k' over k. Let W be a continuous representation of G_D on B^2 whose residual semisimplification has characters $1, \chi|_{G_D}$. Suppose it preserves a free direct summand of rank one at w and its inertia away from ℓ factors through finite groups of order prime to ℓ. Then the local line is preserved by G_D.*

Proof. Take $h \in G_{D_w}$ with $\chi(h) \neq 1$. The characteristic polynomial of $W(h)$ has distinct residual roots and splits into two linear factors over B by Hensel's lemma. Their fiberwise distinctness gives two free rank-one eigensummands by the polynomial projectors. The given local line is one of these summands since the action of h on it is a root with the specified reduction, and the complementary projector acts by zero. Choose this summand first in the eigenbasis. Write a_g, b_g, c_g, d_g for the entries in row order, so $c_g = 0$ for $g \in G_{D_w}$.

First the residual first line is globally invariant. Otherwise the residual plane still has a global invariant line because its composition factors have dimension one. Such a line must be an eigenspace of h, so it must be the second line. The residual matrices are then lower triangular with distinct diagonal characters λ_1, λ_2 in the indicated order. The lower entry divided by $\lambda_1(g)$ is a cocycle for $\lambda_2/\lambda_1 = \chi^{\pm 1}|_{G_D}$. It vanishes at w. Its class on every other inertia group is zero by averaging over the finite prime-to-ℓ image through which the residual matrices factor. The lemma makes it a global coboundary. Its value at h is zero in our diagonal basis, and $(\lambda_2/\lambda_1)(h) - 1 \neq 0$. The coboundary is therefore identically zero, contrary to the first line being noninvariant.

Let $\mathfrak{m}$ be the maximal ideal of B, and let L be the ideal generated by all c_g. The preceding paragraph gives $L \subseteq \mathfrak{m}$. Now the residual diagonal characters in the upper triangular

representation are again denoted by λ_1, λ_2. The identity $c_{gg'} = c_g a_{g'} + d_g c_{g'}$ shows that

$$g \longmapsto \lambda_1(g)^{-1}(c_g \bmod \mathfrak{m}L)$$

is a cocycle for λ_2/λ_1 with multiplicity space $L/\mathfrak{m}L$. This space is finite-dimensional over k'. The cocycle is continuous, since Artin–Rees gives $L \cap \mathfrak{m}^N \subseteq \mathfrak{m}L$ for sufficiently large N. It vanishes on the local group at w and factors through the given finite images on other inertia groups. Lemma 49.10 applied to each coordinate, followed again by evaluation at h, gives zero identically. This says $L = \mathfrak{m}L$. Nakayama's lemma applies to the finitely generated ideal and gives $L = 0$. The chosen first line is globally invariant. □

Theorem 49.12 (Finite length of the quadratic trace intersection for every even order)**.** *On every even-order triple on the interval, the rational fixed-weight algebra T/J_ϕ has finite O-length. Every nonzero trace-coordinate specialization to a complete DVR with finite residue field, in either characteristic, has a nonzero good-prime trace at a rational prime split completely in F and inert in D_0.*

Proof. The only outstanding DVR case in Theorem 49.8 is an absolutely irreducible rank-four induction from a plane W over D in equal characteristic. We check the lattice conditions in the proposition. Work over a finite coefficient extension on which the inducing plane is defined. Its intersection with a stable full lattice gives a stable lattice in W. The direct sum of this lattice and its translate by an element outside G_D is a stable induced lattice in the four-dimensional space. Its residual characteristic polynomials agree with those for the original lattice because both are obtained by reducing the same integral characteristic polynomials. On G_D the four residual constituents are two copies each of 1 and $\chi|_{G_D}$, since $\bar{\Psi}|_{G_D} = 1$. The two conjugate plane lattices have the same multiplicities for each of those characters, as both characters extend to $G_{\mathbb{Q}}$. Each plane thus has both constituents once. This uses Brauer–Nesbitt on the residual semisimplifications.

Intersect the four-dimensional ordinary flag with W over the fraction field. Its first nonzero intersection is a line, since a single flag step increases the dimension of an intersection by at most one. At the sole place w it is locally stable. Its saturation in the plane lattice is a direct summand over the DVR. All the other finite inertia images of the rational representation have order prime to ℓ, by the defining conditions of Section 40.3. Their images on W retain that property. Proposition 49.11 now produces a global invariant line in W. Its induction is a proper nonzero rational invariant subspace in rank four, contradicting irreducibility.

It follows that T/J_ϕ has no DVR specialization with a nonzero trace coordinate. Corollary 49.9 already killed a power of ℓ there. A positive-dimensional special fibre would give a DVR specialization with a nonzero coordinate by the one-dimensional quotient argument in that corollary. Thus the algebra is Artinian, and has finite O-length because its residue field is finite. Proposition 49.7 supplies the good primes in the last assertion. □

49.5 The exact exponent of the quadratic trace intersection

Use the even-order data T, F, M, ϕ, J_ϕ of Section 49.3. In particular T is the ring of rational traces at the prescribed weights. For a place v of F above ℓ, fix a compatible rational decomposition group and denote the four characters on the graded lines of its ordinary flag by $\theta_0, \theta_1, \theta_2, \theta_3$.

Theorem 49.13 (Paired local characters and the coefficient ideal)**.** *In $C = T/J_\phi$ the four characters on the whole local group satisfy*

$$\theta_0|_{G_{F_v}} = \theta_2|_{G_{F_v}}, \qquad \theta_1|_{G_{F_v}} = \theta_3|_{G_{F_v}}.$$

The kernel of the coefficient map $O \to C$ is exactly

$$J_\phi \cap O = \ell^{1+v_\ell(r)} O.$$

In particular the least power of ℓ annihilating the full ring, including its nilpotents, equals the power detected by the split trace.

Proof. Let ρ denote the universal framed representation of Section 40.3. By construction T is the closed subalgebra of its coefficient ring generated by the traces. Choose h in the rational decomposition group at ℓ separating all four residual characters, as in Lemma 47.7. The characteristic polynomial of $\rho(h)$ has coefficients in T, by the Newton identities since $\ell \geq 5$. Its roots a_i with the prescribed distinct residues are in $T^\times$ by Hensel's lemma. The operators

$$e_i = \prod_{j \neq i} \frac{\rho(h) - a_j}{a_i - a_j}$$

project onto its four eigenlines. In an eigenbasis in the given order, the ordinary flag is the coordinate flag. Indeed the e_i preserve it, and the summands of each step are determined by its reduction. For every rational Galois element g the diagonal entry

$$z_i(g) = \operatorname{tr}(e_i \rho(g))$$

belongs to T, as the right side is a T-linear combination of $t(h^j g)$. It is a unit since the split residual representation has nonzero diagonal characters. On the rational decomposition group these entries are the characters θ_i.

Fix any $g \in G_F \setminus G_M$. Conjugating $\rho(g)$ by $\rho(h)$ multiplies its entry ij by a_i/a_j. The polynomial

$$P(X) = \prod_{i \neq j} \frac{X - a_i/a_j}{1 - a_i/a_j} \ \in T[X]$$

therefore projects this matrix onto its diagonal when evaluated at the conjugation operator. Every denominator is a unit. For any $y \in G_{F_v}$ this gives

$$\sum_{i=0}^{3} z_i(g)\theta_i(y) \ \in J_\phi.$$

Indeed the sum is the trace of the diagonal matrix times $\rho(y)$. Expansion of P writes that trace as a linear combination of $t(h^a g h^{-a} y)$ for $0 \leq a \leq 12$. Both F and M are Galois over $\mathbb{Q}$, and ϕ is a rational character. Each conjugate of g thus belongs to $G_F \setminus G_M$, even if h does not fix F. The place v splits in M/F, so $y \in G_M$ and all those arguments stay in the required coset.

Choose $u \in G_{F_v}$ with $\chi(u)^4 \neq 1$ by Theorem 48.5. The residual values of the four $\theta_i(u)$ are $\chi(u), 1, \chi(u), 1$. Consider the two monic polynomials

$$f_{02}(X) = (X - \theta_0(u))(X - \theta_2(u)), \qquad f_{13}(X) = (X - \theta_1(u))(X - \theta_3(u)).$$

Their resultant is a unit because their reductions have disjoint sets of roots. The Chinese remainder theorem gives a polynomial $q \in T[X]$, of degree at most three after division by $f_{02} f_{13}$, congruent to one modulo f_{02} and to zero modulo f_{13}. In the preceding sum replace y successively by $u^j y$, and combine the results with the coefficients of q. Since diagonal characters are multiplicative this keeps precisely the terms with indices $0, 2$. Using $1 - q$ keeps the terms with indices $1, 3$. Thus in C one has

$$z_0(g)\theta_0(y) + z_2(g)\theta_2(y) = 0, \qquad z_1(g)\theta_1(y) + z_3(g)\theta_3(y) = 0.$$

At $y = 1$ the two coefficients in each pair are negatives of one another. They are units. Substitution yields the two claimed character identities.

By Theorem 48.5 the local ramification index of $F/\mathbb{Q}$ divides two. The rational wild inertia acts trivially on this local extension, and its cyclotomic image contains $1 + \ell$. Choose s in it with exactly this cyclotomic value. At s the rational graded characters are the powers of $1 + \ell$ with exponents $r + 1, r, 1, 0$, respectively. The first character identity shows that $(1 + \ell)^r - 1$ belongs to J_ϕ, giving the inclusion $\ell^{1+v_\ell(r)}O \subseteq J_\phi \cap O$. Conversely split evaluation fixes every scalar and sends J_ϕ into this same ideal by Proposition 49.7. This proves equality. □

Corollary 49.14 (The length supported by this exponent on the original fields)**.** *At x on the common interval write $t = t(x)$, and use the common upper bounds n_*, N_* for the degree and conductor in Theorem 15.17. For each prime and weight in the even-order triples take the sum of any collection of images in $U_{\ell,r}$ of modules over T/J_ϕ, via $\mathbb{Z}_\ell$-linear maps. Different characters at the same (ℓ, r) may all occur in the collection. Write $B_{\ell,r}$ for this sum. Then on the actual period fields E_x,*

$$\sum_{\ell>2}(\log \ell) \sum_{2\le r<t} \operatorname{length}_{\mathbb{Z}_\ell} B_{\ell,r} \ll_\psi n_* \log(2N_*)\, t \log^2(2t).$$

For sufficiently large x every $B_{\ell,r}$ has zero image in $R_{\ell,r}$ of Section 6.6.

Proof. The theorem annihilates the entire sum at fixed (ℓ, r) by ℓ^{h_r}, where $h_r = 1 + v_\ell(r)$ is independent of the character and its coefficient field. It is contained in $M_{\ell,r}(E_x)[\ell^{h_r}]$. The Kummer bound used in Theorem 15.17 gives $\dim_{\mathbf{F}_\ell} M_{\ell,r}(E_x)[\ell] \ll_\psi n_* \log(2N_*)$. Each cyclic summand of this finite module contributes at most h_r to its truncated length. For the constant one in h_r, each prime occurs at most $2t/(\ell-1)$ times since $\ell - 1 \mid 2r$. Summing $\log n/(n-1)$ for $2 \le n \le 2t+1$ gives $O(\log^2(2t))$. For each r the other terms contribute at most $\sum_{\ell|r} v_\ell(r) \log \ell = \log r$. These estimates give the display without a multiplicity factor for characters. Finally $\ell \le 2t + 1 \le P$ and $h_r \le 1 + \log_5 t < b_0$ for all sufficiently large x. The defining quotient by $U_{\ell,r}[\ell^{b_0}]$ kills every stated image. □

The corollary places every image from this quadratic trace quotient within the existing cohomology allowance on the same interval. The remaining strict classes would require an argument controlling the other trace components or a construction placing those classes in the stated images.

49.6 Unordered symmetric-cube spectra on the period fields

The equations of Theorem 49.5 use the order of the graded lines. A condition on characteristic polynomials alone also gives a uniform integral bound. It applies to the rational fixed-weight trace ring of each triple of Section 47.3 after restriction to the original period field E_x.

Lemma 49.15 (An integral equation in the characteristic coefficients)**.** *For four roots define*

$$\mathcal{P}(X_1, X_2, X_3, X_4) = \prod_{j=1}^{4} \prod_{\substack{1\le i<k\le 4\\ i,k\ne j}} (X_i X_k - X_j^2).$$

It is an integer polynomial in their four elementary symmetric functions. Its value on the characteristic coefficients of any scalar multiple of a symmetric cube of a two-dimensional matrix is zero over every commutative ring.

Proof. Permutations preserve both the distinguished index j and the unordered pair of endpoints as types of indices, hence permute the twelve factors. The fundamental theorem on symmetric polynomials over $\mathbb{Z}$ gives the first assertion. For a generic two-by-two matrix with independent indeterminate entries, work in a splitting field of its characteristic polynomial over their rational function field. If the two roots are a, b, the four roots after the indicated operations are za^3, za^2b, zab^2, zb^3, with z also indeterminate. The product of the first and third is the square of the second. Thus the value is zero in the function field. The value in terms of characteristic coefficients was a polynomial over $\mathbb{Z}$ in the entries and z, so it was zero in the polynomial ring itself. Specialization proves the identity in all characteristics and in rings with nilpotents. □

For $g \in G_{E_x}$ evaluate $\mathcal{P}$ on the coefficients of the universal characteristic polynomial of g, using the elementary symmetric functions with their usual alternating signs in that polynomial. Denote the value by $p(g)$, and let $J_{\mathrm{sc}}(E_x) = (p(g) : g \in G_{E_x}) \subset T$. These coefficients belong to T by the Newton identities. In particular any specialization whose restricted characteristic polynomials are those of a scalar-twisted symmetric cube kills this ideal, regardless of any identification of flags. The ideal also makes sense for four-dimensional determinants specified only by their characteristic polynomials.

Theorem 49.16 (An integral obstruction from one inertia element)**.** *For $r > 2$ put*

$$D(r) = (r-2)(r-1)(r+1)(r+2)(2r-1)(2r+1), \qquad b_{\ell,r} = 12 + 2v_\ell(D(r)).$$

For $r = 2$ the only possible coefficient prime is five, and put $b_{5,2} = 6$. At every $v \mid \ell$ of E_x there is a single $s \in I_{E_{x,v}}$ for which $p(s)$ is a scalar generating $\ell^{b_{\ell,r}}O$. Thus the quotient $T/J_{\mathrm{sc}}(E_x)$ is annihilated by that power, including at every nonreduced specialization. If $r > 2$, the same assertion holds on restricting further to any auxiliary field whose local ramification index over $E_{x,v}$ is prime to ℓ.

Proof. The rational local ordinary flag gives the four prescribed characters on inertia, hence determines the whole characteristic polynomial there even when the representation is a nonsplit extension. The period field is unramified at ℓ, and both α, ξ are unramified there. Put $u = 1+\ell$. For $r > 2$ choose wild inertia with cyclotomic value u. To see this also after the indicated local extension, observe that the image in the pro-ℓ cyclotomic factor of an inertia subgroup of index prime to ℓ is the whole factor. Its wild inertia still surjects since the tame quotient has order prime to ℓ in every finite quotient. At this element the four roots are u^e for

$$e \in \{0, 1, r, r+1\}.$$

For a factor with middle exponent e_j the valuation is $1 + v_\ell(|e_i + e_k - 2e_j|)$, by the binomial valuation in Theorem 49.5. For middle exponent zero the three differences are $r+1, r+2, 2r+1$. For middle exponent one they are $r-2, r-1, 2r-1$. Reflection $e \mapsto r+1-e$ gives their negatives for the other two middle exponents. All are nonzero, so adding the twelve valuations gives the stated integer.

If $r = 2$, then $\ell \geq 5$ and $\ell - 1 \mid 4$ force $\ell = 5$. The full cyclotomic image on inertia of E_x is $\mathbb{Z}_5^\times$. Choose s with $\langle \kappa_5(s) \rangle = 6$ and $\omega_5(s)$ of order four. The four values, in increasing order of the exponents of $u = 6$, are

$$1, \quad u, \quad -u^2, \quad -u^3.$$

For endpoints of opposite signs the product differs from the middle square already modulo five. For the two positive endpoints the absolute exponent differences with the two possible middle terms are $3, 5$, giving valuations $1, 2$. For the two negative endpoints the differences are $5, 3$,

giving $2, 1$. The sum is six. This computation uses the tame part of rational inertia on the unramified period field. □

Corollary 49.17 (Uniform comparison with the strict length cutoff)**.** *Take the spans of images in $U_{\ell,r}$ on E_x of any collection of $\mathbb{Z}_\ell$-linear maps from modules over the quotients $T/J_{\mathrm{sc}}(E_x)$ on the stated triples, allowing both orders of α. Denote the span at a given (ℓ, r) by $B'_{\ell,r}$. Uniformly on the common intervals,*

$$\sum_{\ell>2}(\log \ell) \sum_{2\le r<t} \operatorname{length}_{\mathbb{Z}_\ell} B'_{\ell,r} \ll_\psi n_* \log(2N_*)\, t \log^2(2t).$$

These spans map to zero in $R_{\ell,r}$ for all sufficiently large x.

Proof. By the theorem the span is contained in $M_{\ell,r}(E_x)[\ell^{b_{\ell,r}}]$. Use the ℓ-torsion rank estimate of Corollary 49.14. The constant twelve in $b_{\ell,r}$ contributes $O(t\log^2(2t))$ to the sum of exponents weighted by $\log \ell$, since $\ell - 1 \mid 2r$. For $r > 2$ prime factorization gives $\sum_{\ell \mid D(r)} v_\ell(D(r)) \log \ell = \log D(r)$, which is $O(\log(2r))$ as this is a fixed polynomial of degree six. The exceptional pair contributes at most $6 \log 5$. This proves the display. Also $b_{\ell,r} = O(\log(2t))$ uniformly, so $b_{\ell,r} < b_0$ and $\ell \le 2t + 1 \le P$ eventually. The quotient defining $R_{\ell,r}$ kills the entire span. □

In particular a characteristic-polynomial congruence with a cube cannot account for higher precision on these fields. The comparison with all of $U_{\ell,r}$ still requires control beyond the trace quotients treated here.

49.7 Polarized ordinary deformations at the two cuspidal points

Use the even-order and odd-order initial points of Section 48.9. Write $L = FK$, $d = [F : \mathbb{Q}]$, and $U = b^{-1}R$ with multiplier $\nu = b^{-2}\mu$. The coefficient field E in this section is a finite extension of $\mathbb{Q}_\ell$. Write J for the fixed alternating form. All places of F above ℓ split in L. Choose one place w above each such v, so that $G_{L_w} = G_{F_v}$. On this group use the full ordinary flag of U.

We describe polarized deformations with fixed polarizing character. Put $\iota(A) = J^{-1}(A^{-1})^{\mathsf{T}}J$ on GL_4, let $e(g)$ be zero on G_L and one on its other coset, and define an action of G_F by

$$g * A = U(g)\, \iota^{\,e(g)}(A)\, U(g)^{-1}.$$

Indeed $\iota^2 = 1$, and $\iota(U(g))$ is a scalar multiple of $U(g)$, so the two operations commute. For a local Artin E-algebra A with residue field E, write a deformation as a continuous cocycle z for this action with values in $1 + \mathrm{Mat}_4(\mathfrak{m}_A)$. Thus $z(gh) = z(g)(g * z(h))$. We identify cocycles by strict changes of basis $z(g) \mapsto Pz(g)(g * P^{-1})$, with P reducing to the identity. On G_L the deformed representation is $z(g)U(g)$. This is the usual fixed polarization condition in the group with outer automorphism given by inverse transpose. At a real conjugation c, the pairing for $U|_{G_L}$ is the symmetric form $JU(c)$ of Proposition 48.25.

Fix a finite set of places containing those above ℓ, the archimedean places, and the ramification of $U, L/F$. Require unramifiedness outside the set. At finite places in the set away from ℓ, require z to be a strict coboundary on inertia. At each chosen $w \mid \ell$ include an invariant full flag of direct summands lifting the indicated one, and transport flags under changes of basis. Let $\epsilon_{v,i}$, $0 \le i \le 3$, be the ratio of the deformed graded character to its original value on local units, read by local reciprocity.

Theorem 49.18 (The polarized weight space and the symplectic equations)**.** *The ratios $\epsilon_{v,i}$ identify the polarized ordinary deformation functor with all tuples of continuous characters of*

local units into $1+\mathfrak{m}_A$*, independently in the four positions and at every* $v \mid \ell$*. Its completed ring over* E *is a power series ring in* $4d$ *variables. The deformations obtained by restricting fixed-multiplier symplectic deformations of* U *over* F *form the smooth closed subspace of codimension* $2d$ *cut out by*

$$\epsilon_{v,0}\epsilon_{v,3} = 1, \qquad \epsilon_{v,1}\epsilon_{v,2} = 1 \qquad (v \mid \ell)$$

on all the local units. On this subspace the symplectic deformation over F *is unique up to strict symplectic conjugacy.*

Proof. Let V be $\mathrm{Mat}_4(E)$ with the derivative of the displayed action. On G_L this is conjugation. At a real conjugation c its action is $X \mapsto -(JU(c))^{-1}X^{\mathsf{T}}(JU(c))$, since $U(c)^2 = 1$. We first prove

$$H^1_f(F, V) = H^1_f(F, V^\vee(1)) = 0.$$

The first group is the full polarized adjoint Selmer group. Over L the representation U is attached to a regular algebraic polarized cuspidal representation and remains absolutely irreducible over $L(\mu_{\ell^\infty})$, by Theorems 48.18 and 48.23. The identification of its polarization is given in Proposition 48.25. Thorne's adjoint vanishing theorem therefore applies directly to V. J. A. Thorne, *On the vanishing of adjoint Bloch–Kato Selmer groups of irreducible automorphic Galois representations*, Theorem 6.1.

For the dual group use the global Selmer dimension formula. Only scalars commute with G_L, and a real conjugation acts by negation on scalars. Thus $H^0(F, V) = 0$. The trace pairing is invariant, also under negative transpose, so $V^\vee \simeq V$. An invariant in $V(1)$ would restrict to a nonzero intertwiner from U to $U(1)$ on G_L. Absolute irreducibility and determinants would give $\kappa_\ell^4 = 1$ there, which is impossible. The finite local condition at $v \nmid \ell$ has dimension $h^0(F_v, V)$ by the unramified cohomology formula. Above ℓ, V is crystalline and has six positive weights per embedding, six negative ones and four zero weights. The difference $h^1_f - h^0$ is $6[F_v : \mathbb{Q}_\ell]$ by the crystalline dimension formula. The fixed Lie algebra at each real place is skew-adjoint for a nondegenerate symmetric form of rank four, of dimension six. The total difference of the two Selmer dimensions is consequently $6d - 6d = 0$. [10, 15]

At $v \mid \ell$ let $\mathfrak{b}$ be the flag-preserving subalgebra, $\mathfrak{n}$ its strictly upper triangular ideal, and $\mathfrak{t} = \mathfrak{b}/\mathfrak{n} = E^4$ the trivial diagonal quotient. The six graded characters in $\mathfrak{n}$ have weights at least two at every embedding. Those in $V/\mathfrak{b}$ are their negatives. Hence $H^0(F_v, V/\mathfrak{b}) = 0$, and local duality gives $H^2(F_v, \mathfrak{n}) = H^2(F_v, \mathfrak{b}) = 0$, since the respective twisted duals have no weight-zero constituents. The ordinary tangent classes inject into $H^1(F_v, V)$ from $H^1(F_v, \mathfrak{b})$, and they surject onto the four additive characters of local units. The latter assertion uses the exact sequence with quotient $\mathfrak{t}$ and extension of an additive unit character to $F_v^\times$ by choosing its value on a uniformizer.

The kernel has dimension $h^0(F_v, \mathfrak{b}) + (10 - 4)[F_v : \mathbb{Q}_\ell]$ by the local Euler characteristic formula. It contains $H^1_f(F_v, \mathfrak{b})$, since the finite condition on each diagonal is unramified. The crystalline dimension formula gives this same dimension for the finite subspace, because $\mathfrak{b}$ has six positive weights. As $h^0(F_v, \mathfrak{b}) = h^0(F_v, V)$, its image is all of $H^1_f(F_v, V)$ by that formula again. Thus the ordinary tangent classes at fixed inertial weight are exactly the finite local condition. For a small extension with square-zero kernel annihilated by the maximal ideal, lift the four diagonal characters using the prescribed units and arbitrary lifts at uniformizers. In a basis of the lifted flag, the multiplication defect of upper triangular lifts lies in a two-cocycle with values in $\mathfrak{n}$ tensored with the kernel. The vanishing of its second cohomology supplies a correction. The choices of cochains can be made continuous in unipotent coordinates. The vanishing of $H^0(F_v, V/\mathfrak{b})$ gives uniqueness of the lifted flag for a fixed representation.

Away from ℓ, undo a strict conjugation to make the inertial cocycle trivial. The permissible Frobenius correction is then fixed by the inertia action. The original inertia has finite image in the action on the Lie algebra. In characteristic zero the finite logarithm identifies invariant strict matrices with invariant nilpotent Lie algebra elements, which lift by averaging over that finite image. A Frobenius correction and the original inertia define a continuous local lift. To see continuity, represent the semidirect product with the outer involution linearly, using the defining module and its dual. The original image is compact, and the new Frobenius differs from the old one by a strict matrix. Its eigenvalues on the successive powers of $\mathfrak{m}_A$ are the original ones, hence units. Its integral powers have compact closure by the binomial formula on Jordan blocks. The same argument on inertia and the unramified quotient extends the lift from the Weil group to the decomposition group. The tangent condition is unramified cohomology. At the real places positive-degree cohomology vanishes by averaging over order two.

For completeness, the obstruction to a global lift lies in $H^2(G_{F,S}, V)$ tensored with the small kernel. It is the multiplication defect of any continuous lifts of the cocycle. The local lifts just constructed show it is locally zero. Poitou–Tate duality identifies the kernel of localization in degree two with the dual of the everywhere locally zero group in degree one for $V^\vee(1)$, which lies in the vanishing dual finite Selmer group. A global lift therefore exists. Its discrepancies from the chosen local conditions are measured modulo the finite conditions in degree one. The localization map onto the sum of those quotients is surjective by the same duality sequence and the dual Selmer vanishing. Correcting the lift by a global class imposes all the conditions with the prescribed weights. Two corrected lifts have difference in $H^1_f(F, V)$ tensored with the kernel and hence are strictly equivalent. Induction over small extensions proves the bijection. Logarithms of the four character ratios give four free parameters per $\mathbb{Z}_\ell$-generator of the local units modulo torsion and their prime-to-ℓ part. Their total number is $4d$.

For a symplectic deformation put $z(g) = U_A(g)U(g)^{-1}$. It takes values in the symplectic group, on which ι acts trivially, hence gives precisely a polarized cocycle of the stated kind. Self-duality of its flags gives the two displayed unit-character equations. Conversely, given a polarized deformation satisfying these equations, use its first two unit characters at every place. Twisting by the fixed b, Theorem 48.27 gives a unique symplectic deformation with these two characters, and twisting back gives the required multiplier ν. Its four polarized unit characters agree with the four given ones. The bijection already proved identifies the two polarized deformations. In logarithmic coordinates the pairwise-sum equations eliminate two distinct entries at each local generator, proving the codimension statement. □

Corollary 49.19 (Local parameters from rational primes split on the interval)**.** *Choose the minimal odd-order fields $H_1 = HC_{w,\ell}$ in Theorem 48.18. For each initial point there are $4d$ distinct good rational primes, all split in every E_x on its original interval, and choices of primes of L above them, such that their Frobenius traces minus their values at the point form formal coordinates for the polarized ring. Among these one may select $2d$ giving coordinates on the symplectic subspace. The rational primes may also be required to split in every auxiliary starting field of the interval and to avoid any fixed finite set.*

Proof. Adjoin to the abelian field $\mathcal{F}$ of Proposition 48.7 the fields $C_{w,\ell}$ for the finitely many odd-order coefficient primes on the interval. The resulting finite Galois field $P/\mathbb{Q}$ contains all the indicated L and E_x. For any chosen $w \mid \ell$ the image of $G_P \cap I_{L_w}$ in the local unit group under reciprocity has finite index. This follows by taking the image of a finite-index closed subgroup of inertia. On a sufficiently small open subgroup of these units the original four characters are distinct powers of the cyclotomic character, since their algebraic weight gaps are positive. Choose $a \in G_P \cap I_{L_w}$ with four distinct original values. Such an element exists since

the cyclotomic image of any finite-index inertia subgroup in G_{L_w} contains an open subgroup of principal units.

Let $\mathcal{T}$ be the completed polarized ring, and center each trace at its constant value. Modulo the square of its maximal ideal the eigenvalues a_i of the universal matrix at a are recovered from its first four power traces. Indeed Newton's identities give the characteristic polynomial and its four distinct residual roots lift uniquely. For $u \in G_P \cap I_{L_w}$ the four traces at $a^j u$, $0 \le j \le 3$, recover all four graded values at u by inversion of the matrix $(a_i^j)_{j,i}$, since the representation is triangular in its flag. Hence the linear parts of the traces on G_P span those of the four unit ratios on this open subgroup. Characters into an E-vector space are determined on such a subgroup, which spans the same local unit space over E. Applying this at every $v \mid \ell$ and using the theorem proves that the trace linear parts span the whole $4d$-dimensional cotangent space.

Every $g \in G_P$ can be approximated by Frobenius elements of primes of L whose underlying rational primes split completely in P. In fact in each finite Galois quotient over $\mathbb{Q}$ refining P, Chebotarev gives a good rational prime with Frobenius conjugate to the image of g. Choosing a prime above it realizes that image itself. It restricts trivially to P, so the residue degree in L is one. Trace and its linear part are continuous in the finite-dimensional coefficient quotient. An E-subspace of the cotangent space is closed, so the good split prime traces also span it. Excluding finitely many primes does not alter the argument. Successively choose distinct primes giving a basis. In a power series ring a tuple with invertible linear part is a system of coordinates, by solving degree by degree for its formal inverse. Its images span the cotangent space of the symplectic quotient, where the same reasoning applies to any basis selected from them. □

These coordinates describe complete local rings over E at the two automorphic starting points. To reach the strict groups in Section 6.6, support must also be propagated to the relevant integral components at the prescribed rational weights.

50 Hecke fibers and local Selmer conditions

50.1 Common Frobenius primes and ramification at finite precision

Use the interval and triples of Section 47.3. Let $\mathcal{L}$ be the finite set of their coefficient primes. For the odd-order case take $F = HC_{w,\ell}$ and in the even-order case take $F = H$, as in Section 49.7. Write $U = b^{-1}R$ for the cuspidal point over F, $L = FK$, and O for a coefficient valuation ring, with uniformizer ϖ and fraction field E. Use the stable lattice with the perfect alternating form of the construction. Let T be its endomorphism lattice with the polarized adjoint action over F from Section 49.7. In particular T has rank sixteen. The statements about its local cohomology below hold also for the fixed-multiplier symplectic Lie lattice.

Proposition 50.1 (Ramified local classes to every precision)**.** *At every finite $v \nmid \ell$ of F, let q_v be the residue cardinality, I_v inertia and Φ_v a lift of arithmetic Frobenius. The operator $q_v - \Phi_v$ on T^{I_v} is invertible over E. Write ϖ^{a_i} for its elementary divisors over O, up to units. For every $n \ge 1$,*

$$\operatorname{length}_O \frac{H^1(F_v, T/\varpi^n T)}{H^1_{\mathrm{unr}}(F_v, T/\varpi^n T)} = \sum_i \min(n, a_i).$$

At either cuspidal point, allowing unrestricted lifts away from ℓ at any fixed finite set of places gives exactly the same ordinary deformation functor on local Artin E-algebras as in Theorem 49.18.

Proof. The inertia image on T has order prime to ℓ. Indeed L/F is unramified at finite places. In the odd-order construction $C_{w,\ell}$ kills the auxiliary character θ, and inertia on the underlying plane away from ℓ has characters $1, \xi$ and zero monodromy. In the even-order construction the inducing extension FD_0/F is unramified, and inertia on each plane has characters $1, \xi$ restricted to that inertia, with zero monodromy. These are the conductor calculations in Theorems 48.18 and 48.23. The order of ξ is prime to ℓ in both cases, and a scalar twist has no effect on conjugation. Thus averaging over the inertia image identifies $(T/\varpi^n T)^{I_v}$ with $T^{I_v}/\varpi^n T^{I_v}$.

Put $A = T/\varpi^n T$ and let I_0 be the kernel of the action of I_v on T. Restriction identifies $H^1(I_v, A)$ with $\mathrm{Hom}_{\mathrm{cts}}(I_0, A)^{I_v/I_0}$ by inflation–restriction and averaging. Wild inertia has residue characteristic different from ℓ. Its intersection with I_0 has no homomorphism to A, and the image of I_0 in tame inertia is an open subgroup of prime-to-ℓ index. Projection to the ℓ-factor of tame inertia consequently identifies its maximal pro-ℓ quotient with $\mathbb{Z}_\ell(1)$. Inertia conjugation on that quotient is trivial and arithmetic Frobenius acts by q_v. The unramified Galois group has ℓ-cohomological dimension one. Inflation–restriction now gives

$$H^1(F_v, A)/H^1_{\mathrm{unr}}(F_v, A) \simeq \ker\bigl(\Phi_v - q_v : A^{I_v} \longrightarrow A^{I_v}\bigr),$$

after a choice of generator of the tame ℓ-factor.

To see that the elementary divisors are finite, restrict first to a place of L over v, where the action on endomorphisms is conjugation. The regular algebraic polarized cuspidal representation giving $U|_{G_L}$ has pure local Weil–Deligne parameter by Barnet-Lamb, Gee, Geraghty and Taylor, Theorem 2.1.1(2). Its nilpotent monodromy is zero by the finite inertia just proved. Thus all Frobenius eigenvalues have the same complex absolute value. Their ratios have absolute value one. As L/F is unramified, this applies to the action of a power of Φ_v of exponent one or two. It follows also for the eigenvalues of Φ_v on T^{I_v}. Since $q_v > 1$, the operator in the proposition has nonzero determinant. Reduction of its Smith normal form modulo ϖ^n gives the length formula. The same argument applies on the stable symplectic Lie subspace. [2]

Over E the same inertia calculation gives $H^1(F_v, T \otimes E) = H^1_{\mathrm{unr}}(F_v, T \otimes E)$. Indeed continuous homomorphisms from compact inertia have bounded image, and after scaling an integral lattice the same tame projection computes them. For the assertion about lifts use induction along small extensions of Artin E-algebras. Once the representation over the quotient has fixed inertia up to strict conjugacy, it has a fixed-inertia lift locally, as shown in Theorem 49.18. The difference from any given unrestricted local lift is a class in $H^1(F_v, T \otimes E)$ tensored with the small kernel. Its restriction to inertia is a coboundary and can be undone by a strict change of basis. This proves the assertion at each place and hence for the global functor with those places allowed. □

Theorem 50.2 (Split primes preserving strict cohomology and detecting weights)**.** *There is one reduced rational residue class on the interval with the following properties. Every prime s in it splits completely in every E_x, in every L just specified and in D_0 when it is used, and satisfies $s \equiv 4 \bmod \ell$ for every $\ell \in \mathcal{L}$. Finitely many primes can be omitted.*

1. *For each indicated (x, ℓ, r) and each finite set of these good primes, allowing ramification above the set leaves $\mathcal{S}_{\ell,r}(E_x)$ and its definition with coefficients $\mathbb{Z}/\ell^n(-r)$ unchanged, for every $n \geq 1$. At all these places the ramified quotient of the preceding proposition is likewise zero to every precision. Every Artin local deformation of the residual point with finite residue field of characteristic ℓ is unramified there.*

2. *At either cuspidal point over F, $4[F : \mathbb{Q}]$ distinct good primes in the class, with choices of places above them, give Frobenius trace coordinates for its full polarized ordinary ring*

over E. Among them $2[F:\mathbb{Q}]$ give coordinates on its symplectic subspace. A single finite list of primes contains such a choice for every indicated triple on the interval.

Proof. Retain $M_*, N^\natural, p_1, p, p', p_3, w$ of Proposition 48.7 and Lemma 48.16. Take D to be the least common multiple of $8N^\natural, M_*, p_1, p, p', p_3, w$, omitting auxiliary factors if their cases are absent. Every member of $\mathcal{L}$ is at least five and is prime to D by the exclusions in Section 47.3 and the progressions defining the auxiliary primes. Prescribe one modulo D and four modulo every $\ell \in \mathcal{L}$. This is a reduced class modulo $D' = D\prod_{\ell\in\mathcal{L}} \ell$. Write $M = \mathbb{Q}(\mu_{D'})$ and $\sigma \in \mathrm{Gal}(M/\mathbb{Q})$ for this class. All the constituents of the common field $\mathcal{F}$ in Proposition 48.7 are fixed by σ. The only ones requiring a check at a coefficient prime are the quadratic subfields of $\mathbb{Q}(\mu_\ell)$, and four is a square modulo ℓ. The fields $C_{w,\ell}$ are fixed since σ is one modulo w. This proves the splitting assertions. Dirichlet's theorem gives infinitely many primes in the class.

At a place of E_x over s the residue size is s. For its unramified module $\mathbb{Z}/\ell^n(-r)$, the kernel in the local calculation above is that of multiplication by $s^{-r} - s$. The congruences $\ell - 1 \mid 2r$ and $s \equiv 2^2 \bmod \ell$ give $s^{-r} - s \equiv -3 \bmod \ell$. The same multiplier acts invertibly on $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-r)$. Consequently every local class is unramified. A global cocycle is then already zero on the added inertia groups, because they act trivially on the coefficients. It factors through the group obtained by killing their closed normal closure. This proves the equality of global groups with the prescribed zero localizations at ℓ in both coefficient systems.

For the initial points the residual diagonal is trivial over $\mathcal{F}$, and b is congruent to one. Thus all eigenvalues of the residual Frobenius at s on U are one. Here Frobenius acts on the Lie space by conjugation since s splits in L. On the reduced endomorphisms, and on the symplectic Lie subspace, all its eigenvalues are one. The determinant of $s - \Phi_v$ is therefore a unit. Moreover consider any lift over an Artin local O-algebra with finite residue field. At a good s the residual inertia is trivial. If inertia first becomes nontrivial modulo consecutive powers of the maximal ideal, its first nonzero term gives a Frobenius-equivariant homomorphism from tame $\mathbb{Z}_\ell(1)$ to the residual matrices tensored with the corresponding graded piece. Wild inertia has zero image there. Such a homomorphism would give an eigenvector of residual conjugation with eigenvalue s. The distinct residue values 1 and 4 exclude it. Induction gives trivial inertia over the whole Artin ring.

It remains to obtain coordinates with the prescribed Frobenius class. Fix a lift $\gamma \in G_\mathbb{Q}$ of σ. The coset γG_M belongs to $G_L \cap G_{D_0}$ when the inducing field is used. We claim there are finitely many s_j in this coset and coefficients $c_j \in E$ with

$$\sum_j c_j U(s_j) = 1_4.$$

In the symmetric cube case the image of G_M acts absolutely irreducibly by the commutator argument of Theorem 48.18, since $M/\mathbb{Q}$ is abelian. Burnside's theorem makes its linear span the full matrix algebra. In the induced case G_M acts on the two inducing summands by absolutely irreducible inequivalent representations, by Theorem 48.23 and its proof on $G_{\mathbb{Q}^{\mathrm{ab}}}$. Its span is the product of the two full matrix algebras. Indeed the two simple quotients have distinct kernels by inequivalence, and the Chinese remainder theorem applies. This argument may first be made over an algebraic closure, since membership in a linear span descends to E. Now $U(\gamma)^{-1}$ lies in the indicated span, also in the induced case because γ fixes D_0. Multiplying a finite linear expansion of this matrix by $U(\gamma)$ proves the claim.

Use the ordinary universal representation modulo the square of the maximal ideal of its E-ring. Let t denote its trace on G_L. At a chosen $w' \mid \ell$, take $h \in I_{L_{w'}} \cap G_M$ whose four original graded values are distinct. Its four lifted eigenvalues a_i are the ordered graded values,

with rank-one spectral projectors P_i, by lifting the distinct roots. Put

$$m_k = \sum_j c_j t(s_j h^k) = \sum_{i=0}^{3} d_i a_i^k, \qquad d_i = \operatorname{tr}\left(\sum_j c_j U_A(s_j) P_i \right),$$

where U_A denotes this first-order representation on G_L. Each d_i reduces to one. If $\prod_i (X - a_i) = X^4 + \sum_{j=0}^{3} b_j X^j$, then $\sum_{j=0}^{3} m_{k+j} b_j = -m_{k+4}$ for $0 \le k \le 3$. The residual determinant of the coefficient matrix is the square of the Vandermonde determinant of the four original values, hence is nonzero. The eight moments therefore determine all b_j, and simple-root lifting determines each a_i. Their linear parts are in the span of the linear parts of the traces on γG_M, because differentiating the linear system and the simple-root equations only uses E-linear combinations at first order.

The images of $I_{L_{w'}} \cap G_M$ in local units have finite index. On an open inertia subgroup the original characters have distinct integral cyclotomic exponents. Choose a there, also in G_M, with cyclotomic value of infinite order. For every $u \in I_{L_{w'}} \cap G_M$ all but finitely many $a^k u$ have four distinct original values, since each pair of characters can coincide for at most one k. The just proved assertion applied to a and $a^k u$ thus determines the linear parts of all four unit-character ratios at u. Taking all the places, Theorem 49.18 shows that the linear parts of traces on this one coset span the cotangent space.

In any finite Galois quotient over $\mathbb{Q}$ refining M, Chebotarev realizes the image of each coset element as Frobenius at a chosen good prime above a rational prime, with any prescribed finite exclusions. Such a rational prime belongs to the residue class because $M/\mathbb{Q}$ is abelian, and has degree one in L by splitting. The trace linear parts are continuous in a finite-dimensional ℓ-adic vector space, and their span is closed. The traces at the indicated primes therefore span the same space. Choosing a basis and inverting its linear part in the formal power series ring proves the coordinate assertion. Restriction to the smooth symplectic quotient allows a subbasis as in Corollary 49.19. The finite union of choices over all triples and fields proves the last assertion. □

These primes can be included in the tame level when constructing ordinary families through the two starting points. The argument controls the cost of adding them to the strict local conditions. The sizes of the original strict groups and the signed integral period comparison in Section 6.6 still require bounds.

50.2 The ordinary Hecke algebra at each cuspidal point

Keep the fields $F, L = FK$, the coefficient ring O and the representation $U = b^{-1}R$ of Section 50.1. We use the automorphic normalization in which $r_\ell(\Pi) = U|_{G_L}$. In this normalization the polarized multiplier attached to an automorphic polarizing character η_F is $\kappa_\ell^{-3} r_\ell(\eta_F)$ in rank four. Here r_ℓ on a character denotes its ℓ-adic avatar with the corresponding reciprocity convention. We may enlarge O to contain the embeddings and character values used below. Write $d = [F : \mathbb{Q}]$.

Lemma 50.3 (A conjugate self-dual twist at the original auxiliary fields)**.** *There is an algebraic Hecke character X of L such that $\Pi' = \Pi \otimes (X \circ \det)$ is regular algebraic, cuspidal and conjugate self-dual. It is unramified at the places above ℓ and at every finite place inert in L/F. It descends to the definite unitary group in four variables over F that is quasi-split at every finite place. Its descent has an ordinary refinement at each place above ℓ. One rational prime from Theorem 50.2 suffices to make all the tame levels used for these descents sufficiently small.*

Proof. The multiplier ν of U is the restriction of a character of $G_{\mathbb{Q}}$, namely $(\det W)^3$ in the odd-order case and $(\kappa_\ell^j \xi)\phi$ in the even-order case. We use these particular extensions. Rational class field theory gives an algebraic Hecke character η of $\mathbb{Q}$ whose avatar is $\kappa_\ell^3 \nu$ on $G_{\mathbb{Q}}$. It is a finite character times an integral norm power. The pairing of Proposition 48.25 and the arithmetic Satake normalization give

$$\Pi^c \simeq \Pi^\vee \otimes ((\eta \circ N_{L/\mathbb{Q}}) \circ \det).$$

Indeed at each good split place this is the equality of the two lists of Frobenius eigenvalues, with the rank-four normalization contributing the third power in the character just chosen. The pairing gives the same identity at all good places of L by applying it to their Frobenius elements. Strong multiplicity one then gives the displayed isomorphism. This choice keeps the specified multiplier even when Π has a quadratic self-twist. [2]

Let δ_K be the character of $K/\mathbb{Q}$. Write the real component of η^{-1} as $\operatorname{sgn}(t)^e |t|^m$. Choose $u \in \{0,1\}$ so that $\delta = \eta^{-1} \delta_K^u$ has real component t^m. We construct a Hecke character λ of K with restriction δ to the rational ideles and with archimedean component z^m at a chosen embedding. At each finite prime of K take the full local units, except above the conductor of the finite part of δ, where we take principal units deep enough that their intersection with the rational units lies in its kernel. Call the resulting product V_0. On the subgroup

$$K^\times \mathbb{A}_{\mathbb{Q}}^\times K_\infty^\times V_0 \ \subset \ \mathbb{A}_K^\times$$

prescribe the trivial character on $K^\times, V_0$ and the two indicated characters on the other factors. To check consistency, a global $x \in K^\times$ appearing in a relation must generate the extension of a rational fractional ideal, by comparing the finite valuations. Such an ideal is principal over $\mathbb{Q}$, and the only units in $K = \mathbb{Q}(\sqrt{-p})$, $p \equiv 7 \bmod 8$, are $1, -1$. Thus x is rational and can be absorbed in the rational idele without changing its character value. The remaining relation holds by the choice of V_0 and by equality of the two archimedean characters on $\mathbb{R}^\times$. The subgroup is open of finite index by ray class finiteness. Extend its character by adjoining coset generators and taking a root of the prescribed value of each first power lying in the preceding subgroup. The result is continuous and has the given algebraic infinity type. Its finite-idele values lie in a number field. Indeed on the initial subgroup these values use rational norm powers, finite character values and powers of elements of K, and the extension uses finitely many further roots.

Now $X = \lambda \circ N_{L/K}$ satisfies

$$X X^c = \delta \circ N_{L/\mathbb{Q}} = \eta^{-1} \circ N_{L/\mathbb{Q}}.$$

The first equality follows from $\lambda(y)\lambda(\bar{y}) = \delta(N_{K/\mathbb{Q}} y)$; the second follows since δ_K is trivial on norms. This gives conjugate self-duality. Twisting preserves cuspidality and the differences between the four weights. The possible finite ramification of X lies over the conductors of ξ, θ, ϕ when these occur and over the discriminant prime p. The former primes all split in K by Lemmas 48.1, 48.4 and 48.16. At p, $L = FK'$ and p splits in K'. All these primes are away from ℓ. The original Π is spherical above ℓ: the plane used for the symmetric cube has prime-to-ℓ level, and the two local inducing planes at the split places of D_0 in the even-order case also have prime-to-ℓ level. The respective unramified Satake transfers and base changes are spherical. Away from ℓ the conductor calculations of Proposition 50.1 and local–global compatibility give spherical components whenever the underlying rational prime is outside those tame levels. This proves the asserted unramifiedness.

The extension L/F is unramified at every finite place and all places above ℓ split, by Theorem 48.2 and Lemma 48.4, also after adjoining $C_{w,\ell}$ to the real base. The parity condition

for the definite unitary group quasi-split at finite places holds since $4d$ is divisible by four. Apply Labesse's descent in Geraghty's Proposition 2.2.7. It gives the indicated general linear components at split places and hyperspecial fixed vectors at the inert places. The crystalline ordinary flag of U stays ordinary after twisting by the avatar of X; this avatar is crystalline at these places since its smooth character is unramified there. The ordinary refinement follows from the level-prime-to-ℓ ordinarity implication in Barnet-Lamb, Gee, Geraghty and Taylor, Section 2.1, remark (7) after Theorem 2.1.1. At each of the ramified split tame places choose a compact open subgroup admitting the given local representation. Choose one further good odd rational prime in the progression of Theorem 50.2. At one place above it use a principal congruence subgroup inside the domain of the matrix logarithm. It is torsion-free, so any finite rational stabilizer of a definite double coset is trivial. This gives the required small levels for all the fields. [16, 2] □

At this level use the ordinary Hecke action on the space of definite unitary forms, with all four diagonal weights variable at the chosen place of L above each $v \mid \ell$. Fix the finite residue-unit diamond character of the starting point. Write Λ for its weight algebra over O. It is the completed group ring of the product of four principal-unit groups at every such v. The local ramification index over $\mathbb{Q}_\ell$ is at most two, since $C_{w,\ell}$ is unramified at ℓ. Thus the principal units have no ℓ-power torsion, as $\ell \geq 5$. The ring Λ is a power series ring in $4d$ variables over O.

Let $\mathbb{T}$ be the Λ-algebra acting through the spherical Hecke operators at the good split places, including the inverse of the last spherical operator. We use the coefficients of the good Frobenius polynomials after undoing X. They give the same algebra by the unramified Satake formula, since the residue cardinalities and the values of the fixed Galois character are units at these places. Let z be the characteristic-zero eigensystem furnished by the lemma, with weight a. Enlarge E so that both are E-rational, and write

$$A = \widehat{\Lambda[1/\ell]}_a, \qquad C = \widehat{\mathbb{T}[1/\ell]}_z.$$

Theorem 50.4 (The completed ordinary Hecke algebra at both starting points)**.** *The maps from the completed weight algebra to C and to the polarized ordinary deformation ring of U identify all three rings. Under these identifications the good split Hecke polynomials, with X removed, are the Frobenius polynomials of the universal deformation. The symplectic locus is cut out by the two unit-character equations of Theorem 49.18 at each $v \mid \ell$.*

Proof. Geraghty's ordinary control theorem, Proposition 2.5.3, Section 2.6 and Lemma 2.6.4, gives a finite free dual ordinary module N over Λ. The small tame level allows the first principal-unit subgroup in Proposition 2.5.3. Projecting onto the desired residue-unit characters is an integral idempotent since their orders are prime to ℓ. The summand is free over the local ring and nonzero at z. At dominant algebraic weights its characteristic-zero specialization is the dual of the corresponding classical ordinary space. The commuting good spherical operators act semisimply there by the classical automorphic decomposition. The subalgebra $\mathbb{T} \subset \mathrm{End}_\Lambda(N)$ is finite and torsion-free over Λ, by Noetherianity and freeness. [16]

Purely algebraic weights strictly dominant at every embedding and with the indicated residue-unit characters detect every element of Λ. Here are details of density in the several weight variables. Choose $4d$ linearly independent integral exponent tuples in the interior of the dominance chamber and multiply them by a common multiple of the residue-unit orders. One can choose such tuples because a nonempty open real cone contains a basis of rational vectors. Add their nonnegative integral combinations to the weight of the starting point, whose

consecutive Hodge gaps are at least two. In logarithmic coordinates the derivative of the resulting weight parameterization is invertible over E: for each F_v, the embedding matrix on a $\mathbb{Q}_\ell$-basis is invertible by separability, and the chosen exponent matrix has full rank. The parameterization therefore gives local analytic coordinates at a, by the logarithm and exponential on a small polydisc. In that polydisc each integral parameter can independently run through positive powers of ℓ tending to zero. Successive use of the one-variable identity theorem makes any analytic function vanishing on this grid zero as a germ. A weight power series injects into its completed local ring at a by localization and the Krull intersection theorem. This proves density.

For every eigensystem at these weights take its semisimple Galois representation on G_L from Geraghty's Proposition 2.7.2 and untwist by $r_\ell(X)$. Include the starting eigensystem. The proposition applies to the classical constituents of the definite group, with the degree-four good Hecke polynomial and with multiplier κ_ℓ^{-3} before untwisting. Thus after untwisting the multiplier on G_L is exactly

$$\kappa_\ell^{-3} r_\ell(XX^c)^{-1} = \kappa_\ell^{-3} r_\ell(\eta \circ N_{L/\mathbb{Q}}) = \nu|_{G_L}.$$

Evaluation embeds $\mathbb{T}$ in the product of the coefficient valuation rings of these classical systems: if all evaluations vanish, the operator acts as zero on each of the classical fibers by semisimplicity, and every matrix entry on N vanishes by weight density. The finite Λ-module $\mathbb{T}$ is compact and its image in this Hausdorff product is closed. Good Frobenius elements at places of L split over F are dense up to choice of places and conjugacy. Indeed any element of G_L is realized in a prescribed finite quotient by Chebotarev over F, using a Galois closure containing L. Taking coordinatewise traces in the compact product therefore defines a continuous $t : G_L \to \mathbb{T}$, agreeing with each classical trace everywhere. The representations are unramified outside a fixed finite set made up of the level and the places above ℓ. [16]

We reconstruct a representation over C. Choose $g_1, \dots, g_{16} \in G_L$ whose matrices at z form a basis, by absolute irreducibility and Burnside's theorem. The trace-pairing matrix $H = (t(g_i g_j))_{ij}$ has determinant nonzero at z. For $h \in G_L$ define $e(h)$ in a free module of rank sixteen over $\mathbb{T}[1/\det H]$ by assigning its coordinates in the basis $e(g_i)$ using the equations with right sides $t(hg_j)$ and coefficient matrix H. Give this module the multiplication $e(g_i)e(g_j) = e(g_i g_j)$. The product is associative, has identity $e(1)$, and $e(h)e(k) = e(hk)$. These identities hold at every classical specialization with $\det H \neq 0$, where the sixteen matrices are a basis. They hold in the localization since the product of the evaluations is faithful: after clearing denominators one further factor of $\det H$ also kills the evaluations with zero determinant.

After extending to C this algebra has special fiber $\mathrm{Mat}_4(E)$. Lift a rank-one idempotent to an idempotent e in the algebra. For example Newton iteration for $Y^2 - Y$ starting at any lift converges in the complete ring, since $2Y - 1$ reduces to an involution and the iterates commute. The left ideal cut out by right multiplication by e is a free module of rank four, as a projective module over the local ring C. Left multiplication identifies the algebra with its endomorphism algebra, since reduction gives that isomorphism and both sides are free of rank sixteen. This gives a representation ρ lifting $U|_{G_L}$, and its trace is t. For the last assertion, the trace of left multiplication on the rank-sixteen algebra is four times the prescribed trace at all the same classical points, hence in the localization. In a matrix algebra it is four times the ordinary matrix trace. The construction is continuous on every finite Artin quotient of C: the trace was continuous in the compact Hecke algebra, and evaluation of a weight variable there has constant term in the maximal ideal of O plus a nilpotent term. It gives a continuous map from Λ to that finite-dimensional E-algebra with its ℓ-adic topology, and hence from a finite Λ-module by using module generators.

Conjugate duality holds over C. The traces of ρ^{c} and $\nu\rho^{\vee}$ agree on all of G_L by their classical evaluations. Two such representations with absolutely irreducible isomorphic special fibers and equal traces are isomorphic. Indeed the trace-pairing basis construction identifies the two matrix algebras and their group elements, and a matrix algebra automorphism over a local ring is inner. The latter follows by choosing a generator for the image of the first diagonal matrix unit and using the other matrix units to carry it to a basis of the module. Thus an invertible pairing $\mathcal{J}$ satisfies $\rho(h)^{\mathsf{T}}\mathcal{J}\rho(chc) = \nu(h)\mathcal{J}$ for $h \in G_L$. Scale it to reduce to $JU(c)$. Its transpose is another such pairing, so is a scalar multiple of it since the matrices span the full algebra. This scalar squares to one and reduces to one, hence is one. A strict change of basis makes the symmetric pairing equal to $JU(c)$: the endomorphism $(JU(c))^{-1}\mathcal{J}$ has a square root congruent to the identity given by the binomial series, self-adjoint for $JU(c)$. With this basis take the polarized cocycle to equal one at c and $\rho(h)U(h)^{-1}$ at $h \in G_L$. The pairing equation verifies the cocycle relation at chc, and $c^2 = 1$ verifies the remaining relation.

At each chosen $w \mid \ell$ let χ_i be the four unit characters in the universal weight, in the order of the initial flag, including the fixed untwist. For clarity, Geraghty’s Corollary 2.7.8 gives for the j-th character before untwisting, $1 \le j \le 4$, the product of κ_{ℓ}^{1-j} and the character which on units x is $\prod_{\tau:w} \tau(x)^{-\lambda_{\tau,5-j}}$ at algebraic weight λ. The four varying factors are precisely the independent slot characters of the weight algebra in his Section 2.6, up to its fixed normalizing characters. Ratios to their values at z therefore give formal coordinates on A. The ordinary flag in the corollary applies at the regular weights used for density. In fact the residue-unit part may be fixed at its algebraic value throughout our grid, so the classical space being used is the ordinary space with Iwahori invariants and this algebraic weight. [16]

Choose h_0 on open inertia with four distinct graded values at the starting point. Put $a_i = \chi_i(h_0)$ and $p_i(Y) = \prod_{j \neq i}(Y - a_j)/(a_i - a_j)$. The denominators are invertible at z. By the classical flags, the characteristic polynomial at h_0 is $\prod_i (Y - a_i)$, and for $g \in G_{L_w}$, $h \in G_L$ the trace of

$$p_j(\rho(h_0))\,\rho(g)\,p_i(\rho(h_0))\,\rho(h)$$

vanishes when $j > i$. Indeed expanding the polynomials writes the expression in group traces with weight coefficients; the identity on the open set of classical points passes to the localization by clearing denominators as above. Nondegeneracy of the trace pairing over C then gives zero for the product before the last factor. The ordered partial sums of the four eigenlines of h_0 are consequently stable. For $u \in I_{L_w}$, interpolation of $t(h_0^k u) = \sum_i a_i^k \chi_i(u)$, $0 \le k \le 3$, and inversion of the Vandermonde matrix identify the graded characters on inertia. This proves the required ordinary condition on the entire completion, even where the four characters have coincident reductions in characteristic ℓ.

At the finite places away from ℓ the unrestricted local lifts just obtained have fixed inertial type on every Artin quotient by Proposition 50.1. The polarized ordinary universal property in Theorem 49.18, applied to these compatible Artin quotients, now gives a homomorphism from its ring to C. In the four unit-character coordinates this is exactly the weight homomorphism $A \to C$. In particular all normalized good spherical coefficients in C are images of elements of A, by taking the characteristic polynomials in the universal representation. Their last coefficients are units. Since $\mathbb{T}$ is generated over Λ by these operators and their indicated inverses, $A \to C$ is surjective. More explicitly $\mathbb{T} \otimes_{\Lambda} A$ is finite over the complete local ring A and is the product of the completed local factors above a. Projection to C is surjective, and all the algebra generators have just been accounted for. Finally $\mathbb{T}$ embeds into a finite free Λ-module. Flatness of localization and completion gives a torsion-free finite module after tensoring with A, and the nonzero factor C is torsion-free as well. Thus $A \to C$ is injective. The claimed symplectic locus is now the one in Theorem 49.18. □

Corollary 50.5 (Hecke relations on the components through the initial point)**.** *Let a Noetherian integral-coefficient family of marked ordinary symplectic representations of G_F have the specified fixed multiplier and have R at a characteristic-zero point z'. Assume continuity on the local Artin specializations at z' and ramification confined to a fixed finite set. Compute the weights and the good split traces on G_L after undoing b. Every polynomial relation over Λ in the good spherical coefficients that vanishes in $\mathbb{T}$ vanishes on each irreducible component through z'.*

Proof. Complete the local ring of the given family at z', extending the coefficient field if necessary. Forgetting the markings and applying symplectic descent as in Theorem 49.18 gives a polarized ordinary deformation at its Artin quotients. Extra tame places are allowed by Proposition 50.1. The preceding theorem identifies its trace and weight specialization with a homomorphism from C. Thus the relation is zero in the completion. Faithful flatness of completion gives zero in the local ring, so its value in the original family is annihilated by an element outside the prime of z'. Every prime contained in that prime also avoids this annihilator and hence contains the value. This applies to all the minimal primes of components passing through the point. □

This proves the trace relations on the components passing through the two characteristic-zero starting points. Propagation to the other components encountered at characteristic ℓ, the strict lengths at the original weights, and the signed period estimate on E_x in Section 6.6 are still open.

50.3 The Hecke fiber at the prescribed rational weights

We specialize the family of Section 50.2 at the weights of the original strict-cohomology problem. Fix one of the primes of K above ℓ, and in defining the unitary level choose above each $v \mid \ell$ of F the place of $L = FK$ lying over that prime. These choices are available at all v, and the original local representations over F_v identify with those at the chosen places. Write $\psi_0, \dots, \psi_3$ for the four ordered inertia characters of the original rational weight, restricted to these places. On open inertia their cyclotomic exponents in order are

$$r+1, \quad r, \quad 1, \quad 0.$$

Their outer and inner products both equal μ on inertia. The Hecke algebra $\mathbb{T}$ uses the untwisted normalization of the preceding section. Its starting point is $U = b^{-1}R$, and the target in its weight space has ordered unit characters $b^{-1}\psi_i$.

Proposition 50.6 (An integral component reaching the prescribed weight)**.** *In Λ impose the two symplectic equations on all local units from Theorem 49.18 and denote the quotient by Λ_{sp}. It is a power series ring over O in $2d$ variables. There is a unique irreducible component of $\mathrm{Spec}(\mathbb{T} \otimes_\Lambda \Lambda_{\mathrm{sp}})$ through the characteristic-zero cuspidal point z. Its reduced coordinate ring B is a finite domain containing Λ_{sp}. Its fiber over the exact weight $(b^{-1}\psi_i)_i$ is nonempty in characteristic zero and is finite over O before inverting ℓ. Throughout B the trace on G_L is invariant under conjugation by $c \in G_F$ and satisfies*

$$t(h) = \nu(h)t(h^{-1}) \qquad (h \in G_L).$$

Proof. Use a basis of the principal units modulo torsion in each slot. For each basis element, the ratios of the two outer characters to their starting values are required to be mutual inverses, and likewise for the two inner characters. These equations eliminate two of the four group-ring

variables by solving for their units $1 + T$. They hold on the whole unit group by continuity and the fixed residue-unit characters. This proves the description of Λ_{sp}.

The completion of the local ring of the indicated tensor product at z is the quotient of the ring in Theorem 50.4 by the same equations. It is the completed characteristic-zero weight ring of Λ_{sp}, a power series domain. Since the local ring injects into this completion, it is a domain. The primes of the tensor product contained in the prime of z correspond to the primes of that local ring. There is thus one minimal prime there, giving the domain B. The inclusion of Λ_{sp} in its own completed local ring at the weight of z shows that this minimal prime has zero contraction. Finiteness follows from finiteness of $\mathbb{T}$ over Λ.

The characters $b^{-1}\psi_i$ have the same reductions as the ordered characters of U on inertia by the residual flag assertions in Theorems 48.18 and 48.23. On prime-to-ℓ torsion, equality after reduction implies equality of the original values. Their two paired products are $b^{-2}\mu = \nu$. Thus the exact tuple defines an O-valued weight of Λ_{sp}. Tensoring B at this weight gives a finite O-algebra. Lying over for the integral injection $\Lambda_{\mathrm{sp}} \subset B$ gives a prime above the kernel of the weight. Its characteristic-zero residue field is finite over E since the fiber is finite. Finally B embeds in its localization at z and in the completion of that localization. On the latter the universal representation on G_L comes from a symplectic one over F with multiplier ν, by Theorems 49.18 and 50.4. The two trace identities there descend along the injection. □

The scalar b can have half-integral cyclotomic exponent. Consequently classical control at the indicated point requires a choice of scalar on the two sets of places of L above ℓ. We make this choice on L itself.

Lemma 50.7 (An anticyclotomic translation of the ordinary Hecke algebra)**.** *There is a continuous ℓ-adic character $\beta : G_L \to O^\times$, unramified away from ℓ, with $\beta^c = \beta^{-1}$ and trivial reduction, with the following properties.*

1. *Its restriction to inertia at the chosen places above ℓ is b. The character $b|_{G_L}/\beta$ is the avatar of an algebraic Hecke character of L.*
2. *Multiplication of the four ordered unit characters by β, and of each good degree-j Frobenius coefficient by $\beta(\mathrm{Frob})^j$, acts by an automorphism of the ordinary Hecke algebra $\mathbb{T}$.*

Proof. The character construction of Lemma 50.3 also gives an algebraic character of K with complex component $z/\bar{z}$, trivial on rational ideles and on all finite local units. Indeed its restriction at infinity to $\mathbb{R}^\times$ is one, and the same ideal argument verifies the relations with $K^\times$ when every finite unit group is taken. Use this character or its inverse and pull it back by $N_{L/K}$. Denote the resulting Hecke character by Y and its avatar by Ξ, with the sign chosen so that Ξ equals κ_ℓ on inertia at the chosen places. To see the restriction, at the two split places of K the algebraic unit factors of its avatar are the two opposite degree-one powers of local reciprocity. Pullback by the local norm gives the rational cyclotomic character or its inverse on the units of L_w. The smooth finite component is unramified. The identity $YY^c = 1$ follows from its trivial rational restriction.

Write $m = 3j$ for the symmetric cube and $m = j$ for the induced representation, so ν has cyclotomic exponent m. Put $a = r + 1 - m$ and $s = a/2 \in \mathbb{Z}_\ell$. For a unit character γ, write $\langle\gamma\rangle$ for its quotient by the Teichmüller lift of its reduction. On principal units at every $v \mid \ell$ the defining equation $b^2 = \mu/\nu$ gives $b^2 = \kappa_\ell^a$, since all finite factors there are unramified or Teichmüller characters. On residue-unit torsion b is trivial since it reduces to one. Taking the unique square root on principal units therefore gives $b = \langle\kappa_\ell\rangle^s$ on all of inertia in the local abelianization. Set

$$\beta = \langle\Xi\rangle^s.$$

The power exists continuously in the abelian pro-ℓ group of principal coefficient units, by taking limits of integer powers. This argument also applies in the presence of ℓ-power torsion. The choices give every local and conjugation assertion about β.

At the chosen places b/β is trivial on inertia. At the conjugate places it has cyclotomic exponent $2s = a$ on principal units. To verify algebraicity explicitly, choose an algebraic character of K whose avatar has exponent zero at the chosen place and a at its conjugate. The same idele construction supplies it: take the integral one-sided infinity type with these exponents, and on the rational ideles take a norm power with the necessary real sign, adjusting the sign by δ_K. Ramification at the discriminant prime is permitted in this construction. After pulling back to L, its quotient into b/β has finite inertia at ℓ and at every other finite place. Away from ℓ this follows also from the square equation for b, since its continuous character there has finite inertial image when its square does. It is unramified outside a finite set. Thus the quotient factors through a finite ray class group of L: take open kernels on units at the finitely many ramified places and full units at the rest, and use global class field theory. It has finite order. This proves the assertion about b/β.

For the automorphism first use Y^k with k an integer divisible by the order of the reduction of Ξ. This character is trivial on $\mathbb{A}_F^\times$ inside $\mathbb{A}_L^\times$, since the norm of a real-base idele to K is its rational norm. Thus Y^k is pulled back from the norm-one torus by $z \mapsto z/\bar{z}$. Local Hilbert 90 gives surjectivity of this map on ideles: at almost all places an integral norm-one unit has a unit preimage, since the extension is unramified and one can adjust a preimage by a base uniformizer. Hilbert 90 over the number fields similarly gives triviality on rational points of the torus. The resulting automorphic character of the torus is trivial on finite maximal compact subgroups at nonsplit places by this unit-preimage argument, and on finite units at split places by unramifiedness of Y. Composing with the determinant of the unitary group therefore twists classical eigensystems at the same tame level. Its base-change character at a split place is Y^k on the chosen factor, as seen on ideles $(x, 1)$. On weights it adds one common exponent to the four slots at each embedding. This preserves dominance and ordinarity (the four graded characters receive the same scalar); our choice of k preserves the residue-unit summand. [16, 2]

On the normalized good Frobenius coefficients and the four unit characters the induced operation is multiplication by the indicated powers of Ξ^k. This respects all relations in $\mathbb{T}$: evaluate them on regular algebraic classical eigensystems and use the faithful product of evaluations in Theorem 50.4. The inverse is given by $-k$. Choose such integers k tending to s in $\mathbb{Z}_\ell$, possible since their required divisibility is prime to ℓ. The resulting scalars $\Xi(g)^k$ converge to $\beta(g)$ for each g. The weight substitutions converge on formal power series, since they send each group generator to itself times a unit congruent to one and converge in the maximal-ideal topology. Passing to the limit in every polynomial relation among the Hecke generators over Λ is legitimate in the finite complete Λ-algebra. This gives the substitution claimed in the lemma, with inverse obtained using $-s$. □

Theorem 50.8 (Classical realizations in the fiber of the original weight)**.** *At every geometric characteristic-zero point y of $\mathbb{T}$ above $(b^{-1}\psi_i)_i$, multiplying its normalized trace by $b|_{G_L}$ gives the trace of a semisimple Galois representation arising from a classical automorphic system over L, with the original weight differences. If this representation is absolutely irreducible, it is attached to a regular algebraic cuspidal representation of $\mathrm{GL}_4(\mathbb{A}_L)$ and is pure at finite places away from ℓ. There is at least one such classical realization, allowing reducible Galois representations, on the component of Proposition 50.6. The number of distinct geometric eigensystems in this whole characteristic-zero fiber is at most the Λ-rank of the ordinary module N of Theorem 50.4.*

Proof. Translate y by the automorphism of Lemma 50.7. In the normalization with X undone,

its four unit characters are now exactly ψ_i at the chosen places. Restoring the fixed algebraic $r_\ell(X)$ shifts all four local exponents by the same integer at each embedding. The exponents have successive gaps $1, r-1, 1$, all at least one since $r \geq 2$. Subtracting the consecutive shifts in the ordinary weight formula of Geraghty's Corollary 2.7.8 thus gives a dominant algebraic coefficient weight. The remaining unit characters are finite order. In particular this is an arithmetic weight even when a in the preceding lemma is odd.

Ordinary control for a dominant weight times a finite-order diamond character gives the classical ordinary space at sufficient depth above ℓ. Every character of the good Hecke algebra over this weight occurs on this space. Indeed its finite module N is faithful over $\mathbb{T}$, so its support is all of $\operatorname{Spec}\mathbb{T}$. Localize at the character and reduce modulo its maximal ideal. Nakayama's lemma gives a nonzero fiber, also after extending scalars. The good operators are semisimple on the classical space and its dual. The dimension of the specialized dual is the constant rank of N, which bounds the number of different characters. Translation preserves that number. [16]

Let D be the semisimple Galois representation attached to the translated classical system in its unitary normalization by Geraghty's Proposition 2.7.2. At good split primes its trace is $r_\ell(X)\beta\, t_y$; Chebotarev gives the identity on G_L. Twist D by the algebraic character whose avatar is $(b/\beta)r_\ell(X)^{-1}$. The resulting representation has trace $b\, t_y$. On the automorphic side this is an algebraic twist of the weak base change of the classical definite unitary system. In the description of that base change used in Proposition 2.7.2, several cuspidal constituents give a direct sum of their Galois representations, and a discrete block of length greater than one contributes the various norm twists of its constituent. These decompositions follow on semisimplifying from the good Frobenius polynomials. Absolute irreducibility therefore forces a single cuspidal constituent of rank four with block length one. Its distinct weights give regularity. The resulting algebraic twist is still polarized over L/F, with its polarizing character multiplied by the product of the twisting character and its conjugate. Purity follows from Theorem 2.1.1(2) of Barnet-Lamb, Gee, Geraghty and Taylor. The existence statement on the specified component follows from the nonempty fiber in Proposition 50.6. [16, 2] □

In particular an absolutely irreducible restriction to G_L of a marked representation at the original weight has this cuspidal realization whenever its normalized traces factor through the indicated Hecke algebra with that weight. Establishing this factorization for every relevant component and bounding the full integral strict lengths are still necessary for the argument in Section 6.6. The integral theta comparison there concerns the periods on the original E_x.

50.4 Regular Frobenius on ordinary curves in positive characteristic

Return to the minimal fields F, L in Section 50.1. Write $D = F$ or $D = L$, $G = G_D$, $G' = G_{L(\mu_{\ell^\infty})}$. Let k be a finite field of characteristic ℓ, $A = k[[t]]$, and $K_1 = k((t))$. Consider a continuous representation $\rho : G \to \mathrm{GL}_4(A)$ unramified away from finitely many places, whose reduction has the four residual characters of the indicated case as its semisimplification. A common scalar twist of those residual characters is allowed. Suppose that over K_1 it has an ordinary flag at the chosen places above ℓ, with ordered continuous unit characters $\theta_{v,i}$ taking values in $A^\times$. For each v let I_v^0 be the subgroup of its inertia on which the full ℓ-adic cyclotomic character and the original four residual diagonal characters are trivial. These inertia groups lie over split places of L/F. Assume that for each pair $i \neq j$, the characters $\theta_{v,i}, \theta_{v,j}$ differ somewhere on I_v^0, and that their reductions agree there. These are the separation hypotheses on the curves in `Development/AutomorphicArithmetic/AutomorphicVarieties/ModularityArgument/ThresholdArguments/DenseThresholdCases/ModularDeformationQuotientRegularAtPoint`.

`lean`, line 26. For a matrix, a regular semisimple spectrum below means that its characteristic polynomial has four distinct roots over the algebraic closure.

Theorem 50.9 (Density of regular spectra on the cyclotomic kernel)**.** *If ρ is absolutely irreducible over K_1, then its restriction to G' is absolutely irreducible. In every nonempty open subset of G', as well as in every nonempty open subset of G, there is an element of regular semisimple spectrum. The local ordered characters are pairwise different and a character in one position is never the product of another with the residual cyclotomic character. At any good prime from the progression of Theorem 50.2, the local representation has no nonzero intertwiner to its cyclotomic twist, also over $\overline{K_1}$.*

Proof. On I_v^0 each quotient of two different local characters has nontrivial image in $1+tk[[t]]$. The latter group is torsion-free: it is pro-ℓ, and $(1+x)^{\ell^m} = 1+x^{\ell^m}$ in this domain. Each of the six kernels is closed of infinite index, hence has empty interior. Their union cannot cover the compact group I_v^0, by the Baire theorem. Choose a outside the union at any one of the places. Then $a \in G'$, $\rho(a)$ has distinct eigenvalues and all its positive powers have distinct eigenvalues. The asserted local character inequalities follow by the same restriction at each place, since the cyclotomic character is one on I_v^0.

The semisimplified residual characters take distinct values at one $g_0 \in G'$, and one of their ratios has order not dividing four. For $F = H$ this follows from Lemma 48.13 and Theorems 48.2 and 48.5. For $F = HC_{w,\ell}$ take the ℓ-th power of the element for H. It fixes $C_{w,\ell}$, and raising residual values to the ℓ-th power preserves distinctness and their prime-to-ℓ orders. In the odd-order case the ratio value $\chi(g_0)$ has order $2m \geq 6$ with $m > 1$ odd, and in the even-order case $\chi(g_0)^4 \neq 1$. Common scalar twists do not affect these comparisons.

Work over $\overline{K_1}$. Restriction to the normal subgroup G' is semisimple: the sum of its simple submodules is nonzero and G-invariant. In each isotypic component multiplicity is one, since multiple copies force repeated eigenvalues of a. The number of components divides four because G permutes them transitively. Its permutation quotient is abelian since $L(\mu_{\ell^\infty})/D$ is abelian, and is regular on the components: a stabilizer in a transitive abelian permutation group fixes every point. The action is continuous. Each component is a sum of eigenlines of $\rho(a)$, and its stabilizer is closed by continuity of the matrices and hence open by finite index. If there were two or four components, this quotient would admit a nontrivial character η of order two. Indexing the components by the quotient, multiplication on each by its value of η would intertwine ρ with $\eta\rho$. Reduction of their characteristic polynomials and Brauer–Nesbitt would then make η permute the four residual characters. It is trivial at g_0, where the values are distinct, so the permutation fixes each character. This forces $\eta = 1$, a contradiction. Thus G' acts irreducibly.

Let H be the Zariski closure of $\rho(G')$ and H° its identity component. The latter is normal under $\rho(G)$. The identity component of the closure of the powers of $\rho(a)$ is a torus in H°, in the eigenbasis of a. Its four weights are distinct: a quotient character trivial on that torus would have finite image on the cyclic group, contrary to our choice of a. A cocharacter of the torus can be chosen outside the finitely many hyperplanes on which two weights have equal pairing. Thus H° contains

$$D(T) = \operatorname{diag}(T^{w_1}, T^{w_2}, T^{w_3}, T^{w_4})$$

as a one-parameter subgroup, with four distinct integers w_i. Restriction to H° is semisimple by the same simple-submodule argument, using the irreducibility on H. The simple constituents are pairwise inequivalent and of equal dimension, since a power of a in H° has simple spectrum and H permutes the components transitively.

We exclude four lines for H°. Otherwise G permutes the four lines transitively, with quotient $P \subset S_4$ and kernel N_0. Over a finite extension take the direct sum of the projections of a stable lattice onto the four lines. This is a stable monomial lattice, since G permutes its summands. The four reduced characters on N_0 are the restrictions of the original residual characters as a multiset, by the characteristic polynomials and Brauer–Nesbitt. They are therefore individually invariant under G. Transitivity on the monomial lines forces them all to be equal. The reduced image of N_0 thus consists of scalars of order prime to ℓ. As $|P|$ divides 24 and $\ell \geq 5$, the entire reduced matrix group has order prime to ℓ. By Maschke's theorem its representation is a direct sum of the four characters, so its image is abelian. Its permutation action on the reduced summands still surjects onto P, making P abelian and regular, of order four. All ratios of the original residual characters are trivial on N_0 and would consequently have order dividing four. This contradicts the ratio at g_0.

It remains to prove that for each $g \in G$ the coset $\rho(g)H^\circ$ contains a matrix with simple spectrum. There are two possible decompositions under H°: one four-dimensional constituent or two planes. If there is one constituent, the matrices in H° span the full matrix algebra by Burnside's theorem. If there are two and g preserves both, the restrictions of H° on each span the full endomorphism algebra of its plane. In either case, in the chosen eigenbasis each diagonal entry is nonzero somewhere on $\rho(g)H^\circ$. The same is true for each diagonal entry of the inverse, by applying this observation to the inverse coset. Since H° is an irreducible variety, we can choose a single x in the coset with all eight entries nonzero. In particular all four diagonal cofactors of x are nonzero.

Order the four w_i increasingly. In the elementary symmetric coefficients e_j of the eigenvalues of $xD(T)$, a principal j-row minor of x is multiplied by the power of T with exponent equal to the sum of its j weights. Thus the T-adic valuations of e_1, e_3, e_4 are w_1, $w_1 + w_2 + w_3$, and $\sum_i w_i$, respectively. If the leading two-row minor is nonzero, the successive slopes of the elementary-coefficient Newton polygon (with points $(j, v_T(e_j))$) are the four distinct w_i, giving distinct roots. If that minor vanishes, $v_T(e_2) \geq w_1 + w_3$. The middle edge then joins positions one and three and has slope $(w_2 + w_3)/2$. Its residual equation is a binomial of degree two with both terms nonzero, since the middle coefficient lies strictly above the edge. Its roots are distinct in characteristic $\ell \neq 2$. More explicitly, after putting $T = u^2$, substituting the middle slope in the polynomial and dividing by its lowest power of u, the nonzero residual roots lift by Hensel's lemma. The outer edges have length one and distinct slopes. This again gives four distinct roots.

If g swaps the two planes, the coset has the block form $\left(\begin{smallmatrix} 0 & B_1 \\ B_2 & 0 \end{smallmatrix}\right)$. Using Burnside on each plane and irreducibility of the coset as a variety, choose x for which every entry of the two blocks is nonzero. Both blocks are invertible. Write D_1, D_2 for the two diagonal blocks of $D(T)$. The characteristic polynomial of $xD(T)$ is

$$Y^4 - \operatorname{tr}(B_1 D_2 B_2 D_1) Y^2 + \det(B_1 B_2) T^{\sum_i w_i}.$$

Indeed eliminating one block in $Y - xD(T)$ gives the two-by-two determinant in Y^2. The least exponent in the trace is the sum of the smaller weight in each plane, attained by a unique pair, with nonzero coefficient. Twice this exponent is strictly smaller than $\sum_i w_i$. The discriminant of the quadratic in Y^2 is therefore nonzero, its two roots are nonzero, and $\ell \neq 2$ gives four distinct roots in Y.

In all cases the discriminant is nonzero for the generic T, so it is nonzero at some element of the coset. For any open subgroup V of G', the Zariski closure of $\rho(V)$ contains H°, since finitely many translates cover H. An open neighborhood in G' or in G contains a translate of such a subgroup of G'. The closure of its image contains the corresponding coset of H°, on

which the discriminant is not identically zero. This proves both density statements. Finally the Frobenius at the auxiliary tame prime in Theorem 50.2 acts on residual endomorphisms with all eigenvalues one, while the cyclotomic factor there is four in k. Hence the determinant of the twisted Frobenius minus one is a unit in A. No local intertwiner can survive even after passing to the algebraic closure of K_1. □

Corollary 50.10 (Simultaneous Frobenius choices at all cyclotomic depths)**.** *Take any finite list of curves satisfying the theorem on the given interval, allowing different fields, uniformizers t_i, and coefficient primes ℓ_i. There exist fixed integers $b_i \geq 0$ with the following property. For every positive integer N, infinitely many good rational primes q split in all original period fields and all the auxiliary starting fields, satisfy $q \equiv 1 \bmod \ell_i^N$ for every i, and admit choices of Frobenius with characteristic-polynomial discriminant of t_i-adic valuation exactly b_i on every curve. Finite exclusions at each depth are permitted.*

Proof. Use one finite abelian field $P/\mathbb{Q}$ containing the period and starting fields as in Corollary 49.19. Let P_∞ be its compositum with all the ℓ_i-power cyclotomic towers in the list and put $H_* = G_{P_\infty}$. For curve i, H_* is normal in $G_i' = G_{L_i(\mu_{\ell_i^\infty})}$. In G_i'/H_* the ℓ_i-primary part is finite. Indeed this abelian quotient embeds by restriction into the product of a finite group and the finitely many $\mathbb{Z}_p^\times$ at the other coefficient primes p. Each such factor has finite ℓ_i-primary part, and its complementary pro-p part has no nontrivial finite ℓ_i-quotient. The same assertions hold for closed subgroups of that product by the primary decomposition of profinite abelian groups.

The compact image $\rho_i(G_i')$ has an open pro-ℓ_i subgroup, by reduction modulo t_i. Its quotient by $\rho_i(H_*)$ is therefore both virtually pro-ℓ_i and a quotient of the preceding abelian group. It is finite. Consequently the image of any open subgroup of H_* has finite index in $\rho_i(G_i')$; its closure in the algebraic group contains the identity component used in the theorem. The same coset argument shows that regular spectra for curve i form a dense open subset of H_*. Openness follows from continuity of the discriminant. Intersect these finitely many dense opens and fix g in the intersection. Take b_i to be the discriminant valuation at g. It is determined by the matrices modulo $t_i^{b_i+1}$.

For a given N adjoin to P the required ℓ_i^N-th roots of unity and the normal closures over $\mathbb{Q}$ of the finite fields cut out by these fixed matrix reductions over each base field. In this finite Galois extension Chebotarev supplies infinitely many good rational primes with Frobenius conjugate to the image of g. Choose a place above each so as to realize that image itself. These primes split in P and in the indicated cyclotomic fields, and the matrix congruences at the same place give the specified valuations on all curves. The precisions $b_i + 1$ were fixed before N was chosen. □

This gives the simple spectra and the separate tame place needed to test the irreducible ordinary curves in characteristic ℓ. Transporting the Hecke relations to all components through these curves also requires control of global cohomology in the patching argument. The discriminant valuations here concern a chosen finite list of curves.

50.5 Dual Selmer tests on irreducible ordinary curves

Use a finite list of the absolutely irreducible curves of Section 50.4. For each one restrict to G_L, write $A = k[[t]]$, $K_1 = k((t))$, $A_m = A/(t^m)$, and put $W = \mathrm{End}_A(A^4)$ with conjugation action through ρ. Set $\omega = \bar{\kappa}_\ell$ and $W_m = W/t^m W$. Work with $G_{L,S}$ for any fixed finite set containing the coefficient places, the ramification of the representation and the infinite places. If necessary enlarge S to contain the ramifications of the common finite field P of Corollary 50.10 and all

coefficient primes in the list. Inflation embeds the cohomology from the smaller set, since the coefficient action already factors through its quotient.

At an element g with $\omega(g) = 1$, define

$$e_g(y) = \operatorname{tr}\big(y(g)\rho(g)\big) \qquad \big(y \in Z^1(G_{L,S}, W_m(1))\big).$$

The notation (1) multiplies conjugation by ω. The evaluation vanishes on coboundaries, by cyclicity of trace. The same definition makes sense with K_1 coefficients. For the even-order case let ϕ cut out $D_0/\mathbb{Q}$, so that $\phi = \tau$ on G_L. Say that the curve has this quadratic self-twist if $\rho \simeq \phi\rho$ over $\overline{K_1}$ on G_L.

Proposition 50.11 (The quadratic exception to tests at completely split primes)**.** *On G_{P_∞} of Corollary 50.10, the curve representation remains absolutely irreducible in the odd-order case and in every even-order case without the indicated quadratic self-twist. In the exceptional case it splits there into two inequivalent absolutely irreducible planes. The subgroup G_{LD_0} preserves both and the other coset of G_{LD_0} in G_L interchanges them. For a cocycle valued in the two off-diagonal matrix blocks, e_g vanishes identically on G_{P_∞}.*

Proof. Write $H_* = G_{P_\infty}$. The proof of Corollary 50.10 shows that $\rho(H_*)$ has finite index in $\rho(G_{L(\mu_{\ell^\infty})})$. It therefore contains a positive power of the matrix $\rho(a)$ with simple spectrum constructed in Theorem 50.9. Restriction of the irreducible representation on G_L to H_* is semisimple, and its constituents have multiplicity one by this simple spectrum. If it is reducible, the abelian quotient G_L/H_* acts regularly and transitively on two or four constituents. A nontrivial quadratic character of this permutation group gives a self-twist of ρ by the sign change on constituents used in Theorem 50.9. This quadratic character survives reduction and must permute the four residual characters by multiplication.

In the odd-order case they are, up to a common twist, $1, \chi, \chi^2, \chi^3$, with χ of order h divisible by $2m \geq 6$, as seen at g_0. A quadratic permutation character must belong to $\langle\chi\rangle$ by looking at the first entry. Its nontrivial possibility adds $h/2$ to the exponents modulo h. This does not preserve the set $\{0, 1, 2, 3\}$: for $h = 6$ the image contains four, and for $h \geq 8$ the image of zero is outside the set. In the even-order case the four are $1, \chi, \phi, \chi\phi$, up to a common twist. Of the three nonidentity candidates only ϕ has order two since $\chi^4 \neq 1$. As ℓ is odd, the quadratic sign over K_1 is uniquely determined by its reduction. This proves the irreducibility assertions.

If there is the self-twist, scale an intertwiner to have square one, after finite coefficient extension. Its square is scalar by Schur's lemma. Its eigenspaces are interchanged by each element with $\phi = -1$, hence are two planes, and G_{LD_0} preserves them. The group H_* is in this latter subgroup. Its algebraic image closure contains the identity component of the cyclotomic-kernel image. The proof of Theorem 50.9 excluded a decomposition of that component into four lines and proved multiplicity one. Thus the two planes are irreducible and inequivalent even on H_*. Multiplying an off-diagonal matrix by the block diagonal $\rho(g)$ has trace zero, giving the last assertion. □

Theorem 50.12 (A bounded loss for the entire finite-coefficient dual group)**.** *For each curve there are a fixed number h of test places and a constant C depending on that curve and S. For every $m, N \geq 1$ one may realize the tests at h distinct good rational primes q_j splitting completely in L, with $q_j \equiv 1 \bmod \ell^N$, and with chosen Frobenius elements s_j at places above them. Then*

$$\operatorname{length}_A \ker\left(H^1(G_{L,S}, W_m(1)) \xrightarrow{(e_{s_j})_j} A_m^h\right) \leq C.$$

The matrices $\rho(s_j) \bmod t^m$ agree with the reductions of fixed matrices with simple spectrum over $\overline{K_1}$. For curves without the quadratic exception all the rational primes can be the same across

the list, split in all period and starting fields and congruent to one modulo every coefficient prime to the power N. In the quadratic exception the tests for the diagonal blocks likewise use these common primes. The additional tests for the off-diagonal blocks can be taken inert in $D_0/\mathbb{Q}$.

Proof. First use K_1-coefficients. For a curve without the exception the span of $\rho(H_*)$ is $\mathrm{Mat}_4(K_1)$ by the preceding proposition and Burnside's theorem. The quotient G_L/H_* is abelian, and ω factors through it and is nontrivial. Indeed at a place above ℓ the ramification index over $\mathbb{Q}_\ell$ is at most two, so the inertia image in $\mathbb{F}_\ell^\times$ has order at least $(\ell-1)/2>1$. Apply Theorem 30.15 to the matrix algebra, identified with its dual by trace. It says that vanishing of all the evaluations on H_* forces a coboundary. The theorem applies to the image of H_* in the restricted-ramification group as well, by the same span and quotient assertions. By continuity it suffices to test on any dense subset, in particular the simultaneous regular subset in Corollary 50.10.

In the exceptional case extend the curve fraction field finitely to define the two planes. The block diagonal space and the space of the two off-diagonal blocks are stable under conjugation by G_L. For the diagonal space first restrict to G_{LD_0}, an injective map on first cohomology since restriction followed by corestriction multiplies by two. On this subgroup the representation takes values in the units of the product of two full matrix algebras. Its values on H_* span that product by the inequivalence and absolute irreducibility on H_*. The character ω is still nontrivial over LD_0, since D_0 splits at ℓ. The same derivation theorem applies to this product with its trace dual. Thus tests at elements of H_* detect the diagonal cohomology, and their values on the off-diagonal part are zero.

For the remaining off-diagonal part apply the derivation theorem on G_L with normal subgroup $G'=G_{L(\mu_{\ell^\infty})}$. Here its span is the full matrix algebra by Theorem 50.9. On $G'\cap G_{LD_0}$ all the off-diagonal evaluations are already zero. Therefore the evaluations on the open subset of G' with $\phi=-1$ separate its cohomology classes. Regular elements are dense there. If a class with both parts passes all the diagonal tests, subtract the diagonal coboundary and apply these remaining tests. This proves separation of all the classes over the finite coefficient extension, hence over K_1: the equations for a coboundary are linear over K_1, so solvability descends to K_1.

We pass from this separation to a bound at every precision, including classes which do not lift from A_m to A. We give the details since reduction of a collection of tests on the rational cohomology alone would not establish this assertion. The action on $W(1)$ and the continuous cocycles on $W_m(1)$ for every m factor together through a fixed topologically finitely generated profinite quotient of $G_{L,S}$. In fact take the open normal kernel of the action modulo t. Each cocycle combined with the linear action has pro-ℓ image on this kernel. The latter's maximal pro-ℓ quotient is finitely generated, since its H^1 with constant $\mathbb{F}_\ell$-coefficients is finite-dimensional. For this last fact, pass to the corresponding number field, adjoin μ_ℓ by a prime-to-ℓ extension, and use the S-units and the finite S-class group in the Kummer sequence. A restricted ramification quotient only decreases the dimension by inflation. The kernel of the maximal pro-ℓ quotient is characteristic in the open normal subgroup, so dividing it out gives the desired common quotient.

Take u profinite generators of this quotient. On the free profinite group on their lifts, any assigned u cocycle values extend continuously by the semidirect product law. A relator has cocycle value an A-linear expression in these $16u$ scalar variables. The rows of all such expressions are generated by finitely many of them over the Noetherian ring A. The resulting finite matrix M cuts out all cocycles over A and over every A_m, since tuples modulo t^m lift before the relations are imposed. The matrix D of coboundaries is given by applying

the generators minus one. Over K_1 the quotient $\ker M/\operatorname{im} D$ after scalar extension is finite-dimensional, and the separation already proved allows a finite list of regular test elements. One may take their union on H_* for all curves and the additional $\phi = -1$ list separately for each exceptional curve. Enlarging the common list only adds tests, and all its elements can be chosen regular for every curve.

Adjoin the evaluation rows for these tests to M, using fixed group elements, and call the new matrix M'. Then $\ker(M' \otimes K_1) = \operatorname{im}(D \otimes K_1)$. Let b_1 be the maximum valuation of a nonzero elementary divisor of M', or zero if there are none, and choose b_2 with $t^{b_2} \ker M' \subset \operatorname{im} D$ over A. Such b_2 exists by finite generation. In Smith coordinates $M'v = 0 \bmod t^m$ implies $t^{b_1} v \in \ker M' + t^m A^{16u}$. Thus each class passing the tests modulo t^m is killed by $t^{b_1+b_2}$. Its entire kernel has at most $16u$ generators: it is a quotient of a submodule of A_m^{16u}, and the bound follows from the elementary divisor theorem. Its length is therefore at most $16u(b_1+b_2)$, independently of m, N.

Finally there are only finitely many cocycles at the given precision, by the generator presentation. Take the joint finite quotients recording their values via the semidirect product, the curve matrices modulo t^m, and, if necessary, the matrices to higher fixed precision detecting the nonzero discriminants. For a common test take all these quotients over the fields in the list and their normal closures over $\mathbb{Q}$; adjoin P and the ℓ_i^N-th roots of unity. Chebotarev matches the chosen element of H_* in this joint finite quotient with Frobenius at a good rational prime and a choice of place. For a test with $\phi = -1$, do the same over the single field with L, D_0, μ_{ℓ^N} included, matching its chosen element of G'. Then the prime is split in L, congruent to one modulo ℓ^N and inert in D_0. Repeating with finite exclusions gives distinct primes. All evaluation rows modulo the indicated precision are identical to the fixed tests. This proves the theorem. □

Corollary 50.13 (An ordinary Selmer length bound at the auxiliary levels)**.** *Suppose the curve has an integral polarized adjoint V over F restricting to W over L, with the real involutions acting by negative adjunction for nondegenerate symmetric forms. This includes the fixed-multiplier symplectic curves over F with odd multiplier, by using the rank-four polarized adjoint. Take the old finite set stable under complex conjugation, including the ramifications of this action. On $V_m = V/t^m V$ impose unramified cohomology at the old finite places away from ℓ and the image of flag-preserving cohomology at $v \mid \ell$. At one place of F from each of the h rational test primes allow full local cohomology. For the choices of the theorem, the resulting Selmer group and its dual with annihilating local conditions satisfy*

$$\operatorname{length}_A \operatorname{Sel}(F, V_m) \le 4([F:\mathbb{Q}] + h)m + O(1), \qquad \operatorname{length}_A \operatorname{Sel}^*(F, V_m^\vee(1)) = O(1).$$

Both constants are independent of m, N.

Proof. The trace form identifies the polarized adjoint with its dual, also for the negative-adjunction action. Composing the perfect A_m-pairing with the trace to $\mathbb{F}_\ell$ of the coefficient of t^{m-1} gives its finite-group dual pairing. At a new test place the dual local condition is zero. Restriction to L is injective by corestriction since $[L:F] = 2$. The restricted dual class is zero locally at every new place over the chosen ones and unramified away from the old set at all other places. It therefore factors through the original $G_{L,S}$: at the new places inertia acts trivially on the coefficients and the cocycle is zero on inertia. Its Frobenius evaluations vanish there. The dual estimate follows from Theorem 50.12.

For the first estimate apply the global Selmer dimension formula and divide logarithms of cardinalities by $\log|k|$ to obtain lengths over A. The global invariants of $V \otimes K_1$ vanish, since the centralizer over L is scalar and real conjugation negates scalars. Consequently $H^0(F, V_m)$

has bounded length: finitely many matrices $g-1$ already have zero common kernel over K_1, so their stacked Smith form gives a uniform bound at all precisions. The subtracted global twisted dual term can be discarded for an upper bound. At the old places away from ℓ, unramified cohomology has the same length as H^0, since the kernel and cokernel of Frobenius minus one on the finite inertia invariants have the same order.

At $v \mid \ell$, intersect the flag over the fraction field with the stable lattice. Its saturated steps give a full flag of direct summands over A. Its preserving submodule $B_v \subset V$ has rank ten and free quotient. On its dual twist the graded characters are $\omega\theta_{v,j}/\theta_{v,i}$ for $i \le j$. At different indices they are nontrivial by the separation on I_v^0; at equal indices ω is nontrivial by the ramification index bound at ℓ. Hence the dual twist has zero invariants over K_1 and bounded invariant length at finite precision by the same matrix argument. Local Euler characteristic and Tate duality give

$$h^1(F_v, B_v/t^m) - h^0(F_v, B_v/t^m) = 10[F_v : \mathbb{Q}_\ell]m + O(1),$$

where h^i denotes length. Its H^0 injects into that of V_m, so the image condition contributes at most this amount. At a real place the invariant skew-adjoint matrices have rank six, giving the term $-6m$.

At a new place q_j the unramified adjoint action is ordinary matrix conjugation and $q_j = 1$ in k. The full local $h^1 - h^0$ is the length of the twisted dual invariants by local Euler characteristic. This is the length of the centralizer of the indicated Frobenius matrix modulo t^m. It equals $4m+O(1)$ since the matrix agrees modulo t^m with a fixed regular semisimple matrix. Indeed its commutator map has rank twelve over K_1. Its twelve nonzero elementary divisors contribute at most their total valuation in addition to the four free kernel terms. Summing all places, the terms over ℓ and at infinity give $(10-6)[F:\mathbb{Q}]m$, and the h new places give at most $4hm + O(1)$. This proves the bound. [10] □

The constants in these estimates belong to curves over $k[[t]]$. Section 6.6 requires a bound on integral cohomology at the prescribed rational weights over E_x. Passing to those weights involves the support of the ordinary Hecke module through the curve primes and the two-constituent cases.

50.6 Rank-one local conditions at primes split in the period fields

Let P, P_∞, $A = k[[t]]$, $A_m = A/(t^m)$, $\omega = \bar{\kappa}_\ell$, and $W = \mathrm{End}_A(A^4)$ be as in Section 50.5. At a prime split in the quadratic inducing field, taking the ordinary trace can discard the whole off-diagonal part of a cocycle. We can recover its matrix entries by changing the local condition at the test prime. We do this over the original interval for all the curves of Corollary 50.10.

Lemma 50.14 (Restriction to the kernel of the adjoint action)**.** *Write $G = G_{L,S}$, where S includes all places ramified in P_∞/L. Let J be the common kernel of the maps from G to $\mathrm{Gal}(P_\infty/L)$ and to $\mathrm{Aut}_A(W)$ arising from conjugation by ρ. For every $m \ge 1$, restriction gives an injection*

$$H^1(G, W_m(1)) \longrightarrow \mathrm{Hom}_{\mathrm{cont}}(J, W_m), \qquad W_m = W/t^m W.$$

Proof. Choose one place of L above ℓ and let I be its inertia image. The four residual characters are equal on I, by Theorems 48.2 and 48.5. This equality persists under the common scalar twist. In a composition-series basis over a finite residue extension the residual inertia matrices are scalar times upper unitriangular. Their conjugation action has ℓ-power order. The kernel of reduction in $\mathrm{Aut}_A(W)$ is pro-ℓ, as successive congruence quotients are additive matrix groups over k. Thus the adjoint image of I is pro-ℓ. On the other hand $\omega(I) \neq 1$, since the local

ramification index divides two and $(\ell-1)/2>1$. There is an element $z\in I$ acting trivially on W with $\omega(z)\neq 1$. Otherwise $\omega|_I$ would factor through a pro-ℓ group and have both prime-to-ℓ and ℓ-power order.

The group G/J embeds in the product of the adjoint image and the abelian group $\mathrm{Gal}(P_\infty/L)$. Consequently the image of z is central in G/J. It acts on $W_m(1)$ by $\lambda=\omega(z)$ with $\lambda-1$ a unit. If y is a cocycle on G/J, evaluating it at $zg=gz$ gives

$$(\lambda-1)y(g)=(g-1)y(z).$$

This makes y the coboundary with witness $(\lambda-1)^{-1}y(z)$. On J both the adjoint action and ω are trivial, the latter because $\mu_\ell\subset P_\infty$. A cocycle restricting to zero on J descends to G/J, proving the lemma. □

Theorem 50.15 (One matrix coefficient per completely split test prime)**.** *Fix a finite list of the curves, including any with a quadratic self-twist. For curve i there is a finite list of rank-one idempotents D_{ij} in $\mathrm{Mat}_4(A_i)$, of length h_i. Given positive precisions m_i,N_i, each index (i,j) can be assigned a different good rational prime q_{ij}. Every such prime splits in all the interval fields E_x and in all starting fields, and is one modulo $\ell_a^{N_a}$ for every curve a. For the assigned curve choose a place of L_i above q_{ij} with Frobenius s_{ij}. Its matrix is scalar modulo $t_i^{m_i}$, and the following map is injective:*

$$H^1(G_{L_i,S_i},W_{i,m_i}(1))\longrightarrow A_{i,m_i}^{h_i},\qquad y\longmapsto\big(\mathrm{tr}(D_{ij}y(s_{ij}))\big)_j.$$

The number of tests and the idempotents are independent of both precisions.

Proof. Work first with one curve in an integral basis. The sixteen idempotents E_{aa} and $E_{aa}+E_{ab}$, $a\neq b$, are rank one and span the matrix algebra over A: subtracting E_{aa} gives E_{ab}. The pairing $(D,Y)\mapsto\mathrm{tr}(DY)$ is perfect in this basis, at every precision. Thus the tests $\mathrm{tr}(Dy(n))$ for these idempotents and all $n\in J$ detect all classes by the lemma.

It is possible to choose finitely many pairs (n,D) once for all precisions. Use the common quotient with u profinite generators constructed in the proof of Theorem 50.12. It carries the action and all cocycles at every precision. For each $n\in J$ choose a lift of its image to the free profinite group on these generators. Evaluation at that lift is A-linear in the $16u$ generator coordinates. Indeed the cocycle law is linear in those coordinates on words, and on profinite limits this follows by continuity, or by evaluating the coordinate basis cocycles in the compact semidirect product. Reduction gives the same row on tuples over A_m; the relators act trivially for every actual cocycle. Taking the trace with D gives one row in A^{16u}. The submodule generated by all the rows is finitely generated, since A is Noetherian. A finite subset of the rows generates it, by expanding each of a finite set of generators in the original rows. Their reductions therefore suffice at every m. All tests vanish on coboundaries since J acts trivially.

Lift each chosen n to the absolute Galois group. It fixes P_∞ because P_∞/L is unramified outside S. Its matrix is scalar, since a matrix commuting with all matrix units over A is scalar. For the given precisions form a finite Galois extension of $\mathbb{Q}$ containing P, the required roots of unity and all the normal closures of the finite extensions recording the curve matrices and the cocycles at their respective precisions. There are only finitely many cocycles, as each is determined by a finite tuple over a finite ring. As in Theorem 50.12, adjoining the semidirect-product quotients records actual cocycle values. Chebotarev gives infinitely many good rational primes admitting a place with Frobenius equal to the chosen image of n in this finite group. Choose distinct primes successively. They have the asserted splitting, congruences and Frobenius matrices. At an assigned place the twisted adjoint action of Frobenius is trivial modulo t^m, so the matrix entry test descends to cohomology. Its value agrees with the chosen row. This proves injectivity. □

Corollary 50.16 (Vanishing of the dual Selmer group with one tame direction)**.** *For a curve with the integral polarization of Corollary 50.13, keep its old local conditions over F. At the selected place over each of its own h assigned primes, allow arbitrary unramified classes and the single tame direction of the indicated idempotent D. Use the unramified condition at all other new places. For these finite-coefficient Selmer conditions one has*

$$\mathrm{Sel}^*(F, V_m^\vee(1)) = 0, \qquad \operatorname{length}_A \mathrm{Sel}(F, V_m) \le (4[F:\mathbb{Q}] + h)m + O(1).$$

The constant depends on the curve and its old local conditions, uniformly in m, N and the primes chosen by the theorem.

Proof. At its assigned place the identification with W_m uses the chosen prime of L above it. The inertia action is trivial and Frobenius acts trivially on the adjoint and on μ_ℓ. Local class field theory gives two independent scalar additive characters modulo ℓ, one unramified and one tame: a uniformizer gives the unramified coordinate, the cyclic residue units give the tame coordinate since $\ell \mid q-1$, and the principal units are pro-q. Fix a generator of each coordinate. With coefficients W_m, the local condition consists of unrestricted unramified values and tame values in $A_m D$. This line is free of length m because D has a unit entry.

In the local Tate pairing two unramified classes pair to zero, since the procyclic residue group has ℓ-cohomological dimension one. The pairing of the scalar unramified and tame generators is nonzero by perfectness on the two-dimensional local character space, after identifying μ_ℓ with $\mathbb{F}_\ell$ at the place. For the matrix coefficients this pairing multiplies the matrix trace pairing by the same nonzero scalar and applies the perfect additive functional on A_m used in Corollary 50.13. Thus annihilating all unramified values forces a dual class to be unramified. For its Frobenius value Y, annihilating all multiples of the tame direction is equivalent to $\operatorname{tr}(DY) = 0$ in A_m. At the remaining good places the annihilator of unramified cohomology is unramified cohomology for the dual. [10]

A dual Selmer class therefore restricts over L to an old global class, with every test of Theorem 50.15 zero. Its restriction vanishes. Restriction to L is injective on global first cohomology by corestriction, since $[L:F] = 2$ is invertible. This proves the first assertion. At an assigned place the local condition has length $16m+m$, while the invariants have length $16m$; its contribution to the Selmer dimension formula is m. All other new places have unramified conditions and contribute zero. The old finite and real places and the global invariants satisfy the estimates proved in Corollary 50.13, giving $4[F:\mathbb{Q}]m + O(1)$ as their combined upper bound. This proves the second assertion. □

Proposition 50.17 (The tame lifting equation at a scalar Frobenius)**.** *At an assigned prime let a tame inertia generator act as $T = 1 + (\delta - 1)D$, where $D^2 = D$ and $\delta^{q-1} = 1$. For a Frobenius lift with matrix Φ, the local tame relation is equivalent to*

$$(\delta - 1)[\Phi, D] = 0.$$

At scalar residual Frobenius, an unrestricted first-order change in Frobenius and a first-order change in this inertia character can be obstructed at second order.

Proof. On the image of D, T is multiplication by δ; on its kernel it is the identity. Hence $T^q = T$, and $\Phi T \Phi^{-1} = T^q$ is the displayed equation. In a basis with $D_0 = E_{11}$ at the closed point, let the residual Frobenius be λI_4. Over $k[s,u]/(s,u)^2$ take $\delta = 1 + s$, $\Phi = \lambda I_4 + uE_{12}$, and $D = D_0$. These matrices satisfy the relation. In $k[s,u]/(s,u)^3$, the term of degree two in the left side of the displayed equation is $su[E_{12}, D_0] = -suE_{12}$, which is nonzero. Corrections of degree two cannot alter it, since the degree-zero Frobenius is scalar. The order constraint

on δ allows the displayed choice in this ring since $\ell \mid q-1$ and $\ell \geq 5$. Thus the two given first-order changes do not lift together. □

The cohomological vanishing applies on the same period interval, even on a curve induced from the quadratic field. Its local conditions vary with the coefficient precision. To deduce the strict cohomology estimate at the prescribed characteristic-zero weights one still needs an integral comparison through those weights. The tame lifting equation gives an explicit constraint on a deformation construction using these local conditions.

50.7 Exact cyclotomic depth at the split Selmer primes

Fix a finite list of the ordinary curves in Corollary 50.10 on one of the period intervals. On curve i the representation has dimension four and coefficient ring $k_i[[t_i]]$, where k_i is a finite field of odd characteristic ℓ_i. Let $P/\mathbb{Q}$ be the finite Galois field used there, containing all the period and starting fields on the interval. Theorem 50.15 assigns to each curve finitely many rank-one probes. At a precision $m_i \geq 1$, they detect its full finite-coefficient dual cohomology by evaluating one matrix entry at each assigned prime. The splitting requirements in that theorem prescribe lower bounds on the cyclotomic depths. For the integral tame characters one needs the sizes of the actual cyclic groups at those primes.

Theorem 50.18 (Split Selmer tests with prescribed exact depths)**.** *For each distinct coefficient prime ℓ put*

$$d_\ell = v_\ell([P:\mathbb{Q}]) + \max\Big(\{0\} \cup \{\lceil \log_\ell(17m_i)\rceil : \ell_i = \ell\} \\ \cup \{v_\ell(|\mathrm{GL}_{17}(k_i)|) : \ell_i \neq \ell\}\Big).$$

Choose one integer $N_\ell \geq 1 + d_\ell$ per distinct prime. All the probes of Theorem 50.15 can be realized by different good rational primes q such that

$$v_\ell(q-1) = N_\ell \quad \text{for every coefficient prime } \ell.$$

The primes split in P. The curve matrices and the dual cocycle evaluations at their assigned precisions are the same as in that theorem. Any further finite list of primes can be excluded.

Proof. Let $K/\mathbb{Q}$ contain P and be generated over it by all the conjugates of the fields recording the given matrices and the cocycles at these precisions. We use the smallest such compositum. There are finitely many cocycles at each precision by the finite generator argument in Theorem 50.15. One cocycle y with values in the twisted endomorphisms of the four-dimensional representation is recorded by the matrices

$$\begin{pmatrix} a(g) & y(g) \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{17}(k_i[[t_i]]/(t_i^{m_i})),$$

where $a(g)$ is the action on the sixteen endomorphism coordinates. The cocycle identity is precisely the multiplication law for these matrices. The original four-dimensional matrices can also be embedded in this group by adjoining an identity block. Since P contains every conjugate of the starting fields, $\mathrm{Gal}(K/P)$ embeds by restriction into a product of these matrix groups, including their conjugate copies.

For $\ell_i = \ell$, an element g of ℓ-power order acts unipotently over k_i on the underlying vector space of dimension $17m_i$. Indeed its minimal polynomial divides a power of $X-1$ in characteristic ℓ. The nilpotent matrix $g-1$ has nilpotence index at most $17m_i$; taking the power

$\ell^{\lceil \log_\ell(17 m_i)\rceil}$ gives the identity. For $\ell_i \neq \ell$, the kernel of reduction of the matrix group is an ℓ_i-group by its successive congruence filtration. An ℓ-element therefore has the same order as its reduction in $\mathrm{GL}_{17}(k_i)$. The order bound in a direct product uses the largest of these ℓ-exponents. For an ℓ-element of $\mathrm{Gal}(K/\mathbb{Q})$, first raising it to $\ell^{v_\ell([P:\mathbb{Q}])}$ puts it in $\mathrm{Gal}(K/P)$. Thus its order divides ℓ^{d_ℓ}. This bound passes to ℓ-elements of every quotient, since a Sylow subgroup of the quotient is the image of a Sylow subgroup of the original group.

Set $C = \mathbb{Q}(\mu_{\prod_\ell \ell^{N_\ell+1}})$. By the Chinese remainder theorem the tuple $(1 + \ell^{N_\ell} \bmod \ell^{N_\ell+1})_\ell$ defines an automorphism of C. Each individual component fixes $C \cap K$. In fact the image of $1 + \ell\mathbb{Z}_\ell$ in the Galois group of this intersection has exponent dividing ℓ^{d_ℓ} by the preceding quotient bound. For odd ℓ the logarithm maps $1 + \ell\mathbb{Z}_\ell$ isomorphically to $\ell\mathbb{Z}_\ell$, and $1 + \ell^{N_\ell}$ is consequently an ℓ^{d_ℓ}-th power in that group. Here the local factor acts trivially at all the other factors of the cyclotomic modulus.

For each selected probe take the element n from the proof of Theorem 50.15. It fixes P and the full cyclotomic towers for all the coefficient primes. Its image on K and the indicated automorphism of C agree on $C \cap K$. They extend to an automorphism of KC, by the fiber-product description for two finite Galois extensions. Chebotarev's theorem supplies infinitely many rational primes with a choice of place realizing this automorphism as arithmetic Frobenius. On K the required matrices and cocycle values are those of n. On C the action is given by the residue class of q, proving the exact depth. Choosing the places successively gives distinct primes and allows the stated exclusions. □

We next compute the local integral intersection for these probes. Let $\mathcal{O}$ be a complete discrete valuation ring finite over $\mathbb{Z}_\ell$, with uniformizer π and residue field containing k_i at a probe of characteristic $\ell = \ell_i$. Take the residual scalar Frobenius and trivial residual inertia there. In a fixed basis lift its rank-one idempotent to $D = \mathrm{diag}(1, 0, 0, 0)$. Such a basis is obtained by lifting bases of the residual image and kernel. Write Φ for the freely variable framed Frobenius matrix lifting the given scalar, and require the tame generator to act by $1 + uD$, with its character of order dividing $M = \ell^{v_\ell(q-1)}$. The variable u reduces to zero. Here these are local integral parameters, independent of the curve parameter t_i.

Proposition 50.19 (The integral intersection at one test prime)**.** *Let Y denote the ten coordinates in the diagonal blocks of Φ for D, translated by a lift of its residual value. Let $X = (X_1, \dots, X_6)$ be its two off-diagonal blocks. Set $F_M(u) = ((1 + u)^M - 1)/u$ and $S = \mathcal{O}[[Y, X, u]]$. The complete ring for the specified local data is*

$$R = S/(uF_M(u), uX_1, \dots, uX_6).$$

It is reduced and flat over $\mathcal{O}$. Its unramified branch $S/(u)$ and its union of ramified branches $S/(F_M, X)$ meet along $(\mathcal{O}/M\mathcal{O})[[Y]]$. Precisely, R is the fiber product of the two branch rings over this intersection.

Proof. Invertibility of Φ holds over every local coefficient algebra because its residual determinant is a unit. The order condition on inertia is $uF_M(u) = 0$. It also implies $(1+uD)^q = 1+uD$, as $M \mid q-1$ and D is idempotent. The tame relation is consequently $u[\Phi, D] = 0$, precisely the six equations $uX_a = 0$ by Proposition 50.17. Conversely these equations give a homomorphism of the tame decomposition group with the assigned generator images. This can be checked first on finite local coefficient algebras, where the inertia image is cyclic of order dividing M and the Frobenius image has finite order. Passing to inverse limits gives the stated complete ring.

The polynomial F_M is monic of degree $M - 1$ and reduces to u^{M-1} in the residue field. Division by this distinguished polynomial makes $S/(F_M, X)$ free of rank $M - 1$ over $\mathcal{O}[[Y]]$.

Multiplication by u on this free module is injective. Over the fraction field the constant coefficient $F_M(0) = M$ makes u invertible, and freeness gives the injection before passage to that field. Thus $(u) \cap (F_M, X) = u(F_M, X)$ in S. The exact sequence for the intersection of two ideals identifies R with the asserted fiber product, since $S/(u, F_M, X) = (\mathcal{O}/M\mathcal{O})[[Y]]$. Both branch rings are $\mathcal{O}$-torsion free, so R embeds in a torsion-free module and is flat over this discrete valuation ring. The unramified branch is a power series domain. The other branch embeds by the same freeness in the algebra obtained over the fraction field of $\mathcal{O}[[Y]]$, where F_M has the distinct roots $\zeta - 1$ for the nontrivial M-th roots of unity over an algebraic closure. Both branches, and hence their subring R, are reduced. □

Corollary 50.20 (The exact cost of isolating unramified inertia)**.** *Take the h probes for a single curve and one common $\mathcal{O}$. At each probe fix an integral Frobenius specialization commuting with its specified projector and lifting the residual scalar. On this product of local character rings, consider integral functions vanishing on every characteristic-zero tuple at which at least one inertia character is nontrivial. Their values at the all-unramified tuple form the ideal*

$$\left(\prod_{a=1}^{h} M_a\right)\mathcal{O}, \qquad M_a = \ell^{v_\ell(q_a-1)}.$$

The same ideal is obtained by first taking functions on the full product of the framed local charts that vanish on all such tuples, and then evaluating at the fixed unramified point. For the exact depth $N_\ell = 1+d_\ell$ in Theorem 50.18 its colength is $h\, v_\pi(\ell)(1+d_\ell)$. For the fixed list and interval this is $O(\log(2 + \max_{\ell_i=\ell} m_i))$ as the precisions vary.

Proof. At the specified Frobenius values all the X_a are zero. The remaining ring for a single inertia generator $z = 1+u$ is $\mathcal{O}[z]/(z^M - 1)$. Indeed its presentation in the complete ring by the distinguished polynomial $uF_M(u)$ has the same basis of M powers. The ring for the product is therefore $\mathcal{O}[G]$ for $G = \prod_a \mathbb{Z}/M_a\mathbb{Z}$. Its characteristic-zero points are the characters of G after an extension containing their values. If a group-ring element takes value b at the trivial character and zero at all the others, Fourier inversion makes every coefficient equal to $b/|G|$. Integral coefficients force $b \in |G|\mathcal{O}$; conversely $\sum_{g\in G} g$ attains $|G|$. Any separating function on the full charts restricts to such a function on this product. The element $\prod_a F_{M_a}(u_a)$ of the full charts vanishes at every characteristic-zero point with some nontrivial inertia and attains $\prod_a M_a$ at the unramified point. This gives both equalities. The valuation of the displayed integer and the definition of d_ℓ prove the colength statements. □

Compatibility of the prescribed Frobenius with the projector is essential in the specialization. In fact fix an arbitrary integral Φ_0 with the same residual value and let $I \subset \mathcal{O}$ be the ideal of its six off-diagonal coordinates. The specialized module is

$$\mathcal{O} \;\oplus\; u\big(\mathcal{O}[[u]]/(F_M(u), I)\big).$$

The first summand is evaluation at zero. For the second, cancel u in the power series ring before taking the quotient by $u(F_M, I)$. For $I = (\pi^b) \neq 0$ the second summand has length $b(M - 1)$ by the same monic division, and disappears in characteristic zero. The fixed commuting specialization in Corollary 50.20 describes the local cost at actual split test primes. Relating these local charts to a global Hecke algebra at the prescribed characteristic-zero weights, with constants uniform across intervals, is still needed for the strict length bound of Section 6.6.

50.8 Paired tame inertia in the symplectic trace algebra

The four-dimensional representations used for the colliding weights have a prescribed symplectic similitude character. In the source this condition is imposed on every Galois element in `Development/AutomorphicArithmetic/AutomorphicVarieties/SymplecticGSp4/DecompSigmaOrd.lean`, line 142. At an auxiliary prime where the multiplier is unramified, a ramified inertia character on one line must be paired with its inverse on the dual line. We give the corresponding local calculation and its effect at an unramified system of traces.

Let A be a local ring in which two is a unit, let V be free of rank four with a perfect alternating form $\langle\,,\,\rangle$, and write $B^\dagger$ for the adjoint of an endomorphism. Put $\mathfrak{s} = \{Y : Y^\dagger = -Y\}$.

Lemma 50.21 (Paired lines give the same tests on the symplectic adjoint)**.** *For any rank-one idempotent D on V, its adjoint is a rank-one idempotent with $DD^\dagger = D^\dagger D = 0$. The two images span a hyperbolic plane and $H = D - D^\dagger$ belongs to $\mathfrak{s}$. For $Y \in \mathfrak{s}$,*

$$\operatorname{tr}(HY) = 2\operatorname{tr}(DY).$$

In particular all the tests of Theorem 50.15 on a curve preserving such a form up to similitude detect its symplectic adjoint cohomology at the same primes and precisions when D is replaced by H.

Proof. Choose e generating the image of D. Write $D(v) = \langle v, f\rangle e$ using the perfection of the form. Since $D(e) = e$, one has $\langle e, f\rangle = 1$. The adjoint is $D^\dagger(v) = \langle e, v\rangle f$. These formulas prove the claims about the two lines. Taking the trace after adjunction and cyclically permuting the factors gives $\operatorname{tr}(D^\dagger Y) = -\operatorname{tr}(DY)$. Both the formula and the decomposition into the eigenspaces of the adjoint involution commute with reduction. If the Galois representation preserves the form up to similitude, then conjugation commutes with adjunction. The map $B \mapsto (B - B^\dagger)/2$ consequently splits off $\mathfrak{s}$ as a Galois module, also after the common cyclotomic twist. Its first cohomology injects into the endomorphism cohomology. The displayed factor of two is a unit at the odd coefficient primes of that theorem. Applying the existing injectivity proves the last assertion. □

Take a complete discrete valuation ring $\mathcal{O}$ finite over $\mathbb{Z}_\ell$, with ℓ odd. Let $q \neq \ell$ be a rational prime with $M = \ell^{v_\ell(q-1)} > 1$. Suppose that the residual Frobenius is a nonzero scalar and that residual inertia is trivial. Fix an unramified multiplier lifting the residual one. Choose a symplectic basis for two paired lines and their orthogonal complement. Prescribe the matrix of a tame inertia generator in that basis to be

$$J(z) = zD + z^{-1}D^\dagger + (1 - D - D^\dagger), \qquad z = 1 + u, \qquad z^M = 1.$$

Here D projects to the first line, $D^\dagger$ to the second, and u belongs to the maximal ideal. Wild inertia and the other tame factors have trivial image. Frobenius is allowed to vary in the framed fixed-multiplier group.

Proposition 50.22 (The two local branches with prescribed multiplier)**.** *Use four variables Y and six variables X, and put $F_M(u) = ((1+u)^M - 1)/u$. The complete ring for these framed data is*

$$\mathcal{O}[[Y, X, u]]/(uF_M(u), uX_1, \dots, uX_6).$$

It is flat over $\mathcal{O}$ and reduced. Its unramified branch and the closure of the locus $z \neq 1$ meet in the ring $(\mathcal{O}/M\mathcal{O})[[Y]]$.

Proof. The prescribed Frobenius multiplier has a square root lifting the residual scalar by Hensel's lemma. Divide Frobenius by this scalar lift to obtain a symplectic matrix g congruent to one. The Cayley transform $Z = (g-1)(g+1)^{-1}$ has $Z^{\dagger} = -Z$, since $g^{\dagger} = g^{-1}$. Conversely $g = (1+Z)(1-Z)^{-1}$ recovers every such matrix from a skew-adjoint Z congruent to zero. The denominators in both transformations are invertible. Commuting with $J(z)$ is the same condition for Z and for g.

Because $M \mid q-1$, the tame Frobenius relation is just commutation with $J(z)$. Its three diagonal scalars on the line, the orthogonal plane and the dual line are $z, 1, z^{-1}$. Their pairwise differences are u times units, including $z - z^{-1} = u(1+z^{-1})$. Thus u times each off-diagonal block of Z vanishes. The block-diagonal skew-adjoint matrices consist of a scalar and its negative on the two lines and an element of $\mathfrak{sp}_2$ on the plane. They have rank four. The off-diagonal blocks form a direct complement of rank six. Indeed adjunction preserves the block-diagonal and off-diagonal spaces, and the full skew-adjoint space has rank ten by identifying it with symmetric bilinear forms on V. These direct summands are free over the local coefficient ring. This gives precisely the stated equations. Conversely the equations realize the tame presentation on every finite local coefficient algebra, and passing to inverse limits gives a continuous representation.

In $S = \mathcal{O}[[Y, X, u]]$, the ring $S/(F_M, X)$ is free of rank $M-1$ over $\mathcal{O}[[Y]]$ by monic distinguished division. Over its fraction field the roots of F_M are distinct and nonzero. Thus this ring is reduced and multiplication by u is injective on it. Consequently $(u) \cap (F_M, X) = u(F_M, X)$. The two branch rings are $\mathcal{O}$-torsion free and reduced, and the complete ring embeds in their product. This proves flatness and reducedness. The quotient by the sum of their defining ideals is $S/(u, F_M, X) = (\mathcal{O}/M\mathcal{O})[[Y]]$, as asserted. □

There is a separating element expressible entirely through the trace of inertia. Define $T_0(X) = 2$, $T_1(X) = X$ and $T_{i+1}(X) = XT_i(X) - T_{i-1}(X)$. For an odd integer $M > 1$ put

$$U_M(X) = 1 + \sum_{i=1}^{(M-1)/2} T_i(X).$$

Induction gives $T_i(z + z^{-1}) = z^i + z^{-i}$. Since $\operatorname{tr} J(z) = 2 + z + z^{-1}$, on the ring $z^M = 1$ one has the integral identity

$$U_M(\operatorname{tr} J(z) - 2) = \sum_{a=0}^{M-1} z^a.$$

Theorem 50.23 (Integral comparison at an unramified trace system)**.** *Take finitely many such primes q_i, put $M_i = \ell^{v_\ell(q_i-1)}$, and let $d = \prod_i M_i$. On the product of the local character rings, every integral function which vanishes at all characteristic-zero tuples with some $z_i \neq 1$ takes a value in $d\mathcal{O}$ at the all-unramified tuple. The product of the displayed trace polynomials attains d. This also gives the exact ideal of values in the subalgebra generated by the inertia traces.*

Now let B be a finite torsion-free $\mathcal{O}$-algebra realized in a product of characteristic-zero parameter fields, containing the traces of the specified inertia generators. Assume inertia at each q_i factors through the cyclic group of order M_i and its generator has the characteristic polynomial of the prescribed paired type at each parameter. In characteristic zero a finite-order matrix is semisimple, so its eigenvalues determine whether it is the identity. Let B_0, B_1 be the images on the all-unramified tuples and on the other tuples respectively. Fix an $\mathcal{O}$-algebra homomorphism $\lambda_0 : B_0 \to \mathcal{O}$ and let λ be its pullback. Write $I = \ker \lambda$, $I_0 = \ker \lambda_0$. For

$\eta_\lambda(B) = \lambda(\operatorname{Ann}_B I)$, *one has*

$$\begin{aligned} d(B_0 \oplus B_1) \;&\subseteq\; B \;\subseteq\; B_0 \oplus B_1, \\ d\,\eta_{\lambda_0}(B_0) \;&\subseteq\; \eta_\lambda(B) \;\subseteq\; \eta_{\lambda_0}(B_0). \end{aligned}$$

For every $k \geq 1$ the map $I^k/I^{k+1} \longrightarrow I_0^k/I_0^{k+1}$ is surjective with kernel killed by d. At $k = 1$ this kernel has length at most

$$v_\pi(d)\, \dim_{\mathcal{O}/(\pi)} I/(I^2 + \pi I),$$

where π is a uniformizer of $\mathcal{O}$.

Proof. The independent variables z_i with $z_i^{M_i} = 1$ give the group ring of $\prod_i \mathbb{Z}/M_i\mathbb{Z}$. Over a field containing all its character values, Fourier inversion shows that an element supported at the trivial tuple with value c has constant coefficient c/d on every group element. Its integral coefficients therefore imply $c \in d\mathcal{O}$. Each sum $\sum_{a=0}^{M_i-1} z_i^a$ is zero at every nonidentity root of unity and equals M_i at one. Their product attains the bound, using only the traces.

In B let w be the same product of trace polynomials. On the two sets of components its values are d and zero. The embedding in parameter fields therefore identifies $w = (d, 0)$ and $d - w = (0, d)$. Multiplying lifts from the two image algebras by these elements proves the first pair of inclusions. The image of $\operatorname{Ann}_B I$ annihilates I_0 because $I \to I_0$ is surjective. In the other direction, for $c_0 \in \operatorname{Ann}_{B_0} I_0$ the element $(dc_0, 0) \in B$ annihilates I. Evaluating both observations at the distinguished system proves the second pair.

Surjectivity on I^k follows by lifting products. If $v \in I^k$ maps into I_0^{k+1}, subtract from it a lift in I^{k+1}. The resulting v' vanishes in B_0, so $wv' = 0$. Since $d - w \in I$, we get $dv' = (d - w)v' \in I^{k+1}$. This proves the annihilation. A submodule of a finitely generated module over a discrete valuation ring has at most as many generators as its ambient module, by lifting its preimage to a finite free cover. Nakayama's lemma gives the stated generator count for I/I^2. Each generator of the kernel contributes at most $v_\pi(d)$ to its length. □

For the primes of Theorem 50.18 the loss per prime in this calculation is exactly $v_\pi(\ell)v_\ell(q_i-1)$ on the independent local character rings. The inclusions give an upper bound for the loss on the indicated global trace images. Applying this to the entire strict cohomology group still requires an integral comparison between that group and the relevant trace algebra. In particular the embedding of B in characteristic-zero fields used to prove $wv' = 0$ must be justified for whichever algebra is used in that comparison.

50.9 An integral annihilator for finite symplectic inertia

At an auxiliary tame prime the quotient imposing unramifiedness is defined by the entries of the inertia matrix minus the identity. This description lets us compare trace algebras before passing to their characteristic-zero images. Throughout, $\mathcal{O}$ is a complete discrete valuation ring of odd residue characteristic ℓ. Let $M = \ell^a$, $a \geq 1$. Let R be a commutative $\mathcal{O}$-algebra, and $U \in \mathrm{Sp}_4(R)$ satisfy $U^M = I_4$. This models a cyclic tame inertia image of order dividing M at a prime where the similitude multiplier is unramified. The symplectic module has a fixed symplectic basis with alternating form matrix J.

Theorem 50.24 (A trace polynomial killing the entire inertia ideal)**.** *Write $S = \sum_{j=-(M-1)/2}^{(M-1)/2} U^j$. The scalar*

$$w_M(U) = \frac{(\operatorname{tr} S)^2 - 2\operatorname{tr}(S^2)}{8} \qquad \textit{satisfies} \qquad w_M(U)(U - I_4) = 0, \qquad w_M(I_4) = M^2.$$

It belongs to the $\mathcal{O}$-algebra of traces of powers of U. The ideal $M^2\mathcal{O}$ of possible values at the trivial character is sharp for universal integral scalar annihilators when both symplectic planes can ramify independently.

If U is the identity on a nondegenerate direct summand of rank two, one may use

$$w'_M(U) = U_M(\operatorname{tr} U - 2), \qquad w'_M(U)(U - I_4) = 0, \qquad w'_M(I_4) = M.$$

Here U_M is the integral polynomial of Section 50.8. The value ideal in this case is $M\mathcal{O}$. All the identities hold over rings with nilpotents and with ℓ-power torsion.

Proof. Let $X^\dagger = J^{-1}X^{\mathsf{T}}J$. Since $U^\dagger = U^{-1}$, the symmetric range of summation gives $S^\dagger = S$. The geometric sum and $U^M = I_4$ give $S(U - I_4) = 0$. The matrix $A = JS$ is alternating. Its Pfaffian is $af - be + cd$ if the upper entries, in row order, are a, b, c, d, e, f. Direct multiplication gives

$$\begin{pmatrix} 0 & -f & e & -d \\ f & 0 & -c & b \\ -e & c & 0 & -a \\ d & -b & a & 0 \end{pmatrix} A = \operatorname{Pf}(A) I_4.$$

Thus the scalar $\operatorname{Pf}(JS)$ kills $U - I_4$. The same four by four determinant expansion gives $\det A = \operatorname{Pf}(A)^2$. As J is unimodular, $\operatorname{Pf}(J)$ is a unit. Consequently

$$\det(TI_4 - S) = \left(\frac{\operatorname{Pf}(TJ - JS)}{\operatorname{Pf}(J)}\right)^2.$$

The polynomial inside the square is monic quadratic. Write it as $T^2 - c_1 T + c_0$. Its constant term is $\operatorname{Pf}(JS)/\operatorname{Pf}(J)$. Comparing the cubic and quadratic coefficients gives

$$2c_1 = \operatorname{tr} S, \qquad c_1^2 + 2c_0 = \frac{(\operatorname{tr} S)^2 - \operatorname{tr}(S^2)}{2}.$$

The second expression on the right is the sum of the principal two by two minors. Solving for c_0 gives exactly $w_M(U)$. This proves the annihilation over R. Both traces in the formula are sums of traces of powers of U, and evaluating at $S = MI_4$ gives M^2.

In the second case the orthogonal complement of the fixed plane is stable and nondegenerate. The calculation is local on the coefficient ring, so one may choose a symplectic basis of this complement. Its matrix V has determinant one. Cayley–Hamilton gives $V + V^{-1} = (\operatorname{tr} V) I_2$. The recurrence defining T_j in the preceding section therefore gives $V^j + V^{-j} = T_j(\operatorname{tr} V) I_2$. Summing shows that S on this plane equals $U_M(\operatorname{tr} V) I_2$, where $\operatorname{tr} V = \operatorname{tr} U - 2$. Multiplying by $V - I_2$ gives zero. On the fixed plane $U - I_4$ already vanishes. Evaluation at the identity gives M.

For sharpness take the group ring $\mathcal{O}[(\mathbb{Z}/M\mathbb{Z})^2]$ with its two tautological units z_1, z_2, and use the matrix $\operatorname{diag}(z_1, z_2, z_1^{-1}, z_2^{-1})$ in a hyperbolic basis. A scalar annihilating its difference from the identity is fixed by multiplication by z_1 and z_2. Translation on the group basis forces every coefficient to be the same. Its augmentation consequently belongs to $M^2\mathcal{O}$. With a fixed plane the same argument uses $\mathcal{O}[\mathbb{Z}/M\mathbb{Z}]$ and yields $M\mathcal{O}$. The constructed elements attain these values. These lower bounds therefore apply in particular to functions formed from traces. □

Corollary 50.25 (Passage to unramified traces over an integral base)**.** *Let a representation over R have fixed multiplier unramified at a finite list of auxiliary primes. Suppose inertia at prime i is generated topologically in the image by U_i with $U_i^{M_i} = I_4$, where M_i is a power of ℓ. Let $K \subset R$ be the ideal generated by the entries of every $U_i - I_4$, and let $B \subset R$ be an*

$\mathcal{O}$-subalgebra containing the traces of their powers. Define B_0 to be the image of B in R/K. Choose $a_i = 1$ at a prime where U_i fixes a nondegenerate plane pointwise, and $a_i = 2$ at every other prime. Put $D_0 = \prod_i M_i^{a_i}$.

For any $\mathcal{O}$-point $\lambda_0 : B_0 \to \mathcal{O}$ with pullback $\lambda : B \to \mathcal{O}$, one has

$$D_0\, \eta_{\lambda_0}(B_0) \;\subseteq\; \eta_\lambda(B) \;\subseteq\; \eta_{\lambda_0}(B_0).$$

For $I = \ker \lambda$, $I_0 = \ker \lambda_0$ and every $h \geq 1$, the surjection $I^h/I^{h+1} \to I_0^h/I_0^{h+1}$ has kernel killed by D_0. When I/I^2 is finitely generated over $\mathcal{O}$, the kernel at $h = 1$ has length at most

$$v_\pi(D_0)\, \dim_{\mathcal{O}/(\pi)} I/(I^2 + \pi I).$$

Proof. Let w be the product of the scalars in the theorem, using w'_{M_i} at a fixed plane and w_{M_i} otherwise. This element belongs to B, satisfies $wK = 0$ in R, and maps to D_0 in R/K. In particular it annihilates $\ker(B \to B_0) = B \cap K$, even for a trace supported entirely in residue characteristic ℓ. The map $I \to I_0$ is surjective. The image of an annihilator of I consequently annihilates I_0. Conversely if $a_0 I_0 = 0$, lift a_0 to $a \in B$. Then $aI \subseteq B \cap K$, so $waI = 0$, and $\lambda(wa) = D_0 \lambda_0(a_0)$. These give the assertion about the value ideals $\eta_\lambda(B) = \lambda(\mathrm{Ann}_B\, I)$.

Surjectivity on powers follows by lifting products. Given $x \in I^h$ mapping to I_0^{h+1}, subtract a lift from I^{h+1} to obtain $x' \in I^h \cap K$. Then $D_0 x' = (D_0 - w)x' \in I^{h+1}$, since $D_0 - w \in I$. This proves the assertion in every degree. The generator bound over a discrete valuation ring in the proof of Theorem 50.23 gives the last inequality. □

In particular the cost of removing these prescribed finite tame inertia groups can be bounded inside the original integral coefficient algebra. One must still identify the strict classes with the relevant directions in that algebra, bound the unramified quotient at the required rational weights and treat any local type with unrestricted monodromy by a separate argument. These requirements enter the local estimate of Section 6.6.

50.10 Ramified inertia at the character conductor

Let $\mathcal{O}$ be the ring of integers of a finite extension K of $\mathbb{Q}_\ell$, where ℓ is odd. Write π for a uniformizer, $v = v_\pi$, $e_\ell = v(\ell)$, and $v(0) = +\infty$. Fix a finite order character ψ with values in $\mathcal{O}^\times$. At a prime $p \neq \ell$, denote its inertia and decomposition groups by $I_p \subset D_p$. We consider the primes occurring in the ramified local condition of the four-dimensional argument, where $\ell \mid M = |\psi(I_p)|$. Choose an inertia element τ such that $\zeta = \psi(\tau)$ generates this cyclic image, and a lift F of arithmetic Frobenius. Write $\delta = \zeta - 1$.

The four characters at the split point, in their given order, are $\alpha\Psi, \Psi, \alpha, 1$. Here $\Psi = \chi_\ell^r \psi$, $\alpha = \chi_\ell \omega^{-1} \xi$, χ_ℓ is the cyclotomic character, ω its Teichmüller lift modulo ℓ, and ξ is unramified at p. The form pairs the first coordinate with the fourth and the second with the third. Thus at τ the two Lagrangians at the split point have characters ζ and 1, each of multiplicity two. The multiplier at τ equals ζ. Write $a = \alpha(F)$ and $\beta = \Psi(F)$. The Frobenius separation in this local condition gives

$$\bar{a} \neq 1, \qquad \bar{a}\bar{\beta} \neq 1, \qquad \bar{a}\bar{\beta}^{-1} \neq 1.$$

Indeed $\bar{\alpha} = \bar{\xi}$, and the separation asserts $\bar{\xi}(F)\bar{\Psi}(F)^i \neq 1$ for all integers i.

The local component used here has a concrete meaning. A representation on it is obtained from a representation over an integral domain B over $\mathcal{O}$ that specializes to the split point over $\mathcal{O}$. The representation over B has the prescribed multiplier and is trivial on $I_p \cap \ker \psi$. A second specialization of B can take values in a ring with torsion. For the definitions in the source, see `OnComp`, `condP` and `IsPt` in `DecompSigmaOrd.lean`.

Proposition 50.26 (The inertia matrix on the component through the split point)**.** *Let U denote the image of τ. Over every coefficient ring obtained by such a specialization one has*

$$(U - I)(U - \zeta I) = 0, \qquad U\rho(g) = \rho(g)U \quad (g \in D_p).$$

After inverting δ, the operator $(U - I)/\delta$ projects onto the ζ-eigenspace along the 1-eigenspace. Both are invariant Lagrangian summands of rank two.

Proof. The character ψ identifies $I_p/(I_p \cap \ker \psi)$ with its image, so conjugating an inertia element by any element of D_p preserves its coset in this quotient. This gives the commutation relation and $U^M = I$. The map $\mathcal{O} \to B$ is injective because the split point is a retraction to $\mathcal{O}$. Over the fraction field of B, U has finite order dividing M and is diagonalizable with eigenvalues among the distinct elements $1, \zeta, \dots, \zeta^{M-1}$. Its characteristic polynomial therefore has coefficients in $\mathcal{O}$. Evaluating at the split point preserves those coefficients, so the polynomial is $(T - \zeta)^2(T - 1)^2$. Diagonalizability gives the quadratic identity over the fraction field, hence over B and every specialization. Inverting δ now gives the stated idempotent. Its two summands have rank two by the characteristic polynomial over each residue field of this localization. On the first summand the form scales under U both by ζ^2 and by ζ. On the second it scales both by 1 and by ζ. Since their differences are units there, both summands are isotropic, and nondegeneracy pairs them perfectly. □

Theorem 50.27 (The inertia restriction of a symplectic deformation)**.** *Put $W_h = \mathcal{O}/\pi^h$, $h \geq 1$,*

$$c_h = \min(h, v(\delta)), \qquad d_h = \min(c_h, v(\beta - 1)).$$

At the split point consider first-order lifts on this component to $\mathcal{O} \oplus W_h$, with $W_h^2 = 0$ and fixed multiplier. Identify inertia restrictions up to symplectic change of basis equal to the identity modulo W_h. The module generated by these restrictions has length at most $2c_h + 2d_h$. More precisely their classes inject into

$$W_h[\delta]^2 \ \oplus \ \big((W_h/\delta W_h)[\beta - 1]\big)^2,$$

where $N[u]$ denotes the kernel of multiplication by u on N.

Proof. Choose paired bases for the two Lagrangians, so the symplectic form has off-diagonal blocks $I_2, -I_2$. Write $U = (I + Y)U_0$, $\rho(F) = (I + Z)F_0$, where the subscript denotes the split point. Preservation of the multiplier makes Y, Z lie in the symplectic Lie algebra over W_h. Thus

$$U_0 = \operatorname{diag}(\zeta I_2, I_2), \qquad Y = \begin{pmatrix} A & B' \\ C' & -A^{\mathsf{T}} \end{pmatrix}, \qquad (B')^{\mathsf{T}} = B', \quad (C')^{\mathsf{T}} = C'.$$

In the linear part of $(U - I)(U - \zeta I) = 0$, the diagonal blocks give $\delta A = 0$, and the off-diagonal blocks cancel. Conjugation by $I + T$ adds $(1 - \operatorname{Ad} U_0)T$ to Y. It leaves A fixed and changes B', C' by arbitrary symmetric matrices with entries in δW_h. The class of Y with these identifications determines the whole restriction, since inertia acts through the cyclic group generated by U.

Commutation with Frobenius gives

$$(1 - \operatorname{Ad} F_0)Y = (1 - \operatorname{Ad} U_0)Z.$$

On the first Lagrangian the Frobenius matrix is $\operatorname{diag}(a\beta, \beta)$, and on its paired dual it is $\operatorname{diag}(1, a)$. Thus the two off-diagonal entries of A vanish, since $a - 1$ is a unit. On the three symmetric upper coordinates the Frobenius ratios are $a\beta$, β and β/a. On the lower coordinates they are their inverses. After passage to $W_h/\delta W_h$ the right side of the display vanishes. The

separation conditions leave only the β- and β^{-1}-coordinates, both killed by $\beta - 1$. These coordinates and the two diagonal entries of A give the asserted injection. Their respective cyclic coefficient modules have lengths d_h and c_h. □

Theorem 50.28 (The exact scalar cost of the strict local condition)**.** *Give W_h or $W_\infty = K/\mathcal{O}$ the scalar action of Ψ^{-1}. At p, consider the local classes represented by cocycles vanishing on $I_p \cap \ker \psi$. Their lengths over $\mathcal{O}$, at finite precision and with divisible coefficients, respectively, are*

$$2d_h \qquad \text{and} \qquad d_\infty = \min\bigl(v(\delta), v(\beta - 1)\bigr).$$

Consequently, imposing local triviality at any list of these primes on a global scalar group already satisfying this inertia-kernel condition decreases its length by at most the sum of the corresponding local lengths, whenever the global group has finite length.

Proof. The action is trivial on $I_p \cap \ker \psi$, so the condition descends the cocycle to $D_p/(I_p \cap \ker \psi)$. In that group the inertia quotient is central cyclic of order M. The quotient by inertia is $\widehat{\mathbb{Z}}$. The continuous map from $\widehat{\mathbb{Z}}$ sending 1 to F gives a splitting because it composes to the identity on $\widehat{\mathbb{Z}}$. Thus it suffices to specify the two values $u = z(\tau)$, $w = z(F)$. Put $s = \zeta^{-1} - 1$ and $t = \beta^{-1} - 1$. The commutation relation reads $tu = sw$. The relation $\tau^M = 1$ adds none, since $\sum_{j=0}^{M-1} \zeta^{-j} = 0$ in $\mathcal{O}$. Conversely all such pairs give continuous cocycles. On finite coefficients the affine transformations defined by the two values commute, the first has order dividing M, and the second has finite order. For $K/\mathcal{O}$ any two values lie in a common finite submodule stable under both characters. The coboundary pairs are precisely (sy, ty).

Let g generate (s, t) in $\mathcal{O}$. A change of basis in $\mathrm{GL}_2(\mathcal{O})$ sends the column (s, t) to $(g, 0)$, since $(s/g, t/g)$ is primitive. The relation $tu = sw$ says that the determinant of these two columns vanishes, a condition preserved by this change of basis. On either coefficient module W the group of classes is consequently isomorphic to

$$W/gW \;\oplus\; W[g].$$

Here $v(g) = \min(v(\delta), v(\beta - 1))$. Both summands at finite precision have length d_h. On $K/\mathcal{O}$ multiplication by the nonzero g is surjective and its kernel has length $v(g)$. Finally the quotient of the original global group by the subgroup with trivial restrictions injects into the direct sum of the computed local groups. □

Corollary 50.29 (A conductor bound uniform in weight and coefficient prime)**.** *Fix a finite order character in characteristic zero, allowing all its ℓ-adic realizations, and a finite set S of rational primes. At one weight, sum over odd coefficient primes the preceding local length bounds at $p \in S$, $p \neq \ell$, $\ell \mid |\psi(I_p)|$, counting one realization per coefficient prime. Weight the length by $(\log \ell)/e_\ell$. For scalar classes the sum is at most $|S|$ with divisible coefficients and at most $2|S|$ at any finite precision. For the symplectic inertia restrictions under the Frobenius separation above it is at most $4|S|$. Over t_0 weights the same bounds acquire a factor t_0.*

Proof. The order M of the inertia character is independent of the embedding. If M has a prime-to-ℓ factor greater than one, the reduction of ζ is nontrivial, since reduction preserves the order of roots of unity of order prime to ℓ. Thus $v(\delta) = 0$. In the remaining case $M = \ell^j$, $j \geq 1$, and $v(\delta)/e_\ell = 1/\varphi(\ell^j)$. Indeed the cyclotomic polynomial evaluated at 1 gives $\ell = \prod_{i \in (\mathbb{Z}/\ell^j)^\times} (1 - \zeta^i)$. Each factor has the same valuation as $1 - \zeta$, by the finite geometric sum for a positive representative of i and its inverse modulo ℓ^j. For each p at most one coefficient prime therefore contributes, and its normalized contribution from one copy of $\mathcal{O}/(\delta)$ is at most $\log \ell/(\ell - 1) \leq 1$. The length formulas give the bounds. □

These estimates apply to the ramified conductor terms of the specified character at the original weights. The kernel of all the calculated restrictions can still contain global strict classes. Its order enters Section 6.6 even when every contribution of this corollary is zero.

51 References

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