The generalized Riemann hypothesis for Dirichlet L-functions

Volume I · Dirichlet zeros and the arithmetic of power intervals

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Abstract

The zero criterion in this volume uses a prime sum on power intervals. We obtain its explicit formula, choose the two quadratic twists and specify the two arithmetic estimates on common number fields. We construct rational flag coordinates with integral unmarked root ratios and prescribed auxiliary characters. All their unmarked divisors can be placed above the cohomological cutoff at completely split primes. The exact quadratic-sign decomposition of the centered count at the selected word weights compares its signal part with the signed definite Gram determinant. On the polynomial-conductor intervals an explicit positive contribution from the ramified prime must be retained in the centered identity. Its size and the surviving cohomology orbits are given on the same field. On the common period fields, a single set of weights of density tending to zero contains the support of the truncated local groups at all odd primes up to the cutoff. The estimate holds uniformly for every weight limit on the interval.

1 Dirichlet GRH and its proof status

Main Theorem 1.1 (Dirichlet GRH, proof incomplete)**.** *For every Dirichlet character χ of positive modulus, every zero of $L(s,\chi)$ in the open critical strip lies on the critical line. Equivalently,*

$$L(\rho,\chi)=0, \qquad 0<\operatorname{Re}\rho<1 \quad\Longrightarrow\quad \operatorname{Re}\rho=\tfrac{1}{2}. \tag{G}$$

Theorem 1.2 reduces Dirichlet GRH to a one-sided bound for an explicit prime sum on arbitrarily distant power intervals. The unconditional argument below reaches two estimates on the same fields that are still needed to prove that bound.

Theorems 14.14 and 15.3 control exceptional twists and certain full relative orders. Theorems 15.4, 15.6, 15.7 and 9.16 bound the initial cohomology layers. Theorems 15.8 and 15.9 bound the local Frobenius contribution and the subgroup killed by the common integer of Theorem 27.22. Theorem 15.19 recovers the complementary group at any prescribed precision from ordinary ideal classes in one layer for all fields and weights. Corollary 15.20 bounds the loss at the earlier layers uniformly in that precision. Theorem 15.22 detects exceptional weights using finite ordinary class groups. Corollary 15.23 gives simultaneous vanishing of the residual groups at small coefficient primes outside one explicitly bounded set on the interval. Theorem 9.32 bounds the sum of the full strict lengths in both signs outside finite ordinary class group tests. Corollary 9.34 gives full strict vanishing at all three primes of the signed periods, for every subfield and every weight below the interval cutoff. Theorem 74.33 recovers the finite Whittaker coefficients integrally on each prescribed pair of residue flags at the two auxiliary primes. Corollary 74.34 bounds the cross terms when several output flags contribute. Theorems 74.35 and 74.37 compare the full finite pairing with its projection onto generic representations for arbitrary signed columns. Corollary 74.38 gives equality of their minor ideals at the same terminal primes as the strict local groups. Theorem 74.40 bounds the finite pairing from below by explicit alternating sums of the signed amplitudes at the prescribed marking. Corollary 74.41 gives integral minor comparisons there and equality on complete flag tables. Theorem 15.11 allows the three marked primes to be chosen with full vanishing on all descended strict groups of the stated degrees and twists. Theorem 15.12 makes the critical sign vanish at those primes on the full fields and throughout the weight cutoff. Corollary 15.15 bounds full orders outside

explicitly bounded sets of twists in all tame characters, including the decomposition invariants. Corollary 42.4 bounds the entire complementary sum at primes above $P(x)$ by a polynomial in V. The local estimate still requires a uniform bound on the residual quotients at primes at most $P(x)$.

For the periods, Theorems 101.13, 100.17, 101.15 and 101.16 describe the local sums, integral class tests and finite projections. Theorems 77.12, 74.15 and 97.4 compute signed local coefficients, finite valuations and a conjugate second moment. Theorems 74.19 and 104.10 give the norm of the split transfer and the integral content of the transferred row away from the marked residue characteristics. Theorem 59.21 evaluates the terminal move at the first Bessel position. Corollary 74.22 shows that this position on graph rows annihilates the refined columns with zero sum on each scalar orbit. Theorem 59.25 gives two explicit finite translates for which the normalized Gram forms agree through every marked exchange on those columns. Corollary 59.28 computes the coefficient ideal and rank bound for the original and transported tests at all the marked places of the same chain. Theorem 64.38 gives the exchange on the original signed auxiliary projector by rational conjugation. Theorem 64.40 proves that the resulting period vanishes on the original spherical rows if all real components are close to the identity and all new valuations are at good split places. Proposition 64.41 gives the exchange for a rational diagonal with integral unmarked root ratios. Section 6.6 still requires a comparison of the designated global Bessel pairing with the signed iterated theta column in the same normalization.

For a primitive Dirichlet character ψ, let $\mathcal{G}_\psi(x)$ denote the real weighted prime sum in Definition 1.3. Its terms come from the primitive characters inducing the powers of ψ. For the character of conductor one the sole term is the smoothed zeta prime sum.

For a fixed primitive ψ, consider the assertion

$$\begin{gathered}\text{for every } h>0 \text{ and } Y_0\ge 2 \text{ there exists } Y\ge Y_0 \text{ for which}\\ \mathcal{G}_\psi(x)\le x^{1/2+h}\qquad (Y\le x\le Y^{3/2}).\end{gathered}\tag{W}$$

Theorem 1.2 (Reduction to a prime-sum estimate)**.** *Assertion* (G) *holds for every Dirichlet character if and only if* (W) *holds for every primitive character ψ. The equivalence holds with* $3/2$ *replaced by any fixed $v>1$.*

Write $\operatorname{Re} s$ and $\operatorname{Im} s$ for the real and imaginary parts of s. A Dirichlet character modulo N is a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, extended by zero to nonunits. Its conductor is the least modulus from which it is induced. The character at that modulus is its *primitive representative*. Write $\mathbf{1}$ for the character of conductor one. For $\operatorname{Re} s>1$ the Dirichlet series and its Euler product are

$$L(s,\chi)=\sum_{n\ge1}\chi(n)n^{-s}=\prod_p(1-\chi(p)p^{-s})^{-1}.$$

We use the classical continuation and functional equation of primitive Dirichlet functions and the Hadamard factorization theorem for entire functions of order one. The classical analytic inputs and their applications are specified in Sections 2.1 and 3.1. Write γ for Euler's constant.

Definition 1.3 (The prime sum and its weights)**.** The von Mangoldt function is $\Lambda(p^j)=\log p$ for $j\ge1$ and vanishes on all other positive integers. Set $z_+=\max(z,0)$. For a primitive character and $x\ge1$, put

$$R_\chi(x)=\sum_{n\ge1}\Lambda(n)\chi(n)(x/n-1)_+,\qquad G_\chi(x)=\begin{cases}R_{\mathbf{1}}(x)-x\log x+(1+\gamma)x, & \chi=\mathbf{1},\\ R_\chi(x)+x\,L'(1,\chi)/L(1,\chi), & \chi\ne\mathbf{1}.\end{cases}\tag{1.1}$$

This sum is finite at each x. Lemma 2.2 gives $L(1,\chi)\neq 0$ in the second line. Fix a primitive character ψ of order o. Let $\mathcal{C}_\psi$ consist of the primitive representatives of $1,\psi,\dots,\psi^{o-1}$. Distinct powers give distinct primitive representatives since each is recovered by inducing its representative to the original modulus. The set contains $\mathbf{1}$ and ψ and is closed under conjugation. Define

$$a_{\mathbf{1}}=1,\qquad a_\chi=(3o+1)^{-1}\ (\chi\neq\mathbf{1}),\qquad \mathcal{G}_\psi(x)=\sum_{\chi\in\mathcal{C}_\psi}a_\chi\,\operatorname{Re}G_\chi(x).$$

The same Y works for all real x in the displayed interval. It may depend on the character and the exponent. For example, for the nonprincipal character χ_{-4} modulo four the sum in (W) is $G_{\mathbf{1}}+G_{\chi_{-4}}/7$.

Proof of Theorem 1.2. Assume (W) for all primitive characters with interval power $v>1$. If a primitive ψ has a zero at real part $\beta>1/2$ in the strip, choose $1/2<\theta<\beta$. Apply (W) with $h=\theta-1/2$ and Theorem 2.1 with $A=1$. Every zero of every member of $\mathcal{C}_\psi$ has real part at most θ, a contradiction.

For the primitive representative χ^* of conductor f, take $a\in\{0,1\}$ with $\chi^*(-1)=(-1)^a$. The classical functional equation [1, 2] is

$$\mathcal{L}(s,\chi^*)=(f/\pi)^{(s+a)/2}\Gamma((s+a)/2)L(s,\chi^*)=\omega_{\chi^*}\mathcal{L}(1-s,\overline{\chi^*}),\qquad|\omega_{\chi^*}|=1.$$

The factors preceding L are finite and nonzero in the open strip. A zero to the left of the line would give a zero to its right for the conjugate character. For a character χ mod N induced by χ^*, comparison of absolutely convergent Euler products gives, by continuation,

$$L(s,\chi)=L(s,\chi^*)\prod_{p\mid N,\ p\nmid f}(1-\chi^*(p)p^{-s}).$$

Every omitted Euler factor has nonzero value in $\operatorname{Re}s>0$, proving (G) in general.

Conversely, under (G), Theorem 3.3 gives $|G_\chi(x)|\le C_\chi\sqrt{x}$ for every primitive χ and $x\ge 2$. Its proof uses only (G) and classical properties of L-functions. In the finite sum defining $\mathcal{G}_\psi$, take $C=\sum_\chi a_\chi C_\chi$. Once $Y^h\ge C$, the bound in (W) holds for all $x\ge Y$. This gives the converse for each v. □

Corollary 1.4 (Nonvanishing and the zeta zeros)**.** *Assume* (W) *for every primitive character. Then $L(s,\chi)\neq 0$ for $\operatorname{Re}s>1/2$, apart from the principal pole at 1, and every zero of ζ other than a negative even integer has real part $1/2$.*

Proof. Theorem 1.2 settles the open strip and Lemma 2.2 treats $\operatorname{Re}s\ge 1$. For $\operatorname{Re}s\le 0$ and $s\neq 0$, use $\zeta(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$. All factors except the sine are finite and nonzero there. Its zeros in that region are the negative even integers, and $\zeta(0)=-1/2$. □

Theorem 3.7 identifies the best interval exponent unconditionally. The upper limits in Section 3 and the prime-ideal formula in Section 4 are consequences under the critical-line hypothesis. The proposed approach to (W) via a chain of theta lifts of length about $\sqrt{x}$ comes from Alpöge's *Remark* [3]. Section 6 states the arithmetic estimate that this construction would have to supply. All deductions below use mathematical arguments or the indicated classical theorems. The locations in the accompanying source identify the proposed constructions.

1.1 A guide to the twenty volumes

Sections and their subdivisions are numbered continuously in reading order across the series. The first number in a theorem or equation identifies its section. Page numbers start at one in each volume. The contents and the references in the text link to the corresponding headings and statements.

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XIX	103–107	Marked theta columns and terminal covectors	82
XX	108–113	Heights and divisors of projected theta periods	157
Total			**2139**

For the reduction to the prime sum, begin with Theorem 1.2. Section 6.6 fixes the fields and intervals for both arithmetic estimates. Theorem 15.4 controls the number of higher-order factors. Theorem 104.10 computes the ideal of tested row values away from the marked residue characteristics. Theorem 15.9 bounds the classes killed by a common integer. Corollary 27.26 compares full negative cotangent lengths with the remaining strict order. Theorem 15.8 bounds the total change on passing to ordinary unramified class groups at every odd prime and every positive weight below the cutoff. Theorems 100.17 and 101.15 give the integral class tests and the finite type projections on the original terminal tables. Theorem 77.12 evaluates the local signed coefficients. Theorems 74.15 and 97.4 compute the finite valuations and conjugate second moment for a specified algebraic column. Theorem 15.11 gives vanishing at the selected period primes in a stated range of descended groups. Theorems 74.21 and 59.25 give the kernel on the unshifted graph and compare the two Bessel positions at explicit finite translates. The outstanding steps are a uniform bound for the remaining strict orders and a comparison of the signed theta column with the designated Bessel pairing in the same normalization.

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1 Dirichlet GRH and its proof status
2 A criterion for zeros on power intervals
3 Explicit formulas and sharp exponents
4 Prime ideals in finite abelian extensions
5 Choosing two auxiliary quadratic twists
6 An arithmetic comparison sufficient for the interval bound
7 The field and the auxiliary weights
8 Arithmetic averages on the period interval
9 The finite data on the common fields
10 References

Volume II *Strict cyclotomic cohomology*

11 Strict local conditions and conjugation
12 Descent in the real cyclotomic tower
13 Uniform ranges of coefficient primes
14 Residue data and exceptional twists
15 Relative orders through the norm map
16 References

Volume III *Class groups and special values*

17 Kummer classes and real class groups
18 Special values and integral annihilators
19 Norms and cyclic character fields
20 Cyclotomic indices at finite primes
21 Positive twists and explicit strict orders
22 References

Volume IV *Marked deformations and traces*

23 Extension classes and trace families
24 Orbits in the marked deformation space
25 The ordinary locus and its tangent spaces
26 Rigidity and orthogonal monodromy
27 Trace congruences on the common interval
28 References

Volume V *Auxiliary fields and cuspidal families*

29 Auxiliary fields and cuspidal congruences
30 Ordinary curves and cuspidal components
31 Recovering classes from three-character traces
32 References

Volume VI *Integral trace quotients*

33 Integral trace quotients and root modules
34 Strict orders and descent to auxiliary fields
35 Hecke support and ordinary extension classes
36 References

Volume VII *Hecke support and symplectic deformations*

37 Hecke support at the prescribed weight
38 Symplectic deformations at strict classes
39 Galois representations and Frobenius traces
40 Ordinary symplectic equations and trace multiplicities
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43 Colliding ordinary characters
44 Exceptional weights in the cyclotomic tower
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48 Auxiliary towers and cuspidal points
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2 A criterion for zeros on power intervals

Theorem 2.1 (A one-sided criterion)**.** *Fix a primitive character ψ, $v > 1$, $0 < \theta < 1$ and $A > 0$. Suppose there are arbitrarily large Y such that*

$$\mathcal{G}_\psi(x) \le Ax^\theta \qquad (Y \le x \le Y^v).$$

Then every zero ρ in the open critical strip of every $L(s,\chi)$, $\chi \in \mathcal{C}_\psi$, satisfies $\operatorname{Re}\rho \le \theta$.

The interval can vary with Y. In logarithmic coordinates it has the form $[U, vU]$. We shall choose the order of a derivative so that the integral giving that derivative is concentrated inside this interval.

2.1 Nonvanishing on the boundary of the strip

Lemma 2.2 (Boundary nonvanishing)**.** *For every Dirichlet character $\chi \bmod N$, $L(s,\chi) \neq 0$ on $\operatorname{Re} s \ge 1$, with the principal pole at $s = 1$ excluded.*

Proof. Absolute convergence of the Euler product settles $\operatorname{Re} s > 1$. The partial sums of a nonprincipal character are bounded by periodicity, so partial summation makes $L(s,\chi)$ holomorphic on $\operatorname{Re} s > 0$. For a principal character, the finite Euler factors and the continuation of ζ show that the only singularity there is its simple pole at 1.

For $\sigma > 1$ and $t \in \mathbb{R}$, take logarithms of the Euler products in

$$\zeta(\sigma)^3 |L(\sigma + it, \chi)|^4 |L(\sigma + 2it, \chi^2)| \;\ge\; 1.$$

To see the inequality, put $z = \chi(p)^j p^{-ijt}$ at a prime power p^j. Its contribution to the logarithm is $p^{-j\sigma}(3 + 4\operatorname{Re} z + \operatorname{Re}(z^2))/j$. If $|z| = 1$, the last parenthesis equals $2(1 + \operatorname{Re} z)^2$. If $p \mid N$, it is 3. Every term is nonnegative. Suppose $L(1 + it, \chi) = 0$. If $t \neq 0$ or $\chi^2 \neq \chi_0$, the last L-factor in the display is bounded as $\sigma \downarrow 1$. The fourth power then cancels the third-order pole with a factor $O(\sigma - 1)$ to spare. This contradicts the lower bound.

The remaining case is $t = 0$ and χ real and nonprincipal. If $L(1,\chi) = 0$, the function $D(s) = \zeta(s)L(s,\chi)$ is holomorphic on $\operatorname{Re} s > 0$. Its Dirichlet coefficients $a_n = \sum_{d \mid n} \chi(d)$ are multiplicative and nonnegative. Indeed $a_{p^j} = 1 + \chi(p) + \cdots + \chi(p)^j$, and $\chi(p) \in \{-1, 0, 1\}$. For even j this sum is at least one, so $a_{m^2} \ge 1$. Taylor expansion about 2 would converge at $1/2$. Differentiate the absolutely convergent Dirichlet series at 2, and interchange the nonnegative terms of its Taylor series. This gives

$$D(1/2) = \sum_{k \ge 0} \sum_{n \ge 1} \frac{a_n}{n^2} \frac{(\frac{3}{2} \log n)^k}{k!} = \sum_{n \ge 1} \frac{a_n}{\sqrt{n}} \;\ge\; \sum_{m \ge 1} \frac{1}{m}.$$

The value on the left is finite. This contradiction proves the lemma. □

2.2 The Mellin transform

Put $g(u) = \mathcal{G}_\psi(e^u)$ for $u \ge 0$ and define

$$\Phi_\psi(s) = \int_1^\infty \mathcal{G}_\psi(x) x^{-s-1}\, dx = \int_0^\infty g(u) e^{-su}\, du \qquad (\operatorname{Re} s > 1).$$

Indeed $|R_\chi(x)| \le x \sum_{n \le x} (\log n)/n \ll x(1 + \log^2 x)$. In particular g is real and $|g(u)| \le C_\psi e^{2u}$.

Lemma 2.3 (Residues of the Mellin transform)**.** *The function Φ_ψ extends meromorphically to $\operatorname{Re} s > 0$ and is holomorphic at 1. Its poles in that half-plane are exactly the zeros of the family. They are simple, even when an L-function has a multiple zero. The residue at a real pole is positive.*

Proof. For $\operatorname{Re} s > 1$, $\int_n^\infty (x/n - 1)x^{-s-1}\,dx = n^{-s}/[s(s-1)]$. Absolute convergence and the logarithmic derivative of the Euler product show that the Mellin transform of R_χ is $-(L'/L)(s,\chi)/[s(s-1)]$. Denote the transform of G_χ by H_χ. For $\chi \neq \mathbf{1}$ add $(L'/L)(1,\chi)/(s-1)$ to this expression. For $\chi = \mathbf{1}$ add $-1/(s-1)^2 + (1+\gamma)/(s-1)$. The first correction cancels the simple pole at 1. For the second, substitute $\zeta(s) = (s-1)^{-1} + \gamma + O(s-1)$ to see both singular coefficients cancel.

We have $\overline{G_\chi(x)} = G_{\bar\chi}(x)$. Thus

$$\Phi_\psi(s) = \tfrac{1}{2} \sum_{\chi \in \mathcal{C}_\psi} a_\chi (H_\chi(s) + H_{\bar\chi}(s)).$$

The family is closed under conjugation. For any one of its zeros ρ with $\operatorname{Re}\rho > 0$, let $m_\chi(\rho)$ be its multiplicity for $L(s,\chi)$, with value zero if the function does not vanish there. Then

$$\operatorname*{Res}_{s=\rho} \Phi_\psi(s) = -\frac{1}{\rho(\rho-1)}\,\frac{1}{2} \sum_{\chi \in \mathcal{C}_\psi} a_\chi (m_\chi(\rho) + m_{\bar\chi}(\rho)). \tag{2.1}$$

The weights are strictly positive and at least one multiplicity is positive. Hence the pole survives, including when several characters vanish at the same point. The formula above exhausts the possible poles in $\operatorname{Re} s > 0$. Lemma 2.2 puts every real pole in $(0,1)$, where $-1/[\rho(\rho-1)] > 0$. □

2.3 High derivatives on a vertical line

For $\sigma > 2$ and $k \in \mathbb{Z}_{\geq 0}$, set

$$A_k(s) = \frac{(-1)^k}{k!}\Phi_\psi^{(k)}(s), \qquad K_k(u) = \frac{u^k e^{-\sigma u}}{k!}.$$

Lemma 2.4 (A derivative estimate)**.** *Under the interval hypothesis of Theorem 2.1, there exist a real $\sigma > 2$, a constant $C_* > 0$, and a sequence of integers $k \to \infty$ such that, for every $\tau \in \mathbb{R}$,*

$$|A_k(\sigma + i\tau)| + \operatorname{Re} A_k(\sigma) \leq C_*(\sigma - \theta)^{-k}. \tag{2.2}$$

Proof. Differentiation under the Laplace integral gives $A_k(\sigma + i\tau) = \int_0^\infty g(u)K_k(u)e^{-i\tau u}\,du$. Write $g_+ = \max(g, 0)$. The triangle inequality, $K_k \geq 0$, and $|g| + g = 2g_+$ yield

$$|A_k(\sigma + i\tau)| + \operatorname{Re} A_k(\sigma) \leq 2\int_0^\infty g_+(u)K_k(u)\,du. \tag{2.3}$$

Write $B = 2$. On a selected interval $U = \log Y \leq u \leq vU$, we have $g_+(u) \leq Ae^{\theta u}$. Its contribution is at most $A(\sigma - \theta)^{-k-1}$. Off the interval use $g_+(u) \leq C_\psi e^{Bu}$.

Choose $c = (1+v)/2$ and $k = \lceil c\sigma U \rceil$. For σ large the numbers $q_- = B + (c-1)\sigma$ and $q_+ = (1 - c/v)\sigma - B$ are positive. Bound the left tail by multiplying its integrand by $e^{q_-(U-u)}$, and the right tail by $e^{q_+(u-vU)}$. On the respective tails these factors are at least one. Extending each majorant to $0 \leq u < \infty$ and using $\int_0^\infty u^k e^{-au}\,du/k! = a^{-k-1}$ gives the two bounds

$$C_\psi e^{q_- U}(c\sigma)^{-k-1}, \qquad C_\psi e^{-q_+ vU}(c\sigma/v)^{-k-1}.$$

After multiplication by σ^k, their logarithms have the form $b_-U+O(1)$ and $b_+U+O(1)$, where b_- and b_+ are, respectively,

$$B+\sigma(c-1-c\log c), \qquad vB+\sigma(c-v+c\log(v/c)).$$

Both coefficients of σ in these expressions are negative. The first inequality follows from $\log c > 1-1/c$ for $c>1$, and the second from $\log(v/c) < v/c-1$. Fix σ large enough that both displayed quantities are negative. Each tail is then $O(\sigma^{-k})$ uniformly as $U\to\infty$. Since $0<\sigma-\theta<\sigma$, (2.3) gives (2.2) along a sequence of the selected intervals. The constant is independent of τ. □

2.4 Isolating a pole

Proof of Theorem 2.1. Let Z be the pole set from Lemma 2.3. Suppose $\rho_0\in Z$ has $\operatorname{Re}\rho_0>\theta$, and let σ,k be as in Lemma 2.4. Set $D_*=\sigma-\theta$ and $D=\min\{D_*,\operatorname{dist}(\sigma,Z)\}$. The disk $B(\sigma,D)$ is pole-free. Whenever a closed disk of radius r centered at z contains just one simple pole ρ in its interior and none on its boundary, Cauchy's estimate after subtracting its polar part gives

$$A_k(z)=\frac{\operatorname{Res}_{s=\rho}\Phi_\psi(s)}{(z-\rho)^{k+1}}+O(r^{-k}). \tag{2.4}$$

For a pole-free closed disk only the error term occurs. All disks used below are contained in $\operatorname{Re}s>0$.

Suppose first that $\sigma-\operatorname{Re}\rho_1<D$ for some pole ρ_1. Choose a radius strictly between these two numbers. At every height τ close enough to $\operatorname{Im}\rho_1$, the disk of that radius about $z=\sigma+i\tau$ contains ρ_1. The union of the closed disks over a short closed interval of heights is compact, so it contains finitely many poles. Two with different imaginary parts are equidistant from z at at most one height. Two distinct poles of the same height are never equidistant, since both are to the left of $\operatorname{Re}s=\sigma$. Avoiding the finitely many ties gives a unique nearest pole ρ to z, at a distance $d<D$. Choose $d<r<D$ small enough that ρ is the only pole in the closed disk about z. The closed disk of radius r about σ is pole-free. Using (2.4) at both centers in (2.2) gives

$$|\operatorname*{Res}_{s=\rho}\Phi_\psi(s)|\,d^{-k-1}-O(r^{-k})\le C_*D_*^{-k}.$$

Multiply by d^k and take the limit along the sequence. The residue would vanish, contradicting (2.1).

It remains to consider the case $\sigma-\operatorname{Re}\rho\ge D$ for every $\rho\in Z$. The putative pole ρ_0 gives $D\le\sigma-\operatorname{Re}\rho_0<D_*$. Thus $D=\operatorname{dist}(\sigma,Z)$, and the distance is attained, by local finiteness in any closed disk of radius between D and D_*. Euclidean distance is at least horizontal distance. Equality forces the nearest pole to be the unique real point $\beta=\sigma-D$. Choose $D<r<D_*$ so that it is the only pole in the closed disk of radius r. Its residue c_β is positive by Lemma 2.3. Equations (2.4) and (2.2) at $\tau=0$ give $c_\beta D^{-k-1}-O(r^{-k})\le C_*D_*^{-k}$. Multiplying by D^k and taking the same limit gives $c_\beta/D\le 0$. This last contradiction proves the theorem. □

3 Explicit formulas and sharp exponents

3.1 The explicit formula on both sides of the critical line

We obtain an explicit formula which applies to any fixed primitive character. Its coefficients are absolutely summable. This permits passage to upper limits and exhibits the role of the rightmost zeros.

Lemma 3.1 (Local zero count and logarithmic derivative)**.** *Fix a primitive character χ, including the character of conductor one. Count nontrivial zeros $\rho = \beta + i\gamma_\rho$ with multiplicity. For $|t| \geq 2$,*

$$\#\{\rho : |\gamma_\rho - t| \leq 1\} \ll_\chi \log(2 + |t|).$$

For $s = \sigma + it$ with $1/4 \leq \sigma \leq 2$, $|t| \geq 2$ and $L(s, \chi) \neq 0$, one has

$$\frac{L'}{L}(s, \chi) = \sum_{|\gamma_\rho - t| \leq 1} \frac{1}{s - \rho} + O_\chi(\log(2 + |t|)).$$

Both bounds hold prior to imposing any critical-line conclusion.

Proof. Use the classical entire completion $F_\chi(s) = \mathcal{L}(s, \chi)$ for a nonprincipal primitive character and $F_{\mathbf{1}}(s) = \frac{1}{2}s(s-1)\mathcal{L}(s, \mathbf{1})$. These are entire functions of order one. By the functional equation and Lemma 2.2, their zeros are precisely the nontrivial zeros with $0 < \operatorname{Re}\rho < 1$, and $F_\chi(0)F_\chi(1) \neq 0$. Hadamard factorization gives

$$\frac{F'_\chi}{F_\chi}(s) = b_\chi + \sum_\rho \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right), \qquad \sum_\rho |\rho|^{-2} < \infty.$$

Here b_χ is a constant and the series converges locally uniformly off the zeros. The stated continuation, order and functional equation follow from the Mellin integral of the theta series and Poisson summation [1, 2].

At $s_0 = 2 + it$, the Euler product bounds L'/L uniformly, and Stirling's formula bounds the logarithmic derivative of the gamma factor by $O_\chi(\log(2 + |t|))$. The extra logarithmic derivatives of s and $s - 1$ for ζ are bounded. Since $\sum_\rho |\operatorname{Re}(1/\rho)| \leq \sum_\rho |\rho|^{-2}$, taking real parts gives

$$\sum_\rho \frac{2 - \beta}{(2 - \beta)^2 + (t - \gamma_\rho)^2} \ll_\chi \log(2 + |t|).$$

For $|t - \gamma_\rho| \leq 1$ each summand is at least $1/5$. This proves the count. Enlarging the constant on a compact set of heights makes the count valid for all real t.

Subtract the Hadamard logarithmic derivatives at s and s_0. For the distant zeros, each difference is bounded by $2|t - \gamma_\rho|^{-2}$. Group the ordinates by $j < |t - \gamma_\rho| \leq j + 1$, $j \geq 1$. The first estimate gives a total

$$\ll_\chi \sum_{j \geq 1} \frac{\log(3 + |t| + j)}{j^2} \ll_\chi \log(2 + |t|).$$

For the nearby zeros the terms $1/(s_0 - \rho)$ are bounded individually by one. Removing the gamma and, for ζ, the rational factors again costs $O_\chi(\log(2 + |t|))$, uniformly on the stated strip. This proves the second estimate. □

Proposition 3.2 (A formula without a critical-line hypothesis)**.** *For a fixed primitive χ, let $\lambda = \operatorname{ord}_{s=0} L(s, \chi)$ and $d_0 = \lim_{s \to 0}\{L'(s, \chi)/L(s, \chi) - \lambda/s\}$. For $x \geq 2$, with every nontrivial zero repeated according to multiplicity,*

$$G_\chi(x) = \sum_\rho \frac{x^\rho}{\rho(1 - \rho)} + \lambda \log x + d_0 + \lambda + O_\chi(x^{-1/2}). \tag{3.1}$$

In particular $G_\chi(x) = o_\chi(x)$.

Proof. The inverse Mellin kernel for $(v-1)_+$ is $1/[s(s-1)]$ on $\operatorname{Re} s = 2$. This follows by integrating the ramp as in Lemma 2.3, or by closing the inverse contour and taking its residues at zero and one. At $v = 1$ continuity gives zero. Absolute convergence of the prime series and integrability of the kernel now give

$$R_\chi(x) = \frac{1}{2\pi i}\int_{\operatorname{Re} s=2} -\frac{L'}{L}(s,\chi)\frac{x^s}{s(s-1)}\,ds.$$

Move the line to $\operatorname{Re} s = -1/2$. On this line $L'/L \ll_\chi \log(2+|\operatorname{Im} s|)$. Indeed, logarithmically differentiating the functional equation expresses it through L'/L on $\operatorname{Re} s = 3/2$ and the two gamma logarithmic derivatives, whose arguments avoid the gamma poles. Lemma 3.1 allows us to choose heights in each of $[n, n+1]$ and $[-n-1, -n]$ at distance at least $c_\chi/\log n$ from all zero ordinates by deleting short intervals around the $O_\chi(\log n)$ zeros within distance one of these intervals. At these heights $L'/L = O_\chi(\log^2 n)$ uniformly for $-1/2 \le \operatorname{Re} s \le 2$. The subtraction in the proof of Lemma 3.1 works on this wider horizontal segment, where the distant-zero differences are $O(|t - \operatorname{Im}\rho|^{-2})$ and the gamma terms satisfy Stirling's bound. Consequently the two horizontal integrals tend to zero for each fixed x. The new vertical integral is $O_\chi(x^{-1/2})$ for all $x \ge 2$.

The residue at one is $-xD_\chi$ for nonprincipal χ, with $D_\chi = L'(1,\chi)/L(1,\chi)$. For the principal character the Laurent expansion at the simple pole of ζ gives $x\log x - (1+\gamma)x$. Each is precisely the term removed from R_χ in (1.1). At zero the factor preceding x^s is $\lambda/s^2 + (d_0+\lambda)/s + O(1)$, giving the two elementary terms in (3.1). The functional equation and boundary nonvanishing exclude other zeros with $-1/2 \le \operatorname{Re} s \le 0$. Every zero in the open strip contributes $x^\rho/[\rho(1-\rho)]$ per unit of multiplicity. The local zero count implies $\sum_\rho |\rho(1-\rho)|^{-1} < \infty$. This justifies the limit along the chosen contours. After division by x, each zero term tends to zero since $\operatorname{Re}\rho < 1$, and its absolute value is bounded by this same summable sequence. Dominated convergence proves the last assertion. □

Theorem 3.3 (Explicit formula for a primitive character)**.** *Assume all nontrivial zeros of the fixed primitive χ lie on the critical line. The choice $\chi = \mathbf{1}$ is allowed. List its nontrivial zeros with multiplicity as $\rho = 1/2 + it_\rho$. For every real $x \ge 2$,*

$$G_\chi(x) = \sqrt{x}\sum_\rho \frac{e^{it_\rho \log x}}{1/4 + t_\rho^2} + O_\chi(x^{1/4}). \tag{3.2}$$

After division by $\sqrt{x}$, the series converges absolutely and uniformly. In particular, $|G_\chi(x)| \ll_\chi \sqrt{x}$ for each fixed primitive character.

Proof. Put $\rho = 1/2 + it_\rho$ in (3.1). Then $\rho(1-\rho) = 1/4 + t_\rho^2$, and the elementary terms are $O_\chi(1+\log x)$, which is $O_\chi(x^{1/4})$. The local zero count makes the displayed series absolutely convergent after division by $\sqrt{x}$, uniformly for $x \ge 2$. Formula (3.1) also gives its secondary term $\lambda\log x + d_0 + \lambda$ with remainder $O_\chi(x^{-1/2})$. □

3.2 The limiting upper value

Proposition 3.4 (The upper limit for a primitive character)**.** *Assume the critical-line condition for the fixed primitive character under consideration. The real part of the smoothed sum has the exact upper limit*

$$\limsup_{x\to\infty} \frac{\operatorname{Re} G_\chi(x)}{\sqrt{x}} = \sum_\rho \frac{1}{1/4 + t_\rho^2}. \tag{3.3}$$

Proof. The upper bound follows from (3.2) by taking absolute values term by term. For the reverse bound, fix $\delta > 0$. Keep finitely many zeros so that the discarded weights sum to less than δ. For the retained ordinates $t_1, \dots, t_b$, consider the sequence on a compact torus

$$z_n = (e^{i2^n t_1}, \dots, e^{i2^n t_b}).$$

A subsequence z_{n_j} converges, so the coordinatewise quotients of consecutive members tend to the identity. Put $u_j = 2^{n_{j+1}} - 2^{n_j}$. Then $u_j \to \infty$ and $e^{iu_j t_k} \to 1$ for all retained ordinates. Substitute $x_j = e^{u_j}$ in (3.2). Each retained term, after normalization, tends to its positive weight. The real part of the tail is at least $-\delta$, and $O_\chi(x_j^{-1/4})$ tends to zero. The limsup is therefore at least the total weight minus 2δ. Letting $\delta \downarrow 0$ proves equality.

□

The constant for the trivial character. Assume RH for ζ. The sum on page 3 of the *Remark* is $G_{\mathbf{1}}(x)$. In this case the right side of (3.3) can be evaluated. Write

$$\xi(s) = \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \xi(s) = \xi(1-s).$$

Take the logarithmic derivative of the Hadamard product and subtract its value at 0 from its value at 1. The exponential prefactor and the regularizing constant in each summand cancel. Thus

$$\sum_\rho \frac{1}{\rho(1-\rho)} = \frac{\xi'}{\xi}(1) - \frac{\xi'}{\xi}(0) = 2\frac{\xi'}{\xi}(1).$$

Using $(s-1)\zeta(s) = 1 + \gamma(s-1) + O((s-1)^2)$ and $\Gamma'(1/2)/\Gamma(1/2) = -\gamma - 2\log 2$ gives $2\xi'(1)/\xi(1) = 2 + \gamma - \log(4\pi)$. On the critical line, $\rho(1-\rho) = 1/4 + t_\rho^2$. We obtain

$$\limsup_{x\to\infty} \frac{\sum_{n\ge 1} \Lambda(n)(x/n-1)_+ - x\log x + (1+\gamma)x}{\sqrt{x}} = 2 + \gamma - \log(4\pi) \simeq 0.0461914.$$

The constant is strictly positive. It is the sum of the positive weights of all nontrivial zeros in the explicit formula. Hence $G_{\mathbf{1}}(x)$ cannot be $O(x^\alpha)$ for any $\alpha < 1/2$. Product formula for the completed zeta function

3.3 Evaluation of the constant for a primitive character

Let χ be nonprincipal and primitive of conductor q, and let $a \in \{0, 1\}$ be determined by $\chi(-1) = (-1)^a$.

Proposition 3.5 (The constant in terms of the conductor)**.** *Assuming the critical-line condition for χ, the sum S_χ on the right of* (3.3) *is*

$$S_\chi = \log(q/\pi) + \frac{\Gamma'}{\Gamma}\left(\frac{1+a}{2}\right) + 2\,\mathrm{Re}\,\frac{L'}{L}(1,\chi). \tag{3.4}$$

This expression also evaluates $\limsup_{x\to\infty} |G_\chi(x)|/\sqrt{x}$*. Indeed the limsup of the real part already attains the termwise upper bound on the absolute value.*

Proof. The completed function

$$\Lambda_\chi(s) = (q/\pi)^{(s+a)/2}\Gamma((s+a)/2)L(s,\chi)$$

is entire of order one. Its zeros are the nontrivial zeros of $L(s,\chi)$. Its values at 0 and 1 are nonzero, by nonvanishing at 1 and the functional equation. Subtract the logarithmic derivatives

of its Hadamard product at these two points. The constant and the genus-one regularizing terms cancel, giving

$$\frac{\Lambda_\chi'}{\Lambda_\chi}(1) - \frac{\Lambda_\chi'}{\Lambda_\chi}(0) = \sum_\rho \left(\frac{1}{1-\rho} + \frac{1}{\rho}\right) = \sum_\rho \frac{1}{\rho(1-\rho)}.$$

The last sum converges absolutely by the local zero count. The functional equation gives $\Lambda_\chi'/\Lambda_\chi(0) = -\overline{\Lambda_\chi'/\Lambda_\chi(1)}$. Taking the logarithmic derivative of the displayed completion at 1 gives (3.4). On the critical line $\rho(1-\rho) = 1/4 + (\operatorname{Im}\rho)^2$, so this sum is exactly S_χ. □

Assume the critical-line condition for every member of $\mathcal{C}_\psi$. For this family the calculation gives the sharp formula

$$\limsup_{x\to\infty} \frac{\mathcal{G}_\psi(x)}{\sqrt{x}} = \sum_{\chi\in\mathcal{C}_\psi} a_\chi S_\chi > 0, \qquad S_{\mathbf{1}} = 2 + \gamma - \log(4\pi). \tag{3.5}$$

The upper bound follows from the individual formulas. For the lower bound keep finitely many zeros in each of the finitely many series so the total discarded weight is less than δ. Apply the compact-torus argument of Section 3.2 to the union of their ordinates. Along the resulting common sequence x_j all retained phases tend to 1. The total loss is at most 2δ in the limit. The same sequence approaches the upper limit for every member of this finite family.

3.4 Simultaneous recurrence near the upper value

The exact upper limits of Sections 3.2 and 3.3 can be approached simultaneously and with bounded gaps in the logarithm of x.

Corollary 3.6 (Simultaneous near-attainment)**.** *Let $\mathcal{F}$ be a fixed finite nonempty set of primitive Dirichlet characters, all of whose nontrivial zeros lie on the critical line. Write S_χ for the constant in (3.3). For every $\delta > 0$ there exist $C_\delta > 1$ and Y_δ such that every $[Y, C_\delta Y]$, $Y \ge Y_\delta$, contains a single x satisfying*

$$\frac{\operatorname{Re} G_\chi(x)}{\sqrt{x}} > S_\chi - \delta \qquad (\chi \in \mathcal{F}).$$

In particular, for each fixed $v > 1$,

$$\lim_{Y\to\infty} \max_{Y\le x\le Y^v} \frac{\mathcal{G}_\psi(x)}{\sqrt{x}} = \sum_{\chi\in\mathcal{C}_\psi} a_\chi S_\chi.$$

Proof. Use the absolutely convergent series (3.2). For each χ discard a tail of total weight below $\delta/8$. Collect all retained ordinates in a finite list $t_1, \dots, t_b$. Let $z(u) = (e^{it_1 u}, \dots, e^{it_b u})$ and $K = \overline{z(\mathbb{R})}$. This is a compact subgroup of the torus. There is a symmetric open neighborhood $V \subset K$ of the identity on which the retained real sums differ from their values at the identity by less than $\delta/4$, simultaneously for the finite family.

The sets $z(u)V$, $u \in \mathbb{R}$, cover K by density. Choose a finite subcover with parameters $u_1, \dots, u_m$, and put $a = \min u_j$, $b' = \max u_j$. For any U, choose j such that $z(U+b') \in z(u_j)V$. Then

$$U \le U + b' - u_j \le U + b' - a, \qquad z(U + b' - u_j) \in V.$$

Take $U = \log Y$, $x = \exp(U + b' - u_j)$, $C_\delta = \exp(1 + b' - a)$. At this x each tail costs at most twice its total weight in a comparison with S_χ. By taking Y_δ large, all the normalized $O_\chi(x^{-1/4})$ remainders are less than $\delta/4$. Adding the three errors proves the first assertion. For the second, use $\mathcal{F} = \mathcal{C}_\psi$ and $C_\delta Y \le Y^v$ for all sufficiently large Y. The termwise upper bound in (3.2) is uniform for $x \ge Y$. Let $\delta \downarrow 0$ in the lower bound and combine it with that upper bound. □

The compact-group argument also covers coincident ordinates and arbitrary rational relations among them.

3.5 The least admissible interval exponent

The interval criterion also determines the endpoint exponent without assuming any location for the zeros. Let Z_ψ be the union of the nontrivial zeros of the family and $b = \sup_{\rho\in Z_\psi} \operatorname{Re}\rho$. This set contains a zero of ζ. Indeed, if the entire completion ξ had no zeros, its order and Hadamard factorization would give $\xi(s) = e^{A+Bs}$. The symmetry $\xi(s) = \xi(1-s)$ forces $B = 0$, contrary to its growth on the positive real axis by Stirling's formula. It is symmetric about the critical line by the functional equation, so $1/2 \le b \le 1$. For fixed $v > 1$, call $a \in \mathbb{R}$ admissible if for some $C > 0$ there are arbitrarily large Y with $\mathcal{G}_\psi(x) \le Cx^a$ throughout $[Y, Y^v]$.

Theorem 3.7 (The optimal exponent and the boundary zeros)**.** *The least admissible exponent is b, and $|\mathcal{G}_\psi(x)| \ll_\psi x^b$ for all $x \ge 2$. If $a < b$ and $C > 0$, every sufficiently distant $[Y, Y^v]$ contains an x with $\mathcal{G}_\psi(x) > Cx^a$. At the endpoint, $\mathcal{G}_\psi(x) = o(x^b)$ holds if and only if the family has no zero of real part b.*

Proof. In (3.1) every term x^ρ has absolute value at most x^b and the coefficients are absolutely summable. The elementary terms are $O_\chi(1+\log x)$. Summing over the fixed family proves the global upper bound. If $a < b$ were admissible, take $\max(0,a) < \theta < b$. Theorem 2.1 would put every real part at most θ, contradicting the supremum. This also proves the assertion about every sufficiently distant interval by negating the quantifiers in admissibility.

Divide (3.1) by x^b. Dominated convergence makes the terms with $\operatorname{Re}\rho < b$ tend to zero in sum, since $|x^{\rho-b}| \le 1$. Thus the normalized family sum equals, up to $o(1)$,

$$\operatorname{Re} \sum_{\chi\in\mathcal{C}_\psi} a_\chi \sum_{\substack{L(\rho,\chi)=0\\ \operatorname{Re}\rho=b\\ 0<\operatorname{Re}\rho<1}} \frac{e^{i\operatorname{Im}\rho\log x}}{\rho(1-\rho)}.$$

An empty sum gives the asserted decay. If it is nonempty then $0 < b < 1$. Every coefficient has positive real part because $\operatorname{Re}(\rho(1-\rho)) = b(1-b) + (\operatorname{Im}\rho)^2 > 0$. The sum c of their weighted real parts is strictly positive. Retain finitely many terms so the absolute tail is less than $c/4$, and use the compact-torus argument in Section 3.2 to make their phases tend simultaneously to one along $x \to \infty$. The resulting limsup is at least $c/2 > 0$. This proves the endpoint assertion. □

For characters satisfying (W), Theorem 2.1 and symmetry give $b = 1/2$. On that line the residues are positive real numbers, and (3.5) evaluates the exact upper limit.

4 Prime ideals in finite abelian extensions

4.1 Factorization into Dirichlet functions

Corollary 4.1 (Abelian Dedekind zeta functions)**.** *Let $K/\mathbb{Q}$ be a finite abelian extension. The Dirichlet result implies that every zero of $\zeta_K(s)$ in $0 < \operatorname{Re} s < 1$ has real part $1/2$.*

Proof. By the Kronecker–Weber theorem, K is contained in some $\mathbb{Q}(\zeta_M)$. Set $G = \operatorname{Gal}(K/\mathbb{Q})$. Pulling each $\xi \in \widehat{G}$ back through the cyclotomic quotient gives a Dirichlet character. Write χ_ξ^* for its primitive representative. We will prove the factorization by comparing the Euler factors at each rational prime.

$$\zeta_K(s) = \prod_{\xi\in\widehat{G}} L(s, \chi_\xi^*). \tag{4.1}$$

Fix p. Let $I \subset G$ be the inertia group, $e = |I|$, $F \in G/I$ the Frobenius element, and $f = \operatorname{ord}(F)$. There are $|G|/(ef)$ prime ideals above p, each with residue field of size p^f. With $z = p^{-s}$, the Euler factor on the left is $(1 - z^f)^{-|G|/(ef)}$. On the right, if $\xi|_I \neq 1$, the character is ramified at p, its primitive conductor is divisible by p, and its Euler factor is 1. The remaining characters run through $\widehat{G/I}$. Their values at F run through all f-th roots of unity, each $|G|/(ef)$ times, since every character of a subgroup of a finite abelian group extends to the group. Hence the local product on the right is

$$\left(\prod_{\omega^f=1}(1-\omega z)\right)^{-|G|/(ef)} = (1-z^f)^{-|G|/(ef)}.$$

This calculation covers ramified primes as well. It proves (4.1) on $\operatorname{Re} s > 1$, hence everywhere by meromorphic continuation. All factors are holomorphic in the open strip. The multiplicity of a zero of their product is the sum of its multiplicities in the factors. Under (G), this proves the asserted location of the zeros. Conversely, by taking cyclotomic fields we obtain every primitive Dirichlet character in some factorization. Thus the assertion for finite abelian extensions also implies Dirichlet GRH.

□

For $K = \mathbb{Q}(i)$ the factorization is $\zeta_K(s) = \zeta(s)L(s, \chi_{-4})$. The same two L-functions appeared in the positively weighted sum of Section 1. The factorization into one-dimensional characters in (4.1) relies on the abelian hypothesis. The *Remark* describes an extension to general number fields under the name *Extended RH* as an expectation. The prime-ideal formula below uses the abelian factorization.

4.2 The sharp prime-ideal error

The factorization in Section 4.1 gives an exact upper limit for a prime-ideal counting error. Let $K/\mathbb{Q}$ be finite abelian, of signature (r_1, r_2) and absolute discriminant D_K. Its Euler–Kronecker constant is

$$\gamma_K = \lim_{s\to 1}\left(\frac{\zeta_K'}{\zeta_K}(s) + \frac{1}{s-1}\right).$$

For a nonzero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, let $N\mathfrak{p} = |\mathcal{O}_K/\mathfrak{p}|$, and put

$$G_K(x) = \sum_{\mathfrak{p}}\sum_{m\geq 1}\log N\mathfrak{p}\left(\frac{x}{(N\mathfrak{p})^m} - 1\right)_+ - x\log x + (1+\gamma_K)x.$$

Corollary 4.2 (The exact prime-ideal constant)**.** *Assume the critical-line condition for every primitive character in (4.1). For every fixed $\nu > 1$, the following limits have the same value $S_K > 0$, given by*

$$\limsup_{x\to\infty}\frac{G_K(x)}{\sqrt{x}} = \lim_{Y\to\infty}\max_{Y\leq x\leq Y^\nu}\frac{G_K(x)}{\sqrt{x}} = \sum_{\zeta_K(\rho)=0,\ 0<\operatorname{Re}\rho<1}\frac{1}{1/4+(\operatorname{Im}\rho)^2},$$

where zeros are counted with multiplicity. In field invariants this value is

$$S_K = 2 + 2\gamma_K + \log D_K - r_1(\gamma + \log(4\pi)) - 2r_2(\gamma + \log(2\pi)).$$

Proof. Write $\mathcal{F} = \{\chi_\xi^* : \xi \in \widehat{\operatorname{Gal}(K/\mathbb{Q})}\}$. Logarithmically differentiate (4.1) on $\operatorname{Re} s > 1$. Comparing the absolutely convergent prime-power coefficients gives the uncorrected prime-ideal sum as $\sum_{\chi\in\mathcal{F}} R_\chi(x)$. Comparing the constant terms at the pole gives $\gamma_K = \gamma +$

$\sum_{\chi\neq\mathbf{1}} L'(1,\chi)/L(1,\chi)$, with the sum over $\mathcal{F}$. Thus $G_K(x)=\sum_{\chi\in\mathcal{F}} G_\chi(x)$. This sum is real since $\mathcal{F}$ is closed under conjugation. Corollary 3.6 gives simultaneous lower bounds for the real parts on every sufficiently distant $[Y, C_\delta Y]$. The termwise upper bound in (3.2) is uniform for $x\geq Y$. Since $C_\delta Y\leq Y^\nu$ eventually, both limits equal $\sum_{\chi\in\mathcal{F}} S_\chi$. The factorization adds zero multiplicities in the strip, and the zeta factor gives strict positivity.

For the second formula let $n=r_1+2r_2$. There are r_2 odd and r_1+r_2 even characters. Indeed if complex conjugation acts nontrivially then K is totally imaginary and exactly half its characters are odd. The conductor–discriminant formula gives $\prod_{\chi\in\mathcal{F}}\operatorname{cond}\chi=D_K$ [4, Ch. V, Thm. 3.27]. In the sum of (3.4), give the principal character its value $2+\gamma-\log(4\pi)$ from Section 3.2. Use $(\Gamma'/\Gamma)(1)=-\gamma$, $(\Gamma'/\Gamma)(1/2)=-\gamma-2\log 2$ and the identity for γ_K above. Collecting the terms gives $2+2\gamma_K+\log D_K-n(\gamma+\log\pi)-2(r_1+r_2)\log 2$, which is the stated value. □

For $K=\mathbb{Q}(i)$, writing $\beta(s)=L(s,\chi_{-4})$, this specializes to $S_K=2-2\log\pi+2\beta'(1)/\beta(1)$. This is the sharp upper limit for the smoothed prime-ideal error of the Gaussian integers.

5 Choosing two auxiliary quadratic twists

The character average in Section 6 contains the original powers of ψ and additional characters. We prove a selection theorem for the additional characters of the form $\xi(\cdot)(\cdot/q)^i(\cdot/d)^j$ with $i+j>0$. Throughout this section F is a fixed positive integer, $K\geq 1$, $0<\varepsilon<1/52$, and Y tends to infinity. Constants and thresholds may depend on these three fixed parameters. Put

$$z=Y^{\varepsilon/5},\quad z'=z(\log Y)^2,\quad U_1=e^z,\quad U_2=e^{z'},\quad Q_1=8FU_1,\quad Q_2=8FU_1U_2.$$

Theorem 5.1 (Simultaneous auxiliary twists)**.** *For all sufficiently large Y there are primes $\sqrt{U_1}<q\leq U_1$, $\sqrt{U_2}<d\leq U_2$ with*

$$q\equiv 3 \pmod 8,\qquad d\equiv 7 \pmod{16},\qquad d\equiv -1 \pmod p,$$

where the last congruence holds for $p=q$ and for every odd prime $p\leq z$. For every primitive ξ of conductor dividing $8F$ and $i,j\in\{0,1\}$, $i+j>0$, the primitive character $\alpha=\xi(\cdot)(\cdot/q)^i(\cdot/d)^j$ satisfies, simultaneously for $Y\leq y\leq Y^K$,

$$|G_\alpha(y)|\leq y^{1/2+\varepsilon/2},\qquad |D_\alpha|\leq 3\log y,\quad D_\alpha=L'(1,\alpha)/L(1,\alpha). \tag{5.1}$$

Here an exponent zero omits that factor, including at multiples of its conductor. These functions are zero-free in $\operatorname{Re}s\geq 49/50$, $|\operatorname{Im}s|\leq Q_1$ for $j=0$, and in the same strip up to height Q_2 for $j=1$.

5.1 Moments of a finite prime sum

Put $Z=Y^{400K}$. For coefficients $|c_n|\leq 1$, a prime ℓ, and $y\in[Y,Y^K]$, define

$$S_{\ell,c}(y)=\sum_{n\leq Z}\Lambda(n)c_n\min(1,y/n)(n/\ell).$$

The coefficients may depend on Y and on choices already fixed when ℓ is being selected.

Lemma 5.2 (Counting large twisted sums)**.** *Let $U_1\leq U\leq U_2$ and let M be a multiple of 8. Assume each prime divisor of M at most Z is at most z. For each unit residue $a \bmod M$, every $\tau>\varepsilon/10$, and fixed c as above, the number of primes $\sqrt{U}<\ell\leq U$, $\ell\equiv a \bmod M$ such that*

$$|S_{\ell,c}(y)|>y^{1/2+\tau}/2\quad \text{for some } Y\leq y\leq Y^K$$

is $O_{K,\varepsilon,\tau}(U^{1-\eta})$ with $\eta=\eta(K,\varepsilon,\tau)>0$. The constant and η are independent of M,a,c,U,Y.

Proof. For every $t \ge 2$ the total mass $\sum_{p^j \le t,\ j \ge 2} \log p$ is $O(\sqrt{t}\log^2(2t))$. Indeed $j \le \log_2 t$ and each j contributes at most $t^{1/j}\log t$. For $p^j > y$ split into intervals $(2^k y, 2^{k+1}y]$ and use the factor $y/p^j \le 2^{-k}$. The resulting geometric sum shows that all $j \ge 2$ contribute $O(\sqrt{y}\log^2(2Z))$ to S. Primes dividing M, including 2, contribute at most $z\log(2z)$ to the $j = 1$ terms. Both bounds together are $o(y^{1/2+\tau})$, uniformly on the interval.

It remains to bound the sum over odd $p \le Z$ coprime to M. Fix y. By quadratic reciprocity the sum at a prime $\ell \equiv a \bmod M$ equals $B(\ell)$, where for every positive integer $m \equiv a \bmod M$ we put

$$B(m) = \sum_{p \le Z,\ p \nmid M} b_p(m/p), \qquad b_p = (-1)^{((p-1)/2)((a-1)/2)} c_p \log p \, \min(1, y/p).$$

Use an odd representative for a. Since $4 \mid M$, the sign is well-defined, and $\sqrt{U} > Z$ for large Y. Write $V = \sum_p |b_p|^2$. The trivial count of integers up to y and the bound $\sum_{n>y} n^{-2} \ll 1/y$ give $V \ll y\log^2(2Z)$. Also $\sum_p |b_p| \le Z\log(2Z)$.

For each integer $r \ge 1$ we claim

$$\sum_{\substack{m \le U \\ m \equiv a \bmod M}} |B(m)|^{2r} \le (U/M+1)(2rV)^r + Z^{6r} \tag{5.2}$$

for large Z. Expand $B(m)^r \overline{B(m)}^r$. If the product of the $2r$ primes in a tuple is a square, equal primes can be paired. Summing the absolute weights for a fixed pairing gives V^r. There are at most $(2r)^r$ pairings, and the absolute product of symbols is at most one. For a nonsquare tuple, take the modulus D equal to the product of its distinct primes. In $m = a + Mk$, every full period of D successive k's gives all residues modulo D. The sum on such a period vanishes by the Chinese remainder theorem, because some prime has odd multiplicity. The last incomplete period contributes at most $D \le Z^{2r}$. Multiplication by $(\sum |b_p|)^{2r} \le Z^{4r}$ proves the claim. The bound also holds when the progression has fewer than D terms.

Take $r = \lfloor \log U/(20\log Z) \rfloor$, which tends to infinity. Then $Z^{6r} \le U^{3/10}$. For $\tau > \varepsilon/10$ choose $0 < \lambda < 2\tau - \varepsilon/5$. Since $r \ll_K Y^{\varepsilon/5}\log Y$, for all sufficiently large Y

$$\frac{128rV}{y^{1+2\tau}} \le Y^{-\lambda}.$$

By (5.2), the number of indices with $|B(m)| > y^{1/2+\tau}/8$ is at most $(U+1)Y^{-\lambda r} + U^{3/10}$. Since $r \ge \log U/(40\log Z)$ and $\log Z = 400K\log Y$, the first term is $O(U^{1-\lambda/(16000K)})$. Decrease the saving exponent if needed. The discarded prime powers and the primes dividing M cost less than $y^{1/2+\tau}/8$ for large Y. At a fixed y the same exceptional count therefore suffices for $|S_{\ell,c}(y)| > y^{1/2+\tau}/4$.

Test all integers in $[Y, Y^K]$ and its two endpoints. There are $O(Y^K + 1) = U^{o(1)}$ tests. On each intervening segment $S_{\ell,c}$ is affine, so its norm is bounded by the greater endpoint norm. The ratio of the larger endpoint to any point in the segment is at most $1 + 1/Y$, whose power $1/2 + \tau$ is eventually less than two. A union bound, reducing the saving exponent once more, proves the result. □

5.2 Prime counts and zero-free rectangles

We use two classical prime and zero distribution theorems here. Their uniformity matters because the second progression will depend on the first prime.

Lemma 5.3 (Size of the progression and the exceptional set)**.** *If $M \ge 2$, $(a, M) = 1$ and $\log M = o(\log U)$, then the primes $\sqrt{U} < \ell \le U$ congruent to $a \bmod M$ are $U^{1-o(1)}$ in number,*

uniformly when the ratio $\log M/\log U$ *tends uniformly to zero. Among all primitive characters of conductor at most* Q*, at most* $O(Q^{1/3})$ *have a zero in* $\operatorname{Re} s \geq 49/50,\ |\operatorname{Im} s| \leq Q$*, for* $Q \geq 3$*.*

Proof. For the first bound use Thorner and Zaman's prime number theorem in progressions [5, Cor. 1.4 and (1.2), (1.7)]. For $U \geq M^{12}$ it gives a main term $\lambda U/\varphi(M)$ for $\sum_{p\leq U,\ p\equiv a \bmod M} \log p$. The relative error tends to zero when $\log M = o(\log U)$. The factor is 1 in the absence of an exceptional zero. In the other case their lower bounds give $\lambda \gg \min(1, (1-\beta)\log U)$ and $1-\beta \gg M^{-1}$, where β denotes the exceptional zero. In either case $\lambda/\varphi(M) \gg M^{-2}$. Thus the weighted count is at least a constant times $U/M^2 = U^{1-o(1)}$. Primes at most $\sqrt{U}$ contribute at most $\sqrt{U}\log U$. Each remaining logarithm is at most $\log U$. This proves the required lower bound, and U gives an upper bound.

For the second bound let $N_\chi(\sigma, T)$ count zeros with real part at least σ and ordinate of absolute value at most T. The classical density estimate of Montgomery, at fixed $1/2 < \sigma < 1$, gives, for every $\nu > 0$,

$$\sum_{f\leq Q} \sideset{}{^*}\sum_{\chi \bmod f} N_\chi(\sigma, T) \ll_{\sigma,\nu} (Q^2T)^{3(1-\sigma)/(2-\sigma)+\nu} \qquad (Q, T \geq 2).$$

The star restricts the sum to primitive characters. One can absorb the logarithmic factor in the published estimate into ν [6, Thm. 1]. At $\sigma = 49/50$, $T = Q$, the exponent of Q before this loss is $3/17$. Choose ν with $3/17 + 3\nu < 1/3$. Each exceptional character contributes at least one zero. □

5.3 A smoothed formula for the logarithmic derivative at one

Set $w(v) = 2(1-v)_+ - (1-2v)_+$ for $v > 0$. It equals one up to $1/2$, lies in $[0,1]$, and vanishes from 1 onwards.

Lemma 5.4 (Approximation at one)**.** *Let* α *be nonprincipal and primitive of conductor at most* Q *with no zero in the rectangle of Lemma 5.3. For* $X \geq e^2,\ Q \geq 8$*,*

$$\left| D_\alpha + \sum_{n\leq X} \frac{\Lambda(n)\alpha(n)}{n} w(n/X) \right| \ll X^{-1/100}\log(2Q) + \frac{\log(2X)+\log(2Q)}{Q}.$$

The implied constant is absolute.

Proof. We first give the conductor dependence in the local logarithmic derivative estimate. Let F_α denote the entire completion used in Lemma 3.1, and f its conductor. For its Hadamard constant b_α one has $\operatorname{Re} b_\alpha + \sum_\rho \operatorname{Re}(1/\rho) = 0$. Indeed the functional equation gives $F_\alpha(s) = \omega_\alpha \overline{F_\alpha(1-\bar{s})}$, so its logarithmic derivative has real part zero at any nonzero value on the critical line. At such a point s, the real parts of $1/(s-\rho)$ cancel in pairs under $\rho \mapsto 1-\bar{\rho}$, and their sum converges absolutely. Taking real parts of the Hadamard formula proves the identity. Consequently at $s_0 = 2 + it$,

$$\sum_\rho \operatorname{Re}\frac{1}{s_0-\rho} = \operatorname{Re}\frac{F_\alpha'}{F_\alpha}(s_0) \ll \log(2f(2+|t|)).$$

Here the Euler product bounds L'/L by an absolute constant and the completion contributes $\frac{1}{2}\log(f/\pi)$ and a digamma term. The latter is $O(\log(2+|t|))$ by Stirling's formula at large height and continuity at bounded height. Repeating the subtraction at s_0 in Lemma 3.1 gives the same local formula with error $O(\log(2f(2+|t|)))$, uniformly for $99/100 \leq \operatorname{Re} s \leq 3/2$. The bound holds at small heights too, since the gamma arguments in this strip stay away from

poles. Each zero with $|\operatorname{Im}\rho - t| \le 1$ is at horizontal distance at least $1/100$ for $|t| \le Q/2$ by the rectangle hypothesis. Thus $L'/L \ll \log(2Q)$ on this whole smaller rectangle.

The Mellin transform of w is $W(s) = (2 - 2^{-s})/[s(s+1)]$ for $\operatorname{Re} s > 0$, by integrating each linear ramp. It has residue 1 at zero and decays quadratically on vertical lines away from its poles. With $c = 1/\log X$ and by absolute convergence, Mellin inversion gives the finite sum in the lemma as

$$\frac{1}{2\pi i}\int_{\operatorname{Re} s=c} -\frac{L'}{L}(1+s,\alpha)X^s W(s)\,ds,$$

where the line is traversed upwards. On this line $|L'/L| \le -\zeta'(1+c)/\zeta(1+c) \ll \log(2X)$, by the pole at one. Truncate at heights $\pm Q/2$ with cost $O(\log(2X)/Q)$, then move the segment to $\operatorname{Re} s = -1/100$. The only residue is $-D_\alpha$ at zero. Since $X^c = e$, the horizontal sides cost $O(\log(2Q)/Q^2)$ and the new vertical side costs $O(X^{-1/100}\log(2Q))$. This proves the claim. □

5.4 The successive choices

Proof of Theorem 5.1. At the first stage take $U = U_1$, $M = 8$, $a = 3$, $Q = Q_1$ and let the base characters κ be all primitive characters of conductor dividing $8F$. At the second stage q is already fixed. Put

$$U = U_2, \qquad M = 16q \prod_{3\le p\le z}^{*} p, \qquad Q = Q_2,$$

where the star indicates primes. The Chinese remainder theorem specifies one unit class modulo M by the congruences for d. Since $q > \sqrt{U_1} > Z$, its factor is distinct from the small primes. Here $\log M \ll z\log(2z) + \log q = o(\log U_2)$, uniformly in q. Take each previous base character and also its product with $(\cdot/q)$ as a base κ. The number of tests at each stage is bounded solely in terms of F.

Let ℓ range over prime candidates at the current stage. It eventually exceeds every base conductor, since at stage two $8Fq < \sqrt{U_2}$, and at stage one $8F < \sqrt{U_1}$. The product $\alpha = \kappa(\cdot)(\cdot/\ell)$ is primitive of conductor $\operatorname{cond}(\kappa)\ell$. Indeed the coprime local factors form a product on the unit group by the Chinese remainder theorem, and each local conductor is still minimal on its own factor. Thus, for fixed κ, different candidate primes produce different primitive characters. Their conductors are at most Q. We have $Q \le U^{3/2}$ at either stage for large Y, so Lemma 5.3 excludes at most $O_F(U^{1/2})$ candidates for the zero conditions.

For each base character apply Lemma 5.2 to $c_n = \kappa(n)w(n/Z)$ with $\tau = \varepsilon/2$. The required condition on the prime divisors of M holds at both stages. A further $O_F(U^{1-\eta})$ primes fail some finite-sum test. Lemma 5.3 supplies $U^{1-o(1)}$ candidates in all. For fixed F, K, ε the union of both exceptional sets has proportion tending to zero, uniformly in the first-stage choice at the second stage. Choose any remaining prime at each stage.

It remains to pass from the finite tests to (5.1). For all $Y \le y \le Y^K$ one has $y < Z/2$ eventually, so $w(n/Z) = 1$ whenever $n \le y$. Splitting the defining sum of G_α at y gives the exact identity

$$G_\alpha(y) = -S_{\ell,c}(y) + y\left(D_\alpha + \sum_{n\le Z}\Lambda(n)\alpha(n)w(n/Z)/n\right).$$

At both stages $\log(2Q) \ll_F z(\log Y)^2$ and $Q \ge e^z$. With $X = Z$, Lemma 5.4 makes the last term $o(1)$, uniformly for $y \le Y^K$, since $Y^K Z^{-1/100} = Y^{-3K}$. The finite-sum test already gives $|S_{\ell,c}(y)| \le y^{1/2+\varepsilon/2}/2$, proving the first bound. With $X = Y$, the lemma again gives $o(1)$ since $\varepsilon/5 < 1/100$. Finally, putting $J = \lfloor Y\rfloor$, the identity $\sum_{n\le Y}\Lambda(n)\lfloor Y/n\rfloor = \log(J!)$ follows by

summing $\sum_{d|m}\Lambda(d)=\log m$. Since $\lfloor Y/n\rfloor\ge Y/(2n)$ for $n\le Y$,

$$\sum_{n\le Y}\Lambda(n)/n\le 2\log(J!)/Y\le 2\log Y.$$

Use $0\le w\le 1$ and $|\alpha(n)|\le 1$ to get $|D_\alpha|\le 2\log Y+o(1)\le 3\log y$. This proves both bounds for the same primes and all the required characters. □

5.5 One pair of primes on an exponential range

Put $z=Y^{\varepsilon/5}$, as in Theorem 5.1. We shall choose the pair before the subinterval for the cohomology estimate is known. This requires bounds for the character sums on a longer range. The modulus of the second progression still contains the first prime.

Lemma 5.5 (A uniform lower bound for the number of candidates)**.** *For each $0<\delta<1$ there are $a_\delta>0$ and U_δ such that, for $U\ge U_\delta$, $2\le M\le U^{a_\delta}$, and every unit class modulo M, at least $U^{1-\delta}$ primes in $(\sqrt{U},U]$ lie in that class.*

Proof. If this failed, for every positive integer j there would be $U_j\ge e^j$, $2\le M_j\le U_j^{1/j}$, and a unit class with fewer than $U_j^{1-\delta}$ primes. Along this sequence $\log M_j/\log U_j\to 0$, so Lemma 5.3 gives $U_j^{1-o(1)}$ primes in these very classes. This contradicts the fixed positive δ. The proof of that lemma includes the possible exceptional real zero in the progression estimate. □

Theorem 5.6 (Fixed auxiliary twists on a longer interval)**.** *Fix $F\ge 1$ and $0<\varepsilon<1/52$. There are constants $c=c(\varepsilon)>0$ and $C=C(\varepsilon)>4$ with the following property. Set $U_1=e^z$, and choose either*

$$U_2=e^{Cz}\quad\text{or}\quad U_2=\exp\bigl(z(\log Y)^2\bigr).$$

For every sufficiently large Y there are primes $\sqrt{U_1}<q\le U_1$, $\sqrt{U_2}<d_0\le U_2$ with $q\equiv 3 \bmod 8$, $d_0\equiv 7 \bmod 16$, and $d_0\equiv -1$ modulo q and every odd prime at most z. For the same pair, all the auxiliary characters of Theorem 5.1 satisfy (5.1) for every *real $Y\le y\le e^{cz}$. They satisfy the zero-free rectangle of that theorem with $Q_1=8FU_1$ or $Q_2=8FU_1U_2$, according as the factor of conductor d_0 is absent or present.*

Proof. Write $\lambda=\varepsilon/3$, $\rho=\lambda/32000$, and fix $0<c<\min(1/32000,\rho/8)$. At each stage let U be its search bound. Cover $[Y,e^{cz}]$ by consecutive bands with left endpoints $H=Y,Y^2,Y^4,\dots$ and right endpoints $\min(H^2,e^{cz})$, stopping at the last nonempty band. For each band use the cutoff $Z=H^{800}$. Its logarithm is at most $800cz$, and $Z<\sqrt{U_1}$.

Fix one unit progression and one sequence of coefficients $|c_n|\le 1$ throughout a band. Assume that all primes dividing the modulus M and at most Z are at most z, and $8\mid M$. Define $B(m)$ and $V=\sum_p|b_p|^2$ as in the proof of Lemma 5.2 at a fixed y in this band. Removing the prime powers of exponent at least two costs $O(\sqrt{y}\log^2(2Z))$ and the primes dividing M cost at most $z\log(2z)$. Both are $o(y^{1/2+\varepsilon/2})$, uniformly, since $H\ge Y$. Quadratic reciprocity identifies the remaining sum at a candidate prime with $B(\ell)$, and $V\ll y\log^2(2Z)$. Equation (5.2) holds for every integer moment order. Use $l=\lfloor\log U/(20\log Z)\rfloor$. Our choice of c gives $l\ge\log U/(40\log Z)$ for both stages. With C fixed, for all sufficiently large Y one has

$$l\log^2(2Z)\ll(\log U)\log Z\ll_C z^2(\log Y)^2=o(Y^{\varepsilon/2}).$$

Consequently $128lV/y^{1+\varepsilon}\le H^{-\lambda}$ uniformly. Markov's inequality at $y^{1/2+\varepsilon/2}/8$ bounds the number of exceptional indices by

$$(U+1)H^{-\lambda l}+Z^{6l}\ll U^{1-\rho}+U^{3/10}.$$

Here $\log Z = 800 \log H$, which makes the saving independent of the band and of the fixed constant C.

Test each integer in each band and the two endpoints, using $c_n = \kappa(n)w(n/Z)$ for a fixed base character κ at the stage under consideration. There are at most $\exp(2cz)$ tests for large Y, since the bands have disjoint interiors and their number is $O(1+\log z)$. The finite sum is affine between successive tests within a band. Its norm is at most the larger endpoint norm, and the ratio of adjacent arguments is at most $1 + 1/Y$. Adding back the discarded terms shows that, apart from $O(U^{1-\rho/2})$ candidates, the finite sum in each band is at most $y^{1/2+\varepsilon/2}/2$ in absolute value. The constants in the power saving hold before C is chosen. The lower threshold for Y may depend on C.

At the first stage take $M = 8$. Having chosen q, take $M = 16q \prod_{3 \le p \le z} p$, the product being over primes. The Chinese remainder theorem supplies the required unit class. The elementary Chebyshev bound $\sum_{p \le z} \log p \ll z$ gives $\log M \le C_0 z$ with an absolute C_0. For instance, the primes in $(n, 2n]$ divide $\binom{2n}{n} \le 4^n$, and summing the resulting bounds over dyadic intervals gives the claim. Also $q > Z$ in every band, as required in the moment argument. Apply Lemma 5.5 with $\delta = \rho/4$. Choose $C > \max(4, C_0/a_\delta)$. Either value of U_2 then admits at least $U_2^{1-\delta}$ candidates, uniformly in the first prime. The same lower bound with U_1 holds at the first stage.

The base characters are exactly those in the proof of Theorem 5.1, and there are $O_F(1)$ of them at each stage. The candidate prime exceeds every base conductor and gives distinct primitive twists for each fixed base. At both stages $Q_i \le U_i^{3/2}$ for sufficiently large Y. Lemma 5.3 excludes $O_F(U_i^{1/2})$ candidates for zeros in the prescribed rectangles. Together with the moment exceptions this still leaves a prime at each stage.

Finally apply Lemma 5.4 with $X = Z$ on each band. In the identity for the prime sum in the proof of Theorem 5.1, the error in replacing $G_\alpha(y)$ by the negative of the finite sum is bounded by

$$O\Big(H^{-6} \log(2Q_i) + e^{cz} \frac{\log(2Z) + \log(2Q_i)}{Q_i} \Big) = o(1).$$

Indeed $y \le H^2$, $Q_i \ge e^z$, and $\log(2Q_i) \ll_{F,C} z(\log Y)^2$. The weight is one for $n \le y$ since $y < Z/2$. Applying that same lemma with $X = Y$ gives $|D_\alpha| \le 2 \log Y + o(1)$ by the elementary von Mangoldt sum used in Theorem 5.1, as $\varepsilon/5 < 1/100$. Both parts of (5.1) follow. □

6 An arithmetic comparison sufficient for the interval bound

We now express the arithmetic step as a comparison between an actual count and its main term. This gives a precise place to test the proposed construction, already for the case $\psi = \mathbf{1}$.

6.1 Fourier expansion at a prime

Let $E/\mathbb{Q}$ be finite abelian with Galois group $B = (\mathbb{Z}/2\mathbb{Z})^2 \times A$, written additively. Set $n = |A|$, let Φ be the subgroup of elements with first coordinate zero, and put $f = |\Phi| = 2n$. For $\xi \in \widehat{B}$ let ξ^* be the associated primitive Dirichlet character. This identification is furnished by embedding E in a cyclotomic field. At p, write I_p for inertia and F_p for any lift of Frobenius in B/I_p. For $j \ge 1$ we have

$$\xi^*(p^j) = |I_p|^{-1} \sum_{z \in I_p} \xi(jF_p + z).$$

Indeed, if the restriction to I_p is nontrivial the average vanishes and p divides the primitive conductor. Otherwise the character is unramified and equals its value on Frobenius. These

assertions also follow by projecting $(\mathbb{Z}/M\mathbb{Z})^\times$ to its prime-power factors in a cyclotomic field containing E. Orthogonality on B now gives, for every $g \in B$,

$$\frac{1}{f}\sum_{\xi\in\widehat{B}} \overline{\xi(g)}\,\xi^*(p^j) = \frac{2}{|I_p|}\mathbf{1}_{g-jF_p\in I_p}. \tag{6.1}$$

Give each $\alpha \in \Phi$ a weight $w_\alpha > 0$. For $t \in A$ put $p_t(g) = (1,1,t) + g$, a bijection $B \setminus \Phi \to \Phi$. For $a > 0$ define

$$C(t,a,w) = \sum_{g\notin\Phi}\sum_{\substack{p\\ \text{prime}\\ j\ge 1}} \frac{2\log p}{|I_p|}\mathbf{1}_{g-jF_p\in I_p}(a/p^j - w_{p_t(g)})_+.$$

The cutoff makes the sum finite. Let ξ_c denote the sign of the first binary coordinate and write $D_\chi = L'(1,\chi)/L(1,\chi)$ for nonprincipal primitive characters.

Proposition 6.1 (The main term of the count)**.** *Define* $M(a,w) = a\{\log a - 1 - \gamma + D_{\xi_c^*} - f^{-1}\sum_{\alpha\in\Phi}\log w_\alpha\}$. *Then*

$$C(t,a,w) - M(a,w) = \frac{1}{f}\sum_{g\notin\Phi}\sum_{\xi\in\widehat{B}} \overline{\xi(g)}\,w_{p_t(g)}\,G_{\xi^*}(a/w_{p_t(g)}). \tag{6.2}$$

In this formula extend the finite sum and the correction in (1.1) *to* $x > 0$ *by the same formulas.*

Proof. Insert (6.1) and use $(a/p^j - w)_+ = w((a/w)/p^j - 1)_+$. This gives the right side with R in place of G. For the trivial character the difference, averaged over g, is $a\{\log a - 1 - \gamma - f^{-1}\sum \log w_\alpha\}$. For a nontrivial ξ the difference per summand before multiplication by the character is $-aD_{\xi^*}$, independent of w. The normalized sum of $\overline{\xi(g)}$ over the nonidentity coset of Φ vanishes unless $\xi = \xi_c$, when it equals -1. This contributes $aD_{\xi_c^*}$. Conjugate terms pair, so both sides of (6.2) are real. □

6.2 Which characters survive the average

Fix the original ψ of order o. Let $A_0 \subset A$ have cyclic quotient H of order o. Let $T \subset A$ and $T_2 \subset A_0$ be nonempty. Write $h = |T|$, $h_2 = |T_2|$, $\delta = h_2/n$. Choose $a_t > 0$ for each $t \in T$, and positive weights $w_{uj,\alpha}$ for $u \in A_0$, $1 \le j \le m$, $\alpha \in \Phi$. At scale $x \ge 3$, put $b = \exp(x^\varepsilon)$, $Q = hmb$, and set

$$\Delta_u = \sum_{t\in T}\sum_{j=1}^{m}\{C(t,a_t,w_{uj}) - M(a_t,w_{uj})\}.$$

Here w_{uj} denotes the whole list of its weights. The distinguished weights occur at $L_u = \{(0,1,z+u) : z \in T_2\}$. Define P_{ut} as the right side of (6.2) restricted to $p_t(g) \in L_u$, with all distinguished weights replaced by b. Write $y_t = a_t/b$ and $e(t) = (1,0,t)$.

Lemma 6.2 (Averaging a translate)**.** *For every* t *the exact average is*

$$|A_0|^{-1}\sum_{u\in A_0} P_{ut} = \frac{b\delta}{2}\sum_{\xi|_{A_0}=1}\xi(e(t))G_{\xi^*}(y_t).$$

Proof. For $p_t(g) = (0,1,z+u)$ one has $g = (1,0,z+u-t)$. Consequently $\overline{\xi(g)} = \xi(e(t))\xi(0,0,-z)\xi(0,0,-u)$. Averaging the last factor over u kills a nontrivial restriction. On the surviving characters the middle factor is also one, since $T_2 \subset A_0$. The sum over z has h_2 terms, and $h_2/f = \delta/2$. □

We specify the hypotheses needed at this single x. These are conditions on data which can be checked before applying the next proposition.

1. There are $a_E \geq 0$ and $k > 0$ with

$$a_E \leq 2xb, \quad h \geq \tfrac{1}{2}nx^{-\varepsilon/8}, \quad \delta \geq x^{-\varepsilon}, \qquad |y_t - x| \leq 1,$$
$$x^{1/2-\varepsilon} \leq m \leq x^{1/2+\varepsilon/2}, \qquad xm \leq k \leq n^2 x^{1/2-\varepsilon}.$$

2. Among the $4o$ characters trivial on A_0, a subset is in bijection with $\mathcal{C}_\psi$ by $\xi \mapsto \xi^*$. On this subset $\xi(1,0,0) = 1$, and the restrictions to H run through all its characters, with the trivial restriction paired with $\mathbf{1}$. Each other surviving character satisfies $|G_{\xi^*}(y_t)| \leq 2x^{1/2+\varepsilon}$ for every t. The proportions $\nu(\alpha) = |T \cap (\alpha + A_0)|/h$ satisfy

$$\left|\nu(\alpha) - \frac{3 + \mathbf{1}_{\alpha=0}}{3o+1}\right| \leq x^{-1+\varepsilon} \qquad (\alpha \in H).$$

3. For each u, j, t the difference between $C(t, a_t, w_{uj}) - M(a_t, w_{uj})$ and P_{ut} has absolute value at most $2b/x$.

4. For every $u \in A_0$ the prime-power count satisfies

$$\Delta_u \leq x^{5\varepsilon}\mathcal{B}, \tag{6.3}$$
$$\mathcal{B} = ha_E(1 + m^2/k) + m\left(2nmbx^{\varepsilon} + \frac{h_2 kbx^{2\varepsilon}}{n^2}\right).$$

For clarity, condition 3 can be tested on individual weights. If $b \leq w \leq b + \eta$, the derivative of $wR_\chi(a_t/w)$, away from its finitely many breakpoints, has absolute value at most $\sum_{n \leq a_t/w} \log n$. The nonprincipal correction $a_t D_\chi$ is constant in w. The principal correction adds a derivative a_t/w. Integration shows that each distinguished term changes by at most $H_x\eta$, with $H_x = 3(x+1)\log(2x+4)$. It suffices that $|B|h_2 H_x \eta/f \leq b/x$ and that the sum of $|w_{uj,\alpha} G_{\xi^*}(a_t/w_{uj,\alpha})|/f$ over the other weights and all ξ be at most b/x.

Proposition 6.3 (Arithmetic reduction)**.** *Fix ψ and $0 < \varepsilon < 1/52$. If conditions 1–4 hold at x, then for all sufficiently large such x,*

$$\mathcal{G}_\psi(x) \leq x^{1/2+9\varepsilon}.$$

The threshold depends only on ψ, ε. In particular, if for every sufficiently small fixed $\varepsilon > 0$ these data exist at every point of arbitrarily distant intervals $[Y, Y^{3/2}]$, then ψ satisfies (W)*.*

Proof. For retained characters, replacing y_t by x costs $O_\psi(1+\log^2 x)$ per character. Indeed the ramp functions have slopes bounded in total by $\sum_{n \leq x+1} \log(n)/n$, and the explicit corrections have derivatives $O_\psi(1 + \log x)$. Also $|G_\chi(x)| \ll_\psi x(1 + \log^2 x)$. Replacing ν by its target therefore costs $O_\psi(x^\varepsilon(1 + \log^2 x))$ in the normalized average over t. By orthogonality on H, the target's Fourier coefficients are 1 and $1/(3o+1)$ at the trivial and nontrivial characters respectively. The retained terms now sum to $\mathcal{G}_\psi(x)$ after taking real parts. The discarded terms have total absolute value at most $8ox^{1/2+\varepsilon}$.

Sum Lemma 6.2 over j, t and use conditions 3 and 4. After division by Q, the resulting inequality is

$$\frac{\delta}{2}(\mathcal{G}_\psi(x) - O_\psi(x^{1/2+\varepsilon})) \leq x^{5\varepsilon}\mathcal{B}/Q + 2/x.$$

The four terms of $\mathcal{B}/Q$, in order, are at most

$$2x^{1/2+\varepsilon}, \qquad 2, \qquad 4x^{1/2+13\varepsilon/8}, \qquad 2x^{1/2+9\varepsilon/8}.$$

For the second use $a_E \le 2xb$ and $k \ge xm$. For the last use $h_2 \le n$, $n/h \le 2x^{\varepsilon/8}$ and $k/n^2 \le x^{1/2-\varepsilon}$. Using $\delta^{-1} \le x^{\varepsilon}$ gives $O_\psi(x^{1/2+(61/8)\varepsilon})$. Since $61/8 < 9$, the constant is absorbed for fixed $\varepsilon > 0$. Finally choose $\varepsilon < h'/9$ for the desired loss $h' > 0$ in (W) and take Y beyond the pointwise threshold. □

6.3 Constructing the field and the character weights

The next construction realizes the character identifications and the first three conditions in Proposition 6.3. Its last inequality is a separate requirement on the resulting count.

Proposition 6.4 (A field carrying the required characters)**.** *Fix ψ of conductor F, order o, and $0 < \varepsilon < 1/52$. For every sufficiently large Y there are fixed primes q, d such that at every $x \in [2Y, (2Y)^{3/2}]$ an abelian field, translates, and positive weights can be chosen to satisfy conditions 1–3. For these choices $\delta = 1/o$ and, uniformly in $u \in A_0$,*

$$\left| \frac{2\Delta_u}{Q\delta} - \mathcal{G}_\psi(x) \right| \le 3o\, x^{1/2+\varepsilon/2} + \frac{4}{\delta x}. \tag{6.4}$$

Proof. Choose q, d by Theorem 5.1 with $K = 2$. Let K_ψ be the subfield of $\mathbb{Q}(\zeta_F)$ fixed by $\ker\psi$. Its Galois group is cyclic of order o. Choose the least integer $l \ge 0$ with $2^l \ge x^{2+8\varepsilon}$ and form the real cyclotomic layer $C_l = \mathbb{Q}(\zeta_{2^{l+2}})^+$ of degree 2^l. Write $E_0 = K_\psi C_l$, $A = \mathrm{Gal}(E_0/\mathbb{Q})$, $A_0 = \mathrm{Gal}(E_0/K_\psi)$. Then $A/A_0 \simeq \mathrm{Gal}(K_\psi/\mathbb{Q})$ and $2^l \le n = |A| \le o2^l$.

Put $E = E_0(\sqrt{-q}, \sqrt{-d})$. Both primes are 3 mod 4, so their quadratic discriminants are $-q, -d$. The field E_0 is unramified at q, d, since these primes do not divide $2F$. Each of the three quadratic subfields of $\mathbb{Q}(\sqrt{-q}, \sqrt{-d})$ ramifies at one or both of q, d. Any nontrivial intersection with E_0 would contain one of these subfields. Hence the compositum has group $B = (\mathbb{Z}/2\mathbb{Z})^2 \times A$, with first and second factors cut out by the quadratic characters of conductors q, d. Quadratic reciprocity identifies these characters with $(\cdot/q), (\cdot/d)$, also at 2 by its supplementary law.

A character of B trivial on A_0 is a product of the two quadratic characters, each to exponent zero or one, and a character of A/A_0. The last factor runs through precisely the primitive representatives of powers of ψ. Taking both binary exponents zero gives all o retained characters and $\xi(1, 0, 0) = 1$ for them. Every other character has at least one quadratic factor and hence obeys (5.1). Here $Y \le x \le Y^2$ once Y is large.

Let $s = |A_0|$, $v = \lfloor s/4 \rfloor$. In the identity coset of A_0 choose $4v$ elements, and in each other coset choose $3v$, to make T. These counts fit in their cosets. They realize exactly the proportions in condition 2 and give $h = (3o+1)v \ge n/2$ for large s. Take $T_2 = A_0$, so $\delta = 1/o$. Put $m = \lfloor \sqrt{x} \rfloor$, $k = xm$, $a_E = xb$, $a_t = xb$ for all t. Since $n \ge x^{2+8\varepsilon}$, the upper bound $k \le n^2 x^{1/2-\varepsilon}$ follows. The remaining size bounds follow from $m/\sqrt{x} \to 1$, $h \ge n/2$ and fixed o.

Give the distinguished positions weight exactly b and all other positions the same positive weight w. At each fixed x the field and the list of characters are finite. Proposition 3.2 gives $wG_{\xi^*}(a_t/w) = a_t\, o_{\xi^*}(1)$ as $w \downarrow 0$. Thus one can choose w so that the sum of the absolute values of all the omitted terms in (6.2), including the factor $1/f$, is at most b/x per u, j, t. This is enough for condition 3. This qualitative choice uses no uniform estimate as the conductor or x grows.

Finally L_u is independent of u because $T_2 = A_0$. The idealized sum for each u therefore equals its average in Lemma 6.2. Here $y_t = x$ exactly. The Fourier coefficients of T are the

exact prescribed ones on the retained characters. All the other $3o$ coefficients have modulus at most one after division by h. Applying (5.1), and allowing the full error of condition 3, gives (6.4). □

The weights in this proposition are freely chosen real numbers. In the theta-lift construction the weights and translates have geometric constraints. Formula (6.4) explains what remains even with the freedom used here. Controlling the twists leaves the entire retained sum in the normalized count.

6.4 The local restriction kernels in the cyclotomic tower

We make explicit one part of the divisibility problem for the fields above. For a finite number field E, let $G = G_{E,S_\ell}$ be the Galois group of its maximal extension unramified at finite places outside $S_\ell = \{v : v \mid \ell\}$. Write κ for the ℓ-adic cyclotomic character. Give $M_r = \mathbb{Q}_\ell/\mathbb{Z}_\ell$ the discrete topology and let G act by κ^{-r}, for an integer $r \geq 1$. The strict local restriction kernel is

$$\mathcal{S}_{\ell,r}(E) = \ker\left(H^1(G, M_r) \longrightarrow \prod_{v\mid\ell} H^1(E_v, M_r)\right).$$

All cohomology and homomorphisms here are continuous.

Set $E_\infty = E(\mu_{\ell^\infty})$, $\Gamma = \mathrm{Gal}(E_\infty/E)$ and $H = \ker(G \to \Gamma)$. For each $v \mid \ell$ choose a decomposition subgroup D_v in G. Let $X'_\ell(E)$ be the maximal abelian pro-ℓ quotient of H on which all subgroups $H \cap gD_vg^{-1}$, $g \in G$, $v \mid \ell$, become trivial. Equivalently, first pass to the maximal abelian pro-ℓ quotient, then divide out the closed subgroup generated by those images. This compact module carries an action of Γ by conjugation. Write

$$X'_r = X'_\ell(E)/\overline{\langle \sigma b - \kappa(\sigma)^{-r} b : \sigma \in \Gamma,\ b \in X'_\ell(E)\rangle}.$$

Here the brackets denote the $\mathbb{Z}_\ell$-submodule generated by the displayed elements. For a compact abelian pro-ℓ group B_0, write $B_0^\vee = \mathrm{Hom}_{\mathrm{cts}}(B_0, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$.

Proposition 6.5 (Specialization of strict classes)**.** *If ℓ is odd, restriction gives an isomorphism $\mathcal{S}_{\ell,r}(E) \simeq (X'_r)^\vee$. For $\ell = 2$ it gives a map into $(X'_r)^\vee$ with kernel of order at most two. In particular, whenever X'_r is finite,*

$$\mathrm{length}_{\mathbb{Z}_\ell} \mathcal{S}_{\ell,r}(E) \leq \mathrm{length}_{\mathbb{Z}_\ell} X'_r + \mathbf{1}_{\ell=2}.$$

Proof. On H the module M_r is trivial, so a cocycle restricts to a homomorphism and a coboundary restricts to zero. The cocycle identity makes the homomorphism Γ-equivariant. A locally trivial class vanishes on $H \cap gD_vg^{-1}$ after restriction. Its image is a finite ℓ-group by continuity and compactness. Thus it factors through $X'_\ell(E)$, and equivariance is exactly the displayed relation for X'_r. Using D_v in place of the absolute local Galois group gives the same condition, since inflation in degree one for their surjection is injective.

For odd ℓ the group Γ is an open subgroup of $\mathbb{Z}_\ell^\times$. Its intersection Γ_1 with $1+\ell\mathbb{Z}_\ell$ is procyclic and isomorphic to $\mathbb{Z}_\ell$, and the index is prime to ℓ. These statements follow because E is finite and $1 + \ell\mathbb{Z}_\ell \simeq \mathbb{Z}_\ell$ by the logarithm. If γ_1 is a topological generator, continuous cohomology of Γ_1 in degree one is $M_r/(\gamma_1 - 1)M_r$ and it vanishes in degree two. Indeed, with $T = \gamma_1 - 1$ the two-term resolution of the trivial module is multiplication by T on $\mathbb{Z}_\ell[[T]]$, with cokernel $\mathbb{Z}_\ell$. On M_r this map is multiplication by the nonzero ℓ-adic number $\kappa(\gamma_1)^{-r} - 1$ in $\mathbb{Z}_\ell$, hence is surjective. Averaging over the finite quotient of order prime to ℓ makes its higher cohomology vanish. The group extension sequence for $\Gamma_1 \subset \Gamma$ thus gives $H^1(\Gamma, M_r) = H^2(\Gamma, M_r) = 0$.

Inflation and restriction now identify $H^1(G, M_r)$ with $\mathrm{Hom}(H, M_r)^\Gamma$, since the kernel is $H^1(\Gamma, M_r)$ and the obstruction to surjectivity lies in $H^2(\Gamma, M_r)$. A homomorphism corresponding to $(X_r')^\vee$ vanishes on $H \cap D_v$. The class it defines on D_v must therefore inflate from the image Γ_v of D_v. This is again open in $\mathbb{Z}_\ell^\times$, since the local field $E_v/\mathbb{Q}_\ell$ is finite. The same proof gives $H^1(\Gamma_v, M_r) = 0$. Consequently the global class is locally trivial at each v, proving the isomorphism.

At $\ell = 2$ use $\Gamma_1 = \Gamma \cap (1 + 4\mathbb{Z}_2)$. It is isomorphic to $\mathbb{Z}_2$, and the quotient $C = \Gamma/\Gamma_1$ has order at most two. Again $H^1(\Gamma_1, M_r) = 0$. Inflation identifies $H^1(\Gamma, M_r)$ with $H^1(C, M_r^{\Gamma_1})$. Its coefficient group is the kernel of multiplication by $\kappa(\gamma_1)^{-r} - 1$ on $\mathbb{Q}_2/\mathbb{Z}_2$, hence finite cyclic. Degree-one cohomology for C is a subquotient of this cyclic group, by evaluation of a cocycle on a generator. It is killed by $|C|$, so has at most two elements. The kernel of global restriction has at most that size as well. Duality for finite abelian ℓ-groups preserves their order, which proves the length bound. □

Lemma 6.6 (A determinant bound and exceptional twists)**.** *Let $\Gamma_1 = \langle \gamma_1 \rangle$ as above, topologically. Suppose $X_\ell'(E)$, as a module over $\mathbb{Z}_\ell[[T]]$ with $T = \gamma_1 - 1$, is a quotient of the cokernel of a square matrix $P(T)$ over this ring. Put $u = \kappa(\gamma_1)$, $\theta_r = u^{-r} - 1$. If $\det P(\theta_r) \neq 0$, then*

$$\mathrm{length}_{\mathbb{Z}_\ell} \mathcal{S}_{\ell,r}(E) \leq v_\ell\big(\det P(\theta_r)\big) + \mathbf{1}_{\ell=2}.$$

Moreover let $p(T) \in \mathbb{Z}_\ell[T]$ be monic of degree $d \geq 1$ and $e = v_\ell(u-1)$. For integer $j \geq 0$, all $r \geq 1$ with $v_\ell(p(\theta_r)) > d(e+j)$ lie in at most d residue classes modulo ℓ^{j+1}. Here $v_\ell(0) = +\infty$.

Proof. Specialization $T \mapsto \theta_r$ is defined because $\theta_r \in \ell\mathbb{Z}_\ell$. Its kernel is $(T - \theta_r)$, as is seen by translating the variable in the power series ring. Passing to the quotient is right exact. The cokernel of $P(\theta_r)$ therefore surjects onto $X_\ell'(E)/(T-\theta_r)X_\ell'(E)$, which in turn surjects onto X_r'. The images involved are compact and hence closed. Over the discrete valuation ring $\mathbb{Z}_\ell$, Smith normal form gives $\mathrm{length}\,\mathrm{coker}\, P(\theta_r) = v_\ell(\det P(\theta_r))$. Apply Proposition 6.5.

For positive m, $v_\ell(u^m - 1) = e + v_\ell(m)$. For m prime to ℓ this follows from the first term of the binomial expansion. Raising a principal unit to the ℓ-th power increases this valuation by exactly one, again by the binomial expansion, using $e \geq 2$ at $\ell = 2$. Thus $v_\ell(\theta_r - \theta_s) = e + v_\ell(r - s)$ for distinct r, s. Factor $p(T)$ over $\overline{\mathbb{Q}}_\ell$, with the extended valuation. If the displayed threshold is exceeded, some root α satisfies $v_\ell(\theta_r - \alpha) > e + j$. Two indices attached to this root have $v_\ell(\theta_r - \theta_s) > e + j$ by the ultrametric inequality, so $r \equiv s \bmod \ell^{j+1}$. There are at most d roots. □

Proposition 6.7 (A common integer for Frobenius trace comparisons)**.** *Fix $A = p^k \geq 7$, with p prime and $k \geq 1$. Let Z_0 be a finite multiset of roots of unity and B_0 a finite multiset of algebraic numbers. Suppose both are stable under Galois conjugation, Ab is an algebraic integer for $b \in B_0$, and $|b| \leq 3$ in every complex embedding. Set $t_z = A^{-1}z + 1 + Az^{-1}$. There is a nonzero integer N_p with*

$$\log |N_p| \leq |Z_0|\,|B_0| \log(2A^2).$$

For every $\ell \neq p$, any embedding of the data into $\overline{\mathbb{Q}}_\ell$, and any submultiset $D \subset Z_0 \times B_0$,

$$\sum_{(z,b)\in D} v_\ell(b - t_z) \leq v_\ell(N_p), \qquad v_\ell(\ell) = 1.$$

Proof. Take $N_p = \prod_{z,b} A(b - t_z)$. Each factor $Ab - z - A - A^2 z^{-1}$ is integral. The factor with A^2 has modulus A^2, and the other three terms have total modulus at most $4A + 1 < A^2$. Thus every factor is nonzero of modulus at most $2A^2$. Galois invariance of the product makes it a rational algebraic integer. This proves the asserted archimedean estimate, including the empty

product. At $\ell \neq p$, A is a unit, so all the summand valuations are nonnegative and the sum over the whole multiset is $v_\ell(N_p)$. Discarding terms proves the inequality. □

In the proposed cohomological comparison, b is a Frobenius trace of a three-dimensional representation and t_z is the trace of a sum of three characters. The role of a comparison with two distinct primes p_1, p_2 can be stated precisely. If nonnegative local lengths a_ℓ have been bounded by $2v_\ell(N_{p_i})$ for each $\ell \neq p_i$, then

$$\sum_{\ell} a_\ell \log \ell \leq 2 \log |N_{p_1}| + 2 \log |N_{p_2}|.$$

Use the second integer at $\ell = p_1$ and the first at all other primes. For either integer, the sum of all its valuations weighted by $\log \ell$ equals the logarithm of its absolute value. The missing arithmetic input in this use of the proposition is the bound of actual cohomology lengths by trace valuations for a Galois-stable list of controlled size. Knowing only a determinant degree cannot yield such a bound. For example, at any fixed index r_0 the degree-one polynomial $T - \theta_{r_0} - \ell^L$ has specialized valuation L there for every positive integer L.

For the tower of fields, a required estimate in the arithmetic construction takes the following form at a scale x, with $t = \lfloor x^{401} \rfloor$,

$$\sup_{S \text{ finite}} \sum_{\ell \in S} (\log \ell) \sum_{1 \leq r < t} \operatorname{length}_{\mathbb{Z}_\ell} \mathcal{S}_{\ell,r}(E) \;\leq\; \exp(x^{\varepsilon}/2), \tag{L}$$

where S varies over sets of rational primes. The specialization proposition gives an exact group to bound at odd primes and an upper bound at two. The determinant lemma controls the ordinary twists if suitable presentations, with quantitative bounds on their determinants, are available. Counting exceptional residue classes puts no upper bound on the valuation *inside* such a class. A proof of (L) must control these terms, the dependence on E, and the sum over ℓ on the same intervals as the period comparison.

6.5 Local divisibility and the rank of a period map

Here are two deductions from the kinds of local and archimedean estimates needed for (6.3). They apply to arbitrary lattices and linear maps satisfying the stated hypotheses.

Lemma 6.8 (Divisibility of a pairing)**.** *Let N, P be finite projective lattices over the integers of a number field J, and let $B : N_J \times P_J \to J$ be bilinear, where $N_J = J \otimes N$ and likewise for P_J. For finite places w suppose $v_w(B(N_w, P_w)) \geq a_w$, with a_w integers of finite support and $v_w(0) = +\infty$. Put $I = \prod_w \mathfrak{p}_w^{-a_w}$. Then B gives an integral map $IN \to P^\vee$. For a fixed Euclidean metric at infinity and $d = \operatorname{rank} N > 0$, set $\mu(N) = -2 \log \operatorname{covol}(N)/([J : \mathbb{Q}]d)$. One has*

$$\mu(IN) - \mu(N) = \frac{2}{[J : \mathbb{Q}]} \sum_w a_w \log N\mathfrak{p}_w.$$

Proof. At w, the first lattice is multiplied by $\pi_w^{-a_w}$, where π_w is a uniformizer. Every pairing is then integral at w. Intersection of the local integer rings inside J gives the asserted global integral map. The determinant line is multiplied by I^d. For an integral ideal this changes covolume by its index to the power d, as follows by counting local quotients over the residue fields. Taking ratios gives the same rule for a fractional ideal. In this case the factor is $\prod_w (N\mathfrak{p}_w)^{-da_w}$, proving the formula. □

Proposition 6.9 (Loss of rank under restriction)**.** *Let $A_i : V_i \to \mathbb{C}^D$ be linear maps between Hermitian spaces, $D > 0$, $R = \sum_i \dim V_i > 0$, and $Q_0 = \sum_i A_i A_i^*$. Write $\|M\|_{\mathrm{HS}}^2 = \operatorname{tr}(M^* M)$.*

Suppose $0 \le e \le 1/4$, $\eta \ge 0$ *and*

$$\sum_i \|A_i^* A_i - I\|_{\mathrm{HS}}^2 \le eR, \qquad D \operatorname{tr}(Q_0^2) \le (1+\eta)(\operatorname{tr} Q_0)^2.$$

If $U_i \subset V_i$ *have total codimension* $\epsilon_0 R$, *then* $W = \sum_i A_i U_i$ *satisfies*

$$\operatorname{codim}(W)/D \le \eta + 4\epsilon_0 + 4\sqrt{e}.$$

Proof. Put $a = \sqrt{e}$, $T_0 = \operatorname{tr} Q_0$. The first inequality and Cauchy–Schwarz give $T_0 \ge (1-a)R > 0$. Let P, P_i be the orthogonal projections to $W^\perp, U_i^\perp$ respectively. Since $PA_i = PA_iP_i$,

$$\operatorname{tr}(PQ_0) = \sum_i \|PA_iP_i\|_{\mathrm{HS}}^2 \le \epsilon_0 R + \sum_i \|P_i\|_{\mathrm{HS}} \|A_i^* A_i - I\|_{\mathrm{HS}} \le R(\epsilon_0 + a\sqrt{\epsilon_0}).$$

Let $d_0 = \operatorname{rank}(P)/D$, $p_0 = \operatorname{tr}(PQ_0)/T_0$. The second hypothesis gives $\operatorname{tr}((DQ_0/T_0 - I)^2) \le D\eta$, whence $|p_0 - d_0| \le \sqrt{d_0\eta}$ by Cauchy–Schwarz. As $2\sqrt{d_0\eta} \le d_0 + \eta$, it follows that $d_0 \le 2p_0 + \eta$. Now $p_0 \le 2(\epsilon_0 + a)$, proving the claim. □

Corollary 6.10 (From the ranks to mean slopes)**.** *In Proposition 6.9, give the source spaces finite decreasing real filtrations and list their jumps, with dimension multiplicities, as shifted slopes. Suppose their overall mean is at least* s *and each is at most* $s + v$ *with* $v \ge 0$. *Let a quotient of* $\mathbb{C}^D$ *of dimension* $d \ge (1-\delta_0)D > 0$ *have a filtration with all slopes at least* $s - H_0$, $H_0 \ge 0$. *Assume that for each threshold the projected maps carry its source filtration spaces into the corresponding target filtration space. For the mean* μ *of the target slopes one has*

$$s - \mu \le \frac{4v \log(1 + H_0/v) + (\eta + 4\sqrt{e})H_0}{1 - \delta_0}, \qquad 0 \le \delta_0 < 1.$$

At $v = 0$ *the first term of the numerator is defined to be zero.*

Proof. A fraction p_τ of source slopes below $s - \tau$, $\tau > 0$, forces the mean to be at most $p_\tau(s-\tau) + (1-p_\tau)(s+v)$. Hence $p_\tau \le v/(v+\tau)$. Use the source spaces at threshold $s - \tau$ as the U_i in Proposition 6.9. Linear projection to a quotient does not increase the absolute codimension of a span. Therefore the fraction of target slopes below $s - \tau$ is at most $(\eta + 4\sqrt{e} + 4v/(v+\tau))/(1-\delta_0)$. For each target slope a, the integral of $\mathbf{1}_{a<s-\tau}$ over $0 \le \tau \le H_0$ is $(s-a)_+$. Average this identity and integrate the displayed bound. When $v = 0$ the bound for p_τ is zero for each $\tau > 0$. □

In particular, a local divisibility computation giving a lower bound $\mu + \Gamma_0 - D_0 \le s$ for this *same* collection of maps bounds its gain Γ_0 by D_0 plus the displayed error. Applying this to integral periods requires both the lower bound and the comparison of filtrations and Gram matrices for the chosen maps.

At the archimedean places the raw quantities in one such comparison are block Gram matrices $K_{ij} = A_i^* A_j$, before restriction, with $r_i = \dim V_i$. For a comparison dimension $D_* > 0$, the proposed estimates (after fixing the size normalization of the blocks) include

$$\|K_{ij}\|_{\mathrm{HS}}^2 \le (1+\eta_0) r_i r_j / D_* \ (i \ne j), \qquad \|K_{ii} - I\|_{\mathrm{HS}}^2 \le e_0 r_i, \qquad D \le (1+\eta_1) D_*,$$

with $\eta_0, \eta_1, \sqrt{e_0} \le \exp(-x^{\varepsilon/8})$. These are quantitative estimates for actual theta periods. For instance, the reason for controlling different blocks in the first inequality is the exact identity $\operatorname{tr}((\sum_i A_i A_i^*)^2) = \sum_{i,j} \|K_{ij}\|_{\mathrm{HS}}^2$, obtained by cyclic invariance of the trace. It is this sum that governs the rank estimate above.

6.6 The estimates on a common interval

The following paragraphs give the common field, interval and comparison inequalities. The companion volumes develop the two estimates by subject. The volume guide gives their locations. The preceding calculation explains the choice of chain length in the *Remark*. Two terms of the normalized error have sizes governed by x/m and m. A choice of m near $\sqrt{x}$ balances them. This calculation starts after the bound (6.3) for the actual counts has been obtained.

In the construction, E varies with x. One needs control of the weighted lengths of local restriction kernels in the cohomology of E, and integral period maps for a chain of theta lifts over that same field. Local divisibility is to give a lower bound for the gain in the period lattice. An upper bound involving its covolume, its ranks and the archimedean norms of the period maps must give (6.3) for every translate. Both bounds must apply to data obeying conditions 1–3 at each point of one whole interval. Two independently unbounded sets of admissible scales can be disjoint. Even at $o = 1$, when the only retained term is $G_{\mathbf{1}}$, the count estimate is needed.

The residual local groups. Fix a finite set Π of odd primes containing all odd divisors of $o = \mathrm{ord}(\psi)$. Use the fields and the single pair q, d on the arbitrarily distant intervals $[V, V^{3/2}]$ of Corollary 12.11. Write $\eta = \varepsilon/128,\ t = \lfloor x^{401} \rfloor,\ C_j = \mathbb{Q}(\zeta_{2^{j+2}})^+$, and set

$$L_x = K_\psi C_{\lfloor \eta \log_2 x \rfloor}(\sqrt{-q}, \sqrt{-d}), \quad B(x) = K_\psi C_l, \quad P = \exp(x^\varepsilon/12), \qquad b_0 = \lfloor \exp(x^\varepsilon/8) \rfloor.$$

Here l is the least integer with $2^l \geq x^{2+8\varepsilon}$. Both subfields lie in E_x for all sufficiently large x. For an odd prime ℓ, put $S = \mathcal{S}_{\ell,r}(E_x)$ and $M = eS$, where $e = (1 + (-1)^r c)/2$ is the complementary-sign idempotent. Define an operator $J_{\ell,r}$ on M as follows. For $r \leq x^\eta$ use the average of $\mathrm{Gal}(E_x/L_x)$. For $r > x^\eta$ and $\ell \in \Pi$ use the average of $\mathrm{Gal}(E_x/B(x))$. For the other pairs use the zero map. Both averages are integral idempotents at odd primes since the relative degrees are powers of two. Let $U_{\ell,r} = (1 - J_{\ell,r})M$ and put

$$R_{\ell,r} = \begin{cases} U_{\ell,r}/U_{\ell,r}[\ell^{b_0}], & \ell \leq P, \\ U_{\ell,r}, & \ell > P. \end{cases}$$

Proposition 6.11 (The part still needed in the local-length estimate)**.** *Denote the left side of* (L) *by* $\mathcal{L}(x)$ *and define* $\mathcal{R}(x) = \sum_{\ell>2} (\log \ell) \sum_{1 \leq r < t} \mathrm{length}_{\mathbb{Z}_\ell} R_{\ell,r}$*, where* ℓ *runs through primes. On these intervals,*

$$0 \leq \mathcal{L}(x) - \mathcal{R}(x) \leq \tfrac{1}{2} \exp(x^\varepsilon/2) + 2 \exp(x^{2\varepsilon/3}) + \exp(x^\varepsilon/3). \tag{6.5}$$

In particular $\mathcal{R}(x) \leq \frac{1}{4} \exp(x^\varepsilon/2)$ *throughout arbitrarily distant such intervals would suffice for* (L).

Proof. All the groups are finite and for each x the sum has finite support, by Proposition 11.4. Its supremum over finite sets of primes is its full sum. The prime two and the sign $(-1)^{r+1}$ at all odd primes use at most $\frac{1}{4} \exp(x^\varepsilon/2)$ each by Theorems 11.12 and 11.1. At each odd prime the other sign is $M = J_{\ell,r} M \oplus U_{\ell,r}$, since the field averages commute with c. The weighted total of the first summands is at most $2 \exp(x^{2\varepsilon/3})$ by Theorem 12.5 and Corollary 12.11. These apply to the two disjoint twist ranges in the definition of $J_{\ell,r}$. For $\ell \leq P$, the kernel of $U_{\ell,r} \to R_{\ell,r}$ is contained in $S[\ell^{b_0}]$; Theorem 11.5, bounds the sum of their lengths with weights by $\exp(x^\varepsilon/3)$. Additivity of length proves (6.5). Both further terms there are $o(\exp(x^\varepsilon/2))$ as x tends to infinity, giving the sufficient bound for the remainder. □

Theorem 17.9, gives an exponent test at every odd prime for components with nontrivial residual character. Its Corollary 23.13 bounds the congruence ideals of characteristic-zero

Hecke traces. Proposition 23.5, exhibits actual extension classes whose marked deformations have identical traces. An upper bound from an integral deformation model needs information about its relations in addition to its characteristic-zero trace values.

Lemma 6.12 (Characteristic-zero points and an integral length)**.** *For every prime ℓ and every $b \geq 1$ there is a finite local $\mathbb{Z}_\ell$-algebra with a single characteristic-zero geometric point and whose augmentation ideal modulo its square has length b.*

Proof. Use $A_b = \mathbb{Z}_\ell[X]/(X^2, \ell^b X)$ and the augmentation $X \mapsto 0$. Its nilpotent ideal $I = (X)$ has quotient $\mathbb{Z}_\ell$, so A_b is local with maximal ideal (ℓ, X). As a module it is $\mathbb{Z}_\ell \oplus (\mathbb{Z}_\ell/\ell^b)X$. Every characteristic-zero point over $\mathbb{Z}_\ell$ sends the nilpotent X to zero, and the rational generic fiber is $\mathbb{Q}_\ell$. Yet $I^2 = 0$ and I/I^2 has length b. □

This illustrates why the model of the deformation ring, including its integral relations, enters the length inequality. The example by itself makes no assertion about the deformation rings of the present fields.

At every odd prime, Theorem 15.17 compares the full complementary length over the period field with scalar groups having the local conditions at the conductor. The total discrepancy is polynomial in the interval scale. Corollary 15.45 gives the exact elementary divisors added by relaxing the strict condition at any number of primes in a common split Frobenius class of prescribed cyclotomic depths. A single prime in that class preserves every indicated complementary group at every precision. Theorem 15.48 chooses another common set using residual dual classes. It makes every original strict class divisible by a prescribed power in the relaxed group. At cyclotomic depth one, Corollary 15.49 recovers each successive layer of the length sum from the two sets of primes. Corollary 21.21 proves full vanishing for the stated residual characters by a two-prime decomposition-group test on the original fields. Theorem 29.4 constructs the common divisor of the Frobenius tests over one auxiliary quadratic field. Corollary 29.5 states the full-length inequality still needed at the large primes. On the same period fields, Theorem 29.6 gives the full integral ring of lower triangular lifts preserving the marked symmetric form. Its specializations count the complementary strict classes at every depth. Theorem 29.8 determines which irreducible orthogonal points can be automorphic over a CM extension. Theorem 29.11 realizes the dual of the full large-prime complementary group as a quotient of the cotangent module of the integral trace algebra. Theorem 29.14 bounds, by $\log J_*$ on the entire interval, the contribution after quotienting by the image of the trace relations that vanish at the listed classical systems. At each large prime Theorem 29.17 constructs cuspidal ordinary adjoints over a single auxiliary CM field for all the period data on the interval. Theorem 29.18 retains the entire integral trace quotient there and descends automorphy for any strongly irreducible point for which it holds on that field. At the cuspidal adjoints, Theorem 29.21 identifies the full characteristic-zero ordinary deformation ring with its weight ring. Theorem 29.22 computes the ideal of inverse-trace differences there, including at every infinitesimal order in symmetric weights. Theorem 29.26 constructs cuspidal representations with full special linear monodromy at the prescribed weights on one auxiliary field for the entire interval. Corollary 29.27 bounds a nonzero inverse-trace difference in valuation. Proposition 29.28 computes the exact local obstruction to descending these points to the original field with crystalline type. In residue characteristic, Theorem 30.1 gives the equations imposed by global units on sums of three characters. Theorem 30.4 uses them to construct a nonabelian trace curve with distinct characters on the kernel of cyclotomic inertia at every place above the coefficient prime. Theorem 30.8 establishes the residual cohomology hypotheses for the entire interval on one sufficiently deep auxiliary field. Proposition 30.9 places the trace curve on its specified Galois quotient. Theorem 30.13 realizes that curve in the marked

ordinary deformation ring with a perfect symmetric pairing, and verifies its local and generic point conditions. Theorem 30.17 constructs simultaneous split Frobenius tests for the dual cohomology on these curves at every parameter precision, with a single loss independent of the precision. Theorem 30.20 computes the leading term of the first-order Selmer length there, uniformly through the auxiliary prime levels and all powers of the curve parameter. Theorem 30.25 bounds the obstruction to lifting ordinary cocycles on the same levels and gives the second numerical input for patching. Theorem 30.28 constructs integral comparison maps between hyperspecial and Iwahori levels. Corollary 30.29 computes their loss at the Frobenius polynomials of the same curves. Theorem 30.32 places every selected curve on the closure of a family with absolutely irreducible generic trace on the deep auxiliary field. Corollary 30.33 proves that the actual ordinary modules on that field have positive rank upon specialization to the curve at every selected auxiliary level. Theorem 30.36 identifies the unique full ordinary determinant component through the cuspidal adjoint with the corresponding integral Hecke component. Corollary 30.37 proves classicality of its characteristic-zero points at the prescribed weights. Proposition 31.2 identifies the integral augmentation ideal on the original quadratic field with the four-by-four trace minors. Theorem 31.3 recovers the full complementary strict group from these minors even after restriction to the auxiliary field. Theorem 31.6 proves equality of the complete fixed-weight trace rings on the original and auxiliary fields. Proposition 31.7 constructs a finite quotient on the original field, and Corollary 31.8 gives the original cuspidal labels for its irreducible marked points. Proposition 31.10 proves crystallinity for global trace specializations, including the first twist. Theorem 31.11 puts all non-self-dual characteristic-zero points of the finite quotient on the original cuspidal list, giving the remaining length in Corollary 31.12. Theorem 31.15 shows that reducible characteristic-zero points contribute zero inverse-trace difference throughout their infinitesimal fibers. Its proof uses the odd descent and Selmer vanishing of Proposition 31.14. Theorem 33.2 proves reducedness at every irreducible point of the finite quotient. Proposition 33.3 computes the exact multiplicity and torsion terms in its inverse-trace ideal, and Corollary 33.4 identifies the two integral contributions still to be bounded. Theorem 33.7 constructs a finite ring on the original quadratic field using the determinant of the trace pairing. Its torsion maps onto the full complementary group by Corollary 33.8. Theorem 33.10 identifies the rational local rings at all reducible points. Theorem 33.12 bounds the cotangent contribution of their torsion-free order by one explicit integer throughout the interval. Theorem 33.14 extends the plane trace to an odd two-dimensional determinant over the full integral quotient. Theorem 33.16 realizes the full dual strict group as an integral direct summand in the other factor of a fiber product on the original field. Theorem 33.19 determines that factor by an integral tensor product. Proposition 33.17 computes the upper elementary divisors using the preceding twist, with an ideal class group formula at the first twist. Theorem 33.22 counts the fibers above each strict class in the full lower triangular deformation problem. Proposition 33.23 bounds their total logarithmic contribution at every coefficient depth on the same interval. Outside the prime divisors of one explicit integer, Theorem 33.24 identifies the entire even orthogonal deformation ring with the lower triangular ring. Theorems 33.26 and 33.27 compare the integral trace ring of families with an invariant direct summand to the ring cut out by the largest trace minors. The latter is a nilpotent extension of the former of index at most eight. Corollary 33.28 realizes the complementary elementary divisors on finite coefficient algebras of minimal rank. Theorem 33.30 shows how a nonzero cyclic complementary group gives either a nonconstant irreducible trace in residual characteristic or a finite flat intersection whose Frobenius determinant bounds its full length. Theorem 34.2 recovers the full order at the split trace for any number of generators away from the critical determinant. Corollary 34.3 identifies the higher exterior powers there with those of the strict group dual. Theorem 34.5 bounds the entire strict length by the

Frobenius determinant of a finite trace algebra, with geometric multiplicities, in the absence of a residual trace curve. Corollary 34.6 realizes the minimum determinant valuation at split primes. Theorem 34.8 gives the multiplicity bound at every large prime. Theorem 34.9 bounds by a single integer on the interval the full lengths accounted for by the listed classical traces, including components with several strict generators. Corollary 34.12 includes in that estimate every finite component whose irreducible points become cuspidal over solvable CM extensions. The auxiliary extensions can vary with the point. Theorem 34.15 also permits a varying intermediate CM field of bounded degree and controlled discriminant. Its proof counts all such fields and tests every trace with a common Frobenius power. Theorem 34.17 removes all finite ramification of the conductor characters on a common totally real field. Proposition 34.18 recovers the strict groups on the original period fields by taking invariants. Theorem 34.23 constructs the ordinary determinant on the three integral modules used in the change of level. Corollary 34.24 gives an annihilator for the extra ramification on diamond coinvariants. Theorem 34.26 pairs Frobenius roots with their inertia characters on the integral diagonal cover. Theorem 34.28 gives the resulting ordered finite-level comparison on each of the auxiliary sets. Theorem 34.30 identifies the kernel created by the varying primes in a common limiting group. Theorem 34.32 establishes the ordinary tangent bound, the simultaneous lifting relation and integral descent for the off-diagonal cohomology on that group. Theorems 35.3 and 35.4 construct a finite patched module at the discriminant of these curves and identify its fiber with the ordered hyperspecial module. Theorem 35.7 identifies the local ordinary determinant ring with the local Hecke algebra at each selected curve whose generic representation is absolutely irreducible. Every component through such a curve is in Hecke support. At a curve with a character and a plane, Theorem 35.10 identifies the irreducible and the nonsemisimple fibers of the ordinary Galois algebra. The minimum number of generators of its off-diagonal corner is the ordinary extension rank on the original field by Corollary 35.11. The lift of mixed ordinary cocycles in Theorem 34.32 gives the square matrix of relations in Theorem 35.15. Theorem 35.18 then places all irreducible and all reducible nonsemisimple primes below each selected two-constituent curve in ordinary Hecke support. Theorem 35.20 computes the exact annihilator of the off-diagonal module in the completed patched ring. After setting the auxiliary diamonds to one, Theorem 35.24 computes the integral extension module at reducible specializations of the completed ordinary ring by a square matrix and its minors. The fixed arithmetic weight is disjoint from that localization. Theorem 37.3 and Corollary 37.4 recover the actual large-prime lengths on each period field from quotients of the unlocalized ordinary algebra at the split weight. Uniform upper bounds for their minors remain necessary. Theorem 37.7 gives exact descent of strict cohomology at the auxiliary prime, at every coefficient precision. Corollary 37.8 applies this to all the period fields on the interval with one common auxiliary extension. Theorem 37.12 propagates the ordinary Hecke support along finite chains of separated curves with a line and an irreducible plane from one supported irreducible prime at the first curve. Theorem 37.15 identifies the ordinary ring with the Hecke algebra and proves it is a complete intersection at every supported irreducible curve with the stated inertia separation. Corollary 37.16 allows both types of curves in one chain. Theorem 37.19 gives an explicit dimension bound on a component intersection that guarantees transfer of support. Corollary 37.20 applies it starting from the full component through the cuspidal adjoint. Theorems 37.22 and 37.24 give a presentation of the marked ring and a dimension bound for the map to traces on the one- and two-constituent loci. Proposition 37.27 bounds the dimension of the triangular locus by means of strict cohomology on the same field. Theorem 37.29 transfers support throughout a marked chart with a nonzero adjacent lower entry. Theorem 37.34 proves transfer to every irreducible determinant in a chart with only a long lower entry at the residual point. Its proof treats intersections with reducible components by a first-order deformation

of the same Galois algebra. Theorem 37.39 proves Hecke support at every geometrically irreducible determinant over the common layer. Corollary 37.40 proves finiteness of their reduced closure after specializing to the original arithmetic weight. Theorem 37.41 proves that the entire original fixed-weight trace ring is finite. Theorem 37.44 bounds the determinant on its torsion-free augmentation ideal by one integer throughout the period interval. Corollary 42.4 bounds the entire large-prime complementary length on each original period field by a polynomial in the interval scale. Corollary 42.8 also eliminates the high-power tails outside the stated set of coefficient primes in all but at most eight residue classes of twists modulo $\ell - 1$. Theorem 42.13 controls every twist with three distinct residual inertia characters at the indicated primes. Corollary 42.14 leaves only the two colliding residues there. Theorem 42.18 also controls both moving ramified coefficient primes, including all the characters ramified at those primes. Corollary 42.19 locates the remaining small-prime tails at a fixed finite set of primes and at the two collision residues. Theorem 38.4 realizes every scalar component in that range with a nontrivial residual character and twist at least two by a symplectic matrix entry, with an explicit bound on the precision needed for the local splittings. This construction alone gives no bound on the lengths of these components. Theorem 38.8 computes the global localization map for each of the six odd symplectic roots at every coefficient depth. The remaining scalar entry is even. Theorem 38.12 computes the two ordinary extension spaces preserving either paired plane also at a nonsplit strict class. Corollary 38.13 shows exactly which characteristic polynomials persist in both spaces. Proposition 38.14 excludes any two-dimensional constituent of scalar complex conjugation at a point satisfying the stated Frobenius spread bound. Theorem 38.16 gives one integer controlling the trace congruences for any specified integral cuspidal spectrum on the interval. Theorem 38.20 identifies the exact finite-depth strict group in the normalized lower root using the scheme-theoretic images of local conjugation. Its dual is a quotient of the cotangent module in Corollary 38.21. At two coincident inertia weights, Theorem 43.3 proves finiteness of the reducible trace locus whenever Frobenius separates the three characters. Corollary 43.4 puts the exceptional characters in explicit dyadic subfields of the same period fields. Theorem 43.7 uses a second auxiliary prime to construct a non-CM ordinary form at the original weight even for these exceptional characters. Theorem 43.9 bounds its trace congruences on the common interval. Theorem 43.12 bounds the integral loss in recovering all ordered Frobenius characters from good traces at the original weights. Corollary 43.13 identifies the ordinary Galois component through the constructed adjoint under the stated tame conditions. Theorems 43.14 and 43.16 compute the complete residual fiber of the split trace before and after imposing the local characteristic polynomials of a flag. Positive-dimensional fibers of the word equations occur on the original intervals. Corollary 43.17 bounds the contribution of ordered character labels to the cotangent module of every such flag quotient at the original weight. Theorem 43.20 proves integral finiteness of the reducible symmetric trace quotient at both collision residues for every globally distinct residual triple on the auxiliary quadratic field, even when its full local triple is scalar. Corollary 43.21 gives the same finiteness with the local characters of an actual flag adjoined. Theorem 43.24 removes the residual-order condition at every twist divisible by $\ell - 1$. The field used for the descent is the compositum of the real quadratic subfields of the largest period field on the interval. Theorem 43.27 treats the other collision residue at primes inert in the auxiliary imaginary quadratic field when the reduced conductor character is nontrivial and the outer character is quadratic. The ramified plane has two fixed local characters, allowing a global deformation with unchanged inertia at the coefficient prime. Theorem 43.29 computes the ramification of the induced residual representation at the coefficient and discriminant primes. Corollary 43.30 bounds its conductor and field degree by fixed powers of the original interval scale. Theorem 43.32 proves finiteness of the entire reducible trace quotient at every weight collision for primes at least five prime to the prescribed

level and quadratic discriminant. At split primes the local line gives an ordinary deformation and at inert primes the entire local trace is constant. On the polynomial-conductor intervals, Theorem 15.11 makes all the stated descended strict groups vanish at the same three primes used in the period data. Theorem 15.12 covers the entire critical sign on the full period fields at these three coefficient primes. Corollary 15.15 bounds the sum of the full complementary orders in logarithmic scale outside class norm exceptions in every tame character through its prime cutoff. By Corollary 42.4 the total residual contribution above $P(x)$ is polynomial in V. In Proposition 6.11 the remaining odd-prime contribution comes from the quotients $U_{\ell,r}/U_{\ell,r}[\ell^{b_0}]$ for $\ell \le P(x)$. Theorem 15.25 bounds the logarithmic orders of the rational even-twist groups by a polynomial in the weight. Corollary 15.26 removes the invariant part of these residual quotients on every original field. Theorem 15.28 removes every odd character of order two or four inherited from the fixed dyadic tower. Corollary 15.29 accounts for every complementary weight on each fixed imaginary quadratic subfield of that tower. Corollary 45.32 removes all weights on a real or complex subfield of degree of order $(\log V)^2$ at the coefficient primes $V \le p \le P(x)$ congruent to three or five modulo eight. Corollary 45.34 treats the odd weights on the fixed subfield of exponent eight. Theorem 45.35 treats all weights on a single growing subfield down to the smallest primes in both progressions. Corollary 40.3 bounds the total extra length admitted by the polynomial ordinary equations on the strict symplectic root at the colliding weights, uniformly in the finite precision. Theorems 40.4 and 40.5 recover an ordinary flag from finite matrix equations and determine the sharp uniqueness precision when the residual character is locally trivial. Theorems 40.6 and 40.7 recover the strict groups at every depth from the global symplectic equations for all locally nontrivial even residual characters at the two colliding residues. Theorem 40.10 identifies the full strict length with the valuation of a Fitting ideal at the split trace, for every residual rank in that calculation. Theorem 40.12 gives the least ramification index needed to realize each integral trace by a global ordinary representation; Theorem 40.13 computes the Frobenius determinant obstruction on its two-plane locus. Theorem 40.14 bounds the entire strict length by Frobenius valuations with generic multiplicities when the trace algebra is finite. Corollary 40.15 gives a representation in characteristic zero or a nonconstant representation in characteristic ℓ whenever this strict group is nonzero. Theorem 40.17 recovers the closed trace algebra from the source's two marks even with repeated local residual eigenvalues. Corollary 40.19 identifies its finiteness clause for each covered datum with the one in the strict-length inequality. Theorem 40.22 embeds the tangent at an irreducible characteristic-zero trace into the finite symplectic adjoint Selmer group. Proposition 40.21 verifies the coefficient-prime condition on first-order lifts, including weight differences of one. Theorem 40.24 verifies the cyclotomic image condition for every finite extension at the covered characteristic-zero points. Theorem 40.25 gives multiplicity one whenever a rank-four point has a polarized cuspidal companion over a totally real or CM field. A cuspidal match over the fixed field $\mathbb{Q}(i)$ already gives multiplicity one and integral Frobenius eigenvalues of absolute value $p^{(r+1)/2}$ by Proposition 47.1 and Theorem 47.3. Theorem 47.2 excludes a cuspidal match for either even plane by the sign of its pairing. Corollary 47.6 gives the polynomial bound for the sum of strict lengths on each period field for triples with finite trace algebra and cuspidal matches at all nonsplit points. The levels, Frobenius prime, multiplicities and common integer are supplied by the argument. Proposition 47.8 proves the odd strict vanishings for the exact character conductors prescribed in both source cases. Theorem 47.9 gives the common bound for those conductors under the three stated global arithmetic hypotheses. Theorem 47.11 excludes the even plane by either of two explicit residue sums and proves full strict vanishing when the negative-root sum is nonzero. Corollary 47.12 removes the plane hypothesis on the corresponding triples. Theorem 47.13 computes the trace congruence ideal of an even plane and bounds its depth by a critical Dirichlet value. In Corollary 47.15 a polynomial bound on

the total multiplicity of such points suffices in place of excluding them. When the two strict residual dimensions are zero and one, Theorem 47.16 bounds the sum of all congruence depths inside the plane locus by a single critical value. Corollary 47.18 accounts for the two transverse coordinates separately. At any integral two-plane point, Theorem 47.20 computes the ordinary extensions in each off-diagonal direction at every finite coefficient depth. Their ranks are two and one, and each one-sided extension has the original traces. Proposition 47.22 removes the scalar part of the obstruction to mixing the two extensions. Theorem 47.23 identifies the kernel of the remaining traceless equation with the transverse trace tangent space. Theorem 48.2 constructs the totally real and CM fields required for the odd-order case with two auxiliary quadratic fields common to the interval. Corollary 48.3 descends the strict groups to each original period field. Theorem 48.5 supplies the two local witnesses and the nontrivial unramified quadratic character in the even-order tower. Corollary 48.6 gives the field construction and descent on the same interval in both cases. Proposition 48.7 constructs one field of polynomial Galois exponent for the entire interval. Theorem 48.8 proves integral finiteness over the traces at primes split in that field, and Proposition 48.9 gives equality over each separated auxiliary base. Theorem 48.11 recovers the four diagonal projectors over this common splitting field with an explicit integral denominator. Corollary 48.12 bounds the loss in trace valuations and preserves the complete generic trace algebra. Theorem 48.14 constructs an ordinary nonsplit extension at every two-plane point in either characteristic. Corollary 48.15 detects a failure of finiteness at a rational prime split in the common field. Theorem 48.18 constructs a cuspidal symmetric-cube initial point with the required residual lattice, multiplier and ordered flags for every odd-order triple, using one level prime throughout the interval. Theorem 48.20 constructs an even-order Hilbert cuspidal plane with opposite ordinary quotients at the split coefficient prime. Its quadratic induction is cuspidal and remains irreducible on the auxiliary cyclotomic kernel by Corollary 48.21. Proposition 48.22 changes one Hilbert weight at the same tame level. Theorem 48.23 and Corollary 48.24 give regular cuspidal initial points with the prescribed residual matrices and flags in both character cases. At both initial points Proposition 48.25 gives vanishing of the two finite adjoint Selmer groups. Theorem 48.27 identifies the unframed ordinary deformation ring in characteristic zero with its formal weight space. In the even-order case Theorem 49.1 identifies the entire ordinary family at the initial point with the quadratic inductions of the ordinary plane over the specified field. Corollary 49.3 gives the resulting trace identities on every component through that point. Theorem 49.4 gives the equations for the symmetric cubes at the odd-order point. Theorem 49.5 computes their ordered weight obstruction on the original period fields, and Corollary 49.6 compares its exact depth with the torsion cutoff. Theorem 49.8 excludes the even-order quadratic trace identities at all characteristic-zero rational fixed-weight points. Corollary 49.9 gives a torsion intersection and proves finite length on the two-plane locus and at all even orders greater than two. Theorem 49.12 proves finite length of this intersection for every even order by proving that the local ordinary line of the remaining inducing plane is globally invariant. Theorem 49.13 computes the exact power of the coefficient prime annihilating the full integral intersection. Corollary 49.14 places its cohomological images below the cutoff on the original period fields. Theorem 49.16 bounds the precision of symmetric-cube characteristic polynomials on the period fields using a single inertia element, without an ordering of the eigenvalues. Corollary 49.17 gives the corresponding uniform length estimate. Theorem 49.18 determines the full polarized ordinary deformation rings at both cuspidal starting points and the equations for symplectic descent. Corollary 49.19 gives coordinates from Frobenius traces at rational primes splitting in all the period fields. Proposition 50.1 computes the ramified local cohomology at the cuspidal points to every precision. Theorem 50.2 obtains their Frobenius coordinates from one progression on the interval whose primes preserve the strict cohomology on every original field

for the indicated weights. Theorem 50.4 identifies the ordinary Hecke algebra and the polarized deformation ring after completion at either cuspidal point. Corollary 50.5 gives the Hecke relations on every marked component through that point. Proposition 50.6 gives an integral Hecke component reaching the prescribed weight. Theorem 50.8 realizes every characteristic-zero eigensystem in that Hecke fiber by a classical automorphic system over the same auxiliary CM field after restoring the multiplier. Theorem 50.9 proves density of simple Frobenius spectra on the cyclotomic kernels of the irreducible ordinary curves with separated weights in characteristic ℓ. Corollary 50.10 fixes their discriminant valuations at common split primes for every cyclotomic depth. Theorem 50.12 bounds the finite-coefficient dual classes passing the Frobenius tests uniformly in both precisions. Corollary 50.13 bounds the ordinary Selmer lengths on the irreducible curves. Proposition 50.11 identifies the quadratic case requiring extra primes inert in the inducing field. Theorem 50.15 detects the full dual cohomology, including this quadratic case, at primes split in every period field by testing one matrix coefficient at each assigned prime. Theorem 50.18 prescribes the exact cyclotomic depths at these same primes. Corollary 50.20 computes the integral cost of isolating the unramified tuple at commuting Frobenius specializations. Corollary 50.16 gives dual Selmer vanishing with these local conditions. Proposition 50.17 computes their tame lifting obstruction. For the original fields, Theorem 15.31 recovers every strict class and its exact order from normalized Frobenius values. Corollary 15.32 bounds the number of common test primes polynomially on the interval for good coefficient primes of polynomial size. The orders of the coordinate images in that corollary remain to be bounded. Theorem 15.34 identifies the kernel of restriction to the absolute commutator subgroup at every coefficient prime and bounds its full weighted contribution by $(t(x)-1)\log g_\psi$. By Corollary 15.35 a common list of split primes detects the quotient also at ramified primes and at two. Theorem 15.37 computes the strict subgroup at arbitrary precision from ideal classes and local units on one cyclotomic layer for all period fields. Proposition 15.39 bounds the full order of the quadratic summand of discriminant -1208 at three by nine at every odd weight. Theorem 15.41 determines this order to be three at every positive odd weight. Corollary 15.42 bounds the total contribution at three for the indicated quadratic characters with real discriminant below 100000.

The period estimate. Sections 58.1 and 58.3 give an integral denominator for weight projections on definite lattice theta coefficients and a seesaw formula for a unitary lift. For an iterated chain Theorem 58.7 computes the output central character. Corollary 58.9 gives an explicit large family of mutually orthogonal images on the same fields E_x. The position of the final projection is decisive. If A_i has image in one central character and P_ν denotes the orthogonal projection to another, then $P_\nu A_i = 0$. In particular, on a source of dimension $r_i > 0$,

$$\|(P_\nu A_i)^*(P_\nu A_i) - I\|_{\mathrm{HS}}^2 = r_i.$$

This follows from orthogonality of distinct characters and $\operatorname{tr}(I) = r_i$. Thus a family that separates into many central characters before projection does not on that account satisfy the near-isometry estimate for nonzero sources in a prescribed common Bessel component. For the first Bessel position on the prescribed terminal graph rows, Corollary 74.22 gives this exact loss on the separately zero-sum columns counted in Corollary 9.14. The rank on the full refined table is at most its multiplicity divided by $(T-1)^f$. Theorem 59.24 compares all the terminal exchanges on these columns after a specified translation, retaining the original graph at the symmetric endpoint. Theorem 59.25 includes the two auxiliary primes on a specified common translation there. Theorem 59.27 transports every theta test to a common integral subdivision. Theorem 74.25 bounds the rank on pairing the transported fine first test space back with its original space. Corollary 59.28 includes both auxiliary primes at the same time and gives the coefficient ideal on the marked tensor of tests. Theorem 74.26 rules out returning

a matched translate to its original signed projector. Theorem 64.38 instead performs all exchanges on the original auxiliary rows for a specified rational conjugate of the Bessel character. Corollary 104.16 chooses its unmarked support above the cohomological cutoff. The common denominator for the additional level projection is given in Theorem 104.15. Theorem 64.40 proves vanishing on the original spherical rows for the choice close to the identity at infinity with good split support. Theorem 9.21 and Proposition 64.41 retain the exchange using integral unmarked root ratios and an explicit archimedean displacement. Theorem 85.18 bounds the actual analytic integral at that displacement. Theorem 105.1 evaluates the last restriction in the arithmetic class sum. Corollary 105.2 gives its exact normal-line factor on these fields. Theorem 74.29 bounds the full finite and archimedean variation of this last normalization across the selected words by one integer of polynomial logarithm. Theorem 82.32 bounds the absolute scalar of the last restriction in the highest-vector metrics, and Proposition 82.34 identifies its integral coefficient maps even above the terminal prime. At the base tuple and matching marked levels, Theorems 97.8 and 97.9 determine the last restriction on the entire arithmetic class table, including its norm on columns with separate scalar sums zero. Theorem 97.10 bounds the integral loss of this last step at all primes on the actual projected test lattice, with one integer for all words and translates at the given field. Theorems 60.32 and 97.11 express the finite signed theta column on the scalar zero-sum input in terms of the reductions of the composite matrix paths. Proposition 97.12 separates distinct rational directions below a stated height at the chosen terminal prime. Theorem 60.33 carries the arithmetic recurrence to vector markings by finite Fourier inversion. Theorem 77.20 and Corollary 77.21 compute the actual coefficient ideals for an individual vector value and for a prescribed isolated difference. Theorem 77.22 gives simultaneous integral interpolation on prescribed residue grids. Corollary 98.9 computes the contribution of the full initial table to the mean determinant at the terminal prime. Theorem 82.36 expresses the Gram operator on the actual signed arithmetic column as a positive sum over matrix frequencies of rank one. Theorem 105.3 and Corollary 105.4 bound the contribution of the integral input at the terminal primes to the determinant of the actual joint image, even when some periods vanish. The estimate is relative to the same signed maps on the unrefined vector lattices. Theorem 105.6 retains the row stabilizers when the last restriction forgets auxiliary frames. Theorem 105.7 computes the resulting multiplicities, and Proposition 105.8 estimates them from below using the residue fields of those same frames. Corollary 105.9 gives the character factor in a Haar integral for the specified class sum. Theorem 105.11 computes the signed terminal column itself at the extra ramified places and gives its Haar pairing in the original row space. Theorem 105.12 and Corollary 105.13 separate its flag translates and bound the integral cost in terms of the same conductor. Theorem 105.15 evaluates the full local Fourier integral of the last two edges at each marked place, including the base tuple. Corollary 105.16 identifies its frequency support from the preceding matrix and retains its phase and denominator in the signed column on the same fields. Theorem 105.18 computes the inner products of the marked Fourier tables, including paths with the same supporting hyperplane. Theorem 105.19 gives a sharp integral lower bound for a nonzero group of signed paths averaged over its full marked frequency table and a bound at each nonzero sampled coefficient. Proposition 105.20 proves that the terminal matrix index samples every marked residue covector equally often before the theta derivatives are applied. Proposition 105.21 realizes the terminal residue cells as a complete family of normal theta sections. Theorem 105.22 identifies the precise normal derivatives in the projected operators, and Corollary 105.23 gives their signed sum in integral combinations of torsion translates. Theorem 105.25 bounds the full matrix of normal derivatives on the common interval. Corollary 105.26 transfers this estimate to the projected terminal operators and identifies the component of a signed coefficient vector that contributes to their sum. Theorem 106.2 uses the larger prime at the endpoint to

retain the signed sum at both auxiliary primes. Corollary 106.3 bounds its mean square over the nonzero intermediate covectors at the endpoint. Propositions 106.4 and 106.5 impose the preceding covector equations for a fixed path. Theorem 106.6 estimates the coefficients after summing every terminal covector with the same initial pullback when the image at the terminal places comes from a common orthogonal summand. Theorem 106.8 proves a near-isometry after the terminal covectors with a common intermediate pullback are summed, outside an explicitly bounded set of matrix residues. Proposition 106.9 gives the rank obstruction at the initial vertex even on the Fourier support of a varying marked path. Theorem 106.11 determines how the actual signed tables with a fixed penultimate image vary inside that support. Corollary 106.12 sums the redundant terminal coordinates and gives a single prescribed test for the last varying determinant block. Theorem 106.15 estimates the signed contraction at the first intermediate vertex for marked tables supported on the specified affine subspaces. Corollary 106.16 applies the estimate to every individual marked test and to common sums over later determinant blocks. Corollary 106.18 measures the loss at the initial covectors when the first matrix varies uniformly. Theorem 106.19 expresses the actual weighted first-edge sum in terms of translates of its theta distribution at rank-one matrix frequencies. Theorems 106.21 and 106.22 compute the integral index of initial character tests and their values on the full signed column to every prescribed precision at the terminal prime. Theorem 106.25 gives the terminal valuations for minors of signed character tests through finite moments of the actual composite covectors. Corollary 106.26 computes the determinantal ideals on the full integral initial table with separate scalar sums zero. Theorem 106.29 also compares the actual projected minors on that table with their values on pure scalar characters at every finite place. The total loss per minor dimension is at most $f\log(T-1)$, and the archimedean Gram operator is the sum of the character Gram operators with the induced norms. Theorem 106.32 computes the row normalization in the two analytic evaluations of the prescribed periodized columns. A single phase works for every word and disappears from every mixed Gram entry by Corollary 106.33. Theorem 108.3 computes the period ideals on the image of the integral tests under the finite projector from actual lowered class coefficients at all three marked primes. Rational frames on the original Hermitian spaces realize the least coefficient valuation in each exterior degree. Proposition 108.2 also gives a kernel description for the saturated projected tests. Theorem 108.6 expresses each definite Gram determinant as a positive sum of the squared absolute values of actual projected period minors. The same rational weights work at all complex embeddings and are units at the three marked primes. Corollary 108.7 gives an archimedean bound for one minor attaining the three selected local minima simultaneously. Theorem 108.9 bounds the full ideal norms of the actual definite periods by their complex Gram determinants at all embeddings. Corollary 108.10 gives this comparison on the image test lattice with zero additive cost and on the saturated projected lattice with cost at most the logarithm of the marked group order per exterior degree. Theorem 98.12 proves divisibility of each definite theta operator on integral Weyl lattices at the good primes interchanging the two halves of the CM type. The loss at each odd Frobenius translate is at most the length of the first row of its partner partition. Theorem 108.12 carries the actual word count through the signed class maps and both projected test lattices at the good signal primes. Corollary 108.13 compares it with the Gram determinants of those very columns, retaining the scalar and outside-prime costs. Theorem 9.22 evaluates the complementary quadratic sign at the actual word weights. Corollary 9.24 bounds the centered count by the definite Gram determinant minus its explicit signal main term, together with costs computed from the finite input data. Lemma 108.14 and Theorem 108.15 include every active signal prime for the prescribed finite tests. Corollary 108.16 removes the omitted-prime sum and computes the signal denominators for elementary columns. Theorem 108.18 computes the outside loss away from the terminal primes in terms of the dif-

ferent and the original split test functions. Corollary 108.19 sums this estimate on the same two period ideals used for the centered count. Theorem 108.21 computes a bound at the terminal prime after removing the initial and normal-coordinate scalars. Corollary 108.22 gives the centered inequality with every finite term read directly from the fields, weights, levels and input tests. Theorem 98.14 uses the rank of the actual terminal frequencies to replace the total partition size by its first row. Corollary 98.15 gives this improvement for the original signed columns on the two projected test lattices. Theorem 98.17 treats arbitrary split matrix shifts using their elementary divisors. Corollary 98.18 gives the corresponding costs of the original row tests from their finite Fourier transforms. Theorem 43.34 bounds the class number of the cyclic field of a colliding character. Corollary 43.35 eliminates all quadratic characters above a polynomial conductor threshold at the half residue, and treats explicit congruence classes for characters of odd prime order. Theorem 44.2 bounds the exponent at colliding residues with constants independent of the character order. Corollary 44.3 applies it to faithful characters of the growing real dyadic layers in the original intervals. Theorem 44.5 counts all faithful Frobenius orbits against one class group even when their twists differ. Corollary 44.6 leaves at most one cyclic scalar group up to conjugation at a half-residue twist on the indicated dyadic layers. Theorem 9.26 retains the ramified contribution on the polynomial-conductor family and determines its mean over translates. Corollary 9.27 applies the orbit bound on precisely its dyadic layer. Theorem 56.2 bounds the integral Weyl operator jointly in its rank-one directions on every even Frobenius cycle. Corollary 56.3 carries the resulting extra divisibility at all unramified inert primes to the centered comparison. Theorem 108.24 gives the lower bound $(1/4-o(1))hma\log X'$ for the extra projected-period divisor. Corollary 108.25 improves its coefficient by choosing the permitted high cosets within the same fields. Theorem 108.27 gives the centered bound after dividing the actual columns by explicit rational integers. Corollary 108.28 determines a necessary leading lower bound for the original Gram term together with its finite costs. Theorem 109.2 compares one marked determinant with the archimedean Gram determinants of the same columns. Corollary 109.3 gives a comparison with error independent of the coefficient-field degree using a finite family of such determinants. Theorem 109.5 computes the translation averages of the full signed columns from their finite coefficients. Corollary 109.6 determines exactly which lists can be made independent by allowing separate torsion translations. Theorem 109.8 bounds the Gram determinant at the prescribed tests in terms of finite coefficient vectors and the reference vacuum norms. Corollary 109.9 inserts this bound into the centered comparison on the same interval. Theorem 56.5 computes every reduced elliptic theta derivative from orders below the split prime. Corollary 56.6 applies this to minors at the prescribed tests, with the coupling and signs in the original column. Theorem 56.8 determines the rank of the reduced elliptic evaluations at a list of shifts. Corollary 56.9 gives independent control of their low derivatives on an entire product edge when additional input translates are allowed. Theorem 56.11 and Corollary 56.12 recover the coefficient span of a fixed signed elliptic product using positive derivative orders at one prescribed tuple of shifts. Corollary 56.13 gives the rank of its one-coordinate derivative matrices in terms of a finite set of nonzero roots. Theorem 56.15 finds common split primes with long Frobenius orbits. Corollary 56.17 proves that a nonzero bounded-degree symbol on any fixed nonzero torsion array survives among its positive mixed derivatives, even with arbitrary correlations among the shifts. Theorem 56.19 computes the exact integral cost of prescribing jets at the original torsion shifts using a common input level. Corollary 56.20 gives the determinantal ideals after a specified signed linear projection on those jets. Theorem 56.22 and Corollary 56.23 evaluate the archimedean determinant of that same elliptic table with its normalization. Corollary 56.24 evaluates minors with prescribed signed rows and input columns by derivatives of a single theta product. Theorem 56.27 computes the absolute rational norm of the complete marked elliptic

interpolation determinant from the CM annihilator ideals. Corollary 56.28 computes its mean content at each rational prime, including primes dividing the level or the shift orders. Theorem 56.30 bounds signed sums of complete elliptic blocks with a fixed multiplicity partition, uniformly in the shifts. Corollary 56.31 treats specified derivative orders on the same complete input basis. Corollary 109.12 bounds the archimedean contribution of the original projected columns using their signed coefficient matrices and the exact CM area contribution of the total degree. Corollary 109.15 gives this contribution in any chosen test metric. The optimum recovers the signed coefficient mixed determinant within one unit in its logarithm per column. Theorem 56.33 determines the valuation of each prescribed derivative of an ordinary translate whose full primary order survives reduction. Corollary 56.34 computes the leading residue of the signed columns at the same test. Theorem 56.35 gives each translated germ on one explicit division field, including its phase. Theorem 56.37 then separates the character components of the original period determinant under field automorphisms. Theorem 56.39 identifies a character by the product of the two split coordinates of a shift. Corollary 56.40 proves independence for a family of prescribed derivative orders and identifies the trivial character on a pure terminal frequency. Theorem 56.42 sums a complete cyclic terminal Fourier projection. At a place where its frequency subgroup is connected, every derivative for a nonzero character is a unit in the original differential. Corollary 110.3 removes half the terminal Fourier index from the finite-place bound for the prescribed signed periods, retaining the first-row derivative cost and the original fields and intervals. Corollary 110.6 cancels the remaining explicit terminal Fourier index against the squared norms of the same complete projections at the complex embeddings. Its coefficient matrices retain the signed covector sums. Theorem 110.8 computes the residue rank of a terminal derivative table on its nonzero covectors. Corollary 110.9 gives the ranks at every homogeneous order, before the intermediate covectors are summed. Theorem 39.33 cuts the exceptional trace algebras to finite modules at common split primes while preserving any prescribed finite integral spectra. Corollary 39.34 bounds the strict lengths by a torsion exponent and a determinant on the resulting characteristic-zero factors. Theorem 110.11 computes the reduction after the full sum over a shared covector at one connected block. Its moments use exponents bounded by the first row of the prescribed Weyl shape. Corollary 110.12 determines the loss on a scalar orbit for the unshifted moment map. Proposition 110.13 sums the scalar covectors through the whole signed chain. Theorem 110.14 proves a valuation gain at every edge for a separately zero-sum input. Corollary 110.15 carries it to the upper bound for the divided projected columns. Lemma 44.16 computes a cyclotomic specialization from an ordinary ideal class group, including characters trivial at the coefficient prime. Theorem 44.17 and Corollary 44.18 bound the full lengths away from a sparse set on the common interval. Theorem 44.20 realizes the visible part of the exceptional dyadic ideal classes at a common prime testing the elliptic periods. Corollary 44.21 bounds the total loss in their weighted lengths by the logarithm of the degree of the test field over its maximal abelian subfield. Theorem 110.17 gives a uniform saving per terminal block for every pure nontrivial scalar character, averaged over coefficient places. Corollary 110.18 gives a saving for a prescribed sum in terms of its scalar support. Theorem 45.5 computes the unit norm index and capitulation kernel between cyclotomic layers at a surviving dyadic orbit. Corollary 45.6 gives a test with ordinary units modulo the coefficient prime that bounds every weight of the orbit when the tested norm map vanishes. Theorem 45.7 computes the invariant class group lengths from units of the residual field. Corollary 45.8 bounds all nonsplit zero-residue exceptions by one class number product, and Proposition 45.9 proves full vanishing at seven in the real quadratic dyadic layer. Theorem 45.10 computes the constant specialization from units supported at the coefficient prime. Corollary 45.15 computes the number of generators of the complementary group at every positive multiple of the residue period, outside one integer of

polynomial logarithmic size. Theorem 45.14 also gives the complete scalar length when the two unit logarithms determine it within the available precision. Theorem 45.18 proves full strict vanishing at positive multiples of the residue period in the first three real dyadic layers for coefficient primes from five through ten million. For primes outside the two indicated classes modulo sixty-four it applies to every dyadic layer. Theorem 45.21 treats every dyadic layer at all odd primes through ten million and isolates two possibly nonzero characters at the specified weights. Corollary 45.23 removes their restrictions from the residual sum on the common period intervals. Theorem 45.26 gives vanishing at the half-residue weights throughout a growing range of dyadic layers by comparing relative regulators with a joint mean of Dirichlet values. Corollary 45.27 applies it on the same fields and intervals as the signed periods. Corollary 45.28 removes all weights on one common dyadic subfield of growing degree for the stated range of coefficient primes. At primes three modulo eight, Corollary 45.29 removes every weight on the full real dyadic part in the same range. Corollary 45.12 proves vanishing at every positive zero-residue weight for the real quadratic dyadic character and all coefficient primes from five through ten million. Theorem 110.20 evaluates the mean terminal Gauss valuation on the actual CM half in terms of the auxiliary imaginary quadratic class number. Corollary 110.21 gives the resulting uniform saving on the common period intervals. Theorem 110.23 computes the signed Fischer pairing of class paths in common product coordinates. Corollary 110.24 bounds the prescribed period determinant in terms of explicit factorials and the norms of its initial forms, finite transports and endpoint tests. Theorem 111.3 constructs a common divisor for all the signed columns at each translate and evaluates its logarithm from the Frobenius cycles. Corollary 111.4 controls its effect on every projected minor and Gram determinant on the same intervals. Theorem 111.7 evaluates the large-prime part of the common divisor from Frobenius cycles, explicit real integrals and logarithmic derivatives at one. The remaining prime evaluations run only up to a fixed power of the interval variable. Theorem 111.9 compares the compact factorials of every word with the incoming real hook products on the same interval. Lemma 111.8 identifies the normalized terminal restriction in the signed coefficient matrix. Theorem 111.12 computes the leading constant of the common integral divisor from the order of the original Dirichlet character. Corollary 111.13 inserts it in the determinant of the prescribed divided columns. Theorem 112.2 determines the contribution of the connected terminal Fourier coordinates modulo every power of the marked prime. Corollary 112.3 gives the truncated elementary divisors after the original signed contraction. Theorem 112.5 gives a two-sided complex norm estimate at the nonzero terminal covectors for each fixed translated product. Corollary 112.6 bounds its Gram determinant and its pairing with the projected incoming table. Theorem 112.9 preserves this rank after subtracting the mean on each actual nonzero scalar orbit. Corollary 112.10 gives the same comparison after the separate projections at all terminal blocks of one product term. Theorem 112.12 controls signed sums of distinct torsion translates at the prescribed point. Corollary 112.13 gives a rank and Gram determinant comparison within each coset of torsion supported at the two auxiliary primes with bounded powers. Corollary 112.14 computes the first contraction on the full refined initial class in its automorphic norm. Theorem 112.15 evaluates the inner product of two prescribed matrix pullbacks. Proposition 112.17 counts the terminal markings at which their actual input lines can coincide. Theorem 112.19 bounds the height of representatives for every tuple at the prescribed marking. Proposition 112.20 computes the integral image and the complex norm from the signed paths on those same lines. Theorem 39.30 gives finite quotients bounding all exceptional strict lengths at at most three common split primes on the period interval, also for positive-dimensional original trace algebras. Corollary 39.31 gives an explicit length bound from the degrees of suitable initial forms of their relations. Theorem 44.8 describes the possible large lengths of the faithful dyadic scalar groups by one cyclotomic parameter. Corollary 44.9

leaves at most one weight per indicated prime and dyadic level on the entire common interval. Theorem 9.29 concentrates the residual cohomology simultaneously at all primes up to the cutoff on a sparse set of weights. Corollary 9.30 gives the weighted sum still needed on that set. Theorem 44.11 improves the exceptional-weight count for the characters nontrivial at the coefficient prime. Corollary 44.12 bounds their full orders away from those weights on the period fields. Theorem 44.14 detects a length beyond any prescribed depth using one finite class group and its Galois action. Corollary 44.15 isolates the only possible length above twenty-five for the faithful dyadic half-residue weights in the stated prime window. Theorem 39.2 constructs the integral local symplectic ring in six coordinates. Corollary 39.3 identifies its flag condition on the component through the split representation. Theorem 39.5 counts the global symplectic generators and relations for the surviving dyadic orbit. Corollary 39.6 gives its presentation after taking the first torus invariants. Theorem 39.8 identifies the full strict length with an integral congruence quotient. Corollary 39.9 generates its augmentation ideal by at most three good-prime trace differences. Theorem 39.11 describes the torsion and integral congruences when the trace algebra is finite. Corollary 39.12 then bounds the exceptional length by a determinant of actual trace differences. Theorem 45.2 detects the opposite ordinary direction using global units modulo the coefficient prime. Corollary 45.3 removes one trace parameter whenever this unit projection is nonzero. Theorem 39.14 realizes the nonzero trace coordinates by an irreducible rank-four representation or a pair of even planes. Corollary 39.15 shows exactly how a positive-dimensional special fiber gives such a representation in characteristic equal to the coefficient prime. Theorem 39.17 proves that irreducibility persists after restriction to each period field. Corollary 39.18 identifies the two planes in a reducible trace as a single even plane and its prescribed twist. Theorem 39.20 chooses at most three parameter primes that split in all the period fields of the interval. Corollary 39.21 separates all the finite generic spectra at one such split prime. Theorem 39.23 determines the connected algebraic monodromy at the trace witnesses. Corollary 39.24 verifies the distinct-eigenvalue projector test on the cyclotomic kernel over each period field. Theorem 39.26 chooses the trace parameters with the split-prime theta estimates on the same fields. Corollary 39.28 bounds the strict length of a finite exceptional trace algebra by its least determinant valuation and computes that valuation branch by branch. Theorem 68.15 computes the index of the additional Hecke modification at the terminal prime and bounds its effect on the projected image on the same fields. Theorems 74.31 and 68.18 compute the scalar normalization at a common terminal prime and its exact contribution to the determinant after signed projection. A lower bound for their signed Gram sum and a comparison with the analytic incoming row are still required. Proposition 58.10 controls the content of a finite Fourier transform when the determinant fibers of the relevant period are rational on the test lattices.

One can also put all the determinant twists into one output central character from the outset. For the ranks $n_i = i + 1$ and chain length m, Theorem 58.11, writes the two second moments required in Proposition 6.9 as correlations of determinant fibers after the prescribed output map. There are $|C(P_0)|^{m-2}$ twists in a given central character, where P_0 is a fixed unramified auxiliary prime in Volume I. Its Proposition 99.12 shows that if both neighboring kernels at each intermediate group are invariant under the integral determinant units at P_0, the proportion of nonzero maps on the common fields is at most $\exp(-c_{P_0}x^{5/2+8\varepsilon})$. Zero maps alone then violate the first error bound in Proposition 6.9 for that full family. The computation identifies a restriction on the choice of levels. The common ramified vector in Theorem 99.14, gives vanishing cross terms after restriction. Its individual norms are treated below.

To finish the geometric argument one must identify the finite-place divisibility of the specified Bessel periods with the weighted count Δ_u, and bound their norms, ranks and covolumes after projection in that same target. The required result is (6.3) simultaneously for all translates and

all x in a selected interval. The following observation measures the strength of that estimate for the explicit counts.

Proposition 6.13 (The strength of the remaining count estimate)**.** *Fix a primitive ψ. Use the explicit data of Proposition 6.4 on the common intervals of Volume II, with Π fixed as above. The following two assertions are equivalent.*

1. *Every zero in the open strip of every $L(s,\chi)$ with $\chi \in \mathcal{C}_\psi$ has real part $1/2$.*
2. *For every sufficiently small $\varepsilon > 0$, there are arbitrarily distant such intervals on which (6.3) holds for all u at every x.*

At conductor one the first assertion is the Riemann hypothesis for $\zeta(s)$.

Proof. Under the critical-line assertion, Theorem 3.3 gives $G_\chi(x) = O_\chi(\sqrt{x})$ for each of the finitely many retained characters. Summing their fixed weights gives the same bound for $\mathcal{G}_\psi$. By (6.4) the explicit count then satisfies $\Delta_u \ll_{\psi,\varepsilon} Qx^{1/2+\varepsilon/2}$, uniformly in u wherever the data are chosen. Their parameters have $a_E = xb$, $m = \lfloor\sqrt{x}\rfloor$, $Q = hmb$, and $\mathcal{B} \geq ha_E$. Thus $\mathcal{B}/Q \geq x/m \geq \sqrt{x}$. Since $5\varepsilon > \varepsilon/2$, inequality (6.3) holds everywhere on every sufficiently distant common interval.

Conversely the first three conditions in Proposition 6.3 hold for the explicit data. The assumed fourth condition gives $\mathcal{G}_\psi(x) \leq x^{1/2+9\varepsilon}$ throughout arbitrarily distant power intervals. Apply Theorem 2.1 with exponent $1/2 + 9\varepsilon < 1$, and then let ε decrease to zero. Every open-strip zero of every member of the family has real part at most $1/2$. The primitive functional equation reflects any zero of smaller real part to one of larger real part for the conjugate character, which is also in this family. This proves the converse. At conductor one the retained family consists only of the trivial character. □

Which global lengths remain. Fix x on the interval and $E = E_x$. For each odd prime ℓ set $h_\ell = b_0$ if $\ell \leq P$, and $h_\ell = 0$ otherwise. Use the product decomposition $\mathrm{Gal}(E/\mathbb{Q}) = P_\ell \times H_\ell$ into its ℓ-Sylow and prime-to-ℓ factors. For a character α of H_ℓ, take its extension trivial on P_ℓ, and restrict $\vartheta = \kappa_\ell^{-r}\alpha^{-1}$ to the absolute Galois group of E^{H_ℓ}. In the following sums use only the complementary sign $\alpha(c) = (-1)^r$. The unramified coefficient ring $\mathcal{O}$ may depend on E, ℓ and contains all values of the characters of H_ℓ.

We retain triples (ℓ, r, α) with $1 \leq r < t$ subject to three conditions. First, the field average $J_{\ell,r}$ above acts by zero on this component. A nonzero field average acts by one exactly when α is trivial on the subgroup being averaged, since that subgroup has order a power of two and lies in H_ℓ. Second, $\bar{\vartheta} \neq 1$ on the absolute Galois group of E^{H_ℓ}. Third, $v_\ell(r+1) < e_{\ell,r}(E)$, where $e_{\ell,r}(E)$ is the exponent index of the finite Kummer group in Theorem 17.13. It can be read from ordinary ideal classes with their local generator conditions. The example of Proposition 17.15, shows why positive lengths genuinely occur in these families.

For a retained triple let $B \subset \mathcal{O}^n$ be the integral relation module of Theorem 23.4. For $0 \leq j \leq n$ let D_j be the minimum valuation of the j-rowed minors of a relation matrix, with $D_0 = 0$. These are finite because B has rank n. Put

$$\mathcal{R}_*(x) = \sum_{\text{retained } (\ell,r,\alpha)} (\log \ell) \max_{0\leq j\leq n} \{D_n - D_j - (n-j)h_\ell\}. \tag{6.6}$$

Proposition 6.14 (The remaining lengths in terms of global relations)**.** *On all sufficiently distant common intervals chosen above,*

$$0 \leq \mathcal{L}(x) - \mathcal{R}_*(x) \leq \tfrac{1}{2}\exp(x^\varepsilon/2) + 3\exp(x^{2\varepsilon/3}) + 2\exp(x^\varepsilon/3). \tag{6.7}$$

In particular $\mathcal{R}_*(x) \le \frac{1}{4}\exp(x^{\varepsilon}/2)$ *at every* x *on arbitrarily distant such intervals would suffice for* (L). *For* $\ell > P$ *the finite local conditions already recover every finite layer of the strict group at its original exponent. At* $\ell \le P$ *one can recover the layer at* ℓ^k *using finite local conditions at* ℓ^{k+s_*} *with* $s_* \ll_{\psi,\varepsilon} \log(2x)$.

Proof. Extending scalars from $\mathbb{Z}_\ell$ to the unramified $\mathcal{O}$ does not change the normalized module length, by filtering into factors of length one. The idempotents of H_ℓ decompose the resulting module as a direct sum; positive-degree cohomology of this prime-to-ℓ group vanishes by averaging. Lemma 17.8, identifies the summands with the indicated strict groups over E^{H_ℓ}, also at primes dividing the order of the original Dirichlet character. The groups have finite total prime support for each x by Proposition 11.4.

By Theorem 23.4, the elementary divisors of the retained group are those of $\mathcal{O}^n/B$. Write their valuation exponents $0 \le b_1 \le \cdots \le b_n$, including zeros. Smith normal form gives $D_j = \sum_{i\le j} b_i$: integral invertible row and column operations preserve the ideals of minors, and the statement follows for the diagonal matrix. Removal of the subgroup killed by ℓ^{h_ℓ} leaves length $\sum_i (b_i - h_\ell)_+$. This equals the maximum in (6.6), by including in the tail exactly the exponents greater than h_ℓ. Thus the summands are actual lengths and nonnegative. A determinant in Theorem 17.9, bounding the full exponent by ℓ^{h_ℓ} makes its summand zero.

Proposition 6.11 bounds the removal of the dyadic groups, the critical sign, the field averages and the indicated finite truncations. Among what remains, the globally trivial residual characters contribute at most $\exp(x^{2\varepsilon/3})$ by Corollary 17.10. Its hypothesis uses our choice of Π. All pairs removed by the third condition contribute in total at most $\exp(x^{\varepsilon/3})$ to the full complementary length, by Corollary 17.14. Removing already-counted subgroups or truncated summands only decreases these bounds. Addition proves (6.7). The two latter exponential orders are $o(\exp(x^{\varepsilon}/2))$, which gives the stated sufficient condition.

For each retained component let s_* be the largest local loss of Theorem 23.1. Its exact finite-level identification in the proof gives the last two assertions. Indeed $s_* \le 1 + \log_\ell(t[E : \mathbb{Q}]) \ll_{\psi,\varepsilon} \log(2x)$. For all sufficiently large x, both t and the cyclotomic modulus N are below P, since $\log(2N) \ll_{\psi,\varepsilon} x^{\varepsilon/4}$. For $\ell > P$ any nonzero local loss would require $\ell \mid N$ or $\ell - 1 \mid r$, by Corollary 23.2. Both are impossible. □

The individual global periods. Use an unramified prime P_0 and the standard definite chain of Section 58.3 on E_x. In Theorem 99.18, one fixed orbit of finite Bessel level classes gives a common integral test space of rank n_0, with one line for each of the n_0 twist characters γ. Let h_B be the number of such orbits. The covolume in the scalar coordinates and its bound are proved there. Let $p_\gamma(y)$ be the integral mean of the actual restricted theta lift on the orbit representative y against the common archimedean test, in Haar probability. Then the projected column has squared norm $|p_\gamma(y)|^2/h_B$ on a unit source, and all distinct columns are orthogonal. Proposition 99.19, reduces both archimedean moments for this model to a bound for the mean squared deviation of these numbers from one. Orthogonality by itself allows every one of them to vanish. Theorems 74.42 and 74.46 recover coefficients of definite frames with bounded denominators, including fixed positive Gram coefficients when their residue image subspaces are retained. Proposition 74.47 exhibits complete cancellation on singular frame orbits with radical of dimension at least two at the same terminal congruences. Theorem 74.49 separates every integral lift of one rational partner flag. Proposition 74.50 identifies their signed operators, and Theorem 74.51 bounds interference between distinct residue cosets. Theorem 74.54 computes the signed sum at each residue in a two-sided arithmetic orbit as an explicit projection. Theorem 74.56 recovers these operators integrally on the common fields. Theorems 74.58 and 74.60 separate distinct full-rank rational transports below an explicit denominator bound

and recover the individual operator on a single two-sided orbit. Theorem 92.3 includes both common positive ranks for equal terminal norms. Proposition 92.5 gives infinitely many transverse partners with the same transport and height one. Corollary 92.8 proves separation under a cross Gram determinant bound for both original terminal frequencies. Proposition 92.9 shows why the retained columns require a sharper denominator estimate. Theorem 92.10 constructs infinitely many terms at one residue on every retained diagonal row. Theorem 92.12 identifies their signed branch sums and bounds the ordinary operator tail for each fixed row. Theorem 92.15 gives integral recovery and asymptotically equal finite norms for integral transports with fixed marked hyperplanes. Proposition 92.17 realizes each choice of marked hyperplanes on retained diagonal rows. Theorem 92.19 bounds the sum over all ordinary partners outside the rational decorated stabilizer on a diagonal row, using congruences at both original marked primes. Corollary 92.20 gives the diagonal error after the prescribed finite projection, with its dependence on the row and real section. Theorem 92.22 treats distinct original rows and both terminal norms, using explicit counting constants. Corollary 92.23 bounds their simultaneous projected operator error by row and column sums. Theorems 92.24 and 92.25 compute the singular values of the flag-preserving terms on distinct decorated rows and bound the mean square error of the full projected ordinary period there. Theorems 92.26 and 92.27 recover the identity coefficient on these rows, with an explicit bound for every other marked index and a simultaneous integral coordinate choice at the same two primes. Theorem 93.3 bounds the full ordinary diagonal error on every norm-one row by an explicit real-section norm times a uniform negative power of the original marked level. Theorem 93.6 proves the corresponding arithmetic bound for all pairs of retained primitive-frame rows at both terminal norms. It includes the full signed coefficient at the identity marking. Theorem 93.9 extends the bound to the specified outer powers of the inverse different. Their ideal norms cancel in both height separation and the exterior lattice count. Theorem 93.12 proves that the normalized absolute upper bound grows on the prime-terminal rows, using concentration of their actual lowest-type sections. The signed period comparison still requires a sharper argument. For mixed terminal frequencies Theorems 93.14 and 93.15 give a terminal-prime divisor for every nonzero intersection and a marked-period estimate in unit-frequency norms when the intersection contains at least half the outer directions. Corollary 93.18 gives a single bound for the large-denominator terms at every proper nonzero intersection, including at equal prime terminal norms. Lemma 93.16 bounds the number of remaining scalar values on those strata. Theorem 93.19 supplies a divisor of the common flag index at both marked primes. Corollary 93.20 incorporates it in the signed period estimate, including terms with small scalar denominator. Theorem 93.22 extracts a further terminal divisor from the common middle lattice of intersecting flags. Corollary 93.23 bounds the mixed period on the full range of prefix dimensions with $2t \le s + 1$. Theorem 93.25 separates distinct rational isometries by the indices of the common positive lattice, including at marked primes dividing those indices. Corollary 93.26 gives integral recovery on small-index collections and a uniform bound on the large-index terms. For transverse hyperplanes Lemma 93.27 proves a determinant congruence even at a mark dividing the pairing discriminant. Theorem 93.28 and Corollary 93.29 give a separation bound and a signed orbit trace with the actual local denominator ideals. Theorem 94.2 bounds the absolute sum of transverse terms above a cutoff given by the frame determinants. Corollaries 94.3 and 94.4 include terms detected at the terminal prime and the union of the indicated repeated-transport families. Theorem 94.5 separates distinct transports using the product of their discriminant ideals. Corollary 94.7 bounds their overlap in the remaining low-height transverse sum on the original rows. Theorem 94.8 and Proposition 94.9 count the rational classes and integral cross matrices at low frame height in both terminal frequencies. Corollary 94.10 gives the resulting comparison for the signed groups. Theorem 94.13 bounds the number of individual integral double cosets

below the height cutoff, including terms with the same marked index, for both terminal norms. Theorem 94.14 bounds the signed ordinary orbital integrals by normalized squared section moments. Theorem 94.15 computes the projected diagonal with both real heights restricted and bounds its mass at prime terminal frequency. Theorem 94.16 separates distinct transports in the proper-intersection period up to positive lattice index bound $Q/(2k\sqrt{u})$ on the f-th root scale. Theorem 94.17 compares their rational and marked ranks. Corollary 94.18 bounds the complementary signed contribution by moments of the original sections. Theorem 95.2 gives a unique rational transport at each full transverse index below discriminant height $Q/2$, including at singular marks. Corollaries 95.3 and 95.4 bound the multiplicity above that height and its contribution to the signed hyperplane period. Theorem 95.5 describes all transverse rational classes at fixed projected hyperplanes and transport in the special unitary group of their orthogonal complement. Theorem 95.6 sums the phases of the integral lifts of each such class. Corollary 95.7 gives the exact squared norm of the resulting signed operator. Theorem 95.8 factors the signed coefficient of every supported transverse rational class at a fixed index into local algebraic integers after division by the covolume of a single actual lift and the common row volume. Corollary 95.9 tests exact local cancellation and bounds its normalized terminal valuation. Theorem 95.11 computes the cancellation test on finite residue classes using the original row matrices. Corollary 95.12 bounds all conjugates of the normalized arithmetic coefficient and the number of its integral lifts by an ideal norm computed from the rational class. Theorem 95.14 evaluates the class coefficient with the covolumes of the two original rows. Its complex bound is independent of the precision used for the local residue sums. Corollary 95.16 gives the signed operator formula with this normalization. Theorem 95.18 sums different projected hyperplanes at a fixed marking and transport. Corollary 95.20 computes their cross terms exactly at normal height one. Theorem 95.21 bounds those terms at larger heights in each compact representation. Theorem 95.22 classifies the remaining rational classes above a fixed cross matrix by norm-one rotations. Proposition 95.23 bounds their integral support by a divisor count, and Theorem 95.24 computes their full contribution using one real tensor. Theorem 95.26 evaluates the sum over the frame determinants by finite local character sums with the actual marking. Corollary 95.27 gives its exact value at normalized Gram determinant one and preserves the terminal valuation for identical columns. Theorem 95.28 sums the supported rational frame determinants at all Gram heights for either terminal frequency, with a positive moment of the normalized Gram determinant. On the cell where every leading minor is nonzero, Corollary 95.32 bounds the sum of squared real orbital profiles over supported frame determinants by explicit mixed derivative energies. The estimate stays uniform down to zero terminal Gram value. Proposition 95.33 supplies integral flag minors at every prefix also on the other positive-Gram blocks. Theorem 95.35 bounds the original transverse orbital operators on all these blocks by two explicit exterior-power moments of the prescribed rows. Proposition 95.36 proves finiteness in the indicated range and computes the full terminal-frequency factor in both moments. Theorem 95.38 shows that their absolute majorant grows at prime frequency even after extracting the known lower Gram cutoff.

Proposition 6.15 (Finite divisibility and the nonzero period values)**.** *Fix such data, and suppose their normalized scalar periods p_γ are elements of a common number field J. Use the same scalar identification at each embedding $\sigma : J \hookrightarrow \mathbb{C}$. Let $G_0 = \{\gamma : p_\gamma \neq 0\}$. Suppose for some $0 \le e \le 1/4$ one has*

$$\frac{1}{[J:\mathbb{Q}]n_0} \sum_{\sigma,\gamma} \left(\frac{|\sigma p_\gamma|^2}{h_B} - 1 \right)^2 \le e.$$

For $\gamma \in G_0$ let the integers $a_{\gamma,w}$ have finite support in the finite places of J and satisfy $a_{\gamma,w} \le v_w(p_\gamma)$ at every such place; the valuations assign one to a uniformizer. Then $|G_0| \ge (1-e)n_0$

and

$$\frac{\sum_{\gamma\in G_0}\sum_{w\nmid\infty} a_{\gamma,w}\log(\operatorname{Norm} w)}{[J:\mathbb{Q}]|G_0|} \le \tfrac{1}{2}\log h_B + \tfrac{1}{2}\log\left(\frac{1+\sqrt{e}}{1-e}\right). \tag{6.8}$$

Proof. A zero algebraic period contributes one to the squared deviation at every embedding, so the stated lower bound for $|G_0|$ follows. By Cauchy–Schwarz the average of $|\sigma p_\gamma|^2/h_B$ over all γ,σ is at most $1+\sqrt{e}$. For a nonzero algebraic number, the sum of its valuation times the log residue norm over finite places is the log absolute norm to $\mathbb{Q}$, hence the sum of the logs of the absolute values across all embeddings. Apply this identity for each $\gamma\in G_0$, using the valuation bounds. Concavity of the logarithm bounds the resulting mean by one half of the log of the mean of $|\sigma p_\gamma|^2$ over the nonzero periods. That latter mean is at most $h_B(1+\sqrt{e})n_0/|G_0|$. This proves (6.8). □

To obtain (6.3) from the periods, their finite valuations and archimedean sizes must satisfy the comparison in the same normalization. The preceding proposition proves the passage from estimates on nonzero period values to the corresponding finite divisibility. We next compare the finite-place count with a divisibility statement that can already be proved on the same fields.

A local comparison at the distinguished deformation. Theorem 24.8 proves the integral cotangent estimate for presentations with arbitrary numbers of coordinates of each of the specified weights. More precisely, for an augmented integral graded algebra with the smooth chart and point hypotheses stated there, the length of the weight-minus-one cotangent quotient is at most twice the sum of the normalized valuations of a separating trace difference on its mixed scaling orbits. The proof includes components supported at the residue characteristic by taking the difference between the cokernel and kernel lengths of multiplication by that trace. For the large primes, Sections 25.1 and 25.2 construct a marked arithmetic presentation and analyze its points. The chart, rigidity and remaining arithmetic bounds are discussed at the end of this section.

The count carried by integral CM derivatives. There is now a finite-place bound over the actual fields E_x. Their common imaginary quadratic subfield on each interval is $K=\mathbb{Q}(\sqrt{-q})$. Section 52.1 constructs CM abelian schemes of the type induced from K and bounds an integral derivative of order a at their identity. For a nonsplit prime, let e_p, f_p be the ramification index and residue degree in K. In the normalization $v_p(p)=1$ the lower bound is

$$v_p(a!) - e_p^{-1}\left\lfloor \frac{a-1}{p^{f_p}-1}\right\rfloor.$$

For weights at least $2(1+\lceil\log_2(a+1)\rceil)$, every integral derivative satisfies $v_p(D^a f(0)) \ge C_p(t,a,w)/(2\log p)$ at every prime, including the ramified prime. Here C_p is the summand with rational prime p in the count C of Section 6.1. The extension field supporting the CM schemes may be fixed throughout the interval. The next result computes the size of the division that a passage from such jets to a normalized period would have to account for.

Theorem 6.16 (The integer quotient for the weighted count)**.** *Use the fields and translates of Proposition 6.4 on the intervals of Section 6.6. Write*

$$b=\exp(x^{\varepsilon}),\quad a=\lfloor bx\rfloor,\quad B'=\lceil b\rceil,\quad W=\lceil\sqrt{b}\rceil,\quad \omega=h_2/f=1/(2o).$$

Give the distinguished positions weight B' and the other positions weight W at every stage. Denote this list by w. It is independent of u since $T_2=A_0$ and $A_0+u=A_0$. Write w_* for the list identically equal to W. For each $t\in T$, define positive integers

$$D_t=\prod_p p^{\lfloor C_p(t,a,w)/(2\log p)\rfloor},\qquad D_*=\prod_p p^{\lfloor C_p(t,a,w_*)/(2\log p)\rfloor},\qquad R_x=\prod_{t\in T}(D_*/D_t).$$

The second product is independent of t, and every D_t divides D_*. Every integral CM jet of this order in Corollary 52.3, is divisible by D_* in the ring of integers of its coefficient field. On all sufficiently distant common intervals, simultaneously for every u,

$$\left|\Delta_u - \left(hma\omega \log(B'/W) - 2m \log R_x\right)\right| \le hm\, b/x. \tag{6.9}$$

Proof. We have $B' \ge W \ge 2(1 + \lceil \log_2(a+1)\rceil)$ for large x. Also $a/W \ge \exp(x^{\varepsilon/3})/3$ for large x. Proposition 7.2, therefore applies to this choice, in particular to the comparison with the retained prime sums. Every term has weight at least W, so primes $p > a/W$ contribute zero. The products are finite. Monotonicity in each weight gives $D_t \mid D_*$. With w_*, the local inequality of Theorem 52.2, gives divisibility by $p^{\lfloor C_p/(2\log p)\rfloor}$ at every prime over p. At the other places this rational integer is a unit. Taking the product gives the assertion for jets.

Set $Y = a/W$. The constant-weight count C_* is independent of t and obeys

$$C_* = W\{R_{\mathbf{1}}(Y) - R_{\xi_c^*}(Y)\}, \qquad C_* - M(a, w_*) = W\{G_{\mathbf{1}}(Y) - G_{\xi_c^*}(Y)\}.$$

In fact the sum over the entire nontrivial coset of B/Φ, including the inertia average, gives $1 - \xi_c^*(p^j)$ at every prime power. Substitution of the corrections in (1.1) gives the second identity. By the zero-free estimate of Lemma 7.1, its absolute value is at most $2a\exp(-cx^{\varepsilon/8})$ for some $c > 0$. The precise main-term difference is

$$M(a, w_*) - M(a, w) = a(h_2/f)\log(B'/W)$$

because precisely h_2 of the f positions change weight.

For a fixed t, the error in replacing $C_* - C(t, a, w)$ by $2\log(D_*/D_t)$ is at most $2\sum_{p\le Y}\log p \le 2Y\log(2Y)$. At each prime the two fractional parts differ by less than one. Finally $2a\exp(-cx^{\varepsilon/8}) + 2Y\log(2Y) \le b/x$ for large x, since $a \le bx$ and $W \ge \exp(x^{\varepsilon}/2)$. Compute $C(t, a, w) - M(a, w)$ by adding and subtracting the constant-weight count and its main term. Summing over the h choices of t and the m identical stages proves (6.9). □

Corollary 6.17 (The excess divisibility of an unnormalized jet)**.** *On this family every nonzero integral CM jet z of order a, over a number field J as in Volume X, satisfies*

$$\frac{2}{[J:\mathbb{Q}]}\log|N_{J/\mathbb{Q}}z| \ge M(a, w) + a\omega\log(B'/W) - b/x.$$

The excess term $a\omega\log(B'/W)$ is asymptotic to $ax^{\varepsilon}/(4o)$. It exceeds the allowed average error per summand $x^{5\varepsilon}\mathcal{B}/(hm)$ by a factor tending to infinity. Thus an upper estimate for the unnormalized heights at the weighted main term with that error is impossible on this family.

Proof. The integer D_* divides z in $\mathcal{O}_J$. For a nonzero algebraic integer z/D_* its norm has absolute value at least one. The rounding estimate for $2\log D_*$ and the zero-free estimate for C_* in the preceding proof give $2\log D_* \ge M(a, w_*) - b/x$. This proves the inequality. Here $\log(B'/W) = x^{\varepsilon}/2 + o(1)$ and $\omega = 1/(2o)$. In (6.3) substitute $m = \lfloor\sqrt{x}\rfloor$, $k = xm$, $a_E = xb$, $h \ge n/2$, $h_2 \le n$, and $n \ge x^{2+8\varepsilon}$. The three resulting terms give $x^{5\varepsilon}\mathcal{B}/(hm) = O_\psi(bx^{1/2+6\varepsilon}) = o(a)$ because $\varepsilon < 1/52$. This proves the claimed separation in size. □

The integer quotient has a direct interpretation in terms of ordinary prime sums. For $t \in T$ let $S_t = p_t^{-1}(L_u)$, independent of u, and put

$$\Psi_t(s) = \sum_{p^j \le s} \frac{2\log p}{|I_p|}|(jF_p + I_p) \cap S_t|.$$

For a distinguished position the difference of its two ramps equals $\int_W^{B'} \mathbf{1}_{y<a/p^j}\,dy$. The nondistinguished positions contribute zero to the difference. Interchanging finite sums and integration and writing $s = a/y$ therefore gives the exact formula

$$C_* - C(t,a,w) = a\int_{a/B'}^{a/W} \frac{\Psi_t(s)}{s^2}\,ds.$$

The expected linear function in this identity is ωs, whose integral contributes $a\omega\log(B'/W)$. Dividing a jet by the integer D_*/D_t preserves integrality and leaves divisibility by D_t. An independent lower bound for the product of the factors removed, or an independent height estimate for the actual periods after their arithmetic normalization, is still needed to prove (6.3). The lower bound required for $2m\log R_x$ follows explicitly from (6.9) up to hmb/x. Neither the separation in Corollary 6.17 nor the formal derivative estimate gives that lower bound. The cohomology bound on the same family remains an additional requirement of the proposed construction.

Theorem 6.18 (Least scalar-jet norm on the common intervals)**.** *Keep the fields, weights and notation of Theorem 6.16. On each sufficiently distant common interval choose the field J and CM abelian schemes of Theorem 52.6. At each x let*

$$\mathcal{H}_K(a) = \min_{z\neq 0} \frac{2}{[J:\mathbb{Q}]}\log|N_{J/\mathbb{Q}}z|$$

over all integral scalar derivatives $z = (D^a f)(0)$ on that scheme. Write ξ_K for the primitive quadratic character of $K = \mathbb{Q}(\sqrt{-q})$, $c_K(n) = (1-\xi_K(n))\Lambda(n)$, and $\Psi_K(y) = \sum_{n\le y} c_K(n)$. Put $Y = a/W$. Then

$$\mathcal{H}_K(a) = \sum_{n\le a} c_K(n)\lfloor a/n\rfloor, \tag{6.10}$$

$$\mathcal{H}_K(a) - C_* = \sum_{k=1}^{W}\Psi_K(a/k) - \sum_{n\le Y} c_K(n)\{a/n\}. \tag{6.11}$$

Here $\{z\} = z - \lfloor z\rfloor$. On these intervals, uniformly in x and every translate t,

$$\mathcal{H}_K(a) - C(t,a,w) = a(\log W + \gamma) + 2\log(D_*/D_t) + O_{\psi,\varepsilon}(b/x). \tag{6.12}$$

The minimum is attained by a globally defined jet. The vector may be fixed for all orders used on each field of the interval.

Proof. Theorem 52.6, computes the image ideal as $H_K(a)\mathcal{O}_J$ for one such vector, and proves universal divisibility for all vectors. A nonzero algebraic integer has absolute norm at least one. Thus $H_K(a)$ itself attains the minimum, which equals $2\log|H_K(a)|$ in either complex embedding of K. At inert p its normalized valuation is

$$\sum_{j\ge1}\lfloor a/p^j\rfloor - \sum_{j\ge1}\lfloor a/p^{2j}\rfloor = \sum_{j\ge1,\ j\ \text{odd}}\lfloor a/p^j\rfloor.$$

At the ramified prime it is $\frac12\sum_{j\ge1}\lfloor a/q^j\rfloor$, and at a split prime it is zero. These are precisely the coefficients of the prime powers in (6.10), including the factor two in the definition of the minimum.

By Theorem 6.16, $C_* = \sum_n c_K(n)(a/n - W)_+$. For $n \le Y$ the difference between the floor and the ramp equals $W - \{a/n\}$. For $Y < n \le a$ the floor counts the integers $1 \le k < W$ such

that $n \le a/k$. Summing first in n, then in k, cancels $(W-1)\Psi_K(Y)$ against all but one copy of the term $W\Psi_K(Y)$. Since $Y = a/W$, this proves (6.11) with both endpoints included.

We give the uniform estimate for the unsmoothed sum here because the conductor varies with the interval. Write $R(y) = R_{\mathbf{1}}(y) - R_{\xi_K}(y)$. By (1.1) and Lemma 7.1, for some $c > 0$ and a real number A_K independent of y,

$$R(y) = y\log y + A_K y + O(y\delta), \qquad \delta = \exp(-cx^{\varepsilon/8}), \qquad y \ge \tfrac{1}{3}\exp(x^{\varepsilon/3}).$$

The constant in the error is uniform on the common family. The coefficients $c_K(n)$ are nonnegative, so $R(y) = \sum_n c_K(n)(y/n-1)_+$ is convex and its right derivative satisfies $yR'_+(y) - R(y) = \Psi_K(y)$, also at integers. For $Y \le y \le a$, both $y(1 \pm \sqrt{\delta})$ are in the range of the estimate for large x, since $Y \asymp x\exp(x^{\varepsilon}/2)$. The two adjacent secant slopes at these three arguments bound $R'_+(y)$. The secants for the main term are $\log y + 1 + A_K + O(\sqrt{\delta})$ by Taylor's formula, and the error terms in the secants are $O(\delta/\sqrt{\delta})$. Hence

$$\Psi_K(y) = y + O(y\sqrt{\delta}) \qquad (Y \le y \le a)$$

uniformly. The constant A_K cancels. This use of convexity only requires the quadratic character of the common subfield.

The last sum in (6.11) lies between zero and $\Psi_K(Y) = O(Y)$. Write $H_W = \sum_{k \le W} 1/k$. The first sum is $aH_W + O(aH_W\sqrt{\delta})$. Since $H_W = \log W + \gamma + O(1/W)$, we have

$$\mathcal{H}_K(a) - C_* = a(\log W + \gamma) + O(Y + aH_W\sqrt{\delta}).$$

Both errors are $o(b/x)$ for fixed $\varepsilon > 0$. The difference between $C_* - C(t,a,w)$ and $2\log(D_*/D_t)$ is at most $2Y\log(2Y) = o(b/x)$ by the prime-by-prime rounding in Theorem 6.16. Adding proves (6.12). □

Corollary 6.19 (What persists after dividing the scalar derivatives)**.** *On these same CM schemes one has the following two statements.*

1. *With D as in the theorem, integral scalar jets divided by their full common factor $H_K(a)$ have every value in $\mathcal{O}_J$ as a normalized derivative, including one. Thus local integrality of this quotient alone imposes no proper ideal of divisibility depending on t.*
2. *Using the smaller divisor D_t, the least normalized log norm $2[J : \mathbb{Q}]^{-1}\log|N(z/D_t)|$ among nonzero jets is*

$$a(\log W + \gamma) + 2\log(D_*/D_t) + O_{\psi,\varepsilon}(b/x).$$

Proof. The first assertion is the image ideal calculation in Volume X. For the second the minimum is $\mathcal{H}_K(a) - 2\log D_t$. Replace $\mathcal{H}_K(a)$ by $C_* + a(\log W + \gamma) + O(b/x)$ as in the proof above and use $0 \le C_* - 2\log D_* \le 2Y\log(2Y)$. □

The remaining arithmetic inequalities. For the large primes $\ell > \exp(x^{\varepsilon}/12)$, Theorems 25.1 and 25.5 realize the strict classes of each complementary character as first-order deformations with a marked inertia element and fixed local matrices. These results exclude the two-dimensional mixed corner on restriction to $G_{\mathbb{Q}(i)}$. Theorem 25.23 realizes the strict classes in weight one at integral lifts with regular tame monodromy at two primes split in every interval field. Proposition 25.22 gives the exact depth to which an unramified lower triangular lift in each higher weight satisfies the same local equation. An integral mixed lift that survives must be absolutely irreducible there, cannot be induced from a character and has three different

absolute values for crystalline Frobenius at ℓ. To use the cotangent estimate, one must still establish the presentation and point hypotheses of Theorem 24.8 for this arithmetic model, as well as the required bound on the surviving traces. The stronger fixed local matrices used in the construction retain the strict first-order classes, but can add relations to the presentation. The actual weighted cohomology bound in Proposition 6.14 also includes the other retained primes.

For the theta comparison, the arithmetic norm in Theorem 6.18 is now attained on the actual fields, while the distinguished Frobenius classes enter only through D_*/D_t. The prescribed Bessel periods in Proposition 6.15 have additional constraints from their kernels and their common projection. A bound for their finite valuations and archimedean size in that normalization, or an independent lower estimate for $2m \log R_x$ sufficient in (6.9), is still required on the common intervals. In particular (6.10) holds even when the twist-dependent lower estimate is unknown. The unconditional implication from these constructions to (G) remains incomplete.

Proposition 6.20 (The exact Frobenius trace ideal of a mixed lift)**.** *For the same large primes, fix an integral mixed lift of the marked model with coefficient discrete valuation ring R. Put $H = G_{\mathbb{Q}(i)}$ and let $U : H \to \mathrm{SL}_3(R)$ be its matrix in the marked basis. Let $d_1 = \Psi, d_2 = 1, d_3 = \Psi^{-1}$ on H; write $t_0(h) = \sum_i d_i(h)$. For distinct indices let $I_{ij} \subset R$ be the ideal generated by the entries $U(h)_{ij}$ for all $h \in H$. Then*

$$(\operatorname{tr} U(h) - t_0(h) : h \in H) = I_{12}I_{21} + I_{13}I_{31} + I_{23}I_{32} = I_{12}I_{21} + I_{13}I_{31}. \tag{6.13}$$

Write $v(I) = \min_{0 \neq z \in I} v_\ell(z)$ and $v_\ell(\ell) = 1$. Outside any prescribed finite set, the minimum valuation of the trace difference at Frobenius elements of degree-one primes of $\mathbb{Q}(i)$ is exactly

$$\min\{v(I_{12}) + v(I_{21}),\ v(I_{13}) + v(I_{31})\}.$$

All four ideals here are nonzero. Each minimum is attained, with infinitely many possible underlying rational primes.

Proof. By Sections 25.1–25.3, the residual representation is diagonal with entries $\bar{d}_i$, the chosen local decomposition group at ℓ has the exact prescribed diagonal, and the restriction to H is absolutely irreducible. Write $g_* \in H$ for the marked inertia element. The three entries $k_i = d_i(g_*)$ are pairwise distinct modulo the maximal ideal. Indeed $\bar{\Psi}(g_*)$ has order $(\ell - 1)/\gcd(\ell - 1, r) > 2$, and the three entries are $\bar{\Psi}(g_*), 1, \bar{\Psi}(g_*)^{-1}$.

Denote by J_0 the left side of (6.13) and by J_d the ideal of the diagonal differences $U(h)_{ii} - d_i(h)$. For $s = 0, 1, 2$, the trace difference at $g_*^s h$ is $\sum_i k_i^s (U(h)_{ii} - d_i(h))$. The Vandermonde determinant is a unit in R. Thus all diagonal differences are R-linear combinations of trace differences, while their sum gives the trace difference. Hence $J_0 = J_d$.

Let $J_c = \sum_{i<j} I_{ij} I_{ji}$. We show first that $J_c \subset J_d$. For a fixed j, let $P_j(X) = \prod_{i \neq j} (X - k_i)/(k_j - k_i) \in R[X]$. Evaluating at $U(g_*)$ gives the projector onto coordinate j. For $i \neq j$ and $h, h' \in H$, the ii entry of $U(h)P_j(U(g_*))U(h')$ is $U(h)_{ij}U(h')_{ji}$. Expand P_j in powers of its variable. Modulo J_d each entry $U(hg_*^s h')_{ii}$ is $d_i(hh')k_i^s$, so the resulting entry is $d_i(hh')P_j(k_i) = 0$. This gives the inclusion for every pair of factors.

Conversely, modulo J_c, the diagonal entries of a product are the products of its diagonal entries. Thus $u_i(h) = U(h)_{ii} \bmod J_c$ is a continuous character to $(R/J_c)^\times$, since these entries reduce to units. Each I_{ij} is nonzero: the span of the orbit of coordinate vector e_j is the whole representation by irreducibility, and so some orbit vector has nonzero i-th entry. All off-diagonal entries reduce to zero, so J_c is a nonzero proper ideal. The characters u_i/d_i are consequently valued in the principal units of a finite local quotient of R, an ℓ-group. On inertia

away from ℓ the marked representation and the reference characters factor through a finite group of order prime to ℓ, by their defining inertia kernels and the large-prime cutoff. Thus each ratio is trivial there. It is trivial on the chosen decomposition group above ℓ by the fixed local matrices. At a conjugate place the same is true, including a place reached by c: the antidiagonal inverse transpose sends the diagonal $\mathrm{diag}(s, 1, s^{-1})$ to itself, and the reference characters come from $G_{\mathbb{Q}}$. Thus each ratio is everywhere unramified. The ordinary ideal class group of the imaginary quadratic field $\mathbb{Q}(i)$ is trivial, so global class field theory kills these characters. Therefore $J_d \subset J_c$, proving the first equality.

The marked symmetry $U(ch^{-1}c^{-1}) = J_3 U(h)^{\mathsf{T}} J_3$ gives $I_{12} = I_{23}$ and $I_{21} = I_{32}$, which proves the other equality. A sum of nonzero ideals in a discrete valuation ring has valuation equal to the least of their valuations. The ideal generated by all the traces thus has the stated valuation, and one of the generating traces attains it by discreteness.

Take such an h and reduce the matrices and reference characters modulo a power of the uniformizer strictly higher than the attained integral valuation. Their kernels contain an open normal subgroup of H; intersect with its conjugate under c to obtain an open subgroup normal in $G_{\mathbb{Q}}$, still contained in H. In the resulting finite Galois extension of $\mathbb{Q}$, the Chebotarev density theorem realizes the conjugacy class of h at infinitely many unramified rational primes outside the excluded set. Since $h \in H$, all such primes split in $\mathbb{Q}(i)$. Choose a prime of the extension whose Frobenius is the specified representative of the conjugacy class. At the prime below it in $\mathbb{Q}(i)$, the evaluated trace is the specified one modulo the chosen power, hence has the same difference valuation. One may fix either consistent Frobenius convention in this argument. □

Corollary 6.21 (The trace on the even alternative)**.** *In the second alternative of Theorem 25.6, write B, C for the ideals generated, respectively, by the upper- and lower-off-diagonal entries of its even two-dimensional representation $\rho(h)$, $h \in H$. The ideal in (6.13) is exactly BC. For any finite collection of mixed points in the same marked model, one integral linear combination of split Frobenius trace differences attains the pointwise minimum of Proposition 6.20 at every point, after finite unramified extension of the base coefficient field if needed.*

Proof. Write a matrix of ρ as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with unit determinant and unit diagonal entries, since its reduction is $\mathrm{diag}(\bar{\Psi}, 1)$. In the trace-zero basis $E_{12}, \mathrm{diag}(1, -1), 2E_{21}$, conjugation of the middle basis vector has first coordinate $-2ab/\det\rho$, and conjugation of E_{12} has middle coordinate $-ac/\det\rho$. The two extreme entries are $-2b^2/\det\rho$ and $-c^2/(2\det\rho)$. Consequently $I_{12} = B, I_{21} = C, I_{13} = B^2, I_{31} = C^2$, using discreteness to attain the least valuation among the squares. Formula (6.13) now gives $BC + B^2C^2 = BC$.

For each point select one Frobenius that attains its minimum, excluding the finitely many bad primes of all the points. Use the union of these finitely many elements. At a fixed point, divide their trace differences by a generator of its trace ideal in a common coefficient extension and reduce. At least one coefficient in this row is nonzero. A combination fails to attain the minimum exactly when its coefficient vector lies on the hyperplane defined by this row. The product of the finitely many nonzero linear forms is a nonzero polynomial. Some vector over the algebraic closure of the residue field avoids all its zeros, and all of its entries lie in a finite residue extension. They lift to the integers of a finite unramified extension of the base coefficient field, enlarged enough to realize the chosen residues in the common extension. Their linear combination has the asserted valuation simultaneously. Trace differences are zero at the diagonal basepoint and are invariant under the scaling action, so this also supplies a separating function on the stated finite set. □

Equations (6.13) and (6.9) concern the same family of fields and intervals. The first computes the individual trace valuations entering the graded cotangent comparison if its arithmetic

presentation hypotheses hold. It gives no uniform upper bound for the off-diagonal ideals' valuations or for their sum over all points, primes and twists. On the other side Theorem 52.8 attains the norm in Theorem 6.18 on a genuine global algebraic theta section of level $3a$ relative to the chosen theta bundle, in one integral frame. Corollary 52.9 controls a projection to inversion parity as well. The normalized Bessel projection in Proposition 6.15 and its distinguished prime counts remain a separate estimate. These are the outstanding arithmetic steps in this route to Main Theorem 1.1.

6.6.1 The CM type in the supplied construction

There are two imaginary quadratic subfields in Proposition 6.4. Write $K_q = \mathbb{Q}(\sqrt{-q})$, $K_d = \mathbb{Q}(\sqrt{-d})$, and let ξ_q, ξ_d denote their primitive quadratic characters. The CM embeddings specified in the source send $\sqrt{-d}$ to $i\sqrt{d}$. The first sign in its Galois group records the action on $\sqrt{-q}$, and the second sign records the action on $\sqrt{-d}$. These conventions are explicit in `Formalism/LocalRingCardinality/DataHost.lean`, line 43 and `Formalism/SharpRestriction/RamificationFrobenius/TotientGaloisClasses/ConjEMemHostFIff.lean`, line 66. The count in Theorem 6.18 uses K_q. The distinction can already be measured at the prime two.

Proposition 6.22 (The two quadratic types at the prime two)**.** *Use $a = \lceil xb \rceil$, $B' = \lceil b \rceil$, $W = \lceil \sqrt{b} \rceil$. For the count indexed by the complement of $\mathrm{Gal}(E/K_q)$, with weights B', W, the term at two satisfies, uniformly for every translate,*

$$\frac{C_{2,q}}{2\log 2} = \frac{2a}{3} + O(B' \log(2 + a/B')).$$

For the count indexed by the complement of $\mathrm{Gal}(E/K_d)$, its term at two is zero. On an abelian scheme of the induced K_d-type over a common coefficient field, a global integral theta section can have odd-order or even-order derivative of the prescribed order a which is a unit at every place above two, in the frame specified below.

Proof. The discriminants are $-q \equiv 5 \bmod 8$ and $-d \equiv 1 \bmod 8$. Thus two is inert in K_q and split in K_d. Although E itself is ramified at two, its inertia maps to the identity in both quadratic quotients. For K_q precisely the odd powers of Frobenius contribute, with total inertia weight one at each power after division by $2\log 2$. Since every weight is between zero and B', the normalized sum lies between $\sum_{j\ \mathrm{odd}} (a/2^j - B')_+$ and $\sum_{j\ \mathrm{odd}} a/2^j = 2a/3$. There are at most $1 + \log_2(a/B')$ indices below the cutoff for large x, and the geometric tail past it is at most $2B'$. This proves the estimate. For K_d every Frobenius coset is contained in $\mathrm{Gal}(E/K_d)$.

Here is the scalar ideal for the second type. For Q a prime power put $F_Q(a) = \sum_{j\ge 1} \lfloor a/Q^j \rfloor$. With splitting and inertia taken in K_d, define

$$H_d(a) = \frac{a!}{\prod_{p\ \mathrm{split}} p^{F_p(a)} \prod_{p\ \mathrm{inert}} p^{F_{p^2}(a)} (\sqrt{-d})^{F_d(a)}}.$$

Take an elliptic curve C_d of maximal CM by $\mathcal{O}_{K_d}$ and form $\mathcal{A}_E = \mathcal{O}_E \otimes_{\mathcal{O}_{K_d}} C_d$. The argument of Sections 52.1–52.3 applies with this field. We give the hypotheses that change. Its only ramified prime is d, of index two and residue cardinality d. At a split prime its formal elliptic factor has height one, including at two. At an inert prime the scalar uniformizer is p with residue cardinality p^2. The local factorial calculation of Lemmas 52.4 and 52.5 consequently gives $v_p(a!) - F_{p^f}(a)/e$ for the attainable valuation at each place, where e, f are the local index and residue degree. The number of elliptic factors is $2n > 1 + \log_2 a$ for large x, so the separate directions used for the nonzero base-p^f digits exist. There are finitely many fields and integer

orders on the bounded interval. The residue extensions and Chinese remainder construction of Theorem 52.6 therefore give one coefficient field J and one tangent vector per scheme for all these orders. The global image ideal is $H_d(a)\mathcal{O}_J$. For any relatively ample symmetric rigidified bundle P, use a section s_0 of P^3 with value one at the identity. Degree-a products in that linear system realize every order-a jet in the frame s_0^a, by Lemma 52.7 and Theorem 52.8. Some section of P^{3a} thus gives $H_d(a)$ itself. At two the denominator contains $2^{F_2(a)}$, with $F_2(a) = v_2(a!)$. The asserted unit property follows. □

Theorem 6.23 (Changing the quadratic type in the centered count)**.** *Keep the fields, the sets $T, T_2 = A_0$, the weights and the integer order of Proposition 6.22 on the common intervals of Section 6.6. Work first in the product coordinates $(i, j, z) \in (\mathbb{Z}/2)^2 \times A$ of Proposition 6.4, in the order q, d, A. For $e = q, d$, use the subgroup fixing K_e for the count C_e and use D_{ξ_e} in its main term M_e. The distinguished Frobenius classes in the two counts at translate t are, respectively,*

$$\{(1, 0, z - t) : z \in A_0\}, \qquad \{(0, 1, z - t) : z \in A_0\}.$$

For $\eta \in \widehat{A/A_0}$, write $\alpha_{ij\eta}$ for the primitive representative of $\xi_q^i \xi_d^j \eta$. Then at each individual t, with $o = |A/A_0|$,

$$\begin{aligned}(C_q - M_q) - (C_d - M_d) \\ = \frac{b}{2o} \sum_{\eta \in \widehat{A/A_0}} \eta(t) \sum_{\substack{i,j \in \{0,1\} \\ i+j=1}} ((-1)^i - (-1)^j) G_{\alpha_{ij\eta}}(x) + O_{\psi,\varepsilon}(b/x).\end{aligned} \tag{6.14}$$

Its absolute value is at most $2bx^{1/2+\varepsilon/2} + O_{\psi,\varepsilon}(b/x)$. These assertions also hold when the original character ψ is odd.

Proof. Let $c_A \in A$ be the restriction of complex conjugation to E_0; it has order dividing two. The change of coordinates

$$(i, j, z) \longmapsto (i, j, z + ic_A)$$

is its own inverse. It sends complex conjugation to $(1, 1, 0)$ and preserves both quadratic signs, the subgroup A and its projection to the real cyclotomic layer. In these coordinates, addition of $(1, 1, t)$ sends the second distinguished set in the theorem to $\{(1, 0, z) : z \in A_0\}$ in the CM subgroup $j = 0$. Its preimage has first sign zero and so has exactly the same coordinates before the change. The first distinguished set is taken in the original coordinates as in Proposition 6.4. This accounts for conjugation without assuming E_0 totally real.

Apply character orthogonality as in (6.2). The sum over the whole complementary coset selects just the trivial character and the quadratic sign with coefficient -1, giving the stated main term for each orientation. For a nondistinguished weight one has $a/W \gg \exp(x^{\varepsilon}/3)$. The bound of Lemma 7.1, times the number $O_\psi(n)$ of terms after division by $2n$, is $O_\psi(na \exp(-cx^{\varepsilon/8})) = o(b/x)$. For the distinguished classes one can replace B' by b with total error $O_{\psi,\varepsilon}(b/x)$, using the integral-weight estimate of Proposition 7.2. Its proof uses only $|B|, |A_0|$, the uniform zero-free bound and the slopes of the ramps, which are the same for the two choices of subgroup.

Summing over $z \in A_0$ annihilates characters nontrivial on A_0. The remaining ones are exactly $\alpha_{ij\eta}$, and the factor from this sum divided by $2n$ is $1/(2o)$. Their conjugate values at the two distinguished sets supply $(-1)^i \eta(t)$ and $(-1)^j \eta(t)$. The difference vanishes for $i = j$. The resulting argument of G is a/b. To replace it by x, note that the slope of the prime sum on $[x, x + 1]$ is bounded by $\sum_{v \le x+1} \Lambda(v)/v \ll \log(2x)$. For the surviving nonprincipal characters their correction has slope $D_{\alpha_{ij\eta}} = O(\log(2x))$ by Theorem 5.1. The change is $O_\psi(\log(2x))$ after multiplication by b, as $0 \le a/b - x \le b^{-1}$. This is $o(b/x)$. Finally the two coefficients per η each have absolute value two, and Theorem 5.1 bounds both functions at x itself. □

Corollary 6.24 (The count and scalar norm for the prescribed type)**.** *In the d-orientation put $D_{t,d}=\prod_p p^{\lfloor C_{p,d}(t,a,w)/(2\log p)\rfloor}$ and define $D_{*,d}$ with the constant weight W. For the minimum normalized logarithmic norm of a nonzero scalar derivative on the K_d-type schemes one has, uniformly in t,*

$$\frac{2}{[J:\mathbb{Q}]}\log|N_{J/\mathbb{Q}}H_d(a)|=\sum_{v\le a}(1-\xi_d(v))\Lambda(v)\lfloor a/v\rfloor,$$

$$\frac{2}{[J:\mathbb{Q}]}\log|N_{J/\mathbb{Q}}H_d(a)|-C_d(t,a,w)=a(\log W+\gamma)+2\log(D_{*,d}/D_{t,d})+O_{\psi,\varepsilon}(b/x). \quad (6.15)$$

The replacement of the first centered count by the second changes Δ_u by $O_\psi(hmbx^{1/2+\varepsilon/2})$ for all u. When $m\le\sqrt{x}$, $a_E=xb$, this is $o(x^{5\varepsilon}\mathcal{B})$.

Proof. The ideal in Proposition 6.22 realizes the minimum by the integrality of the absolute norm. At inert primes its normalized valuation is the sum of $\lfloor a/p^j\rfloor$ for odd j. At d it is half the sum over all j, and at split primes it is zero. This gives the first identity. The constant-weight count is $\sum_v(1-\xi_d(v))\Lambda(v)(a/v-W)_+$ by inertia averaging. Subtract it from the first identity and apply the floor-sum identity (6.11) with K_d. The secant-slope proof following that identity applies because the coefficients are nonnegative and Lemma 7.1, applies to the same ξ_d on these intervals. Thus the difference is $a(\log W+\gamma)+O_{\psi,\varepsilon}(b/x)$. Rounding the exponents for primes at most a/W contributes at most $2(a/W)\log(2a/W)=o(b/x)$ to the difference of the two counts. This gives (6.15). Sum (6.14) over h translates and m stages. Since $\mathcal{B}\ge hxb$, the ratio of the resulting bound to $x^{5\varepsilon}\mathcal{B}$ is $O_\psi(x^{-9\varepsilon/2})$. □

The dyadic calculation rules out transferring the q-count prime by prime to all integral sections of the d-type. The centered count permits the change on the full interval by Theorem 6.23. Its terms with $i=j=0$, including the original Dirichlet characters, persist in each orientation. Estimating their contribution still requires the normalized projected periods or another proof of (6.3). The strict cohomology groups depend on the field E_x itself, which is the same in both counts.

6.6.2 Sparse distinguished embeddings on every translate

The source imposes a restriction on its distinguished set that fails for an entire cyclic subgroup. In integer coordinates, three distinct members s_1,s_2,s_3 must avoid $j(s_2-s_1)=i(s_3-s_1)$ for $0<|i|,|j|\le L$, $i\ne j$. The integer representatives must also satisfy $4Ls<n_0$, where n_0 is the cyclic order. See `Formalism/FrobeniusOrbits/HostThickExists.lean`, line 13 and `Formalism/LocalRingCardinality/DataHost.lean`, line 106. The length in `Development/AutomorphicArithmetic/CyclotomicBounds/ClassNumberBounds/ExplicitConstants/UniformIffAboveThreshold.lean`, line 60 grows logarithmically with x for fixed parameters. We construct such a set while retaining the original prime sum at every translate.

Lemma 6.25 (A sparse set balanced modulo four)**.** *Let s be a power of two tending to infinity. Let $2\le L\le C_0(1+\log s)^{C_1}$ be an integer, with C_0,C_1 fixed. There is a set S of nonnegative integers $<s/(8L)$ with the same number of elements in each residue class modulo four and*

$$|S|\ \ge\ s\exp(-C\sqrt{\log s\ \log(2L+2)})$$

for all sufficiently large s. If $\alpha+\beta+\gamma=0$ and the three coefficients are nonzero integers of absolute value at most $2L$, every solution of $\alpha u+\beta v+\gamma w=0$ in S has $u=v=w$. In $\mathbb{Z}/s\mathbb{Z}$ a translate of S meets each segment of $2L$ successive points of an orbit of order $>2L$ at most twice. It meets each entire orbit of order $\le 8L$ at most once, counting distinct points.

Proof. Choose four low digits

$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 64L + 2, \quad b_3 = (64L)^2 + 3.$$

They represent the four residues modulo four. If all three chosen digits are distinct, the largest exceeds $4L$ times the next largest, so its term cannot be canceled by the other two. If two digits coincide, the zero sum of the coefficients forces all three to coincide. Let $D = 32L(b_3 + 1)$, and let $X = \lfloor s/(16LD) \rfloor$. We first construct a set $S' \subset [0, X)$.

Put $p = \lfloor \sqrt{\log X / \log(2L + 2)} \rfloor$, $M = \lfloor X^{1/p} \rfloor$, and $H = \lfloor M/(8L) \rfloor$. For sufficiently large s we have $p, H \geq 2$. Encode vectors $(z_0, \ldots, z_{p-1})$ with $0 \leq z_i < H$ by $\sum_i z_i M^i < X$. Among the at most $1 + pH^2$ values of $\sum_i z_i^2$, some sphere contains at least $H^p/(1 + pH^2)$ vectors. Use these for S', following Behrend's sphere construction. [7] An integer relation with the stated coefficients gives the same relation in each digit. Indeed reducing modulo M forces the lowest-digit combination to vanish since its absolute value is $< 6LH < M$; division by M repeats the argument. One of the three vectors is then a strict convex combination of the other two unless they coincide. On a sphere this is impossible, by strict convexity of the squared Euclidean norm.

For large s, $H \geq X^{1/p}/(32L)$. The logarithm of the loss from X to $H^p/(1 + pH^2)$ is at most $p \log(32L) + 2 \log M + \log(2p)$, which is $O(\sqrt{\log X \, \log(2L + 2)})$ by the choice of p. Put $S = \{Dv + b_i : v \in S', \ 0 \leq i < 4\}$. All these integers are distinct and $< s/(8L)$ for large s. A relation reduced modulo D has a low-digit combination of magnitude at most $6Lb_3 < D$, so it vanishes over the integers. The low digits coincide, and then the high digits coincide. The residues are exactly balanced. The factor $s/X = O(L^4)$ is absorbed in the displayed loss.

For the orbit assertion, suppose three distinct points with indices i_0, i_1, i_2 occur in a segment of $2L$ consecutive indices, and subtract the translate from their values to obtain representatives $s_0, s_1, s_2 \in S$. Eliminating the common step gives

$$(i_2 - i_0)(s_1 - s_0) \equiv (i_1 - i_0)(s_2 - s_0) \pmod{s}.$$

The difference has absolute value $< s$, so the congruence is an integer equality. Its three coefficients are nonzero differences of indices, each smaller than $2L$, a contradiction. For an orbit of order $b_0' \leq 8L$, distinct elements have circular separation at least $s/b_0' \geq s/(8L)$, larger than the diameter of S. □

Theorem 6.26 (The exact prime sum with a sparse distinguished set)**.** *Fix the fields, a, b, B', W and the K_d-orientation of Theorem 6.23. Take m, k as in Proposition 70.1, and put $a_E = xb$. For each x and any integer L in the range of Lemma 6.25, with $s = |A_0|$, one can choose $T_2 \subset A_0$, $T \subset A$ as follows. The integer representatives of T_2 satisfy that lemma, $h_2 = |T_2|$, $\delta = h_2/n \geq x^{-\varepsilon}$, and the proportions of T in A/A_0 are exactly those of condition 2. Set $y = a/B'$, $Y' = a/W$, $H = A/A_0$, $w_{\mathbf{1}} = 1$ and $w_\eta = 1/(3o + 1)$ for $\eta \in \widehat{H}$ nontrivial. With $\alpha_{ij\eta}$ as in Theorem 6.23, at every $u \in A_0$ one has*

$$\frac{\Delta_u}{hm} = W\big(G_{\mathbf{1}}(Y') - G_{\xi_d}(Y')\big) + \frac{\delta}{2} \sum_{\eta \in \widehat{H}} w_\eta \sum_{i,j=0}^{1} (-1)^j \big(B' G_{\alpha_{ij\eta}}(y) - W G_{\alpha_{ij\eta}}(Y')\big). \tag{6.16}$$

In particular Δ_u is the same for all u, although the distinguished sets themselves vary with u.

Proof. Restriction embeds $A_0 = \mathrm{Gal}(K_\psi C_l / K_\psi)$ in the cyclic group $\mathrm{Gal}(C_l/\mathbb{Q})$, whose order is 2^l. Hence A_0 is cyclic of order $s = n/o \asymp_\psi x^{2+8\varepsilon}$. Fix a generator and identify it with $1 \in \mathbb{Z}/s\mathbb{Z}$.

Its restriction to the cyclotomic layer has the same order. Use the integers of S for T_2 in these coordinates. Since L is at most polylogarithmic, $\log(n/h_2) = O_{\psi,C_0,C_1}(\sqrt{\log s \, \log(2L+2)}) = o(\log x)$. This gives the claimed density for fixed ε.

Include the entire identity coset of A_0 in T. In each nonidentity coset choose a representative and include exactly the three nonzero residue classes modulo $4A_0$ relative to it. There are four classes of equal size since $4 \mid s$. Thus $h = (3o+1)s/4 \geq n/2$, and the normalized character sum $h^{-1}\sum_{t\in T}\eta(t)$ equals w_η for characters of H, by orthogonality on H.

For a character θ of A nontrivial on A_0 the product

$$\left(\sum_{t\in T}\theta(t)\right)\left(\sum_{z\in T_2}\overline{\theta(z)}\right) = 0.$$

If θ is nontrivial on $4A_0$, the first sum vanishes by translation invariance of T. In the remaining case $\theta|_{A_0}$ induces a nontrivial character of $A_0/4A_0$; the second sum vanishes by the four equal multiplicities. Translation by u simply multiplies the second sum by $\overline{\theta(u)}$.

With weight W everywhere, the centered count is the first term of (6.16) at each t. Correcting the weight to B' on the distinguished set changes each Fourier summand by $B'G(y) - WG(Y')$, by (6.2). In the original product coordinates the distinguished Frobenius elements are exactly $(0, 1, z+u-t)$, $z \in T_2$, using the coordinate change in Theorem 6.23. Their character sum gives the product above with the further factor $(-1)^j\overline{\theta(u)}$. Only θ trivial on A_0 survives. Then its normalized first factor is w_η, and its second is h_2. Division by $2n$ and summation over the m identical stages proves the identity with the primitive representatives at every prime, including the ramified primes. □

Corollary 6.27 (The remaining estimate with these embeddings)**.** *For sufficiently large x on the selected intervals, uniformly in the sets just constructed,*

$$\left|\frac{2\Delta_u}{hmb\delta} - \mathcal{G}_\psi(x)\right| \leq 3o\, x^{1/2+\varepsilon/2} + o_{\psi,\varepsilon,C_0,C_1}(1).$$

For fixed ψ, the critical-line assertion for the primitive powers of ψ is equivalent to (6.3) on arbitrarily distant such intervals for every sufficiently small $\varepsilon > 0$.

Proof. At $Y' = a/W$ every $WG_\alpha(Y')$ in the identity is $O(a\exp(-cx^{\varepsilon/8}))$ by Lemma 7.1. Division by $b\delta$ still gives $o(1)$. Replacing $(B'/b)G_\alpha(y)$ by $G_\alpha(x)$ costs $o(1)$ for each surviving character. In fact $|y-x| = O(x/b)$; the slope bound in Theorem 6.23 applies to the auxiliary twists, and the retained characters have fixed conductor with slopes $O_\psi(\log(2x))$ including the principal correction. Their values in this range are $O_\psi(x\log(2x))$, also for the auxiliary twists by the bound on their corrections. The terms $i = j = 0$ now give $\mathcal{G}_\psi(x)$ by conjugate pairing. The other $3o$ terms have weights of absolute value at most one and obey Theorem 5.1.

Under the critical-line assertion the explicit formula gives $\mathcal{G}_\psi(x) = O_\psi(\sqrt{x})$. Since $\delta \leq 1$, $\mathcal{B}/(hmb) \geq x/m \geq \sqrt{x}$, and $5\varepsilon > \varepsilon/2$, (6.27) implies (6.3) for all sufficiently large x. Conversely $k = 2n,\ h \geq n/2,\ m \sim \sqrt{x}$ give $\mathcal{B}/(hmb) = O_\psi(x^{1/2+\varepsilon})$ in (6.3). Using $\delta \geq x^{-\varepsilon}$, the assumed inequality and (6.27) imply $\mathcal{G}_\psi(x) \ll_{\psi,\varepsilon} x^{1/2+7\varepsilon}$ on the whole interval. The interval theorem and functional equation used in Proposition 6.13, with ε arbitrarily small, give the critical-line assertion. □

The sparse sets satisfy the short-orbit and three-point restrictions throughout the cohomology intervals. Equation (6.16) computes their weighted counts for every translate. An upper bound for those counts from the specified projected periods, and the strict local cohomology bound on these fields, are still required.

6.6.3 The numerical field degrees, ranks and densities

We now impose the growth parameters of `Development/AbelianVarietyTheory/Definitions/GlobalTwisting/PairRowIffQdRows.lean`, line 28 and the cardinalities and auxiliary-prime window at lines 78–88 there. The absolute field degree in these conditions is $4n$. The constant factor in $n = [K_\psi C_l : \mathbb{Q}]$ matters for its prescribed upper bound of $2x^{2+8\varepsilon}$.

Proposition 6.28 (Choosing the field degree and transferring strict classes)**.** *On the same selected intervals put $h_0 = 2 + 8\varepsilon$ and take the least index l' with $[K_\psi C_{l'} : \mathbb{Q}] \geq x^{h_0}$. Let $\widetilde{E} = K_\psi C_{l'}(\sqrt{-q}, \sqrt{-d})$ and $\widetilde{n} = [\widetilde{E} : \mathbb{Q}]/4$. For large x,*

$$x^{h_0} \leq \widetilde{n} < 2x^{h_0}, \qquad \widetilde{E} \subset E_x, \qquad [E_x : \widetilde{E}] = 2^e \leq 2o.$$

At every odd prime ℓ and every $r \geq 1$, restriction identifies $\mathcal{S}_{\ell,r}(\widetilde{E})$ with the $\mathrm{Gal}(E_x/\widetilde{E})$-invariants of $\mathcal{S}_{\ell,r}(E_x)$. At two its kernel has $\mathbb{Z}_2$-length at most e. Consequently the full weighted sum of strict lengths up to $r < t = \lfloor x^{401} \rfloor$ over $\widetilde{E}$ is at most that over E_x plus $t \log(2o)$.

Proof. The increasing intersections $K_\psi \cap C_j$ stabilize, since the cyclic field K_ψ has finitely many subfields. The degrees of $K_\psi C_j$ consequently double at each sufficiently large step. Minimality gives the two bounds on $\widetilde{n}$. The index used for E_x already meets this lower bound, since $2^l \geq x^{h_0}$. The ratio of the two degrees is at most $2o$ since $[K_\psi C_l : \mathbb{Q}] \leq o2^l < 2ox^{h_0}$. The independent quadratic factors are disjoint from this tower by ramification at q, d. The relative group is cyclic of the displayed order because it embeds in the cyclic cyclotomic layer.

Write $M = \mathbb{Q}_\ell/\mathbb{Z}_\ell(-r)$. A strict class is an absolute Galois H^1-class unramified outside ℓ at finite places and zero locally above ℓ. Its image by restriction is strict. At odd ℓ the cohomology in positive degrees of the finite relative group on any ℓ-primary module vanishes, by averaging. Inflation–restriction therefore identifies the full H^1 downstairs with the invariants upstairs. If the upstairs class is locally zero above ℓ, its descended restriction is locally zero by injectivity of restriction for each local extension of two-power degree. At any other prime the same argument on inertia proves that the downstairs class is unramified. This establishes the assertion for strict classes. At two the kernel is contained in $H^1(C_{2^e}, M^{G_{E_x}})$. For a cyclic group with generator σ its first cohomology is $\ker(1 + \cdots + \sigma^{2^e-1})/(\sigma - 1)M^{G_{E_x}}$. It is killed by 2^e and is a subquotient of a subgroup of the quasicyclic group $\mathbb{Q}_2/\mathbb{Z}_2$. Such a subquotient of bounded exponent is cyclic, as every finitely generated subgroup is cyclic and the orders of its elements are bounded. Its length is at most e. Summation gives the result, including its uniform additive cost at two. □

From now on write $E = \widetilde{E}$, $n = \widetilde{n}$, $f = 2n$, $A = \mathrm{Gal}(K_\psi C_{l'}/\mathbb{Q})$, and $A_0 = \mathrm{Gal}(K_\psi C_{l'}/K_\psi)$. All analytic large-argument bounds of Lemma 7.1, pass to these fields because their characters are characters of E_x and the primes q, d are the same. The sparse embeddings, cohomology groups and polarized abelian schemes below all use this one field.

Theorem 6.29 (Simultaneous sparse cardinalities at the specified ranks)**.** *Set $\delta_1 = x^{-\varepsilon/8}$, $b = \exp(x^\varepsilon)$, $L_0 = x^\varepsilon$, and choose*

$$m = 2\lceil \sqrt{x\delta_1}/2 \rceil, \qquad k = 2\lceil x^{9/2+10\varepsilon}/2 \rceil, \qquad r_0 = k + m - 1,$$
$$n_W = 8\left\lceil \frac{10r_0}{8}\left(f + \frac{f^2(k-1)^3L_0^2}{m}\right)\right\rceil, \qquad t_{\mathrm{r}} = 8fr_0 + 4 + \lceil x^{377} \rceil n_W - r_0,$$
$$L = 10\lceil 405 \log x \rceil + 2\lceil 30/\varepsilon \rceil + 22.$$

The rank and signature construction and residue-character conclusions of Section 70.1 hold for these ranks, with $t_{\mathrm{r}} < x^{401}$ and $N_m = k + m(t_{\mathrm{r}} + 1) < x^{402}$. The sets in Theorem 6.26 can be

chosen to satisfy simultaneously

$$\tfrac{1}{2}\delta_1 n \le h \le 2\delta_1 n, \qquad h_2 \ge n x^{-\varepsilon}, \qquad \nu(\alpha) = \frac{3 + \mathbf{1}_{\alpha=0}}{3o+1}.$$

The short-orbit and three-point conclusions of Lemma 6.25 hold at this L. The exact identity (6.16) holds for every u at these new densities.

Proof. The rounded k, m are even, r_0 is odd and $8 \mid n_W$, so t_{r} is odd. All the lower rank inequalities of Section 70.1, follow from $2n \le k$, $m^2 \le k$ and $t_{\mathrm{r}} > 3k$ for large x. Also $xm \le k \le n^2 x^{1/2-\varepsilon}$ and q, d each exceed $4t_{\mathrm{r}} r_0 + 1$ eventually, using the fixed initial power window as in that section. For the upper bounds, the leading power of x in $r_0 f^2 (k-1)^3 L_0^2/m$ is $43/2 + (58 + 1/16)\varepsilon < 23$. Thus $n_W = O(x^{23})$, and the formula gives $t_{\mathrm{r}} = O(x^{400})$. The proofs of the rank embeddings, residue characters, Haar norms and Fourier denominators in Volume X use the dimension formulas for the towers for arbitrary such ranks. The logarithms of the finite costs there are still polynomial in x, since f, m, N_m are. The same observation shows that $G = f r_0 (r_0 + t_{\mathrm{r}})$ in Corollary 52.13 is polynomial in x.

Let $s = |A_0| = n/o$, a power of two. Choose the least power of two $D \ge 2/\delta_1$. Then $2/\delta_1 \le D < 4/\delta_1$ and $D \mid s$ for large x. Choose T to consist of four residue classes modulo DA_0 in the identity coset and three in each other coset. Its size is $h = (3o+1)s/D$, which satisfies the two bounds and the asserted proportions. We need a large three-point-free set exactly balanced in all D residues.

For $0 \le v < D$ put $c_v = v + 8LD\, v^2$. These represent the D residues. Suppose $\alpha c_u + \beta c_v + \gamma c_w = 0$ with the coefficients of Lemma 6.25. Reduction modulo $8LD$ gives $\alpha u + \beta v + \gamma w = 0$ over the integers because the absolute value is $< 6LD$. The original relation then gives the same identity in u^2, v^2, w^2. If u, v, w are distinct, the Vandermonde matrix with rows $1, z, z^2$ has nonzero determinant. This contradicts the nonzero coefficient triple. With two equal entries the relations force the third equal also.

Let M be the least multiple of D exceeding $8L(1 + \max c_v)$; thus $M = O(L^2 D^3)$. Take $X = \lfloor s/(16LM) \rfloor$, and construct the high digits $S' \subset [0, X)$ by the sphere proof of Lemma 6.25. Its hypotheses hold because $\log X \asymp_\psi \log x$ and $L = O_\varepsilon(\log x)$. Set $T_2 = \{Mz + c_v : z \in S',\ 0 \le v < D\}$ in the generator coordinates of A_0. Distinct pairs give distinct integers $< s/(8L)$. Reduction modulo M first forces the low-digit relation over the integers, since $6L \max c_v < M$. The preceding argument makes the low digits coincide, and the sphere argument then makes the high digits coincide. The orbit conclusions follow verbatim from the elimination of the common step in Lemma 6.25. Each residue modulo D occurs $|S'|$ times, and

$$\frac{h_2}{n} \gg_\psi \frac{\exp(-C\sqrt{\log s \, \log(2L+2)})}{L^3 D^2} \;\ge\; x^{-\varepsilon/4 - o(1)} \;\ge\; x^{-\varepsilon}$$

for large x. A character of A nontrivial on DA_0 sums to zero on T. One trivial there but nontrivial on A_0 sums to zero on T_2. These are precisely the two cancellations needed in the proof of (6.16), which now applies at every u. □

Proposition 6.30 (The two auxiliary primes in their prescribed interval)**.** *Let $N = \mathrm{lcm}(F, 2^{l'+2}) qd$ and $U = (kb)^8$. For every sufficiently large x on the selected intervals there are distinct primes $g_1, g_2 \in [U, 2U]$, both $1 \bmod N$, with $g_i = \pi_i \bar{\pi}_i$ in $\mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{-d})$. In fact the interval contains $U^{1-o(1)}$ such primes. One may also choose two distinct smaller primes $1 \bmod 15N$ of size $O(N^C)$ for an absolute C. All four split completely in E.*

Proof. Write H_K for the Hilbert class field of K, $C_N = \mathbb{Q}(\zeta_N)$ and $L = H_K C_N$, a Galois extension of $\mathbb{Q}$. The class group of K has odd order. To see this, conjugation acts as inversion.

If the class of an ideal I has order dividing two, then $\bar{I}/I = (a)$ with $a\bar{a} = 1$, because this is a positive rational unit. Write $a = z/\bar{z}$, taking $z = 1 + a$ unless $a = -1$, in which case take $z = \sqrt{-d}$. The fractional ideal zI is conjugation invariant. At a split prime its two exponents agree, and an inert prime itself is rational. The only ramified prime is generated by $\sqrt{-d}$. Consequently every such invariant ideal is principal, proving the claim. Artin reciprocity identifies $\mathrm{Gal}(H_K/K)$ with the class group and identifies conjugation on it with inversion. Since the squaring map on this odd group is onto, commutators with complex conjugation generate the whole group. Thus the largest subextension of $H_K/\mathbb{Q}$ abelian over $\mathbb{Q}$ is K. As $K \subset C_N$, it follows that $H_K \cap C_N = K$ and $\mathrm{Gal}(L/K) = \mathrm{Gal}(H_K/K) \times \mathrm{Gal}(C_N/K)$.

Apply Thorner and Zaman's Chebotarev theorem to $L/\mathbb{Q}$, identity conjugacy class, with abelian subgroup $\mathrm{Gal}(L/K)$. Its maximal relative character conductor norm is at most N^2. Indeed the first factor is unramified. A character on the second factor extends to the abelian group $\mathrm{Gal}(C_N/\mathbb{Q})$; on K it is obtained by the local norm from a Dirichlet character of conductor dividing N. The norm sends $1 + N\mathcal{O}_{K,v}$ to $1 + N\mathbb{Z}_p$, as follows by expanding the local norm or at a split place by multiplying the two coordinates.

The only possible exceptional zero in the relative theorem is simple, real, and attached to a real character of $\mathrm{Gal}(L/K)$. Such a character kills the odd class group, so its zero would also be a zero of ζ_{C_N}. The other Hecke factors of ζ_L are holomorphic there. Factoring ζ_{C_N} over $\mathbb{Q}$, a zero of a nonreal Dirichlet character at a real argument is repeated by its conjugate and therefore cannot give this simple zero. Every real primitive character appearing is a product of powers of ξ_q, ξ_d and a real character with conductor dividing $8F$. This follows from the square subgroup of the two-adic units and the fact that q, d are distinct odd primes prime to $2F$. The finitely many base characters have a fixed zero-free interval next to one. The selected auxiliary twists have no zero in $[49/50, 1]$ by Theorem 5.1. Hence the exceptional term, if present, has its zero bounded away from one and is negligible here.

The theorem gives, uniformly for $Z = U, 2U$,

$$\pi_{\mathrm{split}}(Z) = \frac{\mathrm{Li}(Z)}{[L : \mathbb{Q}]}(1 + o(1)).$$

In fact its two relative errors are bounded by constants times $\exp(-c \log Z/\log(4dN^2))$ and $\exp(-c\sqrt{\log Z})$, since the base degree is two. Here $\log(dN^2) = O_{\psi,\varepsilon}(x^{\varepsilon/5} \log^2(2x) + \log(2x)) = o(\log U)$. Subtract the estimates at the two endpoints. Minkowski's bound represents each ideal class of K by an integral ideal of norm $O(\sqrt{d})$. There are $O(d)$ such ideals since $\zeta_K(2) \le \zeta(2)^2$. Thus $[L : \mathbb{Q}] \ll dN = U^{o(1)}$, and the difference is $U^{1-o(1)}$. A prime counted splits in C_N, so it is 1 mod N, and its prime ideals over K are principal by splitting in H_K. A generator of either has norm g. The two conjugate ideals are coprime. For the two smaller primes apply Linnik's least-prime bound in the progression 1 mod $15N$, and then in the progression 1 mod $15Np_1$, where p_1 is the first prime. Both are polynomial in N, and are below U for large x. Finally $E \subset C_N$. [8, Thm. 1.4 and Thm. 3.1][9, Thm. 1.1] □

Corollary 6.31 (Stability under rounding the orders and weights)**.** *At the fields, ranks and densities just chosen, give each $t \in T$ an integer order a_t with $|a_t/b - x| \le 1$. At each stage and translate suppose the distinguished weights lie in $[b, b + \sqrt{b}]$ and all other weights w satisfy*

$$1 \le w, \qquad a_t/w \ge \tfrac{1}{3} \exp(x^{\varepsilon/3}).$$

Then $|2\Delta_u/(hmb\delta) - \mathcal{G}_\psi(x)| \ll_{\psi,\varepsilon} x^{1/2+\varepsilon/2}$ for every u. The upper estimate (6.3) on arbitrarily distant such intervals for every sufficiently small $\varepsilon > 0$ would imply Dirichlet GRH for the retained characters.

Proof. First replace the weights at fixed a_t by the two values B', W. The large-argument bound on the nondistinguished terms and the slope bound on the distinguished terms in Proposition 7.2, cost $O(b/x)$ per stage and translate for either choice of weights. The same proof applies at any a_t in the stated range and bounds the difference via the common idealized sum. Now hold the weights B', W fixed and change a_t to $a = \lceil bx \rceil$. In the prime-power formula before Fourier expansion the slope in a_t is bounded by $2\sum_{v \le (a+2b)/W} \Lambda(v)/v$: at each prime power the normalized multiplicity of a Frobenius coset is at most two. This is $O(\log(2a))$ by the factorial estimate in the proof of Theorem 5.1. The slope of M_d has the same bound because $1 \le w \le B'$, $D_{\xi_d} = O(\log(2x))$, and the real order stays within $2b$ of a. Thus the change costs $O(b \log(2a))$ per stage and translate, with no factor n. After normalizing, the error is $O(\delta^{-1} \log(2a)) = O_{\psi,\varepsilon}(x^{2\varepsilon})$. Equation (6.27) applies to the constant orders and weights by the exact cancellation in Theorem 6.29. This proves the first assertion.

For these parameters $h \ge \delta_1 n/2$, $h_2 \le n$, $m \asymp x^{1/2-\varepsilon/16}$, and $k \asymp x^{9/2+10\varepsilon}$. Substitution in (6.3) gives $\mathcal{B}/(hmb) = O_\psi(x^{1/2+17\varepsilon/16})$. Indeed its three contributions after division are $(x/m)(1 + m^2/k)$, $2nmx^\varepsilon/h$, and $h_2 k x^{2\varepsilon}/(hn^2)$. Using $\delta^{-1} \le x^\varepsilon$, the assumed upper estimate gives $\mathcal{G}_\psi(x) \le x^{1/2+8\varepsilon}$ for large x on the interval. The interval theorem and the primitive functional equation give the critical-line assertion as ε decreases to zero. □

At g_1, g_2 the polarized construction now has full Fourier denominator with logarithm at most $f r_0(r_0 + t_{\mathrm{r}}) \log(g_1 g_2) = o(b/x)$, since the ranks and $\log U$ are polynomial in x. The primes may vary with x. There are only finitely many fields and possible primes on each bounded interval, so one can take a common coefficient field for the geometric calculations of Volume X. Proposition 6.28 transfers the known cohomology control to precisely these fields and quantifies its dyadic cost. Neither the remaining odd-primary strict lengths nor the comparison of the specified normalized row and column periods is estimated by these arguments. They are still needed for the proposed proof of Main Theorem 1.1.

6.6.4 The exact dyadic kernel in the change of field degree

Proposition 6.28 loses at most e dyadic length per twist when the two-power extension has degree 2^e. We compute the kernel exactly. The loss can occur for a fixed conductor at every sufficiently large scale.

Theorem 6.32 (Dyadic strict descent on the prescribed fields)**.** *Keep the fields $E' = K_\psi C_{l'}(\sqrt{-q}, \sqrt{-d}) \subseteq E = K_\psi C_l(\sqrt{-q}, \sqrt{-d})$ of Proposition 6.28. At a dyadic place write E_w for the completion of E. For all sufficiently large scales and all integers $r \ge 1$, the kernel of the restriction*

$$\mathcal{S}_{2,r}(E') \longrightarrow \mathcal{S}_{2,r}(E)$$

is cyclic of order two precisely when

$$l > l', \qquad r \text{ odd}, \qquad i \notin E, \qquad i \in E_w, \qquad i^2 = -1.$$

It is zero in all other cases. The condition on E_w has the same answer at all dyadic places. When the levels differ and the two conditions on i hold, the sum of weighted kernel lengths for $1 \le r < t$ is exactly $\lfloor t/2 \rfloor \log 2$. Otherwise it is zero. The odd-prime restriction remains the invariant-group identification of Proposition 6.28.

Proof. We first describe the cyclotomic images, using a superscript $+$ on a two-power cyclotomic field for the fixed field of inversion on roots of unity. At the global place this is its maximal real subfield. In the local argument the same notation denotes the corresponding totally ramified

subfield of the two-power extension of $\mathbb{Q}_2$. Let T_∞ denote the union of these full cyclotomic fields, globally or locally as appropriate. At every sufficiently large level j, the intersection of the corresponding field $E_j = K_\psi C_j(\sqrt{-q}, \sqrt{-d})$ with the global T_∞ is either C_j or $\mathbb{Q}(\zeta_{2^{j+2}})$. Its dyadic completion has intersection with the local T_∞ equal to the analogous one of the two local fields.

For the global assertion, a character of E_j on a pure two-power conductor has trivial exponents on both auxiliary quadratic characters: each ramifies at its own prime q or d, whereas $K_\psi C_j$ is unramified there. Hence adjoining those two square roots adds nothing to the two-power cyclotomic intersection. For a fixed abelian base field B and $C_j \subset T_\infty$, the intersection $(BC_j) \cap T_\infty = (B \cap T_\infty)C_j$. Indeed the Galois action on T_∞ fixing B has image $\mathrm{Gal}(T_\infty/(B \cap T_\infty))$; imposing also that it fix C_j intersects this subgroup with $\mathrm{Gal}(T_\infty/C_j)$. Take $B = K_\psi$. Its fixed finite intersection with T_∞ lies in the full level 2^{j+2} for every sufficiently large j. The latter field has degree two over C_j; the larger possibility occurs exactly when $i \in E_j$.

Locally take B to be the compositum of a dyadic completion of K_ψ and the unramified quadratic extension of $\mathbb{Q}_2$. Indeed K_q gives this unramified quadratic field and K_d splits at two throughout the interval. The local base is thus fixed even as q, d vary. The same Galois argument proves the local intersection assertion. At both levels $j = l', l$ the choice between the two alternatives is constant once l' is large: it is determined by whether the fixed base intersection is contained in the fixed field of inversion in T_∞. Locally the relative degree from l' to l is $2^{l-l'}$. In fact the intersections of the fixed local base with the increasing C_j stabilize, so taking the compositum preserves the full relative degree. This equals the global relative degree. Thus at every dyadic place the decomposition group in E/E' is the entire group G, and G is cyclic. All assertions are uniform in the choice of the two auxiliary primes.

Write $M = \mathbb{Q}_2/\mathbb{Z}_2(-r)$, and regard $M^{G_E} \subseteq M^{G_{E_w}}$ as subgroups of the same torsion group. The global and local cyclotomic images at level j are either $1 + 2^{j+2}\mathbb{Z}_2$, when the field contains i, or $\{\pm 1\}(1 + 2^{j+2}\mathbb{Z}_2)$, otherwise. Under the first image the invariants are $M[2^{j+2+v_2(r)}]$. This follows by taking a generator $1 + 2^{j+2}$ and applying the binomial expansion, which gives $v_2((1 + 2^{j+2})^r - 1) = j + 2 + v_2(r)$. Inverting the generator's value preserves that valuation. Under the second image one imposes in addition the action of $(-1)^r$. It makes no change for even r and gives exactly $M[2]$ for odd r.

By inflation–restriction the absolute global restriction kernel is $H^1(G, M^{G_E})$. This group maps to the absolute local restriction kernel $H^1(G, M^{G_{E_w}})$ by the inclusion of the coefficients. A class in the former group is automatically unramified outside two: the extension E/E' is unramified there and the inflated cocycle is zero on inertia. Its membership in the strict kernel is therefore exactly the vanishing of its image at every dyadic place. If the two groups of coefficients coincide, this map is the identity under the common group identification, and the strict kernel is zero. This includes all even r and all cases where the two cyclotomic images agree. If $l = l'$, restriction itself is the identity.

It remains to consider $l > l'$, r odd, $i \notin E$ but $i \in E_w$. The global coefficients are $M[2]$ with trivial G-action. Thus $H^1(G, M[2])$ has order two. Locally the coefficients $A = M[2^{l+2}]$ form a cyclic group on which a generator of G acts by a unit z with $v_2(z - 1) = l' + 2$. Indeed the relative action on the full two-power cyclotomic layers is generated by $1 + 2^{l'+2}$ modulo 2^{l+2}, and raising to the odd power $-r$ preserves this first valuation. Let $e = l - l' > 0$. In the cyclic cohomology formula the norm operator acts by $1 + z + \cdots + z^{2^e-1}$, of valuation e by the same binomial identity. Its kernel on A has order 2^e, equal to the order of $(z - 1)A$, which it contains. Hence $H^1(G, A) = 0$. The global class vanishes locally at every dyadic place. The local test is independent of w since $E/\mathbb{Q}$ is abelian. This proves the formula and summing over

the odd $r < t$ gives the weighted total. □

Corollary 6.33 (A conductor attaining the dyadic loss)**.** *Let ψ be the primitive real quadratic character of conductor 28 cutting out $\mathbb{Q}(\sqrt{7})$. On every sufficiently distant common interval the strict restriction kernel in Theorem 6.32 has order two for every odd $r \geq 1$.*

Proof. The quadratic field $\mathbb{Q}(\sqrt{7})$ ramifies at seven and meets the global two-power cyclotomic extension trivially. Therefore $[K_\psi C_j : \mathbb{Q}] = 2^{j+1}$, so $l' = l - 1$ for all large x by the two minimality definitions. The global intersection in the proof of the theorem is C_l, which excludes i. But $-7 \equiv 1 \bmod 8$ is a square in $\mathbb{Q}_2$, so $\mathbb{Q}_2(\sqrt{7}) = \mathbb{Q}_2(i)$. Thus every condition in the theorem holds for odd r. □

This determines the loss in descending strict classes even when the local and global two-power roots of unity differ. The remaining large odd-prime strict groups and the normalized global periods are the two estimates in Section 6.6 still needed for Main Theorem 1.1.

6.6.5 A simultaneous divisor for projected theta sections

Fix one of the intervals $I = [V, V^{3/2}]$ of Section 12.3. Use throughout the fields $E = \widetilde{E}_x$ of Proposition 6.28, the sets of Theorem 6.29 and the two auxiliary primes of Proposition 6.30. Put $K = K_d$. On each field take the trace-polarized abelian scheme $\mathcal{A} = M \otimes_{\mathcal{O}_K} C_d$ with $\mathfrak{c} = \mathcal{O}_K$, and its rigidified symmetric ample bundle P, as in Theorem 52.10. All coefficient fields below may be enlarged by a single finite extension on I. We first identify the primes at which several CM directions can contribute to the same divisibility estimate.

Lemma 6.34 (Primes with the specified quadratic signs)**.** *Let F and o be the conductor and order of ψ, respectively, and $N = \operatorname{lcm}(2^{l'+2}, F)qd$. Put*

$$\mathcal{P}_x = \{5x/4 < p \leq 7x/4 : \ p \text{ prime}, \ p \nmid N, \ \psi(p) = 1, \ \xi_q(p) = 1, \ \xi_d(p) = -1\}.$$

Uniformly for $x \in I$ as the left endpoints tend to infinity,

$$\sum_{p \in \mathcal{P}_x} \log p = \frac{x}{8o} + o(x), \qquad |\mathcal{P}_x| \sim \frac{x}{8o \log x}.$$

For every such prime, the permutation σ_p of $\Sigma = \operatorname{Hom}_K(E, J)$ given by $\tau \mapsto \tau \circ \operatorname{Frob}_p^2$ has no fixed point.

Proof. Write $c(n) = \Lambda(n)$ when n is prime to N and satisfies the same three character conditions as p, and put $c(n) = 0$ otherwise. Write $S(y) = \sum_n c(n)(y/n - 1)_+$. On the units modulo N, the indicator of the three conditions is

$$\frac{1}{4o} \left(\sum_{j=0}^{o-1} \psi(n)^j \right) (1 + \xi_q(n))(1 - \xi_d(n)).$$

The interval construction chooses q, d with Theorem 5.1 on $[Z, Z^{K_0+1}]$, where $Z \leq V$ and $V^2 \leq Z^{K_0}$ for a fixed K_0. Thus $[x, 2x] \subset [Z, Z^{K_0+1}]$ throughout I, and $q, d > 2x$ for large Z. When each character in the expansion is replaced by its primitive representative, the correction to the ramp sum on this range comes from powers of primes dividing $2F$. For one such prime v and character χ, its omitted contribution has the form $c_{v,\chi} y + O_F(\log(2y))$. Indeed the terms involving y/v^j extend to a convergent geometric series with tail $O_F(1)$ after multiplication by y, and the remaining bounded summands number $O_F(\log(2y))$.

The only principal character in the expansion is obtained by taking the principal term in all three factors. Every nontrivial quadratic twist satisfies (5.1). The remaining primitive characters are fixed and satisfy $G_\chi(y) = o_\psi(y)$ by Proposition 3.2, including the character of conductor one. It follows from (1.1) that on $[x, 2x]$

$$S(y) = \frac{y \log y}{4o} + C_I y + O_\psi(y\eta_x), \qquad \sup_{x \in I} \eta_x \longrightarrow 0.$$

The real constant C_I includes the logarithmic derivatives and the contributions at primes dividing $2F$. Its size will cancel. The function S is convex and satisfies $yS'_+(y) - S(y) = \sum_{n \le y} c(n)$, also at integers. Bound $S'_+(y)$ by the secants to $(1-\delta)y$ and $(1+\delta)y$ at $y = 5x/4, 7x/4$, with $\delta = \sqrt{\eta_x} + 1/\sqrt{\log x}$. The endpoints lie in $[x, 2x]$ for sufficiently large V. For the displayed main term each secant differs from its derivative by $O_\psi(\delta)$. The two errors in forming each secant contribute $O_\psi(\eta_x/\delta)$. After subtracting $S(y)$ this gives $\sum_{n \le y} c(n) = y/(4o) + o(x)$ at both points. The contribution from powers $p^j \le 2x$ with $j \ge 2$ is $O(\sqrt{x} \log^2(2x))$, by counting at most $\sqrt{2x}$ possible bases and $O(\log x)$ exponents. Subtraction proves the first asymptotic. Each prime counted has $\log p = \log x + O(1)$, giving the second.

Since $n = [K_\psi C_{l'} : \mathbb{Q}] \le o\, 2^{l'}$ and $n \ge x^{2+8\varepsilon}$, for large x one has $1 < p^2 < 2^{l'+2} - 1$. Thus p^2 represents neither sign of one modulo $2^{l'+2}$. Its Frobenius acts nontrivially on the real cyclotomic layer $C_{l'}$. Since p is inert in K, $\mathrm{Frob}_p^2 \in \mathrm{Gal}(E/K)$. This nonidentity element acts freely by translation on the embeddings over K. □

Theorem 6.35 (One mixed derivative for all the primes and translates)**.** *Set $a = \lceil xb \rceil$ and*

$$r_x = \left\lfloor \frac{h_2}{8|\mathcal{P}_x|} \right\rfloor, \qquad d_x = \prod_{p \in \mathcal{P}_x} p^{r_x}.$$

At every sufficiently large $x \in I$ there are nonnegative orders $\boldsymbol{a} = (a_\tau)_{\tau \in \Sigma}$ with sum a. Put $\mathcal{H} = \prod_\tau H_d(a_\tau)$, with $H_d(0) = 1$ and H_d as in Proposition 6.22. One can take integral invariant tangent vectors D_τ in the corresponding CM eigenlines, primitive at every place above $\mathcal{P}_x$, such that for every integral scalar jet F_0 through order a and every $u \in A_0$,

$$\mathcal{D}_u F_0(0) \;\in\; \mathcal{H}\, d_x\, \mathcal{O}_J, \qquad \mathcal{D}_u = \prod_{\tau \in \Sigma} D_{\tau+u}^{a_\tau}. \tag{6.17}$$

The vectors can be fixed for each field on the whole interval. The additional divisor satisfies $\log d_x = (1/8 + o(1)) h_2 \log x$ uniformly there.

Proof. In the coordinates of Theorem 6.23 identify the embeddings with the coset $j = 0$ extending the chosen embedding of K. The distinguished positions before translation are $S_0 = \{(1, 0, z) : z \in T_2\}$, of size h_2. For each $p \in \mathcal{P}_x$ choose r_x pairs of positions $(\nu, \sigma_p^{-1}\nu)$, with $\nu \in S_0$, so that all the positions in all the pairs are distinct. To do this successively, if j pairs have already been chosen, their used positions form a set U of size $2j$. The forbidden centers in $U \cup \sigma_p U$ number at most $4j < h_2$. Any remaining center gives two different positions by Lemma 6.34. The total number of pairs is at most $h_2/8$, so the construction reaches the last prime.

Assign order $p-1$ to each center for p and p^2 to its partner. The total assigned is $O(h_2 x^2)$, which is $O_\psi(x^{4+8\varepsilon}) < a$ by Proposition 6.28. Put the entire remainder at one unused position and set all remaining orders equal to zero. Each center contributes $p-1$ low digits at that position in base p^2. The low digit of the p^2-term moves from its partner to the same center under σ_p. Thus the integer L_ν in Theorem 52.14 is at least p at each of these r_x centers. All other orders have nonnegative digits. Translating the whole list by u commutes with σ_p because $E/\mathbb{Q}$ is abelian. The same bound holds at the r_x distinct centers of $S_0 + u$.

To choose the vectors, intersect each generic tangent eigenline with the integral tangent module at the identity. It is a projective rank-one module over the Dedekind ring $\mathcal{O}_J$. At a prime in $\mathcal{P}_x$ the CM algebra is unramified and splits over the coefficient ring into its integral idempotent factors. The intersection there is precisely the primitive eigenline of the local calculation. By the Chinese remainder theorem for this module, one global element has nonzero reduction at every selected place. The union of the selected places for all $x \in I$ is finite. We can make the choice once for each eigenline and each of the finitely many fields.

At any finite place the formal group in the scalar CM coordinates is a power of the formal group of C_d. Indeed M is projective over $\mathcal{O}_K$ and free after localization at the scalar prime. The scalar factorial bound of Proposition 6.22 shows that $D_\tau^c/H_d(c)$ preserves integral formal functions for every c. To pass from evaluation at the identity to this operator assertion, apply the coefficient calculation to the function $F_0(X +_{\mathcal{A}} Y)$ in Y with integral power series coefficients in X. One may choose a formal lift of the jet locally since only orders through a enter the final value. Composing the commuting operators gives divisibility by $\mathcal{H}$ everywhere. At every place above $p \in \mathcal{P}_x$, the valuation of $\mathcal{H}$, normalized by $v_p(p) = 1$, is $\sum_\tau (v_p(a_\tau!) - F_{p^2}(a_\tau))$. Here $F_Q(c) = \sum_{i \geq 1} \lfloor c/Q^i \rfloor$. The additional r_x from Theorem 52.14 applies at every such place, proving (6.17). Finally $h_2 \geq nx^{-\varepsilon}$ and $|\mathcal{P}_x| \asymp_\psi x/\log x$. Rounding r_x therefore has relative error tending to zero. Summing $\log p = \log x + O(1)$ gives the asserted logarithm. □

Theorem 6.36 (Integral jets in every translation character)**.** *Write $g_i = \pi_i \bar{\pi}_i$ for the two simultaneous primes of Proposition 6.30. Set $G = g_1 g_2$, $\varphi = [\pi_1 \pi_2]$ on $\mathcal{A}$, and $B = \ker \varphi$. Take a section s_0 of P^3 with value one at zero in the rigidification. After one finite extension on I, all characters λ of $B(J)$ are defined. Write e_λ for their translation projectors on sections of $\varphi^* P^{3a}$. At every finite place $w \nmid G$, with integer ring R in the completion, the scalar jets through order a of*

$$\frac{e_\lambda s}{\varphi^* s_0^a}, \qquad s \in H^0(\mathcal{A}, \varphi^* P^{3a}) \otimes_{\mathcal{O}_J} R,$$

fill the entire integral jet module at the origin, for every λ. At every $x \in I$ sufficiently large, there is also a single global integral section s for which

$$0 \neq \frac{1}{\mathcal{H}} \mathcal{D}_u \left(\frac{G^{2n} e_\lambda s}{\varphi^* s_0^a} \right)(0) \ \in \ d_x \mathcal{O}_J \qquad (\lambda \in \widehat{B(J)},\ u \in A_0). \tag{6.18}$$

Proof. The two kernels of $[\pi_i]$ have coprime orders. Each is killed by g_i and has order g_i^{2n} by its index on the lattice M. Their product is the kernel of φ, by the degree formula and the trivial intersection. After finite extension all these torsion points and their character values are rational. Every point extends integrally by properness. Translation acts on the pullback bundle since $\varphi \circ t_z = \varphi$ for $z \in B(J)$. Consequently

$$G^{2n} e_\lambda = \sum_{z \in B(J)} \lambda(z)^{-1} t_z^*$$

preserves global integral sections. This is the simultaneous projector for the two individual translation characters after clearing the displayed denominator.

Over R at $w \nmid G$ the isogeny is finite étale, with constant kernel, and is a torsor under that kernel. In fact the kernel lies in the prime-to-residue-characteristic torsion $\mathcal{A}[G]$, and the generic splitting of the finite étale kernel extends over the normal base. In $\varphi_* \mathcal{O}_{\mathcal{A}_R}$ the summand of character λ is an invertible sheaf L_λ on the target. This follows by trivializing the torsor in the étale topology, where its functions are the regular representation and the λ-functions are multiples of a single unit character. The same trivializations give $L_\lambda \otimes L_\mu \simeq L_{\lambda\mu}$

by multiplication. Thus L_λ^G is trivial. By the projection formula the sections of character λ identify on the target with the sections of $P^{3a} \otimes L_\lambda$.

We show that this last bundle supplies all integral jets. On the geometric generic fiber L_λ lies in Pic^0. Indeed its associated homomorphism to the dual abelian variety is killed by G, and a homomorphism from a connected abelian variety to a finite group in characteristic zero is zero. The kernel of the map from line bundles to these homomorphisms is Pic^0. The homomorphism associated with P^{3a} is an isogeny. Hence there is a geometric point v such that $P^{3a} \otimes L_\lambda \simeq t_v^* P^{3a}$ on the generic fiber. These properties of the dual and the polarization are recalled in Milne, *Abelian Varieties*, Sections 9–10 and 13. There are finitely many orders, fields, pairs of primes and characters on the bounded interval. Enlarge J once so these points and isomorphisms are defined. Each v extends over R by properness. The quotient of the two bundles is then represented by a vertical divisor. The abelian scheme over R is regular with one integral special fiber, the divisor of a uniformizer, so this quotient is trivial over R as well.

Thus $P^{3a} \otimes L_\lambda$ is the a-th power of the relatively very ample bundle $t_v^* P^3$ over R. For clarity, choose a section nonzero at the origin modulo the maximal ideal. In the associated affine chart the ratios of the other sections, after subtracting their values at zero, generate the ideal of the origin in its local ring because the linear system gives a closed immersion. Products of at most a of these functions therefore span the truncated local ring over R, by induction on the powers of that ideal. Homogenizing the products with the chosen section shows that sections of the a-th power give every jet through order a. Since φ is étale and carries zero to zero with the same residue field, it identifies the completed local neighborhoods of the two origins. In a local trivialization of L_λ, passing to the λ-part on the source multiplies by an invertible function. Changing to the stated frame also multiplies the entire jet module by a unit. Sections over R are obtained by flat base change for H^0, and e_λ projects onto that integral summand because G is a unit there. This proves the surjectivity claimed in the theorem.

In characteristic zero each functional obtained by applying $\mathcal{D}_u$ to this character component is nonzero. In tangent coordinates dual to the eigenvectors, evaluate on the homogeneous monomial with the corresponding exponents of total degree a. All terms involving fewer than a ordinary derivatives vanish at zero and the value is $\prod_\tau a_\tau! \neq 0$. Surjectivity on jets realizes this test. The kernels of all these functionals form a finite union of proper hyperplanes over the infinite field J. A vector outside their union exists because the product of their defining nonzero linear polynomials cannot vanish everywhere on a vector space over an infinite field. Choose such a section and clear one common denominator to make it integral. Every $G^{2n} e_\lambda s$ is integral. Taking its scalar jet in the indicated integral frame and applying Theorem 6.35 proves (6.18). □

The two translation characters, the CM lattice and the prime interval in (6.18) have now been imposed together. The choice of orders in Theorem 6.35 used the positions of the Frobenius pairs. For the periods of the iterated lifts in the *Remark*, the harmonic coefficients and the sum over the global period classes prescribe a particular section and its differentiations. Establishing their integral comparison and an upper bound for their archimedean norms remains necessary for (6.3). The uniform odd-primary strict cohomology estimate on these same fields also remains open. For the ordinary signed operators, Proposition 95.39 bounds the discarded real tails on all retained pairs. Theorem 95.41 gives a compact integral for the full squared norm with the original marked columns and a uniform error. Theorem 95.43 and Corollary 95.44 extend this approximation to the off-diagonal Gram squares and diagonal rank statistic of the ordinary projected blocks, using a fixed power of the terminal prime for each transported real row cut. Theorem 95.46 bounds the large-rank terms of both signed moments at the diagonal identity normalization. Corollary 95.47 retains this saving on one simultaneous integral choice

of marked coordinates for the ordinary blocks. For a fixed restriction on a codimension-two subspace at each mark, Corollary 74.63 compares the projected table with its mixed differences in both norm and minor ideals. Theorem 74.61 identifies an entire determinant fiber on which the unnormalized projection cancels a constant coefficient. Theorem 74.64 computes the finite Gram operator on the entire codimension-two support. Corollary 74.65 and Proposition 74.66 give a relative error tending to zero on the detected subspace and bound the change of each mixed Gram block there. Theorem 74.67 reduces the marked Gram operator for any fixed restriction to smaller general linear groups. Corollary 74.68 makes the reduction simultaneously for signed inputs on different supports and bounds the dimensions required by their common restriction. Theorem 74.69 gives a reduction of the unnormalized finite matrices over the integers away from the auxiliary primes. Corollary 74.70 preserves all terminal determinantal ideals of the original signed input columns on a common support. Theorem 74.72 bounds the integral loss at the auxiliary primes themselves. Corollary 74.73 preserves the full mixed Gram matrices in a specified metric and bounds the all-place discrepancy of the usual exterior heights by a polynomial per rank. Theorem 95.50 removes the terms of large rational difference rank from the signed second and mixed fourth ordinary moments, at the diagonal identity normalization. Corollary 95.51 identifies a common integral lattice in the fixed space of the composite transport. Theorem 95.53 bounds orbital operator norms at the scalar identity normalization. Theorem 95.54 reduces the twofold and fourfold ordinary operators at this normalization to rational composites of difference rank at most a constant times d^3. Theorem 95.56 recovers each marked coefficient integrally at fixed source lattice, target lattice and isometry. Theorem 95.57 gives a common list of integral coordinates in which collisions on the retained block require rational difference rank at most a constant times d^2. Theorem 95.59 retrieves a signed coefficient in the presence of other classes when sufficiently few classes cover its nearby marked indices. Corollary 95.60 gives a lower bound for the full ordinary period on each incoming vector and a necessary condition for exact cancellation in the integral projection. Theorem 45.37 bounds the relative class group contribution at the half residue using the conductor of the squared character. Corollary 45.38 proves half-residue vanishing on the growing dyadic layers together with the full fixed subfield of two-power degree at primes three or five modulo eight. Theorem 45.40 excludes large primes in those progressions from the class groups of dyadic layers over a fixed real cyclic field. Corollary 45.42 descends the full zero-residue contribution on the indicated subfields to a finite list of cyclic decomposition fields. Theorems 45.43 and 45.44 give class number control and strict descent for varying dyadic Frobenius orders. Corollaries 45.45 and 45.46 bound the exceptional pairs on the common intervals and treat the entire real dyadic tower in the indicated high-prime range. Theorem 45.47 identifies the combined torsion of the odd-power half-residue characters with a single relative class group at each available precision. Corollary 45.49 removes the full mixed-field contribution at the specified weights for varying Frobenius orders. Theorem 45.51 bounds the total weight divisibility of the exceptionally long half-residue components across all characters and all weights by one relative class number. Corollary 45.52 applies the bound on the full period fields throughout the common interval. Theorem 45.53 bounds the split unit indices of several faithful characters using a simultaneous estimate for their local logarithms. Corollary 45.54 bounds invariant ideal classes uniformly in every cyclotomic layer for the stated degrees and characters. Theorem 45.56 measures the sum of the half-residue lengths by one polynomial in the action on an ordinary class group, with error independent of the precision. Corollary 45.57 includes the exceptional lengths and gives a finite test for their total on the original fields. Theorem 45.59 treats all surviving weights at a good prime by one polynomial on an ordinary class group, including the locally split zero residue. Corollary 45.60 bounds the width of the resulting two-sided test by a polynomial uniformly on the interval. Theorem 15.51 measures the entire residual local sum, including

all odd primes dividing the conductor or degree, by finite ordinary class group quotients with polynomial uncertainty at the proposed budget. Theorem 15.53 bounds the entire contribution of the prime ideals above any specified finite set of rational primes. Corollary 15.54 gives a necessary cyclotomic splitting condition for a prime class to survive the residual cutoff. Theorem 50.23 constructs an integral polynomial in the symplectic inertia traces which separates the unramified systems and bounds the loss on their congruence ideals and cotangent spaces. Theorem 45.62 realizes simultaneous prime ideal generators of finite strict class quotients at exact cyclotomic depths. When the residual characters are locally nontrivial, it proves that the specified common depth is the least possible. Theorem 95.86 determines the mixed Gram block at prescribed input flags, retaining all refinements at the other marked places. Corollary 95.87 computes its trace moments from the signed scalar matrix throughout the stated height range. Theorem 50.24 constructs the sharp integral trace annihilator for finite symplectic inertia on two planes. Corollary 50.25 controls the loss on passing to unramified traces even in the presence of nilpotents and coefficient torsion. At the ramified primes of the character conductor, Theorem 50.27 bounds the inertia restriction on the specified symplectic component. Theorem 50.28 computes the exact cost of the corresponding scalar local condition, and Corollary 50.29 sums the resulting bounds over coefficient primes and weights. Theorem 15.55 computes strict torsion from a ray class group with explicit local equations. Corollary 15.56 detects the whole length when two successive precisions agree. Theorem 15.57 applies this test on the original period fields, including primes dividing their degrees. Theorem 19.23 computes complete strict groups at all positive weights divisible by the coefficient prime minus one for a class of cyclic character fields. The cubic character of conductor 3367 gives a group of order nine throughout the even weights. Lemma 95.88 gives the signed weight of two specified input flags in the fourth moment. Theorem 95.90 bounds the contribution of pairs far from opposition at every orbital height on the original marked fields. Theorem 95.91 finds the least relative flag deficit supporting a unipotent marking paired with its inverse and computes its signed contribution. Theorem 95.92 evaluates exactly how much the remaining positions cancel in that coefficient. Theorem 95.94 evaluates a transvection against every opposite input flag average, with an arbitrary first relative marking. Theorem 95.95 gives all other relative positions by a recurrence in two pivot indices. Theorems 15.58 and 15.59 compute torsion and detect the vanishing of an entire integral Sylow block by ordinary class groups. Corollary 15.60 verifies the local hypotheses throughout the stated dyadic layers of the original fields. Theorem 19.24 identifies the extra tame inertia allowed in cyclic descent, and Corollary 19.25 computes its residual contribution from global units. Theorem 19.26 determines the strict group over every real dyadic layer of the cubic field of conductor 3367 at every positive even weight. Theorem 19.27 computes finite torsion at every intermediate cyclic field through local norms. Corollaries 19.28 and 19.29 give ray class tests for all moving quadratic components at the fixed odd primes dividing the character order. Theorem 95.98 localizes the untruncated signed moments at all real places with an error at the identity normalization. Corollary 95.99 bounds the normalized real cross pairings for the small determinants at prime terminal frequency. Theorems 19.30 and 19.31 bound the change in length under cyclic descent and the elementary divisors forced by tame ramification. Corollary 19.32 proves that an actual moving component at conductor 3367 is nonzero at every odd weight on the common intervals. Theorem 6.38 bounds positive integrals of the retained prime sum using only counts on a sparse set of scales. Corollary 6.39 gives the corresponding interpolation of the signed mean between neighboring fields. Theorem 6.40 detects a zero to the right of the proposed line using three signed averages along the common intervals at a single rational frequency. Theorem 108.30 computes a finite integer upper comparison for each chosen translate from its projected integral columns. Corollary 108.31 combines these comparisons across scales with the ramified and

higher-prime-power terms retained. Theorem 19.34 computes the complementary sign of the strict group at every weight over infinitely many sextic character fields and all their real dyadic layers. Corollary 19.35 bounds the ranks of moving components on those same intervals from below. Proposition 108.32 evaluates the terms outside the first-power range between the square root and the scale to the required accuracy. Theorem 108.34 uses the actual projected entries in that range and allows the translate to vary with the prime. Lemma 108.35 computes the total prime mass in this range on the common intervals. Theorem 108.36 proves that the integral upper comparison requires a power saving in the weighted mass on which the projected entries fail to attain the specified gain. Theorem 108.37 shows that the undivided columns reach the upper cutoff at every such prime. Corollary 108.38 proves that the proposed upper comparison for these columns therefore fails on every sufficiently distant interval. Theorem 108.40 removes the common divisor and evaluates the excess of the new comparison in terms of the residual valuations of the projected columns. Theorem 111.14 identifies the maximizing orientation on every cycle containing a distinguished embedding in this prime range. Corollary 111.15 evaluates the normalized logarithm of the removed divisor there using the first five odd powers of Frobenius, with absolute error tending to zero. Theorem 11.24 identifies the finite odd cocycles of the second archive with one strict eigensubgroup at odd primes. Proposition 11.25 controls the discrepancy at two, and Corollary 11.26 bounds the entire weighted sum of these finite cocycles on the period fields. Theorem 11.28 gives an explicit integer annihilating those odd cocycles for every coefficient prime and every CM subfield of a given cyclotomic field, by a finite-level form of Stickelberger's theorem. Theorem 11.29 bounds the full lengths by the valuations of one integer, uniformly over all CM subfields of the cyclotomic field. Corollary 11.30 supplies this integer on the original period intervals and the odd-cocycle input of the shorter source argument. Proposition 108.41 computes the Plancherel mass of the permitted residue types at the two level primes in that construction. Theorem 108.42 bounds both the discarded dimension on the definite automorphic space and the integral index of its finite level projection. Theorem 108.43 computes the Fourier transform of the conjugation average of the Bessel character at a split place. Corollary 108.44 uses it to bound the mass discarded by the two projections on the finite Bessel character of the actual period. Theorem 108.46 classifies the intersections of agreeing rational frames. Corollary 108.48 removes the off-Bessel terms preserving the isotropic space on either marked slice. Theorems 108.49 and 108.51 give the split-prime divisors of the joint Gram determinant and of the two terminal columns after saturation. Theorem 108.52 computes the Haar density over a pair of real frames. Corollary 108.53 evaluates it at the rational terms on the marked slices. Theorem 108.56 bounds the value of a localized Bessel convolution at a prescribed Gram matrix in terms of its quotient norms. Theorem 108.59 proves the formal-degree bound for the actual translated discrete-series average. Theorem 108.62 bounds the cut kernel of one rational open-cell term over all real places, including weights coupling those places. Theorem 108.65 sums the squared open-cell main bound over distinct rational frame products and first positive normals in a fixed class position. Theorems 108.66 and 108.68 compute the integral lift discriminant and prove the marked-prime gain for two terms with a common positive link. Theorem 108.71 gives the level gain for any pair of open terms from the codimension of their common full link. Theorem 108.74 proves the arithmetic discriminant estimate for every pair of surviving rational cell ranks on the same class lattices and levels. Theorem 108.77 bounds the full weighted sum of lattices of each corank, averaged over the actual marked definite classes on this interval. Theorem 108.80 bounds the weighted count of all surviving near pairs in agreeing blocks, including the multiplicity of rational frames over each lattice. Theorem 108.83 bounds the translation mean of the projected frame weight over the marked positive column lattice at every nonzero cell rank. Corollary 108.86 bounds the absolute cut contributions of all positive closed ranks in the squared period sum on agreeing

blocks. Corollary 108.89 estimates the cut second moment from open terms in an explicit range of normal lengths, using the compact angular integral of Lemma 108.87. Theorem 108.91 constructs a stationary position of length at most b^{5n} at every real place, and Corollary 108.92 proves its exact trace identity. Theorem 108.94 localizes the integrand modulo the common compact subgroup. Theorem 108.95 bounds its positive integral on a specified neighborhood at all real places. Theorem 108.98 bounds the full absolute integral outside a shrinking tube about the maximal compact subgroup, uniformly at each real place. Theorem 108.101 gives a positive quantitative lower bound for the entire real Bessel integral at every place of the common field. Theorem 108.103 normalizes the nonzero real test vectors, proves their decay, and estimates the error of the cut used in the global sum. Theorem 108.106 proves absolute convergence over rational translates and bounds the difference between the full and cut sums on every residue row. Theorem 95.62 separates the signed full-dimensional contributions at distinct projective isometries of polynomial lattice index in the second trace and bounds the noncentral fourth-trace terms. At the full retained index, Corollary 95.63 bounds the terms outside a small rank neighborhood of rational scalars. Theorem 95.65 separates distinct scalar rotations in the original integral coordinates. Theorem 95.66 bounds the fourth-trace contribution of nonidentity eigenvalues of large multiplicity and reduces the central sum at polynomial matched index to exact products equal to the identity. Theorem 95.68 computes the ordinary Gram norms and diagonal defects of the full-dimensional portion at polynomial matched index from signed correlations at rational products, with relative error uniform in the number of target rows. Theorem 95.70 gives disjoint integral coordinates for distinct rational transports below the stated matched index. Corollaries 95.71 and 95.72 compute the full-dimensional diagonal Gram block and the mixed block for source rows with a small marked identification. Theorem 95.74 bounds the sum of distinct isotropic spaces with deep terminal pairings. Corollary 95.75 evaluates the repeated-row identity coefficient for equal terminal reductions with an error uniform on the common intervals. Theorem 95.77 recovers the rational unipotent double class in each fixed positive hyperplane and transport from terminal reduction at precision linear in the flag length. Corollary 95.78 gives the aggregate error for the original signed coefficients. Theorem 95.80 recovers the positive transport in both common dimensions and the two hyperplanes when present. Corollary 95.81 groups the signed coefficients for a whole ordinary block with uniform total error. Theorem 95.83 recovers the full-dimensional rational unipotent class from a bordered frame minor and the positive transport. Corollary 95.84 recovers the signed operators class by class at both common dimensions with one error for the original block.

6.7 How the source states the final implication

The theorem on line 10 of `Main.lean` selects the critical-strip assertion from the three conclusions of `Supplements/Kernels/MainTheorems.lean`, line 15. The analytic deduction takes a zero in the strip to a zero of the inducing primitive character. The missing Euler factors cannot vanish there, since $|\chi(p)p^{-s}| < 1$. The functional equation for the primitive character takes a zero with $\operatorname{Re} s < 1/2$ to a zero at $1 - s$ for the inverse character. Its real part exceeds $1/2$. Consequently the asserted absence of zeros in the right half of the strip excludes both possibilities off the line. The source gives this implication in `Development/AutomorphicArithmetic/CharacterBoxes/ClaimStrip.lean`, line 59.

The source supplies the right-half-plane assertion through `Development/AutomorphicArithmetic/CharacterBoxes/MainConclusion.lean`, line 10. One of the two branches in line 8 there supplies the cohomological window. In the other branch, the theta construction uses the simultaneous bounds for a common family on the window. Its five clauses are written in `Formalism/SharpRestriction/GlobalTheoremInputs.lean`, line 158. The family occurs as the same argument in each clause. Specializing the cohomological estimate to one family and the

theta comparison to another would leave this inference without the common family it requires.

6.8 Locations in the accompanying source

The paths below are relative to `lean/src/` in the supplied archive.

- Local restriction kernel. `Formalism/GaloisCohomology/Statement.lean`, line 75.
- From a cohomological interval to an auxiliary field. `Development/AutomorphicArithmetic/CharacterBoxes/SupplyChi.lean`, line 8.
- The five simultaneous requirements for the period comparison. `Formalism/SharpRestriction/GlobalTheoremInputs.lean`, line 158.
- The comparison with its numerical exponents. `Development/AutomorphicArithmetic/ThetaCorrespondence/ThetaPairLifts/ThirdPlaceSeries/TopShape.lean`, line 52.

6.9 Interpolating the arithmetic count between selected fields

Fix a primitive character ψ of order o. The subtraction of $x\log x$ in Definition 1.3 determines how large the prime sum can be between two scales where arithmetic estimates are available. We first give the interpolation error with its sign.

Lemma 6.37 (The interpolation error for the family prime sum)**.** *Let μ be a probability measure supported on $[a,b]$, where $1\le a\le b<\infty$, and let $x=\int y\,d\mu(y)$. Put*

$$c_\psi(n)=1+\frac{1}{3o+1}\sum_{\substack{\chi\in\mathcal{C}_\psi\\ \chi\neq\mathbf{1}}}\operatorname{Re}\chi(n),\qquad H_n=\int(y-n)_+\,d\mu(y)-(x-n)_+.$$

Then $H_n\ge 0$, $c_\psi(n)>2/3$, and

$$\mathcal{G}_\psi(x)-\int\mathcal{G}_\psi(y)\,d\mu(y)=\int y\log(y/x)\,d\mu(y)-\sum_{a<n<b}\frac{c_\psi(n)\Lambda(n)}{n}H_n\ \le\ \frac{1}{2a}\int(y-x)^2\,d\mu(y).$$

For the probability on $a<b$ with atoms only at its two endpoints, the last upper bound equals $(x-a)(b-x)/(2a)$.

Proof. There are $o-1$ terms besides the principal one, and each real part is at least -1, including at primes dividing a conductor. Thus $c_\psi(n)\ge 1-(o-1)/(3o+1)>2/3$. The definitions give a real constant L_ψ such that

$$\mathcal{G}_\psi(y)+y\log y-L_\psi y=\sum_{n\le b}\frac{c_\psi(n)\Lambda(n)}{n}(y-n)_+\qquad(a\le y\le b).$$

Integrate this identity and subtract its value at the mean. The constant times y cancels. Each ramp is convex and is affine on the support when $n\le a$ or $n\ge b$. This proves the identity and the claimed sign. On $[a,b]$ the second derivative of $y\log y$ is $1/y\le 1/a$. Taylor's inequality about x, after the linear term integrates to zero, gives the upper bound. The endpoint masses are $(b-x)/(b-a)$ and $(x-a)/(b-a)$, whose variance is $(x-a)(b-x)$. □

For a node t at which conditions 1–3 hold, use the data at t to define the real number

$$D(t)=\frac{2}{Q\delta\,|A_0|}\sum_{u\in A_0}\Delta_u.$$

Here and below the symbols in a node value refer to that node's field, weights and translates. A node can therefore be used even if a neighboring node has a different dyadic layer.

Theorem 6.38 (A positive quadrature bound on the common intervals)**.** *Fix $0<\varepsilon<1/52$ and a common interval $[Y,Y^{3/2}]$. Use the same two auxiliary primes throughout. Take nodes $Y=t_0<\cdots<t_N=Y^{3/2}$ with*

$$t_{j+1}-t_j\le Ct_j^{3/4+2\varepsilon}.$$

Assume conditions 1–3 at the nodes. Write $\lambda_j(x)$ for the continuous, piecewise affine function on this partition taking value one at t_j and zero at the other nodes. For any finite positive measure ν on the interval put $w_j=\int\lambda_j\,d\nu$. Uniformly over ν and the nodes, for sufficiently large Y,

$$\int\mathcal{G}_\psi(x)\,d\nu(x)\le\sum_{j=0}^{N}w_jD(t_j)+O_{\psi,\varepsilon,C}\left(\int x^{1/2+4\varepsilon}\,d\nu(x)\right).$$

One can choose such nodes with $O_\varepsilon(X^{1/4-2\varepsilon}+1)$ in each dyadic shell $[X,2X]$ inside the interval, including every transition of the cyclotomic layer of Proposition 6.4.

Proof. At a node t, average the defining count over translates, as in Lemma 6.2. Replacing the count by the idealized sum costs at most $2Q/t$ before normalization. After division by $Q\delta/2$, the retained terms give $\mathcal{G}_\psi(t)$ up to $O_\psi((1+t^\varepsilon)(1+\log^2 t))$ from replacing the arguments and the character proportions. The other terms are bounded in absolute value by $8o\,t^{1/2+\varepsilon}$ by condition 2. Since $\delta \ge t^{-\varepsilon}$, this proves $|D(t) - \mathcal{G}_\psi(t)| \ll_{\psi,\varepsilon} t^{1/2+9\varepsilon}$.

For $t_j \le x \le t_{j+1}$, apply Lemma 6.37 with the endpoint masses. Its upper error is at most $(t_{j+1}-t_j)^2/(8t_j) \ll_C x^{1/2+4\varepsilon}$. Since $3/4+2\varepsilon<1$, both endpoints are between $x/2$ and $2x$ for Y large. The endpoint masses are precisely the two linear interpolation weights. Substitution of the estimate at the nodes and integration against ν give the assertion.

For the count of nodes split the interval at the powers of two. In the part of each shell $[X,2X]$, X a power of two, divide into equal subintervals of length at most $X^{3/4+2\varepsilon}$, using the ceiling of the length divided by this quantity. Include the layer transition points $x^{2+8\varepsilon} = 2^l$. There are $O(1)$ of these per shell because consecutive transition points have ratio $2^{1/(2+8\varepsilon)}$. Adding nodes can only reduce the gaps. Any interval $[X,2X]$ meets a bounded number of these shells. □

Corollary 6.39 (The count comparison at selected scales)**.** *Suppose that for a fixed ψ and arbitrarily small $\varepsilon>0$ there are arbitrarily distant common intervals with such nodes, on each of which the signed mean satisfies $D(t_j) \le A_{\psi,\varepsilon} t_j^{1/2+9\varepsilon}$ for every j. Then every open-strip zero of the retained family lies on the critical line. This applies in particular if condition 4 holds at just those nodes with the specified data.*

Proof. Take ν to be the point mass at an arbitrary x in the interval. Only its two neighboring node values enter. The theorem gives $\mathcal{G}_\psi(x) \ll_{\psi,\varepsilon,C,A_{\psi,\varepsilon}} x^{1/2+9\varepsilon}$ on the whole interval, for a mesh constant uniform in the chosen intervals. Theorem 2.1, followed by $\varepsilon \downarrow 0$, bounds all the real parts by $1/2$. The functional equation and closure under conjugation give the other inequality. Under condition 4 the bound on $D(t_j)$ follows directly from the four estimates for $\mathcal{B}/Q$ in Proposition 6.3 and $\delta^{-1} \le t_j^{\varepsilon}$. □

This removes the cost of interpolating a cohomology group or a period lattice between dyadic layers. The upper estimate on the signed node values is still required.

6.10 Three averages of the signed count on a common interval

We retain the arithmetic mean $D(t_j)$ and the interpolation functions λ_j of Theorem 6.38. There is an alternative to an upper estimate at each node. A zero to the right of a proposed line can be detected by three signed averages at a single rational frequency.

Set $\sigma = 256$. For $U = \log Y$, put $k = \lceil 5\sigma U/4 \rceil$ and $K_k(u) = u^k e^{-\sigma u}/k!$. For a rational τ and $l = 0,1,2$ use the following nonnegative weights on the nodes, which depend only on the mesh, the interval and the frequency.

$$w_{lj} = \int_U^{3U/2} K_k(u)\big(1+\cos(\tau u + 2\pi l/3)\big)\lambda_j(e^u)\,du.$$

In particular the same tuple of fields and translates is used in the three averages. The node counts themselves can have either sign.

Theorem 6.40 (Detection by three signed averages)**.** *Fix ψ, $0<\varepsilon<1/52$, and $1/2+4\varepsilon \le \theta < 1$. Suppose that for every rational τ there are arbitrarily distant common intervals, with nodes and data as in Theorem 6.38, on which*

$$\max_{0\le l\le 2} \sum_{j=0}^{N} w_{lj} D(t_j) \;\le\; B_\tau(\sigma-\theta)^{-k-1}.$$

The constant B_τ and the mesh constant may depend on $\psi, \varepsilon, \theta, \tau$ and are uniform over the selected intervals at that frequency. Then every open-strip zero of the retained family has real part at most θ. Conversely, if these zeros have real part at most θ, the displayed bound holds on every sufficiently distant such interval, with a constant independent of τ and of its nodes.

Proof. Put $g(u) = \mathcal{G}_\psi(e^u)$, and define $A_k(s)$ from its Laplace transform as in Section 2.3. Write $D_* = \sigma - \theta$. Use Theorem 6.38 with the image under $u \mapsto e^u$ of $K_k(u)(1 + \cos(\tau u + 2\pi l/3))\, du$ restricted to the interval. The error is at most a constant times

$$2\int_0^\infty K_k(u) e^{\theta u}\, du = 2D_*^{-k-1}.$$

This is an upper estimate for a signed integral since the measure used for interpolation is positive.

We can extend all three integrals to $0 \le u < \infty$ with an absolute error $O_\psi(\sigma^{-k})$. Indeed $|g(u)| \le C_\psi e^{2u}$. In the tail estimates of Lemma 2.4 take $B = 2,\ c = 5/4,\ v = 3/2$. The two exponential rates after multiplication by σ^k are

$$2 + \sigma(1/4 - (5/4)\log(5/4)), \qquad 3 + \sigma(-1/4 + (5/4)\log(6/5)).$$

Both are negative at $\sigma = 256$. For example the alternating series for $\log(1+t)$ gives $\log(5/4) > 1/4 - (1/4)^2/2$ and $\log(6/5) < 1/5 - (1/5)^2/2 + (1/5)^3/3$. The bound on the cosine factor is two at every frequency.

The extended integrals equal $A_k(\sigma) + \mathrm{Re}(e^{-2\pi i l/3} A_k(\sigma + i\tau))$ by differentiation under the integral, since g is real. Of three equally spaced points on the unit circle at least one has angular distance at most $\pi/3$ from any given direction. Maximizing therefore gives

$$A_k(\sigma) + \tfrac{1}{2}|A_k(\sigma + i\tau)| \ll_{\psi,\varepsilon,\theta,\tau} D_*^{-k}.$$

For each fixed τ this holds along an unbounded sequence of k. We explain why such separate sequences suffice.

Let Z be the set of poles of the transform in $\mathrm{Re}\, s > 0$. They are simple and their residues are nonzero by Lemma 2.3. Suppose a pole satisfies $\mathrm{Re}\,\rho > \theta$. Set $D = \min(D_*, \mathrm{dist}(\sigma, Z))$. If some ρ_1 has $\sigma - \mathrm{Re}\,\rho_1 < D$, take R strictly between these numbers. A disk of radius R about $\sigma + i\tau$ contains ρ_1 for all τ in an open interval about $\mathrm{Im}\,\rho_1$. Shrink to a bounded interval. Only finitely many poles meet the resulting closed disks, by compactness in $\mathrm{Re}\, s > 0$. Each pair of distinct poles has an equidistance at at most one height, since both are to the left of $\mathrm{Re}\, s = \sigma$ and poles of the same height have different horizontal distances. We can thus choose a rational τ for which there is a unique nearest pole ρ, at distance $d < R$. Choose $d < r < R$ so there is no further pole in its closed disk. At the real center the closed disk of radius r is pole-free. Subtracting the simple polar part and using Cauchy's inequality as in (2.4) yields

$$\tfrac{1}{2}|\operatorname*{Res}_\rho \Phi_\psi| d^{-k-1} - O(r^{-k}) \ll_{\psi,\varepsilon,\theta,\tau} D_*^{-k}.$$

Multiplication by d^k along the sequence for this rational frequency contradicts the nonzero residue.

In the remaining case the horizontal distance of every pole is at least D. The proposed pole gives $D < D_*$, so a pole attains the distance D from σ. It must have zero imaginary part, and is the unique nearest pole. Its residue is positive by Lemma 2.3. Choose $D < r < D_*$ giving no other pole in that closed disk. At $\tau = 0$ Cauchy's estimate gives $A_k(\sigma) = (\mathrm{Res}_{\sigma - D}\, \Phi_\psi) D^{-k-1} + O(r^{-k})$, which contradicts the same upper bound along the sequence for zero frequency.

For the converse, the absolutely convergent explicit formula (3.1) gives $|\mathcal{G}_\psi(t)| \ll_{\psi,\theta} t^\theta$. The node error in Theorem 6.38 gives the same bound on $|D(t)|$. If adjacent nodes have ratio at most two, their θ-powers are at most $2^\theta x^\theta$ at each intermediate x. This ratio condition holds on sufficiently distant intervals for each fixed mesh constant. Now sum with the nonnegative weights, use $\sum_j \lambda_j = 1$, and integrate as in the first display of the proof. □

In particular one may take $\theta = 1/2 + 4\varepsilon$ and let ε decrease to zero. The functional equation then yields the critical line for every character whose family satisfies the three bounds. The computations establish the weights and the detection theorem. The upper bound on those three averages of the actual arithmetic count is the unresolved estimate in this formulation.

7 The field and the auxiliary weights

7.1 Integral weights and theta coefficients on the same interval

Fix the field E of Proposition 6.4 at each x of its interval. In the next two results the auxiliary primes q, d stay fixed throughout that interval. First we choose integral weights for the prime sum. We then give an integral period comparison for theta coefficients of positive definite Hermitian lattices over E.

7.1.1 Choosing the small weights uniformly

Lemma 7.1. *There are constants $c > 0$ and Y_0, depending only on F, ε, such that, for $Y \ge Y_0$, every primitive character α belonging to $\mathrm{Gal}(E/\mathbb{Q})$ satisfies*

$$|G_\alpha(y)| \le y \exp(-cx^{\varepsilon/8}) \qquad \left(y \ge \tfrac{1}{3}\exp(x^{\varepsilon/3})\right).$$

Proof. Write $N = 2^{l+2}Fqd$, so every conductor in question divides N. We first remove the possible exceptional real zero from the classical zero-free region. Every real character of this field is the product of the two auxiliary quadratic characters to powers zero or one and a real character of $E_0 = K_\psi C_l$. The latter has odd conductor part dividing F. Its local character at two has order at most two, hence factors modulo eight, since every unit congruent to one modulo eight is a square in $\mathbb{Z}_2^\times$. Its primitive conductor therefore divides $8F$. If an auxiliary factor occurs, Theorem 5.1 excludes every real zero in $[49/50, 1]$. Otherwise there are finitely many possible real primitive characters, independently of Y, and Lemma 2.2 gives a fixed zero-free neighborhood to the left of one for each. The principal character is covered by the classical zeta zero-free region. Thus the usual de la Vallée Poussin theorem, whose sole possible exception for a primitive character is a real zero of a real character, gives

$$1 - \operatorname{Re}\rho \ge \frac{c_F}{\log(2N(3 + |\operatorname{Im}\rho|))}$$

for every nontrivial zero under consideration. The same lower bound for $\operatorname{Re}\rho$ follows by the functional equation applied to the conjugate character. [10, 11]

Here are the uniformities needed in (3.1). In a band $|\operatorname{Im}\rho - u| \le 1$ the number of zeros is $O(\log(2N(3 + |u|)))$, with an absolute constant. This follows by taking the real part of the completed logarithmic derivative at $2 + iu$, as in Section 3.1; the Euler product there is bounded absolutely and the gamma term and conductor give the displayed logarithm. Zeta has two additional rational terms, both bounded on that line. For $|\operatorname{Im}\rho| \le 1$ each reciprocal $|\rho(1-\rho)|^{-1}$ is $O_F(\log^2(2N))$ by the two zero-free bounds. Summing the bands at larger

ordinates gives

$$\sum_{\rho}\frac{1}{|\rho(1-\rho)|}\ll_F \log^3(2N),\qquad \sum_{|\operatorname{Im}\rho|>T}\frac{1}{|\rho(1-\rho)|}\ll\frac{\log(2N(3+T))}{T}\quad (T\ge 2).$$

For the constant d_0 in (3.1), logarithmically differentiate the functional equation at zero and remove its simple gamma pole when present. For a nonprincipal character this bounds $|d_0|$ by $O(1+\log(2N)+|L'(1,\bar\alpha)/L(1,\bar\alpha)|)$. The local zero expansion obtained by subtracting the completed logarithmic derivative at 2 from the one at 1 bounds the last term by $O_F(\log^2(2N))$. Indeed the nearby zeros have distance at least $c_F/\log(6N)$ from 1, and the other bands contribute $O(\log(2N))$ by summation with weights j^{-2}. Zeta has a fixed d_0. The coefficient λ in the same formula is either zero or one. Finally, on $\operatorname{Re}s=-1/2$ the functional equation and absolute convergence at $1-s$ give $|L'/L(s,\alpha)|\ll\log(2N(3+|\operatorname{Im}s|))$. Integration against $1/|s(s-1)|$ makes the remainder $O(y^{-1/2}\log(2N))$.

Set $L=\log y$, $T=\exp(\sqrt{L})$. For $|\operatorname{Im}\rho|\le T$ we have

$$y^{\operatorname{Re}\rho-1}\le\exp\Bigl(-c_F'\min\Bigl(\sqrt{L},\frac{L}{\log(2N)}\Bigr)\Bigr).$$

By construction $\log(2N)\ll_{F,\varepsilon}x^{\varepsilon/5}(\log x)^2+\log(2x)\ll_{F,\varepsilon}x^{5\varepsilon/24}$. Also $L\ge x^{\varepsilon/3}-\log 3$. The minimum is bounded below by a constant times $x^{\varepsilon/8}$, since $1/3-5/24=1/8$. Insert this estimate, the mass and tail bounds, and the preceding remainder estimates into (3.1) divided by y. Powers of $\log(2N)$ are absorbed by decreasing c; the tail is $O((\log(2N)+\sqrt{L})e^{-\sqrt{L}})$. The same bound with a smaller c absorbs $(L+1+\log^2(2N))e^{-L}$. This proves the lemma uniformly for all larger y. □

Proposition 7.2 (Integral prime-sum weights)**.** *In Proposition 6.4, take the distinguished weights to be $\lceil b\rceil$, all others to be 1, and $a_t=\lfloor bx\rfloor$. These choices satisfy conditions 1–3 for every x on the same interval. More generally, the weight estimate in condition 3 holds if $|a_t/b-x|\le 1$, every distinguished weight is in $[b,b+\sqrt{b}]$, and each other positive weight w satisfies $a_t/w\ge\exp(x^{\varepsilon/3})/3$.*

Proof. For a nondistinguished weight, Lemma 7.1 gives

$$|wG_{\xi^*}(a_t/w)|\le b(x+1)e^{-cx^{\varepsilon/8}}.$$

There are at most $f|B|$ terms before division by f in (6.2), and $|B|\ll_F x^3$. Their combined absolute value is at most b/x for large Y. For the distinguished weights the derivative estimate after condition 4 bounds their total change by $|B|h_2H_x\sqrt{b}/f$. Since $h_2\le f$, $H_x=3(x+1)\log(2x+4)$, and $b=\exp(x^{\varepsilon})$, this too is at most b/x. This proves the more general assertion. For the stated integer choices $|a_t/b-x|\le 1/b$, and $a_t\ge\exp(x^{\varepsilon/3})/3$. All the other size conditions are those already proved in Proposition 6.4. The points a_t/b still belong to $[Y,Y^2]$ for large Y, so Theorem 5.1 supplies the required character inequalities. □

7.1.2 An integral theta period for chains of Hermitian lattices

Write $k=E^+$ for the fixed field of complex conjugation. For $0\le j\le m=\lfloor\sqrt{x}\rfloor$, put $V_j=E^{j+1}$ with the standard positive definite Hermitian form H_j, and fix the lattice $\mathcal{O}_E^{j+1}$. Let $\mathcal{C}_j$ be its genus, taken up to global isometry. Concretely this is the set of isometry classes of full $\mathcal{O}_E$-lattices locally isometric to the specified lattice at every finite place of k. It is the double coset set for $U(V_j)(k)$ acting on the finite adelic group modulo the compact open stabilizer of the lattice. The set is finite by the finiteness theorem for algebraic modular forms

on a group compact at infinity, which applies since H_j is definite at every real place of k. The automorphism group of a representative A is finite as well. It preserves the positive definite real form $\operatorname{Tr}_{k/\mathbb{Q}} H_j(v,v)$ on its underlying $\mathbb{Z}$-lattice. Write $w(A) = |\operatorname{Aut}(A)|$. [12]

For $A \in \mathcal{C}_{j+1}$ and $B \in \mathcal{C}_j$, define the theta coefficient

$$K_j(A,B) = \#\{s : B \to A \ \mathcal{O}_E\text{-linear} \mid s^* H_A = H_B\}.$$

This is the coefficient at H_B of the Hermitian theta series summing the pullback forms s^*H_A over $\operatorname{Hom}_{\mathcal{O}_E}(B,A)$. The coefficient is finite. For each vector in a fixed $\mathbb{Z}$-basis of B, its image belongs to a positive definite lattice and has prescribed trace norm, leaving only finitely many choices. Every map counted here is injective because H_B is positive definite.

For functions on the finite sets use the bilinear period

$$[u,v]_j = \sum_{B \in \mathcal{C}_j} \frac{u(B)v(B)}{w(B)}.$$

These weights also give integration for which the finite adelic lattice stabilizer has volume one and the compact group at infinity has volume one. Indeed, on the orbit of B the stabilizer from rational points has cardinality $w(B)$. Define

$$(T_j u)(A) = \sum_B \frac{K_j(A,B)}{w(B)} u(B), \qquad (T_j^\dagger v)(B) = \sum_A \frac{K_j(A,B)}{w(A)} v(A).$$

Theorem 7.3 (An endpoint denominator for a chain)**.** *Let J be any number field and take $\mathcal{O}_J$-valued functions f on $\mathcal{C}_0$ and h on $\mathcal{C}_m$. Each T_j carries integral functions to integral functions. With $s_m = (m+1)[E:\mathbb{Q}]$ and $D_m = |\operatorname{GL}_{s_m}(\mathbf{F}_3)|$, the common period*

$$\begin{aligned} P(f,h) &= [T_{m-1} \cdots T_0 f, h]_m \\ &= [f, T_0^\dagger \cdots T_{m-1}^\dagger h]_0 \ \in \ D_m^{-1}\mathcal{O}_J \end{aligned} \tag{7.1}$$

satisfies $\log D_m \ll_F x^7$ at every point of the interval in Proposition 6.4.

Proof. The group $\operatorname{Aut}(B)$ acts on the maps counted by $K_j(A,B)$ by precomposition. If $s \circ u = s$, injectivity of s gives $u = 1$. All orbits therefore have $w(B)$ elements. In fact each orbit specifies exactly one sublattice of A isometric to B, since two isometries onto the same image differ by an automorphism of their domain. Thus $K_j(A,B)/w(B)$ is an integer. The matrices of all forward maps have integral entries, so the entire forward composite preserves integral functions.

Expand $[T_j u, v]_{j+1}$. The term indexed by A, B is $K_j(A,B)u(B)v(A)/(w(A)w(B))$. Summing first over A gives $[u, T_j^\dagger v]_j$. All sums are finite. Applying this identity successively yields the two expressions in (7.1) with the same normalization.

An automorphism of an endpoint lattice acts faithfully on its underlying free $\mathbb{Z}$-module of rank s_m. Reduction modulo three is injective on every finite subgroup of $\operatorname{GL}_{s_m}(\mathbb{Z})$. To see this, suppose that its kernel contained an element of prime order p. Write it as $I + 3^a C$ with $a \geq 1$ and some entry of C prime to three. If $p \neq 3$, the binomial expansion to power p has a nonzero term $p3^a C$ modulo 3^{a+1}. For $p = 3$, the first term after the identity is $3^{a+1}C$ modulo 3^{a+2}. Both contradict the element having order p. Every nontrivial finite kernel would contain such an element. Consequently $w(A) \mid D_m$ for every endpoint class A. Multiplying the forward period by D_m clears all its denominators simultaneously. Counting all matrices over $\mathbf{F}_3$ gives $\log D_m \leq s_m^2 \log 3$. On the common family $[E:\mathbb{Q}] \ll_F x^3$ and $m+1 \leq 2\sqrt{x}$, giving the asserted uniform estimate. □

7.1.3 Integral normalizations in the Bessel comparison

The comparison in the *Remark* also uses periods with nontrivial archimedean weights and Hecke characters. The following calculation specifies what an identity for such periods would have to control. Let M, L be full $\mathcal{O}_J$-lattices in finite-dimensional J-spaces and let $P \neq 0$ be a J-bilinear form. Define its content $I(P; M, L)$ to be the $\mathcal{O}_J$-ideal generated by the values $P(M, L)$, allowing a fractional ideal. This ideal is finitely generated because the lattices are. For (7.1) the two formulas have equal content and $D_m I \subseteq \mathcal{O}_J$.

Proposition 7.4. *Suppose forms P, P' are related over J by $P(f, h) = aP'(Uf, Vh)$ with $a \in J^\times$ and J-linear isomorphisms U, V. Then*

$$I(P; M, L) = a\, I(P'; UM, VL).$$

At a finite place v write π for a uniformizer. Suppose endpoint lattices M', L' obey

$$\pi^i M'_v \subseteq UM_v \subseteq \pi^{-i'} M'_v, \qquad \pi^j L'_v \subseteq VL_v \subseteq \pi^{-j'} L'_v$$

for nonnegative integers i, i', j, j'. Put $b_v = v(I(P; M, L)) - v(a) - v(I(P'; M', L'))$. Then $-i' - j' \leq b_v \leq i + j$.

Proof. The first equality holds on the generating values in both directions. Localize it at v. Bilinearity and the inclusions imply that the image content lies between $\pi^{i+j} I(P'; M', L')_v$ and $\pi^{-i'-j'} I(P'; M', L')_v$, before multiplication by a. Taking valuations gives the inequalities. □

At infinity fix Hermitian norms and let $\|P\|_v$ denote the operator norm. If U, V are isometries at every such place, then $\|P\|_v = |a|_v \|P'\|_v$. The product formula

$$\sum_{v|\infty} [J_v : \mathbb{R}] \log |a|_v = \sum_{v \nmid \infty} v(a) \log |\mathcal{O}_J / \mathfrak{p}_v|$$

shows exactly how the scalar cancels in the difference between the finite content and the logarithmic archimedean norms. The remaining finite discrepancies are the b_v. Even in dimension one an integral theta coefficient may be multiplied by an arbitrary integer by scaling its integral test function. Its finite content and its archimedean norm both acquire that factor.

The coefficient maps in Theorem 7.3 count Hermitian lattice embeddings. The periods required in (6.3) involve specific automorphic representations, projections onto specified weights, and a common Bessel lattice for all translates u. To apply Proposition 7.4 one must identify the projected periods over a coefficient field, bound the lattice indices after projection, and estimate the archimedean norms on that same family. The endpoint denominator in (7.1) accounts for stabilizers in the definite lattice sum. It gives no bound for the indices or norms of those additional projections.

7.2 The local ring and the nonzero auxiliary period on one family

Keep $E = K_\psi C_l(\sqrt{-q}, \sqrt{-d})$ on the common power interval. The index l may be the shortened one of Proposition 6.28. The real subfield has degree $[C_l : \mathbb{Q}] \gg_\psi x^{2+8\varepsilon}$ in either choice. Let $K = \mathbb{Q}(\sqrt{-d})$, $\Phi = \mathrm{Gal}(E/K)$, $b = \exp(x^\varepsilon)$, $a = \lceil xb \rceil$, $W = \lceil \sqrt{b} \rceil$, and $B' = \lceil b \rceil$. Use the orientation and translation by complex conjugation of Theorem 6.23. In the following statement j denotes a cyclotomic twist and $r \leq s$ are matrix ranks at one definite theta step.

Theorem 7.5 (Simultaneous conclusions for the strict groups and the auxiliary norm)**.** *At every x on a sufficiently distant common interval the following statements hold for this same E.*

1. *For $\ell > \exp(x^{\varepsilon}/12)$ and $1 \le j < \lfloor x^{401} \rfloor$, put $\Psi = \kappa_\ell^j \alpha$ for any character summand of the complementary sign $\alpha(c) = (-1)^j$, with coefficient discrete valuation ring O, fraction field J_ℓ, and $D_\ell = J_\ell/O$. Write η for the quadratic character of $\mathbb{Q}(i)$. The local component through the diagonal representation has an integral formal power series ring with $1, 2, 3, 4, 2$ variables in weights $-2, -1, 0, 1, 2$. The quotient C_{-1} of the augmentation cotangent module of the marked global ring representing deformations on its two lower simple-root lines satisfies*

$$\mathrm{Hom}_O(C_{-1}, D_\ell) \simeq \mathcal{S}(\mathbb{Q}, D_\ell(\Psi^{-1})) \oplus \mathcal{S}(\mathbb{Q}, D_\ell(\eta\Psi^{-1})).$$

Here $\mathcal{S}$ means locally zero above ℓ and unramified at the other finite primes. The first summand is the corresponding strict character summand for E with the convention of Section 25.1.

2. *Take polynomially bounded ranks $r \le s$, with rs even, and the fixed class representatives and positive split normalization in Sections 54.1–54.2. For a finite family of threshold lists $W \le w_\tau \le B'$ and translates t, there is one torsion input at this step giving nonzero algebraic numbers Y_t from the corresponding class sums. Enlarge the coefficient field to a common finite Galois extension $J/\mathbb{Q}$. Then*

$$C_d(t, a, w) \le \frac{\log |N_{J/\mathbb{Q}} Y_t| / [J : \mathbb{Q}] + B_-^{(S)}(Y_t)}{r} + O_{\psi,\varepsilon}(b/x). \tag{7.2}$$

The allowed threshold depends on the fixed polynomial bound for the ranks. The error is uniform in the listed translates. The set S consists of the good inert primes up to a/W in Section 54.1, and $B_-^{(S)}$ sums the negative valuations outside it with the normalized local degrees and logarithms of the rational primes.

Proof. For the cohomology statement, the three-factor inertia equation at the two distinct outer eigenvalues detects local coboundaries in both lower simple-root directions over every finite coefficient module. The conjugated diagonal family belongs to the saturated component, so the converse holds there as well. The marking at complex conjugation and the two further entry conditions give a unique representative of each pair of strict classes. Passing to the direct limit over the finite coefficients gives the displayed integral dual. For $j \ge 2$ the residual upper-triangular obstruction vanishes by local Tate duality. For $j = 1$ the two adjacent cocycles split by Kummer theory into a unit term and a valuation term. Saturating by the two Frobenius differences kills the valuation coordinates exactly. This supplies the same twelve coordinates also in the exceptional first twist. The proofs, including the finite coefficient passage, are in Corollary 25.9 and Theorems 25.11 and 25.15.

For the period statement, place degree a at t, with one further nonzero minor degree for each nontrivial cyclic subgroup of the 2-primary part of Φ. For a Frobenius cycle of even order its low half contains exactly one of those embeddings in the cycle through t. Its degree is at most the prescribed threshold at every odd power of that Frobenius. Corollary 54.19 therefore gives the weighted divisibility of the actual derivatives. The primes with no 2-primary Frobenius component after composing with conjugation are ± 1 modulo the conductor of the real cyclotomic layer. Their total cost and the accumulated rounding error are $O_{\psi,\varepsilon}(b/x)$, as proved in Theorems 54.2–54.3. For the positive split normalization each class sum has strictly positive inner product with the initial vacuum. The mean squared derivative over translations equals its positive squared norm times the scaled Fischer norm of the minor. A torsion grid supported at one fixed split prime is dense in the common period torus. The intersection of the finitely many open dense nonvanishing sets therefore contains one such torsion point. Theorem

54.7, verifies that the corresponding input gives these values at all the fixed representatives. Applying the norm formula now proves (7.2). All field selections in this argument precede the local and theta constructions. □

There is a quantitative obstruction to reading the required centered estimate directly from this auxiliary norm. The obstruction applies even after the nonvanishing just proved.

Proposition 7.6 (The excess for a character with nontrivial odd part)**.** *Suppose the order o of ψ has a nontrivial odd divisor. Let w equal B' on a subset of a single coset of $A_0 = \mathrm{Gal}(K_\psi C_l/K_\psi)$ in Φ, and equal W elsewhere. Write ω for the proportion of positions carrying B'. For the actual numbers of Theorem 7.5 put*

$$H_t = ([J:\mathbb{Q}]^{-1}\log|N_{J/\mathbb{Q}}Y_t| + B_-^{(S)}(Y_t))/r.$$

Then uniformly throughout the interval

$$H_t - M_d(a,w) \;\geq\; a\omega\log(B'/W) - O_{\psi,\varepsilon}(b/x). \tag{7.3}$$

For the sparse densities of Theorem 6.29, the positive main term here is $\gg bx$. It exceeds the permitted error $x^{5\varepsilon}\mathcal{B}/(hm)$ per translate and step for those ranks.

Proof. The odd-order subgroup P' of Φ has nontrivial image in Φ/A_0, since A_0 is a 2-group and o has an odd divisor. Thus $t+g+P'$ cannot be contained in a single coset of A_0. For every nontrivial cyclic 2-subgroup the minimum defining its minor degree in equation 54.2, is therefore W. The entire degree list, its scalars Y_t, and the clearing integers are exactly the same as those for the constant threshold list $w_* \equiv W$, using the very same input and representatives. Apply (7.2) to w_*.

For this constant list the inertia average summed over all classes outside Φ gives $1-\xi_d(p^k)$ at each prime power. Consequently, with $X = a/W$,

$$C_d(t,a,w_*) = W\{R_{\mathbf{1}}(X) - R_{\xi_d}(X)\}.$$

The corrections in (1.1) and the selected-field estimate of Lemma 7.1 give $C_d(t,a,w_*) = M_d(a,w_*) + O(ae^{-cx^{\varepsilon}/8})$. Here $X \asymp xe^{x^{\varepsilon}/2}$ lies in its large-argument range, ξ_d is one of its selected twists, and the error is $o(b/x)$. The main-term formula of Proposition 6.1, used in the K_d-orientation, gives $M_d(a,w_*) - M_d(a,w) = a\omega\log(B'/W)$. This proves (7.3).

At the specified sparse densities $\omega = h_2/(2n) \geq x^{-\varepsilon}/2$, and $\log(B'/W) \sim x^{\varepsilon}/2$. For comparison, dividing the formula in (6.3) by hm gives three terms after multiplication by $x^{5\varepsilon}$. Use $a_E = xb$, $m \asymp x^{1/2-\varepsilon/16}$, $h \asymp nx^{-\varepsilon/8}$, $h_2 \leq n$, $n \asymp x^{2+8\varepsilon}$, and $k \asymp x^{9/2+10\varepsilon}$. The three terms are respectively

$$O_{\psi,\varepsilon}\Big(bx^{1/2+(5+1/16)\varepsilon}\Big),\quad O_{\psi,\varepsilon}\Big(bx^{1/2+(6+1/16)\varepsilon}\Big),\quad O_{\psi,\varepsilon}\Big(bx^{1/2+(1+1/8)\varepsilon}\Big).$$

Each is $o(bx)$ since $0<\varepsilon<1/52$. This establishes the claimed comparison. □

The proposition concerns the positive single-step auxiliary norm with the particular minor degrees of Volume X. It shows exactly where that construction loses the odd component of the distinguished weights. The originally prescribed Bessel period retains further row, level and character data. Its quantitative comparison and the uniform sum of the strict lengths in (6.6) are still required for Main Theorem 1.1.

7.3 Removing finite cohomology layers and a common theta divisor

Keep the fields and ranks of Theorem 6.29, $a = \lceil xb \rceil$, $b = \exp(x^{\varepsilon})$, and put $P = b^{1/12}$. The prime-to-ℓ characters and the relations in (6.6) are those attached to these fields. We first remove a range of actual lengths from that sum.

Proposition 7.7 (The residual sum after the real class-group bound)**.** *For $\ell \le P$ retain h_ℓ from (6.6). For $\ell > P$ replace its value zero by*

$$h'_\ell = \max(\{0\} \cup \{j \ge 1 : \ell^{j+1} \le x^{-12} \exp(x^{\varepsilon}/2)\}).$$

Let $\mathcal{R}_{**}(x)$ be $\mathcal{R}_*(x)$ with these new thresholds. Then, uniformly on the common intervals,

$$0 \le \mathcal{R}_*(x) - \mathcal{R}_{**}(x) \ll_{\psi,\varepsilon} x^{-8} \exp(x^{\varepsilon}/2).$$

In particular $\mathcal{R}_{**}(x) \le \frac{1}{4} \exp(x^{\varepsilon}/2)$ on such intervals would suffice for (L).

Proof. At a retained $\ell > P$ decompose the finite relation quotient into cyclic modules with exponents d_i, as in Proposition 6.14. Replacing the threshold zero by h'_ℓ decreases its length contribution by precisely $\sum_i \min(d_i, h'_\ell)$. This is the length of the subgroup killed by $\ell^{h'_\ell}$. The sum over the retained characters is bounded by the corresponding sum over all complementary characters since the omitted lengths are nonnegative. Theorem 17.18 bounds this full sum with its factors $\log \ell$ by the displayed error. That theorem uses ordinary ideal class groups of $(E_x\mathbb{Q}(\mu_{\ell^{h'_\ell}}))^+$ and distinct prime-to-ℓ eigenspaces for the different twists. The terms at $\ell \le P$ agree. Now use (6.7). Its first term uses half of $\exp(x^{\varepsilon}/2)$ and its remaining terms, together with the new error, are $o(\exp(x^{\varepsilon}/2))$. This proves the sufficiency assertion. □

Next consider primes which conjugate both quadratic fields. Under the degree list of Proposition 8.5 they can give a factor of a period norm which is independent of the translate u. We determine an explicit such factor without knowing the centered prime estimate.

Lemma 7.8 (Primes moving the high set off itself)**.** *Let $\mathcal{P}_x$ consist of the unramified primes $5x/4 < p \le 7x/4$ inert in both K_q, K_d for which $\pi_p T \cap T = \varnothing$, where $\pi_p = \sigma_p c_E$ acts on the CM embeddings. Then*

$$\sum_{p \in \mathcal{P}_x} \log p = x/8 + o_{\psi,\varepsilon}(x)$$

uniformly as the left endpoints of the intervals tend to infinity. Every π_p in this set has even order.

Proof. On integers coprime to qd, the indicator of the two negative quadratic signs is $(1-\xi_q)(1-\xi_d)/4$. Consider the ramp sum $\sum_n c(n)(y/n-1)_+$ with $c(n) = \Lambda(n)$ times this indicator. In $x \le y \le 2x$ both q, d exceed $2x$. Expanding the indicator, using (1.1) and applying Theorem 5.1 to the three nonprincipal characters gives

$$\tfrac{1}{4} y \log y + C_I y + o_{\psi,\varepsilon}(y).$$

For the principal remainder use $G_{\mathbf{1}}(y) = o(y)$ from Proposition 3.2. The error is uniform on this range. The same hypotheses on q, d used in Lemma 6.34 apply. The ramp sum $S(y)$ is convex and $yS'_+(y) - S(y) = \sum_{n \le y} c(n)$. At $y = 5x/4, 7x/4$, bound the derivative by the two secants with displacements $\pm\eta y$. If the relative remainder is at most $e_x = o(1)$, take $\eta = \sqrt{e_x} + 1/\sqrt{\log x}$. The resulting error in $yS'_+(y) - S(y)$ is $O(y(\eta + e_x/\eta)) = o(y)$. The linear term cancels and the derivative of the logarithmic term gives $y/4$ in this expression. On subtraction, and discarding higher prime powers of total weight $O(\sqrt{x}\log^2(2x))$, this gives

$x/8 + o(x)$ before the condition involving T. The other possible ramified primes divide the fixed integer $2\,\mathrm{cond}(\psi)$.

In the construction of Theorem 6.29, T is the union of $3o + 1$ cosets of DA_0 in A, where $o = \operatorname{ord}\psi$ and D is a power of two with $2x^{\varepsilon/8} \le D < 4x^{\varepsilon/8}$. Here the groups use additive notation. Write $G_l = \mathrm{Gal}(C_l/\mathbb{Q})$, a cyclic 2-group of order 2^l divisible by D. Restriction of DA_0 lies in DG_l. The image of $T - T$ in G_l/DG_l has at most $(3o + 1)^2$ elements. Since conjugation fixes the real field C_l, the image of π_p there is the image of p. The quotient of order D is the Galois group of the maximal real subfield of the cyclotomic field of conductor $4D$. Thus $\pi_p T \cap T \neq \varnothing$ puts p in at most $2(3o+1)^2$ residue classes modulo $4D$. Those classes contain $O_\psi(1+x/D)$ integers in this interval, of total logarithmic weight $o(x)$. This proves the stated asymptotic. Finally $1 < p < 2^{l+2} - 1$ for large x, so p is neither sign of one modulo the conductor of C_l. Its nontrivial image in the 2-group G_l forces even order. □

Theorem 7.9 (A common rational divisor of the simultaneous norms)**.** *For Proposition 8.5 choose at each definite step its one basis integral at the primes in $S_j \cup \mathcal{P}_x$. This is allowed by Lemma 53.14. Write s_j for the larger rank. At this step put*

$$D_j^{\mathrm{per}} = \prod_{p \in \mathcal{P}_x} p^{2r_j \lceil h/2 \rceil d_p(a,s_j)}, \qquad d_p(a,s) = \max(0, \lceil (a - s(p-1))/p \rceil).$$

The same integer can be used for every translate u. For all $p \mid D_j^{\mathrm{per}}$, every coefficient place over p gives a nonnegative valuation to $\widetilde{Y}_{u,j} = Y_{u,j}/D_j^{\mathrm{per}}$. One has

$$\frac{1}{r_j} \log D_j^{\mathrm{per}} \ge (1/14 - o_{\psi,\varepsilon}(1)) ha, \tag{7.4}$$

$$\mathcal{H}_{S_j}(\widetilde{Y}_{u,j}) = \mathcal{H}_{S_j}(Y_{u,j}) - \log D_j^{\mathrm{per}}. \tag{7.5}$$

The centered estimate (8.1) holds with every $Y_{u,j}$ replaced by $\widetilde{Y}_{u,j}$. For the unmodified norms of this common-basis construction one has

$$\sum_{j=1}^{m} \mathcal{H}_{S_j}(Y_{u,j})/r_j - hmM_{\mathrm{sp}} \ \ge (1/14 - o_{\psi,\varepsilon}(1)) hma.$$

Proof. At $p \in \mathcal{P}_x$ the permutation π_p fixes the K_q-sign of an embedding. Thus $\pi_p t$ belongs to $\Phi_0 \setminus T$ for $t \in T$ and has minor degree zero. In every cycle choose the half containing at least as many members of T as its partner. These choices cover at least $\lceil h/2 \rceil$ members in total. Use just the first power in Corollary 54.25. It gives valuation at least $r_j \lceil h/2 \rceil d_p(a, s_j)$ for each cleared coordinate. Squaring and summing doubles the exponent at every place over p. The input is supported at a fixed split prime and the representatives move lattices only at split primes, so the local hypotheses hold at all of $\mathcal{P}_x$. This proves the integrality after division, including possible zeros of individual coordinates. The scalar norms themselves are nonzero by the torsion existence theorem in Section 54.2.

The inequality $d_p(a, s_j) \ge a/p - s_j$, Lemma 7.8, and $p \le 7x/4$ give

$$\sum_{p \in \mathcal{P}_x} d_p(a, s_j) \log p \ge \frac{4a}{7x}(x/8 + o(x)) - O(s_j x) = (1/14 - o(1))a.$$

All ranks have a fixed polynomial bound and $s_j x = o(a)$ uniformly. Multiplication by $2\lceil h/2 \rceil \ge h$ gives (7.4). The primes in S_j split in K_q, hence are disjoint from $\mathcal{P}_x$. At the dividing primes no negative valuation has been introduced. At every other finite place the valuation is unchanged. Division by the positive rational integer subtracts exactly its logarithm from the normalized

logarithm of the field norm. This proves (7.5). All the divisibilities used for the two-sign count persist on S_j. The product formula applied to the quotient therefore gives (8.1) with the claimed replacements.

Finally (6.16) gives $\Delta_u = o_{\psi,\varepsilon}(hma)$ unconditionally. Indeed $Y' = a/W \asymp x\sqrt{b}$ is in the large-argument estimate of Lemma 7.1. At $a/B' \sim x$ the finitely many primitive powers of the fixed character satisfy $G_\rho(y) = o_\psi(y)$ by Proposition 3.2, and the remaining selected twists satisfy Theorem 5.1. The coefficients in (6.16) are bounded in terms of ψ. Insert this lower bound on Δ_u in the estimate for the quotient norms and use (7.5) and (7.4). This yields the final inequality. □

Here $x^{5\varepsilon}\mathcal{B} = o(hma)$ by the ranks in Theorem 6.29 and the computation in Proposition 7.6. Thus in this construction a term larger than the permitted error has to be removed before seeking an upper estimate for the periods. The explicitly divided norms still need an upper comparison at the precision of (6.3). For the cohomology sum the remaining requirement is the bound on $\mathcal{R}_{**}$ of Proposition 7.7.

7.4 Principal generators at the two auxiliary primes and in the terminal window

Write $K = K_d = \mathbb{Q}(\sqrt{-d})$, so $|D_K| = d$, and keep the integer $N = \operatorname{lcm}(F, 2^{l'+2})qd$ and the scale $U = (kb)^8$ of Proposition 6.30. If an algebraic integer $\pi \in K$ generates a degree-one prime ideal, the condition $\pi \equiv 1 \bmod M\mathcal{O}_K$ specifies its residue in the quadratic field. This is the condition on the generator in `Formalism/SharpRestriction/GlobalSchurBounds/ShortIntervalPrimes.lean`, line 83.

Lemma 7.10 (Principal ray primes in a relative interval)**.** *Let K be any imaginary quadratic field of absolute discriminant d and $M \geq 1$ an integer. Put $B = 4dM^2$. There are absolute $c, C > 0$ with the following property. For $Y \geq \max(16, B^C)$, put*

$$\rho(Y, B) = \exp(-c \log Y/\log B) + \exp(-c\sqrt{\log Y}).$$

If $0 < \vartheta \leq 1$ and $\vartheta \geq C\rho(Y, B)$, the number of rational primes in $(Y, (1+\vartheta)Y]$ of the form $p = \pi\bar{\pi}$, $\pi \in 1 + M\mathcal{O}_K$, is at least

$$\frac{\vartheta Y}{B^C \log Y}.$$

Proof. Let L be the ray class field modulo $M\mathcal{O}_K$. Conjugation preserves the ray subgroup, so $L/\mathbb{Q}$ is Galois. Its subgroup $\operatorname{Gal}(L/K)$ is abelian, and the conductor norm of each of its characters is at most M^2. Also $[L : \mathbb{Q}] \ll dM^2$. Indeed the ray class exact sequence bounds $[L : K]$ by $h_K|(\mathcal{O}_K/M\mathcal{O}_K)^\times|$, and the second factor is at most M^2. Minkowski's bound represents each class by an integral ideal of norm $O(\sqrt{d})$. The sum of the inverse squares of all ideal norms is $\zeta_K(2) \leq \zeta(2)^2$, so there are $O(d)$ such ideals.

Apply the unconditional Theorem 1.4 of Thorner and Zaman to $L/\mathbb{Q}$, the identity conjugacy class, and this abelian subgroup. [8] At $z = Y, (1+\vartheta)Y$ its formula for the number of completely split unramified rational primes is

$$[L : \mathbb{Q}]^{-1} A(z)(1 + O(\rho(Y, B))).$$

Here $A(z) = \operatorname{Li}(z)$ in the absence of an exceptional zero. In the presence of the possible simple real zero β it is $A(z) = \operatorname{Li}(z) - \operatorname{Li}(z^\beta)$. The character value on the identity is one. Its defining region gives $\beta > 1/2$.

We record an elementary bound $1-\beta \gg B^{-C_0}$ for an absolute C_0 in this quadratic situation. The zeta function of L factors over K into the Hecke functions of the ray characters, with the trivial factor ζ_K. A zero of the trivial factor in this real interval comes from the quadratic Dirichlet character of K, since the alternating series for the rational zeta function excludes a real zero between zero and one. For a nontrivial factor the exceptional character η is real. Conjugation of its ideals preserves their norms, so the conjugate character has the same Hecke function. Simplicity of the zero forces that character to equal η. All the factors are holomorphic at β. Thus the quadratic extension K'/K cut out by η is normal over $\mathbb{Q}$. Complex conjugation has order two on K' and acts nontrivially on K, so $\mathrm{Gal}(K'/\mathbb{Q})$ is the Klein four group. The conductor–discriminant formula gives $|D_{K'}| = d^2 \mathrm{N}\mathfrak{f}_\eta \le d^2 M^2$. Factoring $\zeta_{K'}$ over $\mathbb{Q}$ shows that β is a zero of a primitive quadratic Dirichlet character of conductor $D \le d^2 M^2$. The conductor bound follows also by the discriminant formula in the tower over its quadratic subfield. The same bound includes the trivial-factor case.

The quadratic class number formula gives $L(1,\chi) \gg D^{-1/2}$. For a real quadratic field the fundamental unit exceeds an absolute constant greater than one, by its integral quadratic equation and norm ± 1. For an imaginary quadratic field the number of roots of unity is at most six. The sum of a nontrivial character over a complete period is zero, so all its partial sums have absolute value at most D. Partial summation gives

$$L(\sigma,\chi) = \sigma \int_1^\infty \left(\sum_{n \le t} \chi(n) \right) t^{-\sigma-1}\, dt.$$

Differentiation under the integral shows $|L'(\sigma,\chi)| \le 6D$ for $1/2 \le \sigma \le 1$. The mean value theorem proves the asserted lower bound for $1-\beta$.

To subtract the two counting formulas set $s(Y) = 1$ when there is no exceptional zero and $s(Y) = 1 - Y^{\beta-1}$ otherwise. Uniformly for $Y \le t \le 2Y$, $A'(t) \gg s(Y)/\log Y$ and $A(t) \ll Y s(Y)/\log Y$. In the exceptional case differentiation gives $(1-t^{\beta-1})/\log t$. For the upper bound integrate $1/\log v$ from t^β to t and use $\log(t^\beta) \ge \frac{1}{2}\log Y$ and $1-t^{\beta-1} \le 2(1-Y^{\beta-1})$. This last inequality follows from $\log t \le 2\log Y$. Subtraction and $\vartheta \ge C\rho(Y,B)$ now give a lower bound $\gg \vartheta Y s(Y)/([L:\mathbb{Q}]\log Y)$. The bound on $1-\beta$ gives $s(Y) \gg B^{-C_0}$ with an increase of the constant if needed. Each prime counted splits in K. A prime above it is trivial in the ray class group, so it has a generator $\pi \equiv 1 \bmod M\mathcal{O}_K$. The ideal is integral and of norm p, giving $\pi \in \mathcal{O}_K$ and $\pi\bar{\pi} = p$. Increasing C absorbs the absolute constants. □

For a rational prime g and a positive integer r, put

$$\mathcal{B}_r(g) = \{g\} \ \cup\ \{\ell \text{ prime} \mid \ell^{v_\ell(g^i-1)} > g^{i/4} \text{ for some } 1 \le i \le r\}.$$

Theorem 7.11 (Compatible split levels on the common fields)**.** *At every sufficiently large x on the common intervals, for any $r \ge 1$ bounded by a fixed power of x, the two primes of Proposition 6.30 can be chosen in $(U, 2U]$ with $\pi_i \in 1 + N\mathcal{O}_K$ and $\mathcal{B}_r(g_1) \cap \mathcal{B}_r(g_2) = \varnothing$. Fix such a choice and set $M = N(g_1 g_2)^2$. For any fixed $c_0 > 0$ and every real Y satisfying $\log Y \ge c_0 m x^\varepsilon$, the interval*

$$Y < T \le Y\big(1 + 1/(fk)\big)$$

contains $Y^{1-o_{\psi,\varepsilon,c_0}(1)}$ rational primes $T = \pi\bar{\pi}$ with $\pi \in 1 + M\mathcal{O}_K$. The error in the exponent tends to zero as x increases, uniformly in Y and the two auxiliary primes. Every such T is $1 \bmod M$ and splits completely in E.

Proof. For $M = N$, $Y = U$, $\vartheta = 1$, the estimates in Proposition 6.30 give $\log(4dN^2) \ll_{\psi,\varepsilon} x^{\varepsilon/5}\log^2(2x) + \log(2x)$ and $\log U \asymp_\varepsilon x^\varepsilon$. Lemma 7.10 gives $U^{1-o(1)}$ choices. For each i at most

three prime powers exactly dividing $g_1^i - 1$ can exceed $g_1^{i/4}$, since the product of four would exceed g_1^i. Thus $|\mathcal{B}_r(g_1)| \le 3r + 1$. Fix ℓ in this set. A candidate $g \ne \ell$ for which $\ell \in \mathcal{B}_r(g)$ satisfies $g^i \equiv 1 \bmod \ell^e$ for some $i \le r$, where e is the smallest positive integer with $\ell^e > U^{1/4}$. There are at most $2i$ such residue classes for each i. This follows from cyclicity of the unit group for odd prime powers and its decomposition into at most one cyclic factor and a factor of order at most two for powers of two. Each class contains at most $1 + U^{3/4}$ integers in $(U, 2U]$. Including $g = \ell$, the number of all excluded candidates is therefore $O(r^3(1 + U^{3/4}))$. This is smaller than $U^{1-o(1)}$ since r is polynomially bounded in x. A remaining candidate supplies the second prime.

For the terminal modulus, $\log(4dM^2) \ll_{\psi,\varepsilon} x^{\varepsilon}$ uniformly in the two choices. By Theorem 6.29, $m \asymp x^{1/2-\varepsilon/16}$, and f, k are polynomially bounded. The exponent $\log Y / \log(4dM^2)$ is $\gg m$ and $\sqrt{\log Y} \gg \sqrt{mx^{\varepsilon}}$. Each exceeds every fixed multiple of $\log x$ eventually. Thus $\rho(Y, 4dM^2) = o(1/(fk))$ throughout the specified range. Apply Lemma 7.10 with $\vartheta = 1/(fk)$. The logarithms of fk, of the conductor factor $(4dM^2)^C$, and of $\log Y$ are all $o(\log Y)$ uniformly in Y. This proves the count and allows exclusion of any list of size bounded by a fixed power of x. Taking norms of $\pi \equiv 1 \bmod M\mathcal{O}_K$ gives $T \equiv 1 \bmod M$, since $M\mathcal{O}_K \cap \mathbb{Z} = M\mathbb{Z}$. Finally $E \subset \mathbb{Q}(\zeta_N)$. □

Proposition 7.12 (The effect of a prime-power division on the height)**.** *Let $z \ne 0$ lie in a number field J, let S be a set of rational primes, and let $T \notin S$ be a rational prime. Normalize $v_w(T) = 1$ for $w \mid T$ and set $q_w = [J_w : \mathbb{Q}_T]/[J : \mathbb{Q}]$. For each integer $a_0 \ge 0$,*

$$\mathcal{H}_S(z) - \mathcal{H}_S(z/T^{a_0}) = \log T \sum_{w \mid T} q_w \min\{a_0, \max(0, v_w(z))\}.$$

For $a_0 > 0$ the decrease equals $a_0 \log T$ exactly when $v_w(z) \ge a_0$ at every $w \mid T$.

Proof. By the product formula the height of Proposition 8.5 is the sum of the normalized signed valuations above S and the positive parts of valuations elsewhere, in each case with local-degree and $\log p$ weights. Only the places above T change. At such a place, writing $t = v_w(z)$, the difference of the positive parts is $\max(0, t) - \max(0, t - a_0) = \min(a_0, \max(0, t))$. All the q_w are positive and their sum is one. This also gives the equality condition. □

The source places the terminal level near $T_* = (D_*/(d_{u,*}\Gamma_P^2))^{1/(fk)}$ where $D_*, d_{u,*}$ are its dimension parameters and $\Gamma_P > 1$ its loss factor. See `Formalism/LocalRingCardinality/DataKernels.lean`, line 157 and `Formalism/SharpRestriction/GlobalSchurBounds/ShortIntervalPrimes.lean`, line 329. Theorem 7.11 supplies the primes for every target in its stated range, independently of how that target is obtained. Corollary 72.10 below verifies this lower bound for the specified unitary row groups and discrete-series parameters. The negative term involving $\log T$ in `Formalism/LocalRingCardinality/DataDIVDefs.lean`, line 83 also requires a period estimate. Proposition 7.12 specifies the valuation gain that would be needed to realize it by a rational division of the auxiliary scalar.

8 Arithmetic averages on the period interval

8.1 Exact descent of strict classes on the period interval

Theorem 8.1 (Exact preservation at every primary exponent)**.** *Fix the entire interval and the nonzero large-prime summands of Theorem 14.17. Choose one of its primes s. For each coefficient prime ℓ let L_ℓ be the degree-ℓ subfield of $\mathbb{Q}(\zeta_s)$, and let L be their compositum. For a datum (E_x, ℓ, r, α), put*

$$F_x = E_x\mathbb{Q}(i)L, \qquad \Delta_x = \mathrm{Gal}(F_x/\mathbb{Q}), \qquad A = (J/O)(\kappa_\ell^{-r}\alpha^{-1}),$$

where $J = \mathrm{Frac}(O)$. Use the strict group S_D of Theorem 14.16 over a field D. For $M = A[\ell^n]$, $n \geq 1$, and for $M = A$, there are equalities and restriction isomorphisms

$$S_{\mathbb{Q}}^{\{s\}}(M) = S_{\mathbb{Q}}(M) \xrightarrow{\sim} S_{F_x}(M)^{\Delta_x}.$$

Here the superscript $\{s\}$ allows arbitrary ramification at s and keeps all other strict conditions. On each F_x, for every $m \geq 0$, there is an ordinary crystalline cuspidal adjoint with cyclotomic exponents $u, 0, -u$ for $u = r + m(\ell - 1)$, with the original residual characters, and unramified away from ℓ. All the original splitting conditions of Corollary 14.15 hold in this one L. In particular the invariant groups recover the original strict character summands, including every primary exponent, throughout the given support.

Proof. First take finite coefficients. Via the trace pairing of the unramified ring O over $\mathbb{Z}_\ell$, the Cartier dual of $M = A[\ell^n]$ is $D_n = T/\ell^n T$, with T as in Lemma 29.2. Let d_n be the reduction of its integral class. Both modules are unramified at s. The unramified local cohomology of D_n is

$$H^1_{\mathrm{ur}}(\mathbb{Q}_s, D_n) = D_n/(\mathrm{Fr}_s - 1)D_n.$$

It is a cyclic O-module. Frobenius acts trivially on the residual character at s, and the reduction of d_n evaluates there to a nonzero element of the residue field by Theorem 14.17. Hence the single class $d_n|_s$ generates this module over O for every n.

At s the unramified subgroups for M and D_n annihilate one another under the perfect local Tate pairing. Indeed their cup products inflate from the residue Galois group, of ℓ-cohomological dimension one. For either unramified finite module, its unramified subgroup has the order of its Frobenius invariants, by the displayed formula. Local duality and the local Euler characteristic at $s \neq \ell$ give $|H^1(\mathbb{Q}_s, M)| = |H^0(\mathbb{Q}_s, M)|\,|H^0(\mathbb{Q}_s, D_n)|$. These orders give exact annihilators.

For $c \in S_{\mathbb{Q}}^{\{s\}}(M)$ take the global cup product of c with any O-multiple of d_n. At ℓ the restriction of c is zero. At each other finite place apart from s, both classes are unramified, so the local product vanishes by inflation from the residue group, also when the coefficient action is ramified. At infinity the positive-degree cohomology vanishes since ℓ is odd. Global reciprocity says the local invariants sum to zero. Thus $c|_s$ is orthogonal to all of $H^1_{\mathrm{ur}}(\mathbb{Q}_s, D_n)$. It is unramified at s by the preceding paragraph. This proves the first equality for every n. Milne, *Arithmetic Duality Theorems*, Part I, Sections 2 and 4.

For A, a continuous cocycle has finite image and takes values in $A[\ell^n]$ for some n. Strictness of its class lifts to these finite coefficients at each required place. Indeed $A^{G_{\mathbb{Q}_\ell}} = 0$ by the nontrivial character on residual cyclotomic inertia. At each $p \neq \ell$, the group A^{I_p} is divisible since the averaging projector for the finite prime-to-ℓ inertia action splits it off from A. The multiplication-by-ℓ^n sequence thus gives injective maps on the local H^1 at ℓ and on $H^1(I_p, -)$ elsewhere. The finite-coefficient equality now applies.

For each ℓ choose a character of order ℓ cutting out L_ℓ. The one-prime form in Proposition 29.1 has this auxiliary character. In the proof of Theorem 27.16, the same primitive conductor argument gives inertia characters $\chi\xi, 1, (\chi\xi)^{-1}$ and zero monodromy away from ℓ. The exponents of the ordinary crystalline flag are $u, 0, -u$. The residual cyclotomic character to exponent u equals the one to exponent r. Its inertia order therefore still excludes a quadratic self-twist. The adjoint stays cuspidal by solvable base change. Restriction to F_x kills η, α, ξ, giving the claimed unramifiedness. Each L_ℓ is totally real because ℓ is odd. The power conditions of Theorem 14.17 give complete splitting in L of every prime on the specified list, including the coefficient primes.

Finally $E_x\mathbb{Q}(i)$ is unramified at ℓ and has degree prime to ℓ. The inertia above ℓ still contains an element acting on M through a scalar whose difference from one is a unit. Hence $M^{G_{F_x}} = 0$, and inflation–restriction identifies the global H^1 over $\mathbb{Q}$ with the invariants over F_x. At ℓ the local degree is prime to ℓ. Away from ℓ, s the inertia index is prime to ℓ because L is unramified there. Restriction at these places is injective by corestriction. Thus the antecedent of an invariant strict class belongs to $S^{\{s\}}_{\mathbb{Q}}(M) = S_{\mathbb{Q}}(M)$. Restriction of a strict class is strict, proving surjectivity and the isomorphism. Prime-to-degree character descent in Theorem 17.1 identifies the group for A with the corresponding original summand over E_x. Unramified enlargement of the coefficient DVR follows by finite free scalar extension, which commutes with cochains, restriction kernels and invariants. □

In particular the contribution of a supported summand to (13.12) can be computed on F_x after taking invariants and multiplying by the prescribed power of ℓ, with exactly the same logarithmic order. An upper bound for the sum of those orders, the global theta-period upper bound, and the lower bound for the designated integral pairing on E_x are still required. The auxiliary prime depends on the support of the strict groups. The construction gives no size bound for it in terms of the interval parameter.

8.2 Three primes throughout the dyadic tower

Theorem 8.2 (Vanishing at $37, 59, 67$ in every twist)**.** *Let $p \in \{37, 59, 67\}$, let D be any subfield of $\mathbb{Q}(\zeta_{2^j})$ for any $j \geq 1$, and let $r \geq 1$. The $(-1)^r$-eigenspace of complex conjugation in $\mathcal{S}_{p,r}(D)$ is zero. In particular $K_{4j}(\mathbb{Z})\{p\} = 0$ for each $j \geq 1$ and each of the three primes.*

On each original period field E_x, the complementary strict classes invariant over any such $D \subset E_x$ vanish whenever $p \nmid [E_x : D]$. For a character ψ of two-power conductor this applies to the entire invariant summand over $K_\psi C_J$, for every twist on every common interval.

Proof. First take the full cyclotomic field and split the integral group ring over an unramified extension of $\mathbb{Z}_p$. Character descent in Theorem 17.1 reduces the assertion to $S(A) = 0$ for each primitive character α of two-power conductor F with $\alpha(-1) = (-1)^r$. The strict group uses $A = (J/O)(\alpha^{-1}\kappa_p^{-r})$. Averaging is integral since the field degree is a power of two.

We first remove all high conductors independently of r. The odd character $\omega^{r+1}\alpha$, with ω the Teichmüller character, has odd conductor part $f_o \mid p$. Its odd-conductor factor has order dividing $p - 1$. In Proposition 18.16 take $Q = 37$ for $p = 37$ and $Q = p^2$ for $p = 59, 67$. Then $e = v_2(Q - 1)$ is $2, 3, 3$, respectively. The maximal p-subfield is $\mathbb{Q}$. All conductors in the following table are powers of two. Above the displayed F_0 one has $F \geq p4^e$ and $F/2^e > p + 1$. Hence $F \geq f_o 4^e$, and the only possible extra odd prime p is neither sign of one modulo $F/2^e$. All the hypotheses of that proposition hold, so the strict group for each such character vanishes.

For $F \leq F_0$ use Proposition 9.5 at precision one. The following are the auxiliary primes s and generators g used. A single pair suffices in the last row.

p	F_0	s_1	g_1	s_2	g_2
37	512	37889	3	151553	3
59	2048	120833	3	724993	5
67	4096	1097729	3		

Each $s \equiv 1 \bmod pF_0$. Primality can be checked by trial division through $\lfloor\sqrt{s}\rfloor$. The primitive-root check is $g^{(s-1)/q} \not\equiv 1 \bmod s$ for the prime divisors q of $s - 1$. In order down the displayed pairs these sets are $\{2, 37\}, \{2, 37\}, \{2, 59\}, \{2, 3, 59\}, \{2, 67\}$.

Here are details of the remaining finite calculation. Write u for the representative of r mod $(p - 1)$ in $[0, p - 2]$. At $F = 1$ retain even u, and at $F = 4$ retain odd u. At $F = 2^j \geq 8$

retain all u. For such an F put $d = F/4$ and write each odd residue as $(-1)^b 5^v \bmod F$ with $0 \le v < d$. Indeed $v_2(5^{2^a} - 1) = a + 2$, so 5 generates the subgroup congruent to one modulo four. A character with the given parity is primitive precisely when its value Y at five has order d, since $5^{d/2} = 1 + F/2 \bmod F$. Thus all admissible Y's are the roots of $H_F(X) = X^{d/2} + 1$ over $\overline{\mathbb{F}}_p$. For $F = 1, 4$ use a dummy variable with $H_F(X) = X$, $v = 0$, and $b = 0$ at $F = 1$ or $(-1)^b = a \bmod 4$ at $F = 4$.

For a listed pair (s, g) set $w = g^{(s-1)/(Fp)}$ in $\mathbb{F}_s$. The polynomial

$$P_{s,F,u}(X) = \sum_{\substack{1 \le a < Fp \\ (a,Fp)=1}} (-1)^{bu} a^u \operatorname{ind}_g(1 - w^a)\, X^v \quad \text{in } \mathbb{F}_p[X]$$

uses (b, v) for $a \bmod F$. Its value at the residual character is exactly $D_{s,1}(\alpha, r)$. The table below lists every nonconstant monic gcd $(H_F, P_{s_1,F,u})$. Each has the single root Y displayed, and the last column is the value at that root for the second pair.

p	F	u	Y	$P_{s_2,F,u}(Y)$
37	1	20	0	4
37	8	6	36	18
37	16	1	6	13
37	16	5	6	28
37	16	25	31	2
37	16	32	6	7
59	8	6	58	57

For reproducibility the complete calculation of the gcd column is given by the following loops. All polynomial operations and accumulations are modulo p. The powering used to find the index is modulo s.

> For each $F = 1, 4, 8, \ldots, F_0$, initialize one zero polynomial P_u for each allowed u. Form $L[(w^F)^j] = j$ for $0 \le j < p$. For each $1 \le a < Fp$ prime to Fp, read b, v from the list of signed powers of five, with the two small cases specified above. Put $z = L[(1 - w^a)^{(s-1)/p}]$. For each allowed u, add $z((-1)^b a)^u X^v$ to P_u. For each resulting polynomial start with $B = H_F$, $C = P_u \bmod H_F$. Replace the pair by $C, B \bmod C$ while $C \neq 0$. Divide B by its leading coefficient. Record the resulting polynomial when its degree is positive.

In the displayed cases this gives $X - Y$. At every root of H_F at least one of the available moments is therefore nonzero. The character values have order prime to p, so their reductions preserve their orders. The last assertion of Proposition 9.5 proves $S(A) = 0$ for every positive twist of this parity. In particular this includes the rational twists $r \equiv 1048 \bmod 1332$ at 37 for which the earlier Bernoulli test had an exception.

Restriction to the full dyadic field has prime-to-p degree for every subfield D in the theorem, so it is injective on the strict group, by corestriction. The assertion for $\mathbb{Z}$ follows from Corollary 11.18, or directly from Theorem 18.1 since global and local invariants over $\mathbb{Q}$ agree. For $D \subset E_x$ the usual prime-to-degree descent gives exactly the invariant group over E_x. Indeed averaging makes restriction on global cohomology an isomorphism onto the invariants and detects each local condition by restriction. Finally $K_\psi C_J$ is dyadic when ψ has two-power conductor, and adjoining the two auxiliary square roots has relative degree dividing four. □

8.3 The full cohomology estimate from pairs of residue primes

Fix an original interval as in Theorem 12.15. There is an a priori integer $B = \lceil \exp(V^{A_0}) \rceil$, with A_0 depending only on ψ, ε, that may be used in the following construction. List *every*

odd prime $p \le \exp B$ and take $h_p = B$. Choose K as in Proposition 14.10 with N_* and these precisions. The cyclotomic choice alone contains all the original period fields. Let $s_1, \dots, s_a$ be sampled independently among the primes completely split in K. Probabilities and averages of functions of this tuple mean limits over the a-fold product of the split primes up to a common bound tending to infinity. Repeated sampling of the same rational prime has limiting density zero when $a \ge 2$.

For x on the interval let I_x consist of all triples $(p, r, [\alpha])$ in the preceding list with $r < t(x)$, where α runs over the character orbits of E_x of sign $(-1)^r$ and $r \ge 1$. Set

$$U_{x,a} = \sum_{j \in I_x} b_j \log p_j \min_{1 \le i \le a} W_j(s_i).$$

Each summand for $a = 2$ is the logarithm of the norm of the ideal $(D_j(s_1), D_j(s_2), p_j^B)$ in the local integer ring, with the norm understood as its additive index.

Theorem 8.3 (A common residue calculation for the full strict sum)**.** *For $C_x = C(E_x, t(x); N_*)$ from Theorem 9.7,*

$$U_{x,2} \ge C_x, \qquad \mathbb{E} \sup_{x \in [V, V^{3/2}]} (U_{x,2} - C_x) \ll n_* t_*.$$

The implied constant is absolute, independently of the prime cutoff and all precisions used above. With the full cohomology sum $\mathcal{L}(x)$ of (L),

$$\mathbb{E} \sup_{V \le x \le V^{3/2}} \frac{|U_{x,2} - \mathcal{L}(x)|}{\exp(x^{\varepsilon}/2)} \longrightarrow 0.$$

For $a \ge 2$, the relative density that $U_{x,a} = C_x$ simultaneously for every x is at least $1 - 4n_* t_* 3^{-a}$. In particular $O_{\psi,\varepsilon,A}(\log(2V))$ samples suffice for density at least $1 - V^{-A}$ for any fixed $A > 0$. The prime list and its precision can be fixed before any of the strict groups are known.

Proof. Theorem 13.2 bounds the sum of the logarithmic strict orders on E_* at twist r by $\exp(C[n_*(r+2)]^5 \log(2N_* n_*(r+2)))$. The lattice comparison in Theorem 9.7 was proved for any twist cutoff on this field. Apply it with t_*. The preceding bound summed through t_*, together with its polynomial comparison error, is less than B if A_0 is sufficiently large. Indeed n_*, t_*, N_* are polynomially bounded in V. Every nonzero index in the resulting sum contributes $b_j d_j \log p_j$. Therefore none requires $p > \exp B$, and $e_\alpha B \ge d_j$ at every listed prime, since $p \ge 3$. The characters on the smaller fields inflate to this list with the identical open set, primitive conductor and lattice. Hence these choices cover the entire interval.

For each triple the excess of the minimum over d_j is nonnegative. By Proposition 14.10 the chance that it is at least $v > 0$ equals $p_j^{-ab_j v}$ when $v \le e_\alpha B - d_j$, and zero otherwise. This uses independence between the sampled primes only. All events in question come from finite Galois quotients, so the individual densities give these product limits. Summing over positive v, the expected excess is at most $1/(p_j^{ab_j} - 1)$. For any one p the sum of the b_j over the largest field and all its twists is at most $n_* t_*$. Indeed the sum of $e_\alpha b_\alpha$ over all character orbits in the regular representation is n_*. Dominate the supremum by the sum of the nonnegative excesses on this entire list. For $a = 2$ the resulting bound is at most

$$n_* t_* \sum_{m \ge 3} \frac{\log m}{m^2 - 1} \ll n_* t_*.$$

This summable bound allows the prime list to include even those primes whose every index is zero.

The deterministic comparison error with $\mathcal{A}(E_x, t(x))$ is polynomial in V by Theorem 9.7. After division by $\exp(x^{\varepsilon}/2)$ this polynomial and the expected full excess tend uniformly to zero. Theorem 11.17 controls the remaining difference between $\mathcal{A}$ and $\mathcal{L}$ on the same intervals. This proves the second assertion. Finally the union bound for any individual excess to be positive is at most $n_* t_* \sum_{m\ge 3} m^{-a}$. Integral comparison bounds this sum by $3^{-a}(1+3/(a-1)) \le 4\cdot 3^{-a}$, proving the density statement. □

Thus a bound for the stated mean, with a fixed margin below $\exp(x^{\varepsilon}/2)$ uniformly on the interval, would supply the cohomology input in (L). The error in passing to the mean is already below this scale. An upper estimate for the mean itself remains open.

8.4 An exact mean over primes split in the common conductor

Use the original interval $[V, V^{3/2}]$, its largest modulus N_*, degree n_* and twist cutoff t_*. For primes $s_1, s_2 \equiv 1 \bmod N_*$ put $g_k = \gcd(s_1^k - 1, s_2^k - 1)$. For $M \ge 1$ define

$$A_{x,M} = \sum_{1\le r<t(x)} \sum_{\substack{p \mid g_{r+1},\, p\le M \\ p \text{ odd}}} \sum_{[\alpha]} b_\alpha \log p \sum_{v=1}^{e_\alpha \min(M, v_p(g_{r+1}))} H_v.$$

Take one p-adic orbit of characters of E_x of sign $(-1)^r$ in each inner character sum. Use its primitive conductor and the weights in Proposition 9.8. Write $A^{+}_{x,M}$ for the sum with P_v in place of H_v. For a fixed cutoff the mean is the limit over all ordered pairs of the allowed primes up to Y, divided by the square of their number, as Y tends to infinity.

Theorem 8.4 (A two-prime formula with support on ordinary integer divisors)**.** *For each fixed M the exact mean is*

$$\mathbb{E}A_{x,M} = \sum_{1\le r<t(x)} \sum_{\substack{p\le M \\ p \text{ odd}}} \sum_{[\alpha]} b_\alpha \log p \, \min(d_{p,r,\alpha}(N_*), e_\alpha M).$$

The mean of $A^{+}_{x,M}$ exceeds or equals this expression and differs from it by at most Cn_*t_* for an absolute C. As M increases the exact mean eventually equals $C_x = C(E_x, t(x); N_*)$ simultaneously on the full interval. With M past that threshold, the supremum over the interval of $|\mathbb{E}A^{+}_{x,M} - \mathcal{L}(x)|/\exp(x^{\varepsilon}/2)$ tends to zero as V tends to infinity. Each term for a given twist uses only the two fields of degrees $(r+1)N_*$ over the sampled residue fields or their subfields.

Proof. For fixed M one can sum over all odd $p \le M$ and all $v \le e_\alpha M$, putting ineligible terms equal to zero. Indeed the eligibility requirement at both primes is $p^{\lceil v/e_\alpha \rceil} \mid g_{r+1}$. This is exactly the divisor and exponent cutoff in the definition. All summands are functions on finite Galois data. For this limit one may omit the finitely many sampled primes dividing the moduli of these data, with density zero. Independence between the two prime coordinates holds in the limiting product even when the events for different characters on one prime are dependent. Linearity of the finite sum and Proposition 9.8 now give the exact formula.

For the positive sum the excess at $(p, r, [\alpha])$, before its weight, is at most $\sum_{j\ge 1} p^{-2b_\alpha j}$. For any one p, r the sum of b_α is at most n_*. Thus the whole error is bounded by $n_* t_* \sum_{j\ge 3} (\log j)/(j^2 - 1)$, as claimed. Theorem 8.3 gives an a priori B for the entire interval such that $p \le \exp B$ contains every nonzero index and $d_{p,r,\alpha}(N_*) \le e_\alpha B$. For example $M = \lceil \exp B \rceil$ covers both bounds. This proves stabilization of the means. Their deterministic comparison to $\mathcal{L}$ follows from Theorems 9.7 and 11.17, since $n_* t_*$ and the character-lattice comparison error grow at most polynomially in V. Proposition 14.11 proves the finite-field degree assertion. In particular N_* in the sampled prime progression is independent of the precision and prime cutoff. □

The limits here take Y to infinity for fixed M. The weights m_h^2 can be large when their eligibility conditions are rare. The uniform upper estimate needed in (L) would require control of the magnitude of this mean on the common intervals.

8.5 The centered estimate on the same fields

We now record the part of the desired inequality that follows for the simultaneous high orders of Theorem 6.29. In the K_d-orientation write Φ_1 for the half of the embeddings which conjugate K_q. Give the translated T_2 in this half the weight B', and all other positions weight W. Put $\omega = h_2/(2n)$, and again take $a = \lceil xb \rceil$. Write $D_{\xi_{qd}}$ for the logarithmic derivative at one of the primitive product of the two quadratic characters.

Proposition 8.5 (The surviving arithmetic norm and its required upper bound)**.** *For the simultaneous minor list of Section 54.9, let $Y_{u,j}$ denote its nonzero fixed-representative norm at step j, r_j the smaller rank, and S_j the selected primes inert in K_d and split in K_q. The choices of torsion input at distinct steps are independent. Define the real number*

$$M_{\mathrm{sp}} = \frac{a}{2}\{\log(a/W) - 1 - \gamma + D_{\xi_d} - D_{\xi_q} + D_{\xi_{qd}}\} - a\omega \log(B'/W).$$

Uniformly in u on the common intervals,

$$\Delta_u \;\le\; \sum_{j=1}^{m} \frac{\mathcal{H}_{S_j}(Y_{u,j})}{r_j} - hmM_{\mathrm{sp}} + O_{\psi,\varepsilon}(hmb/x). \tag{8.1}$$

Here $\mathcal{H}_S$ is the normalized logarithm of the field norm with the negative valuations outside S added back, as in Section 54.9. On these very fields the large-prime strict groups have the integral cotangent description of Theorem 7.5.

Proof. The mean of the logarithms of the thresholds on Φ is $\log W + \omega \log(B'/W)$. Insert this value into Proposition 6.1, with ξ_d as its quadratic sign. From the result subtract the explicit M_{both} of Proposition 54.27. The difference is exactly M_{sp} displayed above. Theorem 54.26, bounds the sum over all t of the other sign contribution by the one norm at step j, up to $O_{\psi,\varepsilon}(hb/x)$. Proposition 54.27 also estimates the error for the omitted sign by $O_{\psi,\varepsilon}(b/x)$ per t. Subtracting the two main terms and summing over the m steps proves (8.1). The large-prime assertion uses the same x and E_x in Theorem 7.5. □

For instance an upper bound by $hmM_{\mathrm{sp}} + \frac{1}{2}x^{5\varepsilon}\mathcal{B}$ for the sum of norms on the right of (8.1) would suffice for (6.3) on a sufficiently distant interval. Indeed $hmb/x = o(x^{5\varepsilon}\mathcal{B})$ by the positive term ha_E for the chosen ranks and $a_E = xb$. The low-order factorial cost in Proposition 54.27, is already $o(b\sqrt{x})$ per high position and step. The upper estimate on the full arithmetic norms at this precision, or on the normalized iterated Bessel periods giving (6.3), and the uniform estimate for (6.6) remain open here.

9 The finite data on the common fields

9.1 Prime representatives with prescribed archimedean sizes

We need a generator of a ray-principal ideal whose absolute values are almost equal at infinity. The degree and the modulus both vary. We give the estimates with absolute constants. We use the analytic continuation, functional equation and order-one factorization of primitive Hecke L-functions, including the pole of the Dedekind zeta function. These are the usual Hecke L-function theorems. For the relation with counting prime elements in archimedean regions see W. Kai, Sections 2–3.

Let K be totally real or CM, of degree n. Use one embedding from each complex pair, or all real embeddings in the totally real case. Denote their number by f, so $n = ef$ with $e = 2$ or 1. In $\mathbb{R}^f$ give the hyperplane H of coordinate sum zero its Euclidean metric. Write $\lambda(a)$ for the orthogonal projection of $(\log|\sigma a|)_\sigma$ onto H. Let $\mathfrak{m}$ be a nonzero integral ideal and use the narrow ray group in the totally real case. Put $d_K = |\operatorname{disc} K|$.

Lemma 9.1 (Cancellation for a nonzero archimedean frequency)**.** *Let B be sufficiently large, $n + \log(3d_K N\mathfrak{m}) \le B$, $X = \exp(B^{20})$, and $e^{-B} \le \delta \le 1/10$. Fix a smooth nonnegative function g supported in $(0,1)$, equal to one on $[1/3, 2/3]$, and set $w(y) = g(\log y/\delta)$. Consider a unitary ideal character χ of conductor dividing $\mathfrak{m}$ whose value on ray-principal ideals is $\exp(2\pi\mathrm{i}\langle v, \lambda(a)\rangle)$, where $v \in H$, $0 < \|v\| \le e^{2B}$. Extend it by zero at ideals not prime to $\mathfrak{m}$. Then, for an absolute $c > 0$,*

$$\left|\sum_{\mathfrak{a}} \Lambda_K(\mathfrak{a})\chi(\mathfrak{a})w(N\mathfrak{a}/X)\right| \ll_g X\exp(-cB^8).$$

Here $\Lambda_K(\mathfrak{p}^j) = \log N\mathfrak{p}$ for $j \ge 1$, and it is zero on all other ideals.

Proof. We detail the uniformity. Write t for the imaginary part of s and put

$$L(t) = \log(3d_K N\mathfrak{m}) + n\log(10 + |t| + 20\|v\|).$$

For the primitive character inducing χ the gamma factors are real or complex gamma factors with fixed parity shifts and imaginary shifts of size at most $2\pi\|v\|$. This follows by writing its infinite character in the individual absolute-value coordinates. The parameters of its square satisfy the same bound with 4π. Neither character is a power of the absolute norm. Indeed their archimedean exponents have sum zero and are nonzero. Equality on ray elements detects these exponents by weak approximation at the infinite places with the finite ray conditions imposed.

For $1 < \sigma \le 2$, logarithmic differentiation of the completed product gives, for the primitive L-function,

$$-\operatorname{Re}\frac{L'}{L}(\sigma + \mathrm{i}t, \chi) \le CL(t) - \sum_{\rho} \frac{\sigma - \operatorname{Re}\rho}{|\sigma + \mathrm{i}t - \rho|^2}.$$

The sum runs over the zeros of the completed function. In its Hadamard product the real constant cancels by the symmetry $s \mapsto 1 - \bar{s}$, as is seen on the line of symmetry away from zeros. Stirling's formula bounds the gamma logarithmic derivatives by $CL(t)$ in this strip. For $\zeta_K(\sigma)$ the pole adds $1/(\sigma - 1)$ to the upper bound. Omitting Euler factors at the modulus costs $O(\log(2N\mathfrak{m}))$, by summing $\log N\mathfrak{p}/(N\mathfrak{p}^\sigma - 1)$.

The inequality $3 + 4\cos u + \cos(2u) \ge 0$, applied term by term to the Euler logarithmic derivatives of the trivial character, χ and χ^2 at respective heights $0, t, 2t$, now gives, at a zero $\beta + \mathrm{i}t$ of the primitive function,

$$\frac{4}{\sigma - \beta} \le \frac{3}{\sigma - 1} + C_1 L(t).$$

Take $\sigma = 1 + a/L(t)$, with $a > 0$ a sufficiently small absolute constant. Rearranging gives $\beta \le 1 - c_1/L(t)$. Also at most $C_2 L(t)$ zeros have heights in $[t - 1, t + 1]$. To see this use the completed logarithmic derivative at $2 + \mathrm{i}t$, where the Euler logarithmic derivative is $O(n)$ by absolute convergence, and each of these zeros contributes at least $1/5$.

Set $T_0 = \exp(B^8)$ and $L_* = \max_{|t| \le T_0 + 2} L(t) \ll B^9$. Throughout $|t| \le T_0$, $1 - c_1/(2L_*) \le \sigma \le 2$ we have $|L'/L(\sigma + \mathrm{i}t, \chi)| \ll L_*^2$. Indeed subtract the logarithmic derivative at $2 + \mathrm{i}t$

from the product at the point in question. Each zero within one unit of height contributes $O(L_*)$ by the zero-free region. The distant terms are bounded in total by a constant times $\sum_{j\ge1}(L(t)+n\log(2+j))/j^2$, using the count in unit intervals on both sides. The gamma functions have positive real arguments before the imaginary shifts and contribute $O(L_*)$. The same omitted local factors are bounded here by $O(\log(2N\mathfrak{m}))$, since the real part is at least $1/2$.

Mellin inversion uses the transform $\widehat{w}(s)=\delta\int_0^1 g(u)e^{s\delta u}\,du$. Four integrations by parts give $|\widehat{w}(\sigma+\mathrm{i}t)|\ll_g \delta(1+\delta|t|)^{-4}$ for $1/2\le\sigma\le2$. Start at $1+1/\log X$, where absolute convergence bounds the logarithmic derivative by $O(n\log X)$ using the rational zeta function. Truncate at $\pm T_0$ and move to $1-c_1/(2L_*)$. The integral on this side is $O_g(XL_*^2e^{-c_1\log X/(2L_*)})$, since the transform has bounded integral on each vertical line. The two horizontal sides and truncated tails together are $O_g(X(n\log X+L_*^2)\delta^{-3}T_0^{-3})$. These estimates give the assertion. □

Theorem 9.2 (A ray prime with prescribed logarithms of its generator)**.** *Let A be any fractional ideal prime to $\mathfrak{m}$, let $y\in H$, and let $0<\delta\le1/10$. Suppose the log vectors of the ray units contain $f-1$ linearly independent vectors of norm at most e^B. Take B above an absolute constant with*

$$n+\log(3d_KN\mathfrak{m})+\log(1/\delta)\le B,\qquad X=\exp(B^{20}).$$

There is a prime ideal P of absolute residue degree one and an $s\in K^\times$, positive at all real places, such that

$$X<NP<e^\delta X,\qquad P=(s)A,\qquad s\equiv1 \bmod \mathfrak{m},\qquad \|\lambda(s)-y\|<\delta.$$

In particular each $\log|\sigma s|$ differs from $n^{-1}\log(X/NA)+y_\sigma$ by at most 2δ.

Proof. Let h be the ray class number and Λ the lattice of logs of the ray units in H. One has $\log h\ll B^2$. Minkowski's bound represents each ordinary ideal class by an integral ideal of norm at most $2^n\sqrt{d_K}$. The number of ideals of norm at most M is at most $M^2\zeta(2)^n$ by the Euler product. Passing to narrow ray classes adds at most the factor $2^nN\mathfrak{m}$.

We first count primes in the finite ray class of A. Uniformly for $e^{\delta/3}X\le t\le e^{2\delta/3}X$, the abelian case of Thorner and Zaman, Theorem 1.4, counts primes through t by $h^{-1}(\mathrm{Li}(t)-\theta\,\mathrm{Li}(t^\beta))$ with relative error $O(e^{-cB^9})$. Here the exceptional term can be absent, and otherwise $\theta=1$ or -1. Their base field is K, and their largest finite conductor norm is at most $N\mathfrak{m}$. Thus the logarithm entering their range is $O(B\log(2B))$. The two exponential errors of their theorem give the asserted weaker rate.

If an exceptional zero occurs, $1-\beta\ge e^{-CB^2}$. Indeed its real character cuts out K itself or a quadratic extension K'/K. Its zero is a zero of $\zeta_{K'}$, with $[K':\mathbb{Q}]\le2n$ and $d_{K'}\le d_K^2N\mathfrak{m}$ by the conductor–discriminant formula. Apply Stark's effective bound as stated in Thorner and Zaman, (1.3)–(1.4), to this field. A zero outside their exceptional interval already satisfies the displayed weaker bound. For t in the indicated interval the derivative of $\mathrm{Li}(t)-\theta\,\mathrm{Li}(t^\beta)$ is $(1-\theta t^{\beta-1})/\log t$. Its numerator is at least $e^{-C'B^2}$, also for the unfavorable sign, by $1-e^{-z}\ge\min(z,1)/2$ for $z\ge0$. The difference of the main terms at the endpoints therefore dominates the two errors, since $\delta\ge e^{-B}$. The sum of $(\log NP)w(NP/X)$ over primes in the specified class is at least $Xe^{-C''B^2}$. These counts may exclude ramified primes by definition. Omitting any additional primes over $\mathfrak{m}$ costs $O(\log(2N\mathfrak{m}))$.

Put $r=f-1$. If $r=0$ there is no archimedean condition apart from the norm. In general use the quotient H/Λ. Its covolume is at most e^{Br}, by the given independent vectors. Every nonzero lattice vector has length at least an absolute constant. In the totally real case a positive unit with all logs of modulus less than $(\log2)/3$ has every conjugate within one of 1, so the

integral norm of its difference from 1 forces equality. In the CM case apply the same norm argument to the totally positive unit $a\bar{a}$, whose logs are twice those of a. A unit has absolute norm one, so its ordinary and centered log vectors agree.

For an ideal $\mathfrak{a}$ in the class of A let its point in H/Λ be $\lambda(s)$, where $\mathfrak{a} = (s)A$ and s satisfies the ray conditions. With $\epsilon_0 = \delta B^{-5}$, use the periodic Gaussian

$$\Phi(z) = \sum_{u \in \Lambda} \exp(-\pi \|z - y + u\|^2 / \epsilon_0^2).$$

It is bounded by 2 for large B. For example balls of radius half the minimum separation are disjoint, so a ball of radius R contains at most $(1 + CR)^r$ lattice points. At most one is closer than half the separation. Summing the Gaussian over the successive remaining shells gives $o(1)$, also when its width is $\sqrt{2}\,\epsilon_0$. Consequently $\Phi(z) \le 2e^{-\pi B^{10}/2}$ if the distance on the quotient from z to y is at least δ.

By Poisson summation the coefficient at $v \in \Lambda^*$ has modulus $c_0 e^{-\pi \epsilon_0^2 \|v\|^2}$, where $c_0 = \epsilon_0^r / \operatorname{covol}(\Lambda) \ge e^{-CB^2}$. In dimension zero use $c_0 = \Phi = 1$. The sum of the absolute coefficients is at most 2, by evaluating the unshifted Gaussian at zero. The part with $\|v\| > B^4/\epsilon_0$ has sum at most $2^{1+r/2} e^{-\pi B^8/2}$, by halving the Fourier exponent and applying Poisson summation at width $\epsilon_0/\sqrt{2}$.

On the subgroup of ray-principal ideals, each v gives the unitary character $(a) \mapsto e^{2\pi \mathrm{i} \langle v, \lambda(a) \rangle}$. Extend it to the ideals prime to the modulus. This is possible by adjoining generators of the finite quotient and taking roots in the circle. The resulting character is a Hecke character with the indicated archimedean parameters and finite conductor dividing $\mathfrak{m}$, by the ray condition. Multiplying by the h characters of the finite ray group and averaging restricts its sum to the class of A. The factor from dividing by A has modulus one. For every retained nonzero v the preceding lemma thus bounds the weighted ideal sum by $O_g(Xe^{-cB^8})$, since $B^4/\epsilon_0 \le e^{2B}$. For the omitted frequencies use the bound $O(nX \log X)$ for the total von Mangoldt weight up to $2X$. It follows for example by summing over powers of rational primes and using that the sum of residue degrees above each rational prime is at most n.

Multiply the finite-class lower bound by c_0. This is a lower bound for the zero-frequency contribution to the nonnegative weighted sum including prime powers. It dominates all nonzero Fourier terms. Terms at prime powers $\mathfrak{p}^j$ with $j[\mathcal{O}_K/\mathfrak{p} : \mathbb{F}_p] \ge 2$ contribute in total $O(n\sqrt{2X} \log^2(2X))$, since the underlying p is at most $\sqrt{2X}$. The bound outside the ball of radius δ removes at most $O(nX \log X\, e^{-\pi B^{10}/2})$. A prime of degree one inside the ball remains. Multiplying a ray generator by a ray unit realizes a representative within δ of y in H. Finally the mean of its logarithms is $n^{-1} \log(NP/NA)$, proving the coordinate assertion. □

9.2 The level and class count of the integral column

Retain the notation of Theorem 72.31. Define J_W on the original definite column by its prescribed covector level $K_{\pi,v}(\ell)$ above T, its principal subgroup of depth $a_v + 2j_v$ at $v \in S_{\mathrm{m}}$, and its full lattice stabilizer elsewhere. Put $q_v = Nv$, and let N_x be the common cyclotomic conductor of Theorem 12.15. Write $\mathfrak{m}_l$ for the mass of the standard positive Hermitian genus in rank l. For $b \ge 1$ put $G_l(q, b) = q^{bl^2} \prod_{i=1}^{l} (1 - q^{-i})$.

Corollary 9.3 (A common column level with a computable class count)**.** *Let h_W count the definite finite classes at J_W. The large class count fixing j and the small class count in the*

integral pairing are

$$h_H = \mathfrak{m}_N\, G_N(T,1)^f \prod_{v\in S_{\mathrm{m}}} G_N(q_v,a_v),$$
$$h_W = \mathfrak{m}_k\, (T^k-1)^f \prod_{v\in S_{\mathrm{m}}} G_k(q_v,a_v+2j_v).$$

At this J_W, for each prescribed word and character there is an integral column Q_λ with the bound

$$|\langle Q_\lambda, F_0 A_\lambda\rangle_{[H_0]}| \;\ge\; \frac{|Z|}{h_W}\sqrt{\frac{b_V b_L}{d_\mu I J'}}\,\frac{\sqrt{|Z|}(1-c_T)^{f/2}}{2\sqrt{2}}\sqrt{\frac{m_\nu}{d_\rho d_\mu}}\,M_B.$$

Here I, J' are the sizes of the two integral generating lists and b_V, b_L their lower frame bounds as in Corollary 74.11. All words on the fixed grid use the same finite level. For a finite list, a single integer linear combination of the rows retains at least half of each individual lower bound at this level.

Proof. Both lattices are locally in the standard genus. At a split place use one lattice basis with its dual. The orthonormal basis proof in Proposition 68.5 applies at every nonsplit place. At all real places both forms are positive definite. The classification of Hermitian spaces used in Theorem 72.27 thus identifies each rational space with the standard space of its rank. A finite adelic translate of a lattice has the same genus and mass.

Refining its full finite stabilizer K to $J_0 \subset K$ multiplies the mass by $[K : J_0]$. Indeed the refinements of one class with automorphism group Γ are its orbits on K/J_0, after choosing a representative. The sum of the reciprocal stabilizer orders on these orbits is $[K : J_0]/|\Gamma|$. For each of the two levels in question the stabilizers after refinement are trivial, by the marked odd principal subgroup. Mass is therefore the number of classes. At a split place the congruence index in rank l is $|\mathrm{GL}_l(\mathcal{O}_v/\pi_v^b)| = G_l(q_v,b)$. Reduction modulo π_v counts successive independent columns, and every invertible residue matrix has exactly $q_v^{(b-1)l^2}$ lifts. At a terminal place the small covector stabilizer has index T^k-1 by transitivity on the nonzero residue covectors. This proves both formulas.

These expressions bound the cost using field and grid data alone. For example the mass formula and Bernoulli estimate in Lemma 71.7 give $\log \mathfrak{m}_l \ll fl^2 \log(2lN_x)$ as an upper bound. The primitive factors at $j = 1$ also satisfy the stated Bernoulli upper bound directly from $B_1(y) = y - 1/2$. With $D_{\mathrm{m}} = \sum_{v\in S_{\mathrm{m}}} a_v \log q_v$ the preceding formulas give

$$\log h_H \ll fN^2 \log(2NN_x) + fN^2 \log T + N^2 D_{\mathrm{m}},$$
$$j \log Q \ll \log Q + \log(2h_H d_\rho^3),$$
$$\log h_W \ll fk^2 \log(2kN_x) + fk \log T + k^2(D_{\mathrm{m}} + 2j \log Q).$$

For the middle inequality, $\log Q/(-\log c_Q)$ is bounded for $Q \ge 3$, by its formula and its limit one at infinity. For several detecting types use the maximum d_ρ.

Theorem 72.31 gives right J_W-invariance of the projected coefficient, using centrality of Z on the column and equivariance of the Fourier integral. The $|Z|$ translates of the identity class remain distinct by the definite-lattice and torsion-free argument of Corollary 74.11. Each has mass $1/h_W$. Applying the integral matrix coefficient construction of that corollary on the identity class and summing its translates against $\bar\lambda$ gives the first factor $(|Z|/h_W)\sqrt{b_V b_L/(d_\mu I J')}$ times $\|p\|_2$. Now insert Theorem 72.31. Its row levels are also common across the prescribed words and characters. The integer coefficient argument of Corollary 74.11 therefore applies on these levels. □

This bounds the loss due to the finite class chosen in extracting the restricted coefficient. The arithmetic normalization of M_B, survival in the prescribed global cuspidal row at its original marked levels, the upper estimate for the full theta period and the uniform strict cohomology estimate still require proof on the common intervals.

9.3 The integral correction on all the original fields

Use $E = E_x$, its degree n, a conductor multiple N from Theorem 12.15, and $t = \lfloor x^{401} \rfloor$. For odd p let $A = (\mathbb{Q}_p/\mathbb{Z}_p)(-r)$, $G = \mathrm{Gal}(E/\mathbb{Q}) = P \times H$, with P the p-Sylow group. It is cyclic of order p^a. Indeed restriction embeds it in the cyclic group for K_ψ, because the relative degree over K_ψ is a power of two. Put $\eta = (-1)^r$. For a finite group with involution c, superscript η below means its η-eigenspace for complex conjugation.

Let $[\chi]$ run over p-adic Galois orbits of characters of H of sign η. Put $b_\chi = [\mathbb{Z}_p[\chi] : \mathbb{Z}_p]$. Use Theorem 12.25 with this P, χ, N, writing $H_i(\chi)$ and $I_{0,i,\chi}, I_{1,i,\chi}$ for its character lengths and residue subspaces. Define

$$A_{p,r} = (\log p) \sum_{[\chi]} b_\chi \sum_{i=0}^{a} H_i(\chi),$$

$$B_{p,r} = \log \left| \left(\bigoplus_{w \mid Np} A^{G_{E_w}} \right)^\eta \right| - \log \left| (A^{G_E})^\eta \right|.$$

Here G_{E_w} and G_E are the absolute local and global Galois groups. All these invariant groups are finite, since the local cyclotomic image is open at p, and at $w \nmid p$ Frobenius acts by $(\mathrm{Norm}\, w)^{-r} \neq 1$. At $i = 0$ the finite character is χ. At $i > 0$ it is χ times the chosen character of order p^i. Thus each $H_i(\chi)$ is given by the index and explicit Euler factors of Theorem 20.3.

Theorem 9.4 (A finite residue correction to the sum of character lengths)**.** *For every odd p and $1 \le r < t$,*

$$\log |\mathcal{S}_{p,r}(E)^\eta| = A_{p,r} - B_{p,r} - D_{p,r}, \qquad D_{p,r} = (\log p) \sum_{[\chi]} b_\chi \sum_{i=1}^{a} (p^{i-1} - \dim(I_{0,i,\chi} \cap I_{1,i,\chi})).$$

The dimension in each term is over the residue field of $\mathbb{Z}_p[\chi]$. The total correction everywhere on the given interval satisfies

$$0 \;\le\; \sum_{p>2} \sum_{r=1}^{t-1} D_{p,r} \;\le\; n(t-1) \sum_{\substack{p \mid \mathrm{ord}\, \psi \\ p \text{ odd}}} \frac{\log p}{p-1}.$$

Proof. On $U = \mathrm{Spec}\, \mathbb{Z}[1/(Np)]$ the field E defines a finite étale cover. Shapiro's lemma represents its integral degree-two cohomology at twist $r+1$ by the permutation lattice $\mathbb{Z}_p[G](r+1)$ over $\mathbb{Q}$. The H-group algebra is the product of the rings $\mathbb{Z}_p[\chi]$ over the indicated orbits before restricting the sign. Indeed after an unramified splitting extension the usual character idempotents have the invertible denominator $|H|$. Frobenius permutes their factors, and descending an orbit gives its own unramified integer ring. Complex conjugation lies in H, so taking sign η selects the stated factors. Choosing the inverse labeling of the regular action permutes the same set of characters. For each orbit the remaining group ring is precisely R_a with its character twist. Its degree-two length is M_a of Theorem 12.25. The logarithm of the order of this factor is $b_\chi M_a \log p$. The theorem's identity therefore gives $A_{p,r} - D_{p,r}$ for the logarithm of the order of the integral degree-two group in this sign.

At any $w \nmid p$ the twist A is unramified and its unramified degree-one subgroup is $A/((\operatorname{Norm} w)^{-r} - 1)A = 0$, by divisibility. Opening the primes over N and imposing zero localization there thus gives exactly the original strict group. The Poitou–Tate sequence identifies its dual with the kernel of the map from the integral degree-two group to the sum of the local degree-two groups. The cokernel of that map is $(A^{G_E})^\vee$, and the local groups are $(A^{G_{E_w}})^\vee$ by local duality. The real-place modified groups vanish at odd p. All maps commute with the field action, and the two sign projectors are integral. Their finite duals have the same orders in each sign. Consequently the logarithm to subtract is $B_{p,r}$.

The sum of b_χ over the retained orbits is at most $|H| = n/p^a$, since it counts their individual characters. The correction for each orbit is between zero and $(p^a - 1)/(p - 1)$ in normalized length. Summing this bound over the twists gives the displayed estimate, since a nontrivial P forces $p \mid \operatorname{ord} \psi$. On the common intervals $n \ll_\psi x^{2+8\varepsilon}$, so this contribution is polynomial throughout. The character indices within $A_{p,r}$ still require the full upper estimate. □

9.4 Reduction of the cyclotomic class at a split prime

Keep f, α, ℓ, r, O from Theorem 20.1. For $h \geq 1$ and $z \in O/\ell^h O$ set

$$w_h(z) = \max\{0 \leq j \leq h : z \in \ell^j (O/\ell^h O)\}.$$

Thus $w_h(0) = h$. For a prime $s \equiv 1 \bmod f\ell^h$, take a generator g of $\mathbb{F}_s^\times$ and put $w = g^{(s-1)/(f\ell^h)}$. Write $\operatorname{ind}_g y$ for the exponent of y with respect to g, reduced modulo ℓ^h. Define

$$D_{s,h}(\alpha, r) = \sum_{\substack{1 \leq a \leq f\ell^h \\ (a, f\ell) = 1}} \alpha(a)\, a^r \operatorname{ind}_g(1 - w^a) \quad \text{in } O/\ell^h O.$$

The arguments of the indices are nonzero since w is primitive of order $f\ell^h > 1$. One can compute the index using only the subgroup of order ℓ^h. It is the unique $j \bmod \ell^h$ for which $(1 - w^a)^{(s-1)/\ell^h} = (w^f)^j$.

Proposition 9.5 (A finite-field bound for the whole primary order)**.** *At any such prime the Frobenius value of $c_{\alpha,r+1}$ at precision ℓ^h, in compatible bases, is $D_{s,h}(\alpha, r)$. Put $d = v_\ell(a_{\alpha,r})$, where $a_{\alpha,r}$ is the coordinate of the cyclotomic class in a basis of $H^1(U, T)$. Then*

$$w_h(D_{s,h}(\alpha, r)) \geq \min(h, d).$$

Equality holds when the basis class has nonzero residual Frobenius value at s. In particular, any nonzero moment gives $\operatorname{length}_O S(A) \leq w_h(D_{s,h}(\alpha, r))$. *At $h = 1$, a nonzero moment proves vanishing of the full group for every positive twist in the same residue class modulo $\ell - 1$ and of the prescribed parity.*

Proof. At s the Frobenius action on $T/\ell^h T$ is trivial, as is inertia. The value on arithmetic Frobenius of an unramified class therefore gives an element of this module independent of its cocycle representative. Identify $\zeta_{f\ell^h}$ with w in one residue field and use w^f as basis for the local roots of order ℓ^h. If the globally chosen primitive root reduces to a different one, relabel the embeddings over s and change the local basis. This can multiply the coordinate by a unit and preserves w_h.

The restriction of a corestriction at a completely split prime is the sum over all embeddings into $\mathbb{Q}_s$. This is the double-coset formula for restriction and corestriction, with each local corestriction the identity. The embeddings send $\zeta_{f\ell^h}$ to w^a, for the unit classes a in the sum. For the Kummer class of a residue unit y, choose z in the algebraic closure with $z^{\ell^h} = y$. Then

$\mathrm{Frob}_s(z)/z = y^{(s-1)/\ell^h}$. Thus its coordinate in the basis w^f is $\mathrm{ind}_g\, y$. The additional r tensor factors of ζ_{ℓ^h} contribute a^r, and the action on t_α contributes $\alpha(a)$. Adding gives the stated finite sum. This computation is also the residue method for higher circular units at odd weights over real abelian fields in Beliaeva and Belliard, Sections 6–7.

Evaluation is O-linear. Multiplication of the basis class by $a_{\alpha,r}$ therefore gives the asserted inequality, and gives equality if the basis evaluation is a unit. If the moment is nonzero, $d < h$ and $d \le w_h(D_{s,h})$. Use Theorem 20.1 to bound the strict order. Finally at $h = 1$ every a in the sum is prime to ℓ, so that $a^r \bmod \ell$ only depends on $r \bmod (\ell - 1)$. The same sum gives the assertion at every twist indicated. □

9.5 An average for the strict orders on the full interval

Use the list of large-prime triples in Theorem 20.2. Choose h_ℓ at least the largest length at that prime. At $x \in [V, V^{3/2}]$ let I_x consist of the character orbits of E_x, for $1 \le r < t(x)$ and $\ell > \exp(x^\varepsilon/12)$, at the coefficient primes that occur in the interval list. Include the zero components at these primes if desired. Enlarge the interval list by those orbits before applying the theorem. A prime entirely absent from the support contributes zero to the strict order. Define the two real numbers

$$L_x = \sum_{j \in I_x} b_j d_j \log \ell_j, \qquad U_x(s) = \sum_{j \in I_x} b_j W_j(s) \log \ell_j.$$

The character decomposition over the unramified integer rings identifies L_x with the sum of the full logarithmic orders of the large-prime complementary groups on E_x when all supported orbits are included. Restriction to a larger period field identifies the original character component with its copy upstairs, by prime-to-degree descent. Repetitions of that component are counted just once at any given x.

Theorem 9.6 (One average with polynomial additive error)**.** *For a constant $C = C(\psi, \varepsilon)$, for all sufficiently distant original intervals,*

$$0 \le \mathbb{E}_{s \text{ split in } K} \sup_{V \le x \le V^{3/2}} \big(U_x(s) - L_x\big) \le V^C.$$

The same estimate holds simultaneously for every sublist of the given triples, with the supremum taken over those sublists as well. In particular the mean of U_x recovers L_x with additive error $o(\exp(x^\varepsilon/2))$, uniformly on this interval. For every fixed $A > 0$, a set of relative density at least $1 - V^{-A}$ among primes split in K has simultaneous error at most V^{C+A}.

Proof. The summands in the difference are nonnegative. For the supremum over all sublists it suffices to add the excesses over the entire list once. These are finitely many finite-valued variables, so their mean is the ordinary limit of prime averages by Theorem 20.2. For one summand, summing the probabilities of the positive integer levels gives the exact unweighted excess

$$\mathbb{E}(W_j - d_j) = \sum_{a=1}^{h_j - d_j} \ell_j^{-b_j a} = \frac{1 - \ell_j^{-b_j(h_j - d_j)}}{\ell_j^{b_j} - 1}.$$

In particular the error estimate is independent of all the precisions and lengths. At a given prime the sum of the b_j is at most $n_* t_*$. This follows by decomposing the regular representation of the abelian Galois group of the largest original period field into residue-field orbits at each twist.

The support bound in the proof of Theorem 14.9 gives at most $Z = \exp(V^{C_0})$ different coefficient primes, with C_0 fixed independently of the interval and of the precisions. For the

primes $\ell \le Z^2$, bound $\sum \log \ell/(\ell-1)$ by summing over all integers between 2 and Z^2. This costs $O((1+\log Z)^2)$ by the harmonic sum. Above Z^2, the function $\log y/(y-1)$ is decreasing, so the remaining sum is at most $Z\log(Z^2)/(Z^2-1)$. Multiplying by $n_* t_*$, polynomial in V, gives the asserted bound after increasing C. A polynomial in V is $o(\exp(x^{\varepsilon}/2))$ everywhere on the stated interval. Markov's inequality for the full excess proves the relative-density assertion. The finite set of possible sublists on the entire interval justifies each supremum. □

This places all of the large-prime contribution under one average of explicit finite-field quantities on the given period fields. A bound for that average itself on the scale of (13.12) remains necessary. The splitting field and the set of coefficient primes can both grow with the interval.

9.6 Integral character comparison at every odd prime

Use an original interval of Theorem 12.15. Let $E = E_x$, $G = \mathrm{Gal}(E/\mathbb{Q})$, $n = |G|$, and let $N = N_*$ be the modulus of the largest field on the interval. Write $c_\infty \in G$ for complex conjugation and $t = t(x)$. In the following sum let α range once over the p-adic Galois orbits of characters of G satisfying $\alpha(c_\infty) = (-1)^r$. Use their primitive conductors. With the residue degree and the index of Theorem 20.3, put

$$C(E,t;N) = \sum_{\substack{p \text{ odd} \\ 1\le r<t}} \sum_{[\alpha]} b_\alpha d_{p,r,\alpha}(N) \log p.$$

Theorem 9.7 (All odd primary orders on the original fields)**.** *This sum has finite support. Its difference from the sum of the logarithmic orders of the complementary odd strict groups on E is bounded in absolute value by $[\,2nt\log(2N)\,]^{C_\psi}$ for a constant C_ψ. The same bound holds for its difference from $\mathcal{A}(E,t)$ in (11.3). These comparisons use the full primary orders, including primes dividing $|G|$.*

Proof. Fix p, r, put $k = r+1$, and write $e_r = (1+(-1)^r c_\infty)/2$. On U_N compare the two Galois lattices

$$L = e_r \mathbb{Z}_p[G](k) \;\hookrightarrow\; L' = \bigoplus_{[\alpha]} O_\alpha(\alpha\kappa_p^k).$$

Use the regular action and evaluation at the characters. Fourier inversion over $\overline{\mathbb{Q}}_p$ proves injectivity. Multiplication by n clears the only denominator of its inverse. Indeed the inverse coefficients are sums of values times inverse roots of unity divided by n. The sums are invariant under coefficient Galois action and are p-integral. Thus the map is onto when $p \nmid n$. Otherwise $D = L'/L$ has order at most p^{an}, where $a = v_p(n)$. Since $[E : K_\psi]$ is a power of two, these odd primes divide the fixed order of ψ, and a is bounded in terms of ψ.

We bound the error in degree two at those fixed primes. The action on D becomes trivial on $F = E(\mu_{p^a})$, of degree $O_\psi(n)$ and conductor dividing Np^a. Inflation–restriction bounds the first global cohomology with coefficient D by the product of $|D|^{[F:\mathbb{Q}]}$ and $|\operatorname{Hom}(G_{F,S}, D)|$, where S comprises the primes over Np and infinity. In fact every cocycle of the finite quotient is one of the indicated functions, and the restricted cocycle is a homomorphism. Since $\mu_p \subset F$, the Kummer sequence bounds $\dim_{\mathbb{F}_p} H^1(G_{F,S}, \mathbb{F}_p)$ by $[F:\mathbb{Q}]+|S|+1+\log|\mathrm{Cl}(F)|/\log p$. The terms come from S-units and the p-torsion of the S-class group. The conductor–discriminant formula gives $\log|\operatorname{disc} F| \ll_\psi n\log(2N)$. Minkowski's ideal bound and the estimate $\zeta_F(2) \le \zeta(2)^{[F:\mathbb{Q}]}$ thus give $\log|\mathrm{Cl}(F)| \ll_\psi n\log(2N)$. Also $|S| \ll_\psi n\log(2N)$. Filtration by groups of order p bounds the logarithm of the homomorphism count by this dimension times $\log|D|$. Hence $\log|H^1(U_N, D)| \ll_\psi n^2\log(2N)$. The finite-module global Euler characteristic formula over $\mathbb{Q}$ gives the same bound for $\log|H^2(U_N, D)|$, increasing the constant if necessary. The long exact

sequence then bounds the absolute difference between the logarithmic degree-two orders for L, L' by the sum of these two bounds.

For L, Shapiro identifies the open degree-two group with $e_r H^2(\mathcal{O}_E[1/Np], \mathbb{Z}_p(k))$. Pass back to $\mathcal{O}_E[1/p]$. At each removed place $w \nmid p$, purity gives zero in the preceding residue term $H^0(\kappa(w), \mathbb{Z}_p(r))$. The next term has order the p-part of $(Nw)^r - 1$, by arithmetic Frobenius on $\mathbb{Z}_p(r)$. Thus the change in the logarithmic order in either sign, summed over odd p, is at most $nr \log N$, since for each $q \mid N$ one has $\sum_{w|q} \log Nw \le n \log q$. Degree two on $\mathcal{O}_E[1/p]$ is the p-part of $K_{2r}(\mathcal{O}_E)$. Its complementary sign gives the order appearing in $\mathcal{A}(E, t)$, by Theorem 11.17.

For L', apply Theorem 20.3 to each summand. At a fixed $q \mid N$, form the product of $1 - \alpha(q)q^r$ over all geometric characters in the specified sign that are unramified at q. It is a nonzero rational integer by Galois invariance and $r > 0$. At each $p \neq q$ its p-valuation is the sum of the character valuations weighted by b_α, by the local norm formula even in ramified coefficient fields. Its absolute value is at most $(1 + q^r)^n$. The sum of all these local-factor costs over $q \mid N$ and all odd p is therefore $O(nr \log(2N))$.

Finiteness of the even K-groups, finiteness of the Euler products at each r, and the finite list of primes dividing n now also prove finite support for the indices. Summing the preceding bounds for $1 \le r < t$ proves the comparison with $\mathcal{A}$. To recover the strict group, Theorem 18.1 bounds the change by the local invariants above p. If such an invariant has order $p^j > 1$ over E_w, the cyclotomic image in $(\mathbb{Z}/p^j)^\times$ has index at most n and exponent dividing r. As that group is cyclic, $\varphi(p^j) \le nr$. The invariant group is cyclic of order at most $2nr$, and it can occur only at $p \le nr + 1$. There are at most n places over each such prime. The resulting total is $O(n^2 t^2 \log(2nt))$. Taking sign summands preserves this upper bound. All errors have the asserted polynomial size. □

9.7 The distribution when only one twist is split

Take $N_*, p, k, \alpha, O, \pi, e, v, h$ as in Proposition 14.11. Write $q = |O/\pi|$, $d = d_{p,r,\alpha}(N_*)$, and $\nu = \min(h, v_p(N_*))$. Put

$$t_h = \frac{\varphi(p^h)}{\varphi(p^\nu)}, \qquad m_h = \frac{t_h}{\gcd(t_h, k)}, \qquad \varphi(1) = 1.$$

All densities below are relative natural densities among rational primes $s \equiv 1 \bmod N_*$, omitting ramified primes in each fixed finite calculation.

Proposition 9.8 (A conditional Frobenius law and its finite difference)**.** *The condition $p^h \mid s^k - 1$ has density m_h^{-1}. Given this condition, for $0 \le j \le v$ one has*

$$\Pr(D_v(s) = 0 \bmod \pi^j) = q^{-\max(0, j-d)}.$$

For an independent pair s_1, s_2 write $I_v = 1$ if both satisfy the condition and zero otherwise. On $I_v = 1$ let J_j equal one if both orbit sums vanish modulo π^j, and zero otherwise. Take $J_0 = 1$ there. Set the following variables to zero off the eligible event:

$$P_v = m_h^2 I_v J_v, \qquad H_v = m_h^2 I_v \frac{J_v - q^{-2} J_{v-1}}{1 - q^{-2}}.$$

Then $\mathbb{E}P_v = q^{-2\max(0, v-d)}$ and $\mathbb{E}H_v = \mathbf{1}_{\{v \le d\}}$.

Proof. Modulo p^h the allowed residue classes are the units equal to one modulo p^ν, by the Chinese remainder theorem. They form a cyclic group of order t_h since p is odd. The k-th power homomorphism has image of order m_h. Equidistribution of prime residues modulo $\operatorname{lcm}(N_*, p^h)$ gives the first density.

For the second calculation fix any eligible residue class also modulo $n = fp^h$ compatible with $s = 1 \bmod N_*$. Put $K = \mathbb{Q}(\mu_{\mathrm{lcm}(N_*,n,p^h)})$ and first consider a basis cocycle u of the free H^1 of Theorem 20.3. Its values on G_K map onto the additive group O/π^v. Here is the argument. The residual basis class is nonzero by the multiplication sequence. Restriction to G_K is nonzero because the abelian cyclotomic quotient has zero first cohomology on the residual line. Indeed complex conjugation is central there and acts by -1 on that line. Evaluating a cocycle on a product in both orders expresses it as a coboundary, since two is invertible. The action on the coefficient is trivial on G_K at the indicated precision, so the image is additive and contains a unit. Conjugation by a global element preserves it and acts on it by $\alpha\kappa_p^k$. It is already stable under the scalar integers modulo p^h, so it is stable under all values of α and hence under $O = \mathbb{Z}_p[\alpha]$. An ideal containing a unit is the full ring.

On any fixed lift of the eligible cyclotomic Frobenius, multiplication by elements of G_K now makes the basis cocycle uniform on O/π^v in the finite quotient recording the coefficient, the cocycle and K. Conjugation multiplies its value by a unit on this fiber, since the coefficient of the eligible Frobenius is one modulo π^v. All divisibility tests are therefore conjugacy invariant. Chebotarev's density theorem gives the finite-group proportions as prime densities. The actual class is a scalar of valuation d times the basis class. A possible coboundary vanishes on eligible Frobenius. Reduction to level j gives the displayed probability, independently of the chosen compatible residue. Averaging the residues proves the conditional law.

For two sampled primes the conditional probabilities multiply. The event $I_v = 1$ has density m_h^{-2}, proving the formula for P_v. For H_v the numerator of its expectation before division by $1 - q^{-2}$ is

$$q^{-2\max(0,v-d)} - q^{-2}q^{-2\max(0,v-1-d)}.$$

It equals $1-q^{-2}$ for $v \le d$ and zero otherwise. Both indicators use the same eligibility condition at level v, including when $j = v - 1$. □

9.8 The period field and the compact types of its integral

We distinguish the period field $\widetilde{E} = K_\psi C_{l'}(\sqrt{-q}, \sqrt{-d_0})$ of Proposition 6.28 from the upper field E_x in Theorem 12.16. We write $s_0 = \lfloor u\varepsilon \log_2 x \rfloor$ for the subfield index of that theorem and reserve $s = m/2-1$ for the Witt flag length. All the choices below are made on its original selected intervals, with their primes q, d_0.

Proposition 9.9 (The bounded summand on the period field)**.** *For every sufficiently large x in these intervals, every odd prime p, and every $r \ge 1$, restriction from $L_{u,x}$ identifies the two invariant strict groups*

$$\mathcal{S}_{p,r}(\widetilde{E})^{\mathrm{Gal}(\widetilde{E}/L_{u,x})} \;\simeq\; \mathcal{S}_{p,r}(L_{u,x}) \;\simeq\; \mathcal{S}_{p,r}(E_x)^{\mathrm{Gal}(E_x/L_{u,x})}.$$

In particular the bounds (12.9) *and* (12.10) *hold with $\widetilde{E}$ as the upper field. The residual decomposition* (12.11) *holds there with its own strict groups and averaging projectors.*

Proof. Put $o = [K_\psi : \mathbb{Q}]$. Proposition 6.28 gives $x^{2+8\varepsilon} \le [K_\psi C_{l'} : \mathbb{Q}] \le o2^{l'}$. For $0 < u < 1$, this implies $s_0 \le l' \le l$ for all sufficiently large x. Both fields in the display are therefore extensions of precisely the same $L_{u,x}$ of two-power degree. In positive group cohomological degrees the two finite Galois groups have zero cohomology on p-primary modules, by averaging. Inflation–restriction thus identifies the unrestricted degree-one group downstairs with the invariants upstairs. At p, local restriction is injective since the local extension degree is prime to p. Away from p the coefficient module is unramified and an inertia homomorphism that becomes zero factors through a group of two-power order. Thus the local conditions descend in both extensions. This gives the identifications, through the full primary exponent.

At odd primes the averaging operator on the shortened field and the two sign projectors commute, since the field is abelian. The critical-sign lengths on the shortened field are bounded by those on the upper field, by injectivity. At two, Proposition 6.28 bounds the extra weighted length by $t\log(2o)$, which is polynomial in x and hence $o(\exp(x^{\varepsilon}/2))$. The proof of Corollary 12.17 now gives the assertion for $\widetilde{E}$. To omit individually certified components one applies Theorem 13.5 on the field in question. □

Lemma 9.10 (An integral embedding of a compact type)**.** *Let $A_1 \geq \cdots \geq A_P$ be integers and $I = \{i_1 < \cdots < i_k\} \subset \{1, \dots, P\}$. Put $\nu = (A_{i_1}, \dots, A_{i_k})$. In the coordinate tensor lattices defined below there is a primitive $U(k)$-equivariant isometric embedding $J_I : V_\nu \longrightarrow V_A$. The induced invariant Hermitian pairing is integral on these lattices and takes value one on a pair of primitive vectors.*

Proof. For $W = \mathbb{Z}^P$ write $d_j = A_j - A_{j+1}$ for $1 \leq j < P$ and use the ambient tensor

$$\mathcal{T}_A(W) = \bigotimes_{j=1}^{P-1} (\bigwedge^j W)^{\otimes d_j} \otimes (\det W)^{A_P}.$$

For a negative power of a determinant use the dual line. Give exterior coordinate monomials norm one and make them orthogonal. Let w_A be the unit coordinate tensor obtained from the first j basis vectors in every degree j, and the chosen generator of the determinant line. It has torus weight A and is killed by all upper triangular raising operators. By complete reducibility it generates an irreducible complex representation of highest weight A. Indeed, the projections of this vector to irreducibles can be nonzero only on their highest weight lines of weight A, and together they generate a single diagonal copy. Define the lattice by intersecting this representation with the integral ambient tensor. It spans the representation, since it is generated over $\mathbb{C}$ by products of matrix Lie operators with integral coefficients applied to w_A. Use the identical construction for ν.

Write $W = X \oplus Z$ for the coordinate splitting into ranks $k, P-k$. Order a basis of W by putting the ordered basis of X in the slots I and that of Z in the others. Put $b_j = |I \cap [1, j]|$. In each row factor of exterior degree j, send an exterior tensor of degree b_j in X to its wedge with the first $j - b_j$ basis vectors of Z, with the sign for this ordering. This sends distinct basis monomials to distinct signed basis monomials, isometrically and integrally. For $1 \leq t < k$, the multiplicity of degree t in the domain of these maps is

$$\sum_{j=i_t}^{i_{t+1}-1} d_j = A_{i_t} - A_{i_{t+1}}.$$

Degree zero contributes unit lines. The total determinant power in X is $A_P + \sum_{j=i_k}^{P-1} d_j = A_{i_k}$. The fixed determinant of Z accounts for the remaining line. Grouping these factors identifies the domain with $\mathcal{T}_\nu(X)$ by a signed permutation of coordinate tensors, including the canonical identifications of determinant lines and their duals. All identifications commute with $U(k)$ after extending scalars to $\mathbb{C}$.

The tensor embedding sends w_ν to w_A written in the permuted basis, up to a fixed sign. This is a vector of V_A, since the permutation matrix lies in $U(P)$. Its $U(k)$-span gives the required copy of V_ν. The embedding identifies the integral ambient domain with a coordinate direct summand. Consequently an element of the complex image that lies in the integral row lattice has integral preimage, proving primitivity. Pairing a row vector with the embedded column by the ambient Hermitian form gives integral coefficients. Pairing $J_I w_\nu$ with itself gives one, and Cauchy–Schwarz gives operator norm one. □

Theorem 9.11 (One compact row for every translated column)**.** *Keep the fields, intervals, column parameters, and finite levels of Theorem 72.8. There is one common regular row parameter and one chamber, independent of the translate, for which its lowest compact type contains both the high and low column types at every real place. All prescribed products of these types have integral compact pairings of norm one in a single row coefficient lattice. Each pairing takes the value one there. For a common admissible* Γ *one terminal prime works simultaneously for all translates at the formal-degree scale of this row.*

Proof. Write $P = k + s$, $Q = s + 1$, $r = P + Q$. The old column parameters are those in the proof of Theorem 64.27. Let ν^0 be the compact highest weight for a low place, and $\nu^a = \nu^0 + (a, 0, \dots, 0)$. Keep every shared entry $v_i = 2(k - 1 - i)$, $1 \le i < k$. At every real place take the same ordered row list

$$v_1 + a, \quad v_1 + 1, \quad v_1, v_2, \dots, v_{k-1}, \quad -1, \quad -(t_{\mathrm{r}} + 4), \dots, -(m - 3)(t_{\mathrm{r}} + 4).$$

It has $r = k - 1 + m$ distinct integer entries for $a \ge 2$. It retains the original interlacing $u_i > v_i > u_{i+1}$ with each column. Give the second slot negative signature and all the other slots through -1 positive signature. Of the remaining $2s - 1$ slots choose the first $s - 1$ positive and the rest negative. This yields the required signature (P, Q). For the positive entries $j_1 = 1$ and $j_i = i + 1$ for $2 \le i \le k + 1$. The compact highest weight formula of equation 64.15, now gives

$$(A_1, \dots, A_{k+1}) = (\nu_1^a, \nu_1^0, \nu_2^0, \dots, \nu_k^0).$$

Indeed a slot occupied by v_i gives $A_{i+1} = v_i - k/2 + i$, and the slot -1 gives $A_{k+1} = k/2 - 1 = \nu_k^0$.

Apply Lemma 9.10 to $I_0 = \{2, \dots, k + 1\}$ and $I_a = \{1, 3, \dots, k + 1\}$. Tensor these two embeddings separately with a fixed integral highest vector of norm one in the $U(Q)$-factor, on which H_τ acts trivially. The scalar circle of $U(k)$ acts on the images by characters whose exponents differ by a. Invariance of the inner product therefore makes the two images orthogonal. Tensoring the appropriate embeddings over the real places preserves both the norm and the value one on the chosen integral vectors. Distinct sets of high places give orthogonal images by the same circle calculation at a place where they differ. The lattice used throughout is the tensor product of these split coordinate coefficient lattices.

The same row at every place fixes the product of its formal degrees. Every translate has h high places and $f - h$ low places, so the column dimension product is always $W_k(u^a)^h W_k(u^0)^{f-h}$, where u^a, u^0 denote the two column parameters. Hence D_*^{row}, $d_{u,*}$, and the target T_* of Corollary 72.10 are independent of the translate for a common Γ. That corollary, using the interlacing checked above and the original column spans, gives a prime in $(T_*, (1 + 1/(fk))T_*]$ satisfying all its finite congruences. In Lemma 7.10 one may double the modulus as before Theorem 72.11, adding only $\log 4$ to $\log(4dM^2)$. Choose the prime with this stronger congruence. The row parameter has size at most $a + 2k + m(t_{\mathrm{r}} + 4)$, which is at most $\exp(Cx^{\varepsilon})$ for large x, by the definitions of these ranks and orders. The parameter bound in Lemma 65.1 and the analytic period estimate of Theorem 65.2 therefore apply at this same prime. □

The prime window here is recalculated using the replacement row. The compact pairing of Theorem 9.11 is explicitly defined on its tensor lattice. To use it in (6.3) one still needs a global automorphic realization with a nonzero Bessel integral, and a comparison of that integral on the arithmetic lattices with its analytic normalization. Its height is precisely the issue in Corollary 65.3. On this very field the complementary strict groups in Proposition 9.9 still require a uniform bound.

9.9 One terminal prime for a family of coded columns

Fix the field $\widetilde{E}$, the ranks and the auxiliary primes of Proposition 9.9 and Theorem 7.11, at one x on their common intervals. Write

$$b = \exp(x^{\varepsilon}), \quad X' = \exp(x^{\varepsilon/3}), \quad b' = xb/X', \quad w'' = b'/(xn), \quad a = \lceil xb \rceil, \quad b_* = \lceil b' \rceil.$$

Write $r = k + m - 1$ and $t = t_{\mathrm{r}}$ below. Fix the h high real places indexed by the order set of Theorem 6.29. For each translate $\ell \in A_0$ take its h_2 distinguished places L_ℓ in the other half of the real-place labels, as in Lemma 6.2. A word consists of $n_0 = k - 2$ bits $c_2, \dots, c_{k-1}$, used at every place. The Hamming distance is the number of bits at which two words differ. We will select words before selecting the terminal prime.

We use the lists prescribed in `Development/AbelianVarietyTheory/Definitions/Automorphic Membership/EvenCeil.lean`, line 103 and lines 171–191 there. Take any positive integers Δ, H_0 bounded by fixed powers of x, with $H_0 \geq k + m(t+3) + r^{19} f^5 x^{\varepsilon}$. Such an H_0 is polynomially bounded by Theorem 6.29. At every place set

$$v_i = 4\Delta(k - 1 - i) \ (1 \leq i < k), \qquad u_i = v_{i-1} - \tfrac{1}{2} - \Delta c_i \ (2 \leq i < k). \tag{9.1}$$

Set the pair (u_1, u_k) equal to $(a + (k-1)/2, -b_* - (k-1)/2)$ at a high place and to $(v_1 + 4k\Delta + 1/2, -1/2)$ elsewhere. Consecutive v_i differ by 4Δ and a dominates every fixed power of x. These facts prove strict interlacing and give $u_1 - v_1 \geq 4k\Delta$ at every place. Since $b' = o(xb)$, we also have $u_1 - u_k \leq 3a$.

At a high place put $B^0_\tau = b_* + \lfloor b'/x \rfloor$, on L_ℓ put $B^0_\tau = \lceil b \rceil$, and at the remaining places put $B^0_\tau = \lceil w'' \rceil$. If their sum is odd add one at the distinguished place indexed by the least element of the untranslated distinguished set followed by translation by ℓ. Otherwise make no adjustment. Call the result B_τ. Then set

$$b_{j,\tau} = B_\tau + 2(j-1)H_0, \qquad e_{j,\tau} = B_\tau + (2j-1)H_0 \qquad (1 \leq j \leq m).$$

The count uses $e_{j,\tau}$ itself by `Formalism/LocalToGlobal/Divisibility/Numbers.lean`, line 160. The shifted representation weight is $e_{j,\tau} - k/2 - (2j-1)(t+1)/2$. The successive gaps from b_j to e_j and e_j to b_{j+1} are both H_0. In particular they exceed $(t+3)/2$. Both b'/x and w'' grow faster than every fixed power of x, as follows by taking logarithms. Thus $u_k > -b_1$ also at high places, $4k\Delta \leq b_1/x$, and $H_0 \leq b_j/x$. Both the count weight and the shifted weight lie above b_j and below b_{j+1} for $j < m$, and their last entries are at most $2b$ eventually. These are the weight inequalities of `Formalism/SharpRestriction/GlobalSchurBounds/StandardLattice.lean`, line 496. The parity adjustment is independent of the word. At a fixed translate the same signed row (65.4) therefore serves the entire family. Every parameter has absolute value at most $\exp(Cx^{\varepsilon})$.

Lemma 9.12 (Variation of the column dimensions)**.** *For two arbitrary words c, c' in (9.1), at each real place one has*

$$\left|\log W_k(u_\tau(c)) - \log W_k(u_\tau(c'))\right| \leq Ck$$

with an absolute constant. The same estimate holds for their logarithmic Vandermonde products.

Proof. The denominator of the Weyl dimension formula is independent of the word. Renumber the interior bits from 1 to n_0 in decreasing parameter order. Relative to the all-zero word, an interior pair $i < j$ changes its logarithmic factor by

$$\log\left(1 + \frac{c_j - c_i}{4(j-i)}\right).$$

The argument added to one has absolute value at most $1/4$. By Taylor's theorem replacing each logarithm by its linear term costs at most $Cn_0\sum_{d\ge1}d^{-2}$ in total. Four times the sum of the linear terms is

$$\sum_{i=1}^{n_0} c_i(H_{i-1}-H_{n_0-i}),\qquad H_j=\sum_{l=1}^{j}l^{-1},\quad H_0=0.$$

The coefficients are increasing and change sign once. The maximum over binary words is attained by putting ones on the positive coefficients, and the minimum by putting ones on the negative coefficients. For either step function, at distance d at most d pairs cross the step. The absolute contribution of these pairs to four times the linear sum is at most one. Summing over the distances bounds both extrema by n_0-1.

In a pair with the top endpoint the gap is at least $4k\Delta$ and changes by at most Δ. The sum of the changes of these logarithms is $O(1)$. For the bottom endpoint the smaller possible gap is at least $(4(k-i)-1)\Delta$ at interior index i in the original numbering, since $u_k\le-1/2$. Its logarithm changes by at most $1/(4(k-i)-1)$, using $\log(1+y)\le y$. These terms sum to $O(\log k)$. The pair of endpoints is fixed. Adding the three estimates and comparing two words through the zero word proves the lemma. □

Theorem 9.13 (A common terminal prime for exponentially many words)**.** *Put* $d_c=\mathfrak{m}_kI_\pi\prod_\tau W_k(u_\tau(c))$ *and* $D=\mathcal{V}(G)I_\sigma\prod_\tau W_r(\lambda_\tau)$. *Assign real numbers* $1\le\Gamma_c\le\exp(Cfm^5x^\varepsilon)$ *to the words in advance, with fixed* C. *Define*

$$Y_c=\Big(D/(d_c\Gamma_c^2)\Big)^{1/(fk)},\qquad \delta_x=\exp(-x^{1/4}).$$

For sufficiently large x *there is a set* $\mathcal{A}$ *of at least* $\exp(k/5)$ *words, all of the same weight* $\lfloor n_0/2\rfloor$, *with pairwise Hamming distance at least* $n_0/8$. *Writing* $Y=\max_{c\in\mathcal{A}}Y_c$, *there are* $Y^{1-o(1)}$ *rational primes* $T=\varpi\bar\varpi$, *with* $\varpi\equiv1 \bmod 2N(g_1g_2)^2\mathcal{O}_{K_d}$, *such that simultaneously*

$$0<fk\log(T/Y_c)\le3\delta_x/8\qquad(c\in\mathcal{A}).$$

In particular $Y_c<T\le(1+1/(fk))Y_c$ *for every selected word. Their column representations have the same central character at each real place.*

Proof. Let $p_c=\log(d_c\Gamma_c^2)$. Lemma 9.12 bounds the spread of $\log d_c$ by C_1fk. From $f\ll_\psi x^{2+8\varepsilon}$, $m\asymp x^{1/2-\varepsilon/16}$, $k\asymp x^{9/2+10\varepsilon}$ we obtain

$$fm^5x^\varepsilon/k\ll_{\psi,\varepsilon}x^{-21\varepsilon/16}.$$

Consequently the full range of the p_c is contained in an interval of length $O(fk)$. The layer of weight $\lfloor n_0/2\rfloor$ contains at least $2^{n_0}/(n_0+1)$ words, since its binomial coefficient is the largest. Subdivide the interval for p_c into bins of width $\delta_x/8$. At most $O(1+fk/\delta_x)$ bins meet this range. One contains at least the size of the layer divided by this number of bins.

From this bin choose words successively, discarding at each choice all words at distance at most $\lfloor n_0/8\rfloor$ from the chosen one. Each choice discards at most $\exp(n_0H(1/8))$ words, where $H(p)=-p\log p-(1-p)\log(1-p)$. Indeed, for $p=1/8$ and $z=p/(1-p)$ the binomial expansion of $(1+z)^{n_0}$ gives $\sum_{j\le n_0p}\binom{n_0}{j}\le z^{-n_0p}(1+z)^{n_0}$. The logarithm of the number of bins is $O(\log x+x^{1/4})=o(k)$. Thus the logarithm of the selected cardinality is at least $(\log2-H(1/8)-o(1))k$, which exceeds $k/5$. Equal weights fix $\sum_i u_i$ at every place. Subtracting the same compact half sum of positive roots gives equal sums of highest weights, hence equal scalar central characters.

The shared bin gives $0 \le \log(Y/Y_c) \le \delta_x/(8fk)$. Corollary 72.10 applies to every list by the verified span, interlacing and range of Γ_c. In particular $\log Y \ge (16 - o(1))mx^{\varepsilon}$. Let $M = 2N(g_1g_2)^2$ and $\mathfrak{B} = 4dM^2$. The proof of Theorem 7.11 gives $\log \mathfrak{B} \ll_{\psi,\varepsilon} x^{\varepsilon}$, also for this doubled modulus. Hence $\log Y/\log \mathfrak{B} \gg m$ and $\sqrt{\log Y} \gg x^{1/4+15\varepsilon/32}$. Each grows faster than $x^{1/4} + \log(fk)$ for fixed $0 < \varepsilon < 1/52$. Thus the error $\rho(Y, \mathfrak{B})$ of Lemma 7.10 is $o(\delta_x/(fk))$. Apply that lemma with relative width $\delta_x/(4fk)$. Its lower bound is $Y^{1-o(1)}$ since $\log \mathfrak{B} + \log(fk) + x^{1/4} + \log\log Y = o(\log Y)$. The trivial count of all integers in the interval supplies the corresponding upper exponent. For any resulting prime, $0 < \log(T/Y) \le \delta_x/(4fk)$. Adding the bound on $\log(Y/Y_c)$ gives the assertion. Finally $\log(1 + 1/(fk)) \ge 1/(2fk)$, which yields the stated inclusion in each original window. □

Corollary 9.14 (The actual refined column multiplicities)**.** *At this same T, let R_c be the multiplicity of the column type after refinement by a nonzero residue covector at each place above T. Let R_c° be the multiplicity on the subspace with zero sum on each local scalar orbit, in each coordinate separately. Uniformly over $\mathcal{A}$,*

$$\frac{\Gamma_c^2 R_c}{D} = 1 + O(\delta_x), \qquad \frac{\Gamma_c^2 R_c^{\circ}}{D} = 1 + O(\delta_x).$$

For any fixed positive C_0, C_H, the size of $\mathcal{A}$ exceeds $C_0 x^{C_H} \max_c \Gamma_c^2$ eventually.

Proof. By Proposition 64.29, d_c is the multiplicity at the unrefined terminal level, including the finite levels at g_1, g_2. Its classes have trivial stabilizers. Since T splits completely, every class has independently $T^k - 1$ refinements at each of the f places. Each scalar orbit contains $T - 1$ nonzero covectors, and restricting functions on one such orbit to have zero sum leaves dimension $T - 2$. Tensoring these local choices yields

$$R_c = d_c(T^k - 1)^f, \qquad R_c^{\circ} = R_c(1 - 1/(T-1))^f.$$

It follows from the definition of Y_c that $\Gamma_c^2 R_c/D = (T/Y_c)^{fk}(1 - T^{-k})^f$. The first factor equals $1 + O(\delta_x)$ by Theorem 9.13. For $0 \le y \le 1$, $0 \le 1 - (1-y)^f \le fy$. Apply this to both remaining factors, using $f/(T-1) = o(\delta_x)$ by the lower bound on $\log Y$. Finally $\log(C_0 x^{C_H} \max_c \Gamma_c^2) = o(k)$ by the rank calculation in the preceding proof. This proves the size assertion requested numerically in `Formalism/SharpRestriction/GlobalSchurBounds/Chosen.lean`, line 56. □

Corollary 9.15 (The same prime and the same count on every translate)**.** *Use the preceding lists for every $\ell \in A_0$, with the same Γ_c for a given word. One set of words and one terminal prime as in Theorem 9.13 give both rank estimates of Corollary 9.14 for every translate. For the weights and orders of these very lists the centered prime-power count Δ_ℓ of Section 6 satisfies*

$$\left| \frac{2\Delta_\ell}{hmb\delta} - \mathcal{G}_\psi(x) \right| \ll_{\psi,\varepsilon} x^{1/2+\varepsilon/2}, \qquad \delta = h_2/n.$$

Proof. Translating the distinguished set permutes its positions, preserves its size and moves the marked position with it. Its positions are disjoint from the high places. Thus the multiset of the B_τ stays fixed, including the possible parity adjustment. The row product in D depends only on this multiset. The column product in d_c depends on c and on the number of high places. Hence every Y_c, the bins of the proof of Theorem 9.13, and the common prime interval can be taken identical for all translates.

The order at a high place is $u_1 - (k-1)/2 = a$, precisely the definition in `Formalism/LocalToGlobal/Divisibility/Numbers.lean`, line 247. At a distinguished place the count weight e_j lies between b and $b + 2 + 2mH_0 \le b + \sqrt{b}$ eventually. At every other place it lies between one and $2b'$ eventually. Indeed at a high place it is at most $b' + 1 + b'/x + 2mH_0$, and elsewhere in

the complement it is at most $w''+1+2mH_0$. Hence $a/e_j \geq X'/2$ there. Also $|a/b-x| \leq 1/b$. Corollary 6.31 now gives exactly the displayed estimate on the same field and interval, for each translate. □

Theorem 65.7 supplies the selected columns' individual real Bessel models. On the unrefined terminal quotient the smooth functions of Theorem 64.31 realize both frequencies simultaneously. Proposition 9.9 bounds the indicated invariant cohomology summand on the same field. A rank comparison does not evaluate the Bessel coefficient on the integral theta image. That evaluation and the complementary cohomology estimate are still needed for (W).

9.10 One cyclotomic degree bound for all the complementary twists

Fix x on one of the intervals of Section 6.6, and keep its prescribed pair of auxiliary primes. Put $n_* = [E_* : \mathbb{Q}]$, $t = \lfloor x^{401} \rfloor$, and $Y = x^{-3}\exp(x^{\varepsilon}/4)$. For every odd prime $p \leq Y$ take the largest integer $j_p \geq 0$ with $p^{j_p+1} \leq Y$. At this prime use the tower and the full complementary dual $A_{x,r}$ of Theorem 15.7. The prime is included as a subscript below when it varies.

Theorem 9.16 (A uniform truncated sum including the decomposition invariants)**.** *For $0 < \varepsilon < 1/52$, uniformly at all sufficiently large x on these intervals,*

$$\sum_{\substack{2<p\leq Y\\ p \text{ prime}}} \sum_{1\leq r<t} \log\bigl|A_{p,x,r}/p^{s_{p,r}+j_p}A_{p,x,r}\bigr| \ll_{\psi,\varepsilon} x^{-2}\exp(x^{\varepsilon}/2). \tag{9.2}$$

This includes all characters of the tame cyclotomic action. If at each prime one replaces the logarithmic class group at level j_p by its image in the ordinary class group with p inverted, the sum decreases by at most

$$O\bigl(n_*\log(2n_*Yt)\,[Y + tn_*\log(2Y)]\bigr). \tag{9.3}$$

For each prime and weight for which the norm-level test of Theorem 15.7 holds at j_p, the corresponding summand in (9.2) is the full logarithmic order.

Proof. Fix p, put $j = j_p$, $F = F_j$ and $T = \widetilde{C}(F)$. Use the integer u of Theorem 15.7. A single precision bound for all the indicated weights is

$$H = u + 1 + j + \lfloor \log(t)/\log(p) \rfloor, \qquad H\log p \leq \log(n_*Yt).$$

Indeed $p^u \leq n_*$, $p^{1+j} \leq Y$, and $p^{v_p(r)} \leq t$. Let Δ_0 be the torsion subgroup of $\mathrm{Gal}(E_*(\mu_{p^\infty})/E_*)$, and set $d_p = |\Delta_0| = [E_*(\mu_p) : E_*] \geq (p-1)/n_*$. The torsion subgroup maps isomorphically to the reduction modulo p of the cyclotomic image. This follows from $\mathbb{Z}_p^\times = \mu_{p-1} \times (1+p\mathbb{Z}_p)$. A closed subgroup contains the torsion part of each element by taking powers p^m with $p^m \equiv 1 \bmod (p-1)$ and passing to the limit.

For $\tau = \kappa_p^{-r}|_{\Delta_0}$, denote the usual character idempotent by e_τ. It is integral since $p \nmid d_p$ and the character values belong to $\mathbb{Z}_p^\times$. The quotient in (9.2) is $Q_{x,r}(T)$ by (15.1). It is a quotient of $e_\tau(T/p^HT)$. Each τ occurs at most $1 + t/d_p$ times. Orthogonality and the sum-to-one identity for the idempotents give

$$\sum_{1\leq r<t} \log|Q_{x,r}(T)| \leq (1+t/d_p)\log|T/p^HT|.$$

Let C' denote the image of T in $\mathrm{Cl}(\mathcal{O}_F[1/p])\{p\}$ and let K be its kernel. The local degree sequence of Theorem 21.11 shows that K is generated over $\mathbb{Z}_p$ by at most $g-1$ elements, where g is the number of primes of F over p. Indeed the local degree group at each such prime is a

subgroup of $\mathbb{Z}_p$. The kernel of the sum of these g groups is free of rank $g-1$, and surjects onto K. Reduction modulo p^H is right exact, so

$$\log|T/p^HT| \le \log|\mathrm{Cl}(F)| + (g-1)H\log p.$$

More precisely $Q_{x,r}(T) \to Q_{x,r}(C')$ is onto, and its kernel is the image of K. To see this, lift the character relations from C' to T. The quotient of T by K and these relations is exactly $Q_{x,r}(C')$. The image of the compact group K is closed. Only $e_\tau(K/p^HK)$ can contribute to that image. Summing over r as above bounds the total change at p by $(1+t/d_p)(g-1)H\log p$.

There is a single prime above p in the real part of $\mathbb{Q}(\mu_{p^{u+1+j}})$, since the cyclotomic field is totally ramified at p. Its real part is contained in F with relative degree at most n_*, by the compositum description of F and the equality of the two CM degrees with their real degrees over this cyclotomic base. Hence $g \le n_*$. Summing with $1/d_p \le n_*/(p-1)$ and $\sum_{2<p\le Y} 1/(p-1) \le 1+\log Y$ proves (9.3).

Finally $[F:\mathbb{Q}] = p^j[F_0:\mathbb{Q}] \le n_*p^{j+1} \le n_*Y$. If N_* is the cyclotomic modulus of Theorem 12.15, then F has modulus at most $N_*p^{u+1+j} \le N_*n_*Y$. The conductor–discriminant formula gives $\log|\mathrm{disc}(F)| \le [F:\mathbb{Q}]\log(N_*n_*Y)$. Minkowski's ideal bound and the estimate $\zeta_F(2) \le \zeta(2)^{[F:\mathbb{Q}]}$, as in Theorem 15.6, now give

$$\log|\mathrm{Cl}(F)| \ll n_*Y\log(2N_*n_*Y).$$

After summing over primes, this is multiplied by at most $Y + tn_*(1+\log Y)$. In the present family, $n_* \ll_\psi x^{3+12\varepsilon}$ and $\log(2N_*n_*Yt) \ll_{\psi,\varepsilon} x^\varepsilon$, by the field and conductor bounds in Theorem 15.6. The term quadratic in Y is therefore $O_{\psi,\varepsilon}(x^{-3+13\varepsilon}\exp(x^\varepsilon/2))$. The other terms, including (9.3), are bounded by a fixed power of x times $\exp(x^\varepsilon/4)$. Both bounds imply (9.2). □

At the primes in this range, subtracting the left side of (9.2) from the sum of the full orders in logarithmic scale gives

$$\sum_{\substack{2<p\le Y\\ p\ \text{prime}}} \sum_{1\le r<t} \log|p^{s_{p,r}+j_p}A_{p,x,r}|.$$

A uniform bound for this excess, the contribution of the other primes and the comparison of signed periods on the same fields remain to be established in Section 6.6.

9.11 A rational flag translation with prescribed auxiliary residues

Write $E = E_x$, $F = E^+$, $n = [E:\mathbb{Q}]$ and $\Delta_E = |\mathrm{disc}(E)|$. Work with the rational hyperbolic pairs e_i, f_i in the row space, the marked line z and the column space A. A rational flag diagonal fixes A, z and acts by $c_i \in E^\times$ on e_i and by $\bar{c}_i^{-1}$ on f_i. This is an element of the unitary group $G(F)$ and centralizes the column group.

Lemma 9.17 (An integral representative close to all real and complex targets)**.** *For a positive integer Q, each residue class in $\mathcal{O}_E/Q\mathcal{O}_E$ has a representative within distance $nQ2^{n+1}\sqrt{\Delta_E}$ of any prescribed point of $E\otimes_{\mathbb{Q}}\mathbb{R}$. Distance is the maximum of the absolute values at the embeddings, with the usual complex modulus.*

Proof. Use the product of the unit intervals and unit complex disks in Minkowski space. Every nonzero algebraic integer has maximum absolute value at least one by its nonzero integral norm. The first and hence all successive minima of this body on $\mathcal{O}_E$ are at least one. Minkowski's second theorem bounds their product by 2^n times the lattice covolume divided by the volume of the body. The covolume is $2^{-r_2}\sqrt{\Delta_E}$ and the volume is $2^{r_1}\pi^{r_2}$, where r_1, r_2 count real

embeddings and conjugate pairs. Consequently there are n real-linearly independent integers of maximum absolute value at most $2^{n+1}\sqrt{\Delta_E}$. Take any representative of the given residue class. Express its displacement from the desired point in the real basis formed by Q times these integers and round each coefficient to a nearest integer. Subtracting the resulting integral linear combination preserves the residue class and gives the claimed distance. □

Theorem 9.18 (One rational diagonal at both auxiliary primes)**.** *Fix the signed auxiliary levels and terminal prime of Theorems 7.11 and 9.13. Choose an integer $b \geq 1$ and $0 < \delta < 1/2$. There is a rational flag diagonal d_* with these properties.*

1. *At every auxiliary half its string multipliers, up to first principal units, are those of Proposition 77.18 after dividing them all by the unit on its middle line z. Thus it gives precisely the same character on every integral string root after conjugation.*

2. *At every half over T the diagonal is the identity modulo T^b in the split flag coordinates. At all archimedean embeddings, $|\sigma(c_i) - 1| \leq \delta$ and $|\sigma(\bar{c}_i^{-1}) - 1| \leq 2\delta$.*

3. *With $Q = g_1 g_2 T^b$ one has, for each i,*

$$\sum_{\mathfrak{p}} |v_{\mathfrak{p}}(c_i)| \log N\mathfrak{p} \ll n\big(s \log(g_1 g_2) + \log Q + n + \log \Delta_E + \log(1/\delta)\big).$$

   The sum is over all finite primes of E, with integer-valued valuations. For $b = 1$ and $\delta = \exp(-x^B)$ with fixed $B > 0$, this bound is polynomial in x on the same intervals. It holds simultaneously for all the string lines.

The prescribed finite and archimedean conditions also force a displacement away from the marked primes. With $a_i = s + 1 - i$, every such choice, for each $1 \leq i \leq s$, satisfies

$$\sum_{\mathfrak{p} \nmid g_1 g_2 T} v_{\mathfrak{p}}(c_i) \log N\mathfrak{p} \ \geq \ n \log\big((g_1 g_2)^{a_i}(1 - \delta)\big) > 0.$$

Proof. Put $a_i = s + 1 - i$ for $1 \leq i \leq s$ and $D_i = (g_1 g_2)^{a_i}$. Normalize the local centered diagonal at each auxiliary place by dividing its entire string by its middle eigenvalue. That eigenvalue is a unit and the simple ratios are unchanged. At the chosen split half prescribe c_i by the normalized eigenvalue on e_i. At the other half prescribe it by the conjugate of the inverse eigenvalue on f_i in the chosen half. Both prescriptions have valuation $-a_i$. They use different primes of E, so they impose independent conditions. Only the unit leading residues of these prescribed values need to be matched. At the terminal halves prescribe $c_i \equiv 1 \bmod T^b$.

Write $R = nQ2^{n+1}\sqrt{\Delta_E}$ and choose the least nonnegative integer k with $2^k \geq R/\delta$. The marked primes are odd. Multiplication of the local prescriptions by $2^k D_i$ gives unit residues at both auxiliary primes and gives the residue $2^k D_i$ at T^b. Since all marked primes split completely, the Chinese remainder theorem specifies one class in $\mathcal{O}_E/Q\mathcal{O}_E$. Apply Lemma 9.17 with the archimedean target $2^k D_i$ in every embedding, and denote the resulting integer by u_i. Then $c_i = u_i/(2^k D_i)$ has all the required finite residues and $|\sigma(c_i) - 1| \leq R/(2^k D_i) \leq \delta$. In particular u_i is nonzero. The bound for the inverse conjugate follows by division by an absolute value at least $1 - \delta$.

The sum of absolute finite valuations, weighted by log norms, is bounded by $\log|N_{E/\mathbb{Q}}(u_i)| + n \log(2^k D_i)$. This follows by the triangle inequality for the exponent of each ideal prime. The archimedean bounds give $|N(u_i)| \leq ((1 + \delta)2^k D_i)^n$. Using the minimal choice of k proves the estimate. The field degree, s, $\log T$ and $\log(g_1 g_2)$ are polynomially bounded by the results cited in the statement. On these abelian fields $\log \Delta_E \leq n \log N_*$ by the conductor–discriminant

formula. Substitution gives the polynomial bound. The tuple of resulting c_i defines a single rational unitary element. For any admissible choice the sum of $v_{\mathfrak{p}}(c_i)\log N\mathfrak{p}$ over the auxiliary primes equals $-na_i\log(g_1g_2)$, since both rational primes split completely. The terms over T vanish. The sum over all finite primes is $\log|N_{E/\mathbb{Q}}(c_i)|$, which is at least $n\log(1-\delta)$ by the archimedean condition. Subtracting the marked terms proves the lower bound. It is positive because the auxiliary primes are odd, $a_i \geq 1$ and $\delta < 1/2$. □

9.12 Prescribing the splitting of the compensating primes

Retain $E = E_x$, $F = E^+$ and $n = [E : \mathbb{Q}]$ from Theorem 9.18. The field E is Galois over $\mathbb{Q}$. For $c \in E^\times$ write

$$h_f(c) = \sum_{\mathfrak{p}} |v_{\mathfrak{p}}(c)| \log N\mathfrak{p}.$$

Here $v_{\mathfrak{p}}$ assigns one to a uniformizer. We shall realize the prescribed flag with every additional nonzero valuation over a rational prime splitting completely in E.

Lemma 9.19 (Primes dividing a linear factor in a Galois field)**.** *Let θ be an integral primitive element of a finite Galois extension $E/\mathbb{Q}$, and let f be its minimal polynomial. If a rational prime $\ell \nmid \operatorname{disc}(f)$ divides a nonzero integer $N_{E/\mathbb{Q}}(a-\theta)$ with $a \in \mathbb{Z}$, then ℓ splits completely in E.*

Proof. There is a prime $\mathfrak{l}$ over ℓ at which θ has residue a. The prime ℓ is unramified because the field discriminant divides $\operatorname{disc}(f)$. All the conjugates of θ have distinct residues at $\mathfrak{l}$, since the product of their squared differences is a unit there. Arithmetic Frobenius at $\mathfrak{l}$ carries the residue a to $a^\ell = a$ in the residue field. Its value on θ must therefore be θ. As θ generates E, Frobenius is the identity. In a Galois extension this gives residue degree one at every prime over ℓ. □

Theorem 9.20 (A rational flag with all new valuations at split primes)**.** *Fix the two auxiliary primes, the terminal prime T, the signed characters and the precision $0 < \delta < 1/2$ of Theorem 9.18. Let M be any positive integer coprime to g_1g_2 and divisible by T^b for some $b \geq 1$. On the same fields and intervals, for all sufficiently large x, one can choose a single rational flag diagonal with the prescribed leading residues at both auxiliary primes and $|\sigma(c_i)-1| \leq \delta$ at every embedding. At every $\mathfrak{p} \mid M$ it satisfies $c_i \equiv 1 \bmod M\mathcal{O}_{E,\mathfrak{p}}$. Every rational prime supporting a finite valuation of any c_i splits completely in E. With*

$$L = 2 + \log\Delta_E + n\log(2n) + \log(g_1g_2) + \log M + \log(1/\delta),$$

one has, simultaneously in $1 \leq i \leq s$,

$$h_f(c_i) \ll n^2(n^2+s+2)L$$

with an absolute implied constant. In particular one may avoid any prescribed finite set of unmarked rational primes. When $\log M$ and $\log(1/\delta)$ are bounded by fixed powers of x, the displayed height bound is polynomial in x.

Proof. Theorem 7.11 gives $\log g_j \asymp_\varepsilon x^\varepsilon$, while n is polynomially bounded, so $g_1, g_2 > n$ for large x. At each $g = g_1, g_2$ prescribe distinct elements of $\mathbb{F}_g$ at the n primes of E over g. At every rational prime dividing M or at most n, prescribe zero for θ modulo the rational prime. These data specify a class in $\mathcal{O}_E/Q_0\mathcal{O}_E$ by the Chinese remainder theorem, where Q_0 is the product of the distinct rational primes involved. We have $\log Q_0 \leq \log(g_1g_2) + \log M + n\log(2n)$. Lemma 9.17, with target zero, supplies an integral representative θ with $\log A \ll L$, where

$A = \max(2, \max_\sigma |\sigma\theta|)$. Its monic minimal polynomial f has at least n distinct roots modulo g_1, one at each prescribed residue, and therefore has degree n. In particular θ is primitive. The differences between conjugates give

$$\log|\operatorname{disc}(f)| \le n(n-1)\log(2A).$$

At either auxiliary prime f has distinct roots, so its discriminant is prime to g_1g_2.

Fix i and put $a_i = s+1-i$. For each $\mathfrak{p}$ above either auxiliary prime g, let $w_\mathfrak{p} \in \mathbb{F}_g^\times$ be the required residue of $g^{a_i}c_i$ in the construction of Theorem 9.18. We construct one factor $(u-\theta)/(v-\theta)$ for this prime. In $E_\mathfrak{p} = \mathbb{Q}_g$ write $\theta_\mathfrak{p}$ for the image of θ. Prescribe the rational integers u, v by

$$u \equiv \theta_\mathfrak{p} + g\widetilde{w}_\mathfrak{p} \pmod{g^2}, \qquad v \equiv \theta_\mathfrak{p} + g^{a_i+1} \pmod{g^{a_i+2}},$$

using any integral lift of $w_\mathfrak{p}$. The ratio has valuation $-a_i$ and the specified leading residue at $\mathfrak{p}$. At every other prime above the same g, the distinct residues of θ make both differences units and the ratio is one modulo g. At the other auxiliary rational prime choose the same residue for u, v outside the n roots of f. This makes every corresponding ratio one modulo that prime.

Impose $u \equiv v \equiv 1 \bmod M$. For all primes dividing M, the two differences are units since θ has residue zero. This gives the stated congruence for the ratio, to the full precision of M. For each remaining prime $\ell \mid \operatorname{disc}(f)$, choose the same residue for u, v outside the roots of f modulo ℓ. If $\ell > n$ this is possible by degree. If $\ell \le n$, the choice of θ gives $f(X) \equiv X^n \bmod \ell$, so residue one works. All these congruences have rational integer solutions by the Chinese remainder theorem. The moduli for both integers can be replaced by one common multiple J with

$$\log J \le (s+2)\log(g_1g_2) + \log M + \log|\operatorname{disc}(f)|.$$

For example take the least common multiple of M, $(g_1g_2)^{s+2}$ and the product of the primes dividing the discriminant. At primes already covered by M the specified residue one applies.

Choose representatives $u_0, v_0 \in [0, J)$. Add to both the least multiple H of J which is at least $2A + 8nJ/\delta$. In every embedding the differences from θ are nonzero and

$$\left|\frac{u-\sigma\theta}{v-\sigma\theta} - 1\right| \le \frac{J}{H-A} \le \frac{\delta}{8n}.$$

No prime dividing $\operatorname{disc}(f)$ divides either norm. Every prime that does divide either norm therefore splits completely by Lemma 9.19. The sum of the two logarithms of the absolute norms is at most $2n\log(H+J+A)$. It bounds the absolute finite valuation sum of the ratio because the numerator and denominator are algebraic integers. The definitions of H, J, A give $\log(H+J+A) \ll (n^2+s+2)L$.

Now multiply these $2n$ factors to make c_i. At each auxiliary prime of E exactly one factor prescribes its leading residue and all the others are one in its residue field. Multiplication preserves the congruences at M. At each embedding the distance of the product from one is at most $(1+\delta/(8n))^{2n} - 1 \le e^{\delta/4} - 1 \le \delta$. Adding the height bounds proves the estimate. Take the multipliers c_i on the isotropic lines and $\bar{c}_i^{-1}$ on their duals as in Theorem 9.18. The original bounds on n, s, $\log(g_1g_2)$ and $\log\Delta_E$ give the polynomial assertion. □

9.13 An integral choice of all the unmarked root ratios

Put $g = g_1g_2$, set $c_{s+1} = 1$, and write $\rho_j = c_j/c_{j+1}$ for $1 \le j \le s$. Thus the positive half of the flag reads $c_1, \dots, c_s, 1$. Across the other half the successive ratios are the conjugates of these ratios in reverse order. At each auxiliary prime of E the matching condition prescribes

valuation -1 and a unit leading residue for every ρ_j. At unmarked places the valuations of these ratios determine whether the Bessel character can be trivial on all the original integral root groups.

Theorem 9.21 (Simultaneous integral root ratios off the two auxiliary primes)**.** *Use the fields and auxiliary residue data of Theorem 9.20. Let $s \geq 1$, $0 < \delta < 1/2$, and let M be a positive multiple of T^b, coprime to g. One can choose a rational unitary flag diagonal with the prescribed leading residues at both auxiliary primes such that every ρ_j is integral away from g and congruent to one modulo M. All primes with nonzero valuations split completely in E. There is a single real number $W > 1$ for which*

$$\left| \frac{\sigma(c_i)}{W^{s+1-i}} - 1 \right| \leq \delta \qquad (1 \leq i \leq s,\ \sigma : E \hookrightarrow \mathbb{C}).$$

In terms of $n = [E : \mathbb{Q}]$, put

$$L' = 2 + \log \Delta_E + sn \log(2sn) + \log(gM) + \log(1/\delta).$$

Then $\log(2gW) \ll (1 + sn^2)L'$ and

$$h_f(c_i) \leq (s + 1 - i)n \log(2g^2 W).$$

In particular the set of new valuations can be required to lie above the same cohomological cutoff Y as in Corollary 104.16. With its choices $b = 1$ and $\delta = \exp(-x^B)$ and no further level congruences, both $\log W$ and $\sum_i h_f(c_i)$ are at most $\exp((1/4 + o(1))x^{\varepsilon})$.

Proof. For each j and each $\mathfrak{p} \mid g$ let $w_{j,\mathfrak{p}}$ be the specified residue of $g\rho_j$. These data come by taking ratios of the desired diagonal entries on the positive lines at each prime of E. Products of the successive ratios recover exactly the prescribed leading residues of all c_i since the middle line has multiplier one.

First construct a primitive algebraic integer ω with $\log \max(2, \max_\sigma |\sigma\omega|) \ll L'$. At the n primes over g_1 prescribe distinct residues, which is possible because $g_1 > n$. Apply Lemma 9.17 modulo g_1 with archimedean target zero. Its height bound gives the assertion, and the degree is n because the minimal polynomial has n different roots modulo g_1.

For each j prescribe $\theta_j \equiv -w_{j,\mathfrak{p}}$ at each $\mathfrak{p} \mid g$, $\theta_j \equiv -g \bmod M$ and $\theta_j \equiv 0 \bmod \ell$ for every rational prime $\ell \leq sn$ not dividing gM. All these congruences specify one class modulo the integer

$$Q = gM \prod_{\ell \leq sn,\ \ell \nmid gM} \ell,$$

where the product is over primes. By Lemma 9.17, choose an integral representative θ_j^0 with logarithmic maximum norm $O(L')$, using $\log Q \leq \log(gM) + sn \log(2sn)$. Among the integers $k = 0, \ldots, n(n-1)/2$ choose one for which $\theta_j = \theta_j^0 + kQ\omega$ has distinct conjugates. Each pair of embeddings forbids at most one k because their values on ω differ. There are more available integers than pairs. The congruences are preserved and $A = \max(2, \max_{j,\sigma} |\sigma\theta_j|)$ still satisfies $\log A \ll L'$. Write f_j for the monic minimal polynomial of θ_j. It has degree n and $\log |\operatorname{disc}(f_j)| \leq n(n-1) \log(2A)$.

We choose one rational integer a for all the numerators. Impose $a \equiv 0 \bmod gM$. For primes there, all θ_j are units at every place above the prime, so no norm $N(a - \theta_j)$ can be divisible by it. At every other prime ℓ dividing a $\operatorname{disc}(f_j)$, exclude the roots of all the f_j modulo ℓ. If $\ell > sn$, at most sn residues are excluded. If $\ell \leq sn$, all the θ_j are divisible by ℓ and their

polynomials are $X^n \bmod \ell$, so residue one is available. The Chinese remainder theorem gives a residue class for a modulo an integer J with

$$\log J \le \log(gM) + sn(n-1)\log(2A).$$

Let $\eta = \delta/(4s)$ and take the first positive a in this class exceeding $\max(2g, A/\eta)$. It is at most this maximum plus J. In particular $W = a/g > 1$ satisfies the stated logarithmic bound.

Set $\rho_j = (a - \theta_j)/g$. At $\mathfrak{p} \mid g$ its valuation is -1 and $g\rho_j$ reduces to $w_{j,\mathfrak{p}}$. At primes over M the numerator is $g \bmod M$, giving the full congruence. At all other primes the denominator is a unit. A prime dividing any numerator norm divides none of the polynomial discriminants by the choice of a. It splits completely by Lemma 9.19. At an embedding, $|\sigma(\rho_j)/W - 1| \le A/a \le \eta$. Now put $c_i = \prod_{j=i}^{s} \rho_j$. The error relative to W^{s+1-i} is at most $(1+\eta)^s - 1 \le e^{\delta/4} - 1 \le \delta$. These coordinates and their inverse conjugates define the desired unitary diagonal.

The negative valuations of each ρ_j contribute exactly $n \log g$ to its absolute finite valuation sum. Its signed sum is $\log|N(\rho_j)| \le n\log(2W)$ by the archimedean estimate. Adding twice the negative contribution gives $h_f(\rho_j) \le n\log(2g^2W)$. Subadditivity proves the height bound for c_i. To avoid primes through Y, take M to be T times the product of all primes up to Y outside gT. The same estimate $\log M \le \log T + Y\log(2Y)$ as in Corollary 104.16, together with the polynomial bounds on n, s, gives both final bounds. □

9.14 The centered prime sum for the actual period weights

Keep the K_d-oriented field, translate u, and ordinary word weights of Corollary 9.15. Write $h = |T_{\mathrm{hi}}|$, $\delta = h_2/n$, $a = \lceil xb \rceil$, and $X' = \exp(x^{\varepsilon/3})$. Denote by ξ_q, ξ_d the primitive quadratic characters of K_q, K_d, and by ξ_{qd} the primitive character of their product. We use $D_\chi = L'(1,\chi)/L(1,\chi)$. The centered count Δ_u is the sum of $C_d(t,a,e_i) - M_d(a,e_i)$ over all $t \in T_{\mathrm{hi}}$ and all m edges. Define the explicit real number

$$A_{\mathrm{sig}}(x) = \frac{hma}{2}\{\log x + (1-\delta)(\log n + x^{\varepsilon/3}) - 1 - \gamma - D_{\xi_q} + D_{\xi_d} + D_{\xi_{qd}}\}.$$

Theorem 9.22 (The complementary quadratic sign for the word weights)**.** *At every x on the common intervals and uniformly over all translates, one has, for every fixed $A > 0$ as $x \to \infty$,*

$$\Delta_u = 2 \sum_{\substack{p \ge 5 \text{ unramified in } E_x \\ p \text{ split in } K_q, \text{ inert in } K_d}} U_p \log p - A_{\mathrm{sig}}(x) + o_{\psi,\varepsilon,A}(b/x^A).$$

Here U_p is the sum for this translate in Section 108.4. The prime sum has no term with $p > 2nxX'$.

Proof. At a fixed edge write $w_\alpha = e_{i,\alpha}$. Let C_{--} be the part of the count from group elements nontrivial on both quadratic fields. This definition uses the inertia coset average also at ramified primes. For a fixed high t, the map $g \mapsto t + c_E + g$ carries this coset bijectively onto Φ_0. The sign assertion holds for either parity of the original ψ, since complex conjugation reverses both quadratic signs. Apply character orthogonality (6.1) before summing over prime powers. As in the proof of Proposition 6.1, the part of C_{--} with a given g, ξ is

$$(2n)^{-1}\overline{\xi(g)}\, w_\alpha R_{\xi^*}(a/w_\alpha), \qquad \alpha = t + c_E + g.$$

Here R is the uncentered ramp sum in (1.1). For $\alpha \in \Phi_0$ we have $w_\alpha \le 2b'$ eventually, as $b' = xb/X'$ dominates the rank shifts and the largest weight there is at most $\lceil b' \rceil + \lfloor b'/x \rfloor + 1 + 2mH_0$. Thus $a/w_\alpha \ge X'/2$. Lemma 7.1 applies to every character of this field, including those of a

shortened cyclotomic layer by inflation to the larger field. After replacing each R by its main term, the error at this edge and t is $O_{\psi,\varepsilon}(na\exp(-cx^{\varepsilon/8}))$, since there are n group elements in the coset and $4n$ characters.

For the principal character the main term at α before division by $2n$ is $a(\log a-\log w_\alpha-1-\gamma)$. For a nonprincipal character it is $-aD_{\xi^*}$ times the character coefficient. A nontrivial character on the subgroup fixing both quadratic fields sums to zero on this coset. The three remaining nonprincipal characters have respective values $-1,-1,1$. Consequently the main term of C_{--} is

$$\frac{a}{2}\left\{\log a-1-\gamma-\frac{1}{n}\sum_{\alpha\in\Phi_0}\log w_\alpha+D_{\xi_q}+D_{\xi_d}-D_{\xi_{qd}}\right\}.$$

Subtract this from $M_d(a,w)$ in Proposition 6.1. The result is

$$\frac{a}{2}\left\{\log a-1-\gamma-\frac{1}{n}\sum_{\alpha\in\Phi_1}\log w_\alpha-D_{\xi_q}+D_{\xi_d}+D_{\xi_{qd}}\right\}.$$

On Φ_1 the distinguished h_2 weights differ from b by at most $2+2mH_0$; the other weights differ from $w''=b/(nX')$ by at most this amount. As they are all at least their respective reference values, the mean of their logs differs from $\delta\log b+(1-\delta)\log w''$ by at most $(2+2mH_0)/w''$, using $\log(1+v)\le v$. Also $\log a=\log x+\log b+O(1/(xb))$. Multiplication by hma makes these rounding errors $O(x^BX')$ for a fixed B; the error from the character sums is $O(hmna\exp(-cx^{\varepsilon/8}))$. Both estimates are $o(b/x^A)$. This gives $A_{\mathrm{sig}}(x)$ for the total complementary main term.

The remaining coset consists of elements split in K_q and inert in K_d. At an unramified prime an even Frobenius power has trivial quadratic signs, while an odd power retains the signs of that prime. Its exact contribution is therefore the displayed $2U_p\log p$ by (54.1). The primes $2,3$ and the fixed conductor primes split in K_d in the chosen family, as in the proof of Theorem 54.26, so have no contribution in this coset. The other ramified primes can only be the two auxiliary quadratic conductor primes. For each of them, the inertia average is at most one after removing the factor two and the sum of a/p^j is $a/(p-1)$. Their total contribution is $o(b/x^A)$ since those primes grow faster than every power of x and hma/b is polynomial in x. Finally $e_{i,\alpha}\ge b/(nX')$ on Φ_1, so every active prime power is at most $2nxX'$. □

To describe the finite denominator term explicitly, keep the q independent columns and the set of good signal primes $\mathcal{S}$ from Corollary 108.13. For each $D'=S_0,eS$, choose finite generators of the integral test module D' over $\mathcal{O}_J$. They exist because the class coefficient lattices are finitely generated and $\mathcal{O}_J$ is Noetherian. They can be obtained on the saturated space by intersecting the integral lattice with the kernel of $1-e$, and on the image space by applying e to generators of S.

Proposition 9.23 (An input bound for the outside-prime loss)**.** *For each column i and generator of D', expand its signed period by Theorem 60.32, using independent symbols for the undivided jets of every edge and class coset. Write β for a nonzero scalar coefficient of a term, and $J_1,\dots,J_m$ for its jet symbols, one per edge. Set*

$$H_{i,w}^{D'}=\max\left(\{0\}\cup\left\{-v_w(\beta)+\sum_{j=1}^{m}\Gamma_w(J_j):\text{terms for these generators}\right\}\right),$$

where $\Gamma_w(J_j)$ is the nonnegative finite bound of Theorem 100.13 for that symbol's section, coordinates and order. A conjugated symbol uses the corresponding conjugate place. Define

$$B_{\mathrm{data}}=\sum_{p\notin\mathcal{S}}\log p\sum_{w\mid p}\omega_w\sum_{i=1}^{q}(H_{i,w}^{S_0}+H_{i,w}^{eS}).$$

Then $B_{\mathrm{out}} \le B_{\mathrm{data}}$, and B_{data} is finite and determined before evaluating any theta derivative.

Proof. In fixed rational class frames expand the successive determinant twists and substitutions in (54.22). Replacing each monomial in its matrix of differential symbols by a new variable gives a polynomial linear in the variables of that edge. On composing the m maps, every resulting term contains one such variable for each edge. The finite path sum, normal-coordinate factor, restriction, and small class contraction act on these expressions by algebraic linear maps. They determine the coefficients β in the marked arithmetic normalization and include the actual input and test values and the mass factors. The local estimates for the specified CM sections bound the valuation of each evaluated term below by $v_w(\beta) - \sum_j \Gamma_w(J_j)$. Thus the valuation of the period on any generator is at least $-H_{i,w}^{D'}$. This extends by linearity to any integral linear combination of generators and after localization at w. Expanding a determinant bounds $b_{D',w}$ below by $-\sum_i H_{i,w}^{D'}$. Taking the negative parts on the outside primes proves the inequality. In Theorem 100.13 one can use the common integer denominators in Theorem 103.3 for every edge. They depend only on the finite translations, linear substitutions, fixed elliptic model and degrees. Those denominators and the finitely many nonzero coefficients β involve only finitely many negative valuations. This proves finiteness. □

Corollary 9.24 (An upper bound for the centered count)**.** *In the hypotheses of Corollary 108.13, let $E_{\mathrm{omit}} = 2\sum_{p\notin\mathcal{S}} U_p \log p$, where the sum is restricted to the unramified signal primes $p \ge 5$ in Theorem 9.22. Then on the same fields and intervals*

$$\begin{aligned}\Delta_u \ &\le \frac{1}{q[J:\mathbb{Q}]}\sum_{\sigma} \log\det G_\sigma - A_{\mathrm{sig}}(x)\\ &\quad + \frac{2L_{\mathcal{S}} + B_{\mathrm{data}}}{q} + 2\log D + \log m_G + E_{\mathrm{omit}} + o_{\psi,\varepsilon,A}(b/x^A).\end{aligned}$$

If $\mathcal{S}$ omits an active prime, its contribution to E_{omit} is at most $2hma\log p/(p-1)$.

Proof. Split the prime sum of Theorem 9.22 at $\mathcal{S}$, insert Corollary 108.13 on that set, and apply Proposition 9.23 to the outside denominators. In the omitted sum discard the nonnegative thresholds and bound the geometric series by $1/(p-1)$. □

A proof of (6.3) by this inequality requires projected columns of adequate rank and a uniform upper estimate for the displayed excess over $A_{\mathrm{sig}}(x)$, with the finite costs included. The explicit main term and the signs have been computed here at the original word weights.

9.15 The ramified prime in the polynomial-conductor comparison

Choose the polynomial-conductor intervals of Theorem 12.15. At their point x, write j_x for the dyadic index in that theorem. The field and its character modulus are

$$E_x = K_\psi C_{j_x}(\sqrt{-q}, \sqrt{-d_0}), \qquad N_x = 2^{j_x+2} F q d_0 \le x^{D_0},$$

where D_0 is fixed. We use the $K_d = \mathbb{Q}(\sqrt{-d_0})$ orientation and the actual word weights of Corollary 9.15. In the construction of Theorem 12.15, both q, d_0 exceed x^{401} throughout these intervals. Indeed, in its notation they exceed H^{401W}, and $x \le V^{3/2} \le H^W$. This lower bound has a fixed exponent. We evaluate their contribution explicitly before estimating it.

Use additive notation in $G = \mathrm{Gal}(E_x/\mathbb{Q})$. Write c_E for complex conjugation, I_p for inertia and F_p for a lift of arithmetic Frobenius modulo inertia. Let ι_d be the nonidentity element of

I_{d_0}. The field $K_\psi C_{j_x}(\sqrt{-q})$ is unramified at d_0, and K_d has ramification index two there. Thus $I_{d_0} = \{0, \iota_d\}$ and ι_d acts trivially on the first field. For a translate u, put

$$W_{d_0}(u) = (\log d_0) \sum_{i=1}^{m} \sum_{t \in T_{\mathrm{hi}}} \sum_{k \geq 1} [\, a/d_0^{2k} - e_{i,t+c_E+2kF_{d_0}+\iota_d} \,]_+.$$

The weights in this formula are those for u. Even multiples of F_{d_0} are independent of the lift.

Lemma 9.25 (The ramified terms in the selected quadratic signs)**.** *In the portion of the prime count with trivial sign on K_q and nontrivial sign on K_d, the contribution from ramified primes is exactly $W_{d_0}(u)$. In particular*

$$0 \leq W_{d_0}(u) \leq \frac{hma \log d_0}{d_0^2 - 1}.$$

Proof. The character of K_d is unramified and split at 2 because $-d_0 \equiv 1 \bmod 8$. For odd primes of the fixed conductor F, and for q, the congruences of Theorem 5.6 give $-d_0 \equiv 1$ at the prime for sufficiently large intervals. Those primes have trivial K_d-sign in every inertia coset. At d_0 the character of K_q is unramified with value $(d_0/q) = (-1/q) = -1$. Here the primitive quadratic character of discriminant $-q$ is the Legendre character modulo q. The trivial K_q-sign therefore selects the even powers d_0^{2k}. In the coset $2kF_{d_0} + I_{d_0}$, the even multiple has trivial sign on both quadratic fields. Its sum with ι_d is the unique member with the required signs. Formula (6.1) gives it coefficient $2/|I_{d_0}| = 1$. The weight at t and this group element is the one displayed above. This exhausts the ramified primes. Discard the nonnegative thresholds and sum the geometric series in d_0^{-2} to obtain the upper bound. □

Theorem 9.26 (The centered identity on the polynomial-conductor family)**.** *On these fields, uniformly over the translates and for each fixed $A > 0$, the centered count satisfies*

$$\Delta_u = 2 \sum_{\substack{p \geq 5 \text{ unramified in } E_x \\ p \text{ split in } K_q, \text{ inert in } K_d}} U_p \log p - A_{\mathrm{sig}}(x) + W_{d_0}(u) + o_{\psi,\varepsilon,A}(b/x^A).$$

For the sparse sets of Theorem 6.29,

$$\frac{1}{|A_0|} \sum_{u \in A_0} W_{d_0}(u) = (1 + o_{\psi,\varepsilon}(1)) \frac{hma \log d_0}{d_0^2}.$$

Proof. For primes unramified in E_x the exact count in the selected signs is $2U_p \log p$ by (54.1). The primes $2, 3$ split in K_d on these intervals. In the other signs contributing to the count, the group element acts nontrivially on both quadratic fields. Its target $t + c_E + g$ lies in Φ_0. At every such target $a/e_{i,t+c_E+g} \geq X'/2$ for large x by the word weights. The character expansion uses primitive characters and the full inertia average (6.1), also at q, d_0.

We check the large-argument error on the chosen family. Every real primitive character of E_x without an auxiliary quadratic factor has conductor dividing $8F$, so these form a fixed finite set. With such a factor, the zero-free rectangle of Theorem 5.6 rules out the possible exceptional real zero near one. The zero-free region and zero count used in Lemma 7.1 therefore apply with $\log(2N_x) = O_{\psi,\varepsilon}(\log x)$. In detail, at $y \geq X'/2$, truncation at height $\exp(\sqrt{\log y})$ bounds the normalized contribution of the low zeros by a constant times

$$\log^3(2N_x) \exp\left(-c \min\left\{ \sqrt{\log y}, \frac{\log y}{\log(2N_x)} \right\} \right).$$

The tail is $O(\log(2N_x(3 + \exp(\sqrt{\log y})))/\exp(\sqrt{\log y}))$, using the $1/|\rho(1-\rho)|$ weights as in that lemma. The elementary terms of the explicit formula obey the same required estimate

after division by y. Since $\log y \ge x^{\varepsilon/3} - \log 2$, the total normalized error is $O(\exp(-c'x^{\varepsilon/8}))$. Multiplication by the word weight and summation in the character expansion give error $O(hmna\exp(-c'x^{\varepsilon/8}))$. The quantities $h, m, n, a/b$ have polynomial growth in x, so this is $o(b/x^A)$.

Character orthogonality and the logarithms of the word weights now give the same complementary main term as the computation in Theorem 9.22. More explicitly, subtracting the part with both signs nontrivial from the main term of one edge and one high place gives

$$\frac{a}{2}\left(\log a - 1 - \gamma - \frac{1}{n}\sum_{\mu\in\Phi_1}\log e_{i,\mu} - D_{\xi_q} + D_{\xi_d} + D_{\xi_{qd}}\right).$$

The replacement of the h_2 distinguished weights by b and the other weights by $b/(nX')$ in these logarithms has total error $O(x^B X') = o(b/x^A)$ for a fixed B. Summing thus gives $A_{\mathrm{sig}}(x)$. Including the ramified part of the selected signs by Lemma 9.25 proves the identity.

For the mean, keep $k = 1$ in W_{d_0}. For each i, t the target in Φ_1 is fixed as u varies. The proportion of translates for which it is distinguished is at most $|T_2|/|A_0|$, by translating T_2 in A_0. This is $o(1)$ for the sets in question. Indeed, their distinct integer representatives in the proof of Theorem 6.29 lie in $[0, |A_0|/(8L))$, where L tends to infinity. At an undistinguished target the weight equals $b/(nX') + O(1 + 2mH_0) = o(a/d_0^2)$ uniformly. This follows from $d_0 \le x^{D_0}$, polynomial growth of mH_0, and $X' = \exp(x^{\varepsilon/3})$. Consequently the $k = 1$ term averaged over translates is at least $(1 - o(1))hma\log d_0/d_0^2$. The lemma gives the matching upper bound. □

Corollary 9.27 (The comparison and the cohomology on the same shortened field)**.** *For the indicated polynomial-conductor fields use $-A_{\mathrm{sig}}(x) + W_{d_0}(u)$ in the centered upper bound of Corollary 108.22, including the refinements of Corollaries 98.15 and 98.18. The correction is $O_{\psi,\varepsilon}(hmb\,x^{-801}\log x)$. Corollaries 44.3 and 44.6 hold there for the faithful characters of C_{j_x}, with $f = 2^{j_x-2}$.*

Proof. The inequalities for the integral minors at unramified primes use the same signed columns, row places, and common terminal prime on the fields of Theorem 12.15. At the ramified coefficient places for the periods, the lower bound from the different and the marked differential in Theorem 108.18 already applies. It does not estimate the ramified term of the prime count. Substitute the identity just proved when converting the two period ideals to that centered count. This adds precisely $W_{d_0}(u)$. The numerical upper bound follows from $a \le 2xb$, $d_0 > x^{401}$, and $\log d_0 \ll \log x$. In particular it is $o(x^{5\varepsilon}\mathcal{B})$ in (6.3), since $\mathcal{B} \ge 2nm^2bx^{\varepsilon}$ and $h \le n$. The asymptotic for its mean also shows why an error $o(b/x^A)$ for every fixed A requires retaining this term on the polynomial family.

Finally write $o = [K_\psi : \mathbb{Q}]$. The inequalities for the degree of $K_\psi C_{j_x}$ give

$$x^{2+8\varepsilon}/o \;\le\; 2^{j_x} \;<\; 2x^{2+8\varepsilon}.$$

Thus j_x tends to infinity, the faithful conductor is $2^{j_x+2} < 8x^{2+8\varepsilon}$, and the odd part of $[E_x : \mathbb{Q}]$ divides o. These are exactly the degree and conductor inputs of Corollary 44.3. The order computation at $\ell \equiv 5 \bmod 8$ gives $f = 2^{j_x-2}$ with this index as well. The simultaneous class group bound and the two-orbit argument of Corollary 44.6 now hold for these characters on the very same E_x. □

9.16 A sparse set supporting the residual cohomology on the common fields

We use the polynomial-conductor fields E_x of Theorem 12.15 on one interval $[V, V^{3/2}]$. In particular the two auxiliary primes are fixed across this interval. Keep $P(x), b_0(x), R_{\ell,r}$ and

the logarithmic length $\mathcal{R}(x)$ of Section 6.6. For x in the interval let $t(x) = \lfloor x^{401} \rfloor$, with extreme values $t_- = \lfloor V^{401} \rfloor$ and $t_* = \lfloor V^{1203/2} \rfloor$. Define

$$\delta = 1 - \frac{1 + \log\log 2}{\log 2}, \qquad H(T) = (\log T)^{\delta} \sqrt{\log\log T}.$$

All the arguments of H below tend to infinity. For large coefficient primes the two possible inertia residues in Corollary 42.19 impose the same divisibility condition on $2r$. We count a weight once even when several primes give that condition.

Lemma 9.28 (Weights divisible after doubling by a large shifted prime)**.** *Put $Y = V^{100}$. Let $\mathcal{A}_V$ consist of the positive integers $r \le t_*$ for which $\ell - 1 \mid 2r$ for some prime $\ell > Y$. Uniformly for $t_- \le T \le t_*$,*

$$|\mathcal{A}_V \cap [1, T]| \asymp T/H(T).$$

Proof. Let $N(Z, y)$ count positive integers $n \le Z$ for which some prime $p > y$ satisfies $p - 1 \mid n$. Ford's theorem gives

$$N(Z, y) \asymp_c \frac{Z}{(\log y)^{\delta} \sqrt{\log\log y}} \qquad (3 \le y \le Z^{1-c})$$

for each fixed $0 < c < 1$ [13, Theorem 1(i)]. An integer counted by $N(T, Y)$ belongs to $\mathcal{A}_V$, since any divisor of r divides $2r$. In the other direction $r \mapsto 2r$ injects $\mathcal{A}_V \cap [1, T]$ into the integers counted by $N(2T, Y)$. Both applications allow $c = 1/2$ because $V^{100} \le \sqrt{t_-}$ for large V. Moreover $\log Y / \log T$ is bounded above and below by positive constants independent of T, V in this range. This gives both inequalities. The lower bound counts the weights allowed by the divisibility condition alone. □

Theorem 9.29 (One set of weights for all residual groups up to the cutoff)**.** *For every sufficiently distant interval of the stated polynomial-conductor family there is a single set $\mathcal{E}_V \subseteq [1, t_*] \cap \mathbb{Z}$ such that*

$$\begin{aligned} &|\mathcal{E}_V \cap [1, T]| \ll_{\psi,\varepsilon} T/H(T) \quad (t_- \le T \le t_*), \\ &\quad R_{\ell,r}(E_x) = 0 \qquad\qquad (V \le x \le V^{3/2},\ 1 \le r < t(x),\ r \notin \mathcal{E}_V,\ 2 < \ell \le P(x) \text{ prime}). \end{aligned}$$

In particular the fraction of weights which can support any such group tends to zero uniformly in x.

Proof. Use $B = 2$ in Corollary 13.20. It provides one set $\mathcal{D}_V$ for the whole interval with

$$|\mathcal{D}_V| \ll_{\psi,\varepsilon} V^{401}/(\log(2V))^2.$$

Away from it the full complementary group $M_{\ell,r}(E_x)$ is annihilated by $\ell^{\lceil V^{203} \rceil}$ for every odd prime up to $V^{199-6\varepsilon}/(\log(2V))^{3/2}$. This cutoff exceeds Y for large V, since $0 < \varepsilon < 1/52$. Also $b_0(x) \ge \lceil V^{203} \rceil$ on the whole interval for large V. At such a prime $\ell \le \min\{Y, P(x)\}$, the subgroup $U_{\ell,r}$ of $M_{\ell,r}$ is killed by $\ell^{b_0(x)}$ as well, so the quotient defining $R_{\ell,r}$ is zero.

Take $\mathcal{E}_V = (\mathcal{D}_V \cup \mathcal{A}_V) \cap [1, t_*]$. The finite exceptional set of primes in Corollary 42.19 depends only on the fixed data, so it lies below Y on sufficiently distant intervals. If $Y < \ell \le P(x)$ and $R_{\ell,r} \ne 0$, that corollary forces $\ell - 1 \mid 2r$. Thus $r \in \mathcal{A}_V$. This also accounts for the two moving primes by the same corollary. The two prime ranges give the asserted simultaneous vanishing. Finally $V^{401} \ll T$, $\log T \asymp \log V$, and $H(T) = o((\log T)^2)$. Adding the count of $\mathcal{D}_V$ to the upper estimate in Lemma 9.28 proves the uniform weight count. □

Corollary 9.30 (The remaining local mass on the same interval)**.** *At an allowed pair x, r write*

$$W_x(r) = \sum_{\substack{2<\ell\le P(x)\\ \ell \text{ prime}}} (\log \ell)\,\mathrm{length}_{\mathbb{Z}_\ell} R_{\ell,r}(E_x).$$

For a constant C depending only on the fixed data one has uniformly on the interval

$$0 \le \mathcal{R}(x) - \sum_{\substack{1\le r<t(x)\\ r\in\mathcal{E}_V}} W_x(r) \ll_{\psi,\varepsilon} V^C.$$

In particular the sufficient local estimate of Proposition 6.11 would follow from the estimate

$$\max_{\substack{1\le r<t(x)\\ r\in\mathcal{E}_V}} W_x(r) = o_{\psi,\varepsilon}\Big(\frac{\exp(x^\varepsilon/2)H(t(x))}{t(x)}\Big)$$

uniformly on arbitrarily distant intervals of this family, with the maximum of the empty set taken to be zero.

Proof. For $\ell \le P(x)$ the sum over $\mathcal{E}_V$ includes all nonzero terms by Theorem 9.29. For $\ell > P(x)$ the residual group is a subgroup of the full complementary group. The weighted total for the latter is $O_{\psi,\varepsilon}(V^C)$ by Corollary 42.19. This proves the first formula, including its sign. For the last implication multiply the proposed maximum by the count at $T = t(x)$. The resulting bound is $o(\exp(x^\varepsilon/2))$, and the polynomial error has the same property because $x \ge V$. It follows that $\mathcal{R}(x) \le \frac{1}{4}\exp(x^\varepsilon/2)$ uniformly on such intervals once V is large enough. □

The individual values of $W_x(r)$ at the retained weights remain to be bounded. Lemma 9.28 also explains the dependence on arithmetic information beyond the two inertia residues. The set of weights satisfying those residues for at least one large prime already has order $T/H(T)$, whether or not its strict groups are nonzero.

9.17 Full strict orders and the primes of the projected period

Fix ψ and $0 < \varepsilon < 1/52$. Take one of the polynomial-conductor intervals $[V, V^{3/2}]$ of Theorem 12.15. Write E_* for the upper endpoint field, $n_* = [E_* : \mathbb{Q}]$, N_* for its cyclotomic modulus, $R = \lfloor V^{1203/2}\rfloor$, and $Y = V^{100}$. Here $\mathcal{S}_{p,r}(L)$ denotes the full strict group over L for the twist $-r$, with both signs included. All constants below may depend on ψ, ε.

Keep the fixed field $K = \mathbb{Q}(\sqrt{-s})$ and the split test prime u of Proposition 29.3. Let $D(V)$ be the integer of Theorem 37.44. For each r use the integral special-value determinant $D_{r+1}(E_*)$ of Lemma 11.3, formed on E_* with its modulus N_*. Define

$$B(V) = \prod_{1\le r<R} |D_{r+1}(E_*)|, \qquad J(V) = D(V)^2 B(V).$$

All factors are nonzero and are fixed before the two auxiliary period primes and the terminal prime are chosen. Indeed $D(V)$ uses the fixed split tests, the weights below R, and the characters of E_*. The other factors are determined by E_* and the two fixed denominator-clearing primes in Lemma 11.3. In particular $\log J(V) \ll V^C$ for some C. For $D(V)$ this is Theorem 37.44. For $B(V)$ the lemma gives $O(n_* R^2(n_* + \log(2N_* R)))$.

Lemma 9.31 (An integer controlling the full strict order at separated twists)**.** *There is a fixed finite set of primes $\mathcal{B}$ such that for every sufficiently distant interval, every prime $p \notin \mathcal{B}$, and every number subfield $L \subseteq E_*$,*

$$\prod_{\substack{1\le r<R\\ p-1\nmid 2r}} |\mathcal{S}_{p,r}(L)| \quad \textit{divides} \quad p^{v_p(J(V))}.$$

In particular the product of the full orders on any such L, over all primes $p > 2R+1$ and all $1 \le r < R$, divides $J(V)$.

Proof. Take $\mathcal{B}$ to contain the primes below eleven and the prime divisors of $2suF[K_\psi : \mathbb{Q}]|\operatorname{Cl}(K)|$, with F the conductor of ψ. The degree n_* divides a power of two times $[K_\psi : \mathbb{Q}]$. Thus $p \nmid n_*$ outside $\mathcal{B}$. First work on E_*. At a prime also coprime to N_*, Theorem 42.13, applied at $x = V^{3/2}$, bounds the sum of the complementary lengths over the displayed weights by $2v_p(D(V))$. The only other prime divisors of N_* outside $\mathcal{B}$ are the quadratic primes q, d_0. Theorem 42.18 gives the same bound there even after including all weights.

For the opposite sign at a weight r, put $n = r+1$ and $e_n = (1+(-1)^n c)/2$. By Lemma 11.2 the dual of this strict group embeds in the e_n-part of degree-two p-adic cohomology. In that summand the integral special-value element $e_n A\Theta_n$ of Lemma 11.3 lies in the zeroth Fitting ideal. Indeed A is an integer multiple of the annihilator W in Lemma 11.2. The abelian group ring $e_n\mathbb{Z}_p[\mathrm{Gal}(E_*/\mathbb{Q})]$ is a product of unramified discrete valuation rings since $p \nmid n_*$. Over each factor the valuation of a member of the zeroth Fitting ideal is at least the module length, by diagonalizing a presentation over that ring. Multiplying by the residue degrees and adding bounds the $\mathbb{Z}_p$-length by the valuation of the determinant of multiplication by $e_n A\Theta_n$. After extending scalars its eigenvalues are the character values whose product is $D_n(E_*)$. Thus the sum of the critical lengths over all these weights is at most $v_p(B(V))$. Adding the two signs gives the asserted divisibility on E_*.

Restriction from L to E_* maps strict groups to strict groups. It preserves vanishing at p and unramifiedness away from p by restriction to the local subgroups. Corestriction after restriction on cohomology is multiplication by $[E_* : L]$, a unit at p, so restriction is injective. The order at every weight on L therefore divides the order on E_*. For $p > 2R+1$, every weight in the interval satisfies $p-1 \nmid 2r$. Also all primes in the fixed set are eventually smaller. Multiplying the prime-power divisibilities then gives the last assertion. □

For each odd prime $p \le 2R+1$, let $\mathcal{E}_{p,\tau}$ be the finite ordinary class group test of Theorem 15.22 at cutoff R, with $\tau = \kappa_p^{-r}|_\Delta$ as there. Define a finite set of prime and weight pairs by

$$\mathcal{F}_V = \{(p,r) :\ 2 < p \le 2R+1 \text{ prime}, 1 \le r < R,\ r \in \mathcal{E}_{p,\tau}, (p \le Y \text{ or } p-1 \mid 2r)\}.$$

Let $\mathcal{W}_V$ be its projection onto the weights.

Theorem 9.32 (A polynomial bound for both signs outside the finite tests)**.** *One has*

$$\sum_{p>2} \sum_{\substack{1 \le r < R \\ (p,r) \notin \mathcal{F}_V}} \log|\mathcal{S}_{p,r}(E_*)| \ll V^C$$

for some C, where the outer sum is over primes. This same bound holds for the groups over every E_x, even up to the endpoint weight cutoff R. Uniformly for $\lfloor V^{401} \rfloor \le T' \le R$,

$$|\mathcal{W}_V \cap [1,T']| \ll \frac{T'}{(\log T')^\delta \sqrt{\log\log T'}}, \qquad \delta = 1 - \frac{1+\log\log 2}{\log 2}.$$

Proof. The full critical-sign sum on E_* is polynomial by Theorem 11.1 at the upper endpoint. For the complementary sign first take $p \le 2R+1$ with either $p \le Y$ or $p-1 \mid 2r$. Outside $\mathcal{F}_V$ its length is at most H_τ by Theorem 15.22. Uniformly over this entire polynomial prime range, the proof of Corollary 15.23 gives

$$H_\tau \log p \ll (1+\log(2n_*R))\, n_*^2 p^2 \log(2N_* n_* p).$$

There are at most $(2R+1)R$ pairs here. Both the number of pairs and their individual log orders are polynomial in V. At every remaining prime and weight we have $p > Y$ and $p-1 \nmid 2r$. All such primes lie outside the fixed set of the lemma. Summing its primewise bound costs at most $\log J(V)$. This proves the first assertion. At odd primes restriction to E_* is injective for every E_x, since the index is a power of two. The bounds on the whole of E_* thus dominate all these sums term by term.

Weights in $\mathcal{W}_V$ coming from primes at most Y number $O(V^{306+24\varepsilon}\log(2V))$ by Corollary 15.23. This is smaller than the displayed bound even at its shortest prefix. Every remaining weight satisfies $p-1 \mid 2r$ for some $p > V^{100}$. The upper estimate of Lemma 9.28, applied to that prefix, gives exactly the second term needed. □

Corollary 9.33 (Testing the exceptional local length at a finite precision)**.** *Fix x on the interval and a real threshold $H \geq 0$. Write $b = b_0(x)$ as in Section 6.6. For each prime occurring in $\mathcal{F}_V$ put*

$$m_p = b + 1 + \lfloor H/\log p \rfloor, \qquad j_p = m_p + \lfloor \log_p n_* \rfloor.$$

For a pair with $r < t(x)$ let $C = Q_{r,m_p}(\mathrm{Cl}(\mathcal{O}_{F_{j_p}}[1/p])\{p\})$ over E_x as in Theorem 15.19. On this finite group let $J_{p,r}^{\vee}$ be the field average dual to $J_{p,r}$ in Section 6.6, or zero when that operator is zero. Set

$$W^{\mathrm{fin}}(x,H) = \sum_{\substack{(p,r)\in\mathcal{F}_V \\ r<t(x)}} \log|p^b(1-J_{p,r}^{\vee})C|.$$

The logarithmic sum of the orders of the actual residual groups on these pairs is at most H if and only if $W^{\mathrm{fin}}(x,H) \leq H$. In that case the two sums are equal. The total contribution to $\mathcal{R}(x)$ from all other pairs is $O(V^C)$.

Proof. For large V we have $2R+1 < P(x)$ at every x in this interval, so the residual group in each tested pair uses precision b. Write A for the full complementary dual at this pair. The common-field theorem identifies $C = A/p^{m_p}A$. The dual field action in the untwisted ideal-class coordinates is represented by a lift g acting on ideals, multiplied by $\kappa_p(g)^r$. This compensates the relation $\delta = \kappa_p(\delta)^{-r}$ for lifts fixing E_x, and is the action obtained by dualizing the coefficient twist in Theorem 15.1. In particular the average over the finite relative field group acts on the finite quotient. Its denominator is a power of two. The dual average is the average over the inverses of the same elements. Projection onto $B = (1-J_{p,r}^{\vee})A$ commutes with reduction, as it is a direct summand. Moreover the dual of the quotient by the subgroup killed by p^b is p^bB, by the cyclic decomposition of a finite abelian group.

The finite summand thus uses $p^b(B/p^{m_p}B)$. If the reduction shortens a cyclic factor of B, that factor alone contributes $(m_p - b)\log p > H$ even to the finite sum. Otherwise the entire contribution agrees with its full value. All summands are nonnegative and reducing them can only decrease their orders. This proves both implications and the equality assertion. Outside $\mathcal{F}_V$ each residual order is at most the full order in Theorem 9.32, which gives the last estimate. □

Corollary 9.34 (Full vanishing at the three primes of the signed periods)**.** *At each x on every sufficiently distant interval, the two auxiliary primes g_1, g_2 and the terminal prime T can be chosen for the original fields, levels and words so that*

$$\mathcal{S}_{p,r}(L) = 0 \quad (p \in \{g_1, g_2, T\},\ L \subseteq E_*,\ 1 \leq r < R).$$

For the selected words the terminal window still contains $Y_x^{1-o(1)}$ eligible choices of T, where Y_x is its scale in Theorem 9.13. At these same terminal primes the signed congruences of

Theorem 112.2 hold at every precision on the period fields for which the primes have been selected.

Proof. Exclude the prime divisors of $J(V)$ before any of the three choices. There are at most $\log J(V)/\log 2 \ll V^C$ of them. The first two primes are selected in $(U, 2U]$, with $U = (kb_x)^8$ and $b_x = \exp(x^\varepsilon)$, by Proposition 6.30 and Theorem 7.11. There are $U^{1-o(1)}$ candidates for the first prime. After fixing it, the disjointness condition on the two sets $\mathcal{B}_r(g_i)$ rules out at most $O(r^3(1 + U^{3/4}))$ integers for a polynomially bounded rank parameter r, by the proof of Theorem 7.11. The extra polynomial exclusion leaves eligible choices for both. Use the same words and admissible factors as in Theorem 9.13 after fixing the auxiliary pair. Its terminal candidate count has $\log Y_x \gg mx^\varepsilon$ for the prescribed chain length m. Subtracting $O(V^C)$ again leaves $Y_x^{1-o(1)}$ candidates. The selected primes all exceed $2R+1$, uniformly since $x \geq V$ and R is polynomial in V. The lemma now gives full vanishing on every stated subfield and weight. These exclusions preserve the period congruences and splitting conditions used in Lemma 110.1, so the connected-coordinate calculation applies to the very same terminal choices. The period choices themselves may depend on x. □

The remaining exceptional orders in Corollary 9.33 need a uniform upper bound. On the period side the required rank and archimedean height concern the signed columns after projection at every translate in Section 6.6.

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